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Noetherian operators and primary decomposition ☆

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ABSTRACT

Noetherian operators are differential operators that encode primary components of a polynomial ideal. We develop a framework, as well as algorithms, for computing Noetherian operators with local dual spaces, both symbolically and numerically. For a primary ideal, such operators provide an alternative representation to one given by a set of generators. This description fits well with numerical algebraic geometry, taking a step toward the goal of numerical primary decomposition.

1. Introduction

A fundamental problem in computational algebra is *primary decomposition*: given an ideal, find the associated primes, and express the ideal as an intersection of primary components. When the ideal is radical, this corresponds geometrically to decomposing an algebraic variety into a union of irreducible components.

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Algorithms implemented in computer algebra systems (Gianni et al. (1988), Shimoyama and Yokoyama (1996), Eisenbud et al. (1992), Decker et al. (1999)) perform primary decomposition for ideals in polynomial rings by producing a set of ideal generators for each primary component. Although providing generators is the most direct way to represent a primary ideal, in practice it is often infeasible to compute primary decomposition this way, e.g. due to the size of the generators. Thus it makes sense to seek an alternative approach to primary decomposition which can harness the power of numerical methods. The natural setting for this is *numerical algebraic geometry* (Sommese et al. (2005); Sommese and Wampler (2005)), which provides a suite of algorithms for computing with complex algebraic varieties using numerical techniques. For certain tasks, numerical methods may solve problems that are difficult for typical symbolic methods. As an example, numerical irreducible decomposition Sommese et al. (2001) has been used to decompose varieties that were outside the feasible range of symbolic algorithms; see for instance Bates and Oeding (2011); Hauenstein et al. (2018).

In contrast to the description by a set of generators, a primary ideal I can be described by two pieces of data: its minimal prime $\sqrt{I}$ (or geometrically, the variety $\mathbb{V}(I)$), and the *multiplicity structure* of I over $\sqrt{I}$. One can describe the multiplicity structure of I via associated differential operators on $\mathbb{V}(I)$:

Definition 1.1. A set N of differential operators with polynomial coefficients is called a *set of Noetherian operators for* I if $f \in I \iff D \bullet f \in \sqrt{I}\ \forall D \in N$.

The idea of representing a (primary) ideal in a polynomial ring via a dual set of differential operators is both natural and classical, dating back to Macaulay (who introduced inverse systems in Macaulay (1916)) and Gröbner (1938). Since their introduction in Palamodov (1970), Noetherian operators have been sporadically studied in the literature: Oberst (1999), Sturmfels (2002), Nonkań (2013), and Cid-Ruiz et al. (2020). Symbolic algorithms to compute Noetherian operators were developed and implemented in Damiano et al. (2007) and Cid-Ruiz et al. (2020).

Our contribution consists of new algorithms to compute a set of Noetherian operators representing a primary ideal, as well as theoretical results leading up to them. We develop two algorithms: one using exact symbolic computation (Algorithm 1) and the other based on hybrid symbolic-numeric methods of numerical algebraic geometry (Algorithm 6).

Our symbolic algorithm follows a path started by Macaulay (1916) reducing the problem to linear algebra. The potential of this approach is that the body of work in this direction may be adapted to computation of Noetherian operators; for instance, optimizations of the algorithm as in Mourrain (1997) are possible. However, it is important to emphasize that the *classical* Macaulay dual space (inverse system) approach addresses the case of a rational point, to which there is no straightforward reduction in general. We define the *local dual space*, which coincides with Macaulay dual space in the case of a rational point, and develop the theory that underpins an algorithm analogous to the classical case.

Our numerical algorithm may solve problems that are out of reach for purely symbolic techniques (cf. e.g. Example 4.7). Given an ideal with no embedded components, our numerical algorithm combined with numerical irreducible decomposition leads to *numerical primary decomposition* (Algorithm 7): i.e. a numerical description of all components of the ideal, which e.g. enables a probabilistic membership test.

Numerical irreducible decomposition algorithms are efficient but set-theoretic in nature. In contrast, numerical detection of embedded components, studied in Leykin (2008); Krone and Leykin (2017b), is a rather difficult task. Our procedures for describing primary components via Noetherian operators assume that the associated primes of the ideal have already been discovered. Moreover, our algorithms rely on primes being isolated, i.e. not embedded (see Remark 3.5) and therefore do not address the problem of finding embedded components. Nevertheless, it will be interesting to study in the future whether Noetherian operators can make a contribution here.

The paper is organized as follows. Section 2 gives a gentle introduction to Noetherian operators and classical dual spaces. Section 3 generalizes the definition of a dual space to nonrational points and develops theory that leads to a symbolic algorithm based on Macaulay matrices. Section 4 deals with specialization and interpolation of Noetherian operators, leading to a numerical algorithm for computing Noetherian operators as well as an algorithm for numerical primary decomposition. Section 4.1 proposes a hybrid algorithm that uses numerical information to accelerate the symbolic computation of Noetherian operators. Section 5 concludes with general properties of Noetherian operators for nonprimary ideals.

Algorithms are implemented in Macaulay2 (Grayson and Stillman, 2002), and the software is available on GitHub.[1]

2. Preliminaries

Let $\mathbb{K}$ be a field of characteristic 0 and $R := \mathbb{K}[\boldsymbol{x}] = \mathbb{K}[x_1, \dots, x_n]$ a polynomial ring over $\mathbb{K}$. For numerical applications our focus will be on the case $\mathbb{K} = \mathbb{C}$, as implementations of numerical methods generally use floating point approximations of complex numbers to some fixed precision. On the other hand, our symbolic algorithms do not assume that $\mathbb{K}$ is algebraically closed. We often take $\mathbb{K} = \mathbb{Q}$ in examples.

2.1. *Sets of Noetherian operators*

In Definition 1.1 we consider an ideal $I \subseteq R$ and a set N of differential operators in

$$W_R := R\langle \partial_1, \dots, \partial_n \rangle, \quad \text{where } \partial_i := \frac{\partial}{\partial x_i},$$

the noncommutative ring of differential operators with coefficients in R, known as the n-dimensional *Weyl algebra* over R. The differential operators $\partial_1, \dots, \partial_n$ are $\mathbb{K}$-linear endomorphisms of R satisfying the relations $[\partial_i, x_j] := \partial_i x_j - x_j \partial_i = \delta_{ij}$.

Remark 2.1 *(Ideal membership test).* Let $\mathbb{V}(I) \subseteq \mathbb{K}^n$ be the affine variety defined by I. A set of Noetherian operators $N = \{D_1, \dots, D_r\}$ for I gives a probabilistic test for determining if a polynomial f is in I or not, assuming an oracle for sampling a random point from $\mathbb{V}(I)$ (according to some reasonable distribution). The set $\{D_1 \bullet f, \dots, D_r \bullet f\}$ is contained in $\sqrt{I}$ if and only if $f \in I$. If $p \in \mathbb{V}(I)$ is general, then $(D_i \bullet f)(p)$ evaluates to zero for all $i = 1, \dots, r$ if and only if $f \in I$.

If $I = \sqrt{I}$ is radical, then the singleton $\{1\}$ is a set of Noetherian operators for I. The case of most interest is when I is primary, but not radical. In this case, a minimal set of Noetherian operators for I has more than one element. Although such a set need not be unique, its cardinality equals the multiplicity of I over $\sqrt{I}$, which is the ratio $e(I)/e(\sqrt{I})$ of their Hilbert-Samuel multiplicities, see the proof of Theorem 4.1.

Example 2.2. Let $I = ((x + y + 1)^m) \subseteq \mathbb{K}[x, y]$, a primary ideal. Then the sets $N_1 = \{1, \partial_x, \dots, \partial_x^{m-1}\}$ and $N_2 = \{1, \partial_y, \dots, \partial_y^{m-1}\}$ are both minimal sets of Noetherian operators for I.

Note that the generator of I in expanded form consists of $\binom{m+2}{2}$ monomials with integer coefficients that grow with m. On the other hand, both N_1 and N_2 are much simpler expressions of size m; moreover, either of them, together with the radical $\sqrt{I} = (x + y + 1)$, describes the ideal I fully.

For our numerical algorithm one may not even have generators for the radical of I available, which is the case in Example 4.7. Moreover, the input can be a set of generators of an ideal (for which I is a component) which are only available as black-box differentiable evaluation routines. We mention this here to preempt the common assumption in classical computational algebraic geometry that polynomials are always represented as sums of their monomial terms.

[1] `NoetherianOperators` codebase: https://github.com/haerski/NoetherianOperators.

2.2. Dual spaces

We start by reviewing the classical theory of Macaulay dual spaces. The *dual space* R^* is by definition the $\mathbb{K}$-vector space dual of R, i.e. the $\mathbb{K}$-vector space of $\mathbb{K}$-linear functionals $R \to \mathbb{K}$. Let $p = (p_1, \dots, p_n) \in \mathbb{K}^n$ be a $\mathbb{K}$-rational point. The polynomials $\{(\boldsymbol{x} - p)^\alpha := (x_1 - p_1)^{\alpha_1} \cdots (x_n - p_n)^{\alpha_n}\}_{\alpha \in \mathbb{N}^n}$ form a $\mathbb{K}$-basis of R. Let $\mathrm{ev}_p : R \to \mathbb{K}$ denote the evaluation functional at p, and $\mathfrak{m}_p := (x_1 - p_1, \dots, x_n - p_n)$ the maximal ideal in R associated to the point p. Note that ev_p coincides with the natural surjection $R \twoheadrightarrow R/\mathfrak{m}_p \cong \mathbb{K}$.

Post-composing differential operators with the evaluation functional produces new functionals. Let $\partial_{p,i}$ denote the functional $\mathrm{ev}_p \circ \partial_i$, and for a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ let

$$\begin{aligned} \partial_p^\alpha : R &\to \mathbb{K} \\ f &\mapsto (\mathrm{ev}_p \circ \partial_1^{\alpha_1} \circ \cdots \circ \partial_n^{\alpha_n})(f). \end{aligned}$$

The elements of R^* can be expressed as formal power series in the $\partial_{p,i}$, and we write $R^* := \mathbb{K}[\![\partial_p]\!] = \mathbb{K}[\![\partial_{p,1}, \dots, \partial_{p,n}]\!]$. The $\mathbb{K}$-linear span of $\{\partial_p^\alpha\}_{\alpha \in \mathbb{N}^n}$ will be denoted $\mathbb{K}[\partial_p]$.

Definition 2.3. Let $I \subseteq R$ be an ideal. The *orthogonal complement of* I is the $\mathbb{K}$-vector subspace of R^*

$$I^\perp := \{D \in R^* \mid D(f) = 0 \text{ for all } f \in I\}.$$

If $\mathcal{D}$ is a $\mathbb{K}$-vector subspace of R^*, then the *orthogonal complement of* $\mathcal{D}$ is the $\mathbb{K}$-vector subspace of R

$$\mathcal{D}^\perp := \{f \in R \mid D(f) = 0 \text{ for all } D \in \mathcal{D}\}.$$

Proposition 2.4. *For any ideals $I, J \subseteq R$ we have:*

(1) $I \subseteq J \iff I^\perp \supseteq J^\perp$
(2) $(I + J)^\perp = I^\perp \cap J^\perp$
(3) $I^{\perp\perp} = I$.

Note that R^* has a natural R-module structure given by

$$\begin{aligned} f \cdot \Lambda : R &\to \mathbb{K} \\ g &\mapsto \Lambda(fg) \end{aligned}$$

for $f, g \in R$, $\Lambda \in R^*$. The basis $\{(\boldsymbol{x} - p)^\alpha\}_\alpha$ of R acts on $\{\partial_p^\alpha\}_\alpha \subseteq R^*$ in the following way:

$$(x_i - p_i) \cdot \partial_p^\alpha = \alpha_i \partial_{p,1}^{\alpha_1} \cdots \partial_{p,i}^{\alpha_i - 1} \cdots \partial_{p,n}^{\alpha_n}$$

We say that a $\mathbb{K}$-subspace $\mathcal{D} \subseteq R^*$ is *closed* under the R-action if $\mathcal{D}$ is an R-submodule of R^*. In general, $\mathcal{D}$ is an R-submodule of R^* iff $\mathcal{D}^\perp$ is an R-submodule of R, i.e. an ideal of R.

3. A symbolic approach via Macaulay matrices

3.1. Dual spaces at nonrational points

Next, we provide a generalization of dual spaces for nonrational points. For any R-algebra A, set $W_A := A \otimes_R W_R$, where $W_R := R\langle \partial_1, \dots, \partial_n \rangle$ is the Weyl algebra over R (as in Section 2.1). There is a natural action $\_ \bullet \_ \colon W_R \times R \to R$ given by $x_i \bullet f = x_i f$ and $\partial_i \bullet f = \frac{\partial f}{\partial x_i}$, which induces a natural $\mathbb{K}$-bilinear pairing

$$\langle \cdot, \cdot \rangle_A \colon W_A \times R \to A \tag{1}$$

This pairing is A-linear in the first argument, and makes the diagram

$$\begin{array}{ccc} W_R \times R & \xrightarrow{\ \bullet\ } & R \\ \downarrow & & \downarrow \\ W_A \times R & \xrightarrow{\langle\cdot,\cdot\rangle_A} & A \end{array}$$

commute. It can be viewed as follows: for any $f \in R$, $\langle \partial_i, f\rangle_A$ is the image of $\frac{\partial f}{\partial x_i}$ in A. We will often omit the subscript in $\langle\cdot,\cdot\rangle_A$ when A is clear from context.

