A delayed reaction-diffusion viral infection model with nonlinear incidences and cell-to-cell transmission

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Abstract: In this paper, we propose a reaction-diffusion viral infection model with nonlinear incidences, cell-to-cell transmission, and a time delay. We impose the homogeneous Neumann boundary condition. For the case where the domain is bounded, we study the well-posedness, followed by the local stability of homogeneous steady states. We also investigate the threshold dynamics which are shown to be completely characterized by the basic reproduction number. For the case where the domain is the whole Euclidean space, we study the existence of traveling wave solutions by using the cross-iteration method and Schauder's fixed point theorem.

Keywords: reaction-diffusion equation; cell-to-cell transmission; absorption effect; time delay; traveling wave solutions

1 Introduction

Infectious diseases such as cholera, AIDS, and malaria have posed a great threat to human health. In order to study the spread and control of infectious diseases, a large number of mathematical models have been developed[1, 2]. These models have been proved to be a valuable way in understanding the complex interaction between the immune response and virus infection. Spatial diffusion models of virus infection have also been developed. For example, Komarova constructed a virus infection model with a diffusion term to simulate the virus-antibody interaction in order to study the evolutionary competition of split viruses [3]. Based on the classical virus dynamics model (a system consisting of three ordinary differential equations [4, 5]), the random movement of the virus is considered [6]. Nonlinear reaction-diffusion models can describe various physical and biological phenomena. For unbounded domains, traveling wave solutions are important because they can determine the long-term behavior of other solutions, accounting for the transition

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phase between different physical system states, propagation modes and fields of invasive species in population biology [7–9]. The existence of traveling wave solutions in delayed reaction-diffusion systems has attracted great interest [10–13]. Wang et al. [6] proposed an HBV infection model with viral diffusion and proved the existence of traveling wave solutions by the geometric singular perturbation method.

McCluskey and Yang [14] constructed a virus infection model including diffusion, time delay and a general incidence. They studied the global asymptotic stability of the steady state using Lyapunov functional. In [15], Zhang and Xu established the existence of traveling wave solutions for a delayed HBV infection model with the Beddington-DeAngelis incidence using the cross iteration method and the Schauder's fixed point theorem. Viral dynamics and spatial structure have been extensively studied [14, 16]. Considering that virus diffusion consists of random diffusion and chemotactic movement, Wang and Ma [17] proposed a dynamic model of spreading virus infection with nonlinear functional response, chemotaxis and absorption effect,

$$\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= D\Delta u(x,t) + \xi - g(u(x,t), v(x,t))v(x,t) - du(x,t),\\
\frac{\partial \omega(x,t)}{\partial t} &= D\Delta \omega(x,t) + g(u(x,t), v(x,t))v(x,t) - ah(\omega(x,t)),\\
\frac{\partial v(x,t)}{\partial t} &= D_0\Delta v(x,t) + \nabla(v\chi_2(\omega, v)\nabla\omega) + kh(\omega(x,t)) - \mu v(x,t) - g(u(x,t), v(x,t))v(x,t),
\end{aligned} \tag{1.1}$$

where $u(x,t)$, $\omega(x,t)$, and $v(x,t)$ represent the densities of uninfected cells, infected cells, and viruses at location $x \in \Omega \subseteq R^n$ and at time t, respectively. Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$. The Laplacian operator and the diffusion coefficient of both uninfected cells and infected cells are denoted by Δ and D, respectively. D_0 is the free diffusion coefficient of viruses. The term $g(u(x,t), v(x,t))v(x,t)$ represents the infection of uninfected cells by viruses. The same term was subtracted in the third equation in view of viral absorption during infection [18, 19]. The death rate of infected cells depends on the state of infected cells, which is given by the nonlinear removal rate $h(\omega(x,t))$. The term $v\chi_2(\omega, v)\nabla\omega$ describes the chemotactic flux of viruses, where the function $\chi_2(\omega, v)$ represents the chemotactic response. Table 1 summarizes the biological meanings of the other parameters. In [17], the authors mainly studied the well-posedness and linear stability of the model. They showed the threshold dynamics in the absence of chemotaxis. In diffusive viral models, it is needed to consider the intracellular delays in the viral life cycle. The entry of viruses to uninfected cells will trigger a series of events, which will eventually enable infected cells to produce new viruses. The above system (1.1) assumes that this process occurs instantaneously. However, time delays may affect the dynamics [16, 20]. Xu et al. [21] built an HBV infection model with time delay and saturated incidence, and studied the global stability of steady states. In order to explore whether hyperthermia can explain the decline of $CD4^+$ T cells during HIV infection, Wang et al. [22] studied a time periodic reaction-diffusion model with spatial heterogeneity and incubation period.

Table 1: Biological meanings of parameters in (1.1)

Parameter	Biological description
ξ	The production rate of uninfected cells
d	The death rate of uninfected cells
a	The death rate of infected cells
k	The production rate of viruses
μ	The clearance rate of free viruses

Although some studies have shown that cell-to-cell transmission is efficient in viral transmission, many studies have only focused on the cell-free virus infection. During cell-to-cell transmission, viral particles can be simultaneously transferred from infected cells to uninfected cells through virological synapses. Sigal et al. [23] found that this transmission mode can reduce the effectiveness of antiretroviral therapy. Martin et al. [24] showed that the risk of cell-to-cell transmission being affected by neutralizing antibodies or cytotoxic T lymphocytes is low. Wang et al. [25] studied the effect of the infection age and infection ability of infected cells in cell-to-cell transmission. In addition, some studies investigated within-host models that include both cell-free viral infection and cell-to-cell transmission [26–29].

Some studies have shown that viral transmission through cell-to-cell is more effective than cell-free virus infection, because cell-to-cell transmission avoids some biophysical and kinetic obstacles [30, 33]. In order to study the influence of cell-to-cell transmission on virus dynamics, we propose a new mathematical model, combining nonlinear incidences and the two virus transmission modes. Time delay is also included to account for the time for an infected cell to be productive. From model (1.1) without considering the chemotaxis of virus, we obtain the following model

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} =& D\Delta u(x,t) + \xi - f(u(x,t),\omega(x,t))\omega(x,t) - g(u(x,t),v(x,t))v(x,t) - du(x,t),\\ \frac{\partial \omega(x,t)}{\partial t} =& D\Delta \omega(x,t) + e^{-m\tau} f(u(x,t-\tau),\omega(x,t-\tau))\omega(x,t-\tau)\\ &+ e^{-m\tau} g(u(x,t-\tau),v(x,t-\tau))v(x,t-\tau) - ah(\omega(x,t)),\\ \frac{\partial v(x,t)}{\partial t} =& D_0\Delta v(x,t) + kh(\omega(x,t)) - \mu v(x,t) - g(u(x,t),v(x,t))v(x,t),\end{aligned} \tag{1.2}$$

for $t > 0,\ x \in \Omega$, we consider the homogeneous Neumann boundary conditions as follows

$$\frac{\partial u}{\partial \vec{n}} = \frac{\partial \omega}{\partial \vec{n}} = \frac{\partial v}{\partial \vec{n}} = 0, \quad t > 0, \quad x \in \partial\Omega, \tag{1.3}$$

and the initial conditions

$$u(x,\theta)=\phi_1(x,\theta)\geq 0,\ \omega(x,\theta)=\phi_2(x,\theta)\geq 0,\ v(x,\theta)=\phi_3(x,\theta)\geq 0,\ x\in\overline{\Omega},\ \theta\in[-\tau,0], \quad (1.4)$$

where $\frac{\partial}{\partial\vec{n}}$ denotes the outward normal derivative on $\partial\Omega$. $\phi_i(x,\theta)$ $(i=1,2,3)$ are bounded and uniform continuous functions on $\overline{\Omega}\times[-\tau,0]$. The Neumann boundary condition (1.3) assumes that uninfected cells, infected cells and virus particles cannot move across the boundary $\partial\Omega$. Here, cell-to-cell transmission is modeled by the nonlinear incidence function $f(u(x,t),\omega(x,t))\omega(x,t)$. The constant $m\geq 0$ stands for the death rate of infected cells before viral production and hence $e^{-m\tau}$ is the survival probability of a susceptible cell from being infected to viral production.

The functions $f(u,\omega)\omega\in C^1([0,+\infty)\times[0,+\infty),R)$, $g(u,v)v\in C^1([0,+\infty)\times[0,+\infty),R)$, and $h(\omega)\in C^1[0,+\infty)$, are assumed to satisfy the following conditions.

(H_1) $f(u,\omega)\omega\geq 0$ and $g(u,v)v\geq 0$ for $u\geq 0,\ \omega\geq 0$, and $v\geq 0$, and $f(u,\omega)=0$ and $g(u,v)=0$ if and only if $u=0$;

(H_2) $\frac{\partial f(u,\omega)}{\partial u}>0,\ \frac{\partial f(u,\omega)}{\partial \omega}\leq 0$ for $u\geq 0,\ \omega\geq 0$;

(H_3) $\frac{\partial f(u,\omega)\omega}{\partial \omega}>0$ for $u>0,\ \omega\geq 0$;

(H_4) $\frac{\partial g(u,v)}{\partial u}>0,\ \frac{\partial g(u,v)}{\partial v}\leq 0$ for $u\geq 0,\ v\geq 0$;

(H_5) $\frac{\partial g(u,v)v}{\partial v}>0$ for $u>0,\ v\geq 0$;

(H_6) $h(0)=0,\ h'(\omega)>L,\ h''(\omega)>0$, and $\lim\limits_{t\to+\infty}h(\omega)\leq+\infty$ for $\omega\geq 0$, where L is a positive constant;

(H_7) $f(u,\omega)\omega\leq\eta_1 u\omega$ and $g(u,v)v\leq\eta_2 uv$ for $u\geq 0,\ v\geq 0$, and $\omega\geq 0$, where η_1 and η_2 are some positive constants.

The structure of this paper is as follows. First, we assume that Ω is bounded. In Section 2, we study the basic attributes of system (1.2), including the well-posedness of the model and linear stability of two homogeneous steady states. It is also proved that if the basic reproduction number is less than 1, the infection-free steady state is globally asymptotically stable. If the basic reproduction number is greater than 1, the infection is uniformly persistent. Next, when $\Omega=R^n$, we investigate the existence of traveling wave solutions using the cross iteration method and the Schauder's fixed point theorem in Section 3. Section 4 gives a brief summary.

2 A threshold dynamics of (1.2) when Ω is bounded

2.1 Well-posedness of (1.2)

For topological spaces A and B, $C(A,B)$ represents the space of all continuous functions from A to B. Let $X=C(\overline{\Omega},R^3)$ be the Banach space equipped with the supremum norm $\|\cdot\|_X$. Denote

$C = C([-\tau, 0], X)$ to be the Banach space equipped with the norm $\|\phi\| = \max_{\theta \in [-\tau, 0]} \|\phi(\theta)\|_X$. For $\sigma > 0$ and a continuous function $\vartheta : [-\tau, \sigma) \to X$, $\vartheta_t \in C$ is defined by $\vartheta_t(\theta) = \vartheta(t + \theta)$ for $\theta \in [-\tau, 0]$, where $t \in [0, \sigma)$. Let $X_+ = C(\overline{\Omega}, R_+^3)$ and $C_+ = C([-\tau, 0], X_+)$. Then X_+ induces a partial order such that (X, X_+) and (C, C_+) are strongly ordered spaces.

Define $T = (T_1, T_2, T_3) : C_+ \to X$ by

$$
\begin{aligned}
T_1(\phi)(x) &= \xi - f(\phi_1(x,0), \phi_2(x,0))\phi_2(x,0) - g(\phi_1(x,0), \phi_3(x,0))\phi_3(x,0) - d\phi_1(x,0),\\
T_2(\phi)(x) &= e^{-m\tau} f(\phi_1(x,-\tau), \phi_2(x,-\tau))\phi_2(x,-\tau)\\
&\qquad + e^{-m\tau} g(\phi_1(x,-\tau), \phi_3(x,-\tau))\phi_3(x,-\tau) - ah(\phi_2(x,0)),\\
T_3(\phi)(x) &= kh(\phi_2(x,0)) - \mu\phi_3(x,0) - g(\phi_1(x,0), \phi_3(x,0))\phi_3(x,0),
\end{aligned}
$$

for $\phi = (\phi_1, \phi_2, \phi_3)^{\mathrm{T}} \in C_+$ and $x \in \overline{\Omega}$. Obviously, T is Lipschitz continuous in any bounded subset of C_+. System (1.2) can be easily rewritten as the following abstract functional differential equation

$$
\begin{aligned}
&\vartheta_t = \mathcal{A}\vartheta + T(\vartheta_t),\ t > 0,\\
&\vartheta(0) = \phi \in C_+,
\end{aligned}
$$

where $\vartheta = (u, \omega, v)^{\mathrm{T}}$, $\phi = (\phi_1, \phi_2, \phi_3)^{\mathrm{T}}$ and $\mathcal{A}\vartheta = (D\Delta u, D\Delta\omega, D_0\Delta v)^{\mathrm{T}}$.

Lemma 2.1 *For each initial value function $\phi = (\phi_1, \phi_2, \phi_3)^{\mathrm{T}} \in C_+$, system (1.2)-(1.4) has a unique mild solution $\vartheta(\cdot, t, \phi) = (u(\cdot, t, \phi), \omega(\cdot, t, \phi), v(\cdot, t, \phi))$ on $[0, t_\phi)$ with $\vartheta_0(\cdot, \phi) = \phi$, where $t_\phi \le +\infty$. Moreover, $\vartheta_t(\cdot, t, \phi) \in C_+$ for $t \in [0, t_\phi)$ and $\vartheta(\cdot, t, \phi)$ is a classical solution of (1.2) for $t \ge \max(\tau, t_\phi)$.*

Proof. Note that $T(\phi)$ is locally Lipschitzian. It follows from Corollary 8.1.3 in [31] that we only need to show that

$$
\lim_{\varsigma \to 0^+} \mathrm{dist}(\phi(0) + \varsigma T(\phi), C_+) = 0 \quad \text{for} \quad \phi \in C_+. \tag{2.1}
$$

By (H_6) and (H_7), for any $\varsigma \ge 0$, we have

$\phi(x, 0) + \varsigma T(\phi)(x)$

$$
= \begin{pmatrix}
\phi_1(x,0) + \varsigma[\xi - f(\phi_1(x,0), \phi_2(x,0))\phi_2(x,0)\\
-g(\phi_1(x,0), \phi_3(x,0))\phi_3(x,0) - d\phi_1(x,0)]\\
\phi_2(x,0) + \varsigma[e^{-m\tau} f(\phi_1(x,-\tau), \phi_2(x,-\tau))\phi_2(x,-\tau)\\
+e^{-m\tau} g(\phi_1(x,-\tau), \phi_3(x,-\tau))\phi_3(x,-\tau) - ah(\phi_2(x,0))]\\
\phi_3(x,0) + \varsigma[kh(\phi_2(x,0)) - \mu\phi_3(x,0) - g(\phi_1(x,0), \phi_3(x,0))\phi_3(x,0)]
\end{pmatrix}
$$

$$\geq \begin{pmatrix} \phi_1(x,0)(1-\varsigma(\eta_1\phi_2(x,0)+\eta_2\phi_3(x,0)+d)) \\ \phi_2(x,0)(1-\varsigma a h'(\theta_0)),\ (\theta_0\in[0,\omega]) \\ \phi_3(x,0)(1-\varsigma(\mu+\eta_2\phi_1(x,0))) \end{pmatrix}$$

for $x \in \Omega$. This shows that $\phi(0)+\varsigma T(\phi) \in C_+$ when ς is sufficiently small. Thus, (2.1) is proved. It follows from Corollary 4 in [32] that there exists a unique mild solution $\vartheta(\cdot,t,\phi)$ on $[0,t_\phi)$ with $\vartheta_0(\cdot,\phi)=\phi$. Furthermore, $\vartheta(\cdot,t,\phi)$ is a classical solution of (1.2) for $t \geq \max(\tau, t_\phi)$.

Lemma 2.2 *For $\phi \in C_+$, the following description of solutions of system* (1.2) *are valid.*

(i) $u(\cdot,t,\phi)>0$ for $t>0$ and there exists a constant k_0 such that $\lim_{t\to\infty}\inf u(x,t,\phi) \geq \frac{\xi}{k_0+d}$ for $x\in\Omega$.

(ii) Assume that $\omega(\cdot,t_0,\phi)\not\equiv 0$ for some $t_0\geq 0$, then $\omega(x,t,\phi)>0$ and $v(x,t,\phi)>0$ for all $x\in\Omega$ and $t>t_0+\tau$.

(iii) Assume that $v(\cdot,t_0,\phi)\not\equiv 0$ for some $t_0\geq 0$, then $v(x,t,\phi)>0$ and $\omega(x,t+\tau,\phi)>0$ for all $x\in\Omega$ and $t>t_0$.

Proof. (i) According to (H_7), the functions $f(u,\omega)\omega$ and $g(u,v)v$ are continuously differentiable and system (1.2) is point dissipative. Thus there is a constant k_0 such that

$$\frac{\partial u(x,t)}{\partial t} \geq D\Delta u(x,t)+\xi-k_0u(x,t)-du(x,t) \text{ for all large } t. \tag{2.2}$$

By (2.2), there exists small $\zeta>0$ such that $u(x,t)\geq \frac{\xi}{k_0+d}-\zeta$ for all large t. For any solutions of (1.2), we have $\lim_{t\to\infty}\inf u(x,t,\phi)\geq\frac{\xi}{k_0+d}$ by the standard parabolic comparison theorem. This proves statement (i).

