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Nilpotent Centralizers and Good Filtrations

Pramod N. Achar[1] · **William Hardesty**[2]

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Abstract
Let G be a connected reductive group over an algebraically closed field $\Bbbk$. Under mild restrictions on the characteristic of $\Bbbk$, we show that any G-module with a good filtration also has a good filtration as a module for the reductive part of the centralizer of a nilpotent element x in its Lie algebra.

1 Introduction

Let G be a connected reductive group over an algebraically closed field $\Bbbk$ of characteristic $p > 0$, and let H be a connected reductive subgroup. Recall that (G, H) is said to be a *Donkin pair* or a *good filtration pair* if every G-module with a good filtration still has a good filtration when regarded as an H-module.

Now let x be a nilpotent element in the Lie algebra of G, and let $G^x \subset G$ be its stabilizer. If p is good for G, then the theory of *associated cocharacters* is available, and this gives rise to a decomposition

$$G^x = G^x_{\mathrm{red}} \ltimes G^x_{\mathrm{unip}}$$

where G^x_{unip} is a connected unipotent group, and G^x_{red} is a (possibly disconnected) group whose identity component $(G^x_{\mathrm{red}})^\circ$ is reductive (cf. [11, 5.10]). The main result of this paper is the following.

Theorem 1.1 *Let G be a connected reductive group over an algebraically closed field $\Bbbk$ of good characteristic. For any nilpotent element x in its Lie algebra, $(G, (G^x_{red})^\circ)$ is a Donkin pair.*

✉ Pramod N. Achar
pramod@math.lsu.edu

William Hardesty
hardes11@gmail.com

1 Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA

2 School of Mathematics and Statistics, University of Sydney, Camperdown, NSW 2006, USA

Now suppose that $H \subset G$ is a possibly *disconnected* reductive subgroup, i.e., a group whose identity component H° is reductive. If the characteristic of $\Bbbk$ does not divide the order of the finite group H/H°, then the category of finite-dimensional H-modules is a highest-weight category, as shown in [2]. In particular, it makes sense to speak of good filtrations for H-modules, and so the definition of "Donkin pair" makes sense for (G, H).

In order to apply this notion in the case where $H = G^x_{\mathrm{red}}$, we must impose a slightly stronger condition on p: we require it to be *pretty good* in the sense of [9, Definition 2.11]. (In general, this condition is intermediate between "good" and "very good." It coincides with "very good" for semisimple simply connected groups, whereas for GL_n, all primes are pretty good.) This is equivalent to requiring G to be *standard* in the sense of [14, §4]. It follows from [8, Theorem 1.8] and [3, Lemma 2.1] that when p is pretty good for G, it does not divide the order of $G^x/(G^x)^\circ \cong G^x_{\mathrm{red}}/(G^x_{\mathrm{red}})^\circ$ for any nilpotent element x. As an immediate consequence of Theorem 1.1 and Lemma 2.2 below, we have the following result.

Corollary 1.2 *Let G be a connected reductive group over an algebraically closed field $\Bbbk$ of pretty good characteristic. For any nilpotent element x in its Lie algebra, (G, G^x_{red}) is a Donkin pair.*

This corollary plays a key role in the proof of the Humphreys conjecture [1].

The paper is organized as follows: Section 2 contains some general lemmas on Donkin pairs, along with a lengthly list of examples (some previously known, and some new). Section 3 gives the proof of Theorem 1.1. The proof consists of a reduction to the quasi-simple case, followed by case-by-case arguments.

Remark 1.3 It would, of course, be desirable to have a uniform proof of Theorem 1.1 that avoids case-by-case arguments, perhaps using the method of Frobenius splittings. Thanks to a fundamental result of Mathieu [13], Theorem 1.1 would come down to showing that the flag variety of G admits a $(G^x_{\mathrm{red}})^\circ$-canonical splitting. According to a result of van der Kallen [16], this geometric condition is equivalent to a certain linear-algebraic condition (called the "pairing condition") on the Steinberg modules for G and $(G^x_{\mathrm{red}})^\circ$. Unfortunately, for the moment, the pairing condition for these groups seems to be out of reach.

2 Preliminaries

2.1 General Lemmas on Donkin Pairs

We begin with three easy statements about good filtrations.

Lemma 2.1 *Let H be a possibly disconnected reductive group over an algebraically closed field $\Bbbk$. Assume that the characteristic of $\Bbbk$ does not divide $|H/H^\circ|$. An H-module M has a good filtration if and only if it has a good filtration as an H°-module.*

Proof According to [2, Eq. (3.3)], any costandard H-module regarded as an H°-module is a direct sum of costandard H°-modules. Hence, any H-module with a good filtration has a good filtration as an H°-module.

For the opposite implication, suppose M is an H-module that has a good filtration as an H°-module. To show that it has a good filtration as an H-module, we must show that $\mathrm{Ext}^1_H(-, M)$ vanishes on standard H-modules. As explained in the proof of [2, Lemma 2.18], we have

$$\mathrm{Ext}^1_H(-, M) \cong (\mathrm{Ext}^1_{H^\circ}(-, M))^{H/H^\circ},$$

and the right-hand side clearly vanishes on standard H-modules (using [2, Eq. (3.3)] again). □

Lemma 2.2 *Let G be a connected, reductive group, and let $H \subset G$ be a possibly disconnected reductive subgroup. Assume that the characteristic of $\Bbbk$ does not divide $|H/H^\circ|$. Then, (G, H) is a Donkin pair if and only if (G, H°) is a Donkin pair.*

Proof This is an immediate consequence of Lemma 2.1. □

Lemma 2.3 *Let G be a connected, reductive group, and let G' be its derived subgroup. Let $H \subset G$ be a connected, reductive subgroup. Then, (G, H) is a Donkin pair if and only if $(G', (G' \cap H)^\circ)$ is a Donkin pair.*

Proof Let $T \subset B \subset G$ denote a maximal torus and Borel subgroup, respectively, and suppose that G'' is any closed connected subgroup satisfying $G' \subseteq G'' \subseteq G$. Now let $T'' = G'' \cap T$, $B'' = G'' \cap B$, and observe that by [10, I.6.14(1)], we have

$$\mathrm{Res}^G_{G''}\mathrm{Ind}^G_B M \cong \mathrm{Ind}^{G''}_{B''}\mathrm{Res}^B_{B''} M \tag{2.1}$$

for any B-module M. Thus, for any dominant weight $\lambda \in \mathbf{X}(T)^+$ where we set $\lambda'' = \mathrm{Res}^T_{T''}(\lambda) \in \mathbf{X}(T'')^+$, it follows that $\mathrm{Res}^G_{G''}\mathrm{Ind}^G_B(\lambda) \cong \mathrm{Ind}^{G''}_{B''}(\lambda'')$. Thus, (G, G'') is always a Donkin pair. Furthermore, if we let $H' \subseteq H$ be the derived subgroup and $H'' = (G' \cap H)^\circ$, then $H' \subseteq H'' \subseteq H$. We can therefore apply [10, I.6.14(1)] again to show that (H, H'') is a Donkin pair.