Definition 3.1. Let $I \subseteq R$ be an R-ideal, and $P \subseteq R$ a prime ideal with residue field $\kappa(P) := R_P/PR_P$. The *local dual space of I at P*, denoted $D_P[I]$, is the $\mathbb{K}$-vector subspace orthogonal to I with respect to the pairing $\langle\cdot,\cdot\rangle_{\kappa(P)}$, that is

$$D_P[I] := \{D \in W_{\kappa(P)} \mid \langle D, f\rangle_{\kappa(P)} = 0 \text{ for all } f \in I\}.$$

This generalizes the definition of local dual spaces in Krone and Leykin (2017a). Note that by $\kappa(P)$-linearity of $\langle\cdot,\cdot\rangle_{\kappa(P)}$ in the first argument, $D_P[I]$ is also a $\kappa(P)$-vector space. We call the R-module action induced by the $\kappa(P)$-vector space structure on $D_P[I]$ the left R-module action. We can also define a right R-module action on $D_P[I]$ analogous to the R-action on $I^\perp$, namely via

$$\langle D \cdot g, f\rangle = \langle D, gf\rangle$$

for $D \in D_P[I]$ and $f, g \in R$. These actions give $D_P[I]$ the structure of a R-bimodule, cf. Cid-Ruiz et al. (2020) for a treatment along these lines. Analogous to Proposition 2.4, one has:

Proposition 3.2. *Let $I, J \subseteq R$ be ideals and $P \subseteq R$ a prime. Then*

(1) *If $I \subseteq J$, then $D_P[I] \supseteq D_P[J]$*
(2) $D_P[I + J] = D_P[I] \cap D_P[J]$
(3) *The following are equivalent:*
  (a) $D_P[I] \neq 0$
  (b) $1 \in D_P[I]$
  (c) $I \subseteq P$.

Proof. (3)((a) $\Rightarrow$ (b)): This follows from the following facts: (i) $D_P[I]$ is closed under taking brackets $[\cdot, f]$ with any $f \in R$, and (ii) $[\partial_i^m, x_i] = m\partial_i^{m-1}$ for any i, m. Thus if $D_P[I] \neq 0$, then (by iteratively taking brackets) it must also contain a nonzero operator of ∂-degree 0, and hence also $1 \in W_{\kappa(P)}$. $\square$

Remark 3.3. When the prime corresponds to a rational point, Definition 3.1 agrees with the classical dual space: indeed, when $P = \mathfrak{m}_p$, the $\mathbb{K}$-vector spaces $D_P[I]$ and $I^\perp \cap \mathbb{K}[\partial_p]$ are naturally isomorphic.

We will show (Theorem 3.10) that one can obtain Noetherian operators for a primary ideal $Q \subseteq R$ by computing a $\kappa(P)$-basis of $D_{\sqrt{Q}}[Q]$. A natural question that arises is, for a nonprimary ideal, whether one can compute Noetherian operators of a primary component without requiring generators of the primary component. This is indeed the case for isolated components:

Proposition 3.4. *Let $I \subseteq R$ be an ideal, let $P \subseteq R$ be a minimal prime of I, and let Q be the P-primary component of I. Then $D_P[I] = D_P[Q]$.*

Proof. We follow (Gianni and Mora, 1987, Theorem 7.31). The inclusion $D_P[Q] \subseteq D_P[I]$ follows from Proposition 3.2(1). For the opposite inclusion, we use the following characterization of the P-primary component Q:

$$Q = \{f \in R \mid \exists g \in R \setminus P \text{ such that } gf \in I\}. \tag{2}$$

Given $D \in D_P[I]$, we show by induction on the ∂-degree of D that $D \in D_P[Q]$. If the ∂-degree is 0, then $D \in R$. Let $f \in Q$, and choose g as in (2). Then $0 = \langle D, gf\rangle = g\langle D, f\rangle$, hence $\langle D, f\rangle = 0$ (as $g \notin P$).

Now let the ∂-degree of $D \in D_P[I]$ be $d > 0$, and assume the result for ∂-degree $< d$. Let $f \in Q \setminus I$, and choose g as in (2). Note that the operator $\rho := [D, g] = D \cdot g - g \cdot D$ is in $D_P[I]$ and the ∂-degree of ρ is $< d$ (cf. the proof of Proposition 3.2). Then $0 = \langle D, gf\rangle = \langle D \cdot g, f\rangle = g\langle D, f\rangle + \langle \rho, f\rangle$. By induction $\langle \rho, f\rangle = 0$, so $g\langle D, f\rangle = 0$, hence $D \in D_P[Q]$. □

Remark 3.5. Proposition 3.4 relies on P being a minimal prime of I, in multiple ways. If P is an embedded prime of I, then a P-primary component of I is never unique, and no such characterization as in (2) exists. Moreover, in this case $D_P[I]$ is infinite-dimensional as a $\mathbb{K}$-vector space, so there does not exist a finite $\mathbb{K}$-basis of $D_P[I]$ to represent the multiplicity structure of I.

3.2. Zero-dimensional primary ideals

Throughout this subsection, I denotes a zero-dimensional primary ideal in $R = \mathbb{K}[\boldsymbol{x}]$, with $P := \sqrt{I}$.

3.2.1. Primary ideals over a rational point

The simplest case is when $P = \mathfrak{m}_p$ for some $p \in \mathbb{K}^n$. The duality for $\mathfrak{m}_p$-primary ideals is summarized in the following:

Theorem 3.6 *(Marinari et al. (1993), Theorem 2.6). There is a bijection between $\mathfrak{m}_p$-primary ideals $I \subseteq R$ and finite dimensional subspaces $\mathcal{D} \subseteq \mathbb{K}[\partial_p]$ closed under the right R-action. The correspondence is given by $I \mapsto I^\perp$ and $\mathcal{D} \mapsto \mathcal{D}^\perp$. Moreover $\dim_{\mathbb{K}}(I^\perp) = \deg(I) = \dim_{\mathbb{K}}(R/I)$ and $\deg(\mathcal{D}^\perp) = \dim_{\mathbb{K}}(\mathcal{D})$.*

We describe how to obtain a set of Noetherian operators for an $\mathfrak{m}_p$-primary ideal I. First, compute a dual basis $D_1, \dots, D_m$ of $I^\perp$, where $D_i \in \mathbb{K}[\partial_{p,1}, \dots, \partial_{p,n}]$. Let $N_i \in W_R$ be the Weyl algebra element obtained by replacing $\partial_{p,i}$ with ∂_i. Then $\{N_1, \dots, N_m\}$ is a set of Noetherian operators for I: if $f \in I$, then $0 = D_i(f) = (N_i \bullet f)(p)$, which implies $N_i \bullet f \in \mathfrak{m}_p$ for all i. Conversely, if $N_i \bullet f \in \mathfrak{m}_p$ for all i, then $D_i(f) = 0$ for all i, hence $f \in I^{\perp\perp} = I$.

3.2.2. Primary ideals over a nonrational point

Next, assume $P \neq \mathfrak{m}_p$ for any $p \in \mathbb{K}^n$, i.e. P does not correspond to any $\mathbb{K}$-rational point (this happens only when $\mathbb{K}$ is not algebraically closed). If $\overline{\mathbb{K}}$ is the algebraic closure of $\mathbb{K}$, then the extension $P_{\overline{\mathbb{K}}}$ of P to $\overline{\mathbb{K}}[x_1, \dots, x_n]$ is still zero-dimensional and radical, but is no longer prime.

Proposition 3.7. *Let $P \subseteq R$ be a maximal ideal, and $\ell \in R$ a linear form such that $\ell(p) \neq \ell(q)$ for all $p \neq q \in \mathbb{V}(P_{\overline{\mathbb{K}}})$. Then:*

(1) *(Shape lemma) There exist univariate polynomials $g, g_1, \dots, g_n$ over $\mathbb{K}$ such that*

$$P = (g(\ell), x_1 - g_1(\ell), \dots, x_n - g_n(\ell)).$$

Furthermore, $\deg(g_i) < \deg(g) = \deg(P)$.

(2) *For $p \in \mathbb{V}(P_{\overline{\mathbb{K}}})$, the field $\kappa(P) = R/P$ is the smallest extension of $\mathbb{K}$ containing all coordinates of p.*

Proof. (1) See e.g. (Gianni and Mora, 1987, Proposition 1.6) for a proof of the Shape Lemma.
(2) It follows from (1) that

$$\kappa(P) = \frac{\mathbb{K}[x_1, \dots, x_n]}{(g(\ell), x_1 - g_1(\ell), \dots, x_n - g_n(\ell))} \cong \frac{\mathbb{K}[\ell]}{(g(\ell))} =: \mathbb{K}(\beta),$$

where β is a solution to $g(\ell)=0$. Thus $\mathbb{V}(P_{\overline{\mathbb{K}}})$ contains the point $(g_1(\beta),\dots,g_n(\beta))\in\overline{\mathbb{K}}^n$. On the other hand, the maximal ideal in $\overline{\mathbb{K}}[\boldsymbol{x}]$ associated to p contains a linear factor of $g(\ell)$, so any subfield of $\overline{\mathbb{K}}/\mathbb{K}$ containing all coordinates of p contains an isomorphic copy of $\kappa(P)$. □

Let $p\in\mathbb{V}(P_{\overline{\mathbb{K}}})$ be as in Proposition 3.7.2, and let $\mathfrak{m}_p\subseteq\kappa(P)[\boldsymbol{x}]$ be the associated maximal ideal, which is a minimal prime of $P_{\kappa(P)}$ that is now rational over the larger field $\kappa(P)$. Then $\kappa(P)=\kappa(\mathfrak{m}_p)$, so $W_{\kappa(P)}=W_{\kappa(\mathfrak{m}_p)}$. This allows for comparison of local dual spaces at P and $\mathfrak{m}_p$. Note however that even though $\kappa(P)=\kappa(\mathfrak{m}_p)$, there are distinct pairings

$$\langle\_,\_\rangle_{\kappa(P)}\colon \kappa(P)[\partial]\times\mathbb{K}[\boldsymbol{x}]\to\kappa(P)$$
$$\langle\_,\_\rangle_{\kappa(\mathfrak{m}_p)}\colon \kappa(\mathfrak{m}_p)[\partial]\times\kappa(P)[\boldsymbol{x}]\to\kappa(\mathfrak{m}_p)$$

arising over different base rings $\mathbb{K}[\boldsymbol{x}]$ and $\kappa(P)[\boldsymbol{x}]$. However, they do agree on the restriction of $\kappa(P)[\boldsymbol{x}]$ to $\mathbb{K}[\boldsymbol{x}]$, in the sense that $\langle D,f\rangle_{\kappa(P)}=\langle D,f\rangle_{\kappa(\mathfrak{m}_p)}$ for $f\in\mathbb{K}[\boldsymbol{x}]$. In particular:

Lemma 3.8. *The $\kappa(P)$-vector spaces $D_{\mathfrak{m}_p}[I_{\kappa(P)}]$ and $D_P[I]$ are equal.*

Proof. The inclusion $D_{\mathfrak{m}_p}[I_{\kappa(P)}]\subseteq D_P[I]$ is clear since $I=I_{\kappa(P)}\cap\mathbb{K}[\boldsymbol{x}]$. For the other inclusion, let $D\in D_P[I]$. Let $\kappa(P)$ be generated by $\{1,k_2,\dots,k_e\}$ over $\mathbb{K}$. If $I=(f_1,\dots,f_r)$, then $I_{\kappa(P)}$ is also generated by $f_1,\dots,f_r$ in $\kappa(P)[\boldsymbol{x}]$. Hence any element $g\in I_{\kappa(P)}$ is of the form $g=\sum_i g_i f_i$ for some $g_i\in\kappa(P)[\boldsymbol{x}]$, where the g_i themselves are of the form $g_i=\sum_j g_{i,j}k_j$ for some $g_{i,j}\in\mathbb{K}[\boldsymbol{x}]$. Then as $g_{i,j}f_i\in I\subset\mathbb{K}[\boldsymbol{x}]$,

$$\langle D,g\rangle_{\mathfrak{m}_p}=\sum_{i=1}^r\langle D,g_if_i\rangle_{\kappa(\mathfrak{m}_p)}=\sum_{i=1}^r\sum_{j=1}^e k_j\langle D,g_{i,j}f_i\rangle_{\kappa(\mathfrak{m}_p)}=\sum_{i=1}^r\sum_{j=1}^e k_j\langle D,g_{i,j}f_i\rangle_{\kappa(P)}=0.$$ □

Just as in Theorem 3.6, there is also a correspondence theorem given in Marinari et al. (1993) for zero-dimensional primary ideals that are not primary to a rational point.

Proposition 3.9 *(Marinari et al. (1993), Proposition 2.7). Let $P\subseteq\mathbb{K}[\boldsymbol{x}]$ be a maximal ideal, and assume $P\neq\mathfrak{m}_q$ for any $q\in\mathbb{K}^n$. Let $P_{\kappa(P)}$ be the extension of P in $\kappa(P)[\boldsymbol{x}]$, and let $\mathfrak{m}_p$ be a minimal prime of $P_{\kappa(P)}$ (for some $p\in\kappa(P)^n$). There is a bijection between P-primary ideals $I\subseteq\mathbb{K}[\boldsymbol{x}]$ and finite dimensional subspaces $\mathcal{D}\subseteq\kappa(P)[\partial_p]$ closed under the right $\kappa(P)[\boldsymbol{x}]$-action, given by*

$$I\mapsto D_{\mathfrak{m}_p}[I_{\kappa(P)}]\cong\{D\in\kappa(P)[\partial_p]\mid D(f)=0\text{ for all }f\in I_{\kappa(P)}\}\cong Q^{\perp}$$
$$\mathcal{D}\mapsto\{f\in\kappa(P)[\boldsymbol{x}]\mid D(f)=0\text{ for all }D\in\mathcal{D}\}\cap\mathbb{K}[\boldsymbol{x}]\cong\mathcal{D}^{\perp}\cap\mathbb{K}[\boldsymbol{x}],$$

where $Q\subseteq\kappa(P)[\boldsymbol{x}]$ is the $\mathfrak{m}_p$-primary component of $I_{\kappa(P)}$.