(ii) From the third equation of (1.2), we can easily see that $v(\cdot,t_0,\phi)\not\equiv 0$ if $\omega(\cdot,t_0,\phi)\not\equiv 0$ for $t>t_0$ holds. Supposing $v(\cdot,t_0,\phi)\not\equiv 0$. We first claim that $v(\cdot,t)>0$ for $t>t_0$. It follows from Lemma 2.1 that $v(x,t)$ satisfies

$$\begin{aligned} &\frac{\partial v(x,t)}{\partial t} \geq D_0\Delta v(x,t)-\mu v(x,t)-g(u(x,t),v(x,t))v(x,t),\ x\in\Omega,\ t>t_0,\\ &\frac{\partial v(x,t)}{\partial \vec{n}}=0,\ x\in\partial\Omega,\ t>t_0. \end{aligned}$$

By (H_7), there exists sufficiently large $\bar{u}$ such that $g(u(x,t),v(x,t))v(x,t)\leq\eta_2\bar{u}v(x,t)$ and $u(x,t)\leq\bar{u}$. Let $\tilde{v}(x,t)$ be the solution of

$$\begin{aligned} &\frac{\partial \tilde{v}(x,t)}{\partial t} = D_0\Delta \tilde{v}(x,t)-\mu\tilde{v}(x,t)-\eta_2\bar{u}\tilde{v}(x,t),\ x\in\Omega,\ t>t_0,\\ &\frac{\partial \tilde{v}(x,t)}{\partial \vec{n}}=0,\ x\in\partial\Omega,\ t>t_0,\\ &\tilde{v}(x,t_0)=v(x,t_0),\ x\in\overline{\Omega}. \end{aligned}$$

We prove that $\tilde{v}(x,t) > 0$ for $t > t_0$ and $x \in \Omega$ by contradictory methods. Otherwise, $\tilde{v}(x_0,t_1) = 0$ for $x_0 \in \Omega$ and $t_1 > t_0$. It follows from the strong maximum principle (Theorem 1.1.5 in [35]) that $\tilde{v}(x,t) \equiv 0$ for all $t \geq t_0$, which is a contradiction with $\tilde{v}(\cdot,t_0) \not\equiv 0$. According to the parabola comparison theorem (Theorem 7.3.4 in [36]), it can be obtained that $v(x,t) > \tilde{v}(x,t) > 0$ for $t > t_0$ and $x \in \Omega$. This proves the claim. Next, suppose that $\omega(\bar{x},t_2) = 0$ for $\bar{x} \in \Omega$ and $t_2 > t_0 + \tau$. When $\omega(x,t) \geq 0$, we obtain $\frac{\partial \omega(\bar{x},t_2)}{\partial t_2} = 0$. Recall that $h(\omega(\bar{x},t_2)) = h(0) = 0$. From the second equation of (1.2), we can see that

$$\frac{\partial \omega(\bar{x},t_2)}{\partial t_2} = e^{-m\tau} g(u(\bar{x},t_2-\tau), v(\bar{x},t_2-\tau))v(\bar{x},t_2-\tau) > 0.$$

This leads to a contradiction with $u(\bar{x},t_2-\tau) > 0$ and $v(\bar{x},t_2-\tau) > 0$ by (H_1). This proves the claim.

(iii) The proof is similar to that of (ii) and hence is omitted. This completes the proof.

Theorem 2.1 *For any* $\phi = (\phi_1,\phi_2,\phi_3) \in C_+$*, system* (1.2) *has a unique solution* $\vartheta(\cdot,t,\phi) = (u(\cdot,t,\phi), \omega(\cdot,t,\phi), v(\cdot,t,\phi))$ *on* $[0,+\infty)$ *with* $\vartheta_0 = \phi$*, and the solution semiflow* $\Phi(t) = \vartheta(\cdot) : C_+ \to C_+$ *of system* (1.2) *has a global compact attractor in* C_+*.*

Proof. Firstly, let $Z(x,t) = e^{-m\tau}u(x,t-\tau) + \omega(x,t)$ for $x \in \Omega,\ t \in [0,t_\phi)$. It follows from system (1.2), (H_6), and the mean value theorem that

$$\begin{aligned}
\frac{\partial Z(x,t)}{\partial t} &= e^{-m\tau}\frac{\partial u(x,t-\tau)}{\partial t} + \frac{\partial \omega(x,t)}{\partial t} \\
&= D\Delta(e^{-m\tau}u(x,t-\tau) + \omega(x,t)) + e^{-m\tau}\xi - de^{-m\tau}u(x,t-\tau) - ah(\omega(x,t)) \\
&= D\Delta Z(x,t) + e^{-m\tau}\xi - e^{-m\tau}du(x,t-\tau) - ah'(\theta_0)\omega(x,t)\ (\theta_0 \in [0,\omega]) \\
&\leq D\Delta Z(x,t) + e^{-m\tau}\xi - r_1 Z(x,t),
\end{aligned}$$

where $r_1 = \min\{d, aL\}$. From [34], we know that $\frac{e^{-m\tau}\xi}{r_1}$ is the steady state of the global attractive steady state for the scalar parabolic equation

$$\begin{aligned}
&\frac{\partial Z(x,t)}{\partial t} = D\Delta Z(x,t) + e^{-m\tau}\xi - r_1 Z(x,t),\ x \in \Omega,\ t > 0, \\
&\frac{\partial Z(x,t)}{\partial \vec{n}} = 0,\ x \in \partial\Omega,\ t > 0.
\end{aligned}$$

According to the parabola comparison theorem (Theorem 7.3.4 in [36]), it can be obtained that $u(x,t)+\omega(x,t)$ is bounded. From the nonnegativity of $u(x,t)$ and $\omega(x,t)$(Lemma 2.2), we have that $u(x,t)$ and $\omega(x,t)$ of system (1.2) are bounded. Therefore, we assume that there exist sufficiently large $\bar{u}$ and $\bar{\omega}$ such that $0 \leq u(x,t) \leq \bar{u}$ and $0 \leq \omega(x,t) \leq \bar{\omega}$.

Secondly, we let $\bar{v} = \frac{kh(\bar{\omega})}{\mu}$. For any $v(x,t)$, we consider the operator $\mathcal{C}$ as follows

$$\mathcal{C}v(x,t) = v_t - D_0\Delta v - kh(\omega(x,t)) + \mu v(x,t) + g(u(x,t),v(x,t))v(x,t).$$

Clearly,

$$\mathcal{C}\bar{v} = \mu\bar{v} + g(u,\bar{v})\bar{v} - kh(\omega) \geq \mu\bar{v} - kh(\bar{\omega}) = kh(\bar{\omega}) - kh(\bar{\omega}) = 0 = \mathcal{C}v(x,t).$$

It is easy to see that $\frac{\partial \bar{v}}{\partial \vec{n}} = 0$ based on the boundary $\partial\Omega$. Hence, $v = \bar{v}$ is an upper solution of the third equation in system (1.2). From the comparison principle, we have $0 \leq v(x,t) \leq \bar{v}$.

Finally, from the above discussion, we know that the solutions $u(x,t),\ \omega(x,t),\ v(x,t)$ of system (1.2) are bounded on $\bar{\Omega} \times [0, t_\phi)$. By the standard theory of semilinear parabolic systems [37, 38], we can deduce that $t_\phi = +\infty$, otherwise which lead to a contradiction with $\lim\limits_{t \to t_\phi^-} \|\vartheta_t\| = +\infty$. In addition, it can be concluded that the solution semiflow $\Phi(t) = \vartheta_t(\cdot) : C_+ \to C_+$ defined by

$$(\Phi(t)\phi)(x,\theta) = \vartheta(x, t+\theta, \phi) \ \text{ for } \theta \in [-\tau, 0],\ x \in \overline{\Omega},\ t \geq 0,$$

is point dissipative. It follows from Lemma 2.2 in [40] that the solution semiflow $\Phi(t)$ is compact for each $t > \tau$. By Theorem 3.4.8 in [39], $\Phi(t)$ has a global compact attractor in C_+. This completes the proof.

2.2 Linear stability of homogeneous steady states

System (1.2) always has a unique infection-free steady state $E_0 = (u_0, 0, 0) = (\frac{\xi}{d}, 0, 0)$. It follows from the next generation matrix operator [41, 42] that the basic reproduction number of system (1.2) is given by

$$R_0 = \frac{e^{-m\tau} f(u_0, 0)}{ah'(0)} + \frac{e^{-m\tau} kg(u_0, 0)}{a(\mu + g(u_0, 0))}.$$

It represents the expected number of the next generation of newly infected cells produced by a single infected cell in a wholly susceptible population. The proportion of newly infected cells surviving to viral production is $e^{-m\tau}$. Here, $\frac{e^{-m\tau} f(u_0,0)}{ah'(0)}$ represents the total number of newly infected cells produced by a single infected cell. This is the basic reproduction number of the corresponding model with cell-to-cell transmission. $\frac{e^{-m\tau} kg(u_0,0)}{a(\mu+g(u_0,0))}$ is the total number of newly infected cells generated by infection of cells from viruses produced by a single infected cell.

Note that a homogeneous steady state $E^* = (u^*, \omega^*, v^*)$ satisfies

$$\begin{aligned} &\xi - f(u,\omega)\omega - g(u,v)v - du = 0, \\ &e^{-m\tau} f(u,\omega)\omega + e^{-m\tau} g(u,v)v - ah(\omega) = 0, \\ &kh(\omega) - \mu v - g(u,v)v = 0. \end{aligned} \tag{2.3}$$

Through direct calculation, we obtain

$$u = \frac{\xi - ae^{m\tau}h(\omega)}{d},$$
$$v = \frac{f\left(\frac{\xi - ae^{m\tau}h(\omega)}{d}, \omega\right)\omega - ae^{m\tau}h(\omega) + kh(\omega)}{\mu}.$$

It is easy to see that $u > 0$ if and only if $\omega \in (0, h^{-1}(\frac{\xi}{ae^{m\tau}})]$. We substitute u and v into the first equation of system (1.2) and get

$$g\left(\frac{\xi - ae^{m\tau}h(\omega)}{d}, \frac{f\left(\frac{\xi - ae^{m\tau}h(\omega)}{d}, \omega\right)\omega - ae^{m\tau}h(\omega) + kh(\omega)}{\mu}\right)$$
$$\cdot\left(\frac{f\left(\frac{\xi - ae^{m\tau}h(\omega)}{d}, \omega\right)\omega - ae^{m\tau}h(\omega) + kh(\omega)}{\mu}\right) + f\left(\frac{\xi - ae^{m\tau}h(\omega)}{d}, \omega\right)\omega - ae^{m\tau}h(\omega) = 0.$$

Define

$$\begin{aligned} F(\omega) =& g\left(\frac{\xi - ae^{m\tau}h(\omega)}{d}, \frac{f\left(\frac{\xi - ae^{m\tau}h(\omega)}{d}, \omega\right)\omega - ae^{m\tau}h(\omega) + kh(\omega)}{\mu}\right) \\ &\cdot\left(\frac{f\left(\frac{\xi - ae^{m\tau}h(\omega)}{d}, \omega\right)\omega - ae^{m\tau}h(\omega) + kh(\omega)}{\mu}\right) \\ &+ f\left(\frac{\xi - ae^{m\tau}h(\omega)}{d}, \omega\right)\omega - ae^{m\tau}h(\omega). \end{aligned}$$

Clearly,

(i) $F(0) = 0$;

(ii) $F'(0) = f(u_0, 0) + \frac{1}{\mu}(f(u_0, 0)g(u_0, 0) - ae^{m\tau}h'(0)g(u_0, 0) + kh'(0)g(u_0, 0)) - ae^{m\tau}h'(0)$
$= \frac{ae^{m\tau}h'(0)(\mu + g(u_0, 0))}{\mu}(R_0 - 1)$;

(iii) $F(h^{-1}(\frac{\xi}{ae^{m\tau}})) = -\xi < 0$.

It follows from the Intermediate Value Theorem that there exists $\omega^* \in (0, \frac{\xi}{h^{-1}(ae^{m\tau})})$ such that $F(\omega^*) = 0$ if $R_0 > 1$. Next, we prove that there is only a unique homogeneous infection steady state $E^* = (u^*, v^*, \omega^*)$. In fact, this follows from the fact that $F'(\omega^*) < 0$ proved below. By (H_6) and the second and third equations of (2.3), we obtain $\omega^* h'(\omega^*) > h(\omega^*)$, $ae^{m\tau} = \frac{f(u^*, \omega^*)\omega^* + g(u^*, v^*)v^*}{h(\omega^*)}$, and $k = \frac{v^*(\mu + g(u^*, v^*))}{h(\omega^*)}$. Then

$$F'(\omega*) = \left(-\frac{ae^{m\tau}}{d}h'(\omega^*)\frac{\partial g(u^*, v^*)}{\partial u} + \frac{\partial g(u^*, v^*)}{\partial v}\frac{\partial v}{\partial \omega}\right)v^* + \frac{\partial v}{\partial \omega}g(u^*, v^*)$$

$$
\begin{aligned}
&+\left(-\frac{ae^{m\tau}}{d}h'(\omega^*)\frac{\partial f(u^*,\omega^*)}{\partial u}+\frac{\partial f(u^*,\omega^*)}{\partial \omega}\right)\omega^*+f(u^*,\omega^*)-ae^{m\tau}h'(\omega^*)\\
=&-\frac{ae^{m\tau}}{d}h'(\omega^*)\frac{\partial g(u^*,v^*)}{\partial u}v^*+\left(\frac{\partial g(u^*,v^*)}{\partial v}v^*+g(u^*,v^*)\right)\frac{\partial v}{\partial \omega}\\
&+\left(-\frac{ae^{m\tau}}{d}h'(\omega^*)\frac{\partial f(u^*,\omega^*)}{\partial u}+\frac{\partial f(u^*,\omega^*)}{\partial \omega}\right)\omega^*+f(u^*,\omega^*)-ae^{m\tau}h'(\omega^*)\\
=&-\frac{ae^{m\tau}}{d}h'(\omega^*)\frac{\partial g(u^*,v^*)}{\partial u}v^*\\
&+\frac{1}{\mu}\frac{\partial (g(u^*,v^*)v^*)}{\partial v}\left(-\frac{ae^{m\tau}}{d}h'(\omega^*)\frac{\partial f(u^*,\omega^*)}{\partial u}+\frac{\partial f(u^*,\omega^*)}{\partial \omega}\right)\omega^*\\
&+\frac{1}{\mu}\frac{\partial (g(u^*,v^*)v^*)}{\partial v}(f(u^*,\omega^*)-ae^{m\tau}h'(\omega^*)+kh'(\omega^*))\\
&+\left(-\frac{ae^{m\tau}}{d}h'(\omega^*)\frac{\partial f(u^*,\omega^*)}{\partial u}+\frac{\partial f(u^*,\omega^*)}{\partial \omega}\right)\omega^*+f(u^*,\omega^*)-ae^{m\tau}h'(\omega^*)\\
=&-\frac{ae^{m\tau}}{d}h'(\omega^*)\frac{\partial g(u^*,v^*)}{\partial u}v^*\\
&+\left(\frac{1}{\mu}\frac{\partial (g(u^*,v^*)v^*)}{\partial v}+1\right)\left(-\frac{ae^{m\tau}}{d}h'(\omega^*)\frac{\partial f(u^*,\omega^*)}{\partial u}+\frac{\partial f(u^*,\omega^*)}{\partial \omega}\right)\omega^*\\
&+\frac{1}{\mu}\frac{\partial (g(u^*,v^*)v^*)}{\partial v}(f(u^*,\omega^*)-ae^{m\tau}h'(\omega^*)+kh'(\omega^*))+f(u^*,\omega^*)-ae^{m\tau}h'(\omega^*)\\
=&-\frac{ae^{m\tau}}{d}h'(\omega^*)\frac{\partial g(u^*,v^*)}{\partial u}v^*\\
&+\left(\frac{1}{\mu}\frac{\partial (g(u^*,v^*)v^*)}{\partial v}+1\right)\left(-\frac{ae^{m\tau}}{d}h'(\omega^*)\frac{\partial f(u^*,\omega^*)}{\partial u}+\frac{\partial f(u^*,\omega^*)}{\partial \omega}\right)\omega^*\\
&+\frac{1}{\mu}\frac{\partial (g(u^*,v^*)v^*)}{\partial v}\left(f(u^*,\omega^*)-\frac{f(u^*,\omega^*)\omega^*+g(u^*,v^*)v^*}{h(\omega^*)}h'(\omega^*)\right)\\
&+\frac{1}{\mu}\frac{\partial (g(u^*,v^*)v^*)}{\partial v}\frac{v^*(\mu+g(u^*,v^*))}{h(\omega^*)}h'(\omega^*)+f(u^*,\omega^*)-\frac{f(u^*,\omega^*)\omega^*+g(u^*,v^*)v^*}{h(\omega^*)}h'(\omega^*)\\
=&-\frac{ae^{m\tau}}{d}h'(\omega^*)\frac{\partial g(u^*,v^*)}{\partial u}v^*\\
&+\left(\frac{1}{\mu}\frac{\partial (g(u^*,v^*)v^*)}{\partial v}+1\right)\left(-\frac{ae^{m\tau}}{d}h'(\omega^*)\frac{\partial f(u^*,\omega^*)}{\partial u}+\frac{\partial f(u^*,\omega^*)}{\partial \omega}\right)\omega^*\\
&+\frac{1}{\mu}\frac{\partial (g(u^*,v^*)v^*)}{\partial v}\left(f(u^*,\omega^*)-\frac{f(u^*,\omega^*)\omega^*h'(\omega^*)}{h(\omega^*)}+\frac{v^*\mu h'(\omega^*)}{h(\omega^*)}\right)\\
&+f(u^*,\omega^*)-\frac{f(u^*,\omega^*)\omega^*h'(\omega^*)}{h(\omega^*)}-\frac{g(u^*,v^*)v^*h'(\omega^*)}{h(\omega^*)}\\
=&-\frac{ae^{m\tau}}{d}h'(\omega^*)\frac{\partial g(u^*,v^*)}{\partial u}v^*\\
&+\left(\frac{1}{\mu}\frac{\partial (g(u^*,v^*)v^*)}{\partial v}+1\right)\left(-\frac{ae^{m\tau}}{d}h'(\omega^*)\frac{\partial f(u^*,\omega^*)}{\partial u}+\frac{\partial f(u^*,\omega^*)}{\partial \omega}\right)\omega^*\\
&+\left(\frac{1}{\mu}\frac{\partial (g(u^*,v^*)v^*)}{\partial v}+1\right)\left(f(u^*,\omega^*)-\frac{f(u^*,\omega^*)\omega^*h'(\omega^*)}{h(\omega^*)}\right)\\
&+v^*\left(\frac{\partial g(u^*,v^*)}{\partial v}v^*+g(u^*,v^*)\right)\frac{h'(\omega^*)}{h(\omega^*)}-\frac{g(u^*,v^*)v^*h'(\omega^*)}{h(\omega^*)}
\end{aligned}
$$

$$
\begin{aligned}
=& -\frac{ae^{m\tau}}{d}h'(\omega^*)\frac{\partial g(u^*,v^*)}{\partial u}v^* \\
&+\left(\frac{1}{\mu}\frac{\partial(g(u^*,v^*)v^*)}{\partial v}+1\right)\left(-\frac{ae^{m\tau}}{d}h'(\omega^*)\frac{\partial f(u^*,\omega^*)}{\partial u}+\frac{\partial f(u^*,\omega^*)}{\partial \omega}\right)\omega^* \\
&+\left(\frac{1}{\mu}\frac{\partial(g(u^*,v^*)v^*)}{\partial v}+1\right)\left(f(u^*,\omega^*)-\frac{f(u^*,\omega^*)\omega^* h'(\omega^*)}{h(\omega^*)}\right) \\
&+\frac{\partial g(u^*,v^*)}{\partial v}v^{*2}\frac{h'(\omega^*)}{h(\omega^*)} \\
<&0.
\end{aligned}
$$