Now suppose (G, H) is a Donkin pair. In this case, it immediately follows from above that (G, H'') is a Donkin pair. Moreover, if we let $T' = G' \cap T$ and $B' = G' \cap B$, then Eq. 2.1 actually implies that for any $\lambda' \in \mathbf{X}(T')^+$, there exists $\lambda \in \mathbf{X}(T)^+$ with $\lambda' = \mathrm{Res}^T_{T'}(\lambda)$ such that

$$\mathrm{Res}^G_{G'}\mathrm{Ind}^G_B(\lambda) \cong \mathrm{Ind}^{G'}_{B'}(\lambda').$$

In particular,

$$\mathrm{Res}^{G'}_{H''}\mathrm{Ind}^{G'}_{B'}(\lambda') \cong \mathrm{Res}^G_{H''}\mathrm{Ind}^G_B(\lambda)$$

has a good filtration as an H''-module, and hence, (G', H'') is also a Donkin pair.

Conversely, suppose that (G', H'') is a Donkin pair. We can first deduce that (G, H'') is a Donkin pair from the fact that (G, G') is a Donkin pair. Also, by similar arguments as above we can see that for any $\mu'' \in \mathbf{X}(H'' \cap T)^+$, there exists some $\mu \in \mathbf{X}(H \cap T)^+$ with $\mu'' = \mathrm{Res}^{H\cap T}_{H''\cap T}(\mu)$, such that

$$\mathrm{Res}^H_{H''}\mathrm{Ind}^H_{H\cap B}(\mu) \cong \mathrm{Ind}^{H''}_{H''\cap B}(\mu'').$$

This implies that an H-module M has a good filtration if and only if the H''-module $\mathrm{Res}^H_{H''} M$ has a good filtration. Therefore, (G, H) is also a Donkin pair. □

2.2 Examples of Donkin Pairs

The following proposition collects a number of known examples of Donkin pairs. The last five parts of the proposition deal with various examples where G is quasi-simple and simply connected. For pairs of the form (Spin_n, H), it is usually more convenient to describe the image H' of H under the map $\pi : \mathrm{Spin}_n \to \mathrm{SO}_n$. Of course, H can be recovered from H', as the identity component of $\pi^{-1}(H)$. We use the notation that

$$(\mathrm{SO}_n, H')^\sim = (\mathrm{Spin}_n, H).$$

It should be noted that the following proposition does not exhaust the known examples in the literature: for instance, according to [5], there is a Donkin pair of type $(\mathrm{B}_3, \mathrm{G}_2)$, but this example is not needed in the present paper.

Proposition 2.4 *Let G be a connected, reductive group, and let $H \subset G$ be a closed, connected, reductive subgroup. If the pair (G, H) satisfies one of the following conditions, then it is a Donkin pair.*

(1) *$G = H \times \cdots \times H$, and $H \hookrightarrow G$ is the diagonal embedding.*
(2) *H is a Levi subgroup of G.*

For the remaining parts, assume that G is quasi-simple and simply connected.

(3) *G is of simply laced type, and H is the fixed-point set of a diagram automorphism of G:*

$$(\mathrm{A}_{2n-1}, \mathrm{C}_n) = (\mathrm{SL}_{2n}, \mathrm{Sp}_{2n}) \qquad (\mathrm{D}_4, \mathrm{G}_2) = (\mathrm{Spin}_8, \mathrm{G}_2)$$
$$(\mathrm{D}_n, \mathrm{B}_{n-1}) = (\mathrm{SO}_{2n}, \mathrm{SO}_{2n-1})^\sim \qquad (\mathrm{E}_6, \mathrm{F}_4)$$

(4) *Certain embeddings of classical groups:*

$$\left.\begin{matrix}(\mathrm{A}_{2n}, \mathrm{B}_n)\\ (\mathrm{A}_{2n-1}, \mathrm{D}_n)\end{matrix}\right\} = (\mathrm{SL}_r, \mathrm{SO}_r) \qquad (p > 2)$$
$$(\mathrm{A}_{2n-1}, \mathrm{C}_n) = (\mathrm{SL}_{2n}, \mathrm{Sp}_{2n})$$
$$\left.\begin{matrix}(\mathrm{B}_{n+m}, \mathrm{B}_n\mathrm{D}_m)\\ (\mathrm{D}_{n+m}, \mathrm{D}_n\mathrm{D}_m)\\ (\mathrm{D}_{n+m+1}, \mathrm{B}_n\mathrm{B}_m)\end{matrix}\right\} = (\mathrm{SO}_{r+s}, \mathrm{SO}_r \times \mathrm{SO}_s)^\sim \quad (p > 2)$$
$$(\mathrm{C}_{n+m}, \mathrm{C}_n\mathrm{C}_m) = (\mathrm{Sp}_{2n+2m}, \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m})$$

(5) *Certain maximal-rank subgroups of exceptional groups:*

	(E_8, A_2E_6) $(p > 5)$
(E_8, D_8) $(p > 2)$	$(E_8, A_1A_2A_5)$ $(p > 5)$
(E_8, A_1E_7) $(p > 2)$	(E_8, A_3D_5) $(p > 5)$
(E_7, A_1D_6) $(p > 2)$	(E_8, A_4A_4) $(p > 5)$
(F_4, B_4) $(p > 2)$	(F_4, A_3A_1) $(p > 3)$
	(G_2, A_1A_1)

(6) *Certain restricted irreducible representations:*

$$(A_n, A_1) \quad (p > n)$$
$$(A_7, A_2) \quad (p > 3)$$
$$(A_6, G_2) \quad (p > 3)$$

(7) *Tensor product embeddings of classical groups ($p > 2$):*

$$\left.\begin{matrix}(C_{(2n+1)m}, B_n)\\ (C_{2nm}, D_n)\end{matrix}\right\} = (Sp_{2rm}, SO_r) \qquad \left.\begin{matrix}(B_{n+m+2nm}, B_n)\\ (D_{(2n+1)m}, B_n)\\ (D_{nm}, D_n)\end{matrix}\right\} = (SO_{rs}, SO_r)^{\sim}$$

$$(D_{2nm}, C_n) = (SO_{4nm}, Sp_{2n})^{\sim} \qquad (C_{nm}, C_n) = (Sp_{2nm}, Sp_{2n})$$

The details of the embeddings in parts (6) and (7) will be described below.

Proofs for parts (2)–(5) Parts (1) and (2) are due to Mathieu [13] (following earlier work of Donkin [6] that covered most cases). Parts (3) and (4), with the exception of the pair (E_6, F_4), are due to Brundan [5]. The pair (G_2, A_1A_1) in part (5) is also due to Brundan [5]. The pair (E_6, F_4) and the pairs in the first column of part (5) are due to van der Kallen [16]. The pairs in the second column of part (5) are due to Hague–McNinch [7]. □

Proof of part (6) Each pair $(A_n, H) = (SL_{n+1}, H)$ in this statement arises from some $(n + 1)$-dimensional representation of H. Call that representation V. The representations V are as follows:

- (A_n, A_1): the dual Weyl module for SL_2 of highest weight n
- (A_7, A_2): the adjoint representation of PGL_3
- (A_6, G_2): the 7-dimensional dual Weyl module whose highest weight is the short dominant root

According to [5, Lemma 3.2(iv)] or [7, §3.2.6], to prove the claim, we must show that each exterior algebra $\bigwedge^\bullet V$ has a good filtration as an H-module. For (A_n, A_1), this is shown in [7, §3.4.3]. For (A_7, A_2) and (A_6, G_2), explicit calculations using the LiE software package [17] show that the character of $\bigwedge^\bullet V$ is the sum of characters of dual Weyl modules whose highest weights are restricted weights when $p > 3$. □

Proof of part (7) To define the group embeddings in this statement, we will assume that G is either Sp_{2n} or SO_n. However, in the latter case, the proof that (G, H) is a Donkin pair will also imply the corresponding statement for $G = Spin_n$.