Proposition 3.9 links our Definition 3.1 to the classical Macaulay dual spaces: the local dual space of I at P corresponds to a finite dimensional space of linear functionals over a field extension where P contains rational point solutions. The connection to Noetherian operators is described in the following:

Theorem 3.10. *Let $P\subseteq R$ be a maximal ideal, and $I\subseteq R$ a P-primary ideal.*

(1) *If $\{D_i\}_i$ spans $D_P[I]$ (as a $\kappa(P)$-vector space), then any preimages $\{N_i\}_i\subseteq W_R$ of $\{D_i\}_i$ is a set of Noetherian operators for I.*
(2) *Conversely, if $\{N_i\}_i\subseteq W_R$ is a set of Noetherian operators for I, then their images in $W_{\kappa(P)}$ span $D_P[I]$.*

Proof. (1) One has $N_i\bullet f\in P\iff\langle D_i,f\rangle=0$ for $f\in R$. If $f\in I$, then $\langle D_i,f\rangle=0$, so $N_i\bullet f\in P$ for all i. Conversely, if $f\in R\setminus I$, then $f\notin I_{\kappa(P)}$, so $\langle D_i,f\rangle\neq0$ for some i.

(2) Let D_i be the image of N_i in $W_{\kappa(P)}$, and let $\mathcal{D}$ be the $\kappa(P)$-span of $\{D_i\}_i$. Then $\mathcal{D} \subseteq D_P[I]$, and if $\mathcal{D} \neq D_P[I]$, then $\mathcal{D}^\perp \supsetneq D_P[I]^\perp = I$ by Proposition 3.9. Then any $g \in \mathcal{D}^\perp \setminus I$ satisfies $N_i \bullet g \in P$ for all i, which is impossible since $\{N_i\}_i$ are Noetherian operators for I. □

Corollary 3.11. *Let $P \subseteq R$ be a maximal ideal, and $I \subseteq R$ a P-primary ideal. The set $\{N_i\}_i \subseteq W_R$ is a minimal set of Noetherian operators for I if and only if the set of their images $\{D_i\}_i \subseteq W_{\kappa(P)}$ is a basis of $D_P[I]$. Conversely, the set $\{D_i\}_i \subseteq W_{\kappa(P)}$ is a basis of $D_P[I]$ if and only if any preimages $\{N_i\}_i \subseteq W_R$ is a minimal set of Noetherian operators.*

3.3. Positive dimensional primary ideals

Now suppose $I \subseteq R$ is a primary ideal of arbitrary dimension d. Then there exists a set of d variables in R which is algebraically independent in R/I. We refer to these as *independent variables* $\boldsymbol{t} := \{t_1, \ldots, t_d\}$, the remaining variables as *dependent variables* $\boldsymbol{x} := \{x_1, \ldots, x_{n-d}\}$, and write $R = \mathbb{K}[\boldsymbol{t}, \boldsymbol{x}]$. Since we have two types of variables, we also write the Weyl algebra $W_R = R[\partial_{\boldsymbol{t}}, \partial_{\boldsymbol{x}}] := R[\partial_{t_1}, \ldots, \partial_{t_d}, \partial_{x_1}, \ldots, \partial_{x_{n-d}}]$, where $\partial_{x_i}, \partial_{t_j}$ correspond respectively to $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial t_j}$. Note that after a generic linear change of coordinates, every subset of d variables in R is independent in R/I – this can avoid the step of computing an independent set of variables, at the cost of any structure present in the generators of I. Set $U := \mathbb{K}[\boldsymbol{t}] \setminus \{0\}$, and $S := U^{-1}R = \mathbb{K}(\boldsymbol{t})[\boldsymbol{x}]$, the localization of R at the multiplicative set U. Let IS denote the extension of I to S, which is the ideal of S generated by the image of I under the (injective) localization map $R \to S$. For any $f \in S$, there exists $u \in U$ such that $g := uf$ is in R – we call any such g a *lift* of f in R, and extend this notion to the inclusion $W_R \hookrightarrow W_S$ in the natural way.

Lemma 3.12. *Let $\boldsymbol{t}$ be a maximal independent set of variables for I, and $S = \mathbb{K}(\boldsymbol{t})[\boldsymbol{x}]$ as above. Then*

(1) $\dim IS = 0$,
(2) $IS \cap R = I$,
(3) $\sqrt{IS} \cap R = \sqrt{I}$.

Proof. The algebraic independence of $\boldsymbol{t}$ in R/I means exactly that the universal map $\mathbb{K}[\boldsymbol{t}] \to R/I$ is injective. Since $\boldsymbol{t}$ was maximal, none of the dependent variables $\boldsymbol{x}$ are transcendental over $\mathbb{K}[\boldsymbol{t}]$, so localizing at U gives an integral extension $\mathbb{K}(\boldsymbol{t}) \hookrightarrow U^{-1}(R/I) \cong S/IS$. Thus $\dim S/IS = 0$, which is (i). Then (ii) and (iii) follow from the fact that I is primary with $I \cap U = \emptyset$ (so also $\sqrt{I} \cap U = \emptyset$), together with the 1-1 correspondence of primary (resp. prime) ideals in a localization, see (Atiyah and Macdonald, 1969, Proposition 4.8(ii)). □

Since IS is zero-dimensional, we can compute a $\kappa(PS)$-basis of $D_{PS}[IS]$ as in Section 3.2, and recover a set of Noetherian operators for I from this basis:

Proposition 3.13. *Let $I \subseteq R$ be a primary ideal of dimension d, $P = \sqrt{I}$, and $S = \mathbb{K}(\boldsymbol{t})[\boldsymbol{x}]$ where $\boldsymbol{t}, \boldsymbol{x}$ are independent resp. dependent variables for I.*

(1) *If $\{D_i\}_i \subseteq W_S$ is a set of Noetherian operators for IS, then any lift $\{N_i\}_i \subseteq W_R$ of $\{D_i\}_i$ is a set of Noetherian operators for I.*
(2) *Conversely, if $\{N_i\}_i \subseteq W_R$ is a set of Noetherian operators of I whose differential variables involve only $\partial_{\boldsymbol{x}}$ (and not $\partial_{\boldsymbol{t}}$), then their images $\{D_i\}_i \subseteq W_S$ is a set of Noetherian operators for IS.*

Proof. (1) For $f \in S$, one has $N_i \bullet f \in R$ for all $i \iff f \in R$: $\Leftarrow$ follows since the N_i have coefficients in R, and $\Rightarrow$ follows since $1 \in D_{PS}[IS]$ is in the span of $\{D_i\}_i$. Then by Lemma 3.12, $f \in I = IS \cap R \iff N_i \bullet f \in PS \cap R$ for all $i \iff N_i \bullet f \in P$ for all i.

(2) We show that $f \in IS \iff N_i \bullet f \in PS$ for all i. For the forward direction, let $f \in IS$. Then $f = \frac{g}{u}$ for some $g \in I$, $u \in U$. For every i, we have that $\frac{1}{s}$ is a scalar with respect to D_i (since N_i involves only $\partial_{\boldsymbol{x}}$), so $D_i \bullet f = \frac{N_i \bullet g}{u} \in PS$, since $N_i \bullet g \in P$.
Conversely, suppose $f = \frac{g}{u} \in S$ ($g \in R$, $u \in U$) is such that $N_i \bullet f \in PS$ for all i. Then $g = uf$ in R (since $R \hookrightarrow S$ is injective), so $N_i \bullet g = N_i \bullet (uf) = u(N_i \bullet f) \in PS \cap R = P$ for all i. Hence $g \in I$, and thus $f \in IS$. $\square$

3.4. Symbolic algorithms

In this subsection, we present algorithms to symbolically compute bases for local dual spaces, which yields Noetherian operators by Theorem 3.10 and Proposition 3.13. The method is a straightforward adaptation of the classical theory of Macaulay inverse systems involving *Macaulay matrices*.

As usual, we start with the zero-dimensional case. Let I be a zero-dimensional ideal in $R = \mathbb{K}[\boldsymbol{x}]$ and P a minimal prime of I. Given $D \in W_{\kappa(P)}$, we say the *∂-degree* of D is d if D is a degree d polynomial in the ∂-variables with coefficients in $\kappa(P)$. We define the *degree d truncated local dual spaces* as

$$D_P^{(d)}[I] := \{D \in D_P[I] \mid \partial\text{-degree of } D \text{ is} \leq d\}.$$

As in Section 3.1 let $\kappa(P)$ be the residue field at the prime P, let p be a point in $\mathbb{V}(P_{\kappa(P)})$ and $\mathfrak{m}_p$ the maximal ideal associated to p. By Lemma 3.8 and Proposition 3.9,

$$D_P[I] = D_{\mathfrak{m}_p}[I_{\kappa(P)}] = \{D \in \kappa(P)[\partial_p] : D(f) = 0 \text{ for all } f \in I_{\kappa(P)}\}.$$

Both $\{\partial^\alpha\}_{\alpha \in \mathbb{N}^n}$ and $\{\partial_p^\alpha\}_{\alpha \in \mathbb{N}^n}$ are bases for $\kappa(P)[\partial]$, and for $D \in \kappa(P)[\partial]$ the ∂-degree and ∂_p-degree of D are equal.

Fix a ∂-degree d, and let $C := \{\partial^\beta \mid |\beta| \leq d\}$, the set of all ∂-monomials of ∂-degree at most d. Pick a generating set $\{f_1, \ldots, f_r\}$ for I, and let $F := \{\boldsymbol{x}^\alpha f_i \mid i = 1, \ldots, r,\ |\alpha| < d\}$. For a fixed total ordering $\prec$ on ∂-monomials, we define the *degree d Macaulay matrix* M of dimension $|F| \times |C|$, where the rows are indexed by F, and the columns are indexed by C and ordered according to $\prec$. The entry corresponding to the row $\boldsymbol{x}^\alpha f_i$ and column ∂^β of the Macaulay matrix is the value with respect to the pairing Equation (1), i.e.

$$M_{\alpha,i;\beta} = \langle \partial^\beta, \boldsymbol{x}^\alpha f_i \rangle_{\kappa(P)} \in \kappa(P).$$

Any $D = \sum_{|\beta| \leq d} v_\beta \partial^\beta \in W_{\kappa(P)}$ is specified by its coefficient (column) vector $v = (v_\beta)_\beta$. Every entry of Mv is of the form $\langle D, g \rangle$ for some $g \in I$, so every element in the truncated local dual space $D_P^{(d)}[I]$ corresponds to a vector in the kernel of the Macaulay matrix. To show the reverse, we need the following:

Lemma 3.14. *With notation as above,*

$$D_P^{(d)}[I] = D_P[I + P^{d+1}].$$

Proof. First we show that

$$D_P[P^{d+1}] = D_P^{(d)}[0] = \operatorname{span}_{\kappa(P)}\{\partial_p^\beta : |\beta| \leq d\}.$$

Indeed, since $P_{\kappa(P)}$ is a product of maximal ideals, localizing $P_{\kappa(P)}$ at $\mathfrak{m}_p$ gives $D_P[P^{d+1}] = D_{\mathfrak{m}_p}[\mathfrak{m}_p^{d+1}]$, by Lemma 3.8. If $|\alpha| > d$ and $|\beta| \leq d$, then $\langle \partial_p^\beta, (\boldsymbol{x} - p)^\alpha \rangle = 0$, so $D_P^{(d)}[0] \subseteq D_{\mathfrak{m}_p}[\mathfrak{m}_p^{d+1}]$ (as $\mathfrak{m}_p^{d+1}$ is spanned over $\kappa(P)$ by $\{(\boldsymbol{x} - p)^\alpha \mid |\alpha| > d\}$). Conversely, if D has ∂_p-degree $> d$ then it has a nonzero term $c_\alpha \partial_p^\alpha$ with $|\alpha| > d$. Then $\langle D, (\boldsymbol{x} - p)^\alpha \rangle = \alpha! c_\alpha \neq 0$, hence $D \notin D_{\mathfrak{m}_p}[\mathfrak{m}_p^{d+1}]$.

Applying Proposition 3.2(2) then yields

$$D_P[I + P^{d+1}] = D_P[I] \cap D_P[P^{d+1}] = D_P[I] \cap D_P^{(d)}[0] = D_P^{(d)}[I]. \quad \square$$

Proposition 3.15. *With notation as above, let $\{v^{(k)}\}_k$ be a basis of the kernel of the degree d Macaulay matrix, and let $D_k := \sum_\beta v_\beta^{(k)} \partial^\beta$. Then $\{D_k\}_k$ is a basis for the truncated local dual space $D_P^{(d)}[I]$.*

Proof. Let $D \in D_P^{(d)}[I]$. We can write $D = \sum_{|\beta| \le d} v_\beta \partial^\beta$ for some vector $v = (v_\beta)_\beta$. Clearly $v \in \ker M$, so $v = \sum_k c_k v^{(k)}$, which implies $D = \sum_k c_k D_k$.