This proves the fact. Next, we prove that system (1.2) has no homogeneous infected steady state when $R_0 \leq 1$. If $R_0 < 1$, it is obvious that $F'(0) < 0$. Note that $F(0) = 0$ and there exists a sufficiently small $\omega_1 > 0$ such that $F(\omega_1) < 0$ for $\omega_1 \in (0, h^{-1}(\frac{\xi}{ae^{m\tau}})]$. From the above mentioned fact, we can easily see that there is no homogeneous infected steady state when $R_0 < 1$. When $R_0 = 1$, the contradiction method is used to prove that there is no homogeneous infected steady state. Otherwise, we assume that $F(\omega_1)$ has a positive zero say ω_1^*. From $F'(\omega_1^*) < 0$, we conclude that $F(\omega_1) > 0$ for $\omega_1 < \omega_1^*$ close enough to ω_1^*. Thus, $F(\omega_1) < 0$ for $\omega_1 \in (0, h^{-1}(\frac{\xi}{ae^{m\tau}})]$ when $R_0 < 1$. We fix $\omega_1 \in (0, \omega_1^*)$ and select a series of parameters such that $R_0 < 1$ but converges to 1. Clearly, $F(\omega_1)$ converges to $F(\omega_1^*) > 0$, which is a contradiction.

To summarize, we have proved the following result on the existence of homogeneous steady states.

Theorem 2.2 (1) *If $R_0 \leq 1$, then the only homogeneous steady state of system* (1.2) *is the infection-free steady state E_0.*

(2) *If $R_0 > 1$, then besides E_0, system* (1.2) *also has a unique homogeneous infected steady state E^*.*

Next, we investigate the linear stability of the homogeneous steady states.

Theorem 2.3 *If $R_0 < 1$, then the infection-free steady state $E_0 = (u_0, 0, 0)$ is locally asymptotically stable. If $R_0 > 1$, then E_0 is unstable.*

Proof. Denote $N = (u, \omega, v)$, $N_\tau = (u_\tau, \omega_\tau, v_\tau)$, $D_1 = diag(D, D, D_0)$. Taking the linearization of system (1.2) at E_0, we obtain

$$
\frac{\partial N}{\partial t} = D_1 \Delta N + A_1 N + A_2 N_\tau,
$$

where

$$
A_1 = \begin{pmatrix} -d & -f(u_0,0) & -g(u_0,0) \\ 0 & -ah'(0) & 0 \\ 0 & kh'(0) & -\mu - g(u_0,0) \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e^{-m\tau}f(u_0,0) & e^{-m\tau}g(u_0,0) \\ 0 & 0 & 0 \end{pmatrix}.
$$

The corresponding characteristic equation is obtained as follows

$$|\lambda E + D_1 l^2 - A_1 - e^{-\lambda\tau} A_2| = 0, \tag{2.4}$$

where $l \geq 0$ belongs to the set of wave numbers and λ is the characteristic value that determines temporal growth (Theorem 2.3 in [43]). Substituting the matrices $A_1,\ A_2$ and D_1 into (2.4) yields

$$(\lambda + Dl^2 + d)g_1(\lambda, l) = 0,$$

where

$$\begin{aligned} g_1(\lambda, l) = & \ (\lambda + Dl^2 + ah'(0))(\lambda + D_0 l^2 + \mu + g(u_0, 0)) \\ & - ((\lambda + D_0 l^2 + \mu + g(u_0, 0))f(u_0, 0) + kh'(0)g(u_0, 0))e^{-(m+\lambda)\tau}. \end{aligned}$$

The stability of E_0 is determined by the roots of $g_1(\lambda, l) = 0$. For the case $R_0 > 1$, noting $l = 0$ is a wave number, with $l = 0$, we have $g_1(0, 0) = ah'(0)(\mu + g(u_0, 0))(1 - R_0) < 0$ and $\lim_{\lambda \to +\infty} g_1(\lambda, 0) \to +\infty$. Thus, there exists $\lambda_0 > 0$ such that $g_1(\lambda_0, 0) = 0$. Therefore, $g_1(\lambda, 0) = 0$ admits at least one positive real root, which implies that E_0 is unstable when $R_0 > 1$.

Now, assume $R_0 < 1$. Note that, $g_1(\lambda, l) = 0$ can be rewritten as

$$1 = \left[\frac{f(u_0, 0)}{\lambda + Dl^2 + ah'(0)} + \frac{kh'(0)g(u_0, 0)}{(\lambda + Dl^2 + ah'(0))(\lambda + D_0 l^2 + \mu + g(u_0, 0))}\right] e^{-(m+\lambda)\tau}. \tag{2.5}$$

We claim that all roots of $g_1(\lambda, l)$ have negative real parts. Otherwise, there exists l_0 such that there exists λ_0 with $\mathrm{Re}(\lambda_0) \geq 0$ satisfying (2.5) with $l = l_0$. Then

$$\begin{aligned} 1 = & \ \left|\left[\frac{f(u_0, 0)}{(\lambda_0 + Dl_0^2 + ah'(0))} + \frac{kh'(0)g(u_0, 0)}{(\lambda_0 + Dl_0^2 + ah'(0))(\lambda_0 + D_0 l_0^2 + \mu + g(u_0, 0))}\right] e^{-(m+\lambda_0)\tau}\right| \\ \leq & \ \left|\frac{f(u_0, 0)e^{-m\tau}}{(\lambda_0 + Dl_0^2 + ah'(0))} e^{-\lambda_0\tau}\right| + \left|\frac{kh'(0)g(u_0, 0)e^{-m\tau}}{(\lambda_0 + Dl_0^2 + ah'(0))(\lambda_0 + D_0 l_0^2 + \mu + g(u_0, 0))} e^{-\lambda_0\tau}\right| \\ \leq & \ \frac{f(u_0, 0)e^{-m\tau}}{ah'(0)} + \frac{kg(u_0, 0)e^{-m\tau}}{a(\mu + g(u_0, 0))} \\ = & \ R_0, \end{aligned}$$

which leads to a contradiction. Therefore, E_0 is locally asymptotically stable when $R_0 < 1$.

Theorem 2.4 *If $R_0 > 1$, then the homogeneous infected steady state $E^* = (u^*, \omega^*, v^*)$ is locally asymptotically stable.*

Proof. Let $N,\ N_\tau$, and D be the same as those in the proof of Theorem 2.3. Denote $f^* = f(u^*, \omega^*)$, $f_u^* = \frac{\partial f(u^*, \omega^*)}{\partial u}$, $f_\omega^* = \frac{\partial f(u^*, \omega^*)}{\partial \omega}$, $g^* = g(u^*, v^*)$, $g_u^* = \frac{\partial g(u^*, \omega^*)}{\partial u}$, and $g_v^* = \frac{\partial g(u^*, v^*)}{\partial v}$. The linearized system of (1.2) at E^* is

$$\frac{\partial N}{\partial t} = D_1 \Delta N + B_1 N + B_2 N_\tau,$$

where

$$B_1 = \begin{pmatrix} -f_u^*\omega^* - g_u^* v^* - d & -f_\omega^*\omega^* - f^* & -g_v^* v^* - g^* \\ 0 & -ah'(\omega^*) & 0 \\ -g_u^* v^* & kh'(\omega^*) & -\mu - g_v^* v^* - g^* \end{pmatrix},$$

and

$$B_2 = \begin{pmatrix} 0 & 0 & 0 \\ e^{-m\tau}(f_u^*\omega^* + g_u^* v^*) & e^{-m\tau}(f_\omega^*\omega^* + f^*) & e^{-m\tau}(g_v^* v^* + g^*) \\ 0 & 0 & 0 \end{pmatrix}.$$

The characteristic equation is

$$|\lambda E + D_1 l^2 - B_1 - e^{-\lambda\tau} B_2| = 0,$$

or

$$\begin{aligned} &(\lambda + Dl^2 + ah'(\omega^*))[(\lambda + Dl^2 + f_u^*\omega^* + d)(\lambda + D_0 l^2 + \mu + g_v^* v^* + g^*) + (\lambda + D_0 l^2 + \mu) g_u^* v^*] = \\ &(\lambda + Dl^2 + d)[kh'(\omega^*)(g_v^* v^* + g^*) + (f_\omega^*\omega^* + f^*)(\lambda + D_0 l^2 + \mu + g_v^* v^* + g^*)]e^{-(m+\lambda)\tau}, \end{aligned} \tag{2.6}$$

where as before, l belongs to the set of wavenumbers. Now, we use the method of contradiction to prove that all roots of (2.6) have negative real parts which implies that E^* is locally asymptotically stable. Otherwise, there is one root λ_0 with $\mathrm{Re}(\lambda_0) \geq 0$ for some l_0. Then

$$\begin{aligned} 1 = &\left| \frac{(\lambda_0 + Dl_0^2 + d)}{(\lambda_0 + Dl_0^2 + f_u^*\omega^* + d)(\lambda_0 + D_0 l_0^2 + \mu + g_v^* v^* + g^*) + (\lambda_0 + D_0 l_0^2 + \mu) g_u^* v^*} \right| \\ &\cdot \left| \frac{kh'(\omega^*)(g_v^* v^* + g^*)e^{-(m+\lambda_0)\tau}}{\lambda_0 + Dl_0^2 + ah'(\omega^*)} + \frac{(f_\omega^*\omega^* + f^*)(\lambda_0 + D_0 l_0^2 + \mu + g_v^* v^* + g^*)e^{-(m+\lambda_0)\tau}}{\lambda_0 + Dl_0^2 + ah'(\omega^*)} \right| \\ \leq &\left| \frac{(\lambda_0 + Dl_0^2 + d)}{(\lambda_0 + Dl_0^2 + f_u^*\omega^* + d)(\lambda_0 + D_0 l_0^2 + \mu + g_v^* v^* + g^*) + (\lambda_0 + D_0 l_0^2 + \mu) g_u^* v^*} \right| \\ &\cdot \left(\frac{k(g_v^* v^* + g^*)e^{-m\tau}}{a} + \left| \frac{(f_\omega^*\omega^* + f^*)(\lambda_0 + D_0 l_0^2 + \mu + g_v^* v^* + g^*)e^{-m\tau}}{ah'(\omega^*)} \right| \right) \\ = &\left| \frac{(\lambda_0 + Dl_0^2 + d)(\lambda_0 + D_0 l_0^2 + \mu + g_v^* v^* + g^*)}{(\lambda_0 + Dl_0^2 + f_u^*\omega^* + d)(\lambda_0 + D_0 l_0^2 + \mu + g_v^* v^* + g^*) + (\lambda_0 + D_0 l_0^2 + \mu) g_u^* v^*} \right| \\ &\cdot \left(\left| \frac{k(g_v^* v^* + g^*)e^{-m\tau}}{a(\lambda_0 + D_0 l_0^2 + \mu + g_v^* v^* + g^*)} \right| + \frac{(f_\omega^*\omega^* + f^*)e^{-m\tau}}{ah'(\omega^*)} \right) \\ \leq &\left| \frac{(\lambda_0 + Dl_0^2 + d)(\lambda_0 + D_0 l_0^2 + \mu + g_v^* v^* + g^*)}{(\lambda_0 + Dl_0^2 + f_u^*\omega^* + d)(\lambda_0 + D_0 l_0^2 + \mu + g_v^* v^* + g^*) + (\lambda_0 + D_0 l_0^2 + \mu) g_u^* v^*} \right| \\ &\cdot \left(\frac{k(g_v^* v^* + g^*)e^{-m\tau}}{a(\mu + g_v^* v^* + g^*)} + \frac{(f_\omega^*\omega^* + f^*)e^{-m\tau}}{ah'(\omega^*)} \right). \end{aligned}$$

This is impossible as shown below.

On the one hand, note that

$$\frac{f^* e^{-m\tau}\omega^*}{ah(\omega^*)} + \frac{kg^* e^{-m\tau}}{a(\mu+g^*)} = 1.$$

This, combined with

$$\frac{f_\omega^*\omega^* + f^*}{h'(\omega^*)} < \frac{f^*\omega^*}{h(\omega^*)}, \quad \frac{g_v^* v^* + g^*}{\mu + g_v^* v^* + g^*} < \frac{g^*}{\mu + g^*}, \quad h(\omega^*) < h'(\omega^*)\omega^*,$$

from $(H_1) - (H_6)$, gives us

$$\frac{(f_\omega^*\omega^* + f^*)e^{-m\tau}}{ah'(\omega^*)} + \frac{k(g_v^* v^* + g^*)e^{-m\tau}}{a(\mu + g_v^* v^* + g^*)} < 1.$$

On the other hand, we can check that $|(\lambda_0 + Dl_0^2 + f_u^*\omega^* + d)(\lambda_0 + D_0 l_0^2 + \mu + g_v^* v^* + g^*) + (\lambda_0 + D_0 l_0^2 + \mu)g_u^* v^*| > |(\lambda_0 + Dl_0^2 + d)(\lambda_0 + D_0 l_0^2 + \mu + g_v^* v^* + g^*)|$. In fact, denote

$$\begin{aligned}\Lambda_1 = &\ (\lambda_0 + Dl_0^2 + f_u^*\omega^* + d)(\lambda_0 + D_0 l_0^2 + \mu + g_v^* v^* + g^*) + (\lambda_0 + D_0 l_0^2 + \mu)g_u^* v^*,\\ \Lambda_2 = &\ (\lambda_0 + Dl_0^2 + d)(\lambda_0 + D_0 l_0^2 + \mu + g_v^* v^* + g^*).\end{aligned}$$

Let $\lambda_0 = x_0 + iy_0$. Then

$$\begin{aligned}|\Lambda_1|^2 = &\ ((x_0 + Dl_0^2 + f_u^*\omega^* + d)(x_0 + D_0 l_0^2 + \mu + g_v^* v^* + g^*) + (x_0 + D_0 l_0^2 + \mu)g_u^* v^* - y_0^2)^2\\ &+ (2x_0 + (D + D_0)l_0^2 + f_u^*\omega^* + d + \mu + g_v^* v^* + g^* + g_u^* v^*)^2 y_0^2\end{aligned}$$

and

$$\begin{aligned}|\Lambda_2|^2 = &\ ((x_0 + Dl_0^2 + d)(x_0 + D_0 l_0^2 + \mu + g_v^* v^* + g^*) - y_0^2)^2\\ &+ (2x_0 + (D + D_0)l_0^2 + d + \mu + g_v^* v^* + g^*)^2 y_0^2.\end{aligned}$$

Thus,

$$\begin{aligned}|\Lambda_1|^2 - |\Lambda_2|^2 = &\ [(x_0 + Dl_0^2 + f_u^*\omega^* + d)(x_0 + D_0 l_0^2 + \mu + g_v^* v^* + g^*) + (x_0 + D_0 l_0^2 + \mu)g_u^* v^*]^2\\ &- [(x_0 + Dl_0^2 + d)(x_0 + D_0 l_0^2 + \mu + g_v^* v^* + g^*)]^2\\ &+ y_0^2[(f_u^*\omega^* + g_u^* v^*)^2 + 2g_u^* v^*(x_0 + Dl_0^2 + d + g_v^* v^* + g^*) + 2f_u^*\omega^*(x_0 + Dl_0^2 + d)]\\ > &\ 0\end{aligned}$$

as $f_u^* > 0,\ g_u^* > 0$, and $g_v^* v^* + g^* > 0$, which implies that $|\Lambda_1| > |\Lambda_2|$. This completes the proof.

2.3 A threshold dynamics

For convenience, besides X and X_+, we denote $Y := C(\overline{\Omega}, R)$ and $Y_+ = C(\overline{\Omega}, R_+)$.