Let V_1 be a vector space equipped with a nondegenerate bilinear form B_1 satisfying $B_1(v, w) = \varepsilon_1 B_1(w, v)$, where $\varepsilon_1 = \pm 1$, and let $\mathrm{Aut}(V_1, B_1)^\circ$ be the connected group of linear automorphisms of V_1 that preserve B_1. This group is either $\mathrm{SO}_{\dim V}$ or $\mathrm{Sp}_{\dim V}$, depending on ε_1. Let V_2, B_2, and ε_2 be another collection of similar data. Then, $B_1 \otimes B_2$ is a nondegenerate pairing on $V_1 \otimes V_2$, with sign $\varepsilon_1\varepsilon_2$. We obtain an embedding

$$\mathrm{Aut}(V_1, B_1)^\circ \times \mathrm{Aut}(V_2, B_2)^\circ \hookrightarrow \mathrm{Aut}(V_1 \otimes V_2, B_1 \otimes B_2)^\circ.$$

Now restrict to just one factor:

$$\mathrm{Aut}(V_1, B_1)^\circ \hookrightarrow \mathrm{Aut}(V_1 \otimes V_2, B_1 \otimes B_2)^\circ. \tag{2.2}$$

The four kinds of pairs listed in the statement are all instances of this embedding, depending on the signs ε_1 and ε_2. We will now prove that

$$(G, H) = (\mathrm{Aut}(V_1 \otimes V_2, B_1 \otimes B_2)^\circ, \mathrm{Aut}(V_1, B_1)^\circ)$$

is a Donkin pair. Let $r = \dim V_1$ and $s = \dim V_2$.

Suppose first that $\varepsilon_2 = 1$. Then, the embedding (2.2) corresponds to either $(\mathrm{SO}_{rs}, \mathrm{SO}_r)$ or $(\mathrm{Sp}_{rs}, \mathrm{Sp}_r)$. In this case, V_2 admits an orthonormal basis $x_1, \dots, x_s$, where

$$B_2(x_i, x_j) = \delta_{ij}.$$

Then, the group $\mathrm{Aut}(V_1, B_1)^\circ$ preserves each $V_1 \otimes x_i \subset V_1 \otimes V_2$. In this case, the embedding (2.2) can be factored as

$$\mathrm{Aut}(V_1, B_1)^\circ \overset{1}{\hookrightarrow} \underbrace{\mathrm{Aut}(V_1, B_1)^\circ \times \cdots \times \mathrm{Aut}(V_1, B_1)^\circ}_{s \text{ copies}} \overset{4}{\hookrightarrow} \mathrm{Aut}(V_1 \otimes V_2, B_1 \otimes B_2)^\circ.$$

The first map is a diagonal embedding; it results in a Donkin pair by part (1) of the proposition. The second embedding gives a Donkin pair by part (4).

Next, suppose that $\varepsilon_2 = -1$, and assume for now that $\dim V_2 = 2$. Choose a basis $\{x, y\}$ for V_2 such that $B_2(x, y) = 1$. Then, $V_1 \otimes x$ and $V_1 \otimes y$ are both maximal isotropic subspaces of $V_1 \otimes V_2$. Define an action of $\mathrm{GL}(V_1)$ on $V_1 \otimes V_2$ as follows:

$$\begin{aligned} g \cdot (v \otimes x) &= (gv) \otimes x, \\ g \cdot (v \otimes y) &= ((g^{\mathrm{t}})^{-1} v) \otimes y \end{aligned} \qquad \text{for } g \in \mathrm{GL}(V_1),$$

where g^{t} denotes the adjoint operator to g with respect to the nondegenerate form on V_1. This action defines an embedding of $\mathrm{GL}(V_1)$ in $\mathrm{Aut}(V_1 \otimes V_2, B_1 \otimes B_2)^\circ$. In fact, it identifies $\mathrm{GL}(V_1)$ with a Levi subgroup of $\mathrm{Aut}(V_1 \otimes V_2, B_1 \otimes B_2)^\circ$. (This is the usual embedding of GL_r as a Levi subgroup in either SO_{2r} or Sp_{2r}.) The embedding (2.2) then factors as

$$\mathrm{Aut}(V_1, B_1)^\circ \overset{4}{\hookrightarrow} \mathrm{GL}(V_1) \overset{2}{\hookrightarrow} \mathrm{Aut}(V_1 \otimes V_2, B_1 \otimes B_2)^\circ.$$

The first embedding gives a Donkin pair by part (4) of the proposition, and the second by part (2).

Finally, suppose $\varepsilon_2 = -1$ and $s = \dim V_2 > 2$. This dimension must still be even, say $s = 2m$. Choose a basis $x_1, \dots, x_m, y_1, \dots, y_m$ for V_2 such that

$$B_2(x_i, x_j) = B_2(y_i, y_j) = 0, \qquad B_2(x_i, y_j) = \delta_{ij}.$$

Let $V_2^{(i)}$ be the 2-dimensional subspace spanned by x_i and y_i. Then, B_2 restricts to a nondegenerate symplectic form $B_2^{(i)}$ on $V_2^{(i)}$. We factor the map (2.2) as follows:

$$\begin{aligned}\mathrm{Aut}(V_1, B_1)^\circ \overset{1}{\hookrightarrow}\ & \underbrace{\mathrm{Aut}(V_1, B_1)^\circ \times \cdots \times \mathrm{Aut}(V_1, B_1)^\circ}_{m \text{ copies}} \\ \hookrightarrow\ & \mathrm{Aut}(V_1 \otimes V_2^{(1)}, B_1 \otimes B_2^{(1)})^\circ \times \cdots \times \mathrm{Aut}(V_1 \otimes V_2^{(m)}, B_1 \otimes B_2^{(m)})^\circ \\ & \overset{4}{\hookrightarrow} \mathrm{Aut}(V_1 \otimes V_2, B_1 \otimes B_2)^\circ.\end{aligned}$$

Here, the first arrow is a diagonal embedding (part (1) of the proposition); the second arrow is several instances of the embedding from the previous paragraph (since $\dim V_2^{(i)} = 2$); and the last arrow comes from part (4) of the proposition. We thus again obtain a Donkin pair. □

3 Proof of Theorem 1.1

3.1 Reduction to the Quasi-Simple Case

Let G be an arbitrary connected reductive group in good characteristic. For any nilpotent element $x \in \mathrm{Lie}(G)$, there exists a cocharacter $\tau : \mathbb{G}_m \to G$ and a Levi subgroup $L_\tau \subset G$ such that L_τ is the centralizer of the subgroup $\tau(\mathbb{G}_m)$, where $G^x_{\mathrm{red}} = L_\tau \cap G^x$ (cf. [11, 5.10]).

If we let G' be the derived subgroup of G, then any nilpotent element x for G also satisfies $x \in \mathrm{Lie}(G')$, and by [11, 5.9], $\tau(\mathbb{G}_m) \subset G' \subseteq G$. In particular, $(G')^x = G' \cap G^x$ and $L'_\tau = G' \cap L_\tau$ is the centralizer of $\tau(\mathbb{G}_m)$ in G'. Thus,

$$(G')^x_{\mathrm{red}} = G' \cap G^x_{\mathrm{red}}.$$

It now follows from Lemma 2.3 that $(G, (G^x_{\mathrm{red}})^\circ)$ is a Donkin pair if and only if $(G', ((G')^x_{\mathrm{red}})^\circ)$ is a Donkin pair. So we can reduce to the case where G is semisimple.

