FUSION SYSTEMS WITH BENSON-SOLOMON COMPONENTS

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Abstract. The Benson-Solomon systems comprise a one-parameter family of simple exotic fusion systems at the prime 2. The results we prove give significant additional evidence that these are the only simple exotic 2-fusion systems, as conjectured by Solomon. We consider a saturated fusion system $\mathcal{F}$ having an involution centralizer with a component $\mathcal{C}$ isomorphic to a Benson-Solomon fusion system, and we show under rather general hypotheses that $\mathcal{F}$ cannot be simple. Furthermore, we prove that if $\mathcal{F}$ is almost simple with these properties, then $\mathcal{F}$ is isomorphic to the next larger Benson-Solomon system extended by a group of field automorphisms. Our results are situated within Aschbacher's program to provide a new proof of a major part of the classification of finite simple groups via fusion systems. One of the most important steps in this program is a proof of Walter's Theorem for fusion systems, and our first result is specifically tailored for use in the proof of that step. We then apply Walter's Theorem to treat the general Benson-Solomon component problem under the assumption that each component of an involution centralizer in $\mathcal{F}$ is on the list of currently known quasisimple 2-fusion systems.

1. Introduction

This paper is situated within Aschbacher's program to classify a large class of saturated fusion systems at the prime 2, and then use that result to rework and simplify the corresponding part of the classification of the finite simple groups. A saturated fusion system is a category $\mathcal{F}$ whose objects are the subgroups of a fixed finite p-group S, and whose morphisms are injective group homomorphisms between objects such that certain axioms hold. Each finite group G leads to a saturated fusion system $\mathcal{F}_S(G)$, where S is a Sylow p-subgroup of G and the morphisms are the conjugation maps induced by elements of G. Fusion systems which do not arise in this fashion are called *exotic*. While exotic fusion systems seem to be relatively plentiful at odd primes, there is as yet one known family of simple exotic fusion systems at the prime 2. These are the *Benson-Solomon fusion systems* $\mathcal{F}_{\mathrm{Sol}}(q)$ (q an odd prime power) whose existence was foreshadowed in the work of Solomon [Sol74] and Benson [Ben98], and which were later constructed by Levi–Oliver [LO02, LO05] and Aschbacher-Chermak [AC10]. Here for any odd prime power q, the underlying 2-group S of $\mathcal{F}_{\mathrm{Sol}}(q)$ is isomorphic to a Sylow 2-subgroup of $\mathrm{Spin}_7(q)$, all involutions in $\mathcal{F}_{\mathrm{Sol}}(q)$ are conjugate, and the centralizer of an involution is isomorphic to the fusion system of $\mathrm{Spin}_7(q)$.

It has been conjectured by Solomon that the fusion systems $\mathcal{F}_{\mathrm{Sol}}(q)$ are indeed the only simple exotic saturated 2-fusion systems [Gui08, Conjecture 57.12]. Some recent evidence for Solomon's

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conjecture is provided by a project by Andersen, Oliver, and Ventura, who carried out a systematic computer search for saturated fusion systems over small 2-groups and found that each saturated fusion system over a 2-group of order at most 2^9 is realizable by a finite group. (The smallest Benson-Solomon system is based on a 2-group of order 2^{10}.) Theorems within Aschbacher's program can be expected to give yet stronger evidence for Solomon's conjecture, and the results we prove are particularly relevant in this context. In order to explain this, we now summarize a bit more of the background.

The major case distinction in the proof of the classification of finite simple groups is given by the Dichotomy Theorem of Gorenstein and Walter, which partitions the finite simple groups of 2-rank at least 3 into the groups of component type and the groups of characteristic 2-type. A finite group G is said to be of component type if some involution centralizer modulo core in G has a component. Here a *component* is a subnormal subgroup which is quasisimple (i.e. perfect, and simple modulo its center), and the core $O(C)$ of a finite group C is the largest normal subgroup of C of odd order. The largest and richest collection of simple groups of component type are the simple groups of Lie type in odd characteristic. In the classification of finite simple groups, one proceeds by induction on the group order. Thus, if G is a finite group of component type, one assumes that the components of involution centralizers in G are known, and the objective is then to show that the simple group itself is known. More precisely, one usually assumes that a specific quasisimple group K is given as a component in $C_G(t)/O(C_G(t))$ for some involution t of G, and then tries to show that G is known. We refer to such a task as an involution centralizer problem, or a component problem.

As several involution centralizer problems in 1960s and 1970s gave rise to previously unknown sporadic simple groups, this suggests that solving such problems in fusion systems is a good way to search for new exotic 2-fusion systems. Here, we consider an involution centralizer problem in which the component $\mathcal{C}$ in an involution centralizer of $\mathcal{F}$ is a Benson-Solomon system, and our main theorems can be viewed as essentially determining the structure of the "subnormal closure" of $\mathcal{C}$ in $\mathcal{F}$. Thus, we provide the treatment of a problem that has no analogue in the original classification. The results we prove give additional evidence toward the validity of Solomon's conjecture, or at least toward the absence of additional exotic systems arising in some direct fashion from the existence of $\mathcal{F}_{\mathrm{Sol}}(q)$.

Our work is also an important step in Aschbacher's program. We refer to the survey article [AO16] and the memoir [Asc19] for more details on an outline and first steps of his program. Much of the background material is also motivated and collected in Section 2, which can serve as a detailed guide to the proof of the main theorems here for readers not familiar with the classification program. This material has at its foundation many useful constructions from finite group theory that have been established in the context of saturated fusion systems. In particular, these constructions allow one to speak of centralizers of p-subgroups, normal subsystems, simple fusion systems, quasisimple fusion systems, components, and so on. We refer to the standard reference for those constructions [AKO11].

A saturated 2-fusion system $\mathcal{F}$ is said to be of *component type* if some involution centralizer in $\mathcal{F}$ has a component. Aschbacher defines the class of 2-fusion systems of *odd type* as a certain subclass of the fusion systems of component type. The fusion systems of odd type are further partitioned into those of *subintrinsic component type* and those of *J-component type*. The classification of simple fusion systems of subintrinsic component type constitutes the first part of the program. Our first theorem is tailored for use in the proof of Walter's Theorem [Asc20], one of the main steps in the subinstrinsic case. We then apply Walter's Theorem to give a treatment of the general Benson-Solomon component problem in the second main theorem. As a corollary (Corollary 3), we show that if $\mathcal{F}$ is almost simple (that is, the generalized Fitting subsystem $F^*(\mathcal{F})$ is simple) and $\mathcal{F}$ has an involution centralizer with a Benson-Solomon component $\mathcal{C} \cong \mathcal{F}_{\mathrm{Sol}}(q)$, then $F^*(\mathcal{F}) \cong \mathcal{F}_{\mathrm{Sol}}(q^2)$ with the involution inducing an outer automorphism of $F^*(\mathcal{F})$ of order 2.

To state our main theorems in detail, we introduce now some more notation which we explain further in Section 2. Fix a saturated fusion system $\mathcal{F}$ over the 2-group S. Following Aschbacher, we denote by $\mathfrak{C}(\mathcal{F})$ the collection of components of centralizers in $\mathcal{F}$ of involutions in S, roughly speaking. Accordingly, $\mathcal{F}$ is of *component type* if $\mathfrak{C}(\mathcal{F})$ is nonempty. The E-balance Theorem in the form of the Pump-Up Lemma (Section 2.5) allows one to define an ordering on $\mathfrak{C}(\mathcal{F})$, and thus obtain the notion of a *maximal* member of $\mathfrak{C}(\mathcal{F})$. For $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$, we denote by $\mathcal{I}(\mathcal{C})$ the set of involutions t such that $\mathcal{C}$ is a component of $C_{\mathcal{F}}(t)$, roughly speaking, up to replacing $(\mathcal{C}, t)$ by a suitable conjugate in $\mathcal{F}$. Finally a member $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ is said to be *subintrinsic* in $\mathfrak{C}(\mathcal{F})$ if there is $\mathcal{H} \in \mathfrak{C}(\mathcal{C})$ such that $Z(\mathcal{H}) \cap \mathcal{I}(\mathcal{H})$ is not empty. This means in particular that $\mathcal{H}$ itself is in $\mathfrak{C}(\mathcal{F})$, as witnessed by some involution in the center of $\mathcal{H}$.

Theorem 1. *Fix a saturated fusion system $\mathcal{F}$ over a 2-group S and a quasisimple subsystem $\mathcal{C}$ of $\mathcal{F}$ over a fully $\mathcal{F}$-normalized subgroup of S. Assume that $\mathcal{C}$ is a subintrinsic, maximal member of $\mathfrak{C}(\mathcal{F})$ and isomorphic to a Benson-Solomon system. Then $\mathcal{C}$ is a component of $\mathcal{F}$.*

As mentioned above, in the logical structure of Aschbacher's classification program, Theorem 1 is situated within the proof of Walter's Theorem for fusion systems [Asc20]. Walter's Theorem in particular implies that, if a simple saturated 2-fusion system $\mathcal{F}$ has a member of $\mathfrak{C}(\mathcal{F})$ that is the 2-fusion system of a group of Lie type in odd characteristic and not too small, then either $\mathcal{F}$ is the fusion system of a group of Lie type in odd characteristic, or $\mathcal{F} \cong \mathcal{F}_{\mathrm{Sol}}(q)$. One assumption of Walter's Theorem is that each member of $\mathfrak{C}(\mathcal{F})$ is on the list of currently known quasisimple fusion systems, i.e. either one of the Benson-Solomon systems or a fusion system of a finite simple group.

A simple saturated 2-fusion system with an involution centralizer having a Benson-Solomon component would necessarily be exotic, since involution centralizers in fusion systems of groups are the fusion systems of involution centralizers (see also Lemma 2.54). Because of the subintrinsic hypothesis, Theorem 1 does not rule out the possibility of this happening. However, in Section 7, we apply Walter's Theorem for fusion systems to solve the general Benson-Solomon component problem assuming that all members of $\mathfrak{C}(\mathcal{F})$ are on the list of known quasisimple 2-fusion systems.

Theorem 2. *Let $\mathcal{F}$ be a saturated fusion system over the 2-group S. Assume that each member of $\mathfrak{C}(\mathcal{F})$ is known and that some fixed member $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ is isomorphic to $\mathcal{F}_{\mathrm{Sol}}(q)$ for some odd prime power q. Then for each $t \in \mathcal{I}(\mathcal{C})$, there exists a component $\mathcal{D}$ of $\mathcal{F}$ such that one of the following holds.*

(1) $\mathcal{D} = \mathcal{C}$;
(2) $\mathcal{D} \cong \mathcal{C}$, $\mathcal{D}^t \neq \mathcal{D}$, *and $\mathcal{C}$ is diagonally embedded in the direct product $\mathcal{D}\mathcal{D}^t$ with respect to t; or*
(3) $\mathcal{D} \cong \mathcal{F}_{\mathrm{Sol}}(q^2)$, $t \notin \mathcal{D}$, *and* $\mathcal{C} = C_{\mathcal{D}}(t)$.

The automorphism groups and almost simple extensions of the Benson-Solomon systems were determined in [HL18]. The outer automorphism group of $\mathcal{F}_{\mathrm{Sol}}(q)$ is generated by the class of an automorphism uniquely determined as the restriction of a standard Frobenius automorphism of $\mathrm{Spin}_7(q)$ to a Sylow 2-subgroup, and each extension of $\mathcal{F}_{\mathrm{Sol}}(q)$ is uniquely determined by the induced outer automorphism group. So in the situation of Theorem 2(3), for example in the case in which $\mathcal{F}$ is almost simple, the extension $\mathcal{D}\langle t\rangle$ is known and is the expected one.

Corollary 3. *Let $\mathcal{F}$ be a saturated fusion system over the 2-group S such that $\mathcal{D} = F^*(\mathcal{F})$ is simple. Assume that each member of $\mathfrak{C}(\mathcal{F})$ is known, and that some member $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ is isomorphic to $\mathcal{F}_{\mathrm{Sol}}(q)$ for some odd prime power q. Then $\mathcal{D}$ is isomorphic to $\mathcal{F}_{\mathrm{Sol}}(q^2)$. Moreover, for each $t \in \mathcal{I}(\mathcal{C})$, we have $t \notin \mathcal{D}$, some conjugate of t induces a standard field automorphism on $\mathcal{D}$, and $C_{\mathcal{D}}(t) = \mathcal{C}$.*

Corollary 3 follows immediately from Theorem 2 and the following proposition, which extends the results of [HL18]. A more precise statement is found in Proposition 2.45 as one of our preliminary results.

Proposition 4. *Let $\mathcal{F}$ be a saturated fusion system over the 2-group S such that $F^*(\mathcal{F}) = \mathcal{F}_{\mathrm{Sol}}(q^2)$. Then, writing S_0 for Sylow subgroup of $F^*(\mathcal{F})$, all involutions in $S - S_0$ are $\mathcal{F}$-conjugate, and there exists an involution $t \in S - S_0$ such that $C_{F^*(\mathcal{F})}(t) \cong \mathcal{F}_{\mathrm{Sol}}(q)$.*

We now give an outline of the paper. Section 2 provides the requisite background material, much of it due to Aschbacher, together with motivation coming from the group case and some new lemmas needed later on. The various definitions used in later sections are summarized in a large table in Section 2.10. The proof of Theorem 1 begins in Section 3, where we show that a subintrinsic maximal Benson-Solomon component is necessarily a standard subsystem in the sense of Section 2.6. When combined with results of Aschbacher in [Asc19], this allows the consideration of a subsystem $\mathcal{Q}$ which plays the role of the centralizer of $\mathcal{C}$, and with a little more work shows that the Sylow subgroup Q of $\mathcal{Q}$ is either of 2-rank 1 or elementary abelian. Next, in Section 4, we handle the case in which Q is elementary abelian and prove a lemma regarding the 2-rank 1 case. In Section 5, we handle the case in which Q is quaternion using Aschbacher's classification of quaternion fusion packets [Asc17]. Finally, in Section 6 we handle the cyclic case and complete the proof of Theorem 1. We then prove Theorem 2 in Section 7.

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2. Preliminaries

2.1. **Local theory of fusion systems.** Throughout let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. For general background on fusion systems, in particular for the definition of a saturated fusion system, we refer the reader to [AKO11, Chapter I]. In addition to the notations introduced there, we will write $\mathcal{F}^f$ for the set of fully $\mathcal{F}$-normalized subgroups of S. Moreover, we write $\mathcal{E} \leqslant \mathcal{F}$ to indicate that $\mathcal{E}$ is a (not necessarily saturated) subsystem of $\mathcal{F}$. Conjugation-like maps will be written on the right and in the exponent. In particular, if $\mathcal{E}$ is a subsystem of $\mathcal{F}$ over T and $\alpha \in \mathrm{Hom}_{\mathcal{F}}(T, S)$, then $\mathcal{E}^\alpha$ denotes the subsystem of $\mathcal{F}$ over T^α with $\mathrm{Hom}_{\mathcal{E}^\alpha}(P^\alpha, Q^\alpha) = \{\alpha^{-1} \circ \varphi \circ \alpha \colon \varphi \in \mathrm{Hom}_{\mathcal{E}}(P, Q)\}$ for all $P, Q \leqslant T$.

2.1.1. *Local subsystems.* We recall that, for any subgroup X of S, we have the normalizer and the centralizer of X defined. The normalizer $N_{\mathcal{F}}(X)$ is a fusion subsystem of $\mathcal{F}$ over $N_S(X)$, and the centralizer $C_{\mathcal{F}}(X)$ is a fusion subsystem of $\mathcal{F}$ over $C_S(X)$. These subsystems are not necessarily saturated, but if X is fully $\mathcal{F}$-normalized, then $N_{\mathcal{F}}(X)$ is saturated, and if X is fully centralized, then $C_{\mathcal{F}}(X)$ is saturated. Thus, we will often move from a subgroup of S to a fully $\mathcal{F}$-normalized (and thus fully $\mathcal{F}$-centralized) conjugate of this subgroup. In this context it will be convenient to use the following notation, which was introduced by Aschbacher.

Notation 2.1. For a subgroup $X \leqslant S$, denote by $\mathfrak{A}(X)$ or $\mathfrak{A}_{\mathcal{F}}(X)$ the set of morphisms $\alpha \in \mathrm{Hom}_{\mathcal{F}}(N_S(X), S)$ such that $X^\alpha \in \mathcal{F}^f$.

Throughout, we will use often without reference that $\mathfrak{A}(X)$ is non-empty for every subgroup X of S. In fact, the following lemma holds.

Lemma 2.2. *If $X \leqslant S$ and $Y \in X^{\mathcal{F}} \cap \mathcal{F}^f$, then there exists $\alpha \in \mathfrak{A}(X)$ with $X^\alpha = Y$.*

Proof. See e.g. [AKO11, Lemma I.2.6(c)]. □

If $x \in S$, then we often write $C_{\mathcal{F}}(x)$, $N_{\mathcal{F}}(x)$ and $\mathfrak{A}(x)$ instead of $C_{\mathcal{F}}(\langle x \rangle)$, $N_{\mathcal{F}}(\langle x \rangle)$ and $\mathfrak{A}(\langle x \rangle)$ respectively. Similarly, we call x fully centralized (fully normalized), if $\langle x \rangle$ is fully centralized (fully normalized respectively). If x is an involution, then the reader should note that $C_{\mathcal{F}}(x) = N_{\mathcal{F}}(\langle x \rangle)$, and x is fully centralized if and only if $\langle x \rangle$ is fully normalized.

2.1.2. *Normal and subnormal subsystems.* Recall that a subgroup T of S is called *strongly closed* in $\mathcal{F}$ if $P^\varphi \leqslant T$ for every subgroup $P \leqslant T$ and every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$. The following elementary lemma will be useful later on.

Lemma 2.3. *Let T be strongly closed in $\mathcal{F}$ and suppose we are given two $\mathcal{F}$-conjugate subgroups U and U' of S. If $T \leqslant N_S(U)$ and U' is fully normalized, then $T \leqslant N_S(U')$.*

Proof. By Lemma 2.2, there exists $\alpha \in \mathfrak{A}(U)$ such that $U^\alpha = U'$. Then, as T is strongly closed, $T = T^\alpha \leqslant N_S(U)^\alpha \leqslant N_S(U')$ and this proves the assertion. □

A subsystem $\mathcal{E}$ of $\mathcal{F}$ over $T \leqslant S$ is called *normal* in $\mathcal{F}$ if $\mathcal{E}$ is saturated, T is strongly closed, $\mathcal{E}^\alpha = \mathcal{E}$ for every $\alpha \in \mathrm{Aut}_{\mathcal{F}}(T)$, the Frattini condition holds, and a certain technical extra property is fulfilled (see [AKO11, Definition I.6.1]). Here the Frattini condition says that, for every $P \leqslant T$ and every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, T)$, there are $\varphi_0 \in \mathrm{Hom}_{\mathcal{E}}(P, T)$ and $\alpha \in \mathrm{Aut}_{\mathcal{F}}(T)$ such that $\varphi = \varphi_0 \circ \alpha$.

Particularly important cases of normal subsystems include the (unique) smallest normal subsystem of $\mathcal{F}$ over S, which is denoted by $O^{p'}(\mathcal{F})$ (cf. [AKO11, Theorem I.7.7]), and the normal subsystem $O^p(\mathcal{F})$ of $\mathcal{F}$ (cf. [AKO11, Theorem I.7.4]) over the hyperfocal subgroup $\mathfrak{hyp}(\mathcal{F})$ of S. This last theorem shows additionally that there is a one-to-one correspondence between the subgroups T of S containing $\mathfrak{hyp}(\mathcal{F})$ and the saturated fusion systems $\mathcal{E}$ of $\mathcal{F}$ of *p-power index* in $\mathcal{F}$. Here, a subsystem $\mathcal{E}$ over T is said to be of p-power index if $T \geqslant \mathfrak{hyp}(\mathcal{F})$ and $\mathrm{Aut}_{\mathcal{E}}(P) \geqslant O^p(\mathrm{Aut}_{\mathcal{F}}(P))$ for each $P \leqslant T$.

Once normal subsystems are defined, there is then a natural definition of a subnormal subsystem by transitive extension. We will need the following lemma.

Lemma 2.4. *If $\mathcal{E}$ is a subnormal subsystem of $\mathcal{F}$ over T, then every fully $\mathcal{F}$-normalized subgroup of T is also fully $\mathcal{E}$-normalized.*

Proof. In the case that $\mathcal{E}$ is normal in $\mathcal{F}$, this is [Asc08, Lemma 3.4.5]. The general case follows by induction on the length of a subnormal series for $\mathcal{E}$ in $\mathcal{F}$. $\square$

2.1.3. *Normal and subnormal closures.* In Section 5, we will need to work with fusion system analogues of normal and subnormal closures in groups. By [Asc11, Theorem 1], given normal subsystems $\mathcal{E}_i$ over T_i for $i = 1, 2$, one can define a unique normal subsystem $\mathcal{E}_1 \wedge \mathcal{E}_2$ of $\mathcal{F}$ over $T_1 \cap T_2$ which is normal in $\mathcal{E}_1$ and $\mathcal{E}_2$. By iteration of this process, given normal subsystems $\mathcal{E}_1, \ldots, \mathcal{E}_r$, there is a subsystem $\bigwedge_{i=1}^r \mathcal{E}_i$ of $\mathcal{F}$ which plays the same role as the intersection of normal subgroups plays in the theory of groups. For any subgroup $Q \leqslant S$, the *normal closure* of Q in $\mathcal{F}$ is defined as

$$\bigwedge_{\mathcal{E} \trianglelefteq \mathcal{F}, Q \subseteq \mathcal{E}} \mathcal{E}.$$

Set further $\mathrm{sub}_0(\mathcal{F}, Q) = \mathcal{F}$, and for each $i \geqslant 0$, define $\mathrm{sub}_{i+1}(\mathcal{F}, Q)$ to be the normal closure of Q in $\mathrm{sub}_i(\mathcal{F}, Q)$. Then $\mathrm{sub}_{i+1}(\mathcal{F}, Q) \trianglelefteq \mathrm{sub}_i(\mathcal{F}, Q)$ for each $i \geqslant 0$. Since $\mathcal{F}$ is finite, the series is eventually stationary. The *subnormal closure* $\mathcal{F}^\circ$ of Q is defined to be the terminal member of this series.

2.1.4. *Product systems.* We have seen $O^p(\mathcal{F})$ as a particularly important example of a normal subsystem. Conversely, given a normal subsystem $\mathcal{E}$ of $\mathcal{F}$ over T and a subgroup P of S, one may construct the product system $\mathcal{E}P$ of $\mathcal{F}$, which contains $\mathcal{E}$ as a normal subsystem of p-power index. This is a saturated subsystem of $\mathcal{F}$, and furthermore it is the unique saturated subsystem $\mathcal{D}$ of $\mathcal{F}$ over TP such that $O^p(\mathcal{D}) = O^p(\mathcal{E})$. See [Asc11, Section 8], and also [Hen13] for a simplified construction of $\mathcal{E}P$.

Note however that the uniqueness of the construction of the product depends on the ambient system $\mathcal{F}$ in which it is defined, as is seen in the following example.

Example 2.5 ([Hen13, Example 7.4])**.** Let S be a 2-dimensional vector space over $\mathbb{F}_q$ with $q \geqslant 3$ a prime power, let U be a one-dimensional subspace of S, and let $W_1 \neq W_2$ be two complements to U in S. Let $1 \neq \lambda \in \mathbb{F}_q^\times$, and define $\alpha_i \in GL(S)$ to be the transformation which acts as multiplication by λ on U and which is the identity on W_i $(i = 1, 2)$. Then $\alpha_1|_U = \alpha_2|_U$. So if we set $\mathcal{F}_i = \mathcal{F}_S(S \rtimes \langle \alpha_i \rangle)$, then $O^p(\mathcal{F}_1) = \mathcal{F}_U(U \rtimes \langle \alpha_1|_U \rangle) = \mathcal{F}_U(U \rtimes \langle \alpha_2|_U \rangle) = O^p(\mathcal{F}_2)$. Let $\mathcal{E}$ be this subsystem and set $P = W_1$. Then P is central in S, so fully $\mathcal{F}_i$-normalized, and $(\mathcal{E}P)_{\mathcal{F}_1} = \mathcal{F}_1 \neq \mathcal{F}_2 = (\mathcal{E}P)_{\mathcal{F}_2}$.

The following lemma about factorization of morphisms in product systems will be needed later in Lemma 7.5.

Lemma 2.6. *Let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$ over T. Let $P \leqslant S$, $X \leqslant TP$, and $\varphi \in \mathrm{Hom}_{\mathcal{E}S}(X, S)$. Then $\varphi = \psi c_s$ for some $\psi \in \mathrm{Hom}_{\mathcal{E}P}(X, TP)$ and some $s \in S$.*

Proof. We will use the definition of the product given in [Hen13]. By this definition, $\mathcal{E}P = \mathcal{E}(TP)$. Hence, to ease notation we may assume $T \leqslant P$. Setting

$$\mathrm{A}^\circ(Q) := \langle \alpha \in \mathrm{Aut}_{\mathcal{F}}(Q) \colon \alpha \text{ of } p' \text{ order}, [Q, \alpha] \leqslant Q \cap T, \text{ and } \alpha|_{Q \cap T} \in \mathrm{Aut}_{\mathcal{E}}(Q \cap T) \rangle$$

for every $Q \leqslant S$, the definition of the product says that

$$\mathcal{E}S = \langle \mathrm{A}^\circ(Q) \colon Q \leqslant S,\ Q \cap T \in \mathcal{E}^c \rangle_S \quad \text{and} \quad \mathcal{E}P = \langle \mathrm{A}^\circ(Q) \colon Q \leqslant P,\ Q \cap T \in \mathcal{E}^c \rangle_P.$$

Note that $\mathrm{A}^\circ(Q)^{c_s} = \mathrm{A}^\circ(Q^s)$ and $Q^s \cap T \in \mathcal{E}^c$ for every $Q \leqslant S$ with $Q \cap T \in \mathcal{E}^c$ and every $s \in S$. Hence, φ is of the form $\varphi = \psi c_s$ for some $s \in S$ and some morphism $\psi \in \mathrm{Hom}_{\mathcal{E}S}(X, S)$ such that ψ decomposes as a composition of restrictions of automorphisms in $\mathrm{A}^\circ(Q)$ for various $Q \leqslant S$ with $Q \cap T \in \mathcal{E}^c$. (That is, all conjugation homomorphisms induced by elements of S appearing in a decomposition of φ as a morphism in $\mathcal{E}S$ can be moved to the end.) Write $\psi = (\alpha_1|_{X_0})(\alpha_2|_{X_1}) \cdots (\alpha_k|_{X_{k-1}})$ where $X = X_0, \dots, X_k$ and $Q_1, \dots, Q_k$ are subgroups of S, and $\alpha_i \in \mathrm{A}^\circ(Q_i)$ is such that $\langle X_{i-1}, X_i\rangle \leqslant Q_i$ and $X_{i-1}^{\alpha_i} = X_i$. By definition of $\mathrm{A}^\circ(Q_i)$, we have $[Q_i, \beta] \leqslant Q_i \cap T \leqslant Q_i \cap P$ for each $\beta \in \mathrm{A}^\circ(Q_i)$, and hence $\alpha_i|_{Q_i \cap P} \in \mathrm{A}^\circ(Q_i \cap P)$. In particular, as $X_0 = X \leqslant P$ by assumption, we have $X_i \leqslant P$ for $i = 0, 1, \dots, k$. Hence, $\psi\colon X \to P$ is a morphism in $\mathcal{E}P$. □

2.1.5. *Normalizers of p-subgroups in normal subsystems.* We will make use of the following definition and two lemmas concerning local subsystems in product systems later in Lemma 7.4.

Definition 2.7. Let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$ over $T \leqslant S$. Then for any subgroup $P \leqslant S$ such that $P \in (\mathcal{E}P)^f$, we define $N_{\mathcal{E}}(P)$ to be the unique normal subsystem of $N_{\mathcal{E}P}(P)$ over $N_T(P)$ of p-power index.