Conversely, we must show that D_k is in $D_P^{(d)}[I]$ for each k. The set

$$\{\boldsymbol{x}^\alpha f_i \mid |\alpha| < d,\ i = 1, \dots, r\} \cup \{(\boldsymbol{x} - p)^\beta f_i \mid |\beta| \ge d,\ i = 1, \dots, r\}$$

spans $I_{\kappa(P)}$. By construction, $\langle D_k, \boldsymbol{x}^\alpha f_i \rangle = 0$ for all $|\alpha| < d$. Note that each f_i vanishes at p, so $f_i \in \mathfrak{m}_p$. For each j, the term $x_j - p_j$ is also in $\mathfrak{m}_p$. If $|\beta| \ge d$ then $(\boldsymbol{x} - p)^\beta f_i \in \mathfrak{m}_p^{d+1}$. Since the ∂-degree of D_k is at most d, $D_k \in D_{\mathfrak{m}_p}[\mathfrak{m}_p^{d+1}]$ by Lemma 3.14. So $\langle D_k, (\boldsymbol{x} - p)^\beta f_i \rangle = 0$. Therefore $D_k \in D_{\mathfrak{m}_p}[\mathfrak{m}_p^{d+1}] \cap D_P[I] = D_P^{(d)}[I]$. $\square$

It is clear that $D_P^{(1)}[I] \subseteq D_P^{(2)}[I] \subseteq \cdots$, and since the local dual space is finite dimensional, this chain will stabilize to $D_P[I]$ after a finite number of steps. Furthermore, as the $D_P^{(d)}[I]$ are closed under the right R-action, the chain stabilizes when $\dim_{\kappa(P)} D_P^{(d)}[I] = \dim_{\kappa(P)} D_P^{(d+1)}[I]$. In view of Proposition 3.4, we thus arrive at Algorithm 1, which computes Noetherian operators for the P-primary component of I via kernels of successively larger Macaulay matrices. The kernels are represented by matrices with entries in $\kappa(P)$, which we can lift to R to obtain the coefficients of our Noetherian operators. The algorithm computes the local dual space, and then constructs Noetherian operators from a basis thereof, so the output Noetherian operators will depend on a choice of basis of the local dual space. In our Macaulay2 implementation, we always choose a basis in reduced column echelon form.

Algorithm 1 Compute Noetherian operators symbolically in dimension zero.

Input $I = (f_1, \dots, f_r)$ a zero-dimensional ideal, P a minimal prime of I, $\prec$ an ordering on monomials ∂^β
Output A set of Noetherian operators for the P-primary component of I

```
procedure NoetherianOperatorsZero(I, P)
    K ← ∅
    d ← 0                                            ▷ d corresponds to the degree bound
    repeat
        d ← d + 1
        F ← vector with entries x^α f_i, where |α| < d, i = 1, 2, ..., r
        C ← vector with entries ∂^β = ∂_{x_1}^{β_1} ··· ∂_{x_n}^{β_n}, where |β| ≤ d, in the order given by ≺
        M ← the Macaulay matrix with entries ⟨∂^β, x^α f_i⟩_{κ(P)} (rows indexed by F, columns by C)
        K_d ← ker M
    until dim K_d = dim K_{d−1}                      ▷ Stop when the dimension of the kernel stabilizes
    K ← ColReduce(K_d)                               ▷ Rewrites the generators of K_d in a reduced column echelon form
    return preimage of C^T K in W_R                  ▷ Preimage of the surjection W_R → W_{κ(P)}.
end procedure
```

For the general case, if I is positive-dimensional, we can use Proposition 3.13 to reduce to the zero dimensional case, yielding Algorithm 2.

Algorithm 2 Compute Noetherian operators symbolically in positive dimension.

Input $I \subseteq \mathbb{K}[\boldsymbol{t}, \boldsymbol{x}]$ an ideal, where $\boldsymbol{t}, \boldsymbol{x}$ are independent and dependent variables for I respectively, P a minimal prime of I, $\prec$ an ordering on monomials $\partial_{\boldsymbol{x}}^\beta$
Output A set of Noetherian operators for the P-primary component of I

```
procedure NoetherianOperators(I, P)
    S ← K(t)[x]
    K ← NoetherianOperatorsZero(IS, PS)
    return preimage of K in W_R                      ▷ Preimage of the map W_R → W_S induced by localization.
end procedure
```

Remark 3.16. Algorithm 1 describes how to find dual spaces (and therefore Noetherian operators) using Macaulay matrices. As mentioned in the introduction, this is not the only dual space algorithm. We present it here because it is the most general and simplest to describe. The algorithm of Mourrain (1997) instead uses antidifferentiation to find dual space basis elements of each degree from the previous degree elements, and it has better run time when the dimension of the dual space in each degree is low. That paper focuses on the case when the coefficient field is $\mathbb{C}$ but the algorithm can be applied any time the prime P is a rational point. We do not know of a way to generalize it to nonrational points. In our code, the default strategy is antidifferentiation when the point is rational and Macaulay matrices when it is not.

Example 3.17. Consider the 1-dimensional primary ideal $Q = ((x_1^2 - x_3)^2, x_2 - x_3(x_1^2 - x_3)) \subseteq R = \mathbb{Q}[x_1, x_2, x_3]$. Its radical is $P = (x_1^2 - x_3, x_2)$, and we may choose x_1, x_2 as the dependent variables and x_3 as the independent variable. Thus in $S = \mathbb{Q}(x_3)[x_1, x_2]$, QS is a zero-dimensional primary ideal whose radical is PS. In degree 1, the Macaulay matrix has a 2-dimensional kernel. In degree 2, the Macaulay matrix is

$$M = \begin{array}{r} \\ (x_1^2-x_3)^2 \\ (x_2-x_3(x_1^2-x_3)) \\ x_1(x_1^2-x_3)^2 \\ x_1(x_2-x_3(x_1^2-x_3)) \\ x_2(x_1^2-x_3)^2 \\ x_2(x_2-x_3(x_1^2-x_3)) \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & \partial_{x_1} & \partial_{x_2} & \partial_{x_1}^2 & \partial_{x_1}\partial_{x_2} & \partial_{x_2}^2 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 8x_3 & 0 & 0 \\ 0 & -2x_3x_1 & 1 & -2x_3 & 0 & 0 \\ 0 & 0 & 0 & 8x_3x_1 & 0 & 0 \\ 0 & -2x_3^2 & x_1 & -6x_3x_1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2x_3x_1 & 2 \end{bmatrix} \end{array}$$

with entries in S/PS. Performing linear algebra in the field S/PS, we see that the kernel of M is generated by $(1, 0, 0, 0, 0, 0)^T$ and $(0, 1, 2x_1x_3, 0, 0, 0)^T$. Since the dimension of the kernel did not increase, we terminate the loop in Algorithm 1 and conclude that $\{1, \partial_{x_1} + 2x_1x_3\partial_{x_2}\}$ is a set of Noetherian operators for Q.

Contrary to the algorithm in Cid-Ruiz et al. (2020), our algorithm does not go through the *punctual Hilbert scheme*. To make this clear, we perform a parallel computation following (Cid-Ruiz et al., 2020, Algorithm 3.8). Write $\mathbb{F} := \kappa(P)$. The point in the punctual Hilbert scheme corresponding to Q is the ideal

$$I = \langle y_1, y_2 \rangle^2 + \gamma(Q) \cdot \mathbb{F}[y_1, y_2],$$

where γ is the inclusion map

$$\gamma : R \hookrightarrow \mathbb{F}[y_1, y_2], \quad \begin{array}{l} x_1 \mapsto y_1 + x_1 \\ x_2 \mapsto y_2 + x_2 \\ x_3 \mapsto x_3 \end{array}$$

Here $I = (y_1 - 1/(2x_1x_3)y_2, y_2^2)$. A basis for $I^\perp$ can be computed using e.g. the classical Macaulay matrix method. The degree 2 Macaulay matrix is

$$\begin{array}{r} \\ (y_1-1/(2x_1x_3)y_2) \\ y_2^2 \\ y_1(y_1-1/(2x_1x_3)y_2) \\ y_1y_2^2 \\ y_2(y_1-1/(2x_1x_3)y_2) \\ y_2y_2^2 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & \partial_{x_1} & \partial_{x_2} & \partial_{x_1}^2 & \partial_{x_1}\partial_{x_2} & \partial_{x_2}^2 \end{array} \\ \begin{bmatrix} 0 & 1 & \frac{-1}{2x_1x_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & \frac{-1}{2x_1x_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{-1}{x_1x_3} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array},$$

with entries in $\mathbb{F}$, and, as expected, its kernel corresponds to the Noetherian operators $\{1, \partial_{x_1} + 2x_1x_3\partial_{x_2}\}$.

Example 3.18. We compute a primary decomposition using our symbolic algorithm. Consider the rational normal scroll $S(2,2) \subseteq \mathbb{P}^5$ given by the prime ideal

$$P := I_2\left(\begin{bmatrix} x_0 & x_1 & x_3 & x_4 \\ x_1 & x_2 & x_4 & x_5 \end{bmatrix}\right) \subseteq \mathbb{K}[x_0, \dots, x_5]$$

which has codimension 3 and degree 4. We can take x_1, x_3, x_4 as the dependent variables, and x_0, x_2, x_5 as independent variables.

Consider the ideal I generated by the following three polynomials:

$$\begin{aligned}
f_1 &:= x_1^4 - 2x_0x_1^2x_2 + x_0^2x_2^2 + x_1x_2x_3x_4 - x_0x_2x_4^2 - x_1^2x_3x_5 + x_0x_1x_4x_5 \\
f_2 &:= x_1^4 - 2x_0x_1^2x_2 + x_0^2x_2^2 + x_1x_2x_3x_4 - x_1^2x_4^2 - x_0x_2x_3x_5 + x_0x_1x_4x_5 \\
f_3 &:= x_2^2x_3x_4 - x_1x_2x_4^2 + x_4^4 - x_1x_2x_3x_5 + x_1^2x_4x_5 - 2x_3x_4^2x_5 + x_3^2x_5^2
\end{aligned}$$

This ideal was constructed to be a complete intersection defined by suitable linear combinations of generators of P^2. Our goal is to compute a primary decomposition of I. Using Macaulay2 v1.15 on an Intel® Core™ i7-1065G7 CPU, the command `primaryDecomposition I` did not terminate within 9 hours. On the other hand, `minimalPrimes I` quickly returns the primes

$$\begin{aligned}
P_1 &= (x_5, x_4, x_1^2 - x_0x_2), \\
P_2 &= (x_4, x_3, x_1^2 - x_0x_2), \\
P_3 &= (x_2, x_1, x_4^2 - x_3x_5), \\
P_4 &= (x_1, x_0, x_2^2x_3x_4 + x_4^4 - 2x_3x_4^2x_5 + x_3^2x_5^2), \\
P_5 &= (x_4^2 - x_3x_5, x_2x_4 - x_1x_5, x_1x_4 - x_0x_5, x_2x_3 - x_0x_5, x_1x_3 - x_0x_4, x_1^2 - x_0x_2)
\end{aligned}$$

Note that $P_5 = P$ is the prime ideal of the original rational normal scroll. The primes P_i have dimension 3 and degrees (2, 2, 2, 4, 4) respectively. We then run Algorithm 2 for the ideal I and each minimal prime P_i. Noetherian operators for the P_1-primary component of I are

$$\begin{aligned}
N_{1,1} &= 1 \\
N_{1,2} &= x_1\partial_{x_4} + x_2\partial_{x_5} \\
N_{1,3} &= \partial_{x_1} \\
N_{1,4} &= x_1x_3^2\partial_{x_1}^2 + 4x_0^2x_2\partial_{x_4}^2 + 8x_0x_1x_2\partial_{x_4}\partial_{x_5} + 4x_0x_2^2\partial_{x_5}^2 - 8x_0x_3\partial_{x_4}
\end{aligned}$$

For the P_2-primary component, we get Noetherian operators

$$\begin{aligned}
N_{2,1} &= 1 \\
N_{2,2} &= x_1\partial_{x_3} + x_2\partial_{x_4} \\
N_{2,3} &= \partial_{x_1} \\
N_{2,4} &= x_1x_5^2\partial_{x_1}^2 + 4x_0x_2^2\partial_{x_3}^2 + 8x_1x_2^2\partial_{x_3}\partial_{x_4} + 4x_2^3\partial_{x_4}^2 + 8x_1x_5\partial_{x_3}
\end{aligned}$$

For the P_3-primary component, we get Noetherian operators

$$\begin{aligned}
N_{3,1} &= 1 \\
N_{3,2} &= \partial_{x_4} \\
N_{3,3} &= x_4\partial_{x_1} + x_5\partial_{x_2} \\
N_{3,4} &= x_3^2x_5\partial_{x_1}^2 + 2x_3x_4x_5\partial_{x_1}\partial_{x_2} + x_3x_5^2\partial_{x_2}^2 - 2x_0x_4\partial_{x_1} \\
N_{3,5} &= x_3^2x_4x_5\partial_{x_1}^3 + 3x_3^2x_5^2\partial_{x_1}^2\partial_{x_2} + 3x_3x_4x_5^2\partial_{x_1}\partial_{x_2}^2 + x_3x_5^3\partial_{x_2}^3
\end{aligned}$$

$$
\begin{aligned}
&- 6x_0x_3x_5\partial_{x_1}^2 - 6x_0x_4x_5\partial_{x_1}\partial_{x_2} + 6x_3x_4\partial_{x_1}\\
N_{3,6} &= -27x_3^3x_4x_5\partial_{x_1}^4 - 108x_3^3x_5^2\partial_{x_1}^3\partial_{x_2} - 162x_3^2x_4x_5^2\partial_{x_1}^2\partial_{x_2}^2 - 108x_3^2x_5^3\partial_{x_1}\partial_{x_2}^3 - 27x_3x_4x_5^3\partial_{x_2}^4\\
&+ 324x_0x_3^2x_5\partial_{x_1}^3 + 648x_0x_3x_4x_5\partial_{x_1}^2\partial_{x_2} + 324x_0x_3x_5^2\partial_{x_1}\partial_{x_2}^2 + (-324x_0^2x_4 - 648x_3^2x_4)\partial_{x_1}^2\\
&- 648x_3^2x_5\partial_{x_1}\partial_{x_2} + 81x_0^2x_5\partial_{x_4}^2 + 1944x_0x_3\partial_{x_1}
\end{aligned}
$$