Theorem 2.5 *When $R_0 < 1$, the infection-free steady state $E_0 = (u_0, 0, 0)$ of system (1.2) is globally asymptotically stable.*

Proof. We construct the following Lyapunov functional,

$$
\begin{aligned}
L(t) = \int_\Omega \bigg(& e^{m\tau}\omega(x,t) + \frac{a}{k}e^{m\tau}v(x,t) + \int_{t-\tau}^{+\infty} f(u(x,s),\omega(x,s))\omega(x,s)ds \\
& + \int_{t-\tau}^{+\infty} g(u(x,s),v(x,s))v(x,s)ds \bigg) dx.
\end{aligned}
$$

For convenience, we denote $z = z(x,t)$ and $z_\tau = z(x, t-\tau)$ for $z = u, \omega, v$. Calculating the directional derivative of $L(t)$ along the solutions of system (1.2), we obtain

$$
\begin{aligned}
\frac{dL(t)}{dt} &= \int_\Omega \left(e^{m\tau}\frac{\partial \omega}{\partial t} + \frac{a}{k}e^{m\tau}\frac{\partial v}{\partial t} - f(u_\tau,\omega_\tau)\omega_\tau - g(u_\tau, v_\tau)v_\tau \right) dx \\
&= De^{m\tau}\int_\Omega \Delta\omega dx + D_0 e^{m\tau}\frac{a}{k}\int_\Omega \Delta v dx - \int_\Omega v e^{m\tau}\frac{a}{k}(\mu + g(u,v))dx.
\end{aligned}
$$

According to the Divergence Theorem and homogeneous Neumann boundary conditions, we get

$$
\int_\Omega \Delta\omega dx = \int_{\partial\Omega}\frac{\partial \omega}{\partial \vec{n}}dx = 0, \int_\Omega \Delta v dx = \int_{\partial\Omega}\frac{\partial v}{\partial \vec{n}}dx = 0.
$$

Therefore,

$$
\frac{dL(t)}{dt} = -e^{m\tau}\frac{a}{k}\int_\Omega v(\mu + g(u,v))dx.
$$

Hence, $\frac{dL(t)}{dt} \leq 0$ for all $u(x,t)$, $\omega(x,t)$, $v(x,t) \geq 0$. Moreover, $\frac{dL(t)}{dt} = 0$ if and only if $v = 0$. This, together with system (1.2), implies that the Largest invariant set $M_0 \subseteq M = \{(u,\omega,v) \in C_+ | \frac{dL(t)}{dt} = 0\}$ is the singleton $\{E_0\}$. According to LaSalle's Invariance Principle, the infection-free steady state E_0 is globally asymptotically stable when $R_0 < 1$. This completes the proof.

Recall that the linearized system of (1.2) at the infection-free steady state E_0 is

$$
\begin{aligned}
\frac{\partial u_1}{\partial t} &= D\Delta u_1 - du_1 - f(u_0,0)u_2 - g(u_0,0)u_3, \\
\frac{\partial u_2}{\partial t} &= D\Delta u_2 - ah'(0)u_2 + e^{-m\tau}f(u_0,0)u_2 + e^{-m\tau}g(u_0,0)u_3, \\
\frac{\partial u_3}{\partial t} &= D_0\Delta u_3 + kh'(0)u_2 - (\mu + g(u_0,0))u_3,
\end{aligned}
\tag{2.7}
$$

with the boundary conditions

$$
\frac{\partial u_1}{\partial \vec{n}} = \frac{\partial u_2}{\partial \vec{n}} = \frac{\partial u_3}{\partial \vec{n}} = 0 \text{ for } x \in \partial\Omega, \quad t > 0.
$$

It follows from system (2.7) that we can combine the last two equations into a cooperative system. Substituting $u_2(x,t) = e^{\lambda t}\phi_1(x)$ and $u_3(x,t) = e^{\lambda t}\phi_2(x)$ into u_2, u_3, we get the following characteristic problem

$$\begin{cases} \lambda\phi_1(x) = D\Delta\phi_1(x) + (e^{-m\tau} f(u_0, 0) - ah'(0))\phi_1(x) + e^{-m\tau} g(u_0, 0)\phi_2(x), \\ \lambda\phi_2(x) = D_0\Delta\phi_2(x) + kh'(0)\phi_1(x) - (\mu + g(u_0, 0))\phi_2(x), \\ \frac{\partial\phi_1(x)}{\partial\vec{n}} = \frac{\partial\phi_2(x)}{\partial\vec{n}} = 0 \text{ for } x \in \partial\Omega, \quad t > 0. \end{cases} \tag{2.8}$$

The uniform persistence of system (1.2) is elicited by applying the following results.

Lemma 2.3 *The eigenvalue problem of* (2.8) *has a principal eigenvalue* $\lambda_0(D, D_0, u_0)$ *associated with a strongly positive eigenvector.*

Lemma 2.4 $R_0 - 1$ *has the same sign as* λ_0.

Lemma 2.5 *If* $R_0 > 1$*, then there exists* $\varepsilon_0 > 0$ *such that for any* $\phi \in C_+$ *with* $\phi_2 \neq 0$*, the solution* $u(t, x, \phi)$ *of system* (1.2) *satisfies*

$$\limsup_{t\to+\infty} \|u(t, \cdot, \phi) - (u_0, 0, 0)\|_{C_+} \geq \varepsilon_0.$$

Proof. It follows from the second equation of system (1.2) and (H_6) that

$$\frac{\partial\omega}{\partial t} \geq D\Delta\omega - ah(\omega) \geq D\Delta\omega - ah'(\bar{\omega})\omega.$$

By the parabola maximum principle, we obtain

$$\omega(x, t) > 0,\ t > 0,\ x \in \overline{\Omega}. \tag{2.9}$$

Since $R_0 > 1$, we have $\lambda_0 > 0$ by Lemma 2.4. Given $\varepsilon \in (0, u_0]$, we let $\lambda_0(\varepsilon)$ be the principal eigenvalue of the following elliptic eigenvalue problem

$$\begin{cases} \lambda\phi_1(x) = D\Delta\phi_1(x) + (e^{-m\tau} f(u_0 - \varepsilon, \varepsilon) - ah'(\varepsilon))\phi_1(x) + e^{-m\tau} g(u_0 - \varepsilon, \varepsilon)\phi_2(x), \\ \lambda\phi_2(x) = D_0\Delta\phi_2(x) + kh'(0)\phi_1(x) - (\mu + g(u_0 + \varepsilon, 0))\phi_2(x), \\ \frac{\partial\phi_1(x)}{\partial\vec{n}} = \frac{\partial\phi_2(x)}{\partial\vec{n}} = 0 \text{ for } x \in \partial\Omega, \quad t > 0. \end{cases}$$

Clearly, we have $\lim_{\varepsilon\to0^+} \lambda_0(\varepsilon) = \lambda_0$. Therefore, there exists a sufficiently small number $\varepsilon_0 \in (0, u_0]$ such that $\lambda_0(\varepsilon_0) > 0$. Now, we prove the result with contradictive arguments. Assume that there exists $\phi \in X_+$ with $\phi_2 \neq 0$ such that

$$\limsup_{t\to+\infty} \|u(t, \cdot, \phi) - (u_0, 0, 0)\|_{X_+} < \varepsilon_0.$$

Then there is $T_1 \geq 0$ such that

$$u_0 - \varepsilon_0 < u(x,t) < \varepsilon_0 + u_0, \quad \omega(x,t) < \varepsilon_0, \quad v(x,t) < \varepsilon_0 \ \text{ for } \ t > T_1 - \tau, \quad x \in \overline{\Omega}.$$

It follows that for $t \geq T_1$,

$$\begin{aligned}
\frac{\partial \omega}{\partial t} &\geq D\Delta\omega + \left(e^{-m\tau} f(u_0 - \varepsilon_0, \varepsilon_0) - ah'(\varepsilon_0)\right)\omega + e^{-m\tau} g(u_0 - \varepsilon_0, \varepsilon_0) v, \\
\frac{\partial v}{\partial t} &\geq D_0 \Delta v + kh'(0)\omega - [\mu + g(u_0 + \varepsilon_0, 0)]v.
\end{aligned}$$

We consider

$$\begin{aligned}
\frac{\partial \nu_1}{\partial t} &= D\Delta\nu_1 + \left(e^{-m\tau} f(u_0 - \varepsilon_0, \varepsilon_0) - ah'(\varepsilon_0)\right)\nu_1 + e^{-m\tau} g(u_0 - \varepsilon_0, \varepsilon_0)\nu_2, \\
\frac{\partial \nu_2}{\partial t} &= D_0 \Delta\nu_2 + kh'(0)\nu_1 - [\mu + g(u_0 + \varepsilon_0, 0)]\nu_2.
\end{aligned}$$

It follows from (2.9) with $(\nu_1(x,\theta), \nu_2(x,\theta)) = (\omega(x,\theta), v(x,\theta))$ for $(x,\theta) \in \overline{\Omega} \times [T_1 - \tau, T_1]$ and $\frac{\partial \nu_1}{\partial \vec{n}} = \frac{\partial \nu_2}{\partial \vec{n}} = 0$ and $\lambda_0(\varepsilon_0) > 0$ that

$$\lim_{t \to +\infty} \nu_1(x,t) = \lim_{t \to +\infty} \nu_2(x,t) = +\infty.$$

By the Comparison Theorem, we have

$$(\omega(x,t), v(x,t)) \geq (\nu_1(x,t), \nu_2(x,t)) \text{ for } t \geq T_1.$$

This leads to a contradiction.

Next, we are ready to establish the uniform persistence of system (1.2).

Theorem 2.6 *If $R_0 > 1$, then there exists $\delta > 0$ such that for any nonnegative solution $u(t,x,\phi)$ with $\phi_2 \neq 0$,*

$$\lim_{t \to +\infty} \inf \omega(x,t) \geq \delta, \qquad \lim_{t \to +\infty} \inf v(x,t) \geq \delta$$

uniformly for $x \in \overline{\Omega}$.

Proof. Define

$$W = \{\phi = (\phi_1, \phi_2, \phi_3) \in C_+ : \phi_2 \not\equiv 0 \ \text{ and } \ \phi_3 \not\equiv 0\}$$

and

$$\partial W := C_+ \setminus W = \{\phi \in C_+ : \phi_2 \equiv 0 \ \text{ or } \ \phi_3 \equiv 0\}.$$

It follows from Lemma 2.1 that W is a positive invariant set. Next, we define

$$M_\partial := \{\phi \in C_+ : \Phi(t)\phi \in \partial W, \quad t \geq 0\}.$$

Let $\omega(\phi)$ be the omega-limit set of the orbit of $\Phi(t)$ through $\phi \in C_+$ and $M_1 := \{(u_0, 0, 0)\}$. Then we show $\cup_{\phi \in M_\partial} \omega(\phi) \subset M_1$. In fact, for any $\phi \in M_\partial$, we have $u_t(\phi) \in \partial W$. For $t \geq 0$, we have that either $\omega(t, \phi) \equiv 0$ or $v(t, \phi) \equiv 0$. We claim that $\omega(t, \phi) \equiv 0$ for all $t \geq 0$. Otherwise, $\omega(t, \phi) \not\equiv 0$ for some $t_1 \geq 0$. Then by Lemma 2.2 (ii) that $\omega(t, \phi) > 0$ and $v(t, \phi) > 0$ for $t > t_1 + \tau$, a contradiction to $\omega(t, \phi) \equiv 0$ or $v(t, \phi) \equiv 0$ for all $t \geq 0$. This proves the claim. The claim, together with the second equation of (1.2) and Lemma 2.2 (iii), gives $v(t, \phi) \equiv 0$ for $t > 0$. Then it follows from the first equation of (1.2) that $\lim_{t \to \infty} u(t, x) = u_0$ uniformly for $x \in \overline{\Omega}$. Therefore, we have $\cup_{\phi \in M_\partial} \omega(\phi) \subset M_1$.

Define a continuous function $p : C_+ \to [0, \infty)$ by

$$p(\phi) = \min\{\min_{x \in \overline{\Omega}} \phi_2(x, 0), \ \ \min_{x \in \overline{\Omega}} \phi_3(x, 0)\} \text{ for } \phi \in C_+.$$

One can easily see that $p^{-1}(0, +\infty) \subset W$. If $p(\phi) = 0$ where $\phi \in W$ or $p(\phi) > 0$, then $p(\Phi(t)(\phi)) > 0$ for all $t > 0$. Thus, p is a generalized distance function for the semiflow $\Phi(t)$. Note that $\Phi(t)(\phi)$ converges to E_0 in M_∂ and $\{E_0\}$ is an isolated invariant set in C_+, and $W^s(E_0) \cap W = \emptyset$, where $W^s(E_0)$ is the stable set of E_0. Moreover, there is no cycle in M_∂ from E_0 to E_0. By Theorem 3 in [46], there is a $\delta > 0$ such that $\min\{p(\psi)\} > \delta$ for any $\phi \in W$. This completes the proof.

3 Existence of travelling wave solutions when $\Omega = R^n$

Spatial effect plays an important role in studying the propagation speed of infectious diseases. For cooperative systems, some researchers have proved that the spreading speed is equal to the minimum wave speed [44]. For some non-cooperative systems, it has been shown that the spreading speed is also equal to the minimum wave speed [45]. However, it is still unknown if there is a similar result for other non-cooperative systems, including virus infection models. It was found through numerical simulations that the virus propagation speed may be greater than its minimum propagation speed [43]. Under such circumstances, it is extremely difficult to calculate the spread speed of the virus. However, one can study the traveling wave solutions, which can be used as a function of measurable parameters for quantitative prediction. In this section, we study the existence of traveling wave solutions of (1.2) when $\Omega = R^n$.

3.1 Preliminaries

We adopt some notations for the standard ordering in R^3. For $u = (u_1, u_2, u_3)^{\mathrm{T}}$ and $v = (v_1, v_2, v_3)^{\mathrm{T}}$, we denote $u \le v$ if $u_i \le v_i,\ i = 1, 2, 3;\ \ u < v$ if $u \le v$ but $u \ne v$; and $u \ll v$ if $u \le v$ but $u_i \ne v_i,\ i = 1, 2, 3$. If $u \le v$, we also denote $(u, v] = \{\omega \in R^3 : u < \omega \le v\}$, $[u, v) = \{\omega \in R^3 : u \le \omega < v\}$ and $[u, v] = \{\omega \in R^3 : u \le \omega \le v\}$. We represent the Euclidean norm in terms of $|\cdot|$ in R^3 and the supremum norm in terms of $\|\cdot\|$ in $C([-\tau, 0], R^3)$.

For convenience, we assume that uninfected cells, infected cells and viruses have the same diffusion coefficient D_0. We also assume that $f(u, \omega)\omega = \beta u\omega,\ g(u, v)v = \alpha uv$, and $h(\omega) = \omega$. Thus, the model becomes

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= D_0\Delta u + \xi - \beta u(x,t)\omega(x,t) - \alpha u(x,t)v(x,t) - du(x,t),\\
\frac{\partial \omega}{\partial t} &= D_0\Delta \omega + e^{-m\tau}\beta u(x,t-\tau)\omega(x,t-\tau) + e^{-m\tau}\alpha u(x,t-\tau)v(x,t-\tau) - a\omega(x,t),\\
\frac{\partial v}{\partial t} &= D_0\Delta v + k\omega(x,t) - \mu v(x,t) - \alpha u(x,t)v(x,t).
\end{aligned}
$$

To facilitate the calculation, we introduce the dimensionless variables by letting

$$
\begin{gathered}
\bar{u} = \frac{d}{\xi}u,\quad \bar{v} = \frac{d}{\xi}v,\quad \bar{\omega} = \frac{d}{\xi}\omega,\quad \bar{t} = dt,\quad \bar{\beta} = \frac{\beta\xi}{d^2},\quad \bar{\alpha} = \frac{\alpha\xi}{d^2},\\
\bar{a} = \frac{a}{d},\quad \bar{k} = \frac{k}{d},\quad \bar{\mu} = \frac{\mu}{d},\quad \bar{D_0} = \frac{D_0}{d},\quad \bar{d}_m = \frac{m}{d},\quad \bar{\tau} = d\tau.
\end{gathered}
$$

After dropping the bars on $u,\ v,\ \omega,\ t,\ \beta,\ \alpha,\ a,\ k,\ \mu,\ D_0$ and τ, we get

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= D_0\Delta u + 1 - \beta u(x,t)\omega(x,t) - \alpha u(x,t)v(x,t) - u(x,t),\\
\frac{\partial \omega}{\partial t} &= D_0\Delta \omega + e^{-d_m\tau}\beta u(x,t-\tau)\omega(x,t-\tau) + e^{-d_m\tau}\alpha u(x,t-\tau)v(x,t-\tau) - a\omega(x,t),\\
\frac{\partial v}{\partial t} &= D_0\Delta v + k\omega(x,t) - \mu v(x,t) - \alpha u(x,t)v(x,t).
\end{aligned}
\tag{3.1}
$$

Recall that system (3.1) always has a unique infection-free steady state $E_0 = (1, 0, 0)$. In addition, when $R_0 > 1$, where $R_0 = \frac{\beta e^{-d_m\tau}}{a} + \frac{\alpha k e^{-d_m\tau}}{a(\mu+\alpha)}$, it also has a unique homogeneous infected steady state $E^* = (u^*,\ \omega^*,\ v^*)$, where

$$
\begin{aligned}
u^* &= \frac{-(\mu\beta + k\alpha - \alpha a e^{d_m\tau}) + \sqrt{(\mu\beta + k\alpha - \alpha a e^{d_m\tau})^2 + 4\alpha\beta\mu a e^{d_m\tau}}}{2\alpha\beta},\\
\omega^* &= \frac{2\alpha\beta + \mu\beta + k\alpha - \alpha a e^{d_m\tau} - \sqrt{(\mu\beta + k\alpha - \alpha a e^{d_m\tau})^2 + 4\alpha\beta\mu a e^{d_m\tau}}}{2\alpha\beta a e^{d_m\tau}},\\
v^* &= \frac{k(2\alpha\beta + \mu\beta + k\alpha - \alpha a e^{d_m\tau} - \sqrt{(\mu\beta + k\alpha - \alpha a e^{d_m\tau})^2 + 4\alpha\beta\mu a e^{d_m\tau}})}{\alpha a e^{d_m\tau}(\mu\beta - k\alpha + \alpha a e^{d_m\tau} + \sqrt{(\mu\beta + k\alpha - \alpha a e^{d_m\tau})^2 + 4\alpha\beta\mu a e^{d_m\tau}})}.
\end{aligned}
$$

The objective of this section is to find traveling wave solutions connecting the infection-free steady state E_0 and the infection steady state E^*. In order to simplify the mathematical analysis, the following changes are made to the variables $\hat{u}(x,t) = 1 - u(x,t),\ \hat{\omega}(x,t) = \omega(x,t),\ \hat{v}(x,t) = v(x,t)$. We remove the hats and get

$$\begin{aligned}\frac{\partial u}{\partial t} =& D_0\Delta u - u(x,t) + \beta(1-u(x,t))\omega(x,t) + \alpha(1-u(x,t))v(x,t),\\ \frac{\partial \omega}{\partial t} =& D_0\Delta\omega + e^{-d_m\tau}\beta(1-u(x,t-\tau))\omega(x,t-\tau)\\ &+ e^{-d_m\tau}\alpha(1-u(x,t-\tau))v(x,t-\tau) - a\omega(x,t),\\ \frac{\partial v}{\partial t} =& D_0\Delta v + k\omega(x,t) - \mu v(x,t) - \alpha(1-u(x,t))v(x,t).\end{aligned} \tag{3.2}$$

System (3.2) always has a unique infection-free steady state $E_0(0,0,0)$. When $R_0 > 1$, it also has a unique homogeneous infected steady state $E^*(1-u^*, \omega^*, v^*)$.