Notice that the above definition makes sense. For if $P \in (\mathcal{E}P)^f$, $N_{\mathcal{E}P}(P)$ is saturated. Moreover, $\mathfrak{hyp}(N_{\mathcal{E}P}(P)) \leqslant \mathfrak{hyp}(\mathcal{E}P) \leqslant T$ and thus $\mathfrak{hyp}(N_{\mathcal{E}P}(P)) \leqslant N_T(P)$ with $N_T(P)$ is strongly closed in $N_{\mathcal{E}P}(P)$. So by [AKO11, Theorem I.7.4], there exists a unique normal subsystem of $N_{\mathcal{E}P}(P)$ over $N_T(P)$ of p-power index.

As before, the definition of $N_{\mathcal{E}}(P)$ depends on the fusion system $\mathcal{F}$, since $\mathcal{E}P = (\mathcal{E}P)_{\mathcal{F}}$ depends on $\mathcal{F}$. For instance, in Example 2.5, $P = W_1$ is normal in $\mathcal{F}_1$ but not in $\mathcal{F}_2$. So $N_{\mathcal{E}}(P)_{\mathcal{F}_1}$ is the unique normal subsystem of $\mathcal{F}_1$ over $N_U(P) = U$ of index a power of p, namely $\mathcal{E}$. But $N_{\mathcal{F}_2}(P) = \mathcal{F}_S(S)$, so $N_{\mathcal{E}}(P)_{\mathcal{F}_2}$ is the unique normal subsystem of $\mathcal{F}_S(S)$ over U of p-power index, namely $\mathcal{F}_U(U)$. Hence, $N_{\mathcal{E}}(P)_{\mathcal{F}_1}$ and $N_{\mathcal{E}}(P)_{\mathcal{F}_2}$ are not equal, and indeed are not even isomorphic to each other.

We write $N_{\mathcal{E}}(P)_{\mathcal{F}}$ for $N_{\mathcal{E}}(P)$ if we want to make clear that we formed $N_{\mathcal{E}}(P)$ inside of $\mathcal{F}$. If $p = 2$ and t is an involution, then we write $C_{\mathcal{E}}(t) = C_{\mathcal{E}}(t)_{\mathcal{F}}$ for $N_{\mathcal{E}}(\langle t\rangle)$.

Lemma 2.8. *Let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$ over $T \leqslant S$. If $P \in \mathcal{F}^f$, then $P \in (\mathcal{E}P)^f$ and $N_{\mathcal{E}}(P)$ is normal in $N_{\mathcal{F}}(P)$.*

Proof. Let $P \in \mathcal{F}^f$ and fix and $\mathcal{E}P$-conjugate Q of P with $Q \in (\mathcal{E}P)^f$. By construction of $\mathcal{E}P$, we have $TP = TQ$. Let $\alpha \in \mathfrak{A}(Q)$ with $Q^\alpha = P$. As T is strongly closed, we have $N_T(Q)^\alpha \leqslant N_T(P)$. So $N_{TP}(Q)^\alpha = N_{TQ}(Q)^\alpha = (N_T(Q)Q)^\alpha \leqslant N_T(P)P = N_{TP}(P)$. Hence, $|N_{TP}(Q)| \leqslant |N_{TP}(P)|$. As Q is fully $\mathcal{E}P$-normalized, it follows that P is fully $\mathcal{E}P$-normalized.

By [Hen13, Theorem 1], $O^p(N_{\mathcal{E}P}(P))N_T(P)$ is the unique saturated subsystem $\mathcal{D}$ of $N_{\mathcal{E}P}(P)$ over $N_T(P)$ with $O^p(\mathcal{D}) = O^p(N_{\mathcal{E}P}(P))$. Looking at the construction of normal subsystems of p-power index given in [AKO11, Theorem I.7.4], one observes that $O^p(N_{\mathcal{E}}(P)) = O^p(N_{\mathcal{E}P}(P))$. Thus, $O^p(N_{\mathcal{E}P}(P))N_T(P)$ equals $N_{\mathcal{E}}(P)$. Hence, if $P \in \mathcal{F}^f$, our notation is consistent with the one introduced by Aschbacher in [Asc11, 8.24], where it is proved that $N_{\mathcal{E}}(P)$ is normal in $N_{\mathcal{F}}(P)$. □

Lemma 2.9. *Let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$ over $T \leqslant S$. Fix $P \leqslant S$ such that $P \in (\mathcal{E}P)^f$, and let $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_T(P)P, S)$. Then $P^\varphi \in (\mathcal{E}P^\varphi)^f$, $N_T(P)^\varphi = N_T(P^\varphi)$ and $\varphi|_{N_T(P)}$ induces an isomorphism from $N_{\mathcal{E}}(P)$ to $N_{\mathcal{E}}(P^\varphi)$.*

Proof. It is sufficient to show that $(N_T(P)P)^\varphi = N_T(P^\varphi)P^\varphi$ and $N_{\mathcal{E}P}(P)^\varphi = N_{\mathcal{E}P^\varphi}(P^\varphi)$. For if this is true then, as T is strongly closed, $N_T(P)^\varphi = N_T(P^\varphi)$. So, since φ induces an isomorphism from $N_{\mathcal{E}P}(P)$ to $N_{\mathcal{E}P^\varphi}(P^\varphi)$, it will take the unique normal subsystem of $N_{\mathcal{E}P}(P)$ over $N_T(P)$ of p-power index to the unique normal subsystem of $N_{\mathcal{E}P^\varphi}(P^\varphi)$ over $N_T(P^\varphi)$ of p-power index.

By [Asc19, 1.3.2], we have $\mathcal{F} = \langle \mathcal{E}S, N_\mathcal{F}(T)\rangle$. Hence, it is sufficient to prove the claim in the case that φ is a morphism in $N_\mathcal{F}(T)$ or a morphism in $\mathcal{E}S$.

If φ is a morphism in $N_\mathcal{F}(T)$, then φ extends to $\alpha \in \mathrm{Hom}_\mathcal{F}(TP, TP^\varphi)$ with $T^\alpha = T$. Notice that $P^\alpha = P^\varphi$, and $\alpha\colon TP \to TP^\varphi$ is an isomorphism of groups, which induces by the construction of $\mathcal{E}P$ and $\mathcal{E}P^\varphi$ in [Hen13] an isomorphism from $\mathcal{E}P$ to $\mathcal{E}P^\varphi$. So $P^\varphi = P^\alpha \in (\mathcal{E}P^\varphi)^f$ and $\varphi = \alpha|_{N_T(P)P}$ induces an isomorphism from $N_{\mathcal{E}P}(P)$ to $N_{\mathcal{E}P^\varphi}(P^\varphi)$. Hence, the assertion holds if φ is a morphism in $N_\mathcal{F}(T)$.

Assume now that φ is a morphism in $\mathcal{E}S$. By the construction of $\mathcal{E}S$ and $\mathcal{E}P$ in [Hen13], φ is the composition of a morphism in $\mathcal{E}P$ and a morphism in $\mathcal{F}_S(S)$. As $\mathcal{F}_S(S) \leqslant N_\mathcal{F}(T)$ and the assertion holds if φ is a morphism in $N_\mathcal{F}(T)$, we may thus assume without loss of generality that φ is a morphism in $\mathcal{E}P$. However, then $TP = TP^\varphi$, $\mathcal{E}P = \mathcal{E}P^\varphi$ and it follows from $P \in (\mathcal{E}P)^f$ that $(N_T(P)P)^\varphi = N_{TP}(P)^\varphi = N_{TP^\varphi}(P^\varphi) = N_T(P^\varphi)P^\varphi$. So φ induces an isomorphism from $N_{\mathcal{E}P}(P)$ to $N_{\mathcal{E}P}(P^\varphi) = N_{\mathcal{E}P^\varphi}(P^\varphi)$. So the assertion holds if φ is a morphism in $\mathcal{E}P$ and thus also if φ is a morphism in $\mathcal{E}S$. As argued above, this shows that the lemma holds. □

2.2. **Automorphisms and extensions of fusion and linking systems.** At several later points, we will need to construct various extensions of fusion systems and to determine the structure of extensions where they arise. For example, if $\mathcal{F}$ is a saturated fusion system over S and $\mathcal{E}$ is a normal subsystem of $\mathcal{F}$, then we want to be able to construct certain subsystems of $\mathcal{F}$ containing $\mathcal{E}$ and determine their structure from the structure of $\mathcal{E}$. In the category of groups, this is a relatively painless process when the normal subgroup is quasisimple. However, in fusion systems there are technical difficulties that necessitate in many cases the consideration of linking systems associated to $\mathcal{F}$ and $\mathcal{E}$.

We refer to [AKO11, Section III.4] or [AOV12] for the definition of an abstract linking system as used here, and for more details on automorphisms of fusion and linking systems. Fix a linking system $\mathcal{L}$ for $\mathcal{F}$ with object set Δ and structural functors δ and π, which we write on the left of their arguments. The group of automorphisms of $\mathcal{F}$ is defined by

$$\mathrm{Aut}(\mathcal{F}) = \{\alpha \in \mathrm{Aut}(S) \mid \mathcal{F}^\alpha = \mathcal{F}\}.$$

Then $\mathrm{Aut}_\mathcal{F}(S)$ is normal in $\mathrm{Aut}(\mathcal{F})$, and the quotient $\mathrm{Aut}(\mathcal{F})/\mathrm{Aut}_\mathcal{F}(S)$ is denoted $\mathrm{Out}(\mathcal{F})$.

An automorphism of $\mathcal{L}$ is an equivalence $\alpha\colon \mathcal{L} \to \mathcal{L}$ that is both *isotypical* and *sends inclusions to inclusions*. Since we do not use those conditions explicitly, we refer to [AKO11, Section III.4] for their precise meanings. Each automorphism of $\mathcal{L}$ is indeed an automorphism of the category $\mathcal{L}$, not merely a self-equivalence. We shall write $\mathrm{Aut}(\mathcal{L})$ for the group of automorphisms of $\mathcal{L}$. There is always a *conjugation map*

$$c\colon \mathrm{Aut}_\mathcal{L}(S) \longrightarrow \mathrm{Aut}(\mathcal{L})$$

which sends an element $\gamma \in \mathrm{Aut}_\mathcal{L}(S)$ to the functor $c_\gamma \in \mathrm{Aut}(\mathcal{L})$ defined on objects by $P \mapsto P^\gamma := P^{\pi(\gamma)}$. For a morphism $\varphi \in \mathrm{Mor}_\mathcal{L}(P, R)$, the map c sends φ to φ^γ, namely the morphism

$$\varphi^\gamma := \gamma^{-1}|_{P^\gamma, P} \circ \varphi \circ \gamma|_{R, R^\gamma} \in \mathrm{Mor}_\mathcal{L}(P^\gamma, R^\gamma),$$

where, for example, $\gamma|_{R,R^\gamma}$ is the *restriction* of γ, uniquely determined by the condition that $\delta_{R,S}(1) \circ \gamma = \gamma|_{R,R^\gamma} \circ \delta_{R^\gamma,S}(1)$ in $\mathcal{L}$. The image of c in $\mathrm{Aut}(\mathcal{L})$ is a normal subgroup of $\mathrm{Aut}(\mathcal{L})$, and

$$\mathrm{Out}(\mathcal{L}) := \mathrm{Aut}(\mathcal{L})/\{c_\gamma \mid \gamma \in \mathrm{Aut}_{\mathcal{L}}(S)\}$$

is the group of outer automorphisms of $\mathcal{L}$.

There are natural maps $\tilde{\mu}\colon : \mathrm{Aut}(\mathcal{L}) \to \mathrm{Aut}(\mathcal{F})$ and $\mu\colon \mathrm{Out}(\mathcal{L}) \to \mathrm{Out}(\mathcal{F})$ which, at least when $\Delta = \mathcal{F}^c$, fit into a commutative diagram

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
Z(\mathcal{F}) \arrow[r, "\text{incl}"] \arrow[d, equal] & Z(S) \arrow[r] \arrow[d, "\delta_S"] & \widehat{Z}^1(\mathcal{O}(\mathcal{F}^c), \mathcal{Z}_{\mathcal{F}}) \arrow[r] \arrow[d, "\widetilde{\lambda}"] & \varprojlim^1(\mathcal{Z}_{\mathcal{F}}) \arrow[r] \arrow[d, "\lambda"] & 1 \\
Z(\mathcal{F}) \arrow[r] & \mathrm{Aut}_{\mathcal{L}}(S) \arrow[r, "c"] \arrow[d, "\pi_S"] & \mathrm{Aut}(\mathcal{L}) \arrow[r] \arrow[d, "\widetilde{\mu}"] & \mathrm{Out}(\mathcal{L}) \arrow[r] \arrow[d, "\mu"] & 1 \\
1 \arrow[r] & \mathrm{Aut}_{\mathcal{F}}(S) \arrow[r] \arrow[d] & \mathrm{Aut}(\mathcal{F}) \arrow[r] \arrow[d] & \mathrm{Out}(\mathcal{F}) \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd} \tag{2.10}$$

with all rows and columns exact. Here, $\widehat{Z}^i(\mathcal{O}(\mathcal{F}^c), \mathcal{Z}_{\mathcal{F}})$ denotes a certain subgroup of the group of normalized cocycles of the center functor defined on the orbit category of $\mathcal{F}$-centric subgroups [AKO11, pp.186,174], and $\varprojlim^i \mathcal{Z}_{\mathcal{F}}$ denotes the corresponding cohomology group. The diagram is an updated version of the one appearing in [AKO11, p.186]. The last two columns were not known to be exact until Chermak's proof of the uniqueness of centric linking systems [Che13]. For example, by [AKO11, Proposition III.5.12] the cokernel of the map μ injects into $\lim^2 \mathcal{Z}_{\mathcal{F}}$, which is zero by [Oli13, Theorem 3.4] or [GL16, Theorem 1]. Then using a diagram chase like that given in a five lemma for groups, one sees that the penultimate column is also exact.

Lemma 2.11. *Let $\mathcal{F}$ be a saturated fusion system over S with associated centric linking system $\mathcal{L}$, and suppose that $\mu\colon \mathrm{Out}(\mathcal{L}) \to \mathrm{Out}(\mathcal{F})$ is injective. Then $\ker(\tilde{\mu}) = \{c_{\delta_S(z)} \mid z \in Z(S)\}$ consists of automorphisms of $\mathcal{L}$ induced by conjugation by elements of $Z(S)$.*

Proof. By assumption on μ, we see from (2.10) that $\lim^1(\mathcal{Z}_{\mathcal{F}}) = 0$ by the exactness of the third column. The assertion follows from exactness of the top row (2.10), together with commutativity of the square containing $Z(S)$ and $\mathrm{Aut}(\mathcal{L})$. $\square$

In the situation where $\mathcal{F}$ is realized by a finite group G with Sylow subgroup S, there are maps which compare certain automorphism groups of G with the automorphism groups of $\mathcal{L}$ and $\mathcal{F}$. For example, there is a group homomorphism $\tilde{\kappa}_G\colon \mathrm{Aut}(G,S) \to \mathrm{Aut}(\mathcal{L})$, where $\mathrm{Aut}(G,S)$ is the subgroup of $\mathrm{Aut}(G)$ consisting of those automorphisms which normalize S. Then $\tilde{\kappa}_G$ sends the image of $N_G(S)$ to $\mathrm{Im}(c) \leqslant \mathrm{Aut}(\mathcal{L})$, and so induces a homomorphism $\kappa\colon \mathrm{Out}(G) \to \mathrm{Out}(\mathcal{L})$.

Definition 2.12. A finite group G with Sylow subgroup S is said to *tamely realize* $\mathcal{F}$ if $\mathcal{F} \cong \mathcal{F}_S(G)$ and the map $\kappa\colon \mathrm{Out}(G) \to \mathrm{Out}(\mathcal{L})$ is split surjective. The fusion system $\mathcal{F}$ is said to be *tame* if it is tamely realized by some finite group.

From work of Andersen-Oliver-Ventura and Broto-Møller-Oliver, the fusion systems of all finite simple groups at all primes are now known to be tamely realized by some finite group [AO16, Section 3.3]. To give one example of the importance of tameness for getting a hold of extensions of fusion systems of finite quasisimple groups, we mention the following result of Oliver that will be useful later.

Theorem 2.13. *Let $\mathcal{F}$ be a saturated fusion system over the finite p-group S and let $\mathcal{E}$ be a normal subsystem over the subgroup $T \leqslant S$. Assume that $F^*(\mathcal{F}) = O_p(\mathcal{F})\mathcal{E}$ with $\mathcal{E}$ quasisimple and that $\mathcal{E}$ is tamely realized by the finite group H. Then $\mathcal{F}$ is tamely realized by a finite group G such that $F^*(G) = O_p(G)H$.*

Proof. This is Corollary 2.5 of [Oli16]. See also a correction and a strengthening to the results of [Oli16] in [Oli21]. □

2.3. **Components and the generalized Fitting subsystem.** Aschbacher [Asc11, Chapter 9] introduced components and the generalized Fitting subsystem $F^*(\mathcal{F})$ of $\mathcal{F}$. By analogy with the definition for groups, a component is a subnormal subsystem of $\mathcal{F}$ which is quasisimple. Here $\mathcal{F}$ is called quasisimple if $O^p(\mathcal{F}) = \mathcal{F}$ and $\mathcal{F}/Z(\mathcal{F})$ is simple. By [Asc11, 9.8.2,9.9.1], the generalized Fitting subsystem of $\mathcal{F}$ is the central product of $O_p(\mathcal{F})$ and the components of $\mathcal{F}$. Moreover, for every set J of component of $\mathcal{F}$, there is a well-defined subsystem $\Pi_{\mathcal{C}\in J}\mathcal{C}$, which is the central product of the components in J. Writing $E(\mathcal{F})$ for the central product of all components of $\mathcal{F}$, $F^*(\mathcal{F})$ is the central product of $O_p(\mathcal{F})$ with $E(\mathcal{F})$. We will use the following lemma.

Lemma 2.14. *If $\mathcal{C}$ is a component of $\mathcal{F}$ over T then the following hold:*

(a) $\mathcal{C}$ is normal in $F^*(\mathcal{F})$.
(b) If $\gamma \in \mathrm{Hom}_{\mathcal{F}}(T, S)$, then $\mathcal{C}^\gamma$ is a component of $\mathcal{F}$.

Proof. By definition of a component, $\mathcal{C}$ is subnormal and thus saturated. As mentioned above, by [Asc11, 9.8.2,9.9.1], $F^*(\mathcal{F})$ is the central product of $O_p(\mathcal{F})$ (more precisely $\mathcal{F}_{O_p(\mathcal{F})}(O_p(\mathcal{F}))$) and the components of $\mathcal{F}$. It is elementary to check that each of the central factors in a central product of saturated fusion systems is normal. Hence, every component of $\mathcal{F}$ is normal in $F^*(\mathcal{F})$ and (a) holds.

For the proof of (b) let $S_0 \leqslant S$ such that $F^*(\mathcal{F})$ is a fusion system over S_0. The Frattini condition (applied to the normal subsystem $F^*(\mathcal{F})$) says that we can factorize γ as $\gamma = \gamma_0 \circ \alpha$ with $\gamma_0 \in \mathrm{Hom}_{F^*(\mathcal{F})}(T, S_0)$ and $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S_0)$. By (a), $\mathcal{C}^{\gamma_0} = \mathcal{C}$ and thus $\mathcal{C}^\gamma = \mathcal{C}^\alpha$. As $F^*(\mathcal{F})$ is a normal subsystem, α induces an automorphism of $F^*(\mathcal{F})$. Thus, $\mathcal{C}^\alpha$ is normal in $F^*(\mathcal{F})$ as $\mathcal{C}$ is normal in $F^*(\mathcal{F})$. So $C^\gamma = \mathcal{C}^\alpha$ is subnormal in $\mathcal{F}$. Hence, $\mathcal{C}^\gamma$ is a component of $\mathcal{F}$, since $\mathcal{C}^\gamma \cong \mathcal{C}$ is quasisimple. □

Lemma 2.15. *Let $\mathcal{F}$ be a saturated fusion system which is the central product of saturated subsystems $\mathcal{F}_1, \ldots, \mathcal{F}_n$. If $\mathcal{C}$ is a component of $\mathcal{F}$, then there exists $i \in \{1, 2, \ldots, n\}$ such that $\mathcal{C}$ is a component of $\mathcal{F}_i$.*

Proof. Assume that $\mathcal{C}$ is a component of $\mathcal{F}$ which, for all $i = 1, \ldots, n$, is not a component of $\mathcal{F}_i$. Let $\mathcal{C}$ be a subsystem on $T \leqslant S$, and let $\mathcal{F}_i$ be a subsystem on S_i for $i = 1, \ldots, n$. Since $\mathcal{F}$ is the central product of $\mathcal{F}_1, \ldots, \mathcal{F}_n$, each of the subsystems $\mathcal{F}_1, \ldots, \mathcal{F}_n$ is normal in $\mathcal{F}$. So for each $i = 1, \ldots, n$, it follows from [Asc11, 9.6] and the assumption that $\mathcal{C}$ is not a component of

$\mathcal{F}_i$ that T centralizes S_i. As $S = \Pi_{i=1}^n S_i$, this yields that T centralizes S and is thus abelian. Now [Asc11, 9.1] yields a contradiction to $\mathcal{C}$ being quasisimple. $\square$

2.4. **Components of involution centralizers.** Suppose now that $\mathcal{F}$ is a saturated fusion system over a 2-group S. If $\mathcal{F}$ is of component type, then in analogy to the group theoretical case, one wants to work with components of involution centralizers (or more generally with components of normalizers of subgroups of S). In fusion systems, the situation is slightly more complicated than in groups, since only components of saturated fusion systems are defined. Therefore, we can only consider components of normalizers of *fully normalized* subgroups. It makes sense to work also with conjugates of such components. Following Aschbacher [Asc19, Section 6] we will use the following notation.

Notation 2.16. If $\mathcal{C}$ is a quasisimple subsystem of $\mathcal{F}$ over T, then define the following sets:

- $\mathcal{X}(\mathcal{C})$ is the set of subgroups or elements X of $C_S(T)$ such that $C_{\mathcal{F}}(X)$ contains $\mathcal{C}$.
- $\tilde{\mathcal{X}}(\mathcal{C})$ is the set of subgroups or elements X of S such that $\mathcal{C}^\alpha$ is a component of $N_{\mathcal{F}}(X^\alpha)$ for some $\alpha \in \mathfrak{A}(X)$.
- $\mathcal{I}(\mathcal{C})$ is the set of involutions in $\tilde{\mathcal{X}}(\mathcal{C})$.

If we want to stress that these sets depend on $\mathcal{F}$, we write $\mathcal{X}_{\mathcal{F}}(\mathcal{C})$, $\tilde{\mathcal{X}}_{\mathcal{F}}(\mathcal{C})$ and $\mathcal{I}_{\mathcal{F}}(\mathcal{C})$ respectively. Moreover, we write $\mathfrak{C}(\mathcal{F})$ for the set of quasisimple subsystems $\mathcal{C}$ of $\mathcal{F}$ such that $\mathcal{I}(\mathcal{C})$ is nonempty.

Lemma 2.17. *Let $\mathcal{C}$ be a quasisimple subsystem of $\mathcal{F}$ over T and $X \in \tilde{\mathcal{X}}(\mathcal{C})$. Then for any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\langle X, T\rangle, S)$ the following hold:*

(a) If $X^\varphi \in \mathcal{F}^f$, then $\mathcal{C}^\varphi$ is a component of $N_{\mathcal{F}}(X^\varphi)$.
(b) We have $X^\varphi \in \tilde{\mathcal{X}}(\mathcal{C}^\varphi)$.

Proof. Assume first $X^\varphi \in \mathcal{F}^f$. Let $\alpha \in \mathfrak{A}(X)$ such that $\mathcal{C}^\alpha$ is a component of $N_{\mathcal{F}}(X^\alpha)$. By Lemma 2.2, there exists $\beta \in \mathfrak{A}(X^\alpha)$ such that $X^{\alpha\beta} = X^\varphi$. Then $N_S(X^\alpha)^\beta = N_S(X^\varphi)$ and β induces an isomorphism from $N_{\mathcal{F}}(X^\alpha)$ to $N_{\mathcal{F}}(X^\varphi)$. So $\mathcal{C}^{\alpha\beta}$ is a component of $N_{\mathcal{F}}(X^\varphi)$. As $X^{\alpha\beta} = X^\varphi$, the map $\beta^{-1}\alpha^{-1}\varphi$ is a morphism in $N_{\mathcal{F}}(X^\varphi)$. Moreover $\mathcal{C}^{\alpha\beta}$ is conjugate to $\mathcal{C}^\varphi$ under $\beta^{-1}\alpha^{-1}\varphi$. Thus, $\mathcal{C}^\varphi$ is a component of $N_{\mathcal{F}}(X^\varphi)$ by Lemma 2.14. This proves (a). If we drop the assumption that $X^\varphi \in \mathcal{F}^f$ and pick $\alpha \in \mathfrak{A}(X^\varphi)$, then applying (a) with $\varphi\alpha$ in place of φ gives that $(\mathcal{C}^\varphi)^\alpha = \mathcal{C}^{\varphi\alpha}$ is a component of $N_{\mathcal{F}}(X^{\varphi\alpha})$. This gives (b). $\square$

Lemma 2.18. *Let $\mathcal{C}$ be a quasisimple subsystem of $\mathcal{F}$ over T and let $X \in \tilde{\mathcal{X}}(\mathcal{C})$ be a subgroup of S. Suppose we are given $Y \in \mathcal{F}^f$ satisfying $[X, Y] \leqslant X \cap Y$ and $\mathcal{C} \leqslant N_{\mathcal{F}}(Y)$. Then $X \in \tilde{\mathcal{X}}_{N_{\mathcal{F}}(Y)}(\mathcal{C})$. In particular, if X has order 2, then $\mathcal{C} \in \mathfrak{C}(N_{\mathcal{F}}(Y))$.*

Proof. Let $\beta \in \mathfrak{A}_{N_{\mathcal{F}}(Y)}(X)$ so that $X^\beta \in N_{\mathcal{F}}(Y)^f$. Let $\alpha \in \mathfrak{A}(X^\beta)$. Then by [Asc10, 2.2.1,2.2.2], we have that $Y^\alpha \in N_{\mathcal{F}}(X^{\beta\alpha})^f$, $(N_S(Y) \cap N_S(X^\beta))^\alpha = N_S(Y^\alpha) \cap N_S(X^{\beta\alpha})$, and α induces an isomorphism from $N_{N_{\mathcal{F}}(Y)}(X^\beta)$ to $N_{N_{\mathcal{F}}(X^{\beta\alpha})}(Y^\alpha)$. By Lemma 2.17(a), we have that $\mathcal{C}^{\beta\alpha}$ is a component $N_{\mathcal{F}}(X^{\beta\alpha})$. So by [Asc19, 2.2.5.2], $\mathcal{C}^{\beta\alpha}$ is a component of $N_{N_{\mathcal{F}}(X^{\beta\alpha})}(Y^\alpha)$. As α induces an isomorphism from $N_{N_{\mathcal{F}}(Y)}(X^\beta)$ to $N_{N_{\mathcal{F}}(X^{\beta\alpha})}(Y^\alpha)$, this implies that $\mathcal{C}^\beta$ is a component of $N_{N_{\mathcal{F}}(Y)}(X^\beta)$. This proves $X \in \tilde{\mathcal{X}}_{N_{\mathcal{F}}(Y)}(\mathcal{C})$ and the assertion follows. $\square$

2.5. **Pumping up.** Crucial in the classification of finite simple groups of component type is the Pump-Up Lemma, which leads to the definition of a maximal component. As we explain in more detail in the next subsection, such maximal components have very nice properties generically, which ultimately allow one to pin down the group if the structure of a maximal component is known.

The main purpose of this section is to state the Pump-Up Lemma for fusion systems. However, to give the reader an intuition, we briefly want to describe the Pump-Up Lemma for groups. Let G be a finite group. To avoid technical difficulties which do not play a role in the context of fusion systems, we assume that none of the 2-local subgroups of G has a normal subgroup of odd order. The results we state here are actually true for all almost simple groups, but to show this one would have to use the B-theorem whose proof is extremely difficult. Avoiding the necessity to prove the B-theorem is one of the major reasons why it is hoped that working in the category of fusion systems will lead to a simpler proof of the classification of finite simple groups.