For the P_4-primary component, we get Noetherian operators

$$N_{4,1} = 1$$

For the P-primary component, we get Noetherian operators

$$
\begin{aligned}
N_{5,1} &= 1,\\
N_{5,2} &= \partial_{x_4},\\
N_{5,3} &= \partial_{x_3},\\
N_{5,4} &= \partial_{x_1},\\
N_{5,5} &= 2x_0x_5\partial_{x_1}\partial_{x_3} + x_0x_2\partial_{x_3}^2 + x_2x_4\partial_{x_1}\partial_{x_4},\\
N_{5,6} &= x_2^4x_5^2\partial_{x_1}^2 + (-8x_2^5x_4 + 4x_2^3x_4x_5^2 + 32x_0x_2^2x_5^3 - 8x_0x_5^5)\partial_{x_1}\partial_{x_3}\\
&+ (-4x_2^5x_5 + 16x_2^3x_4x_5^2 + 2x_2^3x_5^3 - 4x_2x_4x_5^4)\partial_{x_1}\partial_{x_4} + (4x_2^4x_5^2 - x_2^2x_5^4)\partial_{x_4}^2,\\
N_{5,7} &= -x_2^4x_4x_5\partial_{x_1}^2 + (8x_0x_2^4x_5 - 32x_0x_2^2x_4x_5^2 - 4x_0x_2^2x_5^3 + 8x_0x_4x_5^4)\partial_{x_1}\partial_{x_3}\\
&+ (4x_2^5x_4 - 2x_2^3x_4x_5^2 - 16x_0x_2^2x_5^3 + 4x_0x_5^5)\partial_{x_1}\partial_{x_4} + (8x_0x_2^3x_5^2 - 2x_0x_2x_5^4)\partial_{x_3}\partial_{x_4},\\
N_{5,8} &= (-8x_2^{11}x_4x_5^3 - 8x_0x_2^8x_5^6 + 6x_0x_2^6x_5^8 - x_0x_2^4x_5^{10})\partial_{x_1}^3\\
&+ (96x_0x_2^{11}x_5^3 + 96x_0x_2^9x_4x_5^4 - 48x_0x_2^9x_5^5 - 120x_0x_2^7x_4x_5^6 + 48x_0x_2^5x_4x_5^8\\
&- 6x_0x_2^3x_4x_5^{10})\partial_{x_1}^2\partial_{x_3}\\
&+ (384x_0x_2^{10}x_4x_5^3 - 96x_0x_2^8x_4x_5^5 + 384x_0^2x_2^7x_5^6 - 384x_0^2x_2^5x_5^8 + 120x_0^2x_2^3x_5^{10}\\
&- 12x_0^2x_2x_5^{12})\partial_{x_1}\partial_{x_3}^2\\
&+ (128x_0x_2^{13}x_4 - 384x_0^2x_2^{10}x_5^3 - 512x_0^2x_2^8x_4x_5^4 + 288x_0^2x_2^8x_5^5\\
&\quad + 768x_0^2x_2^6x_4x_5^6 - 48x_0^2x_2^6x_5^7 - 416x_0^2x_2^4x_4x_5^8 + 96x_0^2x_2^2x_4x_5^{10} - 8x_0^2x_4x_5^{12})\partial_{x_3}^3\\
&+ (48x_2^{12}x_4x_5^2 - 24x_2^{10}x_4x_5^4 + 48x_0x_2^9x_5^5 - 60x_0x_2^7x_5^7 + 24x_0x_2^5x_5^9 - 3x_0x_2^3x_5^{11})\partial_{x_1}^2\partial_{x_4}\\
&+ (384x_0x_2^{10}x_5^4 + 384x_0x_2^8x_4x_5^5\\
&\quad - 96x_0x_2^8x_5^6 - 384x_0x_2^6x_4x_5^7 + 120x_0x_2^4x_4x_5^9 - 12x_0x_2^2x_4x_5^{11})\partial_{x_1}\partial_{x_3}\partial_{x_4}\\
&+ (192x_0x_2^{13}x_5 - 576x_0x_2^{11}x_4x_5^2 + 432x_0x_2^9x_4x_5^4 - 768x_0^2x_2^8x_5^5\\
&\quad - 72x_0x_2^7x_4x_5^6 + 1152x_0^2x_2^6x_5^7 - 624x_0^2x_2^4x_5^9 + 144x_0^2x_2^2x_5^{11} - 12x_0^2x_5^{13})\partial_{x_3}^2\partial_{x_4}\\
&+ (96x_2^{11}x_4x_5^3 - 24x_2^9x_4x_5^5 + 96x_0x_2^8x_5^6 - 96x_0x_2^6x_5^8 + 30x_0x_2^4x_5^{10} - 3x_0x_2^2x_5^{12})\partial_{x_1}\partial_{x_4}^2\\
&+ (192x_2^{14}x_4 - 192x_0x_2^{11}x_5^3 - 384x_0x_2^9x_4x_5^4 + 144x_0x_2^9x_5^5 + 576x_0x_2^7x_4x_5^6\\
&\quad - 24x_0x_2^7x_5^7 - 312x_0x_2^5x_4x_5^8 + 72x_0x_2^3x_4x_5^{10} - 6x_0x_2x_4x_5^{12})\partial_{x_3}\partial_{x_4}^2\\
&+ (64x_2^{14}x_5 - 64x_0x_2^9x_5^5 + 96x_0x_2^7x_5^7 - 52x_0x_2^5x_5^9 + 12x_0x_2^3x_5^{11} - x_0x_2x_5^{13})\partial_{x_4}^3\\
&+ (24x_2^{10}x_5^4 + 24x_2^8x_4x_5^5 - 18x_2^6x_4x_5^7 + 3x_2^4x_4x_5^9)\partial_{x_1}^2\\
&+ (768x_2^{13}x_4 - 96x_2^{11}x_4x_5^2 + 768x_0x_2^{10}x_5^3 + 48x_2^9x_4x_5^4\\
&\quad - 672x_0x_2^8x_5^5 + 216x_0x_2^6x_5^7 - 48x_0x_2^4x_5^9 + 6x_0x_2^2x_5^{11})\partial_{x_1}\partial_{x_3}
\end{aligned}
$$

$$+ (384x_2^{13}x_5 + 384x_2^{11}x_4x_5^2 - 144x_2^{11}x_5^3$$
$$- 432x_2^9x_4x_5^4 + 48x_2^9x_5^5 + 204x_2^7x_4x_5^6 - 54x_2^5x_4x_5^8 + 6x_2^3x_4x_5^{10})\partial_{x_1}\partial_{x_4}$$

From this we deduce that if Q is the P-primary component of I, then the multiplicity of Q over P is 8. Note that this is consistent with the fact that $2(4) + 2(4) + 2(6) + 4(1) + 4(8) = \deg I = 4^3$. Furthermore, as the set of Noetherian operators of Q contains the set of Noetherian operators of P^2, namely $\{1, \partial_{x_1}, \partial_{x_3}, \partial_{x_4}\}$, we see that Q is strictly contained in P^2. We can also see that the P_2-primary component is radical.

4. A numerical approach via interpolation

Keeping notation from Section 3.3, let $I \subseteq \mathbb{K}[\boldsymbol{t}, \boldsymbol{x}]$ be a primary ideal of dimension d, where $\boldsymbol{t}$ and $\boldsymbol{x}$ are sets of independent and dependent variables for I respectively. Let $\{N_1, \dots, N_m\}$ be a set of Noetherian operators for I as in Proposition 3.13, and write

$$N_i := \sum_\alpha f_{\alpha,i}(\boldsymbol{t}, \boldsymbol{x})\partial_{\boldsymbol{x}}^\alpha.$$

Fix a point $(\boldsymbol{t}_0, \boldsymbol{x}_0) \in \mathbb{V}(I)$ on the variety of I. We denote by $N_i(\boldsymbol{t}_0, \boldsymbol{x}_0)$ the *specialized Noetherian operator*

$$N_i(\boldsymbol{t}_0, \boldsymbol{x}_0) = \sum_\alpha f_{\alpha,i}(\boldsymbol{t}_0, \boldsymbol{x}_0)\partial_{\boldsymbol{x}}^\alpha \in \mathbb{K}[\partial_{\boldsymbol{x}}].$$

Theorem 4.1. *Assume $\mathbb{K} = \overline{\mathbb{K}}$. Let $\{N_1, \dots, N_m\}$ be a minimal set of Noetherian operators of a primary ideal I, and let $(\boldsymbol{x}_0, \boldsymbol{t}_0) \in \mathbb{V}(I)$. If $\boldsymbol{t}_0$ is general, then*

$$\operatorname{span}_{\mathbb{K}}\{N_1(\boldsymbol{t}_0, \boldsymbol{x}_0), \dots, N_m(\boldsymbol{t}_0, \boldsymbol{x}_0)\} = D_{\mathfrak{m}_{(\boldsymbol{t}_0,\boldsymbol{x}_0)}}[I + (\boldsymbol{t} - \boldsymbol{t}_0)].$$

Proof. We first show that $D_{\mathfrak{m}_{(\boldsymbol{t}_0,\boldsymbol{x}_0)}}[(\boldsymbol{t} - \boldsymbol{t}_0)] = \mathbb{K}[\partial_{\boldsymbol{x}}]$. The inclusion $\supseteq$ is clear. For the opposite inclusion, we first note that every element $D \in D_{\mathfrak{m}_{(\boldsymbol{t}_0,\boldsymbol{x}_0)}}[(\boldsymbol{t} - \boldsymbol{t}_0)]$ can be written in the form

$$D = \sum_{\alpha,\beta} c_{\alpha,\beta}\partial_{\boldsymbol{x}}^\alpha\partial_{\boldsymbol{t}}^\beta,$$

where $c_{\alpha,\beta} \in \mathbb{K}$, $\alpha \in \mathbb{N}^{n-d}$, $\beta \in \mathbb{N}^d$, and only finitely many of the $c_{\alpha,\beta}$ are nonzero. We need to show that for all β such that $\beta_1 + \dots + \beta_d > 0$ we have $c_{\alpha,\beta} = 0$. Assume this is not the case. Since the local dual space is closed under the right R-action, we can repeatedly act on D from the right with elements of the form $(t_i - (\boldsymbol{t}_0)_i)$ and $(x_i - (\boldsymbol{x}_0)_i)$ to obtain an operator $D' \in D_{\mathfrak{m}_{(\boldsymbol{t}_0,\boldsymbol{x}_0)}}[(\boldsymbol{t} - \boldsymbol{t}_0)]$ that has degree 1 in $\partial_{\boldsymbol{t}}$-variables and degree 0 in $\partial_{\boldsymbol{x}}$-variables. More precisely, we get an operator

$$D' = c_0 + \sum_{i=1}^d c_i\partial_{t_i},$$

where $c_j \in \mathbb{K}$, $j = 0, \dots, d$, and $c_i \neq 0$ for at least one $i = 1, \dots, d$. In this case however, we have

$$\langle D', (t_i - (\boldsymbol{t}_0)_i)\rangle = c_i \neq 0,$$

which is a contradiction.

With this, Proposition 3.2 yields that

$$D_{\mathfrak{m}_{(\boldsymbol{t}_0,\boldsymbol{x}_0)}}[I + (\boldsymbol{t} - \boldsymbol{t}_0)] = D_{\mathfrak{m}_{(\boldsymbol{t}_0,\boldsymbol{x}_0)}}[I] \cap D_{\mathfrak{m}_{(\boldsymbol{t}_0,\boldsymbol{x}_0)}}[(\boldsymbol{t} - \boldsymbol{t}_0)] = D_{\mathfrak{m}_{(\boldsymbol{t}_0,\boldsymbol{x}_0)}}[I] \cap \mathbb{K}[\partial_{\boldsymbol{x}}].$$

Since $\boldsymbol{t}_0 \in \mathbb{K}^d$ is general, the specializations $\{N_1(\boldsymbol{t}_0, \boldsymbol{x}_0), \dots, N_m(\boldsymbol{t}_0, \boldsymbol{x}_0)\}$ are $\mathbb{K}$-linearly independent in $D_{\mathfrak{m}_{(\boldsymbol{t}_0,\boldsymbol{x}_0)}}[I + (\boldsymbol{t} - \boldsymbol{t}_0)]$. Thus, to prove the theorem, it suffices to show that $\dim_{\mathbb{K}} D_{\mathfrak{m}_{(\boldsymbol{t}_0,\boldsymbol{x}_0)}}[I + (\boldsymbol{t} - \boldsymbol{t}_0)] = m$, where $m = m(I, P)$ is the multiplicity of I over P.