To prove the existence of traveling wave solutions of system (3.2), we study the following general reaction-diffusion system with time delay

$$\begin{aligned}\frac{\partial u}{\partial t} &= D_0\Delta u + f_1(u_t(x), \omega_t(x), v_t(x)),\\ \frac{\partial \omega}{\partial t} &= D_0\Delta\omega + f_2(u_t(x), \omega_t(x), v_t(x)),\\ \frac{\partial v}{\partial t} &= D_0\Delta v + f_3(u_t(x), \omega_t(x), v_t(x)),\end{aligned} \tag{3.3}$$

where $t \in R,\ x \in \Omega = R^3,\ D_0 > 0,\ f_i \in C([-\tau,0], R^3) \to R\ (i = 1,2,3)$ is continuous and $u_t(x) \in C([-\tau,0], R)$ is given by $u_t(x)(s) = u(t+s, x),\ s \in [-\tau, 0]$, where for any fixed $x \in \Omega$, $\omega_t(x)$ and $v_t(x)$ are defined similarly. We also assume that the reaction term of system (3.3) satisfies the following the partial quasi-monotonicity (PQM) conditions [49, 50].

(PQM) There are three positive constants $\beta_i > 0\ (i = 1,2,3)$ such that

$$\begin{aligned}&f_{c1}(\phi_1, \varphi_1, \psi_1) - f_{c1}(\phi_2, \varphi_2, \psi_2) + \beta_1(\phi_1(0) - \phi_2(0)) \geq 0,\\ &f_{c2}(\phi_1, \varphi_1, \psi_1) - f_{c2}(\phi_1, \varphi_2, \psi_2) + \beta_2(\varphi_1(0) - \varphi_2(0)) \geq 0,\\ &f_{c2}(\phi_1, \varphi_1, \psi_1) - f_{c2}(\phi_2, \varphi_1, \psi_1) \leq 0,\\ &f_{c3}(\phi_1, \varphi_1, \psi_1) - f_{c3}(\phi_2, \varphi_2, \psi_2) + \beta_3(\psi_1(0) - \psi_2(0)) \geq 0,\end{aligned}$$

where $\phi_i,\ \varphi_i,\ \psi_i \in C([-\tau,0], R),\ i = 1,2$, with $0 \leq \phi_2(s) \leq \phi_1(s) \leq M_1,\ 0 \leq \varphi_2(s) \leq \varphi_1(s) \leq M_2$, $0 \leq \psi_2(s) \leq \psi_1(s) \leq M_3,\ s \in [-\tau, 0]$.

A traveling wave solution of system (3.3) is of the form $u(x,t) = \phi(x \cdot e + ct),\ \omega(x,t) = \varphi(x \cdot e + ct),\ v(x,t) = \psi(x \cdot e + ct)$, where $\phi,\ \varphi,\ \psi \in C^2(R,R)$, $c > 0$ is the wave speed, and e is a unit vector in R^n. Substituting these expressions of $u,\ \omega$, and v into (3.3) and denoting the traveling wave coordinate $x \cdot e + ct$ by t, we get the corresponding wave equation

$$\begin{aligned} D_0\phi''(t) - c\phi'(t) + f_{c1}(\phi_t, \varphi_t, \psi_t) = 0, \\ D_0\varphi''(t) - c\varphi'(t) + f_{c2}(\phi_t, \varphi_t, \psi_t) = 0, \\ D_0\psi''(t) - c\psi'(t) + f_{c3}(\phi_t, \varphi_t, \psi_t) = 0, \end{aligned} \tag{3.4}$$

where $\phi_t(\zeta) = \phi(\zeta + t)$, $\varphi_t(\zeta) = \varphi(\zeta + t)$, $\psi_t(\zeta) = \psi(\zeta + t)$, the functions $f_{ci}(\phi, \varphi, \psi) : X_c = C([-c\tau, 0], R^3) \to R$ $(i = 1, 2, 3)$ are defined by $f_{ci}(\phi, \varphi, \psi) = f_i(\phi^c, \varphi^c, \psi^c)$, and

$$\phi^c(s) = \phi(cs),\ \varphi^c(s) = \varphi(cs),\ \psi^c(s) = \psi(cs),\ s \in [-\tau, 0],\ i = 1, 2, 3.$$

Based on [47, 48], we propose the following assumptions

(A_1) $f_i(0, 0, 0) = f_i(k_1, k_2, k_3) = 0,\ i = 1, 2, 3.$

(A_2) There are three positive constants $L_i > 0$ $(i = 1, 2, 3)$ such that

$$|f_i(\Phi) - f_i(\Psi)| \le L_i \|\Phi - \Psi\|$$

for $\Phi = (\phi_1, \varphi_1, \psi_1),\ \Psi = (\phi_2, \varphi_2, \psi_2) \in C([-\tau, 0], R^3)$ with $0 \le \phi_i(s) \le M_1,\ 0 \le \varphi_i(s) \le M_2,\ 0 \le \psi_i(s) \le M_3,\ s \in [-\tau, 0],\ M_j \ge k_j$ $(i = 1, 2.\ j = 1, 2, 3)$ are positive constants.

(A_1) implies that $(0, 0, 0)$ and (k_1, k_2, k_3) are two steady states of (3.3). Without loss of generality, we assume the boundary conditions for traveling wave solutions,

$$\begin{aligned} \lim_{t \to -\infty} (\phi(t), \varphi(t), \psi(t)) = (\phi_-, \varphi_-, \psi_-) = (0, 0, 0), \\ \lim_{t \to +\infty} (\phi(t), \varphi(t), \psi(t)) = (\phi_+, \varphi_+, \psi_+) = (k_1, k_2, k_3), \end{aligned} \tag{3.5}$$

and seeking for traveling wave solutions to connect the two steady states.

In order to apply the Schauder's fixed point theorem, we consider the continuity of operators. For this purpose, a topology is introduced in $C(R, R^3)$. Let $\mu_0 > 0$ and equipped $C(R, R^3)$ with the exponential decay norm given by

$$|\Phi|_{\mu_0} = \sup_{t \in R} e^{-\mu_0 |t|} |\Phi(t)|_{R^3}.$$

Define

$$B_{\mu_0}(R, R^3) = \{\Phi \in C(R, R^3) : |\Phi|_{\mu_0} < \infty\}.$$

It is easy to show that $(B_{\mu_0}(R, R^3), |\cdot|_{\mu_0})$ is a Banach space.

For system (3.3), we give the definitions of upper and lower solutions as follows.

Definition 3.1 *A pair of continuous functions* $\bar{\rho}=(\bar{\phi},\bar{\varphi},\bar{\psi})$ *and* $\underline{\rho}=(\underline{\phi},\underline{\varphi},\underline{\psi})$ *are called a pair of upper and lower solutions of system* (3.3) *if there is a finite set of multiple points* $\Lambda=\{t_1,t_2,\cdots,t_m\}$ *such that* $\bar{\rho}$ *and* $\underline{\rho}$ *are twice continuously differentiable in* $R\setminus\Lambda$*, they are essentially bounded on* $R\setminus\Lambda$*, and*

$$\begin{aligned}
&D_0\bar{\phi}''(t)-c\bar{\phi}'(t)+f_{c1}(\bar{\phi}_t,\bar{\varphi}_t,\bar{\psi}_t)\le 0,\ a.e.\ in\ R,\\
&D_0\bar{\varphi}''(t)-c\bar{\varphi}'(t)+f_{c2}(\underline{\phi_t},\bar{\varphi}_t,\bar{\psi}_t)\le 0,\ a.e.\ in\ R,\\
&D_0\bar{\psi}''(t)-c\bar{\psi}'(t)+f_{c3}(\bar{\phi}_t,\bar{\varphi}_t,\bar{\psi}_t)\le 0,\ a.e.\ in\ R,
\end{aligned}$$

and

$$\begin{aligned}
&D_0\underline{\phi}''(t)-c\underline{\phi}'(t)+f_{c1}(\underline{\phi_t},\underline{\varphi_t},\underline{\psi_t})\ge 0,\ a.e.\ in\ R,\\
&D_0\underline{\varphi}''(t)-c\underline{\varphi}'(t)+f_{c2}(\bar{\phi}_t,\underline{\varphi_t},\underline{\psi_t})\ge 0,\ a.e.\ in\ R,\\
&D_0\underline{\psi}''(t)-c\underline{\psi}'(t)+f_{c3}(\underline{\phi_t},\underline{\varphi_t},\underline{\psi_t})\ge 0,\ a.e.\ in\ R.
\end{aligned}$$

3.2 The existence of traveling wave solutions for system (3.3)

In this subsection, we study the nonlinear reaction term of system (3.3) that satisfies (PQM). In addition to this, we also assume that a pair of upper and lower solutions satisfy the following properties

$$\begin{aligned}
&(P_1)\quad (0,0,0)\le(\underline{\phi}(t),\ \underline{\varphi}(t),\ \underline{\psi}(t))\le(\bar{\phi}(t),\ \bar{\varphi}(t),\ \bar{\psi}(t))\le(M_1,M_2,M_3),\quad t\in R.\\
&(P_2)\lim_{t\to-\infty}(\underline{\phi}(t),\underline{\varphi}(t),\underline{\psi}(t))=(0,0,0),\quad \lim_{t\to+\infty}(\bar{\phi}(t),\ \bar{\varphi}(t),\bar{\psi}(t))=(k_1,k_2,k_3).
\end{aligned}$$

Let $C_K(R,R^3)=\{(\phi,\varphi,\psi)\in C(R,R^3):(0,0,0)\le(\phi(s),\varphi(s),\psi(s))\le(M_1,M_2,M_3),\ s\in R\}$. For the constants $\beta_1,\ \beta_2,\ \beta_3>0$ in (PQM), define $H:C_K(R,R^3)\to C(R,R^3)$ by

$$\begin{aligned}
H_1(\phi,\varphi,\psi)(t)&=f_{c1}(\phi_t,\varphi_t,\psi_t)+\beta_1\phi(t),\\
H_2(\phi,\varphi,\psi)(t)&=f_{c2}(\phi_t,\varphi_t,\psi_t)+\beta_2\varphi(t),\\
H_3(\phi,\varphi,\psi)(t)&=f_{c3}(\phi_t,\varphi_t,\psi_t)+\beta_3\psi(t).
\end{aligned}$$

The operators H_1,H_2 and H_3 have the following properties.

Lemma 3.1 *Assume that* (A_1) *and* (PQM) *hold. Then*

$$\begin{aligned}
&H_2(\phi_1,\varphi_2,\psi_2)(t)\le H_2(\phi_1,\varphi_1,\psi_1)(t),\ H_2(\phi_1,\varphi_1,\psi_1)(t)\le H_2(\phi_2,\varphi_1,\psi_1)(t),\\
&H_2(\phi_1,\varphi_1,\psi_1)(t)\ge H_2(\phi_1,\varphi_2,\psi_1)(t)
\end{aligned}$$

for $(\phi_i,\varphi_i,\psi_i)\in C_K(R,R^3)$ *with* $0\le\phi_2(s)\le\phi_1(s)\le M_1,\ \ 0\le\varphi_2(s)\le\varphi_1(s)\le M_2,\ \ 0\le\psi_2(s)\le\psi_1(s)\le M_3,\ s\in R,\ i=1,2.$

Proof. By (PQM), we calculate

$$H_2(\phi_1,\varphi_1,\psi_1)(t)-H_2(\phi_1,\varphi_2,\psi_2)(t)=f_{c2}(\phi_1,\varphi_1,\psi_1)-f_{c2}(\phi_1,\varphi_2,\psi_2)+\beta_2(\phi_1(0)-\phi_2(0))\geq 0,$$

$$H_2(\phi_2,\varphi_1,\psi_1)(t)-H_2(\phi_1,\varphi_1,\psi_1)(t)=f_{c2}(\phi_2,\varphi_1,\psi_1)-f_{c2}(\phi_1,\varphi_1,\psi_1)\geq 0,$$

$$H_2(\phi_1,\varphi_1,\psi_1)(t)-H_2(\phi_1,\varphi_2,\psi_1)(t)=f_{c2}(\phi_1,\varphi_1,\psi_1)-f_{c2}(\phi_1,\varphi_2,\psi_1)+\beta_2(\varphi_1(0)-\varphi_2(0))\geq 0.$$

This completes the proof.

Lemma 3.2 *Assume that (A_1) and (PQM) hold. Then for any $(0,0,0)\leq(\phi,\varphi,\psi)\leq(k_1,k_2,k_3)$, we can obtain*

(i) $H_1(\phi,\varphi,\psi)(t)\geq 0,\ H_3(\phi,\varphi,\psi)(t)\geq 0,\ t\in R.$

(ii) $H_1(\phi_2,\varphi_2,\psi_2)(t)\leq H_1(\phi_1,\varphi_1,\psi_1)(t),\ H_3(\phi_2,\varphi_2,\psi_2)(t)\leq H_3(\phi_1,\varphi_1,\psi_1)(t)$ *for* $t\in R$ *with* $0\leq\phi_2(s)\leq\phi_1(s)\leq M_1,\ \ 0\leq\varphi_2(s)\leq\varphi_1(s)\leq M_2,\ \ 0\leq\psi_2(s)\leq\psi_1(s)\leq M_3.$

From the definitions of H_1, H_2 and H_3, system (3.4) can be rewritten as follows.

$$D_0\phi''(t)-c\phi'(t)-\beta_1\phi(t)+H_1(\phi,\varphi,\psi)(t)=0,$$

$$D_0\varphi''(t)-c\varphi'(t)-\beta_2\varphi(t)+H_2(\phi,\varphi,\psi)(t)=0,$$

$$D_0\psi''(t)-c\psi'(t)-\beta_3\psi(t)+H_3(\phi,\varphi,\psi)(t)=0.$$

Let

$$\lambda_1=\frac{c-\sqrt{c^2+4\beta_1D_0}}{2D_0},\quad \lambda_2=\frac{c+\sqrt{c^2+4\beta_1D_0}}{2D_0},$$

$$\lambda_3=\frac{c-\sqrt{c^2+4\beta_2D_0}}{2D_0},\quad \lambda_4=\frac{c+\sqrt{c^2+4\beta_2D_0}}{2D_0},$$

$$\lambda_5=\frac{c-\sqrt{c^2+4\beta_3D_0}}{2D_0},\quad \lambda_6=\frac{c+\sqrt{c^2+4\beta_3D_0}}{2D_0}.$$

Define $F=(F_1,F_2,F_3):C_K(R,R^3)\to C(R,R^3)$ by

$$F_1(\phi,\varphi,\psi)(t)=\frac{1}{D_0(\lambda_2-\lambda_1)}\left[\int_{-\infty}^{t}e^{\lambda_1(t-s)}H_1(\phi,\varphi,\psi)(s)ds+\int_t^{\infty}e^{\lambda_2(t-s)}H_1(\phi,\varphi,\psi)(s)ds\right],$$

$$F_2(\phi,\varphi,\psi)(t)=\frac{1}{D_0(\lambda_4-\lambda_3)}\left[\int_{-\infty}^{t}e^{\lambda_3(t-s)}H_2(\phi,\varphi,\psi)(s)ds+\int_t^{\infty}e^{\lambda_4(t-s)}H_2(\phi,\varphi,\psi)(s)ds\right],$$

$$F_3(\phi,\varphi,\psi)(t)=\frac{1}{D_0(\lambda_6-\lambda_5)}\left[\int_{-\infty}^{t}e^{\lambda_5(t-s)}H_3(\phi,\varphi,\psi)(s)ds+\int_t^{\infty}e^{\lambda_6(t-s)}H_3(\phi,\varphi,\psi)(s)ds\right]$$

for $(\phi,\varphi,\psi)\in C_K(R,R^3)$. It is easy to conclude that $F_1(\phi,\varphi,\psi),F_2(\phi,\varphi,\psi)$ and $F_3(\phi,\varphi,\psi)$ satisfy

$$D_0F_1''(\phi,\varphi,\psi)-cF_1'(\phi,\varphi,\psi)-\beta_1F_1(\phi,\varphi,\psi)+H_1(\phi,\varphi,\psi)=0,$$

$$D_0F_2''(\phi,\varphi,\psi)-cF_2'(\phi,\varphi,\psi)-\beta_2F_2(\phi,\varphi,\psi)+H_2(\phi,\varphi,\psi)=0,$$

$$D_0F_3''(\phi,\varphi,\psi)-cF_3'(\phi,\varphi,\psi)-\beta_3F_3(\phi,\varphi,\psi)+H_3(\phi,\varphi,\psi)=0.$$

Similar to Lemma 3.1 and Lemma 3.2, we have the following lemma to explain some properties of F.