Let t be an involution of G. If $O(G) = 1$, then the L-balance theorem of Gorenstein and Walter gives that $E(C_G(t)) \leqslant E(G)$, where $E(G)$ denotes the product of the components of G. Further analysis shows that a component C of $C_G(t)$ lies in $E(G)$ in a particular way. Namely, either C is a component of G, or there exists a component D of G such that $D = D^t$ and C is a component of $C_D(t)$, or there exists a component D of G such that $D \neq D^t$ and $C = \{dd^t : d \in D\}$ is the homomorphic image of D under the map $d \mapsto dd^t$. If one applies this property to the centralizer of a suitable involution a rather than to the whole group G, then one obtains the Pump-Up Lemma. More precisely, consider two commuting involutions t and a centralized by a quasisimple subgroup C which is a component of $C_G(t)$, and thus of $C_{C_G(a)}(t)$. The result stated above yields immediately that one of the following holds:

(1) C is a component of $C_G(a)$.
(2) There exists a component D of $C_G(a)$ such that $D = D^t$ and C is a component of $C_D(t)$.
(3) There exists a component D of $C_G(a)$ such that $D \neq D^t$ and $C = \{dd^t : d \in D\}$ is a homomorphic image of D.

This statement is known as the Pump-Up Lemma. If (2) holds then D is called a *proper pump-up* of C. The component C is called *maximal* if it has no proper pump-ups.

We now state a similar result for fusion systems, which was formulated by Aschbacher. Again, the statement is slightly more complicated than the statement for groups, since we need to pass from an involution a to a fully centralized conjugate of a for the centralizer to be saturated.

Lemma 2.19 ([Asc19, 6.1.11]). *Let $\mathcal{F}$ be a saturated fusion system over a 2-group S and let $\mathcal{C}$ be a quasisimple subsystem of $\mathcal{F}$ on T. Suppose we are given two commuting involutions $t, a \in S$ such that $t \in \mathcal{I}(\mathcal{C})$ and $\langle t, a\rangle \in \mathcal{X}(\mathcal{C})$. Fix $\alpha \in \mathfrak{A}(a)$, and set $\bar{a} = a^\alpha$, $\bar{t} = t^\alpha$, and $\bar{\mathcal{C}} = \mathcal{C}^\alpha$. Then one of the following holds:*

(1) (trivial) $\bar{\mathcal{C}}$ is a component of $C_{\mathcal{F}}(\bar{a})$, so $a \in \mathcal{I}(\mathcal{C})$,
(2) (proper) there is $\zeta \in \mathrm{Hom}_{C_{\mathcal{F}}(\bar{a})}(C_S(\langle \bar{a}, \bar{t}\rangle), C_S(\bar{a}))$ and a $\bar{t}^\zeta$-invariant component $\mathcal{D}$ of $C_{\mathcal{F}}(\bar{a})$ such that $\bar{\mathcal{C}}^\zeta$ is a component of $C_{\mathcal{D}\langle \bar{t}^\zeta\rangle}(\bar{t}^\zeta)$, and we have $\bar{\mathcal{C}}^\zeta \neq \mathcal{D}$,
(3) (diagonal) there is a component $\mathcal{D}$ of $C_{\mathcal{F}}(\bar{a})$ such that $\mathcal{D} \neq \mathcal{D}^{\bar{t}}$, $\bar{\mathcal{C}} \leqslant \mathcal{D}_0 := \mathcal{D}\mathcal{D}^{\bar{t}}$, and $\mathcal{C}$ is a homomorphic image of $\mathcal{D}$.

Definition 2.20. Let $\mathcal{F}$ be a saturated 2-fusion system and $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$.

- Whenever the hypotheses of Lemma 2.19 occur, and $\mathcal{D}$ satisfies (2) of the conclusion, then $\mathcal{D}$ is a *proper pump-up* of $\mathcal{C}$.
- $\mathcal{C}$ is called *maximal* (or a *maximal component*) if it has no proper pump-ups.

2.6. **Standard components.** We explain now in more detail how maximal components play a role in pinning down the structure of a finite simple group G, and in how far these ideas carry over to fusion systems. As in the previous subsection, we start by explaining the basic ideas for groups. For that, assume again that G is a finite group in which no involution centralizer has a non-trivial normal subgroup of odd order.

Write $\mathfrak{C}(G)$ for the set of components of involution centralizers of G. Using the Pump-Up Lemma, one can choose $C \in \mathfrak{C}(G)$ such that every element $D \in \mathfrak{C}(G)$ which maps homomorphically onto C is maximal. For such C, Aschbacher's component theorem says basically that, with some "small" exceptions, either C is a homomorphic image of a component of G, or the following two conditions hold:

(C1') C does not commute with any of its conjugates; and
(C2') if t is an involution centralizing C, then C is a component of $C_G(t)$.

Assuming that (C1') and (C2') hold and $C/Z(C)$ is a "known" finite simple group, the structure of G is determined case by case from the structure of C. The problem of classifying G from the structure of such a subgroup C is usually referred to as a *standard form problem*. The key to solving such a standard form problem is that properties (C1') and (C2') imply that the centralizer $C_G(C)$ is a tightly embedded subgroup of G and thus has (by various theorems in the literature) a very restricted structure if G is simple. Here a subgroup K of G of even order is called *tightly embedded* in G if $K \cap K^g$ has odd order for any element $g \in G - N_G(K)$. A *standard subgroup* of G is a quasisimple subgroup C of G such that C commutes with none of its conjugates, $K := C_G(C)$ is tightly embedded in G, and $N_G(C) = N_G(K)$. If C is a component of an involution centralizer which satisfies properties (C1') and (C2'), then it is straightforward to prove that C is a standard subgroup. So if G is simple, then with some small exceptions, Aschbacher's component theorem implies that there exists a standard subgroup C of G.

We will now explain the theory of standard components of fusion systems, which Aschbacher [Asc19] has developed roughly in analogy to the situation for groups as far as this seems possible. For the remainder of this subsection let $\mathcal{F}$ be a saturated fusion system over a 2-group S, and let $\mathcal{C}$ be a quasisimple subsystem of $\mathcal{F}$ on T. The situation for fusion systems is significantly more complicated, most importantly since the definition of a standard component of a group involves a statement about its centralizer, and the centralizer of $\mathcal{C}$ in $\mathcal{F}$ is currently only defined in certain special cases. For example, Aschbacher has defined the normalizer and the centralizer of a component of a fusion system [Asc19, Sections 2.1 and 2.2]. In particular, if $\mathcal{C}$ is a component of $C_{\mathcal{F}}(t)$ for a fully centralized involution t, then $C_{C_S(t)}(\mathcal{C})$ is defined inside $C_{\mathcal{F}}(t)$. If $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$, then this allows us to define a subgroup of S which centralizes $\mathcal{C}$, dependent on an involution $t \in \mathcal{I}(\mathcal{C})$.

Notation 2.21 (cf. (6.1.15) in [Asc19])**.** Let $\mathcal{C}$ be a quasisimple subsystem of $\mathcal{F}$ over T. If $t \in \mathcal{I}(\mathcal{C})$ and $\alpha \in \mathfrak{A}(t)$, then define

$$P_{t,\alpha} := C_{C_S(t^\alpha)}(\mathcal{C}^\alpha) \cap C_S(t)^\alpha$$

and

$$Q_t := Q_{t,\alpha} = P_{t,\alpha}^{\alpha^{-1}}$$

By [Asc19, 6.6.16.1], $Q_{t,\alpha} \leqslant C_S(t)$ is independent of the choice of α and so Q_t is indeed well-defined. With this definition in place, one can formulate conditions on $\mathcal{C}$ which roughly correspond to conditions (C1') and (C2'). If $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ fulfills such conditions, then $\mathcal{C}$ is called *terminal*. The precise definition is given below in Definition 2.23.

Notation 2.22 (cf. (6.1.17) and (6.2.7) in [Asc19])**.** Let $\mathcal{C}$ be a quasisimple subsystem of $\mathcal{F}$ over T.

- $\Delta(\mathcal{C})$ is the set of $\mathcal{F}$-conjugates $\mathcal{C}_1$ of $\mathcal{C}$ such that, writing T_1 for the Sylow of $\mathcal{C}_1$, we have $T_1^{\#} \subseteq \tilde{\mathcal{X}}(\mathcal{C})$ and $T^{\#} \subseteq \tilde{\mathcal{X}}(\mathcal{C}_1)$.
- $\rho(\mathcal{C})$ is the set of pairs $(t^{\varphi}, \mathcal{C}^{\varphi})$ such that $t \in \mathcal{I}(\mathcal{C})$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\langle t, T\rangle, S)$.

Definition 2.23 ([Asc19, Definition 8.1.1])**.** A subsystem $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ over T is called *terminal* if the following conditions hold:

(C0) $T \in \mathcal{F}^f$,
(C1) $\Delta(\mathcal{C}) = \varnothing$, and
(C2) If $(t_1, \mathcal{C}_1) \in \rho(\mathcal{C})$, then $Q_{t_1}^{\#} \subseteq \tilde{\mathcal{X}}(\mathcal{C}_1)$.

In this definition, property (C2) corresponds roughly to property (C2') above. Moreover, assuming (C2), property (C1) should be thought of as roughly corresponding to property (C1') above. By Lemma 2.17(b), for any $(t_1, \mathcal{C}_1) \in \rho(\mathcal{C})$, we have $t_1 \in \mathcal{I}(\mathcal{C}_1)$ and so Q_{t_1} is well-defined, i.e. the statement in property (C2) makes sense.

Aschbacher proved a version of his component theorem for fusion systems [Asc19, Theorem 8.1.5]. Suppose $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ is such that every $\mathcal{D} \in \mathfrak{C}(\mathcal{F})$ mapping homomorphically onto $\mathcal{C}$ is maximal. The component theorem for fusion systems states essentially that, with some small exceptions, either $\mathcal{C}$ is the homomorphic image of a component of $\mathcal{F}$, or $\mathcal{C}$ is terminal. This statement is similar to the statement of the component theorem in the group case. However, it is not clear that the centralizer of a terminal component is defined and "tightly embedded" in $\mathcal{F}$. This makes it more complicated to define standard subsystems. We will work with Aschbacher's definition of a standard subsystem, which we state next.

Definition 2.24 ([Asc19, Section 9.1])**.** Let $\mathcal{C}$ be a quasisimple subsystem of $\mathcal{F}$ over $T \in \mathcal{F}^f$. Then $\mathcal{C}$ is a *standard subsystem* of $\mathcal{F}$ if the following four conditions are satisfied:

(S1) $\tilde{\mathcal{X}}(\mathcal{C})$ contains a unique maximal (with respect to inclusion) member Q.
(S2) For each $1 \neq X \leqslant Q$ and $\alpha \in \mathfrak{A}(X)$, we have $\mathcal{C}^{\alpha} \trianglelefteq N_{\mathcal{F}}(X^{\alpha})$.
(S3) If $1 \neq X \leqslant Q$ and $\beta \in \mathfrak{A}(X)$ with $X^{\beta} \leqslant Q$, then $T^{\beta} = T$.
(S4) $\mathrm{Aut}_{\mathcal{F}}(T) \leqslant \mathrm{Aut}(\mathcal{C})$.

If $\mathcal{C}$ satisfies conditions (S1),(S2),(S3), then $\mathcal{C}$ is called *nearly standard*.

Remark 2.25. In the above definition, the first condition (S1) says essentially that the centralizer of $\mathcal{C}$ in S is well-defined. Namely, the unique maximal member Q of $\tilde{\mathcal{X}}(\mathcal{C})$ should be thought of as this centralizer. Given a standard subsystem $\mathcal{C}$ of $\mathcal{F}$, this allows Aschbacher [Asc19, Definition 9.1.4] to define a saturated subsystem $\mathcal{Q}$ of $\mathcal{F}$ over Q which plays the role of the centralizer of $\mathcal{C}$ in $\mathcal{F}$. More precisely, $\mathcal{Q}$ centralizes $\mathcal{C}$ in the sense that $\mathcal{F}$ contains a subsystem which is a central product of $\mathcal{Q}$ and $\mathcal{C}$ (cf. [Asc19, 9.1.6.1]). Also, by [Asc19, 9.1.6.2], $\mathcal{Q}$ is a tightly embedded as defined in the next subsection (cf. Definition 2.30). We will refer to $\mathcal{Q}$ as the *centralizer* of $\mathcal{C}$ in $\mathcal{F}$.

In general, it is difficult to get control of $C_S(T)$ when T is the Sylow subgroup of a member $\mathcal{C}$ of $\mathfrak{C}(\mathcal{F})$. However, $C_S(T) \leqslant N_S(Q)$ when $\mathcal{C}$ is standard. This inclusion gives much needed leverage, as is shown in Lemma 2.28 below. Since the method of proof of that lemma is more widely applicable, we next state and prove a more general result which we feel is of independent interest. In the proof of Proposition 2.26, we reference a *normal pair* of linking systems $\mathcal{L} \trianglelefteq \mathcal{L}_1$ as defined in [AOV12, Definition 1.27]. Also, we take the opportunity to write certain maps on the left-hand side of their arguments.

Proposition 2.26. *Let $\mathcal{F}$ be a saturated fusion system over the 2-group S, and let $\mathcal{F}_1$ be a saturated subsystem over $S_1 \leqslant S$. Assume that $\mathcal{C}$ is a perfect normal subsystem of $\mathcal{F}_1$ over $T \in \mathcal{F}^f$ having associated centric linking system $\mathcal{L}$ such that*

(i) $C_S(T) \leqslant S_1$, and
(ii) $\mu\colon \operatorname{Out}(\mathcal{L}) \to \operatorname{Out}(\mathcal{C})$ is injective (see Section 2.2).

Then $C_S(T) = C_{S_1}(\mathcal{C})Z(T)$.

Proof. By assumption, $\mathcal{C}$ is normal in $\mathcal{F}_1$, so we may form the product system $\mathcal{C}_1 := \mathcal{C}S_1$ in this normalizer, as in [Asc11, Chapter 8] or [Hen13]. Then $O^2(\mathcal{C}_1) = O^2(\mathcal{C}) = \mathcal{C}$ since $\mathcal{C}$ is perfect, so by [AOV12, Proposition 1.31(a)], there is a normal pair of linking systems $\mathcal{L} \trianglelefteq \mathcal{L}_1$ associated to the pair $\mathcal{C} \trianglelefteq \mathcal{C}_1$ in which $\operatorname{Ob}(\mathcal{L}_1) = \{P \leqslant S_1 \mid P \cap T \in \mathcal{C}^c\}$. Note that not only is $\mathcal{L}$ is a subcategory of $\mathcal{L}_1$, but the structural functors δ, π for $\mathcal{L}$ are the restrictions of the functors for $\mathcal{L}_1$ by definition of an inclusion of linking systems. Because of this, we write δ, π also for the structural functors for $\mathcal{L}_1$.

Now by the definition of a normal pair of linking systems [AOV12, Definition 1.27(iii)], the conjugation map $c\colon \operatorname{Aut}_{\mathcal{L}}(T) \to \operatorname{Aut}(\mathcal{L})$ lifts to a map $\operatorname{Aut}_{\mathcal{L}_1}(T) \to \operatorname{Aut}(\mathcal{L})$, which we also denote by c. So the existence of the pair $\mathcal{L} \trianglelefteq \mathcal{L}_1$ allows one to define a homomorphism $\nu\colon S_1 \to \operatorname{Aut}(\mathcal{L})$ given by the composition $S_1 \xrightarrow{\delta_T} \operatorname{Aut}_{\mathcal{L}_1}(T) \xrightarrow{c} \operatorname{Aut}(\mathcal{L})$. This map has kernel

$$\ker(\nu) = C_{S_1}(\mathcal{C}) \tag{2.27}$$

by [Sem15, Theorem A].

We can now prove the assertion. Clearly $C_{S_1}(\mathcal{C})Z(T) \leqslant C_S(T)$. For the reverse inclusion, fix $s \in C_S(T)$. Then ν is defined on s by (i). The map $\tilde{\mu}\colon \operatorname{Aut}(\mathcal{L}) \to \operatorname{Aut}(\mathcal{C})$ is more precisely defined by the equation $t^{\tilde{\mu}(\varphi)} = \delta_T^{-1}(\delta_T(t)^{\varphi_T})$ for all $\varphi \in \operatorname{Aut}(\mathcal{L})$ and all $t \in T$. Using this for $\varphi = \nu(s) = c_{\delta_T(s)}$, we obtain for all $t \in T$ that $t^{\tilde{\mu}(\nu(s))} = \delta_T^{-1}(\delta_T(s)^{-1} \circ \delta_T(t) \circ \delta_T(s)) = \delta_T^{-1}(\delta_T(t^s)) = t^s = t$, where the last equality uses $s \in C_S(T)$. The automorphism $\tilde{\mu}(\nu(s)) \in \operatorname{Aut}(\mathcal{C})$ is thus trivial. Hence by Lemma 2.11 and assumption on μ, $\nu(s) = c_{\delta_T(z)} = \nu(z)$ for some $z \in Z(T)$. It follows that $\nu(sz^{-1})$ is the identity on $\mathcal{L}$. Hence, $sz^{-1} \in C_{S_1}(\mathcal{C})$ by (2.27), so $s \in C_{S_1}(\mathcal{C})Z(T)$, which completes the proof. $\square$

Lemma 2.28. *Let $\mathcal{F}$ be a saturated fusion system over the 2-group S. Suppose $\mathcal{C}$ is a standard subsystem of $\mathcal{F}$ over T with centralizer $\mathcal{Q}$ over Q. Let $\mathcal{L}$ be a centric linking system associated to $\mathcal{C}$. If $\mu\colon \operatorname{Out}(\mathcal{L}) \to \operatorname{Out}(\mathcal{C})$ is injective, then $C_S(T) = QZ(T)$.*

Proof. As Q is fully $\mathcal{F}$-normalized by [Asc19, 9.1.1], $N_{\mathcal{F}}(Q)$ is a saturated fusion system over $N_S(Q)$. Further, (S2) says that $\mathcal{C}$ is normal in $N_{\mathcal{F}}(Q)$. Finally, by [Asc19, Proposition 5], $\mathcal{Q}$ is normal in $N_{\mathcal{F}}(T)$ and $C_{N_S(Q)}(\mathcal{C}) = Q$. In particular, $C_S(T) \leqslant N_S(Q)$. Thus, if μ is injective, then $C_S(T) = C_{N_S(Q)}(\mathcal{C})Z(T) = QZ(T)$ by Proposition 2.26. □

When considering involution centralizer problems for fusion systems, we generally need to verify in each individual case that the terminal component we consider is standard. In contrast with the group case, this is not a straightforward task. Indeed, in some cases a terminal component is provably not standard. However, it develops that many of these technically difficult configurations are ultimately unnecessary to treat, because they arise for example in almost simple fusion systems that are not simple, or in wreath products of a simple system with an involution. Thus, we will usually need assumptions that go beyond supposing merely that the structure of the terminal component we consider is known, but also information about the embedding of that subsystem in the ambient system. Such stronger assumptions can be made if one classifies, as was proposed by Aschbacher, the simple "odd systems" rather than the simple fusion systems of component type (cf. [Asc19]). The hypothesis in Theorem 1 that $\mathcal{C}$ is subintrinsic should be seen in this context.

Definition 2.29. Let $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$. Then $\mathcal{C}$ is said to be *subintrinsic in* $\mathfrak{C}(\mathcal{F})$ if there exists $\mathcal{H} \in \mathfrak{C}(\mathcal{C})$ such that $\mathcal{I}_{\mathcal{F}}(\mathcal{H}) \cap Z(\mathcal{H}) \neq \varnothing$.

It follows in a fairly straightforward way from results of Aschbacher that a subintrinsic Benson-Solomon component $\mathcal{C}$ is terminal. Rather than use the component theorem for fusion systems, it is more convenient in our case to show that $\mathcal{C}$ is terminal using a part of the argument for [Asc19, Theorem 7.4.14], which is a major ingredient of the proof of the component theorem. As suggested above, a nontrivial amount of work is then required to go on and show that $\mathcal{C}$ is standard; see Section 3.

2.7. **Tightly embedded subsystems and tight split extensions.** Recall from the previous subsection that a subgroup K of a finite group G is called *tightly embedded* if K has even order and $K \cap K^g$ has odd order for every $g \in G \backslash N_G(K)$. This definition does not translate well to fusion systems as it is, but there exist suitable reformulations. It follows from Aschbacher [Asc19, 0.7.1] that a subgroup K of G of even order is tightly embedded if and only if the following two conditions hold:

(T1') K is normalized by $N_G(X)$ for every non-trivial 2-subgroup X of K.
(T2') For every involution x of K, $x^G \cap K = x^{N_G(K)}$.

If K is tightly embedded and Q is a Sylow 2-subgroup of K, then note furthermore that $N_G(Q) \leqslant N_G(K)$ and $N_G(K) = KN_G(Q)$ by a Frattini argument. This leads to a definition of tightly embedded subsystem of saturated fusion systems at arbitrary primes.

Definition 2.30 (cf. [Asc19, Definition 3.1.2])**.** Let $\mathcal{F}$ be a saturated fusion system on a p-group S, and let $\mathcal{Q}$ be a saturated subsystem of $\mathcal{F}$ on a fully normalized subgroup Q of $\mathcal{F}$. Then $\mathcal{Q}$ is *tightly embedded* in $\mathcal{F}$ if it satisfies the following three conditions:

(T1) For each $1 \neq X \in \mathcal{Q}^f$ and each $\alpha \in \mathfrak{A}(X)$,

$$O^{p'}(N_{\mathcal{Q}}(X))^{\alpha} \text{ is normal in } N_{\mathcal{F}}(X^{\alpha}).$$

(T2) For each $X \leqslant Q$ of order p,

$$X^{\mathcal{F}} \cap Q = X^{\mathrm{Aut}_{\mathcal{F}}(Q)\mathcal{Q}}$$

where $X^{\mathrm{Aut}_{\mathcal{F}}(Q)\mathcal{Q}} := \{X^{\alpha\varphi} \mid \alpha \in \mathrm{Aut}_{\mathcal{F}}(Q),\ \varphi \in \mathrm{Hom}_{\mathcal{Q}}(X\alpha, Q)\}$.

(T3) $\mathrm{Aut}_{\mathcal{F}}(Q) \leqslant \mathrm{Aut}(\mathcal{Q})$.

When working with standard subsystems later on, we will need the following lemma on tightly embedded subsystems.

Lemma 2.31. *Let $\mathcal{F}$ be a saturated fusion system on S, and suppose $\mathcal{Q}$ is a tightly embedded subsystem of $\mathcal{F}$ on an abelian subgroup Q of S. Then $\mathcal{F}_Q(Q)$ is tightly embedded in $\mathcal{F}$.*

Proof. As Q is abelian, by Alperin's fusion theorem (cf. [AKO11, Theorem I.3.6]), the following holds:

(*) The p-group Q, and thus the subsystem $\mathcal{F}_Q(Q)$, is normal in any saturated fusion system on Q.

Let $1 \neq X \leqslant Q$ and $\alpha \in \mathfrak{A}(X)$. By (*), we have $Q = N_Q(X) \trianglelefteq N_{\mathcal{Q}}(X)$ and thus $Q = O^{p'}(N_{\mathcal{Q}}(X))$. As $\mathcal{Q}$ is tightly embedded, it follows $N_Q(X)^\alpha = Q^\alpha = O^{p'}(N_{\mathcal{Q}}(X))^\alpha \trianglelefteq N_{\mathcal{F}}(X^\alpha)$. So (T1) holds for $\mathcal{F}_Q(Q)$.

Let $X \leqslant Q$ be of order p. Again using (*), we have $Q \trianglelefteq \mathcal{Q}$. So every morphism in $\mathcal{Q}$ extends to an element of $\mathrm{Aut}_{\mathcal{Q}}(Q) \leqslant \mathrm{Aut}_{\mathcal{F}}(Q)$, and this implies $X^{\mathrm{Aut}_{\mathcal{F}}(Q)\mathcal{Q}} = X^{\mathrm{Aut}_{\mathcal{F}}(Q)}$. Hence, as $\mathcal{Q}$ is tightly embedded, $X^{\mathcal{F}} \cap Q = X^{\mathrm{Aut}_{\mathcal{F}}(Q)\mathcal{Q}} = X^{\mathrm{Aut}_{\mathcal{F}}(Q)} = X^{\mathrm{Aut}_{\mathcal{F}}(Q)\mathcal{F}_Q(Q)}$. This shows that (T2) holds for $\mathcal{F}_Q(Q)$. Clearly (T3) holds for $\mathcal{F}_Q(Q)$. $\square$

To exploit the existence of standard subsystems, it is useful in many situations to study certain kinds of extensions involving tightly embedded subsystems. We summarize the main definitions:

Definition 2.32. Let $\mathcal{F}_0$ be a fusion system on a 2-group S_0.

- A *split extension* of $\mathcal{F}_0$ is a pair $(\mathcal{F}, U)$, where
  - $\mathcal{F}$ is a saturated fusion system over a 2-group S,
  - $\mathcal{F}_0$ is normal in $\mathcal{F}$,
  - $O^2(\mathcal{F}) = O^2(\mathcal{F}_0)$, and
  - U is a complement to S_0 in S.
- The split extension $(\mathcal{F}, U)$ is *tight* if $\mathcal{F}_U(U)$ is tightly embedded in $\mathcal{F}$.
- A *critical split extension* is a tight split extension in which U is a four group.
- $\mathcal{F}_0$ is said to be *split* if there exists no nontrivial critical split extension of $\mathcal{F}_0$; that is, for each such extension $(\mathcal{F}, U)$, the fusion system $\mathcal{F}$ is the central product of $\mathcal{F}$ with $C_S(\mathcal{F}_0)$.

Suppose $\mathcal{F}$ is a saturated 2-fusion system and $\mathcal{C}$ is a standard component with centralizer $\mathcal{Q}$ on Q. If $\mathcal{C}$ is split, then by [Asc19, Theorem 8], $\mathcal{C}$ is either a component of $\mathcal{F}$, or Q is elementary abelian, or the 2-rank of Q equals 1. We show in Lemma 2.49 that the Benson–Solomon fusion systems are split. So after showing that a component $\mathcal{C}$ as in Theorem 1 is standard, we know that, unless $\mathcal{C}$ is a component of $\mathcal{F}$, its centralizer Q in S is either elementary abelian or quaternion or cyclic. Accordingly, these are the cases we will treat.

Lemma 2.33. *Let $\mathcal{C}$ be a quasisimple saturated fusion system over the 2-group T, and let $(\mathcal{F}, U)$ be a critical split extension of $\mathcal{C}$ over the 2-group S. Then*

(a) $\mathrm{Aut}_{\mathcal{F}}(U) = 1$ *and so* $N_{\mathcal{F}}(U) = C_{\mathcal{F}}(U)$*; and*
(b) $\langle u \rangle \in \mathcal{F}^f$ *and* $C_{\mathcal{F}}(u) = C_{\mathcal{F}}(U)$ *for each* $1 \neq u \in U$*.*

Proof. By definition of critical split extension, U is a four subgroup of S tightly embedded in $\mathcal{F}$ and a complement to T in S. Also, $O^2(\mathcal{F}) = O^2(\mathcal{C}) = \mathcal{C}$, as $\mathcal{C}$ is quasisimple. Since $O^2(\mathcal{F}) = \mathcal{C}$, this means $\mathfrak{hyp}(\mathcal{F}) = T$. Since $S/\mathfrak{hyp}(\mathcal{F}) \cong U$ is abelian, we see from [AKO11, Lemma I.7.2] that also $\mathfrak{foc}(\mathcal{F}) = T$. Thus, $\mathrm{Aut}_{\mathcal{F}}(U) = 1$, since otherwise $T \cap U = \mathfrak{foc}(\mathcal{F}) \cap U \geqslant [U, \mathrm{Aut}_{\mathcal{F}}(U)] > 1$, which is not the case. This proves the first assertion in (a), and the second then follows from the definitions of the normalizer and centralizer systems.