Set $R_0 := R_{\mathfrak{m}_{(\boldsymbol{t}_0,\boldsymbol{x}_0)}}$, the localization of R at the maximal ideal $\mathfrak{m}_{(\boldsymbol{t}_0,\boldsymbol{x}_0)}$, $I_0 := IR_0$, $P_0 := PR_0$, and

$$J_0 := (I + (\boldsymbol{t} - \boldsymbol{t}_0))R_0 = I_0 + (\boldsymbol{t} - \boldsymbol{t}_0)R_0,$$

which is primary to the maximal ideal in R_0. Then

$$\dim D_{\mathfrak{m}_{(\boldsymbol{t}_0,\boldsymbol{x}_0)}}[I + (\boldsymbol{t} - \boldsymbol{t}_0)] = \dim D_{\mathfrak{m}_{(\boldsymbol{t}_0,\boldsymbol{x}_0)}}[J_0] = \dim_{\mathbb{K}} R_0/J_0$$

by Theorem 3.6. On the other hand, $(\boldsymbol{t} - \boldsymbol{t}_0)R_0$ is a parameter ideal for R_0/I_0 and R_0/P_0. By generality of $\boldsymbol{t}_0$ again, Bertini's Theorem gives that $\boldsymbol{t} - \boldsymbol{t}_0$ forms a regular sequence on R_0/I_0, and $P_0 + (\boldsymbol{t} - \boldsymbol{t}_0)$ is radical, which implies $P_0 + (\boldsymbol{t} - \boldsymbol{t}_0) = \mathfrak{m}_{(\boldsymbol{t}_0,\boldsymbol{x}_0)}$. Thus, for general $\boldsymbol{t}_0$, (Eisenbud, 2013, Exercise 12.11(d),(e)) implies that

$$m(I, P) = m(I_0, P_0) = \frac{e((\boldsymbol{t} - \boldsymbol{t}_0), R_0/I_0)}{e((\boldsymbol{t} - \boldsymbol{t}_0), R_0/P_0)} = e((\boldsymbol{t} - \boldsymbol{t}_0), R_0/I_0) = \dim_{\mathbb{K}} R_0/J_0$$

as desired (here $e(\mathfrak{q}, M)$ is the Hilbert-Samuel multiplicity of the parameter ideal $\mathfrak{q}$ on a module M). $\square$

Using the above result we obtain a numerical algorithm that computes Noetherian operators specialized at points, described in Algorithm 3. This algorithm is very similar to the symbolic algorithm for computing Noetherian operators, the only difference being that the Macaulay matrix is evaluated at a point. The column reduction in step 11 is used to construct a basis consistent with the one computed in the symbolic algorithm. More precisely, for a fixed ordering $\prec$, the numerical matrix $K(p)$ in Algorithm 3 is precisely the symbolic matrix K in Algorithm 1 evaluated at the point p. Thus if the output of NoetherianOperators(I, P) is $\{N_1(\boldsymbol{t}, \boldsymbol{x}), \dots, N_m(\boldsymbol{t}, \boldsymbol{x})\}$, then the output of NoetherianOperatorsAtPoint$(I, (\boldsymbol{t}_0, \boldsymbol{x}_0))$ will be $\{N_1(\boldsymbol{t}_0, \boldsymbol{x}_0), \dots, N_m(\boldsymbol{t}_0, \boldsymbol{x}_0)\}$. In general, Algorithm 3 will be faster than Algorithm 2, as computations in the former are done in the base field $\kappa(\mathfrak{m}_{(\boldsymbol{t}_0,\boldsymbol{x}_0)}) = \mathbb{K}$ rather than in $\kappa(P)$, which is an extension of the rational function field in $\boldsymbol{t}$.

Algorithm 3 Compute specializations of Noetherian operators at a point.

Input $I \subseteq \mathbb{K}[\boldsymbol{t}, \boldsymbol{x}]$ an ideal, where $\boldsymbol{t}, \boldsymbol{x}$ are independent and dependent variables for I respectively, P a minimal prime of I, $\prec$ an ordering on monomials $\partial_{\boldsymbol{x}}^{\gamma}$, and $p \in \mathbb{V}(P)$
Output A set of Noetherian operators for the P-primary component of I, specialized at p

1: **procedure** NoetherianOperatorsAtPoint(I, p)
2: $K \leftarrow \emptyset$
3: $d \leftarrow 0$ ▷ d corresponds to the degree bound
4: **repeat**
5: $d \leftarrow d + 1$
6: $F \leftarrow$ vector with entries $\boldsymbol{x}^{\alpha}\boldsymbol{t}^{\beta} f_i$, where $|\alpha + \beta| < d$, $i = 1, 2, \dots, r$
7: $C \leftarrow$ vector with entries $\partial_{\boldsymbol{x}}^{\gamma}$, where $|\gamma| \le d$, in the order given by $\prec$
8: $M \leftarrow$ the Macaulay matrix with entries $(\partial_{\boldsymbol{x}}^{\gamma} \bullet (\boldsymbol{x}^{\alpha}\boldsymbol{t}^{\beta} f_i))(p)$ (rows indexed by F, columns by C)
9: $K_d \leftarrow \ker M$
10: **until** $\dim K_d = \dim K_{d-1}$ ▷ Stop when the dimension of the kernel stabilizes
11: $K(p) \leftarrow$ ColReduce(K_d) ▷ Rewrites generators of K_d in reduced column echelon form
12: **return** $C^T K(p)$
13: **end procedure**

Remark 4.2. Suppose ideal $I \subseteq R = \mathbb{K}[\boldsymbol{t}, \boldsymbol{x}]$ has a visible P-primary component Q, and a procedure is given for choosing a random point in $\mathbb{V}(P)$. Algorithm 3 gives a probabilistic algorithm for checking if a set $N = \{N_1(\boldsymbol{t}, \boldsymbol{x}), \dots, N_m(\boldsymbol{t}, \boldsymbol{x})\}$ is a set of Noetherian operators for Q as follows. Choose a random point $p \in \mathbb{V}(P)$ and use the algorithm to compute a set of Noetherian operators N' for Q specialized at p. Then check if N' has the same span as $N(p) = \{N_1(p), \dots, N_m(p)\}$.

To see that this algorithm works, if N is a set of Noetherian operators then $N(p)$ must match N' for any choice of $p \in \mathbb{V}(P)$. Conversely, suppose that N is not a set of Noetherian operators for Q. Then either there is $f \in Q$ and N_i such that $N_i \bullet f \notin P$ or there is $f \notin Q$ such that $N_i \bullet f \in P$ for all $i = 1, \ldots, m$. In the former case, since P is prime, the set of points p for which $N_i \bullet f(p) \neq 0$ is Zariski dense in $\mathbb{V}(P)$. For a randomly selected point $p \in \mathbb{V}(P)$, $N_i(p)$ is not the specialization at p of a Noetherian operator for Q. In the latter case $f \notin Q$ implies $f \notin Q R_{\mathfrak{m}_p}$ for any $p \in \mathbb{V}(P)$ since Q is primary. Because f is orthogonal to $N(p)$, $N(p)$ is not the specialization at p of a set of Noetherian operators for Q.

4.1. A hybrid approach

One of the bottlenecks in the performance of Algorithm 2 is working with a large Macaulay matrix over the field S/PS, where $S = \mathbb{Q}(\boldsymbol{t})[\boldsymbol{x}]$. On the other hand, Algorithm 3 performs the same computations over $\mathbb{C}$, which is much faster. In particular, Algorithm 3 computes an evaluated set of Noetherian operators, which reveals the monomial support in $\partial_{\boldsymbol{x}}$ of a valid set of Noetherian operators. This information can then be used to optimize the size of the Macaulay matrix and symbolically produce a set of Noetherian operators in a fraction of the time taken by Algorithm 2.

Let $I = (f_1, \ldots, f_r) \subseteq \mathbb{K}[\boldsymbol{t}, \boldsymbol{x}]$ be unmixed, P a minimal prime, and $p \in \mathbb{V}(P) \subseteq \mathbb{K}^n$ a rational point on the variety of P. Let $N' = \{D_1(p), \ldots, D_m(p)\}$ be the output of Algorithm 3, i.e. a reduced set of specialized Noetherian operators. Let $D_i(p) = \sum_{\beta \in B} c_{i,\beta}(p) \partial_{\boldsymbol{x}}^{\beta}$, where $B \subset \mathbb{N}^n$ is finite and let $d_i = \deg D_i$ be the $\partial_{\boldsymbol{x}}$-degree of the operator. Clearly, the vector $(c_\beta(\boldsymbol{x}, \boldsymbol{t}))_{|\beta| \le d}$ is in the kernel of the degree d Macaulay matrix M_d.

Let M be the submatrix of M_d obtained by keeping only columns corresponding to $\partial_{\boldsymbol{x}}^{\beta}$ with $\beta \in B$. The vector $(c_\beta(\boldsymbol{x}, \boldsymbol{t}))_{\beta \in B}$ is in the kernel of M, and because the operators are reduced, the kernel is one-dimensional. Thus in order to find a symbolic representation of the operator $D_i(\boldsymbol{x}, \boldsymbol{t})$, it suffices to find the kernel of the matrix M over $\kappa(PS)$.

One can further optimize the procedure by starting with fewer rows than necessary, and adding rows until the kernel becomes one-dimensional. Since rows are indexed by $\boldsymbol{x}^{\alpha} f_j$ for all $|\alpha| < d_i$ and $j = 1, 2, \ldots, r$, one could for example run the algorithm for $|\alpha| < 0, 1, \ldots, d_i$ until the dimension of the kernel is 1. This method is implemented in Algorithm 4.

Algorithm 4 Hybrid computation of Noetherian operators.

Input $I = (f_1, \ldots, f_r)$ an unmixed ideal, P a minimal prime of I, a point $p \in \mathbb{V}(P)$
Output A set of Noetherian operators for the P-primary component of I

procedure HybridNoetherianOperators(I, P, p)
  $N' \leftarrow$ NoehterianOperatorsAtPoint(I,p)
  $N \leftarrow \emptyset$
  for all $D' \in N'$ **do**
    $C \leftarrow$ vector with entries $\partial_{\boldsymbol{x}}^{\beta} = \partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n}$ for each $\partial_{\boldsymbol{x}}^{\beta}$ appearing in D'.
    $d \leftarrow 0$
    repeat
      $d \leftarrow d + 1$
      $R \leftarrow$ vector with entries $\boldsymbol{x}^{\alpha} f_i$, where $|\alpha| < d$, $i = 1, 2, \ldots, r$
      $M \leftarrow$ the matrix with entries $\langle \partial_{\boldsymbol{x}}^{\beta}, \boldsymbol{x}^{\alpha} f_i \rangle_{\kappa(PS)}$ (rows indexed by F, columns by C)
      $K \leftarrow \ker M$
    until $\dim K = 1$
    $D \leftarrow C^T K$
    $N \leftarrow N \cup \{D\}$
  end for
  return lift of N in W_R
end procedure

Example 4.3. Consider the primary ideal $I = (x^2, y^2 - xt) \subseteq \mathbb{Q}[t, x, y]$, and $S = \mathbb{Q}(t)[x, y]$. Let $p = (1, 0, 0)$. Algorithm 3 reveals that the reduced set of Noetherian operators specialized at p are $\{1, \partial_y, \frac{1}{2}\partial_y^2 + \partial_x, \frac{1}{6}\partial_y^3 + \partial_x \partial_y\}$. To find the operator corresponding unevaluated operator corresponding to $\frac{1}{6}\partial_y^3 + \partial_x\partial_y$ for example, it suffices to find the kernel of the following submatrix of the Macaulay matrix

$$\begin{array}{c}
 \\
x^2 \\
y^2-xt \\
x(x^2) \\
x(y^2-xt) \\
y(x^2) \\
y(y^2-xt)
\end{array}
\begin{array}{c}
\begin{array}{cc} \partial_x\partial_y & \partial_y^3 \end{array} \\
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
-t & 6
\end{bmatrix}
\end{array}$$

over $\kappa(\sqrt{I}S)$. The kernel is 1-dimensional and generated by $(1, t/6)$, so we conclude that $\frac{1}{6}\partial_y^3 + \partial_x\partial_y$ is the Noetherian operator $\frac{t}{6}\partial_y^3 + \partial_x\partial_y$ evaluated at the point $(0, 0, 1)$. We repeat the procedure with all other operators to obtain the complete set of Noetherian operators $\{1, \partial_y, \frac{t}{2}\partial_y^2 + \partial_x, \frac{t}{6}\partial_y^3 + \partial_x\partial_y\}$.

If we had used Algorithm 2 we would have had to compute the kernel of the degree 4 Macaulay matrix, which has size (40×15).

Example 4.4. Consider the Noetherian operators $N_{5,1}, \ldots, N_{5,8}$ for the P-primary component in Example 3.18. The largest Noetherian operator has degree 3, which means that we have to compute the kernel of the degree 4 Macaulay matrix, which has dimensions (252×35). Over the field $\kappa(PS) = S/PS$, where $S = \mathbb{Q}(x_0, x_2, x_5)[x_1, x_3, x_4]$, this takes about 2 minutes. In contrast, computing the kernel of the evaluated Macaulay matrix over $\mathbb{C}$ takes about 0.4 seconds.

Following Algorithm 4, we note that the largest matrix we need to deal with has dimensions (12×13), which allows us to obtain the same Noetherian operators in about 1 second.

4.2. Reconstructing a set of Noetherian operators from sampled points

Given an ideal I and an oracle for sampling points on an isolated component V of $\mathbb{V}(I)$, we seek to produce a set of Noetherian operators describing the primary ideal Q corresponding to V. One way to supply such an oracle is via numerical irreducible decomposition (Sommese et al., 2001) to construct a *witness set* for each isolated component. The witness set for V can then be used to sample points on V, as described in Sommese et al. (2005).