Suppose now that $\pi : G \twoheadrightarrow \bar{G}$ is an isogeny (i.e., surjective with finite central kernel), where G is an arbitrary connected reductive group in good characteristic. Then, by [11, Proposition 2.7(a)], π induces a bijection between the nilpotent elements in $\mathrm{Lie}(G)$ and those in $\mathrm{Lie}(\bar{G})$, and for any nilpotent element $x \in \mathrm{Lie}(G)$, we have $\pi(G^x) = \bar{G}^{\pi(x)}$. Moreover, by similar arguments as above, we can also deduce that $\pi(G^x_{\mathrm{red}}) = \bar{G}^{\pi(x)}_{\mathrm{red}}$ (cf. [11, 5.9]). In particular,

$$\pi((G^x_{\mathrm{red}})^\circ) = \pi(G^x_{\mathrm{red}})^\circ = (\bar{G}^{\pi(x)}_{\mathrm{red}})^\circ,$$

since any surjective morphism of algebraic groups takes the identity component to the identity component.

Let $H = (G^x_{\mathrm{red}})^\circ$ and $\bar{H} = (\bar{G}^{\pi(x)}_{\mathrm{red}})^\circ$, and note that for any $\bar{G}$-module M, there is a natural isomorphism

$$\mathrm{Res}^G_H \mathrm{Res}^{\bar{G}}_G M \cong \mathrm{Res}^{\bar{H}}_H \mathrm{Res}^{\bar{G}}_{\bar{H}} M.$$

From this, we can see that if (G, H) is a Donkin pair, then $(\bar{G}, \bar{H})$ must also be a Donkin pair, since it is straightforward to check that a $\bar{G}$-module M (resp. an $\bar{H}$-module N) has a good filtration if and only if $\mathrm{Res}^{\bar{G}}_{G} M$ (resp. $\mathrm{Res}^{\bar{H}}_{H} N$) has a good filtration. This allows us to reduce to the case where G is semisimple and simply connected.

Finally, suppose that that $G = G_1 \times G_2$ where G_1, G_2 are connected reductive groups in good characteristic. Let $x = (x_1, x_2) \in \mathrm{Lie}(G_1) \oplus \mathrm{Lie}(G_2)$ be an arbitrary nilpotent element. We can immediately see that

$$(G^x_{\mathrm{red}})^\circ = ((G_1)^{x_1}_{\mathrm{red}})^\circ \times ((G_2)^{x_2}_{\mathrm{red}})^\circ.$$

It now follows from the general properties of induction for direct products (see [10, I.3.8]) that $(G, (G^x_{\mathrm{red}})^\circ)$ is a Donkin pair if and only if $(G_1, ((G_1)^{x_1}_{\mathrm{red}})^\circ)$ and $(G_2, ((G_2)^{x_2}_{\mathrm{red}})^\circ)$ are Donkin pairs. Therefore, by the well-known fact that any simply connected semisimple group is a direct product of quasi-simple simply connected groups, we can reduce the proof of Theorem 1.1 to the case where G is quasi-simple.

3.2 Proof for Classical Groups

We now prove the theorem for the groups GL_n, Sp_n, and Spin_n. For the last case, we will actually describe the group $(G^x_{\mathrm{red}})^\circ$ and its embedding in G for SO_n instead, but the proof of the Donkin pair property will also apply to Spin_n.

Let x be a nilpotent element in the Lie algebra of one of GL_n, Sp_n, or SO_n. Let $\mathbf{s} = [s_1^{r_1}, s_2^{r_2}, \ldots, s_k^{r_k}]$ be the partition of n that records the sizes of the Jordan blocks of x. (This means that x has r_1 Jordan blocks of size s_1, and r_2 Jordan blocks of size s_2, etc.) The vector space $V = \Bbbk^n$ can be decomposed as

$$V = V^{(1)} \oplus V^{(2)} \oplus \cdots \oplus V^{(k)}$$

where each $V^{(i)}$ is preserved by x, and x acts on $V^{(i)}$ by Jordan blocks of size s_i. (Thus, $\dim V^{(i)} = r_i s_i$.) When G is Sp_n or SO_n, the nondegenerate bilinear form on V restricts to a nondegenerate form of the same type on each $V^{(i)}$.

The description of $(G^x_{\mathrm{red}})^\circ$ in [12, Chapter 3] shows that it factors through the appropriate embedding below:

$$\begin{aligned}
\mathrm{GL}(V^{(1)}) \times \cdots \times \mathrm{GL}(V^{(k)}) &\hookrightarrow \mathrm{GL}(V)\\
\mathrm{Sp}(V^{(1)}) \times \cdots \times \mathrm{Sp}(V^{(k)}) &\hookrightarrow \mathrm{Sp}(V)\\
\mathrm{SO}(V^{(1)}) \times \cdots \times \mathrm{SO}(V^{(k)}) &\hookrightarrow \mathrm{SO}(V)
\end{aligned}$$

All three of these embeddings give Donkin pairs: in the case of GL_n, it is an inclusion of a Levi subgroup (Proposition 2.4(2)); and in the case of Sp_n or SO_n, it falls under Proposition 2.4(4).

We can therefore reduce to the case where x has Jordan blocks of a single size. Suppose from now on that $\mathbf{s} = [s^r]$. Then, there exists a vector space isomorphism

$$V \cong V_1 \otimes V_2$$

where $\dim V_1 = r$ and $\dim V_2 = s$, and such that x corresponds to $\mathrm{id}_{V_1} \otimes N$, where $N : V_2 \to V_2$ is a nilpotent operator with a single Jordan block (of size s).

Suppose now that $G = \mathrm{GL}(V)$. Then, according to [12, Proposition 3.8], we have $(G^x_{\mathrm{red}})^\circ \cong \mathrm{GL}(V_1)$. Choose a basis $\{v_1, \ldots, v_s\}$ for V_2. The embedding of $(G^x_{\mathrm{red}})^\circ$ in G factors as

$$\mathrm{GL}(V_1) \hookrightarrow \mathrm{GL}(V_1 \otimes v_1) \times \cdots \times \mathrm{GL}(V_1 \otimes v_s) \hookrightarrow \mathrm{GL}(V).$$

The first map above is a diagonal embedding (Proposition 2.4(1)), and the second is the inclusion of a Levi subgroup (Proposition 2.4(2)), so $(G, (G^x_{\mathrm{red}})^\circ)$ is a Donkin pair in this case.

Next, suppose $G = \mathrm{Sp}(V)$ or $\mathrm{SO}(V)$. According to [12, Proposition 3.10], both V_1 and V_2 can be equipped with nondegenerate bilinear forms B_1 and B_2 such that $B_1 \otimes B_2$ agrees with the given bilinear form on V. Moreover, $(G^x_{\mathrm{red}})^\circ = \mathrm{Aut}(V_1, B_1)^\circ$. We are thus in the setting of Proposition 2.4(7).

3.3 Proof for E_8

When x is distinguished, $(G^x_{\mathrm{red}})^\circ$ is the trivial group; and when $x = 0$, $(G^x_{\mathrm{red}})^\circ = G$. For all remaining nilpotent orbits, we rely on the very detailed case-by-case descriptions of $(G^x_{\mathrm{red}})^\circ$ given in [12, Chapter 15]. In each case, that description shows that the embedding $(G^x_{\mathrm{red}})^\circ \hookrightarrow G$ factors as a composition of various cases from Proposition 2.4.