Now by definition of tight embedding, U is fully normalized in $\mathcal{F}$. Fix $1 \neq u \in U$. By (T2) and part (a), it follows that $u^{\mathcal{F}} \cap U = \{u\}$. However, (3.1.5) of [Asc19] says that $\langle u \rangle$ has a fully $\mathcal{F}$-normalized $\mathcal{F}$-conjugate in U, so $\langle u \rangle \in \mathcal{F}^f$. Then taking α to be identity in (T1), we see that U is normal in $N_{\mathcal{F}}(\langle u \rangle) = C_{\mathcal{F}}(u)$, so that $C_{\mathcal{F}}(u) \leqslant N_{\mathcal{F}}(U) = C_{\mathcal{F}}(U)$ by (a). This completes the proof of (b), as the other inclusion is clear. □

2.8. **The fusion system of** $\mathrm{Spin}_7(q)$ **and** $\mathcal{F}_{\mathrm{Sol}}(q)$**.** Our main references for $\mathcal{F}_{\mathrm{Sol}}(q)$ and for 2-fusion systems of $\mathrm{Spin}_7(q)$ are [LO02, LO05, COS08, AC10, HL18].

We follow Section 4 of Aschbacher and Chermak fairly closely [AC10] within this subsection, except that it will be convenient to restrict the choice of the finite fields $\mathbf{F}_q$ over which the systems in question are defined, and to make small changes to notation. For concreteness, we consider a fixed but arbitrary nonnegative integer l, and set $q_l = 5^{2^l}$. Except for the fact that l is in the role of "k", we adopt the notation in [AC10, Section 4], as follows.

Let $\tilde{\mathbf{F}}$ be an algebraic closure of the field with 5 elements (thus, we take $p = 5$ in [AC10, Section 4]), and let $\mathbf{F}$ be the union of the subfields of the form $\mathbf{F}_{5^{2^n}}$ in $\tilde{\mathbf{F}}$. Let $H = \mathrm{Spin}_7(\mathbf{F})$, let T be a maximal torus of H, and let T_{2^∞} be the 2-power torsion subgroup of T.

Let W be the subgroup of H defined on page 911 of [AC10], let W_S be the subgroup of W defined on page 915 of [AC10], and set $S = T_{2^\infty} W_S$. Thus, $S \cap W = W_S$. The subgroup B of H is defined just before Lemma 4.4 of [AC10] as the normalizer of the unique normal four subgroup U of S (see Notation 2.37). Finally, the group K is defined at the top of page 918 as a certain semidirect product of the connected component B^0 of B with a subgroup $\langle y, \tau \rangle \cong S_3$, such that $\tau \in B$ is of order 2, and such that y is of order 3 and permutes transitively the involutions in U. A free amalgamated product $G = H *_B K$ having Sylow 2-subgroup S is then defined by an amalgam in [AC10, Section 5], which is ultimately constructed at the top of page 923. Here, Sylow p-subgroups of infinite groups are defined in [AC10, Definition 1.4] generalizing properties of Sylow p-subgroups of finite groups in a natural way.

Let ψ be the Frobenius endomorphism of H as defined in (4.2.2) of [AC10] and inducing the 5-th power map on T, and set $\psi_l = \psi^{2^l}$ for each integer $l \geqslant 0$.

Theorem 2.34. *For any choice of integer* $l \geqslant 0$*, the automorphism* ψ_l *of* H *lifts uniquely to an automorphism* σ_l *of the group* G *that commutes with* y*, and hence an automorphism that leaves* K *invariant. Moreover,* $C_S(\sigma_l)$ *is then a finite Sylow* 2*-subgroup of* $C_G(\sigma_l)$*, and* $\mathcal{F}_{C_S(\sigma_l)}(C_G(\sigma_l))$ *is isomorphic to* $\mathcal{F}_{\mathrm{Sol}}(q_l)$*.*

Proof. The first statement here is shown in Lemma 5.7 of [AC10] and in the paragraph before it. The second part is Theorem A(3) of [AC10]: it is shown in [AC10, Lemma 7.4(b)] that $C_G(\sigma_l)$ is isomorphic to $C_H(\sigma_l) *_{C_B(\sigma_l)} C_K(\sigma_l)$ and in [AC10, Lemma 7.5(b)] that $C_S(\sigma_l)$ is a

Sylow 2-subgroup of $C_G(\sigma_l)$. Ultimately it is then shown in Theorem 9.9 that $\mathcal{F}_{C_S(\sigma_l)}(C_G(\sigma_l))$ is isomorphic to the fusion system $\mathcal{F}_{\mathrm{Sol}}(q_l)$ defined by Levi and Oliver in [LO02, LO05]. $\square$

For the automorphism σ_l of G given by Theorem 2.34 and for any subgroup $X \leqslant G$, we write X_{σ_l} for $C_X(\sigma_l)$.

Lemma 2.35. *Let σ_l be the unique lift of $\psi_l = \psi^{2^l}$ to G. Then for each such $l \geqslant 0$,*

(a) X and X_{σ_l} are invariant under σ_0 for each $X \in \{H, B, K, S, W, T\}$,
(b) σ_l centralizes W, and
(c) $S_{\sigma_l} = (T_{2^\infty})_{\sigma_l} W_S$, and S_{σ_l} is a Sylow 2-subgroup of H_{σ_l}.

Proof. Except for the case $X = K$, all parts of (a) follow from the definitions in [AC10, Section 4], since the statement just expresses invariance of X under ψ for $X \leqslant H$. By uniqueness of the lift in Theorem 2.34, $\sigma_0^{2^l} = \sigma_l$, so $[\sigma_0, \sigma_l] = 1$ in particular. Hence, $(K_{\sigma_l})^{\sigma_0} = C_{K^{\sigma_0}}(\sigma_l) = K_{\sigma_l}$. Part (b) is proved in [AC10, Lemma 4.3], while part (b) is proved in [AC10, Lemma 4.9]. $\square$

Thus, as H is of universal type,

$$H_{\sigma_l} = C_H(\sigma_l) = C_H(\psi_l) = \mathrm{Spin}_7(q_l),$$

and so part (a) of the lemma says for example that $(H_{\sigma_l})_{\sigma_0} = \mathrm{Spin}_7(5)$. Also note that ψ_l acts on T as $t \mapsto t^{q_l}$ (by definition of ψ), so T_{σ_l} is a split maximal torus of H_{σ_l}, isomorphic to $(C_{q_l-1})^3$.

Notation 2.36 ($\mathcal{F}_{\mathrm{Spin}}(q)$ and $\mathcal{F}_{\mathrm{Sol}}(q)$)**.** Fix $l \geqslant 0$ and set $\sigma = \sigma_l$ for short. Write

$$\mathcal{H}_\sigma := \mathcal{F}_{S_\sigma}(H_\sigma) = \mathcal{F}_{\mathrm{Spin}}(q) \quad \text{and} \quad \mathcal{F}_\sigma := \mathcal{F}_{S_\sigma}(G_\sigma) = \mathcal{F}_{\mathrm{Sol}}(q).$$

Let $Z := Z(S_\sigma)$, a group of order 2. Write z for the involution in Z.

Thus, by Theorem A(3) of [AC10], $\mathcal{F}_\sigma$ is isomorphic to the exotic fusion system defined by Levi and Oliver in [LO02, LO05], and $C_{\mathcal{F}_\sigma}(z) = \mathcal{H}_\sigma$.

We continue to set up notation for some common subgroups of S_σ, and we recall the various parts of the set up appearing in [AC10, §4] that are needed later.

Notation 2.37 (Some subgroups of S_σ)**.** Set $k := l + 2$, and set $T_{2^k} := (T_{2^\infty})_\sigma \cong (C_{2^k})^3$. This is the 2^k-torsion subgroup of T_{2^∞} and a Sylow 2-subgroup of the finite abelian group T_σ. Let $w_0 \in W_S$ be the element of order 2 fixed in [AC10, Lemma 4.3]. Thus, w_0 is centralized by σ by Lemma 2.35(b) and w_0 inverts T_{2^k}. The 2-group S_σ has a sequence of elementary abelian subgroups

$$1 < Z < U < E < A,$$

each of index 2 in the next, with $Z = Z(S_\sigma)$ as above, U the unique normal four subgroup of S_σ, $E = \Omega_1(T_{2^k})$, and $A = E\langle w_0 \rangle$, an elementary abelian subgroup of order 16. We also set $R_\sigma = C_{S_\sigma}(E) = T_{2^k}\langle w_0 \rangle = T_{2^k}A$.

We adopt Notation 2.36 and 2.37 for the remainder of this subsection.

The following lemma collects a number of properties of these subgroups and their automorphism groups.

Lemma 2.38. *The following hold.*

(a) For each $k_0 \geqslant 2$, $T_{2^{k_0}}$ is the unique homocyclic abelian subgroup of S of rank 3 and exponent 2^{k_0}, and $T_{2^{k_0}}$ is inverted by w_0.
(b) T_{2^k} is $\mathcal{F}_\sigma$-centric, $S_\sigma/T_{2^k} \cong C_2 \times D_8$, and $\mathrm{Aut}_{\mathcal{F}_\sigma}(T_{2^k}) \cong C_2 \times GL_3(2)$.
(c) R_σ is characteristic in S_σ, and $\mathrm{Out}_{\mathcal{F}_\sigma}(R_\sigma) \cong GL_3(2)$.
(d) A is an elementary abelian subgroup of S_σ of maximum order, and so S_σ has 2-rank 4.
(e) $\mathrm{Aut}_{\mathcal{F}_\sigma}(X) = \mathrm{Aut}(X)$ for $X \in \{Z, U, E, A\}$.

Proof. By Lemma 4.9(b) of [AC10], $T_{2^2} \leqslant T_{2^{k_0}}$ is the unique homocyclic subgroup of S of rank 3 and exponent 4. Moreover, for the maximal torus T of H, we have $T = C_H(T_{2^2})$, and $S_\sigma \cap T = T_{2^k}$ is of rank 3 and of exponent 2^k. This shows that T_{2^2}, and more generally, $T_{2^{k_0}} = \Omega_{k_0}(T_{2^k})$ for $2 \leqslant k_0 \leqslant k$ is the unique subgroup of S of its isomorphism type. Also, w_0 inverts T by [AC10, Lemma 4.3(a)]. This completes the proof of (a). Again as $T = C_H(T_{2^2})$, one has $C_{H_\sigma}(T_{2^2}) = T_\sigma = T_{2^k} \times O(T_\sigma)$, and it follows that T_{2^k} is $\mathcal{F}_\sigma$-centric. The second statement in part (b) follows from [AC10, Lemma 4.3(c)], while the third is the content of [AC10, Theorem 5.2].

For the proof of (c), note first that T_{2^k} is characteristic in S by (a). So also $R_\sigma = C_S(\Omega_1(T_{2^k}))$ is characteristic in S. Finally, as T_{2^k} is fully $\mathcal{F}_\sigma$-normalized by (a) and as R_σ/T_{2^k} is of order 2 and induces $O_2(\mathrm{Aut}_{\mathcal{F}_\sigma}(T_{2^k}))$ on T_{2^k}, the restriction map $\rho\colon \mathrm{Aut}_{\mathcal{F}_\sigma}(R_\sigma) \to \mathrm{Aut}_{\mathcal{F}_\sigma}(T_{2^k})$ is surjective by the Extension Axiom. Let $\varphi \in \ker(\rho)$. Then by [BLO03, Lemma A.8] and the first statement in (b), φ is conjugation by an element of $Z(T_{2^k}) = T_{2^k}$. It follows that $\ker(\rho) = \mathrm{Aut}_{T_{2^k}}(R_\sigma)$ is of index 2 in $\mathrm{Inn}(R_\sigma)$. Thus, $\mathrm{Out}_{\mathcal{F}_\sigma}(R_\sigma) \cong \mathrm{Aut}_{\mathcal{F}_\sigma}(T_{2^k})/O_2(\mathrm{Aut}_{\mathcal{F}_\sigma}(T_{2^k})) \cong GL_3(2)$ by the last statement in (b).

Now as $E = \Omega_1(T_{2^k})$ is elementary abelian of order 8 by (a), and w_0 inverts T_{2^k}, it follows that A is elementary abelian of order 16. There are no elementary abelian subgroups of S of rank 5 by [AC10, Lemma 7.9(a)], so (d) holds. Finally, we refer to Lemma 3.1 of [LO02] for the $\mathcal{F}_\sigma$-automorphism groups of $X \in \{Z, U, E, A\}$, where A is denoted "E^*". □

Lemma 2.39. *All involutions in S_σ are $\mathcal{F}_\sigma$-conjugate.*

Proof. This is a direct consequence of the construction of these systems [LO02, Theorem 2.1] and can be seen as follows. Each involution in $H_\sigma - Z$ has -1-eigenspace of dimension 4 on the orthogonal space for $H_\sigma/Z(H_\sigma)$ by [LO02, Lemma A.4(b)]. It follows from this that H_σ has two conjugacy classes of involutions, namely the classes in Z and in $S_\sigma - Z$. In $\mathcal{F}_\sigma$ these two classes become fused by construction (c.f. Lemma 2.38(e)). □

Lemma 2.40. *Let $\mathcal{F} \in \{\mathcal{F}_\sigma, \mathcal{H}_\sigma\}$, and let $\mathcal{L}$ be the centric linking system for $\mathcal{F}$. Then the natural map $\mu\colon \mathrm{Out}(\mathcal{L}) \to \mathrm{Out}(\mathcal{F})$ is an isomorphism.*

Proof. This follows from [LO02, Lemma 3.2] and the obstruction sequence in [AKO11, Proposition 5.12] (that is, from (2.10) above). □

The choice of $q_l = 5^{2^l}$ is motivated by the next two lemmas, especially Lemma 2.41(a).

Lemma 2.41. *Let $\mathcal{H}_q$ be the 2-fusion system of $\mathrm{Spin}_7(q)$ for some odd q, let $l+3$ be the 2-adic valuation of q^2-1, and set $H_\sigma = \mathrm{Spin}_7(q_l)$ as above. Then the following hold.*

(a) $\mathcal{H}_q$ is tamely realized by H_σ.
(b) With R_σ as in Notation 2.37, each automorphism of H_σ that normalizes S_σ and centralizes R_σ is conjugation by an element of $E = Z(R_\sigma)$.

Proof. Since $q^2 - 1$ and $q_l^2 - 1$ have the same 2-adic valuation, the fusion systems $\mathcal{H}_q$ and $\mathcal{H}_\sigma$ are isomorphic by [BMO12, Theorem A(a,c)]. The composition $\mathrm{Out}(H_\sigma) \to \mathrm{Out}(\mathcal{H}_\sigma)$ of μ with κ (see Section 2.2) is an isomorphism with $q_l = 5^{2^l}$ by [BMO19, Proposition 5.16]. Thus, $\mathcal{H}_q$ is tamely realized by H_σ by Lemma 2.40 and the definition of tame (Definition 2.12).

Set $k = l + 2$ as before. For the sake of convenience, we make appeals to [BMO19, §5] also for (b). Note that by choice of q_l, H_σ satisfies Hypotheses 5.1(III.1) of that reference. Let α be an automorphism of H_σ that normalizes S_σ and centralizes R_σ. Since $R_\sigma \geqslant T_{2^k}$, α centralizes T_{2^k}. Thus, by [BMO19, Lemma 5.9], $\alpha \in \mathrm{Inndiag}(H_\sigma) = \mathrm{Inn}(H_\sigma)\,\mathrm{Aut}_T(H_\sigma)$, and so there is $h \in H_\sigma$ and $t \in T$ such that α is conjugation by ht. Then also $h \in C_H(T_{2^k}) = T$, with the last equality by [BMO19, Lemma 5.3(a)], so that $ht \in T$. However, R_σ contains the element w_0 inverting T_{2^k}, c.f. Lemma 2.38(a), and so it follows that $ht \in \Omega_1(T_{2^k}) = Z(R_\sigma)$. □

Lemma 2.42. *The following hold.*

(a) The collection $\{\mathcal{F}_{\mathrm{Sol}}(q_l) \mid l \geqslant 0\}$ gives a nonredundant list of the isomorphism types of the 2-fusion systems $\mathcal{F}_{\mathrm{Sol}}(q)$ as q ranges over odd prime powers.

(b) The collection $\{\mathcal{F}_{\mathrm{Spin}}(q_l) \mid l \geqslant 0\}$ gives a nonredundant list of the isomorphism types of the 2-fusion systems $\mathcal{F}_{\mathrm{Spin}}(q)$ as q ranges over odd prime powers.

Proof. Part (a) is the content of [COS08, Theorem B]. For each odd prime power q, the fusion system of $\mathrm{Spin}_7(q)$ is isomorphic to some fusion system in the given collection by Lemma 2.41(a). Then (b) follows as a Sylow 2-subgroup of $\mathrm{Spin}_7(q_l)$ has order 2^{10+3l} by Lemma 2.38(a,b). □

The next lemma shows that $\mathcal{F}_\sigma$ has just one more essential subgroup in addition to the essential subgroups of $\mathcal{H}_\sigma$.

Lemma 2.43. *Let $P \in \mathcal{F}_\sigma^e$ be an essential subgroup. Then one of the following holds.*

(a) $\mathrm{Aut}_{\mathcal{F}_\sigma}(P) = \mathrm{Aut}_{\mathcal{H}_\sigma}(P)$ and P is $\mathcal{H}_\sigma$-essential, or

(b) $P = C_{S_\sigma}(U)$, $\mathrm{Aut}_{\mathcal{F}_\sigma}(P) = \langle \mathrm{Aut}_{\mathcal{H}_\sigma}(P), c_y \rangle$, and $\mathrm{Out}_{\mathcal{F}_\sigma}(P) \cong S_3$.

Proof. Recall that an essential subgroup in a fusion system is in particular both centric and radical. In [LS19], the centric radical subgroups and their outer automorphism groups in $\mathcal{H}_\sigma$ and $\mathcal{F}_\sigma$ are explicitly tabulated. From Tables 1 and 4 there, the only outer automorphism groups having a strongly embedded subgroup are S_3 and a Frobenius group of order $3^2 \cdot 2$. In all cases, either P is essential in $\mathcal{H}_\sigma$ and $\mathrm{Out}_{\mathcal{F}_\sigma}(P) = \mathrm{Out}_{\mathcal{H}_\sigma}(P)$ so that (a) holds, or $P = C_{S_\sigma}(U)$ and $\mathrm{Out}_{\mathcal{F}_\sigma}(P) \cong S_3$. In the latter case, $\mathrm{Aut}_{\mathcal{F}_\sigma}(P)$ is generated by $\mathrm{Aut}_{\mathcal{H}_\sigma}(P)$ and c_y essentially by the construction of y in Section 5 of [AC10], but it is difficult to find a precise statement of this claim. So instead, we appeal to [LO05] where $C_{S_\sigma}(U)$ is denoted $S_0(q^\infty)$ on p. 2400, and where c_y is explicitly constructed as the automorphism $\widehat{\gamma}_u$ of $C_{S_\sigma}(U)$ in [LO05, Definition 1.6]. There, Γ_n is used to denote $\mathrm{Aut}_{\mathcal{F}_\sigma}(C_{S_\sigma}(U))$ when $n = 2^l$. □

The following generation statements will be needed in the process of showing that a subintrinsic maximal Benson-Solomon subsystem is standard. The generation statement of Lemma 2.44(a) is the one which is obtained by the construction by Levi and Oliver in [LO05].

Lemma 2.44. *The following hold.*

(a) $\mathcal{F}_\sigma$ is generated by $\mathcal{H}_\sigma$ and $\mathrm{Aut}_{\mathcal{F}_\sigma}(C_{S_\sigma}(U))$.

(b) $\mathcal{F}_\sigma$ is generated by $\mathcal{H}_\sigma$ and $N_{\mathcal{F}_\sigma}(R_\sigma)$.

Proof. Part (a) follows from Lemma 2.43 and the Alperin-Goldschmidt fusion theorem [AKO11, Theorem I.3.6]. As $U \leqslant E$, $R_\sigma = C_{S_\sigma}(E) \leqslant C_{S_\sigma}(U)$. Further, T_{2^k} is abelian and weakly $\mathcal{F}_\sigma$-closed by Lemma 2.38(a), hence also $R_\sigma = C_{S_\sigma}(\Omega_1(T_{2^k}))$ is weakly $\mathcal{F}_\sigma$-closed. Thus, each element of $\mathrm{Aut}_{\mathcal{F}_\sigma}(C_{S_\sigma}(U))$ restricts to normalize R_σ, and hence lies in $N_{\mathcal{F}_\sigma}(R_\sigma)$. This shows that (b) is a consequence of (a). □

The next lemma augments the results of [HL18] on automorphisms and extensions of the Benson-Solomon systems.

Proposition 2.45. *Let $\mathcal{D}$ be a saturated fusion system over the 2-group D such that $F^*(\mathcal{D}) = \mathcal{F}_\sigma = \mathcal{F}_{\mathrm{Sol}}(q_l)$.*

(a) All involutions in $D - S_\sigma$ are $C_{\mathcal{D}}(z)$-conjugate, hence $\mathcal{D}$-conjugate.

(b) If $f \in D - S_\sigma$ is a fully $\mathcal{D}$-centralized involution, then

$$C_{\mathcal{F}_\sigma}(f) = \mathcal{F}_{\sigma_{l-1}} \cong \mathcal{F}_{\mathrm{Sol}}(q_{l-1}).$$

Proof. The almost simple extensions of $\mathcal{F}_\sigma$ were determined in [HL18]. By [HL18, Theorem 3.10, Theorem 4.3], we have $O^2(\mathcal{D}) = \mathcal{F}_\sigma$. We may further fix a complement F to S_σ in D such that F is cyclic of order 2^{l_0} with $1 \leqslant l_0 \leqslant l$, and such that the conjugation action of F on S_σ is the restriction of the conjugation action of a group of field automorphisms of H_σ to S_σ. We may thus assume that $|F| = 2$, and that F is generated by the restriction of σ_{l-1} to S_σ. Write f for the automorphism $\sigma_{l-1}|_{H_\sigma}$ of H_σ, and also write f for its restriction to S_σ. Note that σ_{l-1} normalizes H_σ and S_σ by Lemma 2.35(a). We also rely on Lemma 2.35(a) at many places in the proof below, without explicitly saying so. By Lemma 2.41(a) and Theorem 2.13,

(2.46) $\qquad C_{\mathcal{D}}(z)$ is the fusion system of the extension $H_\sigma\langle f\rangle$,

a semidirect product.

Recall $k = l + 2$ as before, and let $H_1 = N_H(H_\sigma)$. Then $H_1/Z(H_\sigma) \cong \mathrm{Inndiag}(H_\sigma)$ by [GLS98, Lemma 2.5.9(b)], and hence $H_1 = H_\sigma N_T(H_\sigma)$. As $\mathrm{Outdiag}(H_\sigma)$ has order 2, we may fix $t \in N_T(H_\sigma) - H_\sigma$ with order 2^{k+1} and powering to z, so that $H_1 = H_\sigma\langle t\rangle$. As σ_{l-1} normalizes H_σ and T, it restricts to an automorphism of H_1. Set $g = \sigma_{l-1}|_{H_1}$, so that g has order 4, and $g|_{H_\sigma} = f$ has order 2. Set $J_1 := H_1\langle g\rangle$, $J := H_\sigma\langle f\rangle$, $\widehat{J}_1 = J_1/C_{J_1}(H_\sigma)$, and $\widetilde{J} = J/Z(H_\sigma)$. Note that $C_{J_1}(H_\sigma) = \langle g^2, z\rangle$. Since g^2 centralizes H_σ, $\langle g^2\rangle$ is normal in $H_\sigma\langle g\rangle$, and $H_\sigma\langle g\rangle/\langle g^2\rangle \cong H_\sigma\langle f\rangle$ via an isomorphism which sends $g\langle g^2\rangle$ to f. Hence, there is an isomorphism $\widehat{H}_\sigma\langle\hat{g}\rangle \to \widetilde{H}_\sigma\langle\tilde{f}\rangle = \widetilde{J}$ which is the identity on $\widehat{H}_\sigma = H_\sigma/Z(H_\sigma) = \widetilde{H}_\sigma$ and which sends $\widehat{g}$ to $\widetilde{f}$.

By [GLS98, Theorem 4.9.1(d)], $\mathrm{Inndiag}(H_\sigma) \cong \widehat{H}_1$ acts transitively on the involutions in $\widehat{H}_\sigma\hat{g} - \widehat{H}_\sigma$, and so each involution in $\widehat{H}_\sigma\hat{g}$ is $\widehat{H}_\sigma$-conjugate to $\hat{g}$ or $\hat{g}^{\hat{t}}$. As $t^g = t^{\sigma_{l-1}} = t^{5^{2^{l-1}}}$ and $5^{2^{l-1}} - 1$ has 2-adic valuation $l+1 = k-1$, we see that there is an element $u \in \langle t^{2^{k-1}}\rangle \leqslant H_\sigma$ of order 4, such that $[g,u] = 1$, $u^2 = z$, and $t^g = tu$. Then $g^t = gu^{-1}$, so that $\hat{g}^{\hat{t}} = \hat{g}\hat{u}^{-1}$. From the isomorphism $\widehat{H}_\sigma\langle\widehat{g}\rangle \cong \widetilde{H}_\sigma\langle\tilde{f}\rangle$, we conclude that each involution in $\widetilde{H}_\sigma\langle\tilde{f}\rangle$ is $\widetilde{H}_\sigma$-conjugate to either of $\tilde{f}$ or $\tilde{f}\tilde{u}^{-1}$. However, the two preimages of $\tilde{f}\tilde{u}^{-1}$ in $H_\sigma\langle f\rangle$ are fu^{-1} and $fu = fu^{-1}z$, both of which are of order 4 as $[f,u] = 1$. Thus, all four subgroups of $H_\sigma\langle f\rangle$ which contain $\langle z\rangle$ and are not contained in H_σ are H_σ-conjugate. Since $\langle f, z\rangle$ is such a four subgroup, it is enough to show that f is H_σ-conjugate to fz. But $f^s = fz$ where $s = t^2 \in H_\sigma$. This completes the proof that all involutions in $H_\sigma f - H_\sigma$ are H_σ-conjugate, and this implies (a).

It remains to prove (b). We keep the notation from above, writing f for $\sigma_{l-1}|_{H_\sigma}$ and for $\sigma_{l-1}|_{S_\sigma}$.

We first prove that $\langle f\rangle$ itself is fully $\mathcal{D}$-centralized. By Lemma 2.35(c), the 2-group S_σ has a decomposition $S_\sigma = T_{2^k}W_S$ such that f centralizes W_S and acts on T_{2^k} via the map $t \mapsto t^{q_{l-1}}$. In particular $C_{S_\sigma}(f) = T_{2^{k-1}}W_S$ is a Sylow 2-subgroup of $H_{\sigma_{l-1}}$. The centralizer $C_{H_\sigma}(f) = H_{\sigma_{l-1}}$ is isomorphic to $\mathrm{Spin}_7(q_{l-1})$ by [GLS98, Theorem 4.9.1(a)], so $C_{C_D(z)}(f) = C_D(f) = C_{S_\sigma}(f)\langle f\rangle$ is a Sylow 2-subgroup of $C_{H_\sigma\langle f\rangle}(f)$. Hence, f is fully $C_{\mathcal{D}}(z)$-centralized by (2.46), and

$$C_{C_{\mathcal{D}(z)}}(f) \cong \mathcal{H}_{\sigma_{l-1}} \times \langle f\rangle, \tag{2.47}$$

by [AKO11, I.5.4]. However, all involutions in $D - S_\sigma$ are $C_{\mathcal{D}}(z)$-conjugate by (a), so two involutions in $D - S_\sigma$ are $C_{\mathcal{D}}(z)$-conjugate if and only if they are $\mathcal{D}$-conjugate. This completes the proof that $\langle f\rangle$ is fully $\mathcal{D}$-centralized.

Next, since $\langle f\rangle$ is fully $\mathcal{D}$-centralized, $C_{\mathcal{D}}(f)$ is saturated. Lemma 2.8 then shows $C_{\mathcal{F}_\sigma}(f)$ is normal in $C_{\mathcal{D}}(f)$, so that $C_{C_{\mathcal{F}_\sigma}(f)}(z)$ is normal in $C_{C_{\mathcal{D}}(f)}(z)$ by Lemma 2.8 applied with $C_{\mathcal{D}}(z)$ in the role of $\mathcal{F}$. Observe that $C_{C_{\mathcal{F}_\sigma}(f)}(z)$ has Sylow group $S_{\sigma_{l-1}} = C_{S_\sigma}(f)$. Consequently, we have

$$C_{C_{\mathcal{F}_\sigma}(f)}(z) = \mathcal{H}_{\sigma_{l-1}}. \tag{2.48}$$

from (2.47).