Lemma 3.3 *Assume that* (A_1) *and* (PQM) *hold. For any* $(0,0,0) \le (\phi,\varphi,\psi) \le (M_1,M_2,M_3)$, *we obtain*

$$
\begin{aligned}
&F_1(\phi_2,\varphi_2,\psi_2)(t) \le F_1(\phi_1,\varphi_1,\psi_1)(t),\ \ F_2(\phi_1,\varphi_2,\psi_2)(t) \le F_2(\phi_1,\varphi_1,\psi_1)(t),\\
&F_2(\phi_1,\varphi_1,\psi_1)(t) \le F_2(\phi_2,\varphi_1,\psi_1)(t),\ \ F_3(\phi_2,\varphi_2,\psi_2)(t) \le F_3(\phi_1,\varphi_1,\psi_1)(t)
\end{aligned}
$$

for $(\phi,\varphi,\psi) \in C_K(R,R^3)$ with $0 \le \phi_2(s) \le \phi_1(s) \le M_1,\ \ 0 \le \varphi_2(s) \le \varphi_1(s) \le M_2,\ \ 0 \le \psi_2(s) \le \psi_1(s) \le M_3$.

Now, we put forward the following profile set to seek the traveling wave solutions of system (3.3).

$$
\Gamma((\underline{\phi},\underline{\varphi},\underline{\psi}),(\bar{\phi},\bar{\varphi},\bar{\psi})) = \begin{cases} (i)\ (\phi(t),\varphi(t),\psi(t)) \in C(R,R^3); \\ (ii)\ \underline{\phi}(t) \le \phi(t) \le \bar{\phi}(t),\ \ \underline{\varphi}(t) \le \varphi(t) \le \bar{\varphi}(t),\ \ \underline{\psi}(t) \le \psi(t) \le \bar{\psi}(t). \end{cases}
$$

Obviously, $\Gamma((\underline{\phi},\underline{\varphi},\underline{\psi}),(\bar{\phi},\bar{\varphi},\bar{\psi}))$ is non-empty, convex, closed and bounded.

Lemma 3.4 *Assume that* (A_2) *holds. Then* $F = (F_1,F_2,F_3) : C_K(R,R^3) \to C(R,R)$ *is continuous in point of the norm* $|\cdot|_{\mu_0}$ *in* $B_{\mu_0}(R,R^3)$.

Proof. We pick $\mu_0 > 0$ such that $\mu_0 < \min\{-\lambda_1,\lambda_2,-\lambda_3,\lambda_4,-\lambda_5,\lambda_6\}$. For any $\epsilon_1 > 0$, we let $\sigma_1 < \frac{\epsilon_1}{L_1 e^{\mu_0 c\tau} + \beta_1}$. Let $\Phi = (\phi_1,\varphi_1,\psi_1)$ and $\Psi = (\phi_2,\varphi_2,\psi_2) \in C_K(R,R^3)$ with

$$
|\Phi - \Psi|_{\mu_0} = \sup_{t\in R} |\Phi(t) - \Psi(t)| e^{-\mu_0|t|} < \sigma_1.
$$

Direct calculations yield

$$
\begin{aligned}
&|H_1(\phi_1,\varphi_1,\psi_1) - H_1(\phi_2,\varphi_2,\psi_2)| e^{-\mu_0|t|}\\
\le\ & |f_1(\phi_{1t},\varphi_{1t},\psi_{1t}) - f_1(\phi_{2t},\varphi_{2t},\psi_{2t})| e^{-\mu_0|t|} + \beta_1|\phi_1 - \phi_2|_{\mu_0}\\
\le\ & L_1\|\Phi_t - \Psi_t\|_{X_c} e^{-\mu_0|t|} + \beta_1|\phi_1 - \phi_2|_{\mu_0}\\
\le\ & L_1 \sup_{s\in[-c\tau,0]} |\Phi(s+t) - \Psi(s+t)| e^{-\mu_0|t+s|} \sup_{s\in[-\tau,0]} e^{\mu_0|t+s|} e^{-\mu_0|t|} + \beta_1|\phi_1 - \phi_2|_{\mu_0}\\
\le\ & L_1|\Phi - \Psi|_{\mu_0} e^{-\mu_0|t|} e^{\mu_0|t|} e^{\mu_0 c\tau} + \beta_1|\Phi - \Psi|_{\mu_0}\\
\le\ & (L_1 e^{\mu_0 c\tau} + \beta_1)|\Phi - \Psi|_{\mu_0}\\
\le\ & \epsilon_1.
\end{aligned}
$$

Next, we claim that $F_1 : C_K(R,R^3) \to C(R,R)$ is continuous with respect to the norm $|\cdot|_{\mu_0}$. If $t > 0$, then we have

$$
|F_1(\phi_1,\varphi_1,\psi_1) - F_1(\phi_2,\varphi_2,\psi_2)| e^{-\mu_0|t|}
$$

$$
\begin{aligned}
&= \frac{1}{D_0(\lambda_2-\lambda_1)}\left[\int_{-\infty}^{t} e^{\lambda_1(t-s)}|H_1(\phi_1,\varphi_1,\psi_1)(s)-H_1(\phi_2,\varphi_2,\psi_2)(s)|ds\right]e^{-\mu_0 t}\\
&\quad+\frac{1}{D_0(\lambda_2-\lambda_1)}\left[\int_{t}^{\infty} e^{\lambda_2(t-s)}|H_1(\phi_1,\varphi_1,\psi_1)(s)-H_1(\phi_2,\varphi_2,\psi_2)(s)|ds\right]e^{-\mu_0 t}\\
&\le \frac{\epsilon_1}{D_0(\lambda_2-\lambda_1)}\left[e^{\lambda_1 t}\int_{-\infty}^{0} e^{-(\lambda_1+\mu_0)s}ds+e^{\lambda_1 t}\int_0^t e^{(\mu_0-\lambda_1)s}ds+e^{\lambda_2 t}\int_t^{\infty} e^{(\mu_0-\lambda_2)s}ds\right]e^{-\mu_0 t}\\
&= \frac{\epsilon_1}{D_0(\lambda_2-\lambda_1)}\left[\frac{2\mu_0}{\lambda_1^2-{\mu_0}^2}e^{(\lambda_1-\mu_0)t}+\frac{\lambda_2-\lambda_1}{(\mu_0-\lambda_1)(\lambda_2-\mu_0)}\right]\\
&\le \frac{\epsilon_1}{D_0(\lambda_2-\lambda_1)}\left[\frac{2\mu_0}{\lambda_1^2-{\mu_0}^2}+\frac{\lambda_2-\lambda_1}{(\mu_0-\lambda_1)(\lambda_2-\mu_0)}\right].
\end{aligned}
$$

If $t<0$, we obtain

$$
\begin{aligned}
&|F_1(\phi_1,\varphi_1,\psi_1)-F_1(\phi_2,\varphi_2,\psi_2)|e^{-\mu_0|t|}\\
&= \frac{1}{D_0(\lambda_2-\lambda_1)}\left[\int_{-\infty}^{t} e^{\lambda_1(t-s)}|H_1(\phi_1,\varphi_1,\psi_1)(s)-H_1(\phi_2,\varphi_2,\psi_2)(s)|ds\right]e^{\mu_0 t}\\
&\quad+\frac{1}{D_0(\lambda_2-\lambda_1)}\left[\int_{t}^{\infty} e^{\lambda_2(t-s)}|H_1(\phi_1,\varphi_1,\psi_1)(s)-H_1(\phi_2,\varphi_2,\psi_2)(s)|ds\right]e^{\mu_0 t}\\
&\le \frac{\epsilon_1}{D_0(\lambda_2-\lambda_1)}\left[e^{\lambda_1 t}\int_{-\infty}^{t} e^{-(\lambda_1+\mu_0)s}ds+e^{\lambda_2 t}\int_t^0 e^{-(\lambda_2+\mu_0)s}ds+e^{\lambda_2 t}\int_0^{\infty} e^{(\mu_0-\lambda_2)s}ds\right]e^{\mu_0 t}\\
&= \frac{\epsilon_1}{D_0(\lambda_2-\lambda_1)}\left[\frac{2\mu_0}{\lambda_2^2-{\mu_0}^2}e^{(\mu_0+\lambda_2)t}+\frac{\lambda_1-\lambda_2}{(\mu_0+\lambda_1)(\lambda_2+\mu_0)}\right]\\
&\le \frac{\epsilon_1}{D_0(\lambda_2-\lambda_1)}\left[\frac{2\mu_0}{\lambda_2^2-{\mu_0}^2}+\frac{\lambda_1-\lambda_2}{(\mu_0+\lambda_1)(\lambda_2+\mu_0)}\right].
\end{aligned}
$$

In summary, F_1 is continuous. Similarly, we can prove that $F_2,\ F_3 : C_K(R,R^3)\to C(R,R)$ are continuous. Thus, we see that $F=(F_1,F_2,F_3)$ is continuous with respect to the norm $|\cdot|_{\mu_0}$ in $B_{\mu_0}(R,R^3)$. This completes the proof.

Lemma 3.5 *Assume that* (A_1) *and* (PQM) *hold. Then*

$$F(\Gamma((\underline{\phi},\underline{\varphi},\underline{\psi}),(\bar{\phi},\bar{\varphi},\bar{\psi})))\subset\Gamma((\underline{\phi},\underline{\varphi},\underline{\psi}),(\bar{\phi},\bar{\varphi},\bar{\psi})).$$

Proof. For any (ϕ,φ,ψ) with $(\underline{\phi},\underline{\varphi},\underline{\psi})\le(\phi,\varphi,\psi)\le(\bar{\phi},\bar{\varphi},\bar{\psi})$, it follows from Lemma 3.3 that

$$
\begin{aligned}
F_1(\underline{\phi},\underline{\varphi},\underline{\psi}) &\le F_1(\phi,\varphi,\psi)\le F_1(\bar{\phi},\bar{\varphi},\bar{\psi}),\\
F_2(\bar{\phi},\underline{\varphi},\underline{\psi}) &\le F_2(\phi,\varphi,\psi)\le F_2(\underline{\phi},\bar{\varphi},\bar{\psi}),\\
F_3(\underline{\phi},\underline{\varphi},\underline{\psi}) &\le F_3(\phi,\varphi,\psi)\le F_3(\bar{\phi},\bar{\varphi},\bar{\psi}).
\end{aligned}
$$

By the definition of the upper and lower solution, we obtain

$$D_0\bar{\phi}''(t)-c\bar{\phi}'(t)-\beta_1\bar{\phi}(t)+H_1(\bar{\phi},\bar{\varphi},\bar{\psi})(t)\le 0. \tag{3.6}$$

From Lemma 3.2, we select $(\phi, \varphi, \psi) = (\bar{\phi}, \bar{\varphi}, \bar{\psi})$, and denote $\bar{\phi}_1(t) = F_1(\bar{\phi}, \bar{\varphi}, \bar{\psi})(t)$ such that

$$D_0\bar{\phi}_1''(t) - c\bar{\phi}_1'(t) - \beta_1\bar{\phi}_1(t) + H_1(\bar{\phi}, \bar{\varphi}, \bar{\psi})(t) = 0. \tag{3.7}$$

Letting $y(t) = \bar{\phi}_1(t) - \bar{\phi}(t)$ and combining (3.6) and (3.7), we get the following inequality

$$\begin{aligned}
&D_0(\bar{\phi}_1''(t) - \bar{\phi}''(t)) - c(\bar{\phi}_1'(t) - \bar{\phi}'(t)) - \beta_1(\bar{\phi}_1(t) - \bar{\phi}(t)) \geq 0,\\
&D_0 y''(t) - cy'(t) - \beta_1 y(t) \geq 0.
\end{aligned}$$

From Lemma 3.3 in [12], it can be concluded that $y(t) \leq 0$, which implies that $F_1(\bar{\phi}, \bar{\varphi}, \bar{\psi}) \leq \bar{\phi}$. Similarly, we can obtain $F_1(\underline{\phi}, \underline{\varphi}, \underline{\psi}) \geq \underline{\phi}$, $F_2(\bar{\phi}, \underline{\varphi}, \underline{\psi}) \geq \underline{\varphi}$, $F_2(\underline{\phi}, \bar{\varphi}, \bar{\psi}) \leq \bar{\varphi}$, $F_3(\underline{\phi}, \underline{\varphi}, \underline{\psi}) \geq \underline{\psi}$, and $F_3(\bar{\phi}, \bar{\varphi}, \bar{\psi}) \leq \bar{\psi}$. This completes the proof.

Lemma 3.6 *Assume that* (PQM) *holds. Then* $F : \Gamma((\underline{\phi}, \underline{\varphi}, \underline{\psi}), (\bar{\phi}, \bar{\varphi}, \bar{\psi})) \to \Gamma((\underline{\phi}, \underline{\varphi}, \underline{\psi}), (\bar{\phi}, \bar{\varphi}, \bar{\psi}))$ *is compact.*

Proof. By Lemma 4.6 in [47], we can get the proof of this lemma.

Theorem 3.1 *Assume that* $(A_1), (A_2)$ *and* (PQM) *hold. If there is a pair of upper and lower solutions* $\bar{\rho}(t) = (\bar{\phi}(t), \bar{\varphi}(t), \bar{\psi}(t))$ *and* $\underline{\rho}(t) = (\underline{\phi}(t), \underline{\varphi}(t), \underline{\psi}(t))$ *satisfying the conditions* (P_1) *and* (P_2)*, then system* (3.3) *has a traveling wave solution satisfying* (3.5).

Proof. It follows from Lemma 3.4-3.6 that we claim $F(\Gamma((\underline{\phi}, \underline{\varphi}, \underline{\psi}), (\bar{\phi}, \bar{\varphi}, \bar{\psi}))) \subset \Gamma((\underline{\phi}, \underline{\varphi}, \underline{\psi}), (\bar{\phi}, \bar{\varphi}, \bar{\psi}))$ and F are compact. There exists a fixed point $(\phi^*(t), \varphi^*(t), \psi^*(t)) \in \Gamma((\underline{\phi}, \underline{\varphi}, \underline{\psi}), (\bar{\phi}, \bar{\varphi}, \bar{\psi}))$ by Schauder's fixed point theorem, which gives a solution of (3.3). Next, in order to prove that the solution is a traveling wave solution, it is necessary to verify the asymptotic boundary conditions (3.5).

By (P_2) and the obvious fact

$$(0, 0, 0) \leq (\underline{\phi}, \underline{\varphi}, \underline{\psi}) \leq (\phi^*(t), \varphi^*(t), \psi^*(t)) \leq (\bar{\phi}, \bar{\varphi}, \bar{\psi}) \leq (M_1, M_2, M_3),$$

we obtain

$$\begin{aligned}
&\lim_{t\to-\infty}(\phi^*(t), \varphi^*(t), \psi^*(t)) = (0, 0, 0),\\
&\lim_{t\to+\infty}(\phi^*(t), \varphi^*(t), \psi^*(t)) = (k_1, k_2, k_3).
\end{aligned}$$

Therefore, the fixed point $(\phi^*(t), \varphi^*(t), \psi^*(t))$ satisfies the asymptotic boundary conditions. This completes the proof.

3.3 The existence of traveling wave solutions for system (3.2)

In this subsection, we will use the results in subsection 3.2 to prove the existence of the traveling wave solutions of (3.2). From (3.4), we have

$$
\begin{aligned}
D_0\phi''(t) - c\phi'(t) + f_{c1}(\phi_t, \varphi_t, \psi_t) = 0,\\
D_0\varphi''(t) - c\varphi'(t) + f_{c2}(\phi_t, \varphi_t, \psi_t) = 0,\\
D_0\psi''(t) - c\psi'(t) + f_{c3}(\phi_t, \varphi_t, \psi_t) = 0,
\end{aligned}
$$

where

$$
\begin{aligned}
&f_{c1}(\phi_t, \varphi_t, \psi_t) = -\phi(t) + \beta(1-\phi(t))\varphi(t) + \alpha(1-\phi(t))\psi(t),\\
&f_{c2}(\phi_t, \varphi_t, \psi_t) = e^{-d_m\tau}\beta(1-\phi(t-c\tau))\varphi(t-c\tau) + e^{-d_m\tau}\alpha(1-\phi(t-c\tau))\psi(t-c\tau) - a\varphi(t),\\
&f_{c3}(\phi_t, \varphi_t, \psi_t) = k\varphi(t) - \mu\psi(t) - \alpha(1-\phi(t))\psi(t).
\end{aligned}
$$

The following asymptotic boundary conditions are satisfied.

$$
\lim_{t\to-\infty}(\underline{\phi}(t), \underline{\varphi}(t), \underline{\psi}(t)) = (0,0,0), \quad \lim_{t\to+\infty}(\bar{\phi}(t), \bar{\varphi}(t), \bar{\psi}(t)) = (1-k_1, k_2, k_3),
$$

where $k_1 = u^*,\ k_2 = \omega^*,\ k_3 = v^*$.