Another instance in which such an oracle can be obtained is when the variety V of interest is expressed as the image of a known rational map from another variety W for which one has a witness set, $\varphi : W \dashrightarrow V$ (cf. *pseudo-witness set* from Hauenstein and Sommese (2010)). In this case points sampled from W can be mapped forward to points on V. In particular when $W = \mathbb{K}^m$, sampling points on $\mathbb{K}^m$, and therefore on V, is trivial.

As in Algorithm 2, let $I \subseteq \mathbb{K}[\boldsymbol{t}, \boldsymbol{x}]$ be P-primary, $S = \mathbb{K}(\boldsymbol{t})[\boldsymbol{x}]$, and K a basis for the kernel of the Macaulay matrix over S in Algorithm 1. The entries of K are coefficients of elements in $D_{PS}[IS]$, which live in the residue field $\kappa(PS) = S/PS$, and are represented by polynomials in $\boldsymbol{x}$ with coefficients which are rational functions in $\boldsymbol{t}$. On the other hand, entries of $K(p)$ in Algorithm 3 are evaluations of the aforementioned rational functions at a point p, and live in $\mathbb{K}$. We now seek to recover K from a sampled set of evaluations $K(p_1), \ldots, K(p_\ell)$, via interpolation of rational functions.

Example 4.5. Let $I = (x^2, y^2 - tx)$ be an ideal in $\mathbb{C}[t, x, y]$. Here t is an independent variable, and x, y are dependent. We sample four points $(1, 0, 0), (2, 0, 0), (3, 0, 0), (4, 0, 0)$ on the variety $\mathbb{V}(I)$. Running Algorithm 3 gives four differential operators with constant coefficients for each point, shown in Table 1.

Interpolating each coefficient, we conclude that the coefficient of ∂_x in the third operator and the coefficient of $\partial_x\partial_y$ in the fourth one can be both chosen to be $\frac{1}{t}$. Hence we get a set of four Noetherian operators

$$1, \quad \partial_y, \quad \frac{1}{2}\partial_y^2 + \frac{1}{t}\partial_y, \quad \frac{1}{6}\partial_y^3 + \frac{1}{t}\partial_x\partial_y,$$

Table 1
Specialized Noetherian operators at different points.

(t, x, y)	Operator 1	Operator 2	Operator 3	Operator 4
$(1, 0, 0)$	1	∂_y	$\frac{1}{2}\partial_y^2 + \partial_x$	$\frac{1}{6}\partial_y^3 + \partial_x\partial_y$
$(2, 0, 0)$	1	∂_y	$\frac{1}{2}\partial_y^2 + \frac{1}{2}\partial_x$	$\frac{1}{6}\partial_y^3 + \frac{1}{2}\partial_x\partial_y$
$(3, 0, 0)$	1	∂_y	$\frac{1}{2}\partial_y^2 + \frac{1}{3}\partial_x$	$\frac{1}{6}\partial_y^3 + \frac{1}{3}\partial_x\partial_y$
$(4, 0, 0)$	1	∂_y	$\frac{1}{2}\partial_y^2 + \frac{1}{4}\partial_x$	$\frac{1}{6}\partial_y^3 + \frac{1}{4}\partial_x\partial_y$

and after clearing denominators, we get

$$1, \quad \partial_y, \quad t\partial_y^2 + 2\partial_y, \quad t\partial_y^3 + 6\partial_x\partial_y.$$

This result is confirmed to be correct by computing Noetherian operators symbolically, e.g. via Algorithm 2.

The interpolation procedure is described as follows: we wish to find a rational function $\frac{f(\boldsymbol{t},\boldsymbol{x})}{g(\boldsymbol{t},\boldsymbol{x})}$ such that $f(p_i)/g(p_i) = c_i$ for all $i = 1, \ldots, \ell$. Choose an ansatz for f, g of the form $f = \sum_{(\alpha,\beta)\in A} f_{\alpha,\beta}\boldsymbol{t}^\alpha\boldsymbol{x}^\beta$, and $g = \sum_{(\alpha,\beta)\in B} g_{\alpha,\beta}\boldsymbol{t}^\alpha\boldsymbol{x}^\beta$, where $A, B \subseteq \mathbb{Z}^n_{\geq 0}$, with the $f_{\alpha,\beta}, g_{\alpha,\beta}$ to be determined. Then for each point p_i we get a linear equation

$$\sum_{(\alpha,\beta)\in A} f_{\alpha,\beta}\, p_i^{(\alpha,\beta)} - c_i \sum_{(\alpha,\beta)\in B} g_{\alpha,\beta}\, p_i^{(\alpha,\beta)} = 0, \tag{3}$$

where $p_i^{(\alpha,\beta)}$ is the monomial $\boldsymbol{t}^\alpha\boldsymbol{x}^\beta$ evaluated at the point p_i. Since in (3) we are solving $f(p_i) - c_i g(p_i) = 0$, a possible solution obtained from the algorithm may correspond to a rational function f/g where both $f, g \in \sqrt{I}$. For this reason, we remove the solutions where the numerator or the denominator vanishes on a generic point in $\mathbb{V}(I)$. This method is described in Algorithm 5.

Algorithm 5 Multivariate rational function interpolation.

Input A sequence of points $p = (p_i)$ and values $v = (v_i)$; row vectors $\vec{n}, \vec{d}$ specifying the monomials appearing in the numerator and denominator
Output A rational function f/g such that $\frac{f(p_i)}{g(p_i)} = v_i$ for all i, and where f and g have monomial support in $\vec{n}$ and $\vec{d}$ respectively

procedure RationalInterpolation($p, v, \vec{n}, \vec{d}$)
  $N \leftarrow$ matrix, whose ith row is the vector $\vec{n}$ evaluated at p_i
  $D \leftarrow$ matrix, whose ith row is the vector $-v_i\vec{d}$ evaluated at p_i
  $M \leftarrow \begin{pmatrix} N & D \end{pmatrix}$
  $K \leftarrow \ker(M)$
  for all columns k in K **do**
    $k_f \leftarrow$ first Length($\vec{n}$) entries of k
    $k_g \leftarrow$ last Length($\vec{d}$) entries of k
    $f \leftarrow \vec{n}x_f$
    $g \leftarrow \vec{d}x_g$
    if $f(p_0) = 0$ or $g(p_0) = 0$ **then**
      remove column k from K
    end if
  end for
  if K is empty **then**
    return error ▷ No suitable rational functions found
  end if
  $x_f \leftarrow$ first Length($\vec{n}$) entries of any vector in K
  $x_g \leftarrow$ last Length($\vec{d}$) entries of any vector in K
  $f \leftarrow \vec{n}x_f$
  $g \leftarrow \vec{d}x_g$
  return $\frac{f}{g}$
end procedure

Remark 4.6. One has freedom to choose any plausible ansatz for f, g. For instance one can take all rational functions in $\boldsymbol{t}$ and $\boldsymbol{x}$ with degrees of numerators and denominators bounded by some constant k. Then any sufficiently large k is guaranteed to capture the operators we seek. This is the method used in our Macaulay2 implementation.

Other types of ansatzes for coefficients of operators are possible: for instance, one can choose a generating set of monomials in $\boldsymbol{x}$ for the residue field $\kappa(PS)$ as an extension of $\mathbb{K}(\boldsymbol{t})$, together with a degree bound on numerators and denominators of rational functions in $\boldsymbol{t}$.

Combining the subroutines in Algorithms 3 and 5, we obtain Algorithm 6, the main numerical algorithm for computing Noetherian operators. The algorithm takes as input an ideal and an oracle for sampling points on $\mathbb{V}(I)$, and outputs a set of Noetherian operators with interpolated rational function coefficients.

Algorithm 6 Compute Noetherian operators numerically via interpolation.

Input $I \subseteq \mathbb{K}[\boldsymbol{t}, \boldsymbol{x}]$ an ideal, $p = (p_i)$ a sequence of points in $\mathbb{V}(P)$ where P is an isolated prime of I
Output A set of Noetherian operators for the P-primary component of I

procedure NumericalNoetherianOperators(I, p)
    for all $i = 1, 2, \ldots$ **do**
        $\mathcal{N}_i \leftarrow$ NoetherianOperatorsAtPoint(I, p_i)
    end for
    for all terms $c_\alpha \partial^\alpha$ appearing in elements of $\mathcal{N}_1$ **do** ▷ $c_\alpha \in \mathbb{K}$
        $v_i \leftarrow c_\alpha$ for the corresponding term $c_\alpha \partial^\alpha$ in $\mathcal{N}_i$ for all i
        $d \leftarrow 0$
        repeat
            $\vec{n} \leftarrow$monomials $\boldsymbol{x}^\alpha \boldsymbol{t}^\beta$ such that $|\alpha + \beta| \le d$.
            $\vec{d} \leftarrow$ monomials $\boldsymbol{t}^\gamma$ such that $|\gamma| \le d$
            $f_\alpha / g_\alpha \leftarrow$RationalInterpolation($p, v, \vec{n}, \vec{d}$)
            $d \leftarrow d + 1$
        until interpolation succeeds
    end for
    return the set of operators $\mathcal{N}_1$ in which each term $c_\alpha \partial^\alpha$ is replaced by $\frac{f_\alpha}{g_\alpha} \partial^\alpha$
end procedure

Finally, combining Algorithm 6 with an existing numerical irreducible decomposition procedure yields Algorithm 7, a numerical primary decomposition algorithm for unmixed ideals.

Algorithm 7 Numerical primary decomposition for unmixed ideals.

Input $I \subseteq \mathbb{K}[\boldsymbol{t}, \boldsymbol{x}]$ an unmixed ideal
Output A list of irreducible components of $V(I)$ and a set of Noetherian operators for each primary component of I

procedure NumericalPrimaryDecomposition(I)
    $NV \leftarrow$ NumericalIrreducibleDecomposition(I)
    output $\leftarrow \{\}$
    for W in NV **do**
        $p \leftarrow$ sample(W)
        $N \leftarrow$ NumericalNoetherianOperators(I, p)
        output $\leftarrow$ append(output, $\{W, N\}$)
    end for
    return output
end procedure

Example 4.7. Next, we illustrate a numerical primary decomposition using Algorithm 6. Let J be the ideal of the K3 carpet over the scroll $S(3,3) \subseteq \mathbb{P}^7$, i.e.

$$\begin{aligned} J := (&x_1^2 - x_0x_2, x_1x_2 - x_0x_3, x_2^2 - x_1x_3, x_2y_0 - 2x_1y_1 + x_0y_2, x_3y_0 - 2x_2y_1 + x_1y_2, \\ &x_2y_1 - 2x_1y_2 + x_0y_3, x_3y_1 - 2x_2y_2 + x_1y_3, y_1^2 - y_0y_2, y_1y_2 - y_0y_3, y_2^2 - y_1y_3) \end{aligned}$$

in the ring $\mathbb{Q}[x_0, \dots, x_3, y_0, \dots, y_3]$. Let I be the ideal of a generic complete intersection of quadrics containing the carpet, generated by 5 random $\mathbb{Q}$-linear combinations of the 10 generators of J.

Neither `primaryDecomposition I` nor `minimalPrimes I` terminated within 9 hours. However, a numerical irreducible decomposition reveals that I has two minimal primes, of dimension 3 and degrees (6, 20) respectively. We then run Algorithm 7 on the witness sets.

Let Q be the component primary to the degree 6 minimal prime of I. We obtain

$$N_{Q,1} = 1$$
$$N_{Q,2} = \frac{x_3 y_1}{0.333333}\partial_{y_0} + \frac{x_2 y_3}{0.5}\partial_{y_1} + x_3 y_3 \partial_{y_2}$$

as Noetherian operators for Q. The component primary to the degree 20 minimal prime (which defines a generic link of the K3 carpet) has Noetherian operators $\{1\}$, i.e. is radical. For timing: the numerical irreducible decomposition took under 2 seconds, and computing Noetherian operators took under a second in total.

As the degree 6 minimal prime obtained from the numerical irreducible decomposition is the scroll $S(3,3)$ (being of minimal degree), i.e. $\sqrt{Q} = \sqrt{J}$, a natural question that arises is whether Q is in fact equal to J. We may verify this by directly computing Noetherian operators of J using Algorithm 2, obtaining

$$N_{J,1} = 1$$
$$N_{J,2} = 3y_1\partial_{y_0} + 2y_2\partial_{y_1} + y_3\partial_{y_2}$$

Although the Noetherian operators for J and the Noetherian operators for Q look different, the coefficients are equal up to multiplication by x_3 on the minimal prime of interest, which is the scroll (note that $x_3 y_2 - y_3 x_2$ lies in $\sqrt{J}$). This confirms that $Q = J$.

Example 4.8. One can also run Algorithm 7 on Example 3.18: using a reasonable number of points quickly yields partial information about the Noetherian operators displayed above, such as the multiplicity. The caveat is that some of the rational functions have large degree (for example denominators of $N_{1,6}$ have degree 6), so interpolating those coefficients will take correspondingly longer times.

5. General properties of Noetherian operators

Thus far, we have focused our attention on primary ideals. As we have seen, this is enough for the purpose of numerical primary decomposition, cf. Algorithm 7. Nonetheless, Definition 1.1 makes sense for arbitrary (i.e. not necessarily primary) ideals. In this last section, to expand the theoretical framework of Noetherian operators, we discuss various properties and behaviors of Noetherian operators for arbitrary ideals. First, we record how Noetherian operators vary under linear coordinate changes.