Now Lemma 2.9 shows that it suffices to determine $C_{\mathcal{F}_\sigma}(f)$ in order to finish the proof of (b). For if $f' \in D - S_\sigma$ is another involution fully centralized in $\mathcal{D}$, then any element φ of $\mathfrak{A}_{\mathcal{D}}(f)$ with $f^\varphi = f'$ will induce an isomorphism $C_{\mathcal{F}_\sigma}(f) \to C_{\mathcal{F}_\sigma}(f')$ by that lemma.

We argue next within the Aschbacher-Chermak amalgam to show that $C_{\mathcal{D}}(f)$ contains $\mathcal{F}_{\sigma_{l-1}}$. Now $\mathcal{F}_{\sigma_{l-1}}$ is generated by $\mathcal{H}_{\sigma_{l-1}}$ and $\mathrm{Aut}_{\mathcal{F}_{\sigma_{l-1}}}(C_{S_{\sigma_{l-1}}}(U))$ from Lemma 2.44(a). So to prove that $C_{\mathcal{D}}(f)$ contains $\mathcal{F}_{\sigma_{l-1}}$, we are reduced via (2.47) to a verification that $C_{\mathcal{D}}(f)$ contains $\mathrm{Aut}_{\mathcal{F}_\sigma}(C_{S_{\sigma_{l-1}}}(U))$. Now y (in the notation of Theorem 2.34) acts on $C_{S_{\sigma_{l-1}}}(U)$. Viewing y in the semidirect product of $K_\sigma\langle\sigma_{l-1}\rangle|_{K_\sigma}$ and applying Theorem 2.34 with σ_{l-1} in the role of σ_l, we see that y centralizes σ_{l-1}, so that c_y extends to the automorphism c_y of $C_{S_{\sigma_{l-1}}}(U)\langle f\rangle$ that centralizes f. So as $\mathrm{Aut}_{\mathcal{F}_{\sigma_{l-1}}}(C_{S_{\sigma_{l-1}}}(U))$ is generated by $\mathrm{Aut}_{\mathcal{H}_{\sigma_{l-1}}}(U)$ and c_y by Lemma 2.43(b), it follows that $\mathrm{Aut}_{\mathcal{F}_{\sigma_{l-1}}}(C_{S_{\sigma_{l-1}}}(U))$ is contained in $C_{\mathcal{D}}(f)$. Lemma 2.44(a) with σ_{l-1} in the role of σ shows that $C_{\mathcal{D}}(f)$ contains $\mathcal{F}_{\sigma_{l-1}}$. We have $\mathcal{D} = \mathcal{F}_\sigma\langle f\rangle$, since $\mathcal{F}_\sigma = O^2(\mathcal{D})$. So also $C_{\mathcal{F}_\sigma}(f) \geqslant O^2(C_{\mathcal{D}}(f)) \geqslant O^2(\mathcal{F}_{\sigma_{l-1}}) = \mathcal{F}_{\sigma_{l-1}}$. This shows that $C_{\mathcal{F}_\sigma}(f)$ contains $\mathcal{F}_{\sigma_{l-1}}$.

Finally, we complete the proof by appealing to Holt's Theorem for fusion systems [Asc17, Theorem 2.1.9]. The systems $C_{\mathcal{F}_\sigma}(f)$ and $\mathcal{F}_{\sigma_{l-1}}$ both have Sylow group $C_{S_\sigma}(f)$. Further, all involutions in $\mathcal{F}_{\sigma_{l-1}}$ are conjugate by (a), and so $z^{C_{\mathcal{F}_\sigma}(f)} = z^{\mathcal{F}_{\sigma_{l-1}}}$ for the involution $z \in Z(S_{\sigma_{l-1}})$. Moreover, we showed in (2.48) that $C_{C_{\mathcal{F}_\sigma}(f)}(z) = \mathcal{H}_{\sigma_{l-1}} \leqslant \mathcal{F}_{\sigma_{l-1}}$. This completes the verification of the hypotheses of Holt's Theorem, and by that theorem we have $C_{\mathcal{F}_\sigma}(f) = \mathcal{F}_{\sigma_{l-1}}$. □

We close this section by verifying that the Benson-Solomon systems are split. This allows one, via Theorem 8 of [Asc19], to severely restrict the Sylow subgroup of the centralizer of a Benson-Solomon standard subsystem.

Lemma 2.49. *$\mathcal{F}_\sigma$ is split.*

Proof. Let $(\mathcal{F}, V)$ be a critical split extension of $\mathcal{F}_\sigma$, where $\mathcal{F}$ is a saturated fusion system over any finite 2-group S. Let $\mathcal{L}_\sigma$ be the centric linking system for $\mathcal{F}_\sigma$. Note that $S/C_S(\mathcal{F}_\sigma)S_\sigma$ embeds in $\mathrm{Out}(\mathcal{L}_\sigma)$ by [Sem15, Theorem A], while $\mathrm{Out}(\mathcal{L}_\sigma)$ is cyclic of 2-power order by [HL18, Theorem 3.10]. Hence

$$V \cap C_S(\mathcal{F}_\sigma)S_\sigma > 1. \tag{2.50}$$

Write $V = \langle u, v\rangle$ with $u \in V \cap C_S(\mathcal{F}_\sigma)S_\sigma$. Then either $V \cap C_S(\mathcal{F}_\sigma) > 1$, or there exist elements $c \in C_S(\mathcal{F}_\sigma)$ and $1 \neq t \in S_\sigma$ such that $u = ct$.

Assume the former case, and set $X = V \cap C_S(\mathcal{F}_\sigma) > 1$. As X commutes with V and S_σ and $S = S_\sigma V$, we have $X \leqslant Z(S)$. Thus, $C_{\mathcal{F}}(X)$ is a saturated subsystem over S which contains $\mathcal{F}_\sigma$, so $C_{\mathcal{F}}(X) = \mathcal{F} = N_{\mathcal{F}}(X)$. Applying condition (T1) in Definition 2.30 with $Q = V$, $\mathcal{Q} = \mathcal{F}_V(V)$, $X = V \cap C_S(\mathcal{F}_\sigma)$, and $\alpha = \mathrm{id}_X$, we see that V is normal in $\mathcal{F}$, and hence that $V \leqslant Z(\mathcal{F})$ by Lemma 2.33(a). Therefore, $\mathcal{F}$ is the central product of $V = C_S(\mathcal{F}_\sigma)$ and $\mathcal{F}_\sigma$.

Consider the latter case. As $C_S(\mathcal{F}_\sigma) \cap S_\sigma \leqslant Z(\mathcal{F}_\sigma) = 1$ and u is an involution, t is an involution. Then, as $\mathcal{F}_\sigma$ has one class of involutions by Lemma 2.39, u is $\mathcal{F}$-conjugate to $cz \in Z(S)$. However, $\langle u\rangle$ is itself fully $\mathcal{F}$-centralized by Lemma 2.33(a), and so $u \in Z(S)$. As $\langle v\rangle$ is fully $\mathcal{F}$-centralized and $C_{\mathcal{F}}(u) = C_{\mathcal{F}}(v)$ by Lemma 2.33, we have $v \in Z(S)$. But then, using Lemma 2.40 to see that Proposition 2.26 applies, we have $V \leqslant Z(S) = C_S(S_\sigma) = C_S(\mathcal{F}_\sigma)Z(S_\sigma)$ by that proposition applied with $\mathcal{F}_1 = \mathcal{F}$, so that $C_S(\mathcal{F}_\sigma)S_\sigma = VS_\sigma = S$. Thus, $\mathcal{F}$ is the central product of $C_S(\mathcal{F}_\sigma)$ with $\mathcal{F}_\sigma$ in this case as well. □

2.9. **The known quasisimple groups and quasisimple 2-fusion systems.** In this section, we define the class Chev[large] of 2-fusion systems that appears in Walter's Theorem and which is needed in Section 7, and we prove two lemmas about components and involution centralizers in known almost quasisimple groups that will be used in Sections 5 and 7. By a *known finite simple group* we mean a finite group isomorphic to one of the groups appearing in the statement of the classification of finite simple groups. By a *known quasisimple group* we mean a quasisimple covering of a known finite simple group. Such coverings are listed in [GLS98, Section 6.1]. An *almost simple group* is a finite group whose generalized Fitting subgroup is simple. Similarly, an *almost quasisimple group* is a finite group whose generalized Fitting subgroup is quasisimple.

Definition 2.51. Let p be an odd prime. Chev(p) is the class of quasisimple groups of Lie type in characteristic p, i.e, the quasisimple groups K for which there is a simple algebraic group $\overline{L}$ over $\overline{\mathbb{F}}_p$ and a Steinberg endomorphism σ such that $K/Z(K) \cong C_{\overline{L}}(\sigma)'$ (cf. [GLS98, Definition 2.2.8]), and

$$\begin{aligned}\mathrm{Chev}^*(p) = \mathrm{Chev}(p) &- \{L_2(q) \mid q = p^a, q \geqslant 5\}\\ &- \{{}^2G_2(q) \mid q = 3^{2a+1}, a \geqslant 1\}\\ &- \{{}^2G_2(3)'\}.\end{aligned}$$

Thus, the groups $SL_2(q)$ for $q = p^a \geqslant 5$ lie in Chev$^*(p)$ (but $SL_2(3)$ does not). The above working definition of Chev$^*(p)$ appears to be consistent with that in [Asc20] and [Asc17], but our main interest is in Chev[large] below.

We use analogous terminology for the class of known simple, quasisimple, almost simple, and almost quasisimple 2-fusion systems, respectively. The class of *known simple 2-fusion systems* consists of the fusion systems of simple groups whose 2-fusion systems are simple, together with the Benson-Solomon fusion systems. As was shown in [Asc17, Theorem 5.6.18], the simple groups whose 2-fusion systems fail to be simple are precisely the *Goldschmidt groups*, namely the finite simple groups which have a nontrivial strongly closed abelian 2-subgroup. By a result of Goldschmidt, a Goldschmidt group is either a group of Lie type in characteristic 2 of Lie rank 1, or has abelian Sylow 2-subgroups. For example, $\Omega_7(q)$ is not a Goldschmidt group, so its 2-fusion system is simple for any odd prime power q.

In the proof of Walter's Theorem, and in Section 7 below, the 2-fusion systems of a certain subclass of $\mathrm{Chev}^*(p)$ consisting of "large" groups play an important role.

Definition 2.52 ([Asc20, Definition 1.1]). Let Chev^* be the union of $\mathrm{Chev}^*(p)$ as p ranges over the odd primes. Chev[large] is the class of quasisimple 2-fusion systems $\mathcal{K}$ such that $\mathcal{K} = \mathcal{F}_2(K)$ for some finite group K in the collection

$$\begin{aligned}\mathrm{Chev}^* &- \{L_3^\epsilon(q) \mid q \equiv \pm 3 \pmod 8,\ \epsilon = \pm 1\}\\ &- \{G_2(q) \mid q \equiv \pm 3 \pmod 8\}\\ &- \{P\Omega_n^\epsilon(q) \mid q \equiv \pm 3 \pmod 8,\ \epsilon = \pm 1,\ 5 \leqslant n \leqslant 8\}\\ &- \{\Omega_6^+(q) \mid q \equiv \pm 3 \pmod 8\}.\end{aligned}$$

For example, the 2-fusion system of $\Omega_7(q)$, $q \equiv \pm 3 \pmod 8$ is excluded from Chev[large], while the 2-fusion system of $\mathrm{Spin}_7(q)$ is a member of Chev[large] for each odd prime power q.

Lemma 2.53. *Let G be a finite group with $F^*(G)$ a known quasisimple group. Then for each involution $t \in G$ and each component $\bar{L}$ of $C_G(t)/O(C_G(t))$, $\bar{L}$ is a known quasisimple group.*

Proof. This follows from the determination of the conjugacy classes of involutions and their centralizers in the known almost quasisimple groups in [GLS98]. More precisely, see Tables 4.5.1-4.5.3, Section 4.9, and Corollary 3.1.4 of [GLS98] for these data with regard to the members of Chev, Table 5.3 of [GLS98] for the sporadic groups and their covers, and Section 5.2 of [GLS98] for the alternating groups and their covers. □

The next lemma says that each component of the 2-fusion system of a group G with $O(G) = 1$ is the fusion system of a component of the group provided the components of G are known finite quasisimple groups. It was suggested to us by Aschbacher.

Lemma 2.54. *Let G be a finite group with $O(G) = 1$. Assume that for each component K of G, $K/Z(K)$ is a known finite simple group. Let $S \in \mathrm{Syl}_2(G)$ and let $\mathcal{C}$ be a component of $\mathcal{F}_S(G)$. Then there exists a component K of G such that $\mathcal{C} = \mathcal{F}_{S\cap K}(K)$.*

Proof. Set $\mathcal{F} = \mathcal{F}_S(G)$ and $\mathcal{E} = \mathcal{F}_{S\cap F^*(G)}(F^*(G))$. Suppose $\mathcal{C}$ is a subsystem of $\mathcal{F}$ on T. As $F^*(G)$ is normal in G, the subsystem $\mathcal{E}$ is normal in $\mathcal{F}$ by [AKO11, Proposition I.6.2].

Assume first that $\mathcal{C}$ is not a component of $\mathcal{E}$. Write J for the set of components of $\mathcal{F}$ which are not a component of $\mathcal{E}$, and set $\mathcal{D} = \Pi_{\mathcal{C}'\in J}\mathcal{C}'$. Then by [Asc11, 9.13], $\mathcal{F}$ contains a subsystem $\mathcal{DE}$ which is the central product of $\mathcal{D}$ and $\mathcal{E}$. As $\mathcal{C} \in J$, this implies in particular that $\mathcal{E} \leqslant C_{\mathcal{F}}(T)$. Since $\mathcal{E} = \mathcal{F}_{S\cap F^*(G)}(F^*(G))$, it follows now from [HS15, Theorem B] that $T \leqslant C_S(F^*(G)) \leqslant C_G(F^*(G)) = Z(F^*(G))$. In particular, T is abelian, which by [Asc11, 9.1] yields a contradiction to $\mathcal{C}$ being quasisimple. Thus, we have shown that $\mathcal{C}$ is a component of $\mathcal{E}$.

As $O(G) = 1$, $F^*(G)$ is the central product of $O_2(G)$ and the components of G. Thus, $\mathcal{E}$ is a central product of $\mathcal{F}_{O_2(G)}(O_2(G))$ and the subsystems of the form $\mathcal{F}_{S\cap K}(K)$ where K is a component of G. Since $\mathcal{C}$ is not the fusion system of a 2-group, it follows now from Lemma 2.15

that $\mathcal{E}$ is a component of $\mathcal{F}_{S\cap K}(K)$ for some component K of G. Set $\overline{K} := K/Z(K)$. As $O(G) = 1$, we have that $Z(K) \leqslant S$ and $Z(K)$ is contained in the centre of $\mathcal{F}_{S\cap K}(K)$. Moreover, $\mathcal{F}_{S\cap K}(K)/Z(K) = \mathcal{F}_{\overline{K\cap S}}(\overline{K})$. Recall that $\overline{K}$ is a "known" finite simple group. So by [Asc17, Theorem 5.6.18], $\mathcal{F}_{\overline{S\cap K}}(\overline{K})$ is either simple or $\overline{S \cap K}$ is normal in $\overline{K}$. As the fusion system $\mathcal{F}_{S\cap K}(K)$ contains $\mathcal{C}$ as a component, it is not constrained by [Asc11, 9.9.1]. It follows thus from [Hen19, Lemma 2.10] that $\mathcal{F}_{\overline{K\cap S}}(\overline{K})$ is not constrained, which excludes the case that $\overline{S \cap K}$ is normal in $\mathcal{F}_{\overline{S\cap K}}(\overline{K})$. Hence, $\mathcal{F}_{S\cap K}(K)/Z(K) = \mathcal{F}_{\overline{S\cap K}}(\overline{K})$ is simple. In particular, $Z(K) = Z(\mathcal{F}_{S\cap K}(K))$. As K is quasisimple, we have $K = O^2(K)$. Therefore, it follows from Puig's hyperfocal subgroup theorem [Pui00, §1.1] and [AKO11, Corollary I.7.5] that $O^2(\mathcal{F}_{S\cap K}(K)) = \mathcal{F}_{S\cap K}(K)$. So $\mathcal{F}_{S\cap K}(K)$ is quasisimple and thus, by [Asc11, 9.4], we have $\mathcal{C} = \mathcal{F}_{S\cap K}(K)$. □

2.10. **Summary of preliminary definitions.** For the convenience of the reader, we summarize in a quick-reference table the most important definitions from Section 2 for following the later arguments.

Table 1: Summary of Section 2

Notation/Property	**Assumption**	**Description**	**Reference**
$\mathfrak{A}(X) = \mathfrak{A}_{\mathcal{F}}(X)$	$X \leqslant S$	set of $\alpha \in \mathrm{Hom}_{\mathcal{F}}(N_S(X), S)$ with $X^\alpha \in \mathcal{F}^f$	Notation 2.1
$N_{\mathcal{E}}(P)$	$\mathcal{E} \trianglelefteq \mathcal{F}$, $P \leqslant S$ with $P \in (\mathcal{E}P)^f$	unique normal subsystem of $N_{\mathcal{E}P}(P)$ over $N_T(P)$ of p-power index	Definition 2.7
component of $\mathcal{F}$		quasisimple subnormal subsystem of $\mathcal{F}$	Section 2.3
$E(\mathcal{F})$		central product of the components of $\mathcal{F}$	
$F^*(\mathcal{F})$		central product of $O_p(\mathcal{F})$ and $E(\mathcal{F})$	
$\mathcal{X}(\mathcal{C}) = \mathcal{X}_{\mathcal{F}}(\mathcal{C})$	$\mathcal{C}$ is a quasisimple subsystem of $\mathcal{F}$ over $T \leqslant S$	set of elements or subgroups X of $C_S(T)$ with $\mathcal{C} \leqslant C_{\mathcal{F}}(X)$	Notation 2.16
$\tilde{\mathcal{X}}(\mathcal{C}) = \tilde{\mathcal{X}}_{\mathcal{F}}(\mathcal{C})$		set of elements or subgroups X of S such that $\mathcal{C}^\alpha$ is a component of $N_{\mathcal{F}}(X^\alpha)$ for some $\alpha \in \mathfrak{A}(X)$	
$\mathcal{I}(\mathcal{C}) = \mathcal{I}_{\mathcal{F}}(\mathcal{C})$		set of involutions in $\tilde{\mathcal{X}}(\mathcal{C})$	

Table 1: Summary of Section 2 (continued from previous page)

Notation/Property	**Assumption**	**Description**	**Reference**
$\mathfrak{C}(\mathcal{F})$		set of quasisimple subsystems $\mathcal{C}$ of $\mathcal{F}$ with $\mathcal{I}(\mathcal{C}) \neq \varnothing$	Notation 2.16
$\mathcal{D}$ proper pump-up of $\mathcal{C}$	$\mathcal{C} \in \mathfrak{C}(\mathcal{F})$, $\mathcal{D}$ a subsystem of $\mathcal{F}$	hypothesis and conclusion (2) of the "Pump-Up Lemma" 2.19 hold	Definition 2.20
$\mathcal{C}$ maximal	$\mathcal{C} \in \mathfrak{C}(\mathcal{F})$	no proper pump-up of $\mathcal{C}$	
$P_{t,\alpha}$	$\mathcal{C} \in \mathfrak{C}(\mathcal{F})$, $t \in \mathcal{I}(\mathcal{C})$, $\alpha \in \mathfrak{A}(t)$	$P_{t,\alpha} := C_{C_S(t^\alpha)}(\mathcal{C}^\alpha) \cap C_S(t)^\alpha$	Notation 2.21
$Q_t := Q_{t,\alpha}$		$Q_{t,\alpha} := P_{t,\alpha}^{\alpha^{-1}}$	
$\Delta(\mathcal{C})$	$\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ a subsystem over $T \leqslant S$	set of $\mathcal{F}$-conjugates $\mathcal{C}_1$ of $\mathcal{C}$ over some $T_1 \leqslant S$ such that $T_1^\# \subseteq \tilde{\mathcal{X}}(\mathcal{C})$ and $T^\# \subseteq \tilde{\mathcal{X}}(\mathcal{C}_1)$	Notation 2.22
$\rho(\mathcal{C})$		set of all $(t^\varphi, \mathcal{C}^\varphi)$ where $t \in \mathcal{I}(\mathcal{C})$ and $\varphi \in \mathrm{Hom}_\mathcal{F}(\langle t, T \rangle, S)$	
$\mathcal{C}$ terminal		(C1) $T \in \mathcal{F}^f$, (C2) $\Delta(\mathcal{C}) \neq \varnothing$, (C3) $(t_1, \mathcal{C}_1) \in \rho(\mathcal{C}) \Longrightarrow Q_{t_1}^\# \subseteq \tilde{\mathcal{X}}(\mathcal{C}_1)$	Definition 2.23
$\mathcal{C}$ nearly standard		(S1) $\tilde{\mathcal{X}}(\mathcal{C})$ contains a unique maximal member Q (S2) $\mathcal{C}^\alpha \trianglelefteq N_\mathcal{F}(X^\alpha)$ for each $1 \neq X \leqslant Q$ and $\alpha \in \mathfrak{A}(X)$ (S3) $T^\beta = T$ for each $1 \neq X \leqslant Q$ and $\beta \in \mathfrak{A}(X)$ with $X^\beta \leqslant Q$	Definition 2.24
$\mathcal{C}$ standard		$\mathcal{C}$ nearly standard and (S4) $\mathrm{Aut}_\mathcal{F}(T) \leqslant \mathrm{Aut}(\mathcal{C})$ (ensures existence of a "centralizer" of $\mathcal{C}$, in many cases fulfilled if $\mathcal{C}$ terminal)	
$\mathcal{C}$ subintrinsic		there is $\mathcal{H} \in \mathfrak{C}(\mathcal{C})$ with $\mathcal{I}_\mathcal{F}(\mathcal{H}) \cap Z(\mathcal{H}) \neq \varnothing$	Definition 2.29

Table 1: Summary of Section 2 (continued from previous page)

Notation/Property	**Assumption**	**Description**	**Reference**
$\mathcal{Q}$ tightly embedded	$\mathcal{Q}$ subsystem over $Q \in \mathcal{F}^f$	(T1) $O^{p'}(N_{\mathcal{Q}}(X))^{\alpha}$ is normal in $N_{\mathcal{F}}(X^{\alpha})$ for each $1 \neq X \in \mathcal{Q}^f$ and $\alpha \in \mathfrak{A}(X)$ (T2) $X^{\mathcal{F}} \cap Q = X^{\mathrm{Aut}_{\mathcal{F}}(Q)\mathcal{Q}}$ for each X of order p. (T3) $\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{Aut}(\mathcal{Q})$ ((T1)–(T3) hold if $\mathcal{Q}$ is the "centralizer" of standard subsystem)	Definition 2.30

3. Subintrinsic maximal Benson-Solomon components

We assume the following hypothesis throughout this section.

Hypothesis 3.1. Let $\mathcal{F}$ be a saturated fusion system over the 2-group S, and let q be an odd prime power. Suppose $\mathcal{C} \cong \mathcal{F}_{\mathrm{Sol}}(q)$ is a subsystem of $\mathcal{F}$ over the subgroup $T \in \mathcal{F}^f$. Let $z \in Z(T)$ be the involution, set $\mathcal{H} = C_{\mathcal{C}}(z)$. Assume $\mathcal{C}$ is maximal in $\mathfrak{C}(\mathcal{F})$ and $z \in \mathcal{I}_{\mathcal{F}}(\mathcal{H})$, but that $\mathcal{C}$ is not a component of $\mathcal{F}$.

As we explain in detail in Remark 3.3 below, this hypothesis amounts to assuming that the pair $(\mathcal{F}, \mathcal{C})$ is a counterexample to Theorem 1. The purpose of this section is to prove the following theorem.

Theorem 3.2. *Assume Hypothesis 3.1. Then $\mathcal{C}$ is standard.*

Proof. This is the content of Lemmas 3.8 and 3.9 below. □

By Lemma 2.42, we may take $q = 5^{2^l}$ for a unique $l \geqslant 0$ in Hypothesis 3.1. Since we will be working in this section with the internal structure of $\mathcal{C} \cong \mathcal{F}_{\mathrm{Sol}}(q)$ in some detail, we shall make the following identifications in order to make it simpler to apply the results of Section 2.8. Here, $\sigma = \sigma_l$ is the automorphism of the Aschbacher-Chermak free amalgamated product appearing in Theorem 2.34.

Hypothesis 3.1	Identify with	Reference in §§2.8	Note
T	S_σ	Lemma 2.35(c)	T not to be confused with a maximal torus
$\mathcal{C}$	$\mathcal{F}_\sigma$	Notation 2.36	
$\mathcal{H}$	$\mathcal{H}_\sigma$	Notation 2.36	

We reiterate that the symbol S is now used for the Sylow group in the ambient system $\mathcal{F}$ of Hypothesis 3.1. (Previously, it stood for the Sylow 2-subgroup of the Aschbacher-Chermak free amalgamated product.)

Remark 3.3. Apart from the assumption that $\mathcal{C}$ is not a component of $\mathcal{F}$, Hypothesis 3.1 is a restatement of the hypotheses of Theorem 1. The assumption $z \in \mathcal{I}(\mathcal{H}) = \mathcal{I}_{\mathcal{F}}(\mathcal{H})$ in Hypothesis 3.1 is equivalent to the condition that $\mathcal{C}$ is subintrinsic in $\mathfrak{C}(\mathcal{F})$, because z is the unique involution in $Z(\mathcal{H})$ and $\mathfrak{C}(\mathcal{C}) = \{\mathcal{H}\}$. To see the latter property recall that every involution in T is $\mathcal{C}$-conjugate to z by Lemma 2.39 and that z is fully normalized, as $z \in Z(T)$. Therefore, each member of $\mathfrak{C}(\mathcal{C})$ is $\mathcal{C}$-conjugate to a component of $\mathcal{H}$ by Lemma 2.2 and Lemma 2.17. As $\mathcal{H}$ is quasisimple, $\mathcal{H}$ is by [Asc11, 9.4] the only component of $\mathcal{H}$. Hence, every member of $\mathfrak{C}(\mathcal{C})$ is $\mathcal{C}$-conjugate to $\mathcal{H}$. For each $\alpha \in \mathrm{Hom}_{\mathcal{F}}(T,T)$, $z^\alpha = z$ as $Z(T) = \langle z\rangle$ is characteristic in T, and so $\mathcal{H}^\alpha = \mathcal{H}$. Hence, $\mathcal{H}$ is the only $\mathcal{C}$-conjugate of $\mathcal{H}$ and $\mathfrak{C}(\mathcal{C}) = \{\mathcal{H}\}$.