These factorizations are shown in Tables 1 and 2. Here is a brief explanation of the notation used in these tables. Nearly all groups mentioned are semisimple, and they are recorded in the tables by their root systems. However, the notation "T_1" indicates a 1-dimensional torus; this is used to indicate a reductive group with a 1-dimensional center. In a few cases, nonstandard names for root systems—such as B_1 or C_1, in place of A_1—are used when it is convenient to emphasize the role of a certain classical group. The notation D_1 (meant to evoke SO_2) is occasionally used as a synonym for T_1.

Finally, we remark that there are two orbits—labeled by A_6 and by A_6A_1—where the information given in [12] is insufficient to finish the argument. In each of these cases, $(G^x_{\mathrm{red}})^\circ$ contains a copy of A_1 that is the centralizer of a certain copy of G_2 inside E_7, and [12] does not give further details on this embedding $A_1 \hookrightarrow E_7$. However, according to [15, §3.12], this A_1 is in fact (the derived subgroup of) a Levi subgroup of E_7.

3.4 Proof for E_7 and E_6

Recall that if $H \subseteq G$ is a closed subgroup, then there is a natural embedding $\mathcal{N}_H \hookrightarrow \mathcal{N}_G$ of nilpotent cones. We also recall that a subgroup $L \subseteq G$ is a Levi subgroup if and only if it is the centralizer of a torus $S \subset G$.

Lemma 3.1 *Let G be a reductive group, $S \subseteq G$ a torus with $L = C_G(S)$ a Levi subgroup, and suppose $x \in \mathcal{N}_L \subseteq \mathcal{N}_G$ is such that $S \subseteq (G^x_{red})^\circ$. Then, $(L^x_{red})^\circ$ is a Levi subgroup of $(G^x_{red})^\circ$.*

Table 1 Nilpotent centralizers in E_8

Orbit	$(G^x_{\mathrm{red}})^\circ$
A_1	$E_7 \overset{2}{\hookrightarrow} E_8$
A_1^2	$B_6 \overset{2}{\hookrightarrow} B_1B_6 \overset{4}{\hookrightarrow} D_8 \overset{5}{\hookrightarrow} E_8$
A_2	$E_6 \overset{2}{\hookrightarrow} A_2E_6 \overset{5}{\hookrightarrow} E_8$
A_1^3	$A_1F_4 \overset{3}{\hookrightarrow} A_1E_6 \overset{2}{\hookrightarrow} E_8$
A_2A_1	$A_5 \overset{2}{\hookrightarrow} A_1A_2A_5 \overset{5}{\hookrightarrow} E_8$
A_3	$B_5 \overset{2}{\hookrightarrow} B_2B_5 \overset{4}{\hookrightarrow} D_8 \overset{5}{\hookrightarrow} E_8$
A_1^4	$C_4 \overset{7}{\hookrightarrow} D_8 \overset{5}{\hookrightarrow} E_8$
$A_2A_1^2$	$A_1B_3 = B_1B_3 \overset{1}{\hookrightarrow} B_1B_1B_1B_3 \overset{4}{\hookrightarrow} D_8 \overset{5}{\hookrightarrow} E_8$
A_2^2	$G_2G_2 \overset{3}{\hookrightarrow} D_4D_4 \overset{4}{\hookrightarrow} D_8 \overset{5}{\hookrightarrow} E_8$
A_3A_1	$A_1B_3 = C_1B_3 \overset{7}{\hookrightarrow} D_2B_3 \overset{2}{\hookrightarrow} B_2D_2B_3 \overset{4}{\hookrightarrow} D_8 \overset{5}{\hookrightarrow} E_8$
A_4	$A_4 \overset{2}{\hookrightarrow} E_8$
D_4	$F_4 \overset{3}{\hookrightarrow} E_6 \overset{2}{\hookrightarrow} E_8$
$D_4(a_1)$	$D_4 \overset{2}{\hookrightarrow} E_8$
$A_2A_1^3$	$A_1G_2 \overset{6}{\hookrightarrow} A_1A_6 \overset{2}{\hookrightarrow} A_1E_7 \overset{5}{\hookrightarrow} E_8$
$A_2^2A_1$	$A_1G_2 \overset{2}{\hookrightarrow} G_2G_2 \overset{3}{\hookrightarrow} D_4D_4 \overset{4}{\hookrightarrow} D_8 \overset{5}{\hookrightarrow} E_8$
$A_3A_1^2$	$A_1B_2 = B_1C_2 \overset{7}{\hookrightarrow} B_1D_4 \overset{2}{\hookrightarrow} B_1D_4B_2 \overset{4}{\hookrightarrow} D_8 \overset{5}{\hookrightarrow} E_8$
A_3A_2	$B_2T_1 = B_2D_1 \overset{7}{\hookrightarrow} B_2D_3 \overset{2}{\hookrightarrow} B_2D_3B_2 \overset{4}{\hookrightarrow} D_8 \overset{5}{\hookrightarrow} E_8$
A_4A_1	$A_2T_1 \overset{2}{\hookrightarrow} A_4A_4 \overset{5}{\hookrightarrow} E_8$
D_4A_1	$C_3 \overset{2}{\hookrightarrow} F_4 \overset{3}{\hookrightarrow} E_6 \overset{2}{\hookrightarrow} E_8$
$D_4(a_1)A_1$	$A_1^3 \overset{2}{\hookrightarrow} E_8$
A_5	$A_1G_2 \overset{3}{\hookrightarrow} A_1D_4 \overset{2}{\hookrightarrow} E_8$
D_5	$B_3 \overset{2}{\hookrightarrow} B_3B_4 \overset{4}{\hookrightarrow} D_8 \overset{5}{\hookrightarrow} E_8$
$D_5(a_1)$	$A_3 \overset{2}{\hookrightarrow} A_3D_5 \overset{5}{\hookrightarrow} E_8$
$A_2^2A_1^2$	$B_2 \overset{7}{\hookrightarrow} B_7 \overset{4}{\hookrightarrow} D_8 \overset{5}{\hookrightarrow} E_8$
$A_3A_2A_1$	$A_1A_1 \overset{1}{\hookrightarrow} A_1(A_1A_1) \overset{6}{\hookrightarrow} A_1(A_4A_2) \overset{2}{\hookrightarrow} A_1E_7 \overset{5}{\hookrightarrow} E_8$
A_3^2	$C_2 \overset{7}{\hookrightarrow} D_8 \overset{5}{\hookrightarrow} E_8$
$A_4A_1^2$	$A_1T_1 \overset{1}{\hookrightarrow} A_1A_1T_1 \overset{2}{\hookrightarrow} A_4 \overset{2}{\hookrightarrow} E_8$
A_4A_2	$A_1A_1 \overset{1}{\hookrightarrow} A_1(A_1A_1A_1) \overset{6}{\hookrightarrow} A_1(A_1A_2A_3) \overset{2}{\hookrightarrow} A_1E_7 \overset{5}{\hookrightarrow} E_8$
D_4A_2	$\tilde{A}_2 \overset{2}{\hookrightarrow} F_4 \overset{3}{\hookrightarrow} E_6 \overset{2}{\hookrightarrow} E_8$