Proposition 5.1. *Let $R := \mathbb{K}[x_1, \dots, x_n] = \mathbb{K}[\boldsymbol{x}]$, and let φ be a $\mathbb{K}$-linear automorphism of R given by $\varphi(\boldsymbol{x}) := A\boldsymbol{x}$ for some $A \in GL_n(\mathbb{K})$. Define a $\mathbb{K}$-linear automorphism of the Weyl algebra $W_R = \mathbb{K}[\boldsymbol{x}]\langle\partial\rangle$ by*

$$\psi : \begin{pmatrix} \boldsymbol{x} \\ \partial \end{pmatrix} \mapsto \begin{pmatrix} A\boldsymbol{x} \\ (A^{-1})^T \partial \end{pmatrix}.$$

If $I \subseteq R$ is an ideal, and $D_1, \dots, D_r$ is a set of Noetherian operators for I, then $\psi(D_1), \dots, \psi(D_r)$ is a set of Noetherian operators for $\varphi(I) \subseteq R$.

Proof. For $f \in R$, one has

$$\begin{aligned} f \in \varphi(I) &\iff \varphi^{-1}(f) \in I \iff D_i \bullet \varphi^{-1}(f) \in \sqrt{I} \quad \forall i = 1, \dots, r \\ &\iff \varphi(D_i \bullet \varphi^{-1}(f)) \in \sqrt{\varphi(I)} \quad \forall i = 1, \dots, r, \end{aligned}$$

since $\sqrt{\varphi(I)} = \varphi(\sqrt{I})$, as φ is a $\mathbb{K}$-linear automorphism of R. Writing $D_i = \sum_\alpha p_\alpha \partial^\alpha$, we have $\varphi(D_i \bullet \varphi^{-1}(f)) = \varphi((\sum_\alpha p_\alpha \partial^\alpha) \bullet \varphi^{-1}(f)) = \sum_\alpha \varphi(p_\alpha)\varphi(\partial^\alpha \bullet \varphi^{-1}(f))$, so it suffices to show that $\varphi(\partial^\alpha \bullet \varphi^{-1}(f)) = \psi(\partial^\alpha) \bullet f$ for any $f \in R$. By linearity, it suffices to check this when $f = \boldsymbol{x}^\beta$ is a monomial, i.e. we must show $\varphi(\partial^\alpha \bullet \varphi^{-1}(\boldsymbol{x}^\beta)) = \psi(\partial^\alpha) \bullet \boldsymbol{x}^\beta$ for all $\alpha, \beta \in \mathbb{N}^n$.

We first consider the case where α, β are standard basis vectors, i.e. $\partial^\alpha = \partial_{x_j}$ and $\boldsymbol{x}^\beta = x_i$ for some $i, j \in \{1, \dots, n\}$. Then $\varphi\left(\partial_{x_j} \bullet \varphi^{-1}(x_i)\right) = \varphi\left(\partial_{x_j} \bullet \sum_{k=1}^n (A^{-1})_{i,k} x_k\right) = \varphi\left((A^{-1})_{i,j}\right) = (A^{-1})_{i,j} = \left(\sum_{k=1}^n (A^{-1})_{k,j} \partial_{x_k}\right) \bullet x_i = \psi(\partial_{x_j}) \bullet x_i$.

To show that this extends to arbitrary β, note that both $\varphi\left(\partial_{x_j} \bullet \varphi^{-1}(\_)\right)$ and $\psi(\partial_{x_j}) \bullet (\_)$ are both differential operators, which must satisfy the product rule, so if these agree on every variable x_i then they agree on every monomial $\boldsymbol{x}^\beta$. To extend to arbitrary α, note that ψ preserves multiplication in W by definition, so

$$\begin{aligned}
\varphi\left(\partial_{x_j}\partial_{x_k} \bullet \varphi^{-1}(_)\right) &= \varphi\left(\partial_{x_j} \bullet \varphi^{-1}\varphi\left(\partial_{x_k} \bullet \varphi^{-1}(_)\right)\right) \\
&= \varphi\left(\partial_{x_j} \bullet \varphi^{-1}(\psi(\partial_{x_k}) \bullet (_))\right) \\
&= \psi(\partial_{x_j}) \bullet \psi(\partial_{x_k} \bullet (_)) \\
&= \psi(\partial_{x_j})\psi(\partial_{x_k} \bullet (_)) \\
&= \psi(\partial_{x_j}\partial_{x_k}) \bullet (_)
\end{aligned}$$

hence inductively $\varphi\left(\partial^\alpha \bullet \varphi^{-1}(\_)\right) = \psi(\partial^\alpha) \bullet (\_)$ for any α. $\square$

Next, we give a construction for a global set of Noetherian operators for an unmixed ideal:

Proposition 5.2. *Let I be an unmixed ideal, with a minimal primary decomposition $I = q_1 \cap \dots \cap q_r$, and let N_i be a set of Noetherian operators for q_i for $i = 1, \dots, r$. For $D \in \bigcup_i N_i$, choose $h_D \in \bigcap\limits_{D \notin N_j} \sqrt{q_j} \setminus \bigcup\limits_{D \in N_i} \sqrt{q_i}$. Then $N := \{h_D D \mid D \in \bigcup_i N_i\}$ is a set of Noetherian operators for I.*

Proof. First, note that if $\bigcap\limits_{D \notin N_j} \sqrt{q_j} \subseteq \bigcup\limits_{D \in N_i} \sqrt{q_i}$ for some D, then $\bigcap\limits_{D \notin N_j} \sqrt{q_j} \subseteq \sqrt{q_i}$ for some i by prime avoidance, and then $\sqrt{q_j} \subseteq \sqrt{q_i}$ for some $i \neq j$, contradicting the unmixedness assumption on I. Thus choices of h_D always exist.

Suppose $f \in I$, and choose $D \in \bigcup_i N_i$. For any i with $D \in N_i$, we have $f \in q_i \implies D \bullet f \in \sqrt{q_i}$. By choice of h_D, this implies $h_D D \bullet f \in \left(\bigcap\limits_{D \in N_i} \sqrt{q_i}\right) \cap \left(\bigcap\limits_{D \notin N_j} \sqrt{q_j}\right) = \sqrt{I}$.

Conversely, suppose $f \notin I$. Then WLOG $f \notin q_1$, so there exists $D_1 \in N_1$ such that $D_1 \bullet f \notin \sqrt{q_1}$. Since also $h_{D_1} \notin \sqrt{q_1}$ and $\sqrt{q_1}$ is prime, this means $h_{D_1} D_1 \bullet f \notin \sqrt{q_1}$, and thus $h_{D_1} D_1 \bullet f \notin \sqrt{I}$. $\square$

Finally, we consider the question of recovering $\sqrt{I}$ from the data of I and Noetherian operators for I. Fix a finite generating set G of I and a set of Noetherian operators N of I. We consider the ideal $N(G) := (D \bullet g \mid D \in N,\ g \in G)$ obtained by applying operators in N to the generating set G. Note that since G generates I, one has $N(G) = (D \bullet f \mid D \in N,\ f \in I)$ – in particular, $N(G)$ does not depend on the choice of G, and one always has $N(G) \subseteq \sqrt{I}$ by definition. However, even if I is primary, $N(G)$ need not equal $\sqrt{I}$:

Example 5.3. Let $I = ((xy - z^2)^2) \subseteq \mathbb{C}[x, y, z]$. Then $N = \{1, \partial_y\}$ is a set of Noetherian operators of I. Applying N to the single generator of I yields $N(G) = ((xy - z^2)^2, 2x(xy - z^2))$, which is strictly contained in $\sqrt{I} = (xy - z^2)$.

However, the issue in Example 5.3 was that $N(G)$ was not unmixed (whereas radical ideals are evidently unmixed), which turns out to be the only obstruction:

Proposition 5.4. *If $I = (G)$ is primary, and N is a set of Noetherian operators for I constructed as in Proposition 3.13(1), then the unmixed part of $N(G)$ is $\sqrt{I}$.*

Proof. Let $P = \sqrt{I}$. Since WLOG 1 is in the $\mathbb{K}$-span of N, we have $I \subseteq N(G) \subseteq P$, which implies that $\sqrt{N(G)} = P$. Let Q be the unmixed part of $N(G)$, which is the P-primary component of $N(G)$.

First consider the case $\dim I = 0$, so that $Q = N(G)$, the dual space $D_P[P]$ is spanned by $\{1\}$, and $N \subseteq D_P[I]$. Suppose that $Q \neq P$, so that $\dim_{K(P)} D_P[Q] > 1$, hence $D_P[Q]$ contains a nonzero element p of ∂-degree ≥ 1. For each $D \in N$, the operator $p \circ D$ is an element of $D_P[I]$ (since $D \bullet f \in Q$ for all $f \in I$). Choosing some $D \in N$ of maximal degree gives that $p \circ D$ is outside the linear span of N. Therefore $D_P[I]$ is strictly larger than the span of N, contradicting Corollary 3.11.

If now I is primary of any dimension, then by the same procedure as in Section 3.3 we may invert a maximal set of independent variables to obtain a zero-dimensional ideal IS. By Proposition 3.13(2), the set N gives a set of Noetherian operators for IS. Then the reasoning in the zero-dimensional case above shows that $QS = PS$, which implies $Q = P$. $\square$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

Atiyah, Michael F., Macdonald, Ian G., 1969. Introduction to Commutative Algebra. Addison Wesley.
Bates, Daniel J., Oeding, Luke, 2011. Toward a Salmon conjecture. Exp. Math. 20 (3), 358–370.
Cid-Ruiz, Yairon, Homs, Roser, Sturmfels, Bernd, 2020. Primary ideals and their differential equations. arXiv preprint. arXiv:2001.04700.
Damiano, Alberto, Sabadini, Irene, Struppa, Daniele C., 2007. Computational methods for the construction of a class of Noetherian operators. Exp. Math. 16 (1), 41–53.
Decker, Wolfram, Greuel, Gert-Martin, Pfister, Gerhard, 1999. Primary decomposition: algorithms and comparisons. In: Algorithmic Algebra and Number Theory. Springer, pp. 187–220.
Eisenbud, David, 2013. Commutative Algebra: with a View Toward Algebraic Geometry, vol. 150. Springer Science & Business Media.
Eisenbud, David, Huneke, Craig, Vasconcelos, Wolmer, 1992. Direct methods for primary decomposition. Invent. Math. 110 (1), 207–235.
Gianni, Patrizia, Mora, Teo, 1987. Algebraic solution of systems of polynomial equations using Groebner bases. In: International Conference on Applied Algebra, Algebraic Algorithms, and Error-Correcting Codes. Springer, pp. 247–257.
Gianni, Patrizia, Trager, Barry, Zacharias, Gail, 1988. Gröbner bases and primary decomposition of polynomial ideals. J. Symb. Comput. 6 (2–3), 149–167.
Grayson, Daniel R., Stillman, Michael E., 2002. Macaulay2, a software system for research in algebraic geometry.
Gröbner, Wolfgang, 1938. Über eine neue idealtheoretische Grundlegung der algebraischen Geometrie. Math. Ann. 115 (1), 333–358.
Hauenstein, Jonathan D., Sherman, Samantha N., Wampler, Charles W., 2018. Exceptional Stewart–Gough platforms, Segre embeddings, and the special Euclidean group. SIAM J. Appl. Algebra Geom. 2 (1), 179–205.
Hauenstein, Jonathan D., Sommese, Andrew J., 2010. Witness sets of projections. Appl. Math. Comput. 217 (7), 3349–3354.
Krone, Robert, Leykin, Anton, 2017a. Eliminating dual spaces. J. Symb. Comput. 79, 609–622.
Krone, Robert, Leykin, Anton, 2017b. Numerical algorithms for detecting embedded components. J. Symb. Comput. 82, 1–18.
Leykin, Anton, 2008. Numerical primary decomposition. In: International Symposium on Symbolic and Algebraic Computation, pp. 165–172.
Macaulay, F.S., 1916. The Algebraic Theory of Modular Systems. Cambridge Mathematical Library. Cambridge University Press, Cambridge. Revised 1994 reprint of the 1916 original. With an introduction by Paul Roberts.
Marinari, M.G., Möller, H.M., Mora, T., 1993. Gröbner bases of ideals defined by functionals with an application to ideals of projective points. Appl. Algebra Eng. Commun. Comput. 4 (2), 103–145.
Mourrain, B., 1997. Isolated points, duality and residues. In: Algorithms for Algebra, Eindhoven, 1996. J. Pure Appl. Algebra 117/118, 469–493.
Nonkań, Ibrahim, 2013. The Weyl algebra and Noetherian operators. Afr. Diaspora J. Math. 16 (1), 59–69.
Oberst, Ulrich, 1999. The construction of Noetherian operators. J. Algebra 222 (2), 595–620.

Palamodov, Viktor P., 1970. Linear Differential Operators with Constant Coefficients. Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete. Springer-Verlag.

Shimoyama, Takeshi, Yokoyama, Kazuhiro, 1996. Localization and primary decomposition of polynomial ideals. J. Symb. Comput. 22 (3), 247–277.

Sommese, Andrew J., Verschelde, Jan, Wampler, Charles W., 2001. Numerical decomposition of the solution sets of polynomial systems into irreducible components. SIAM J. Numer. Anal. 38 (6), 2022–2046.

Sommese, Andrew J., Verschelde, Jan, Wampler, Charles W., 2005. Introduction to numerical algebraic geometry. In: Solving Polynomial Equations: Foundations, Algorithms, and Applications. Springer Berlin Heidelberg, Berlin, Heidelberg, pp. 301–337.

Sommese, Andrew J., Wampler II, Charles W., 2005. The Numerical Solution of Systems of Polynomials. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ.

Sturmfels, Bernd, 2002. Solving Systems of Polynomial Equations. CBMS Regional Conference Series in Mathematics, vol. 97. American Mathematical Society, Providence, RI. Published for the Conference Board of the Mathematical Sciences, Washington, DC.