Lemma 3.4. *Let $\alpha \in \mathrm{Hom}_{\mathcal{F}}(T,S)$. Then $\mathcal{C}^\alpha$ is maximal in $\mathfrak{C}(\mathcal{F})$, $\mathcal{H}^\alpha = C_{\mathcal{C}^\alpha}(z^\alpha)$, $z^\alpha \in \mathcal{I}(\mathcal{H}^\alpha)$, and $\mathcal{C}^\alpha$ is not a component of $\mathcal{F}$.*

Proof. By Lemma 2.14(b), $\mathcal{C}^\alpha$ is not a component of $\mathcal{F}$ as $\mathcal{C}$ is not a component of $\mathcal{F}$. As $T \in \mathcal{F}^f$, it follows from [Asc19, 6.2.13] that $\mathcal{C}^\alpha$ is maximal in $\mathfrak{C}(\mathcal{F})$. As α induces an isomorphism from $\mathcal{C}$ to $\mathcal{C}^\alpha$, we have $\mathcal{H}^\alpha = C_{\mathcal{C}^\alpha}(z^\alpha)$. Since $z \in \mathcal{I}(\mathcal{H})$, Lemma 2.17(b) gives $z^\alpha \in \mathcal{I}(\mathcal{H}^\alpha)$. □

Lemma 3.5. *$\mathcal{C}$ is terminal in $\mathfrak{C}(\mathcal{F})$.*

Proof. By Hypothesis 3.1, $\mathcal{C}$ is maximal and subintrinsic in $\mathfrak{C}(\mathcal{F})$. Condition (C0) in Definition 2.23 is $T \in \mathcal{F}^f$, which holds by Hypothesis 3.1. Assume $\Delta(\mathcal{C}) \neq \varnothing$. Then in the notation of [Asc19, 6.1.17], the set $\mathcal{C}^\perp = \Delta(\mathcal{C}) \cup \{\mathcal{C}\} \neq \mathcal{C}$. Since $Z(\mathcal{C}) = 1$, Hypothesis 7.2.1 of [Asc19] holds. By [Asc19, Theorem 7.2.5], $\mathcal{C}$ is a component of $\mathcal{F}$, contrary to hypothesis. Thus, $\Delta(\mathcal{C}) = \varnothing$, i.e (C1) is verified. Since $m(T) = 4$ by Lemma 2.38(d), Theorem 7.4.14 of [Asc19] shows that $\Delta(\mathcal{C}) = \varnothing$ (where $\Delta(\mathcal{C})$ is defined as in Notation 2.22).

It remains to verify (C2). Let $t \in \mathcal{I}(\mathcal{C})$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\langle t, T\rangle, S)$ so that $(t^\varphi, \mathcal{C}^\varphi) \in \rho(\mathcal{C})$. Fixing $1 \neq a \in Q_{t^\varphi}$, we need to show that $a \in \tilde{\mathcal{X}}(\mathcal{C}^\varphi)$. Note that $a \in \mathcal{X}(\mathcal{C}^\varphi)$. So if $\tilde{a}$ is the unique involution in $\langle a\rangle$ and $\tilde{a} \in \tilde{\mathcal{X}}(\mathcal{C}^\varphi)$, then $a \in \tilde{\mathcal{X}}(\mathcal{C}^\varphi)$ by [Asc19, 6.1.5]. So we may assume without loss of generality that a is an involution. Fix $\alpha \in \mathfrak{A}(a)$. It remains to show that $\mathcal{C}^{\varphi\alpha}$ is a component of $C_{\mathcal{F}}(a^\alpha)$ and thus $a \in \tilde{\mathcal{X}}(\mathcal{C}^\varphi)$.

Note first that, by definition of Q_{t^φ}, $\mathcal{C}^\varphi \leqslant C_{\mathcal{F}}(a)$ and thus $\mathcal{C}^{\varphi\alpha} \leqslant C_{\mathcal{F}}(a^\alpha)$. By Lemma 2.17(b) applied with $(\langle t\rangle, \varphi\alpha)$ in place of (X, φ), we have $t^{\varphi\alpha} \in \tilde{\mathcal{X}}(\mathcal{C}^{\varphi\alpha})$. Moreover, $[t^\varphi, a] = 1$ by definition of Q_{t^φ} and thus $[t^{\varphi\alpha}, a^\alpha] = 1$. Hence, Lemma 2.18 yields $\mathcal{C}^{\varphi\alpha} \in \mathfrak{C}(C_{\mathcal{F}}(a^\alpha))$.

We will argue next that $\mathcal{C}^{\varphi\alpha}$ is subintrinsic in $\mathfrak{C}(C_{\mathcal{F}}(a^\alpha))$. By Lemma 3.4, we have $C_{\mathcal{C}^{\varphi\alpha}}(z^{\varphi\alpha}) = \mathcal{H}^{\varphi\alpha}$ and $z^{\varphi\alpha} \in \mathcal{I}(\mathcal{H}^{\varphi\alpha})$. Recall that $\mathcal{H}^{\varphi\alpha} \leqslant \mathcal{C}^{\varphi\alpha} \leqslant C_{\mathcal{F}}(a^\alpha)$. In particular, $[T^{\varphi\alpha}, a^\alpha] = 1$ and thus $[z^{\varphi\alpha}, a^\alpha] = 1$. Hence, by Lemma 2.18 applied with $(\langle z^{\varphi\alpha}\rangle, \langle a^\alpha\rangle, \mathcal{H}^{\varphi\alpha})$ in place of $(X, Y, \mathcal{C})$, we have $z^{\varphi\alpha} \in \mathcal{I}_{C_{\mathcal{F}}(a^\alpha)}(\mathcal{H}^{\varphi\alpha})$. As $z^{\varphi\alpha} \in Z(\mathcal{H}^{\varphi\alpha})$, this implies that $\mathcal{C}^{\varphi\alpha}$ is indeed subintrinsic in $\mathfrak{C}(C_{\mathcal{F}}(a^\alpha))$ as we wanted to prove.

As we have verified that $\mathcal{C}^{\varphi\alpha}$ is a subintrinsic member of $\mathfrak{C}(C_{\mathcal{F}}(a^\alpha))$, it follows now from [Asc20, 1.9.2] applied with $C_{\mathcal{F}}(a^\alpha)$ in the role of $\mathcal{F}$ and with $\mathcal{C}^{\varphi\alpha}$ in the role of $\mathcal{M}$ that $\mathcal{C}^{\varphi\alpha}$ is contained in some component of $C_{\mathcal{F}}(a^\alpha)$. Since $\mathcal{C}^\varphi$ is maximal in $\mathfrak{C}(\mathcal{F})$ by Lemma 3.4, it follows from Lemma 2.19 applied with $(t^\varphi, \mathcal{C}^\varphi)$ in place of the pair $(t, \mathcal{C})$ of that lemma that $\mathcal{C}^{\varphi\alpha}$ is a component of $C_{\mathcal{F}}(a^\alpha)$. As argued above this shows (a). □

For the remainder of this section, we adopt the following notation for certain subgroups of T. After this, we will not need to refer explicitly to any additional notation from in Section 2.8.

Notation below	Subgroup	Reference in §§2.8	Description
T_{2^k}	T_{2^k}	Notation 2.37	unique homocyclic subgroup of T of rank 3 and exponent 2^k
E	E	Notation 2.37	$\Omega_1(T_{2^k})$
R	R_σ	Notation 2.37	$R = C_T(E) = T_{2^k}\langle w_0\rangle$ for an involution w_0 inverting T_{2^k}

Lemma 3.6. *The following hold:*

(a) The subgroup E is characteristic in R and thus $\mathrm{Aut}_{\mathcal{F}}(R)$-invariant. We have $\mathrm{Aut}_{\mathcal{F}}(R) = C_{\mathrm{Aut}_{\mathcal{F}}(R)}(E)\,\mathrm{Aut}_{\mathcal{C}}(R)$ and $O_2(\mathrm{Aut}_{\mathcal{F}}(R)) = C_{\mathrm{Aut}_{\mathcal{F}}(R)}(E)$.

(b) We have $N_S(R) = N_S(T) = C_S(E)T$.

(c) R is fully $\mathcal{F}$-normalized.

(d) We have $\mathrm{Out}_{\mathcal{F}}(R) = O_2(\mathrm{Out}_{\mathcal{F}}(R)) \times \mathrm{Out}_{\mathcal{C}}(R)$ and $O_2(\mathrm{Aut}_{\mathcal{F}}(R)) = \mathrm{Aut}_{C_S(E)}(R)$. In particular, $O^2(\mathrm{Aut}_{\mathcal{F}}(R)) = O^2(\mathrm{Aut}_{\mathcal{C}}(R))$.

Proof. **(a)**: It follows from Lemma 2.38(a) that T_{2^k} and $E = \Omega_1(T_{2^k})$ are characteristic in R. Set $C := C_{\mathrm{Aut}_{\mathcal{F}}(R)}(E)$. Observe that $\mathrm{Aut}_{\mathcal{F}}(R)/C$ embeds into $\mathrm{Aut}(E) \cong GL_3(2)$. As $\mathrm{Aut}_{\mathcal{C}}(R)/\mathrm{Inn}(R) \cong GL_3(2)$ and $C_{\mathrm{Aut}_{\mathcal{C}}(R)}(E) = \mathrm{Inn}(R)$, it follows that $\mathrm{Aut}_{\mathcal{F}}(R) = C\,\mathrm{Aut}_{\mathcal{C}}(R)$. By Lemma 2.38(a), T_{2^k} is homocyclic of rank 3 and exponent 2^k. So as T_{2^k} is characteristic in R, for every $1 \leqslant i < k$, the map $\Omega_{i+1}(T_{2^k})/\Omega_i(T_{2^k}) \to \Omega_i(T_{2^k})/\Omega_{i-1}(T_{2^k}), x\Omega_i(T_{2^k}) \mapsto x^2\Omega_{i-1}(T_{2^k})$ is an isomorphism of $\mathrm{Aut}_{\mathcal{F}}(R)$-modules. So in particular, C acts trivially on $\Omega_{i+1}(T_{2^k})/\Omega_i(T_{2^k})$ for all $1 \leqslant i < k$. As $|R/T_{2^k}| = 2$, C acts also trivially on R/T_{2^k}. Hence, C is a 2-group and thus contained in $O_2(\mathrm{Aut}_{\mathcal{F}}(R))$. As E is an irreducible $\mathrm{Aut}_{\mathcal{F}}(R)$-module, it follows $C = O_2(\mathrm{Aut}_{\mathcal{F}}(R))$. This shows (a).

(b): As R is characteristic in T by Lemma 2.38(c), we have $N_S(T) \leqslant N_S(R)$. By (a), $C\,\mathrm{Aut}_T(R)$ is the unique Sylow 2-subgroup of $\mathrm{Aut}_{\mathcal{F}}(R)$ containing $\mathrm{Aut}_T(R)$. As $\mathrm{Aut}_S(R)$ is a 2-group containing $\mathrm{Aut}_T(R)$, it follows $\mathrm{Aut}_S(R) \leqslant C\,\mathrm{Aut}_T(R)$ and thus $N_S(R) \leqslant C_S(E)T \leqslant C_S(z)$. Let now $x \in C_S(E) \leqslant C_S(z)$. As $z \in \mathcal{I}(\mathcal{H})$, there exists $\alpha \in \mathfrak{A}(z)$ such that $\mathcal{H}^\alpha$ is a component of $C_{\mathcal{F}}(z^\alpha)$. Then $x^\alpha \in C_S(E^\alpha) \leqslant C_S(z^\alpha)$ and $(\mathcal{H}^\alpha)^{x^\alpha}$ is a component of $C_{\mathcal{F}}(z^\alpha)$ by Lemma 2.14. So by [Asc11, 9.8.2], either $\mathcal{H}^\alpha = (\mathcal{H}^\alpha)^{x^\alpha}$, or $\mathcal{H}^\alpha$ and $(\mathcal{H}^\alpha)^{x^\alpha}$ form a commuting product. In the latter case, $E^\alpha = (E^\alpha)^{x^\alpha} \leqslant Z(\mathcal{H}^\alpha)$, a contradiction to $Z(\mathcal{H}^\alpha) = \langle z^\alpha\rangle$. Hence, $\mathcal{H}^\alpha = (\mathcal{H}^\alpha)^{x^\alpha}$ and thus $(T^x)^\alpha = (T^\alpha)^{x^\alpha} = T^\alpha$. This implies $x \in N_S(T)$. So we have shown that $C_S(E) \leqslant N_S(T)$ and thus $N_S(R) \leqslant C_S(E)T \leqslant N_S(T) \leqslant N_S(R)$. This yields (b).

(c): Let $\gamma \in \mathfrak{A}(R)$. Recall from (b) that $T \leqslant N_S(T) = N_S(R)$. So in particular, as $T \in \mathcal{F}^f$, we have $T^\gamma \in \mathcal{F}^f$ and $N_S(T)^\gamma = N_S(T^\gamma)$. Thus, along with $T^\gamma \in \mathcal{F}^f$, Lemma 3.4 says that Hypothesis 3.1 holds for $\mathcal{C}^\gamma$, z^γ, and $\mathcal{H}^\gamma$ in place of $\mathcal{C}$, z, and $\mathcal{H}$. So we can apply (b) with R^γ and T^γ in place of R and T to obtain $N_S(R^\gamma) = N_S(T^\gamma)$. This gives $N_S(T^\gamma) = N_S(T)^\gamma = N_S(R)^\gamma \leqslant N_S(R^\gamma) = N_S(T^\gamma)$ and therefore $N_S(R)^\gamma = N_S(R^\gamma)$. Since R^γ is fully normalized, it follows that R is fully normalized. This shows (c).

(d): By (c) and the Sylow axiom, $\mathrm{Aut}_S(R) \in \mathrm{Syl}_2(\mathrm{Aut}_{\mathcal{F}}(R))$ and so $C = O_2(\mathrm{Aut}_{\mathcal{F}}(R)) \leqslant \mathrm{Aut}_S(R)$. Thus, $C = \mathrm{Aut}_{C_S(E)}(R)$. By (b), $[C_S(E), T] \leqslant C_S(E) \cap T = C_T(E) = R$. Hence,

$[C, \mathrm{Aut}_T(R)] \leqslant \mathrm{Inn}(R)$. For each $\alpha \in \mathrm{Aut}_{\mathcal{C}}(R)$,

$$[C, \mathrm{Aut}_T(R)^\alpha] = [C^{\alpha^{-1}}, \mathrm{Aut}_T(R)]^\alpha = [C, \mathrm{Aut}_T(R)]^\alpha,$$

because C is normalized by $\mathrm{Aut}_{\mathcal{C}}(R)$. So as $\mathrm{Aut}_{\mathcal{C}}(R) = \langle \mathrm{Aut}_T(R)^{\mathrm{Aut}_{\mathcal{C}}(R)} \rangle$ by Lemma 2.38(c), the previous equation gives

$$[C, \mathrm{Aut}_{\mathcal{C}}(R)] = [C, \langle \mathrm{Aut}_T(R)^{\mathrm{Aut}_{\mathcal{C}}(R)} \rangle] = \langle [C, \mathrm{Aut}_T(R)]^{\mathrm{Aut}_{\mathcal{C}}(R)} \rangle \leqslant \mathrm{Inn}(R).$$

This together with (a) implies that (d) holds. □

Notice that $N_S(T) \leqslant C_S(z)$ as z is the unique central involution of T. Hence, if $\alpha \in \mathfrak{A}(z)$, then $T^\alpha \in \mathcal{F}^f$ as $T \in \mathcal{F}^f$. So by Lemma 3.4, replacing $(\mathcal{C}, T, z)$ by $(\mathcal{C}^\alpha, T^\alpha, z^\alpha)$, we may assume that z is fully centralized. Moreover, we set

$$V_R := RC_S(R) \quad \text{and} \quad Q_0 := C_S(T).$$

Lemma 3.7. *The following hold.*

(a) We have $\mathcal{H} \leqslant C_{\mathcal{F}}(Q_0)$.
(b) We have $C_S(R) = EC_S(T)$, and hence $V_R = RC_S(T)$.
(c) $N_{N_{\mathcal{F}}(R)}(V_R)$ is a constrained fusion system and $N_{\mathcal{C}}(R) \leqslant N_{N_{\mathcal{F}}(R)}(V_R)$.
(d) Let G_R be a model for $N_{N_{\mathcal{F}}(R)}(V_R)$ and $N := C_{G_R}(V_R/R)$. Then $N_1 := \langle T^N \rangle = O^2(N)R$ is a model for $N_{\mathcal{C}}(R)$.
(e) We have $Q_0 = \langle z \rangle \times Q$ where $Q = C_{Q_0}(N_1)$ with N_1 as in (d).
(f) If Q is as in (e), then Q is the unique largest subgroup of S centralized by $\mathcal{C}$. More precisely, $\mathcal{C} \leqslant C_{\mathcal{F}}(Q)$, and $X \leqslant Q$ for all $X \leqslant S$ with $\mathcal{C} \leqslant C_{\mathcal{F}}(X)$.
(g) If Q is as in (e), then Q is the unique largest member of $\tilde{\mathcal{X}}(\mathcal{C})$.

Proof. **(a), (b)**: Recall that $E = Z(R)$. As $R \leqslant T$, clearly $EC_S(T) \leqslant C_S(R)$, so for (b) we must show the other inclusion. Since $z \in R$, we have $C_S(R) = C_{C_S(z)}(R) \leqslant C_S(z)$. Now by our choice of notation, $\langle z \rangle$ is fully $\mathcal{F}$-centralized, so $C_{\mathcal{F}}(z)$ is a saturated fusion system on $C_S(z)$. By Hypothesis 3.1, $\mathcal{H}$ is a component of $C_{\mathcal{F}}(z)$. The normalizer of a component is constructed in [Asc19, §2.1], and thus, we may form $N_{C_{\mathcal{F}}(z)}(\mathcal{H})$ over the 2-group $N_S(T) = N_{C_S(z)}(T)$. By Lemma 3.6(b), $C_S(R) \leqslant N_S(R) = N_S(T)$, so we may form the product system $\widehat{\mathcal{H}} := \mathcal{H}C_S(R)$ as in [Asc11, Chapter 8] or [Hen13] in the normalizer $N_{C_{\mathcal{F}}(z)}(\mathcal{H})$. Thus $\widehat{\mathcal{H}}$ is a saturated subsystem of $C_{\mathcal{F}}(z)$ with $O^2(\widehat{\mathcal{H}}) = O^2(\mathcal{H}) = \mathcal{H} = E(\widehat{\mathcal{H}})$. Since $\mathcal{H}$ is tamely realized by $H = \mathrm{Spin}_7(5^{2^l})$ by Lemma 2.41(a), Theorem 2.13 gives an extension $\widehat{H} = HC_S(R)$ of H that tamely realizes $\widehat{\mathcal{H}}$. By Lemma 2.41(b), each automorphism of H normalizing T and centralizing R is conjugation by an element of E. Hence, $Q_0 \leqslant C_S(R) \leqslant EC_S(H) \leqslant EC_S(T)$. This implies $C_S(R) = EC_S(T)$ and $Q_0 = C_E(T)C_S(H) = \langle z \rangle C_S(H) = C_S(H)$. The first property gives (b), and the latter property yields (a).

(c): Since R is fully normalized by Lemma 3.6(c), $N_{\mathcal{F}}(R)$ is saturated. Note that V_R is weakly closed and thus fully normalized in $N_{\mathcal{F}}(R)$. So $N_{N_{\mathcal{F}}(R)}(V_R)$ is saturated. Clearly $N_{N_{\mathcal{F}}(R)}(V_R)$ is constrained, as V_R is a centric normal subgroup of this fusion system.

We show next that $N_{\mathcal{C}}(R) \leqslant N_{N_{\mathcal{F}}(R)}(V_R)$. Let $R \leqslant P \leqslant T$ and $\varphi \in \mathrm{Aut}_{N_{\mathcal{C}}(R)}(P)$. By Alperin's fusion theorem [AKO11, Theorem I.3.6], it is enough to show that φ extends to an element of $\mathrm{Aut}_{\mathcal{F}}(PV_R)$ normalizing V_R. Let $\alpha \in \mathfrak{A}(P)$ and observe that $\varphi^\alpha \in \mathrm{Aut}_{\mathcal{F}}(P^\alpha)$. By (b), $V_R = RC_S(T) \leqslant PC_S(P)$. Thus $V_R^\alpha \leqslant P^\alpha C_S(P^\alpha) \leqslant N_{\varphi^\alpha}$. As P^α is fully normalized, it follows from the extension axiom that φ^α extends to a morphism $\psi\colon P^\alpha V_R^\alpha \to S$ in $\mathcal{F}$. Note that $R^{\alpha\psi} = (R^\alpha)^{\varphi^\alpha} = R^\alpha$ as $R^\varphi = R$. Since R is fully normalized and thus fully centralized, we have $C_S(R)^{\alpha\psi} = C_S(R^{\alpha\psi}) = C_S(R^\alpha) = C_S(R)^\alpha$ and thus $V_R^{\alpha\psi} = R^\alpha C_S(R)^\alpha = V_R^\alpha$. So $\psi \in \mathrm{Aut}_{\mathcal{F}}(P^\alpha V_R^\alpha)$ extends φ^α and normalizes V_R^α. Hence, $\hat{\varphi} := \psi^{\alpha^{-1}} \in \mathrm{Aut}_{\mathcal{F}}(PV_R)$ extends φ and normalizes V_R. This proves (c).

(d): Let now G_R and N be as in (d), and set $N_1 := \langle T^N\rangle$. (The model G_R for $N_{N_{\mathcal{F}}(R)}(R)$ exists and is unique up to isomorphism by [AKO11, Proposition III.5.8]. Moreover, $C_{G_R}(V_R) \leqslant V_R$.) Note that $S_0 := N_S(R) \in \mathrm{Syl}_2(G_R)$. By (b), $[V_R, T] \leqslant R$. As V_R and R are normal in G_R, it follows that $[V_R, \langle T^{G_R}\rangle] \leqslant R$ and thus $N_1 \leqslant \langle T^{G_R}\rangle \leqslant N$. Let $P \leqslant T$ be essential in $N_{\mathcal{C}}(R)$. As $\mathrm{Out}_{\mathcal{C}}(R) \cong GL_3(2)$, we observe that $R \leqslant P$, $P/R \cong C_2 \times C_2$ and $\mathrm{Out}_{N_{\mathcal{C}}(R)}(P) \cong GL_2(2) \cong S_3$. In particular, $\mathrm{Aut}_{N_{\mathcal{C}}(R)}(P) = \langle \mathrm{Aut}_T(P)^{\mathrm{Aut}_{N_{\mathcal{C}}(R)}(P)}\rangle$. Since $\mathrm{Aut}_{N_{\mathcal{C}}(R)}(P) \leqslant \mathrm{Aut}_{G_R}(P)$ by (c), it follows that $\mathrm{Aut}_{N_{\mathcal{C}}(R)}(P) \leqslant \mathrm{Aut}_{\langle T^{G_R}\rangle}(P) \leqslant \mathrm{Aut}_N(P)$. Now we conclude similarly that $\mathrm{Aut}_{N_{\mathcal{C}}(R)}(P) \leqslant \langle \mathrm{Aut}_T(P)^{\mathrm{Aut}_N(P)}\rangle \leqslant \mathrm{Aut}_{N_1}(P)$. As P was arbitrary, the Alperin–Goldschmidt Fusion Theorem yields that $N_{\mathcal{C}}(R) \leqslant \mathcal{F}_{S_0\cap N_1}(N_1)$.

Note that $N/C_N(R)$ embeds into $\mathrm{Aut}_{\mathcal{F}}(R)$. As $C_{G_R}(V_R) \leqslant V_R$, and $C_N(R)$ centralizes V_R/R and R, $C_N(R)$ is a normal 2-subgroup of N. So it follows from Lemma 3.6(d) that $N/O_2(N) \cong \mathrm{Out}_{\mathcal{C}}(R) \cong GL_3(2)$ and $O_2(N) = C_N(E) \leqslant C_{S_0}(E)$. Using Lemma 3.6(b), we conclude that $O_2(N) \leqslant C_S(E) \leqslant N_S(T)$ and thus $[O_2(N), T] \leqslant C_T(E) = R$. Since $O_2(N)$ and R are normal in N, this implies $[O_2(N), N_1] \leqslant R$. In particular, noting $O_2(N_1) = O_2(N) \cap N_1$ and setting $\overline{N} := N/R$, it follows that $\overline{O_2(N_1)}$ is abelian. Observe that $T/(T \cap O_2(N)) = T/R$ is isomorphic to a Sylow 2-subgroup of $GL_3(2)$. Thus, $TO_2(N)/O_2(N)$ is a Sylow 2-subgroup of $N/O_2(N)$ and so $TO_2(N)$ is a Sylow 2-subgroup of N. In particular, $TO_2(N_1) = (TO_2(N)) \cap N_1$ is a Sylow 2-subgroup of N_1. Note that $T \cap O_2(N_1) = T \cap O_2(N) = C_T(E) = R$. Thus $\overline{T}$ is a complement to $\overline{O_2(N_1)}$ in the Sylow 2-subgroup $\overline{TO_2(N_1)}$ of $\overline{N_1}$. So by a Theorem of Gaschütz [KS04, Theorem 3.3.2], there exists a complement $\overline{N_0}$ of $\overline{O_2(N_1)}$ in $\overline{N_1}$. We choose a preimage N_0 of such a complement $\overline{N_0}$ with $R \leqslant N_0 \leqslant N_1$. As $N/O_2(N) \cong GL_3(2)$ is simple, we have $N = O_2(N)N_1 = O_2(N)N_0$. Since $O_2(N) \cap N_0 = O_2(N_1) \cap N_0 = R$ and $\overline{O_2(N)}$ is centralized by $\overline{N_1}$, it follows $\overline{N} = \overline{O_2(N)} \times \overline{N_0}$. In particular, $N_0 = O^2(N)R$ is normal in G_R. As $N_{\mathcal{C}}(R) \leqslant \mathcal{F}_{S_0\cap N_1}(N_1) \leqslant \mathcal{F}_{S_0\cap N}(N)$, we have $\mathfrak{hyp}(N_{\mathcal{C}}(R)) \leqslant \mathfrak{hyp}(\mathcal{F}_{S_0\cap N}(N)) \leqslant O^2(N)$. Hence $T = \mathfrak{hyp}(N_{\mathcal{C}}(R))R \leqslant O^2(N)R = N_0$. In particular, $N_0 = N_1$, $O_2(N_1) = R$, $T \in \mathrm{Syl}_2(N_1)$ and $N_1/R \cong GL_3(2)$. We show next that $N_{\mathcal{C}}(R) = \mathcal{F}_T(N_1)$. We have seen already that $N_{\mathcal{C}}(R) \leqslant \mathcal{F}_T(N_1)$. If P is essential in $\mathcal{F}_T(N_1)$, then it follows from $N_1/R \cong GL_3(2)$ that $R \leqslant P \leqslant T$, $P/R \cong C_2 \times C_2$ and $\mathrm{Out}_{N_1}(P) \cong GL_2(2)$. As $GL_2(2) \cong \mathrm{Out}_{N_{\mathcal{C}}(R)}(P) \leqslant \mathrm{Out}_{N_1}(P)$, it follows that $\mathrm{Aut}_{N_1}(P) = \mathrm{Aut}_{\mathcal{C}}(P)$. Hence, we have $N_{\mathcal{C}}(R) = \mathcal{F}_T(N_1)$. Since $C_{N_1}(O_2(N_1)) \leqslant N_1 \cap C_N(E) = N_1 \cap O_2(N) = O_2(N_1)$, we conclude that N_1 is a model for $N_{\mathcal{C}}(R)$. This completes the proof of (d).

(e): We consider now the action of $N_1/R \cong GL_3(2)$ on $U_R := C_S(R) = C_{V_R}(R)$. Note that $E = Z(R)$ is central in U_R and recall that $U_R = EC_S(T)$ by (b). In particular, $U_R/\Phi(C_S(T))$ is elementary abelian and thus $\Phi(U_R) \leqslant \Phi(C_S(T))$. If $E \cap \Phi(U_R)$ were non-trivial, then we would have $E \leqslant \Phi(U_R)$ as N_1 acts irreducibly on E. So it would follow that $E \leqslant C_S(T)$ contradicting $E \not\leqslant Z(T)$. This shows that $E \cap \Phi(U_R) = 1$. Set

$$\widetilde{U_R} = U_R/\Phi(U_R).$$

As $\widetilde{U_R} = \widetilde{E}\widetilde{C_S(T)}$ is elementary abelian, there is a complement to $\widetilde{E}$ in $\widetilde{U_R}$ which lies in $\widetilde{C_S(T)}$. So by a Theorem of Gaschütz [KS04, Theorem 3.3.2], applied in the semidirect product $N_1 \ltimes \widetilde{U_R}$, there exists a complement $\widetilde{Q}$ to $\widetilde{E}$ in $\widetilde{U_R}$ which is normalized by N_1. We choose the preimage Q of $\widetilde{Q}$ such that $\Phi(U_R) \leqslant Q \leqslant U_R$.