Proof We clearly have $C_{G^x}(S) = G^x \cap L = L^x$. Moreover, the identity component $(L^x)^\circ$ must be contained in $(G^x)^\circ \cap L = C_{(G^x)^\circ}(S)$. But since the centralizer of a torus in a connected group is connected, we actually have

$$(L^x)^\circ = C_{(G^x)^\circ}(S).$$

Table 2 Nilpotent centralizers in E_8, continued

Orbit	$(G^x_{\text{red}})^\circ$
$D_4(a_1)A_2$	$A_2 \overset{6}{\hookrightarrow} A_7 \overset{2}{\hookrightarrow} E_8$
A_5A_1	$A_1A_1 \overset{2}{\hookrightarrow} A_1G_2 \overset{3}{\hookrightarrow} A_1D_4 \overset{2}{\hookrightarrow} A_1E_7 \overset{5}{\hookrightarrow} E_8$
D_5A_1	$A_1A_1 = B_1C_1 \overset{7}{\hookrightarrow} B_1D_2 \overset{2}{\hookrightarrow} B_4B_1D_2 \overset{4}{\hookrightarrow} D_8 \overset{5}{\hookrightarrow} E_8$
$D_5(a_1)A_1$	$A_1A_1 \overset{6}{\hookrightarrow} A_1\tilde{A}_2 \overset{2}{\hookrightarrow} A_1F_4 \overset{3}{\hookrightarrow} A_1E_6 \overset{2}{\hookrightarrow} A_1E_7 \overset{5}{\hookrightarrow} E_8$
A_6	$A_1^2 \overset{2}{\hookrightarrow} A_1E_7 \overset{5}{\hookrightarrow} E_8$
D_6	$B_2 \overset{2}{\hookrightarrow} B_2B_5 \overset{4}{\hookrightarrow} D_8 \overset{5}{\hookrightarrow} E_8$
$D_6(a_1)$	$A_1A_1 = D_2 \overset{2}{\hookrightarrow} D_2D_6 \overset{4}{\hookrightarrow} D_8 \overset{5}{\hookrightarrow} E_8$
$D_6(a_2)$	$A_1A_1 = D_2 \overset{2}{\hookrightarrow} D_2D_6 \overset{4}{\hookrightarrow} D_8 \overset{5}{\hookrightarrow} E_8$
E_6	$G_2 \overset{3}{\hookrightarrow} D_4 \overset{2}{\hookrightarrow} E_8$
$E_6(a_1)$	$A_2 \overset{2}{\hookrightarrow} A_2E_6 \overset{5}{\hookrightarrow} E_8$
$E_6(a_3)$	$G_2 \overset{3}{\hookrightarrow} D_4 \overset{2}{\hookrightarrow} E_8$
$A_4A_2A_1$	$A_1 \overset{1}{\hookrightarrow} A_1A_1A_1 \overset{6}{\hookrightarrow} A_1A_2A_3 \overset{2}{\hookrightarrow} E_7 \overset{2}{\hookrightarrow} E_8$
A_4A_3	$A_1 = B_1 \overset{7}{\hookrightarrow} B_7 \overset{4}{\hookrightarrow} D_8 \overset{5}{\hookrightarrow} E_8$
D_5A_2	T_1
$D_5(a_1)A_2$	$A_1 = B_1 \overset{7}{\hookrightarrow} B_4 \overset{2}{\hookrightarrow} B_4B_3 \overset{4}{\hookrightarrow} D_8 \overset{5}{\hookrightarrow} E_8$
A_6A_1	$A_1 \overset{2}{\hookrightarrow} E_8$
E_6A_1	$A_1 \overset{2}{\hookrightarrow} G_2 \overset{3}{\hookrightarrow} D_4 \overset{2}{\hookrightarrow} E_8$
$E_6(a_1)A_1$	T_1
$E_6(a_3)A_1$	$A_1 \overset{2}{\hookrightarrow} G_2 \overset{3}{\hookrightarrow} D_4 \overset{2}{\hookrightarrow} E_8$
A_7	$A_1 = C_1 \overset{7}{\hookrightarrow} D_8 \overset{5}{\hookrightarrow} E_8$
D_7	$A_1 = B_1 \overset{2}{\hookrightarrow} B_1B_6 \overset{4}{\hookrightarrow} D_8 \overset{5}{\hookrightarrow} E_8$
$D_7(a_1)$	T_1
$D_7(a_2)$	T_1
E_7	$A_1 \overset{2}{\hookrightarrow} A_1E_7 \overset{5}{\hookrightarrow} E_8$
$E_7(a_1)$	$A_1 \overset{2}{\hookrightarrow} A_1E_7 \overset{5}{\hookrightarrow} E_8$
$E_7(a_2)$	$A_1 \overset{2}{\hookrightarrow} A_1E_7 \overset{5}{\hookrightarrow} E_8$
$E_7(a_3)$	$A_1 \overset{2}{\hookrightarrow} A_1E_7 \overset{5}{\hookrightarrow} E_8$
$E_7(a_4)$	$A_1 \overset{2}{\hookrightarrow} A_1E_7 \overset{5}{\hookrightarrow} E_8$
$E_7(a_5)$	$A_1 \overset{2}{\hookrightarrow} A_1E_7 \overset{5}{\hookrightarrow} E_8$

Next, consider the semidirect product decomposition $(G^x)^\circ = (G^x_{\text{red}})^\circ \ltimes G^x_{\text{unip}}$. Let $g \in G^x$, and write it as $g = (g_r, g_u)$, with $g_r \in (G^x_{\text{red}})^\circ$, $g_u \in G^x_{\text{unip}}$. If g centralizes S, then g_r and g_u must individually centralize S as well. In other words,

$$C_{(G^x)^\circ}(S) = C_{(G^x_{\text{red}})^\circ}(S) \ltimes C_{G^x_{\text{unip}}}(S). \tag{3.1}$$

Here, $C_{(G^x_{\text{red}})^\circ}(S)$ is a connected reductive group, and $C_{G^x_{\text{unip}}}(S)$ is a normal unipotent group (which must be connected, because $C_{(G^x)^\circ}(S)$ is connected). We conclude that Eq. 3.1 is a Levi decomposition of $(L^x)^\circ$. In particular, we see that $(L^x_{\mathrm{red}})^\circ = C_{(G^x_{\mathrm{red}})^\circ}(S)$. □

We now let G denote the simple, simply connected group of type E_8. If G_0 is the simple, simply connected group of type E_7 or E_6, then as explained in [12, Lemma 11.14], there is a simple subgroup H of type A_1 or A_2, respectively, such that $G_0 = C_G(H)$. Moreover, by [12, 16.1.2], there exists a torus $S \subset H \subset G$ such that G_0 is the derived subgroup of the Levi subgroup $L = C_G(S)$. Explicitly, let $\alpha_1, \ldots, \alpha_8$ be the simple roots for G, labelled as in [4], and let α_0 be the highest root. The groups H and G_0 can be described as follows.

	simple roots for H	simple roots for L or G_0
$G_0 = \mathrm{E}_7$	$-\alpha_0$	$\alpha_1, \ldots, \alpha_7$
$G_0 = \mathrm{E}_6$	$\alpha_8, -\alpha_0$	$\alpha_1, \ldots, \alpha_6$

(3.2)

Now it is explained in [12, 16.1.1], that if $x \in \mathcal{N}_{G_0} = \mathcal{N}_L \subset \mathcal{N}_G$, then the subgroup $(G^x_{\mathrm{red}})^\circ$ must contain a conjugate of H. Without loss of generality we can assume that x is chosen so that $H \subseteq (G^x_{\mathrm{red}})^\circ$. Hence, we can also assume that $S \subseteq (G^x_{\mathrm{red}})^\circ$. Thus, by Lemma 3.1, $(L^x_{\mathrm{red}})^\circ$ is a Levi subgroup of $(G^x_{\mathrm{red}})^\circ$ and we also have

$$(L^x_{\mathrm{red}})^{\circ\prime} \subseteq ((G^x_0)_{\mathrm{red}})^\circ \subseteq (L^x_{\mathrm{red}})^\circ.$$

By Lemma 2.3, Proposition 2.4(2) and Section 3.3 we deduce that $(G, (G^x_0)_{\mathrm{red}}))^\circ)$ is a Donkin pair.