Lemma 3.7 *The nonlinear reaction term of system* (3.2) *satisfies* (A_1), (A_2), *and (PQM).*

Proof. For any $\phi_i,\ \varphi_i,\ \psi_i \in C([-\tau,0],R),\ i=1,2$, with $0 \le \phi_2(s) \le \phi_1(s) \le M_1,\ 0 \le \varphi_2(s) \le \varphi_1(s) \le M_2,\ 0 \le \psi_2(s) \le \psi_1(s) \le M_3,\ s \in [-\tau,0]$, we obtain

$$
\begin{aligned}
& f_{c1}(\phi_{1t}, \varphi_{1t}, \psi_{1t}) - f_{c1}(\phi_{2t}, \varphi_{2t}, \psi_{2t})\\
=\ & -\phi_1(0) + \beta(1-\phi_1(0))\varphi_1(0) + \alpha(1-\phi_1(0))\psi_1(0)\\
& + \phi_2(0) - \beta(1-\phi_2(0))\varphi_2(0) - \alpha(1-\phi_2(0))\psi_2(0)\\
\ge\ & -(\phi_1(0)-\phi_2(0)) - \beta\varphi_1(0)(\phi_1(0)-\phi_2(0)) - \alpha\psi_1(0)(\phi_1(0)-\phi_2(0))\\
=\ & -(\phi_1(0)-\phi_2(0)) - (\beta\varphi_1(0)+\alpha\psi_1(0))(\phi_1(0)-\phi_2(0))\\
\ge\ & -(1+\beta M_2+\alpha M_3)(\phi_1(0)-\phi_2(0)).
\end{aligned}
$$

Let $\beta_1 = 1+\beta M_2+\alpha M_3 > 0$. This implies that $f_{c1}(\phi_{1t}, \varphi_{1t}, \psi_{1t}) - f_{c1}(\phi_{2t}, \varphi_{2t}, \psi_{2t}) + \beta_1(\phi_1(0) - \phi_2(0)) \ge 0$.

For $f_{c2}(\phi, \varphi, \psi)$, we have

$$
\begin{aligned}
& f_{c2}(\phi_{1t}, \varphi_{1t}, \psi_{1t}) - f_{c2}(\phi_{1t}, \varphi_{2t}, \psi_{2t})\\
=\ & e^{-d_m\tau}\beta(1-\phi_1(-c\tau))\varphi_1(-c\tau) + e^{-d_m\tau}\alpha(1-\phi_1(-c\tau))\psi_1(-c\tau) - a\varphi_1(0)\\
& - e^{-d_m\tau}\beta(1-\phi_1(-c\tau))\varphi_2(-c\tau) - e^{-d_m\tau}\alpha(1-\phi_1(-c\tau))\psi_2(-c\tau) + a\varphi_2(0)
\end{aligned}
$$

$$
\begin{aligned}
&\geq \; e^{-d_m\tau}\beta(1-\phi_1(-c\tau))\varphi_2(-c\tau)+e^{-d_m\tau}\alpha(1-\phi_1(-c\tau))\psi_2(-c\tau)-a(\varphi_1(0)-\varphi_2(0)) \\
&\quad -e^{-d_m\tau}\beta(1-\phi_1(-c\tau))\varphi_2(-c\tau)-e^{-d_m\tau}\alpha(1-\phi_1(-c\tau))\psi_2(-c\tau) \\
&= \; -a(\varphi_1(0)-\varphi_2(0)).
\end{aligned}
$$

Let $\beta_2 = a$. Then $f_{c2}(\phi_1,\varphi_1,\psi_1) - f_{c2}(\phi_1,\varphi_2,\psi_2) + \beta_2(\varphi_1(0)-\varphi_2(0)) \geq 0$.

Similarly, the following results are obtained

$$
\begin{aligned}
&f_{c2}(\phi_{1t},\varphi_{1t},\psi_{1t}) - f_{c2}(\phi_{2t},\varphi_{1t},\psi_{1t}) \\
= \; &e^{-d_m\tau}\beta(1-\phi_1(-c\tau))\varphi_1(-c\tau)+e^{-d_m\tau}\alpha(1-\phi_1(-c\tau))\psi_1(-c\tau)-a\varphi_1(0) \\
&-e^{-d_m\tau}\beta(1-\phi_2(-c\tau))\varphi_1(-c\tau)-e^{-d_m\tau}\alpha(1-\phi_2(-c\tau))\psi_1(-c\tau)+a\varphi_1(0) \\
= \; &e^{-d_m\tau}\beta(\phi_2(-c\tau)-\phi_1(-c\tau))\varphi_1(-c\tau)+e^{-d_m\tau}\alpha(\phi_2(-c\tau)-\phi_1(-c\tau))\psi_1(-c\tau) \\
\leq \; &0.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&f_{c3}(\phi_{1t},\varphi_{1t},\psi_{1t}) - f_{c3}(\phi_{2t},\varphi_{2t},\psi_{2t}) \\
= \; &k\varphi_1(0)-\mu\psi_1(0)-\alpha(1-\phi_1(0))\psi_1(0)-k\varphi_2(0)+\mu\psi_2(0)+\alpha(1-\phi_2(0))\psi_2(0) \\
= \; &k(\varphi_1(0)-\varphi_2(0))-\mu(\psi_1(0)-\psi_2(0))-\alpha(\psi_1(0)-\psi_2(0))+\alpha(\phi_1(0)\psi_1(0)-\phi_2(0)\psi_2(0)) \\
\geq \; &-(\mu+\alpha)(\psi_1(0)-\psi_2(0)).
\end{aligned}
$$

Let $\beta_3 = \mu+\alpha > 0$. We derive that $f_{c3}(\phi_1,\varphi_1,\psi_1) - f_{c3}(\phi_2,\varphi_2,\psi_2) + \beta_3(\psi_1(0)-\psi_2(0)) \geq 0$. This completes the proof.

Next, we construct a pair of upper and lower solutions of system (3.2). Let $c^* = 2\sqrt{D_0K_0}$, where

$$
K_0 = \max\left\{ \frac{\beta k_2 + \alpha k_3 - 1 + k_1}{1-k_1},\; \frac{k_2(\beta-a)+\alpha k_3}{k_2},\; \frac{kk_2}{k_3} \right\}.
$$

If $R_0 > 1$ and $c > c^*$, then there exists $\lambda_0 \in [\lambda_-, \lambda_+]$ such that

$$
D_0\lambda_0^2 - c\lambda_0 + K_0 \leq 0,
$$

where

$$
\lambda_- = \frac{c-\sqrt{c^2-4D_0K_0}}{2D_0}, \quad \lambda_+ = \frac{c+\sqrt{c^2-4D_0K_0}}{2D_0}.
$$

We select appropriate $\varepsilon_i > 0$ $(i = 1,2,3)$ that satisfy the following inequalities,

$$
\begin{aligned}
&(\beta-a)(k_2+\varepsilon_1)+\alpha M_3 < 0, \\
&k\varepsilon_1 - \mu\varepsilon_2 - \alpha k_1\varepsilon_2 + \frac{\alpha(k_3+\varepsilon_3)}{1-k_1} < 0.
\end{aligned}
$$

Then, for $\lambda > 0$ we define continuous functions $\bar{\rho}(t) = (\bar{\phi}(t), \bar{\varphi}(t), \bar{\psi}(t))$ and $\underline{\rho}(t) = (\underline{\phi}(t), \underline{\varphi}(t), \underline{\psi}(t))$ as follows,

$$\bar{\phi}(t) = \min\{(1-k_1)e^{\lambda_0 t}, 1, 1-k_1+\frac{1}{1-k_1}e^{-\lambda t}\}, \quad \bar{\varphi}(t) = \min\{k_2 e^{\lambda_0 t}, k_2+\varepsilon_1 e^{-\lambda t}\},$$
$$\bar{\psi}(t) = \min\{k_3 e^{\lambda_0 t}, k_3+\varepsilon_2 e^{-\lambda t}\}, \quad \underline{\phi}(t) = \max\{0, 1-k_1-\frac{2-k_1}{k_1}e^{-\lambda t}\},$$
$$\underline{\varphi}(t) = \max\{0, k_2-\frac{2k_2}{k_1}e^{-\lambda t}\}, \quad \underline{\psi}(t) = \max\{0, k_3-\frac{2k_3}{k_1}e^{-\lambda t}\}.$$

We see that $M_1 = \sup_{t\in R}\bar{\phi} > 1-k_1$, $M_2 = \sup_{t\in R}\bar{\varphi} > k_2$, $M_3 = \sup_{t\in R}\bar{\psi} > k_3$, $\bar{\rho}(t) = (\bar{\phi}(t), \bar{\varphi}(t), \bar{\psi}(t))$ and $\underline{\rho}(t) = (\underline{\phi}(t), \underline{\varphi}(t), \underline{\psi}(t))$ satisfy (P_1) and (P_2).

Lemma 3.8 *There exists $\bar{\lambda}^* > 0$ such that, for any $\lambda \in (0, \bar{\lambda}^*)$, $\bar{\rho}(t) = (\bar{\phi}(t), \bar{\varphi}(t), \bar{\psi}(t))$ is an upper solution of system* (3.2).

Proof. For $\bar{\phi}(t)$, let us consider three cases.

Case 1. $(1-k_1)e^{\lambda_0 t} \le \min\{1, 1-k_1+\frac{1}{1-k_1}e^{-\lambda t}\}$. Then $\bar{\phi}(t) = (1-k_1)e^{\lambda_0 t}$ and $\bar{\varphi}(t) = k_2 e^{\lambda_0 t}$, $\bar{\psi}(t) = k_3 e^{\lambda_0 t}$. It follows that

$$\begin{aligned}
& D_0\bar{\phi}''(t) - c\bar{\phi}'(t) - \bar{\phi}(t) + \beta(1-\bar{\phi}(t))\bar{\varphi}(t) + \alpha(1-\bar{\phi}(t))\bar{\psi}(t) \\
\le\ & D_0\lambda_0^2(1-k_1)e^{\lambda_0 t} - c\lambda_0(1-k_1)e^{\lambda_0 t} + \beta k_2 e^{\lambda_0 t} + \alpha k_3 e^{\lambda_0 t} - (1-k_1)e^{\lambda_0 t} \\
=\ & (1-k_1)e^{\lambda_0 t}\left(D_0\lambda_0^2 - c\lambda_0 + \frac{\beta k_2 + \alpha k_3 - 1 + k_1}{1-k_1}\right) \\
\le\ & 0.
\end{aligned}$$

Case 2. $1 \le \min\{(1-k_1)e^{\lambda_0 t}, 1-k_1+\frac{1}{1-k_1}e^{-\lambda t}\}$. Then $\bar{\phi}(t) = 1$. We have

$$D_0\bar{\phi}''(t) - c\bar{\phi}'(t) - \bar{\phi}(t) + \beta(1-\bar{\phi}(t))\bar{\varphi}(t) + \alpha(1-\bar{\phi}(t))\bar{\psi}(t) = -1 < 0.$$

Case 3. $1-k_1+\frac{1}{1-k_1}e^{-\lambda t} \le \min\{(1-k_1)e^{\lambda_0 t}, 1\}$. Then $\bar{\phi}(t) = 1-k_1+\frac{1}{1-k_1}e^{-\lambda t}$.

$$\begin{aligned}
& D_0\bar{\phi}''(t) - c\bar{\phi}'(t) - \bar{\phi}(t) + \beta(1-\bar{\phi}(t))\bar{\varphi}(t) + \alpha(1-\bar{\phi}(t))\bar{\psi}(t) \\
\le\ & \frac{D_0\lambda^2}{1-k_1}e^{-\lambda t} + \frac{c\lambda}{1-k_1}e^{-\lambda t} - (1-k_1) - \frac{1}{1-k_1}e^{-\lambda t} \\
& + \beta M_2\left(k_1 - \frac{1}{1-k_1}e^{-\lambda t}\right) + \alpha M_3\left(k_1 - \frac{1}{1-k_1}e^{-\lambda t}\right) \\
\le\ & e^{-\lambda t}\left[\frac{D_0\lambda^2}{1-k_1} + \frac{c\lambda}{1-k_1} + k_1 e^{\lambda t} - \frac{1}{1-k_1}\right. \\
& \left. + \beta M_2 e^{\lambda t}\left(k_1 - \frac{1}{1-k_1}e^{-\lambda t}\right) + \alpha M_3 e^{\lambda t}(k_1 - \frac{1}{1-k_1}e^{-\lambda t})\right] \\
=\ & I_1(\lambda).
\end{aligned}$$

Note that $I_1(0) = k_1 - \frac{1}{1-k_1} + \beta M_2\left(k_1 - \frac{1}{1-k_1}\right) + \alpha M_3\left(k_1 - \frac{1}{1-k_1}\right) = \left(k_1 - \frac{1}{1-k_1}\right)(1 + \beta M_2 + \alpha M_3) < 0$. From the continuity of $I_1(\lambda)$, there exists $\lambda_1^* > 0$ such that $I_1(\lambda_1) < 0$ for $\lambda \in (0, \lambda_1^*)$. Hence,

$$D_0\bar{\phi}''(t) - c\bar{\phi}'(t) - \bar{\phi}(t) + \beta(1-\bar{\phi}(t))\bar{\varphi}(t) + \alpha(1-\bar{\phi}(t))\bar{\psi}(t) \leq 0,\ \lambda \in (0, \lambda_1^*).$$

For $\bar{\varphi}(t)$, we consider the following two cases.

Case 4. $k_2e^{\lambda_0 t} \leq k_2 + \varepsilon_1 e^{-\lambda t}$. Then $\bar{\varphi}(t) = k_2e^{\lambda_0 t}$, $\bar{\varphi}(t-c\tau) = k_2e^{\lambda_0(t-c\tau)} \leq k_2e^{\lambda_0 t}$, $\bar{\psi}(t-c\tau) = k_3e^{\lambda_0(t-c\tau)} \leq k_3e^{\lambda_0 t}$. It is seen that

$$\begin{aligned}
& D_0\bar{\varphi}''(t) - c\bar{\varphi}'(t) + e^{-d_m\tau}\beta(1-\underline{\phi}(t-c\tau))\bar{\varphi}(t-c\tau) + e^{-d_m\tau}\alpha(1-\underline{\phi}(t-c\tau))\bar{\psi}(t-c\tau) - a\bar{\varphi}(t) \\
\leq\ & D_0k_2\lambda_0^2e^{\lambda_0 t} - c\lambda_0k_2e^{\lambda_0 t} + \beta k_2e^{\lambda_0 t} + \alpha k_3e^{\lambda_0 t} - ak_2e^{\lambda_0 t} \\
=\ & k_2e^{\lambda_0 t}\left(D_0\lambda_0^2 - c\lambda_0 + \frac{k_2(\beta-a)+\alpha k_3}{k_2}\right) \\
\leq\ & 0.
\end{aligned}$$

Case 5. $k_2 + \varepsilon_1 e^{-\lambda t} \leq k_2e^{\lambda_0 t}$. Then $\bar{\varphi}(t) = k_2 + \varepsilon_1 e^{-\lambda t}$. It follows that

$$\begin{aligned}
& D_0\bar{\varphi}''(t) - c\bar{\varphi}'(t) + e^{-d_m\tau}\beta(1-\underline{\phi}(t-c\tau))\bar{\varphi}(t-c\tau) + e^{-d_m\tau}\alpha(1-\underline{\phi}(t-c\tau))\bar{\psi}(t-c\tau) - a\bar{\varphi}(t) \\
\leq\ & D_0\lambda^2\varepsilon_1e^{-\lambda t} + c\lambda\varepsilon_1e^{-\lambda t} + \beta(k_2 + \varepsilon_1e^{-\lambda(t-c\tau)}) + \alpha M_3 - a(k_2 + \varepsilon_1e^{-\lambda t}) \\
\leq\ & D_0\lambda^2\varepsilon_1e^{-\lambda t} + c\lambda\varepsilon_1e^{-\lambda t} + \beta(k_2 + \varepsilon_1e^{\lambda c\tau}) + \alpha M_3 - a(k_2 + \varepsilon_1e^{-\lambda t}) \\
=\ & I_2(\lambda).
\end{aligned}$$

As $I_2(0) = (\beta - a)(k_2 + \varepsilon_1) + \alpha M_3 < 0$, there exists $\lambda_2^* > 0$ such that $I_2(0) < 0$ for all $\lambda \in (0, \lambda_2^*)$.

For $\bar{\psi}(t)$, we distinguish two cases again.