As $[U_R, N_1] \leqslant [V_R, N] \leqslant R$, we have $[Q, N_1] \leqslant [U_R, N_1] \leqslant U_R \cap R = Z(R) = E$. In particular, $[\widetilde{Q}, N_1] \leqslant \widetilde{Q} \cap \widetilde{E} = 1$. So $[Q, N_1] \leqslant \Phi(U_R) \cap E = 1$. Recalling $Q_0 = C_S(T)$, we conclude $Q \leqslant C_{Q_0}(N_1)$. Observe that Q has index 2 in $Q_0 = C_S(T)$ as $\widetilde{E} \cap \widetilde{C_S(T)} = \langle \widetilde{z} \rangle$ has order 2. Hence, since $[z, N_1] \neq 1$, it follows $Q = C_{Q_0}(N_1)$ and $Q_0 = \langle z \rangle \times Q$. This proves (e).

(f): By (a), Q_0 centralizes $\mathcal{H}$, and by Lemma 2.44(b), we have $\mathcal{C} = \langle \mathcal{H}, N_{\mathcal{C}}(R) \rangle$. So if $X \leqslant Q_0 = C_S(T)$, then X contains $\mathcal{C}$ in its centralizer if and only if it contains $N_{\mathcal{C}}(R)$ in its centralizer. As $N_{\mathcal{C}}(R) = \mathcal{F}_T(N_1)$ by (d) and Q is centralized by N_1, clearly every subgroup of Q contains $N_{\mathcal{C}}(R)$ in its centralizer.

Fix $X \leqslant C_S(T)$ with $N_{\mathcal{C}}(R) \leqslant C_{\mathcal{F}}(X)$. To complete the proof of (f), we need to show that $X \leqslant Q$. To prove this let Θ be the set of all pairs (Y, φ) such that $RX \leqslant Y \leqslant V_R$, $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Y)$, $[Y, \varphi] \leqslant R$, $\varphi|_X = \mathrm{id}_X$, and $\varphi|_R \in \mathrm{Aut}_{\mathcal{C}}(R)$ has order 7. As $\mathrm{Aut}_{\mathcal{C}}(R)/\mathrm{Inn}(R) \cong GL_3(2)$, there exists an element φ_0 of order 7 in $\mathrm{Aut}_{\mathcal{C}}(R)$. As $N_{\mathcal{C}}(R) \leqslant C_{\mathcal{F}}(X)$, φ_0 extends to an automorphism $\varphi \in \mathrm{Aut}_{\mathcal{F}}(RX)$ with $\varphi|_X = \varphi_0$, and for such φ we have $(RX, \varphi) \in \Theta$. Thus $\Theta \neq \varnothing$ and we may fix $(Y, \varphi) \in \Theta$ such that $|Y|$ is maximal.

Assume first that $Y = V_R$. Then φ is a morphism in $N_{N_{\mathcal{F}}(R)}(V_R)$ and thus realized by conjugation with an element of G_R. Recall that $H_1 = O^2(H)R$ is normal in G_R and contains T. Hence, $Q = C_{V_R}(H_1)$ is normal in G_R and thus φ-invariant. As $[V_R, \varphi] \leqslant R$ by definition of Θ, it follows $[Q, \varphi] \leqslant R \cap Q = 1$. As $U_R = EC_S(T) = EQ$ and $\varphi|_R$ acts fixed-point-freely on $E^{\#}$, it follows $Q = C_{U_R}(\varphi)$. By definition of Θ, we have $\varphi|_X = \mathrm{id}_X$ and thus $X \leqslant C_{U_R}(\varphi) = Q$. So $X \leqslant Q$ if $Y = V_R$.

Assume now $Y < V_R$. Recall from above that $N_{\mathcal{F}}(R)$ is saturated. So we can fix $\alpha \in \mathfrak{A}_{N_{\mathcal{F}}(R)}(Y)$. Then $\varphi^\alpha \in \mathrm{Aut}_{\mathcal{F}}(Y^\alpha)$ and $[Y^\alpha, \varphi^\alpha] \leqslant R$ as $[Y, \varphi] \leqslant R$ by definition of Θ. Recall also that $\varphi|_R \in \mathrm{Aut}_{\mathcal{C}}(R)$ has order 7. By Lemma 3.6(d), we have $O^2(\mathrm{Aut}_{\mathcal{F}}(R)) = O^2(\mathrm{Aut}_{\mathcal{C}}(R))$. So we can conclude that $\varphi^\alpha|_R = (\varphi|_R)^\alpha \in O^2(\mathrm{Aut}_{\mathcal{F}}(R))^\alpha = O^2(\mathrm{Aut}_{\mathcal{F}}(R)) \leqslant \mathrm{Aut}_{\mathcal{C}}(R)$. As $N_{\mathcal{C}}(R) = \mathcal{F}_T(N_1)$ by (d), there exists thus $n \in N_1$ with $\varphi^\alpha|_R = c_n|_R$. Set $\psi := c_n|_{V_R} \in \mathrm{Aut}_{\mathcal{F}}(V_R)$. As $N_1 \leqslant N$, we have $[V_R, \psi] \leqslant R$. In particular, as $R \leqslant Y^\alpha \leqslant V_R$, we have $(Y^\alpha)^\psi = Y^\alpha$. Thus, $\chi := (\psi|_{Y^\alpha})^{-1} \circ \varphi^\alpha \in \mathrm{Aut}_{\mathcal{F}}(Y^\alpha)$ is well-defined. Observe also that $\chi|_R = \mathrm{id}_R$ and $[Y^\alpha, \chi] \leqslant R$, as $[Y^\alpha, \psi] \leqslant [V_R, \psi] \leqslant R$ and $[Y^\alpha, \varphi^\alpha] \leqslant R$. So χ is an element of $C_{\mathrm{Aut}_{\mathcal{F}}(Y^\alpha)}(R) \cap C_{\mathrm{Aut}_{\mathcal{F}}(Y^\alpha)}(Y^\alpha/R)$, which is a normal 2-subgroup of $\mathrm{Aut}_{N_{\mathcal{F}}(R)}(Y^\alpha)$. Since $Y^\alpha \in N_{\mathcal{F}}(R)^f$, the Sylow axiom yields that $\mathrm{Aut}_{N_S(R)}(Y^\alpha)$ is a Sylow 2-subgroup of $\mathrm{Aut}_{N_{\mathcal{F}}(R)}(Y^\alpha)$. Hence, there exists $s \in N_{C_S(R)}(Y^\alpha)$ with $\chi|_{Y^\alpha} = c_s|_{Y^\alpha}$. So $\varphi^\alpha = \psi|_{Y^\alpha} \circ c_s|_{Y^\alpha}$ extends to $\rho = \psi \circ c_s|_{V_R} \in \mathrm{Aut}_{\mathcal{F}}(V_R)$. Since $[V_R, \psi] \leqslant R$, the automorphism ρ acts on V_R/R in the same way as $c_s|_{V_R}$. So writing m for the order of s, we have $[V_R, \rho^m] \leqslant R$. Moreover, ρ^m extends $(\varphi^\alpha)^m$. Since $Y < V_R$, we have $Y < W := N_{V_R}(Y)$. Note that $R \leqslant Y^\alpha \leqslant W^\alpha \leqslant V_R$, so $[W^\alpha, \rho^m] \leqslant [V_R, \rho^m] \leqslant R$ and $\rho^m|_{W^\alpha} \in \mathrm{Aut}_{\mathcal{F}}(W^\alpha)$. Therefore, $\hat{\varphi} := (\rho^m|_{W^\alpha})^{\alpha^{-1}} \in \mathrm{Aut}_{\mathcal{F}}(W)$ with $[W, \hat{\varphi}] \leqslant R^{\alpha^{-1}} = R$. Moreover, $\hat{\varphi}|_R = (\varphi|_R)^m \in \mathrm{Aut}_{\mathcal{C}}(R)$ has order 7, as $\varphi|_R \in \mathrm{Aut}_{\mathcal{C}}(R)$ has order 7 and m is a power of 2. Also $\hat{\varphi}|_X = (\varphi|_X)^m = \mathrm{id}_X$ as $\varphi|_X = \mathrm{id}_X$. This shows $(W, \hat{\varphi}) \in \Theta$. As $|W| > |Y|$ and $(Y, \varphi) \in \Theta$ was chosen such that $|Y|$ is maximal, this contradicts the assumption that $Y < V_R$. So ultimately, we have shown that

$Y = V_R$. As argued in the previous paragraph, this yields $X \leqslant Q$, and thus completes the proof of (f).

(g): Since $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$, we may fix an involution $t \in \mathcal{I}(\mathcal{C})$. We have $\mathcal{I}(\mathcal{C}) \subseteq \tilde{\mathcal{X}}(\mathcal{C}) \subseteq \mathcal{X}(\mathcal{C})$ by definition of these collections, so $\langle t \rangle \in \tilde{\mathcal{X}}(\mathcal{C})$, and also $\langle t \rangle \leqslant Q \leqslant \mathcal{X}(\mathcal{C})$ by (f). Therefore, $Q \in \tilde{\mathcal{X}}(\mathcal{C})$ by [Asc19, 6.1.5]. $\square$

Lemma 3.8. *$\mathcal{C}$ is nearly standard.*

Proof. By Lemma 3.5(a), $\mathcal{C}$ is terminal in $\mathfrak{C}(\mathcal{F})$. By Lemma 3.7(g), the collection $\tilde{\mathcal{X}}(\mathcal{C})$ has a unique maximal member. Hence, $\mathcal{C}$ is nearly standard by [Asc19, Proposition 7]. $\square$

Lemma 3.9. $\mathrm{Aut}_{\mathcal{F}}(T) \leqslant \mathrm{Aut}(\mathcal{C})$.

Proof. Let $\alpha \in \mathrm{Aut}_{\mathcal{F}}(T)$ and note that $\alpha \in C_{\mathcal{F}}(z)$. Recall that z was chosen to be fully normalized. Thus, as $z \in \mathcal{I}(\mathcal{H})$, $\mathcal{H}$ is a component of $C_{\mathcal{F}}(z)$. It follows from [Asc11, 9.7] that there is a unique component of $C_{\mathcal{F}}(z)$ with Sylow group T, so that $\mathcal{H}^\alpha = \mathcal{H}$ by Lemma 2.14(b). Since T is fully $\mathcal{F}$-normalized by Hypothesis 3.1, α extends to an automorphism $\tilde{\alpha}$ of $Q_0T = C_S(T)T = C_S(R)T$ with the last equality by Lemma 3.7(b). From Lemma 2.38(c), R is characteristic in T, so we have that $R^\alpha = R$, and hence that $\tilde{\alpha}$ normalizes $C_S(R)$. Thus, $\alpha \in N_{N_{\mathcal{F}}(V_R)}(R)$, a model for which is, by definition, G_R. We may therefore choose $g \in N_{G_R}(T)$ such that $\alpha = c_g|_T$. As $N := C_{G_R}(V_R/R)$ is a normal subgroup of G_R, g leaves invariant $O^2(N)R = \langle T^N \rangle$, which is a model for $N_{\mathcal{C}}(R)$ by Lemma 3.7(d), whence α normalizes $N_{\mathcal{C}}(R)$. Thus, $\alpha \in \mathrm{Aut}(\langle \mathcal{H}, N_{\mathcal{C}}(R) \rangle) = \mathrm{Aut}(\mathcal{C})$, the equality coming from the generation statement of Lemma 2.44(b), and now the assertion follows as α was chosen arbitrarily. $\square$

4. The centralizer of $\mathcal{C}$ and the elementary abelian case

We operate from now until just before the end of Section 6 under the following hypothesis and notation, although we will often state it again for emphasis.

Hypothesis 4.1. Let $\mathcal{F}$ be a saturated fusion system over the 2-group S, and let q be an odd prime power. Suppose $\mathcal{C} \cong \mathcal{F}_{\mathrm{Sol}}(q)$ is a subsystem of $\mathcal{F}$ over the subgroup $T \in \mathcal{F}^f$. Assume $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ is standard in $\mathcal{F}$, but that $\mathcal{C}$ is not a component of $\mathcal{F}$.

By Theorem 3.2 (and Remark 3.3), if $\mathcal{C}$ is a Benson-Solomon system which is maximal *and subintrinsic* in $\mathfrak{C}(\mathcal{F})$, then Hypothesis 4.1 holds or $\mathcal{C}$ is a component of $\mathcal{F}$. But we have not assumed that $\mathcal{C}$ is subintrinsic in $\mathfrak{C}(\mathcal{F})$ in Hypothesis 4.1 and there is no obvious reason that Hypothesis 4.1 implies this property. This means that the results of Section 3 are generally not applicable in Sections 4-6.

Notation 4.2. Let $\mathcal{Q}$ be the centralizer of $\mathcal{C}$ (Remark 2.25), and let Q be the Sylow group of $\mathcal{Q}$.

Thus, in the case where $\mathcal{C}$ is subintrinsic in $\mathfrak{C}(\mathcal{F})$, the group Q was ultimately constructed in Lemma 3.7(f,g). For future reference, we record the following lemma.

Lemma 4.3. *The following hold.*

(a) $\mathcal{Q}$ is tightly embedded in $\mathcal{F}$, and
(b) $C_S(T) = QZ(T)$.

Proof. Since $\mathcal{C}$ is assumed to be a standard subsystem in Hypothesis 4.1, the subsystem $\mathcal{Q}$ exists and is saturated by [Asc19, 9.1.4, 9.1.5]. Then $\mathcal{Q}$ is a tightly embedded subsystem of $\mathcal{F}$ by [Asc19, 9.1.6.2]. Part (b) then follows from a combination of Lemmas 2.28 and 2.40. $\square$

Lemma 4.4. *One of the following holds.*

(a) Q is elementary abelian, or
(b) Q is of 2-rank 1.

Proof. This is a direct consequence of Hypothesis 4.1, Lemma 2.49, and [Asc19, Theorem 8]. $\square$

Proposition 4.5. *Assume Hypothesis 4.1. Then Q has 2-rank 1.*

Proof. The subsystem $\mathcal{Q}$ is tightly embedded by Lemma 4.3(a). Assume that Q has 2-rank larger than 1. Then by Lemma 4.4, Q is elementary abelian and $|Q| > 2$. Moreover, by Lemma 2.31, $\mathcal{F}_Q(Q)$ is tightly embedded in $\mathcal{F}$. By [Asc19, 9.4.11], we can fix $P \in Q^{\mathcal{F}}$ such that $P \leqslant N_S(Q)$ and $P \neq Q$. By [Asc19, 3.1.8], we have

$$P \cap Q = 1.$$

As $\mathcal{C}$ is standard, we have $\mathcal{C} \trianglelefteq N_{\mathcal{F}}(Q)$. In particular, we can form the product $\mathcal{C}P$ inside of $N_{\mathcal{F}}(Q)$. As Q is normal in $N_{\mathcal{F}}(Q)$, we have $Q \notin P^{\mathcal{C}P}$. Furthermore, if $\alpha \in \mathrm{Hom}_{\mathcal{C}P}(P, TP)$ then α induces the identity on PT/T by the construction of $\mathcal{C}P$ in [Hen13] and since $P \cong Q$ is abelian. So $TP = TP^{\alpha}$. Hence, replacing P by a suitable $\mathcal{C}P$-conjugate of P, we may assume

$$P \in (\mathcal{C}P)^f.$$

Then by [Asc19, Theorem 3.4.2], $\mathcal{F}_P(P)$ is tightly embedded in $\mathcal{C}P$.

By [HL18, Theorem 3.10], $\mathrm{Out}(\mathcal{C})$ is cyclic. Note that $N_S(Q)$ induces automorphisms of $\mathcal{C}$ via conjugation as $\mathcal{C} \trianglelefteq N_{\mathcal{F}}(Q)$. Moreover, the elements of $N_S(Q)$ inducing inner automorphisms of $\mathcal{C}$ are precisely the elements in $TC_S(T)$. Thus, $N_S(Q)/TC_S(T)$ is cyclic. Writing z for the unique involution in $Z(T)$, by Lemma 4.3(b), $C_S(T) = \langle z \rangle Q$ and so $TC_S(T) = TQ$. Since $P \cong Q$ is elementary abelian of 2-rank at least 2, it follows that $P \cap (TQ) \neq 1$. Let $1 \neq x \in P \cap (TQ)$ and write $x = uv$ with $u \in T$ and $v \in Q$. Note that u and v commute. As x is an involution, it follows that u and v have order at most 2. If $u = 1$ then $x = v \in P \cap Q$ contradicting $P \cap Q = 1$. Hence u is an involution. Let $\alpha \in \mathrm{Hom}_{\mathcal{C}P}(C_{TP}(x), TP)$ such that $x^{\alpha} \in (\mathcal{C}P)^f$. We proceed now in several steps to reach a contradiction.

Step 1: We show that $x^{\alpha} \in C_S(T)$ and $x^{\alpha} = zv^{\alpha}$ with $v^{\alpha} \in Q$.

For the proof note first that, as $\mathcal{C} \trianglelefteq N_{\mathcal{F}}(Q)$, we have $T \trianglelefteq N_S(Q)$ and thus $Z(T) = \langle z \rangle \trianglelefteq N_S(Q)$. Hence, z is central in $N_S(Q)$ and thus fully centralized in $\mathcal{C}P$. As $u \in T$ is an involution and all involutions in T are $\mathcal{C}$-conjugate by Lemma 2.39, the element u is $\mathcal{C}P$-conjugate to z. Hence, there exists $\varphi \in \mathrm{Hom}_{\mathcal{C}P}(C_{TP}(u), TP)$ such that $u^{\varphi} = z$. Note that $x, v \in C_{TP}(u)$, since $x = uv$ and u and v commute. We obtain $x^{\varphi} = zv^{\varphi}$, where $v^{\varphi} \in Q \leqslant C_S(T)$, as $v \in Q$ and φ is a morphism in $N_{\mathcal{F}}(Q)$. Since $z \in Z(T)$, it follows $T \leqslant C_S(x^{\varphi})$. Recall that α was chosen such that $x^{\alpha} \in (\mathcal{C}P)^f$. Thus, using Lemma 2.3, we can conclude that $T \leqslant C_S(x^{\alpha})$ and so $x^{\alpha} \in C_S(T)$. Note that $u, v \in C_{TP}(x)$ as u and v commute. Moreover, since α is a morphism in $N_{\mathcal{F}}(Q)$, we have $v^{\alpha} \in Q \leqslant C_S(T)$. So $x^{\alpha} = u^{\alpha}v^{\alpha}$ and $u^{\alpha} = x^{\alpha}(v^{\alpha})^{-1} \in C_S(T)$. As T is strongly closed in $N_{\mathcal{F}}(Q)$, we have $u^{\alpha} \in T$ and thus $u^{\alpha} \in Z(T) = \langle z \rangle$. As $u \neq 1$, it follows $u^{\alpha} = z$ and $x^{\alpha} = zv^{\alpha}$ with $v^{\alpha} \in Q$. This completes Step 1.

Step 2: We show $C_{\mathcal{C}}(z) \leqslant C_{\mathcal{C}P}(x^\alpha)$.

For the proof, we may assume that $x^\alpha \neq z$. By definition of Q, we have $\mathcal{C} \leqslant C_{\mathcal{F}}(Q)$. By Step 1, $x^\alpha \in Q\langle z\rangle$. Therefore $C_{\mathcal{C}}(z) \leqslant C_{\mathcal{F}}(Q\langle z\rangle) \leqslant C_{N_{\mathcal{F}}(Q)}(x^\alpha)$. Let $D \in C_{\mathcal{C}}(z)^{fc}$ and let $\chi \in \mathrm{Aut}_{C_{\mathcal{C}}(z)}(D)$ be an arbitrary element of odd order. Then χ extends to some $\hat{\chi} \in \mathrm{Aut}_{N_{\mathcal{F}}(Q)}(D\langle x^\alpha\rangle)$ with $(x^\alpha)^{\hat{\chi}} = x^\alpha$. The order of $\hat{\chi}$ equals the order of χ and is therefore odd. As $x^\alpha \neq z$ is by Step 1 an involution centralizing T, we have $x^\alpha \notin T$ and thus $(D\langle x^\alpha\rangle) \cap T = D$. Moreover, clearly $[D\langle x^\alpha\rangle, \hat{\chi}] \leqslant D$ and $\hat{\chi}|_D = \chi$ is a morphism in $\mathcal{C}$. By [BLO03, Lemma 6.2], we have $D \in \mathcal{C}^c$. So it follows from the definition of $\mathcal{C}P$ in [Hen13] that $\hat{\chi}$ is a morphism in $\mathcal{C}P$. Hence, χ is a morphism in $C_{\mathcal{C}P}(x^\alpha)$. By Alperin's fusion theorem [AKO11, Theorem I.3.6], $C_{\mathcal{C}}(z)$ is generated by the collection of automorphism groups $\mathrm{Aut}_{C_{\mathcal{C}}(z)}(D)$ as D ranges over $C_{\mathcal{C}}(z)^{fc}$. But $\mathrm{Aut}_{C_{\mathcal{C}}(z)}(D) = O^2(\mathrm{Aut}_{C_{\mathcal{C}}(z)}(D))\,\mathrm{Aut}_T(D)$ since $\mathrm{Aut}_T(D)$ is a Sylow 2-subgroup of $\mathrm{Aut}_{C_{\mathcal{C}}(z)}(D)$ for such D, so $C_{\mathcal{C}}(z)$ is generated by $\mathrm{Inn}(T)$ together with $O^2(\mathrm{Aut}_{C_{\mathcal{C}}(z)}(D))$ as D ranges over $C_{\mathcal{C}}(z)^{fc}$. As $T \leqslant C_S(x^\alpha)$, it follows that $C_{\mathcal{C}}(z) \leqslant C_{\mathcal{C}P}(x^\alpha)$.

Step 3: We show that $P^\alpha \trianglelefteq C_{\mathcal{C}P}(x^\alpha)$ and $P^\alpha \cap T \leqslant \langle z\rangle$.

As remarked above, $\mathcal{F}_P(P)$ is tightly embedded in $\mathcal{C}P$. Hence, it follows from (T1) that $P^\alpha \trianglelefteq N_{\mathcal{C}P}(\langle x^\alpha\rangle) = C_{\mathcal{C}P}(x^\alpha)$. In particular, as $C_{\mathcal{C}}(z) \leqslant C_{\mathcal{C}P}(x^\alpha)$ by Step 2, it follows that $P^\alpha \cap T$ is strongly closed in $C_{\mathcal{C}}(z)$. As $P^\alpha \cap T$ is abelian, [AKO11, Corollary I.4.7] gives that $P^\alpha \cap T$ is normal in $C_{\mathcal{C}}(z)$. Since $C_{\mathcal{C}}(z)/\langle z\rangle$ is the 2-fusion system of $\Omega_7(q)$, which is not a Goldschmidt group, $C_{\mathcal{C}}(z)/\langle z\rangle$ is simple by [Asc17, Theorem 5.6.1] (see also [AOV17, Proposition 1.17]). Therefore, $P^\alpha \cap T \leqslant \langle z\rangle$ as required.

Step 4: We show that $[T, P^\alpha] = 1$.

As $C_{\mathcal{C}}(z) = O^2(C_{\mathcal{C}}(z))$, we have

$$T = \mathfrak{hyp}(C_{\mathcal{C}}(z)) = \langle [Y,\beta] \colon Y \leqslant T,\ \beta \in \mathrm{Aut}_{C_{\mathcal{C}}(z)}(Y) \text{ of odd order}\rangle.$$

Let $Y \leqslant T$ and $\beta \in \mathrm{Aut}_{C_{\mathcal{C}}(z)}(Y)$ of odd order. We will show that $[Y, \beta, P^\alpha] = 1$, which is sufficient to complete Step 4. By Step 2, $C_{\mathcal{C}}(z) \leqslant C_{\mathcal{C}P}(x^\alpha)$. As $P^\alpha \trianglelefteq C_{\mathcal{C}P}(x^\alpha)$ by Step 3, we can thus extend β to $\hat{\beta} \in \mathrm{Aut}_{\mathcal{C}P}(YP^\alpha)$ with $(P^\alpha)^{\hat{\beta}} = P^\alpha$. By the definition of $\mathcal{C}P$ in [Hen13] and since P is abelian, we have $[P^\alpha, \hat{\beta}] \leqslant P^\alpha \cap T \leqslant \langle z\rangle$, where the last inclusion uses Step 3. In particular, $[P^\alpha, \hat{\beta}, Y] = 1$. As $P^\alpha \trianglelefteq C_{\mathcal{C}P}(x^\alpha)$ and T centralizes x^α by Step 1, T normalizes P^α. Hence, again using Step 3, we conclude $[Y, P^\alpha] \leqslant [T, P^\alpha] \leqslant T \cap P^\alpha \leqslant \langle z\rangle$ and so $[Y, P^\alpha, \hat{\beta}] = 1$. It follows now from the Three-Subgroup-Lemma that $[Y, \beta, P^\alpha] = [\hat{\beta}, Y, P^\alpha] = 1$. This finishes Step 4.

Step 5: We now derive the final contradiction.

By Step 4, we have $P^\alpha \leqslant C_S(T)$. As we saw above, $C_S(T) = Q\langle z\rangle$ and thus Q has index 2 in $C_S(T)$. Since $|P^\alpha| = |Q| > 2$, it follows $P^\alpha \cap Q \neq 1$. However, as $Q \notin P^{\mathcal{C}P}$, the subgroup P^α is an $\mathcal{F}$-conjugate of Q not equal to Q. Hence, by [Asc19, 3.1.8], we have $P^\alpha \cap Q = 1$. This contradiction completes the proof. □

We are thus left with the case that Q has 2-rank 1, i.e. is either cyclic or quaternion. We end this section with a lemma which handles a residual situation occurring in this context. It will be needed both in Section 5 to exclude the quaternion case and in Section 6 to handle the cyclic case. When Q is of 2-rank 1, the unique involution in Q lies in $\mathcal{I}(\mathcal{C})$ by (S2). This explains the choice of notation for it below.

Lemma 4.6. *Assume Hypothesis 4.1 with Q of 2-rank 1. Let t be the unique involution in Q and fix a subnormal subsystem $\mathcal{F}_0$ of $\mathcal{F}$ over $S_0 \leqslant S$ such that $t \in S_0$. Write z for the unique involution in $Z(T)$. Then the following hold:*

(a) $\langle t \rangle$ is fully $\mathcal{F}_0$-normalized.
(b) If $[T, C_{S_0}(t)] \neq 1$, then $\mathcal{C}$ is a component of $C_{\mathcal{F}_0}(t)$. Moreover,

$$\Omega_1(C_{S_0}(C_{S_0}(t))) = \Omega_1(Z(C_{S_0}(t))) = \langle t, z \rangle.$$

(c) Assume $Q \leqslant S_0$ and $\mathcal{C} \leqslant C_{\mathcal{F}_0}(t)$. If $\langle t \rangle \leqslant Z(S_0)$, then $\langle t \rangle$ is not weakly $\mathcal{F}_0$-closed in $Z(S_0)$.

Proof. As $\mathcal{Q}$ is tightly embedded from Lemma 4.3(a), there is a fully $\mathcal{F}$-normalized $\mathcal{F}$-conjugate of $\langle t \rangle$ in Q by [Asc19, 3.1.5]. It follows that $\langle t \rangle$ is fully $\mathcal{F}$-normalized, since t is the unique involution in Q. So (a) follows from 2.4. In particular, $C_{\mathcal{F}_0}(t)$ is saturated.

In the proof of (b) and (c), we will use that $\mathcal{C}$ is normal in $C_{\mathcal{F}}(t)$ by (S2). In particular $\mathcal{C}$ is a component of $C_{\mathcal{F}}(t)$. In addition, we will use that $C_S(T) = \langle z \rangle Q$ from Lemma 4.3(b).

For the proof of (b) assume that $[T, C_{S_0}(t)] \neq 1$. By (a) and [Asc11, 8.23.2], $C_{\mathcal{F}_0}(t)$ is subnormal in $C_{\mathcal{F}}(t)$. So by [Asc11, 9.6], $\mathcal{C}$ is a component of $C_{\mathcal{F}_0}(t)$ as $[T, C_{S_0}(t)] \neq 1$. In particular, $T \leqslant C_{S_0}(t)$. As $\mathcal{C}$ is normal in $C_{\mathcal{F}}(t)$, we have $T \trianglelefteq C_S(t)$ and in particular, $z \in Z(C_{S_0}(t))$. As $C_S(T) = \langle z \rangle Q$, we obtain $\langle t, z \rangle \leqslant \Omega_1(Z(C_{S_0}(t))) \leqslant \Omega_1(C_{S_0}(C_{S_0}(t))) \leqslant \Omega_1(C_S(T)) = \langle t, z \rangle$ and this implies that (b) holds.