Finally, to show that $(G_0, ((G^x_0)_{\mathrm{red}})^\circ)$ is a Donkin pair, it will be sufficient to show that every fundamental tilting module for G_0 is a summand of the restriction of a tilting module for G. In more detail, let $\pi : \mathbf{X}_G \twoheadrightarrow \mathbf{X}_{G_0}$ be the map on weight lattices. It is well known that if λ is a dominant weight for G, then the G_0-tilting module $T_{G_0}(\pi(\lambda))$ occurs as a direct summand of $\mathrm{Res}^G_{G_0}(T_G(\lambda))$. So it is enough to show that every fundamental weight for G_0 occurs as $\pi(\lambda)$ for some dominant G-weight λ. Let $\varpi_1, \ldots, \varpi_8$ be the fundamental weights for G. A short calculation with the table (3.2) shows that $\pi(\varpi_1), \ldots, \pi(\varpi_7)$ are precisely the fundamental weights for E_7, and that $\pi(\varpi_1), \ldots, \pi(\varpi_6)$ are the fundamental weights for E_6.

Remark 3.2 One can also prove the theorem for E_7 and E_6 directly by writing down the embedding of each centralizer, as we did for E_8. Here is a brief summary of how to carry out this approach. Let G be of type E_8, and let G_0 and H be as in the discussion above. As explained in [12, §16.1], we have

$$((G^x_0)_{\mathrm{red}})^\circ = C_{(G^x_{\mathrm{red}})^\circ}(H).$$

The computation of $C_{(G^x_{\mathrm{red}})^\circ}(H)$ is explained in [12, §16.1.4], and the results are recorded in Tables 3, 4, and 5, following the same notational conventions as in the E_8 case.

Table 3 Nilpotent centralizers in E_7

Orbit	$(G^x_{\mathrm{red}})^\circ$
A_1	$D_6 \overset{2}{\hookrightarrow} E_7$
A_1^2	$A_1B_4 = B_1B_4 \overset{4}{\hookrightarrow} D_6 \overset{2}{\hookrightarrow} E_7$
A_2	$A_5 \overset{2}{\hookrightarrow} E_7$
A_2A_1	$A_3T_1 \overset{2}{\hookrightarrow} E_7$
$(A_1^3)'$	$F_4 \overset{3}{\hookrightarrow} E_6 \overset{2}{\hookrightarrow} E_7$
$(A_1^3)''$	$A_1C_3 \overset{3}{\hookrightarrow} A_1A_5 \overset{2}{\hookrightarrow} E_7$
A_3	$A_1B_3 = B_1B_3 \overset{4}{\hookrightarrow} D_5 \overset{2}{\hookrightarrow} E_7$
A_1^4	$C_3 \overset{7}{\hookrightarrow} D_6 \overset{2}{\hookrightarrow} E_7$
$A_2A_1^2$	$A_1A_1A_1 = B_1A_1B_1 \overset{7}{\hookrightarrow} B_4A_1B_1 \overset{4}{\hookrightarrow} A_1D_6 \overset{5}{\hookrightarrow} E_7$
A_2^2	$A_1G_2 \overset{3}{\hookrightarrow} A_1D_4 \overset{1}{\hookrightarrow} A_1A_1A_1D_4 = A_1D_2D_4 \overset{4}{\hookrightarrow} A_1D_6 \overset{5}{\hookrightarrow} E_7$
A_4	$A_2T_1 \overset{2}{\hookrightarrow} E_7$
D_4	$C_3 \overset{3}{\hookrightarrow} A_5 \overset{2}{\hookrightarrow} E_7$
$D_4(a_1)$	$A_1A_1A_1 \overset{2}{\hookrightarrow} E_7$
$A_2A_1^3$	$G_2 \overset{6}{\hookrightarrow} A_6 \overset{2}{\hookrightarrow} E_7$
$A_2^2A_1$	$A_1A_1 \overset{1}{\hookrightarrow} (A_1A_1A_1)G_2 \overset{3}{\hookrightarrow} A_1D_2D_4 \overset{4}{\hookrightarrow} A_1D_6 \overset{5}{\hookrightarrow} E_7$
$(A_3A_1)'$	$B_3 \overset{4}{\hookrightarrow} D_4 \overset{2}{\hookrightarrow} E_7$
$(A_3A_1)''$	$A_1A_1A_1 = C_1A_1B_1 \overset{7}{\hookrightarrow} D_2A_1B_1 \overset{4}{\hookrightarrow} D_2A_1D_2 \overset{4}{\hookrightarrow} A_1D_4 \overset{2}{\hookrightarrow} E_7$
$A_3A_1^2$	$A_1A_1 \overset{1}{\hookrightarrow} A_1(A_1A_1) \overset{2}{\hookrightarrow} A_1A_3 \overset{2}{\hookrightarrow} E_7$
A_3A_2	$A_1T_1 = B_1D_1 \overset{7}{\hookrightarrow} B_1D_3 \overset{2}{\hookrightarrow} B_1D_3B_1 \overset{4}{\hookrightarrow} D_6 \overset{2}{\hookrightarrow} E_7$
A_4A_1	T_2
D_4A_1	$C_2 \overset{3}{\hookrightarrow} A_3 \overset{2}{\hookrightarrow} E_7$
$D_4(a_1)A_1$	$A_1A_1 \overset{2}{\hookrightarrow} E_7$
$(A_5)'$	$G_2 \overset{3}{\hookrightarrow} D_4 \overset{2}{\hookrightarrow} E_7$
$(A_5)''$	$A_1A_1 \overset{1}{\hookrightarrow} A_1(A_1A_1A_1) \overset{2}{\hookrightarrow} E_7$
D_5	$A_1A_1 \overset{1}{\hookrightarrow} A_1(A_1A_1) \overset{2}{\hookrightarrow} E_7$
$D_5(a_1)$	$A_1T_1 \overset{2}{\hookrightarrow} E_7$
$A_3A_2A_1$	$A_1 \overset{1}{\hookrightarrow} A_1A_1 \overset{6}{\hookrightarrow} A_4A_2 \overset{2}{\hookrightarrow} E_7$
A_4A_2	$A_1 \overset{1}{\hookrightarrow} A_1A_1A_2 \overset{6}{\hookrightarrow} A_1A_2A_3 \overset{2}{\hookrightarrow} E_7$
A_5A_1	$A_1 \overset{2}{\hookrightarrow} G_2 \overset{3}{\hookrightarrow} D_4 \overset{2}{\hookrightarrow} E_7$