Case 6. $k_3e^{\lambda_0 t} \leq k_3 + \varepsilon_2 e^{-\lambda t}$. Then $\bar{\psi}(t) = k_3e^{\lambda_0 t}$, $\bar{\varphi}(t) \leq k_2e^{\lambda_0 t}$, $\bar{\phi}(t) = 1$. We obtain that

$$\begin{aligned}
& D_0\bar{\psi}''(t) - c\bar{\psi}'(t) + k\bar{\varphi}(t) - \mu\bar{\psi}(t) - \alpha(1-\bar{\phi}(t))\bar{\psi}(t) \\
\leq\ & D_0\lambda_0^2k_3e^{\lambda_0 t} - c\lambda_0k_3e^{\lambda_0 t} + kk_2e^{\lambda_0 t} \\
=\ & k_3e^{\lambda_0 t}\left(D_0\lambda_0^2 - c\lambda_0 + \frac{kk_2}{k_3}\right) \\
\leq\ & 0.
\end{aligned}$$

Case 7. $k_3 + \varepsilon_2 e^{-\lambda t} \leq k_3e^{\lambda_0 t}$. Then $\bar{\psi}(t) = k_3 + \varepsilon_2 e^{-\lambda t}$, $\bar{\phi}(t) = 1 - k_1 + \frac{1}{1-k_1}e^{-\lambda t}$, $\bar{\varphi}(t) = k_2 + \varepsilon_1 e^{-\lambda t}$. We have

$$\begin{aligned}
& D_0\bar{\psi}''(t) - c\bar{\psi}'(t) + k\bar{\varphi}(t) - \mu\bar{\psi}(t) - \alpha(1-\bar{\phi}(t))\bar{\psi}(t) \\
\leq\ & D_0\lambda^2\varepsilon_2e^{-\lambda t} + c\lambda\varepsilon_2e^{-\lambda t} + k(k_2 + \varepsilon_1e^{-\lambda t})
\end{aligned}$$

$$
\begin{aligned}
& -\mu(k_3+\varepsilon_2 e^{-\lambda t}) - \alpha\left(k_1 - \frac{1}{1-k_1}e^{-\lambda t}\right)(k_3+\varepsilon_2 e^{-\lambda t}) \\
= \; & D_0\lambda^2\varepsilon_2 e^{-\lambda t} + c\lambda\varepsilon_2 e^{-\lambda t} + kk_2 + k\varepsilon_1 e^{-\lambda t} \\
& -\mu k_3 - \mu\varepsilon_2 e^{-\lambda t} - \alpha k_1 k_3 - \alpha k_1\varepsilon_2 e^{-\lambda t} + \frac{\alpha k_3 e^{-\lambda t}}{1-k_1} + \frac{\alpha\varepsilon_2 e^{-\lambda t}}{1-k_1}e^{-\lambda t} \\
= \; & D_0\lambda^2\varepsilon_2 e^{-\lambda t} + c\lambda\varepsilon_2 e^{-\lambda t} + k\varepsilon_1 e^{-\lambda t} - \mu\varepsilon_2 e^{-\lambda t} - \alpha k_1\varepsilon_2 e^{-\lambda t} + \frac{\alpha k_3 e^{-\lambda t}}{1-k_1} + \frac{\alpha\varepsilon_2 e^{-\lambda t}}{1-k_1}e^{-\lambda t} \\
= \; & I_3(\lambda).
\end{aligned}
$$

Since $I_3(0) = k\varepsilon_1 - \mu\varepsilon_2 - \alpha k_1\varepsilon_2 + \frac{\alpha(k_3+\varepsilon_2)}{1-k_1} < 0$, there exists a $\lambda_3^* > 0$ such that $I_3(\lambda) < 0$ for all $\lambda \in (0, \lambda_3^*)$.

Thus, taking $\bar{\lambda}^* = \min(\lambda_1^*, \lambda_2^*, \lambda_3^*)$, we have shown that $\bar{\rho}(t) = (\bar{\phi}(t), \bar{\varphi}(t), \bar{\psi}(t))$ is an upper solution of system (3.2) for $\lambda \in (0, \bar{\lambda}^*)$. □

Lemma 3.9 *There exists* $\underline{\lambda}^* > 0$ *such that, for* $0 < \lambda < \underline{\lambda}^*$, $\underline{\rho}(t) = (\underline{\phi}(t), \underline{\varphi}(t), \underline{\psi}(t))$ *is a lower solution of system* (3.2)*.*

Proof. For $\underline{\phi}(t)$, we can prove it in two cases.

Case 1. $1 - k_1 - \frac{2-k_1}{k_1}e^{-\lambda t} \le 0$. Then $\underline{\phi}(t) = 0$. It is clear that

$$
\begin{aligned}
& D_0\underline{\phi}''(t) - c\underline{\phi}'(t) - \underline{\phi}(t) + \beta(1-\underline{\phi}(t))\underline{\varphi}(t) + \alpha(1-\underline{\phi}(t))\underline{\psi}(t) \\
= \; & \beta\underline{\varphi}(t) + \alpha\underline{\psi}(t) \\
\ge \; & 0.
\end{aligned}
$$

Case 2. $0 \le 1 - k_1 - \frac{2-k_1}{k_1}e^{-\lambda t}$. Then $\underline{\phi}(t) = 1 - k_1 - \frac{2-k_1}{k_1}e^{-\lambda t}$, $\underline{\varphi}(t) = k_2 - \frac{2k_2}{k_1}e^{-\lambda t}$, $\underline{\psi}(t) = k_3 - \frac{2k_3}{k_1}e^{-\lambda t}$. We have

$$
\begin{aligned}
& D_0\underline{\phi}''(t) - c\underline{\phi}'(t) - \underline{\phi}(t) + \beta(1-\underline{\phi}(t))\underline{\varphi}(t) + \alpha(1-\underline{\phi}(t))\underline{\psi}(t) \\
= \; & -D_0\frac{2-k_1}{k_1}\lambda^2 e^{-\lambda t} - c\frac{2-k_1}{k_1}\lambda e^{-\lambda t} - (1-k_1) + \frac{2-k_1}{k_1}e^{-\lambda t} \\
& + \beta\left(k_1 + \frac{2-k_1}{k_1}e^{-\lambda t}\right)\left(k_2 - \frac{2k_2}{k_1}e^{-\lambda t}\right) + \alpha\left(k_1 + \frac{2-k_1}{k_1}e^{-\lambda t}\right)\left(k_3 - \frac{2k_3}{k_1}e^{-\lambda t}\right) \\
\ge \; & -D_0\frac{2-k_1}{k_1}\lambda^2 e^{-\lambda t} - c\frac{2-k_1}{k_1}\lambda e^{-\lambda t} - (1-k_1) \\
& + \frac{2-k_1}{k_1}e^{-\lambda t} + \beta k_1\left(k_2 - \frac{2k_2}{k_1}e^{-\lambda t}\right) + \alpha k_1\left(k_3 - \frac{2k_3}{k_1}e^{-\lambda t}\right) \\
= \; & -D_0\frac{2-k_1}{k_1}\lambda^2 e^{-\lambda t} - c\frac{2-k_1}{k_1}\lambda e^{-\lambda t} + \frac{2-k_1}{k_1}e^{-\lambda t} - 2\beta k_2 e^{-\lambda t} - 2\alpha k_3 e^{-\lambda t} \\
= \; & I_4(\lambda).
\end{aligned}
$$

As $I_4(0) = \frac{2-k_1}{k_1} - 2\beta k_2 - 2\alpha k_3 = \frac{2-k_1}{k_1} - \frac{2(1-k_1)}{k_1} = 1 > 0$, there exists a $\lambda_4^* > 0$ such that $I_4(\lambda) > 0$ for all $\lambda \in (0, \lambda_4^*)$.

For $\underline{\varphi}(t)$, we also discuss it in two cases.

Case 3. $k_2 - \frac{2k_2}{k_1}e^{-\lambda t} \le 0$. Then $\underline{\varphi}(t) = 0$. We get

$$\begin{aligned}
& D_0\underline{\varphi}''(t) - c\underline{\varphi}'(t) + e^{-d_m\tau}\beta(1-\bar{\phi}(t-c\tau))\underline{\varphi}(t-c\tau) + e^{-d_m\tau}\alpha(1-\bar{\phi}(t-c\tau))\underline{\psi}(t-c\tau) - a\underline{\varphi}(t) \\
= \ & e^{-d_m\tau}\alpha(1-\bar{\phi}(t-c\tau))\underline{\psi}(t-c\tau) \\
\ge \ & 0.
\end{aligned}$$

Case 4. $0 \le k_2 - \frac{2k_2}{k_1}e^{-\lambda t}$. Then $\underline{\varphi}(t) = k_2 - \frac{2k_2}{k_1}e^{-\lambda t}$, $\bar{\phi}(t-c\tau) = 1-k_1+\frac{1}{1-k_1}e^{-\lambda(t-c\tau)}$, $\underline{\psi}(t-c\tau) = k_3 - \frac{2k_3}{k_1}e^{-\lambda(t-c\tau)}$. We have

$$\begin{aligned}
& D_0\underline{\varphi}''(t) - c\underline{\varphi}'(t) + e^{-d_m\tau}\beta(1-\bar{\phi}(t-c\tau))\underline{\varphi}(t-c\tau) + e^{-d_m\tau}\alpha(1-\bar{\phi}(t-c\tau))\underline{\psi}(t-c\tau) - a\underline{\varphi}(t) \\
= & -D_0\frac{2k_2}{k_1}\lambda^2 e^{-\lambda t} - c\frac{2k_2}{k_1}\lambda e^{-\lambda t} + e^{-d_m\tau}\beta\left(k_1 - \frac{1}{1-k_1}e^{-\lambda(t-c\tau)}\right)\left(k_2 - \frac{2k_2}{k_1}e^{-\lambda(t-c\tau)}\right) \\
& + e^{-d_m\tau}\alpha\left(k_1 - \frac{1}{1-k_1}e^{-\lambda(t-c\tau)}\right)\left(k_3 - \frac{2k_3}{k_1}e^{-\lambda(t-c\tau)}\right) - a\left(k_2 - \frac{2k_2}{k_1}e^{-\lambda t}\right) \\
= & -D_0\frac{2k_2}{k_1}\lambda^2 e^{-\lambda t} - c\frac{2k_2}{k_1}\lambda e^{-\lambda t} - a\left(k_2 - \frac{2k_2}{k_1}e^{-\lambda t}\right) \\
& + e^{-d_m\tau}\beta\left(k_1k_2 - 2k_2e^{-\lambda(t-c\tau)} - \frac{k_2}{1-k_1}e^{-\lambda(t-c\tau)} + \frac{2k_2}{k_1(1-k_1)}e^{-\lambda(t-c\tau)}e^{-\lambda(t-c\tau)}\right) \\
& + e^{-d_m\tau}\alpha\left(k_1k_3 - 2k_3e^{-\lambda(t-c\tau)} - \frac{k_3}{1-k_1}e^{-\lambda(t-c\tau)} + \frac{2k_3}{k_1(1-k_1)}e^{-\lambda(t-c\tau)}e^{-\lambda(t-c\tau)}\right) \\
= & -D_0\frac{2k_2}{k_1}\lambda^2 e^{-\lambda t} - c\frac{2k_2}{k_1}\lambda e^{-\lambda t} + e^{-d_m\tau}\beta\left(-\frac{k_2}{1-k_1}e^{-\lambda(t-c\tau)} + \frac{2k_2}{k_1(1-k_1)}e^{-\lambda(t-c\tau)}e^{-\lambda(t-c\tau)}\right) \\
& + e^{-d_m\tau}\alpha\left(-\frac{k_3}{1-k_1}e^{-\lambda(t-c\tau)} + \frac{2k_3}{k_1(1-k_1)}e^{-\lambda(t-c\tau)}e^{-\lambda(t-c\tau)}\right) \\
= & -D_0\frac{2k_2}{k_1}\lambda^2 e^{-\lambda t} - c\frac{2k_2}{k_1}\lambda e^{-\lambda t} \\
& + \frac{e^{-\lambda(t-c\tau)}}{k_1(1-k_1)}\left[e^{-d_m\tau}\beta(-k_1k_2 + 2k_2e^{-\lambda(t-c\tau)}) + e^{-d_m\tau}\alpha(-k_1k_3 + 2k_3e^{-\lambda(t-c\tau)})\right] \\
= & -D_0\frac{2k_2}{k_1}\lambda^2 e^{-\lambda t} - c\frac{2k_2}{k_1}\lambda e^{-\lambda t} + \frac{e^{-\lambda(t-c\tau)}}{k_1(1-k_1)}a\left(-k_2 + \frac{2k_2}{k_1}e^{-\lambda t}\right) \\
= & \ I_5(\lambda).
\end{aligned}$$

In the above, we have used

$$e^{-d_m\tau}\beta k_1\left(k_2 - \frac{2k_2}{k_1}e^{-\lambda(t-c\tau)}\right) + e^{-d_m\tau}\alpha k_1\left(k_3 - \frac{2k_3}{k_1}e^{-\lambda(t-c\tau)}\right) - a\left(k_2 - \frac{2k_2}{k_1}e^{-\lambda t}\right) = 0$$

to obtain

$$e^{-d_m\tau}\beta(k_1k_2 - 2k_2e^{-\lambda(t-c\tau)}) + e^{-d_m\tau}\alpha(k_1k_3 - 2k_3e^{-\lambda(t-c\tau)}) = a\left(k_2 - \frac{2k_2}{k_1}e^{-\lambda t}\right).$$

Then $I_5(0) = \frac{a}{k_1(1-k_1)}(-k_2 + \frac{2k_2}{k_1}) = \frac{ak_2(2-k_1)}{k_1^2(1-k_1)} > 0$ implies that there exists $\lambda_5^* > 0$ such that $I_5(\lambda) > 0$ for $\lambda \in (0, \lambda_5^*)$.

For $\underline{\psi}(t)$, the following two situations are discussed separately.

Case 5. $k_3 - \frac{2k_3}{k_1}e^{-\lambda t} \leq 0$. Then $\underline{\psi}(t) = 0$. It is clear that

$$D_0\underline{\psi}''(t) - c\underline{\psi}'(t) + k\underline{\varphi}(t) - \mu\underline{\psi}(t) - \alpha(1-\underline{\phi}(t))\underline{\psi}(t) = k\underline{\varphi}(t) \geq 0.$$

Case 6. $0 \leq k_3 - \frac{2k_3}{k_1}e^{-\lambda t}$. Then $\underline{\psi}(t) = k_3 - \frac{2k_3}{k_1}e^{-\lambda t}$, $\underline{\phi}(t) = 1-k_1-\frac{2-k_1}{k_1}e^{-\lambda t}$, $\underline{\varphi}(t) = k_2 - \frac{2k_2}{k_1}e^{-\lambda t}$. It is seen that

$$\begin{aligned}
& D_0\underline{\psi}''(t) - c\underline{\psi}'(t) + k\underline{\varphi}(t) - \mu\underline{\psi}(t) - \alpha(1-\underline{\phi}(t))\underline{\psi}(t) \\
= & -D_0\frac{2k_3}{k_1}\lambda^2 e^{-\lambda t} - c\frac{2k_3}{k_1}\lambda e^{-\lambda t} + k\left(k_2 - \frac{2k_2}{k_1}e^{-\lambda t}\right) \\
& -\mu\left(k_3 - \frac{2k_3}{k_1}e^{-\lambda t}\right) - \alpha\left(k_1 + \frac{2-k_1}{k_1}e^{-\lambda t}\right)\left(k_3 - \frac{2k_3}{k_1}e^{-\lambda t}\right) \\
= & -D_0\frac{2k_3}{k_1}\lambda^2 e^{-\lambda t} - c\frac{2k_3}{k_1}\lambda e^{-\lambda t} + k\left(k_2 - \frac{2k_2}{k_1}e^{-\lambda t}\right) - \mu\left(k_3 - \frac{2k_3}{k_1}e^{-\lambda t}\right) \\
& -\alpha k_1 k_3 + 2\alpha k_3 e^{-\lambda t} - \frac{\alpha k_3(2-k_1)}{k_1}e^{-\lambda t} + \frac{2\alpha k_3(2-k_1)}{k_1^2}e^{-\lambda t}e^{-\lambda t} \\
= & -D_0\frac{2k_3}{k_1}\lambda^2 e^{-\lambda t} - c\frac{2k_3}{k_1}\lambda e^{-\lambda t} - \frac{\alpha k_3(2-k_1)}{k_1}e^{-\lambda t} + \frac{2\alpha k_3(2-k_1)}{k_1^2}e^{-\lambda t}e^{-\lambda t} \\
= & \ I_6(\lambda).
\end{aligned}$$

As $I_6(0) = -\frac{\alpha k_3(2-k_1)}{k_1} + \frac{2\alpha k_3(2-k_1)}{k_1^2} > -\frac{\alpha k_3(2-k_1)}{k_1} + \frac{2\alpha k_3(2-k_1)}{k_1} = \frac{\alpha k_3(2-k_1)}{k_1} > 0$, there exists a $\lambda_6^* > 0$ such that $I_6(\lambda) > 0$ for all $\lambda \in (0, \lambda_6^*)$.

Let $\underline{\lambda}^* = \min(\lambda_4^*, \lambda_5^*, \lambda_6^*)$. Then we have shown that $\underline{\rho}(t) = (\underline{\phi}(t), \underline{\varphi}(t), \underline{\psi}(t))$ is a lower solution of system (3.2) for $\lambda \in (0, \underline{\lambda}^*)$.

Applying Lemma 3.8 - 3.9, we have the following theorem.

Theorem 3.2 *Let $R_0 > 1$. For every $c > c^*$ and any value of $\tau \geq 0$, system* (3.2) *always has a traveling wave solution with speed c connecting the infection-free steady state $E_0 = (0,0,0)$ and the unique homogeneous infected steady state $E^* = (1-u^*, \omega^*, v^*)$.*

4 Conclusion

In this paper, we developed a dynamic model of virus infection with nonlinear functional response, diffusion, absorption due to infection, and time delay. We also considered two viral transmission mechanisms: cell-to-cell transmission and cell-free infection. When the domain is bounded, we studied the well-posedness of the model and discussed the linear stability of the homogeneous steady states of the model under homogeneous Neumann boundary conditions. More precisely, it is proved that if the basic reproduction number is less than unity then the disease-free steady state is globally asymptotically stable while if the basic reproduction number is larger than unity then the infection is uniformly persistent. When the domain is the whole space, by using the cross iteration method and Schaueder's fixed point theorem, we attributed the existence of traveling wave solutions to the existence of a pair of upper and lower solutions. Furthermore, when the basic reproduction number is larger than unity by constructing a pair of upper and lower solutions, we obtained the existence of traveling wave solutions connecting the disease-free steady state and the homogeneous infected steady state.

We discussed how the speed of spread in space affects the spread of cells and viruses. We studied the existence of the wave speed c^*, which is dependent on the diffusion coefficient. Moreover, the two modes of virus transmission affect the minimum wave speed. A natural question is whether the wave speed c^* is the minimum wave speed $c_{\min}$. For $0 < c < c_{\min}$, there is no traveling wave solution connecting the two steady states. According to the linear theory [51], the minimum wave speed is usually the asymptotic propagation speed, but in general the relation between the minimum wave speed and asymptotic propagation speed remains to be further investigated.

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