3.5 Proof for F_4

Let G be the simple, simply connected group of type E_8. Then, [12, Lemma 11.7] implies that G contains a simple subgroup H of type G_2, and that its centralizer $G_0 = C_G(H)$ is a simple group of type F_4. The embeddings of centralizers of

Table 4 Nilpotent centralizers in E_7, continued

Orbit	$(G^x_{\mathrm{red}})^\circ$
D_5A_1	$A_1 \overset{1}{\hookrightarrow} A_1A_1 \overset{2}{\hookrightarrow} E_7$
$D_5(a_1)A_1$	$A_1 \overset{6}{\hookrightarrow} \tilde{A}_2 \overset{2}{\hookrightarrow} F_4 \overset{3}{\hookrightarrow} E_6 \overset{2}{\hookrightarrow} E_7$
A_6	$A_1 \overset{2}{\hookrightarrow} E_7$
D_6	$A_1 \overset{1}{\hookrightarrow} A_1A_1 \overset{2}{\hookrightarrow} E_7$
$D_6(a_1)$	$A_1 \overset{2}{\hookrightarrow} E_7$
$D_6(a_2)$	$A_1 \overset{2}{\hookrightarrow} E_7$
E_6	$A_1 \overset{1}{\hookrightarrow} A_1A_1A_1 \overset{2}{\hookrightarrow} E_7$
$E_6(a_1)$	T_1
$E_6(a_3)$	$A_1 \overset{1}{\hookrightarrow} A_1A_1A_1 \overset{2}{\hookrightarrow} E_7$

nilpotent elements for $G_0 = F_4$ can then be computed using the method explained in Remark 3.2. One caveat is that the name (i.e., the Bala–Carter label) of a nilpotent orbit usually changes when passing from F_4 to E_8. The correspondence between these names is given in [12, Proposition 16.10].

Table 5 Nilpotent centralizers in E_6

Orbit	$(G^x_{\mathrm{red}})^\circ$
A_1	$A_5 \overset{2}{\hookrightarrow} E_6$
A_1^2	$B_3T_1 \overset{4}{\hookrightarrow} D_4T_1 \overset{2}{\hookrightarrow} E_6$
A_2	$A_2A_2 \overset{2}{\hookrightarrow} E_6$
A_1^3	$A_2A_1 \overset{1}{\hookrightarrow} (A_2A_2)A_1 \overset{2}{\hookrightarrow} E_6$
A_2A_1	$A_2T_1 \overset{2}{\hookrightarrow} E_6$
A_3	$B_2T_1 \overset{4}{\hookrightarrow} D_3T_1 \overset{2}{\hookrightarrow} E_6$
$A_2A_1^2$	$A_1T_1 = B_1T_1 \overset{7}{\hookrightarrow} B_4T_1 \overset{4}{\hookrightarrow} D_5T_1 \overset{2}{\hookrightarrow} E_6$
A_2^2	$G_2 \overset{3}{\hookrightarrow} D_4 \overset{2}{\hookrightarrow} E_6$
A_3A_1	$A_1T_1 = C_1T_1 \overset{7}{\hookrightarrow} D_2T_1 \overset{2}{\hookrightarrow} E_6$
A_4	$A_1T_1 \overset{2}{\hookrightarrow} E_6$
D_4	$A_2 \overset{1}{\hookrightarrow} A_2A_2 \overset{2}{\hookrightarrow} E_6$
$D_4(a_1)$	T_2
$A_2^2A_1$	$A_1 \overset{1}{\hookrightarrow} A_1A_1A_1 \overset{2}{\hookrightarrow} E_6$
A_4A_1	T_1
A_5	$A_1 \overset{2}{\hookrightarrow} E_6$
D_5	T_1
$D_5(a_1)$	T_1

Table 6 Nilpotent centralizers in F_4

Orbit	$(G^x_{\mathrm{red}})^\circ$
A_1	$C_3 \overset{2}{\hookrightarrow} F_4$
$\tilde{A}_1$	$A_3 \overset{2}{\hookrightarrow} A_3\tilde{A}_1 \overset{5}{\hookrightarrow} F_4$
$A_1\tilde{A}_1$	$A_1\tilde{A}_1 \overset{6}{\hookrightarrow} A_1\tilde{A}_2 \overset{2}{\hookrightarrow} F_4$
A_2	$\tilde{A}_2 \overset{2}{\hookrightarrow} F_4$
$\tilde{A}_2$	$G_2 \overset{3}{\hookrightarrow} D_4 \overset{4}{\hookrightarrow} B_4 \overset{5}{\hookrightarrow} F_4$
B_2	$A_1A_1 \overset{2}{\hookrightarrow} B_4 \overset{5}{\hookrightarrow} F_4$
$A_2\tilde{A}_1$	$\tilde{A}_1 \overset{2}{\hookrightarrow} F_4$
$\tilde{A}_2A_1$	$A_1 \overset{2}{\hookrightarrow} G_2 \overset{3}{\hookrightarrow} D_4 \overset{4}{\hookrightarrow} B_4 \overset{5}{\hookrightarrow} F_4$
B_3	$A_1 \overset{6}{\hookrightarrow} \tilde{A}_2 \overset{2}{\hookrightarrow} F_4$
C_3	$A_1 \overset{2}{\hookrightarrow} F_4$
$C_3(a_1)$	$A_1 \overset{2}{\hookrightarrow} F_4$

We remark that in some cases, the book [12] does not quite give enough details about embeddings of subgroups to establish our result, but in these cases, the relevant details can be found in [15, §3.16]. Here is an example illustrating this. The F_4-orbit labelled $\tilde{A}_1$ corresponds (by [12, Proposition 16.10]) to the E_8-orbit labelled A_1^2. Let x be an element of this orbit. We have see that in E_8, $(G^x_{\mathrm{red}})^\circ = B_6$, which embeds in the Levi subgroup $D_7 \subset E_8$. The group $(G^x_{\mathrm{red}})^\circ$ has a subgroup of type D_3B_3, which embeds in $D_3D_4 \subset D_7 \subset D_8$. The explicit construction of $H = G_2$ in [15, §3.16] shows that it is contained in the second factor in each of $D_3B_3 \subset D_3D_4$. It follows that $D_3 = A_3$ is contained in $((G_0^x)_{\mathrm{red}})^\circ = C_{(G^x_{\mathrm{red}})^\circ}(H)$, and then a dimension calculation shows that in fact $((G_0^x)_{\mathrm{red}})^\circ = A_3$.

The results of these calculations are recorded in Table 6.

3.6 Proof for G_2

In this case, there are only two nilpotent orbits that are neither distinguished nor trivial. From the classification, both of these orbits meet the maximal reductive subgroup $A_1\tilde{A}_1 \subset G_2$, and an argument explained in [12, §16.1.4] shows that if x belongs to either of these orbits, then the reductive part of its centralizer in $A_1\tilde{A}_1$ is equal to the reductive part of its centralizer in G_2. See Table 7.

Table 7 Nilpotent centralizers in G_2

Orbit	$(G^x_{\mathrm{red}})^\circ$
A_1	$\tilde{A}_1 \overset{2}{\hookrightarrow} A_1\tilde{A}_1 \overset{5}{\hookrightarrow} G_2$
$\tilde{A}_1$	$A_1 \overset{2}{\hookrightarrow} A_1\tilde{A}_1 \overset{5}{\hookrightarrow} G_2$

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Declarations

Conflict of Interest The authors declare no competing interests.

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