For the proof of (c) assume now that $Q \leqslant S_0$, $\mathcal{C} \leqslant C_{\mathcal{F}_0}(t)$, and $\langle t \rangle \leqslant Z(S_0)$ is weakly $\mathcal{F}_0$-closed in $Z(S_0)$. Then in particular, $S_0 = C_{S_0}(t)$ and $C_{\mathcal{F}_0}(t)$ is saturated. As $T \leqslant C_{S_0}(t)$ is nonabelian, (b) gives that $\mathcal{C}$ is a component of $C_{\mathcal{F}_0}(t)$ and $\Omega_1(Z(S_0)) = \langle t, z \rangle$. As $\mathcal{C}$ is normal in $C_{\mathcal{F}}(t)$, one easily checks that $\mathcal{C}$ is $C_{\mathcal{F}_0}(t)$-invariant (using the equivalent definition of $\mathcal{F}$-invariant subsystems given in [AKO11, Proposition I.6.4(d)]). Hence, by a theorem of Craven [Cra11], $\mathcal{C} = O^{p'}(\mathcal{C})$ is normal in $C_{\mathcal{F}_0}(t)$. We proceed now in several steps to reach a contradiction.

Step 1: We show that $\langle t \rangle$ is not weakly $\mathcal{F}_0$-closed. Suppose this is false. Then for every essential subgroup P of $\mathcal{F}_0$, we have $t \in Z(S_0) \leqslant C_{S_0}(P) \leqslant P$ and t is fixed by $\mathrm{Aut}_{\mathcal{F}_0}(P)$. So by Alperin's Fusion Theorem [AKO11, Theorem I.3.6], we have $t \in Z(\mathcal{F}_0)$. Hence, $\mathcal{C}$ is a component of $C_{\mathcal{F}_0}(t) = \mathcal{F}_0$. As $\mathcal{F}_0$ is subnormal in $\mathcal{F}$, it follows that $\mathcal{C}$ is subnormal in $\mathcal{F}$ and thus a component of $\mathcal{F}$, contradicting Hypothesis 4.1. This completes Step 1.

As shown in Step 1, there exists an $\mathcal{F}_0$-conjugate f of t with $f \neq t$. Fix such f from now on.

Step 2: We show that $f \notin QT$ and t is weakly $\mathcal{F}_0$-closed in QT.

Assuming $f \in QT$, we would have $f \in \Omega_1(QT) \leqslant T\langle t \rangle$. So $f \in T$ or $f = ut$ with $u \in T$. In the latter case, since $t \in Q \leqslant C_S(T)$ and f is an involution, u is an involution. By Lemma 2.39, all involutions in T are $\mathcal{C}$-conjugate. Moreover $\mathcal{C} \leqslant C_{\mathcal{F}_0}(t)$. So if $f \in T$, then f is $\mathcal{F}_0$ conjugate to z, and if $f = ut$ for some involution $u \in T$, then f is $\mathcal{F}_0$-conjugate to zt. In both cases we get a contradiction to the assumption that t is $\mathcal{F}_0$-closed in $Z(S_0)$. So $f \notin QT$. Because of the arbitrary choice of f, this completes Step 2.

As $\mathcal{C}$ is normal in $C_{\mathcal{F}_0}(t)$, we can form the product system $\mathcal{C}\langle f \rangle$ (as defined in [Hen13]) in $C_{\mathcal{F}_0}(t)$ over the 2-group $T\langle f \rangle$.

Step 3: We show that f is $\mathcal{C}\langle f \rangle$-conjugate to every element of the coset fE.

Note first that $F^*(\mathcal{C}\langle f\rangle) = \mathcal{C}$. As all involutions in $T\langle f\rangle - T$ are $\mathcal{C}\langle f\rangle$-conjugate by Proposition 2.45(a), we see that indeed f is $\mathcal{C}\langle f\rangle$-conjugate to every element of fE.

Step 4: We derive the final contradiction.

Since $\mathcal{C}$ is normal in $C_{\mathcal{F}}(t)$ and $S_0 \leqslant C_S(t)$, S_0 induces automorphisms of $\mathcal{C}$ by conjugation. As $C_S(T) = Q\langle z\rangle$ and $\mathrm{Aut}(\mathcal{C})$ is cyclic by [HL18, Theorem 3.10], it follows that $QT = TC_S(T)$ is normal in S_0 and S_0/QT is cyclic. Now let $\alpha \in \mathfrak{A}_{\mathcal{F}_0}(f)$ with $f^\alpha = t$. Then α is defined on $\langle f\rangle E$ and, hence t is $\mathcal{F}_0$-conjugate to every member of the coset tE^α by Step 3. Since E^α is of 2-rank 3, while S_0/QT is cyclic, it follows that $E^\alpha \cap QT \neq 1$. For $1 \neq e \in E^\alpha \cap QT$, t is conjugate to $te \in QT$. This contradicts Step 2. □

5. The quaternion case

In this section we show, assuming Hypothesis 4.1, that Q is not quaternion using Aschbacher's classification of quaternion fusion packets [Asc17]. When combined with Proposition 4.5, the results of this section reduce to the case in which Q is cyclic, which is handled in Section 6.

The Classical Involution Theorem identifies the finite simple groups which have a *classical involution*, that is, an involution whose centralizer has a component (or solvable component) isomorphic to $SL_2(q)$ (or $SL_2(3)$) [Asc77a, Asc77b]. With the exception of M_{11}, the simple groups having a classical involution are exactly the groups of Lie type in odd characteristic other than $L_2(q)$ or ${}^2G_2(q)$, where the $SL_2(q)$ components in involution centralizers are fundamental subgroups generated by the center of a long root subgroup and its opposite.

In a group with a classical involution, the collection of these $SL_2(q)$ subgroups satisfies special fusion theoretic properties that were identified and abstracted by Aschbacher in [Asc77a, Hypothesis Ω]. More recently, Aschbacher has formulated these conditions in fusion systems in the definition of a *quaternion fusion packet*, and his memoir [Asc17] classifies all such packets.

Definition 5.1. A *quaternion fusion packet* is a pair $\tau = (\mathcal{F}, \Omega)$, where $\mathcal{F}$ is a saturated fusion system on a finite 2-group S, and Ω is an $\mathcal{F}$-invariant collection of subgroups of S such that

- (QFP1) There exists an integer m such that for all $K \in \Omega$, K has a unique involution $z(K)$ and is nonabelian of order m.
- (QFP2) For each pair of distinct $K, J \in \Omega$, $|K \cap J| \leqslant 2$.
- (QFP3) If $K, J \in \Omega$ and $v \in J - Z(J)$, then $v^{\mathcal{F}} \cap C_S(z(K)) \subseteq N_S(K)$.
- (QFP4) If $K, J \in \Omega$ with $z = z(K) = z(J)$, $v \in K$, and $\varphi \in \mathrm{Hom}_{C_{\mathcal{F}}(z)}(\langle v\rangle, S)$, then either $v^\varphi \in J$ or v^φ centralizes J.

We assume the following hypothesis until the last result in this section. (See Section 2.1.3 for a discussion of normal and subnormal closures in fusion systems.)

Hypothesis 5.2. Hypothesis 4.1 and Notation 4.2 hold with Q quaternion. Let t be the unique involution in Q. Set $\Omega = Q^{\mathcal{F}}$, denote by $\mathcal{F}^\circ$ the subnormal closure of Q in $\mathcal{F}$ over the subgroup $S^\circ \leqslant S$, and set $\Omega^\circ = Q^{\mathcal{F}^\circ}$.

A tightly embedded subsystem with quaternion Sylow 2-subgroups, such as the centralizer system $\mathcal{Q}$ in Hypothesis 5.2, always yields a quaternion fusion packet in a straightforward way.

Lemma 5.3. $(\mathcal{F}, \Omega)$ *is a quaternion fusion packet.*

Proof. We go through the list of axioms. (QFP1) holds by definition of Ω. Note that $\Omega \subseteq \mathcal{P}^*$ in the sense of Definition 3.1.9 of [Asc19]. Hence, by [Asc19, 3.1.12.2], $K \cap J = 1$ for each pair of distinct $K, J \in \Omega$. This shows that (QFP2) holds, and that any element of S centralizing $z(K)$ must normalize K, so that (QFP3) also holds. Finally, under the hypotheses of (QFP4), $K = J$ in the current situation. Fix $1 \neq v \in K$ and $\varphi \in \mathrm{Hom}_{C_{\mathcal{F}}(z(K))}(\langle v\rangle, S)$. Then $z(K) \in \langle v\rangle$, and $z(K)^\varphi = z(K)$. Also, $\langle v\rangle \in \mathcal{P}$, and $K \in \mathcal{P}^*$, in the sense of Definition 3.1.9 of [Asc19]. Since $\langle v\rangle^\varphi \cap K > 1$, we see from [Asc19, 3.1.14] (applied with $\langle v\rangle$, φ, and K in the role of P, ψ, and R) that $\langle v\rangle^\varphi \leqslant K$. This shows that (QFP4) holds. □

Lemma 5.4. *Let $\mathcal{F}_0$ be a subnormal subsystem of $\mathcal{F}$ over the subgroup $S_0 \leqslant S$. Assume that $Q \leqslant S_0$, and that $\mathcal{C} \leqslant C_{\mathcal{F}_0}(t)$. Then $Q^{\mathcal{F}_0} \neq \{Q\}$.*

Proof. Suppose on the contrary that $Q^{\mathcal{F}_0} = \{Q\}$. Then Q is normal in S_0, and so $t \in Z(S_0)$. Let α be a morphism in $\mathcal{F}_0$ with $t^\alpha \in Z(S_0)$. By the extension axiom, we may assume that α is defined on Q, and then $Q^\alpha = Q$ by assumption, so that $t^\alpha = t$. This shows that $\langle t\rangle$ is weakly $\mathcal{F}_0$-closed in $Z(S_0)$, contradicting Lemma 4.6(c). □

Lemma 5.5. *$\mathcal{C}$ is a component of $C_{\mathcal{F}^\circ}(t)$. In particular, $\mathcal{C}$ is contained in $C_{\mathcal{F}^\circ}(t)$.*

Proof. For each $i \geqslant 0$ set $\mathcal{F}_i := \mathrm{sub}_i(\mathcal{F}, Q)$ and write S_i for the Sylow of $\mathcal{F}_i$. Recall from Section 2.1.3 that $\mathcal{F}_{i+1} \trianglelefteq \mathcal{F}_i$ for each $i \geqslant 0$, and $\mathcal{F}^\circ$ is by definition the terminal member of this series. By Lemma 4.6(a), $\langle t\rangle$ is fully normalized in $\mathcal{F}_i$ for $i \geqslant 0$, so $C_{\mathcal{F}_i}(t)$ is saturated for each i.

We assume now that the assertion is false. As $\mathcal{C}$ is normal in $C_{\mathcal{F}}(t)$ by (S2), there exists $i \geqslant 0$ such that $\mathcal{C}$ is not a component of $C_{\mathcal{F}_{i+1}}(t)$. Fix the smallest such i. Then $\mathcal{C}$ is a component of $C_{\mathcal{F}_i}(t)$. By Lemma 5.4, we have that $Q^{\mathcal{F}_i} \neq \{Q\}$. Fix $Q' \in Q^{\mathcal{F}_i} - \{Q\}$. As $\mathcal{Q}$ is tightly embedded in $\mathcal{F}$ (Lemma 4.3(a)), we have $Q \cap Q' = 1$ by [Asc19, 3.1.12.2], and we have

$$Q' \leqslant C_{S_i}(Q) \leqslant C_{S_i}(t)$$

by [Asc19, 3.3.5]. By definition of $\mathcal{F}_{i+1}$, we have $Q' \leqslant S_{i+1}$ and thus $Q' \leqslant C_{S_{i+1}}(t)$. As $C_S(T) = QZ(T)$ by Lemma 4.3(b), it follows $[Q', T] \neq 1$, and thus $[T, C_{S_{i+1}}(t)] \neq 1$. Hence, $\mathcal{C}$ is a component of $C_{\mathcal{F}_{i+1}}(t)$ by Lemma 4.6(b). This contradicts the choice of i. □

Lemma 5.6. *The pair $(\mathcal{F}^\circ, \Omega^\circ)$ is a quaternion fusion packet, $\mathcal{F}^\circ$ is the normal closure of Q in $\mathcal{F}^\circ$, and $\mathcal{F}^\circ$ is transitive on Ω°.*

Proof. Note that $(\mathcal{F}^\circ, \Omega^\circ)$ is a quaternion fusion packet by Lemma 5.3 and [Asc17, Lemma 6.4.2.1]. Recall that $\mathcal{F}^\circ$ is the subnormal closure of Q in $\mathcal{F}$. So the second statement follows from the definition of subnormal closure, while the third holds by definition of Ω°. □

Now remove the standing assumption that Hypothesis 5.2 holds.

Proposition 5.7. *Assume Hypothesis 4.1. Then Q is cyclic.*

Proof. We argue by contradiction, so that Q is quaternion by Proposition 4.5. Hence, Hypothesis 5.2 holds, and so we adopt the notation there. By Lemma 5.6, the pair $(\mathcal{F}^\circ, \Omega^\circ)$ satisfies the hypotheses of Theorem 1 of [Asc17]. From that theorem, one of the following holds: either

(1) $t \in Z(\mathcal{F}^\circ)$, or
(2) $t \in O_2(\mathcal{F}^\circ) - Z(\mathcal{F}^\circ)$, or

(3) there is a finite group G with Sylow 2-subgroup S° such that $\mathcal{F}^\circ = \mathcal{F}_{S^\circ}(G)$, and one of the following holds,
  (a) S° has 2-rank at most 3, or
  (b) $G \in \mathrm{Chev}^*(p)$ for some odd prime p, or
  (c) G is quasisimple with $Z(G)$ a 2-group, and $G/Z(G) \cong Sp_6(2)$ or $\Omega_8^+(2)$.

Observe that in all cases,

$$\mathcal{C} \text{ is a component of } C_{\mathcal{F}^\circ}(t) \tag{5.8}$$

by Lemma 5.5.

In Case (1), $\mathcal{C}$ is a component of $C_{\mathcal{F}^\circ}(t) = \mathcal{F}^\circ$. Hence $\mathcal{C}$ is a component of $\mathcal{F}$ since $\mathcal{F}^\circ$ is subnormal in $\mathcal{F}$, contrary to Hypothesis 4.1. In Case (2), the hypotheses of [Asc17, Theorem 2] hold for $(\mathcal{F}^\circ, \Omega^\circ)$, and then by [Asc17, Lemma 6.7.3], we have that $\mathcal{F}^\circ$ is constrained. Thus, $C_{\mathcal{F}^\circ}(t)$ is also constrained, and hence $\mathcal{C} \leqslant E(C_{\mathcal{F}^\circ}(t)) = 1$, a contradiction.

Case (3)(a) yields a contradiction, since $QT \leqslant S^\circ$ is of 2-rank 5 by Lemma 2.38(d). In Case (3)(b), note that $C_{\mathcal{F}^\circ}(t)$ is the fusion system of $C_G(t)/O(C_G(t))$ by [AKO11, I.5.4]. By (5.8) and Lemma 2.53, the hypothesis of Lemma 2.54 hold, and so there is a component K of $C_G(t)/O(C_G(t))$ such that $\mathcal{C}$ is the 2-fusion system of K by that lemma. This contradicts the fact that $\mathcal{C}$ is exotic [LO02, Proposition 3.4].

In Case 3(c) we may assume that Case (2) does not hold, so that $t \notin Z(\mathcal{F}^\circ)$. Then $t \notin Z(G)$. As $Sp_6(2)$ and $\Omega_8^+(2)$ are of characteristic 2-type and as $t \notin Z(G)$, we have that $C_G(t)$ is of characteristic 2. Hence $C_{\mathcal{F}^\circ}(t)$ is constrained. We therefore obtain the same contradiction here as in Case (2). □

6. The cyclic case and the proof of Theorem 1

In this section, we finish the proof of Theorem 1. Using Theorem 3.2 and Proposition 5.7, one quickly reduces to the case that $\mathcal{C}$ is standard and the centralizer Q of $\mathcal{C}$ in S is cyclic. We therefore assume the following hypothesis and notation for most of this section.

Hypothesis 6.1. Hypothesis 4.1 and Notation 4.2 hold with Q cyclic. Write $Z(T) = \langle z \rangle$, $\Omega_1(Q) = \langle t \rangle$, $S_t = C_S(t)$, and $\mathcal{F}_t = C_{\mathcal{F}}(t)$.

Lemma 6.2. *Assume Hypothesis 4.1. Then the following hold.*

(a) $\langle t \rangle \in \mathcal{F}^f$,
(b) $C_S(T) = Q\langle z \rangle$,
(c) $\Omega_1(C_S(S_t)) = \Omega_1(Z(S_t)) = \langle t, z \rangle$,
(d) if $t \in Z(S)$, *then* $\langle t \rangle$ *is not weakly* $\mathcal{F}$*-closed in* $Z(S)$, *and*
(e) t *is not* $\mathcal{F}$*-conjugate to* z.

Proof. Parts (a), (c), and (d) follow from Lemma 4.6 applied with $\mathcal{F}_0 = \mathcal{F}$, while (b) is just a recollection of Lemma 4.3(b).

It remains to prove (e). As Q is cyclic, we have $\mathcal{Q} = Q$. By (T1) in the definition of tight embedding (Definition 2.30), we have $Q = \mathcal{Q} \trianglelefteq \mathcal{F}_t$. Further, $Q = C_{S_t}(\mathcal{C})$ by [Asc19, 9.1.6.3]. Write quotients by Q with bars. Note that $C_{\bar{S}_t}(\bar{\mathcal{C}})$ is trivial by [Lyn15, Lemma 1.14], and $\bar{\mathcal{C}} \cong \mathcal{C}$. Thus, $F^*(\bar{\mathcal{F}}_t) = \bar{\mathcal{C}}$ is isomorphic to a Benson-Solomon system. By [HL18, Theorem 4.3], this quotient is therefore a split extension of $\bar{\mathcal{C}}$ by a 2-group of outer automorphisms, and in particular,

$O^2(\bar{\mathcal{F}}_t) = \bar{\mathcal{C}}$. It follows that $O^2(\mathcal{F}_t) \leqslant Q\mathcal{C}$. Since $O^2(Q\mathcal{C}) = \mathcal{C}$ and since $O^2(O^2(\mathcal{F}_t)) = O^2(\mathcal{F}_t)$, we have that $O^2(\mathcal{F}_t) = \mathcal{C}$. Hence, t is fully normalized and not in the hyperfocal subgroup of $\mathcal{F}_t$, while z^α is contained in the hyperfocal subgroup of $\mathcal{H}^\alpha \leqslant C_{\mathcal{F}}(z^\alpha)$ for every $\alpha \in \mathfrak{A}(\langle z\rangle)$. Thus, t and z are not $\mathcal{F}$-conjugate. □

Lemma 6.3. *If Hypothesis 6.1 holds, then $\mathcal{C}$ is not subintrinsic in $\mathfrak{C}(\mathcal{F})$.*

Proof. Assume on the contrary that $\mathcal{C}$ is subintrinsic in $\mathfrak{C}(\mathcal{F})$. As argued in Remark 3.3, this means that $z \in \mathcal{I}_{\mathcal{F}}(\mathcal{H})$ where $\mathcal{H} = C_{\mathcal{C}}(z)$.

Assume first that $t \notin Z(S)$. Then $S_t < S$, so that $S_t < N_S(S_t)$. Fix $a \in N_S(S_t) - S_t$. Then $t^a = tz$ and $z^a = z$ by Lemma 6.2(c,e).

As $z \in \mathcal{I}_{\mathcal{F}}(\mathcal{H})$, we may pick $\alpha \in \mathfrak{A}(z)$ such that $\mathcal{H}^\alpha$ is a component of $C_{\mathcal{F}}(z^\alpha)$. Since $z^a = z$, we may define $\overline{a} := a^\alpha \in C_S(z^\alpha)$. Then $(\mathcal{H}^\alpha)^{\overline{a}}$ is a component of $C_{\mathcal{F}}(z)$ on $(T^\alpha)^{\overline{a}} = (T^a)^\alpha$. However, if $(\mathcal{H}^\alpha)^{\overline{a}} \neq \mathcal{H}^\alpha$, then since Sylow subgroups of distinct components commute, we would have $[(T^a)^\alpha, T^\alpha] = 1$ and thus $T^a \leqslant C_{S_t}(T) \leqslant Q\langle z\rangle$ by Lemma 6.2(b), and we would be forced to conclude that T is abelian. Since this is not the case, $\overline{a}$ normalizes T^α, which implies that a normalizes T. Hence, by (S4) in Definition 2.24, conjugation by a restricts to an automorphism of $\mathcal{C}$. As $t \in \mathcal{F}^f$ by Lemma 6.2(a), it follows from [Asc19, 9.1.6.3] that $Q = C_{S_t}(\mathcal{C})$. Thus, a acts also on $Q = C_{S_t}(\mathcal{C})$, so that $t^a = t$. This contradicts the choice of a.

We have shown that $t \in Z(S)$. So Lemma 6.2(c) yields $V := \Omega_1(Z(S)) = \Omega_1(Z(S_t)) = \langle t, z\rangle$, while Lemma 6.2(e) says that t is not $\mathcal{F}$-conjugate to z. Notice that $\mathrm{Aut}_{\mathcal{F}}(V)$ is by the Sylow axiom of odd order, since S centralizes V. As $\mathrm{Aut}(V) \cong S_3$ and every element of $\mathrm{Aut}(V)$ of order 3 acts transitively on $V^\#$, it follows that $\mathrm{Aut}_{\mathcal{F}}(V) = 1$. If t is $\mathcal{F}$-conjugate to an element of $Z(S)$ under an $\mathcal{F}$-morphism α, then by Lemma 2.2, α can be assumed to be an $\mathcal{F}$-automorphism of S, which thus restricts to an element of $\mathrm{Aut}_{\mathcal{F}}(V)$. This shows that $\langle t\rangle$ is weakly closed in $Z(S)$, contradicting Lemma 6.2(d). □

We are now in the position to complete the proof of Theorem 1, so we now drop the standing assumption from the beginning of Section 4 that Hypothesis 4.1 holds.

Proof of Theorem 1. If $\mathcal{F}$ is a counterexample to Theorem 1, then we may choose the notation such that Hypothesis 3.1 holds (cf. Remark 3.3). So by Theorem 3.2, Hypothesis 4.1 holds. Adopt Notation 4.2. Thus, Proposition 5.7 yields that Q is cyclic, so that Hypothesis 6.1 holds. However, now Lemma 6.3 yields a contradiction to the assumption that $\mathcal{C}$ is subintrinsic. □

7. General Benson-Solomon components

In this section, we apply Walter's Theorem for fusion systems [Asc20, Theorem] to treat the general Benson-Solomon component problem under the assumption that all components in involution centralizers are on the list of currently known quasisimple 2-fusion systems, and we thus complete the proof of Theorem 2. We stress that the proof of Walter's theorem relies in turn on our Theorem 1.

Throughout this section, let $\mathcal{F}$ be a saturated fusion system over a 2-group S.

The reader is referred to Section 2.9 for more information on the class of known quasisimple fusion systems, as well as for the definition of the subclass Chev[large]. The following theorem is essentially a restatement of Theorem 2. For if Theorem 7.1(2) holds, then it follows from [Asc11, 10.11.3] that $\mathcal{C}$ is diagonally embedded in $\mathcal{D}\mathcal{D}^t$ with respect to t.

Theorem 7.1. *Let $\mathcal{F}$ be a saturated fusion system over the 2-group S. Assume that all members of $\mathfrak{C}(\mathcal{F})$ are known and that some fixed member $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ is isomorphic to $\mathcal{F}_{\mathrm{Sol}}(q)$ for some odd q. Then for each $t \in \mathcal{I}(\mathcal{C})$, there exists a component $\mathcal{D}$ of $\mathcal{F}$ such that one of the following holds.*

(1) $\mathcal{D} = \mathcal{C}$;
(2) $\mathcal{D} \cong \mathcal{C}$, $\mathcal{D}^t \neq \mathcal{D}$, $t \in (\mathcal{D}\mathcal{D}^t\langle t\rangle)^f$ and $\mathcal{C} = C_{\mathcal{D}\mathcal{D}^t}(t)$; or
(3) $\mathcal{D} \cong \mathcal{F}_{\mathrm{Sol}}(q^2)$, $t \notin \mathcal{D}$, $t \in (\mathcal{D}\langle t\rangle)^f$ and $\mathcal{C} = C_{\mathcal{D}}(t)$.

It is worth remarking on a few technical points: if $t \in S$ normalizes a component $\mathcal{D}$ of $\mathcal{F}$, then $\mathcal{D}$ is normal in $E(\mathcal{F})\langle t\rangle$ and thus we may form $\mathcal{D}\langle t\rangle$ inside of $E(\mathcal{F})\langle t\rangle$. Moreover, if t is fully $\mathcal{D}\langle t\rangle$-normalized, then $C_{\mathcal{D}}(t) = C_{\mathcal{D}}(t)_{E(\mathcal{F})\langle t\rangle}$ is defined (cf. Definition 2.7). Whenever we write $C_{\mathcal{D}}(t)$, we mean implicitly that t normalizes $\mathcal{D}$. If $\mathcal{D}$ is a component of $\mathcal{F}$ with $\mathcal{D} \neq \mathcal{D}^t$, then similarly $\mathcal{D}\mathcal{D}^t$ is normal in $E(\mathcal{F})\langle t\rangle$, thus we may form $\mathcal{D}\mathcal{D}^t\langle t\rangle = (\mathcal{D}\mathcal{D}^t\langle t\rangle)_{E(\mathcal{F})\langle t\rangle}$ and, if $t \in (\mathcal{D}\mathcal{D}^t\langle t\rangle)^f$, then also $C_{\mathcal{D}\mathcal{D}^t}(t) = C_{\mathcal{D}\mathcal{D}^t}(t)_{E(\mathcal{F})\langle t\rangle}$.

The remainder of this section is devoted to the proof of Theorem 7.1. As Walter's theorem for fusion systems is applied twice in the proof, we restate that theorem here.

Theorem 7.2 (Walter's Theorem for fusion systems, [Asc20])**.** *Suppose that all members of $\mathfrak{C}(\mathcal{F})$ are known and that there exists $\mathcal{L} \in \mathfrak{C}(\mathcal{F})$ such that $\mathcal{L}$ is in* Chev[large]*. Let $t \in \mathcal{I}(\mathcal{L})$ such that t is fully $\mathcal{F}$-centralized. Then there exists a component $\mathcal{D}$ of $\mathcal{F}$ such that one of the following holds.*

(1) $\mathcal{D} \in$ Chev[large] and $\mathcal{L} \in \mathrm{Comp}(C_{\mathcal{D}}(t))$;
(2) $\mathcal{L} = E(C_{\mathcal{D}\mathcal{D}^t}(t))$ is a homomorphic image of $\mathcal{D}$, so $\mathcal{D} \in$ Chev[large];
(3) $\mathcal{L}$ is the 2-fusion system of $\mathrm{Spin}_7(q)$, $\mathcal{L} = C_{\mathcal{D}}(t)$, and $\mathcal{D} = \mathcal{F}_{\mathrm{Sol}}(q)$ for some odd prime power q; or
(4) $\mathcal{L}$ is the 2-fusion system of $SL_2(9)$, $\mathcal{L} \in \mathrm{Comp}(C_{\mathcal{D}}(t))$, and $\mathcal{D}$ is the 2-fusion system of a finite group D such that either $D \cong 2A_n$ or $D/Z(D) \cong L_3(4)$ and $Z(\mathcal{L}) \leqslant \Phi(Z(D))$.

As in the statement of Walter's Theorem, $\mathcal{L}$ in this section is always some type of fusion subsystem, and not a linking system. We will not need to make any explicit reference to linking systems in this section.

We continue now with three technical lemmas that will be needed in the proof of Theorem 7.1.

Lemma 7.3. *Assume all members of $\mathfrak{C}(\mathcal{F})$ are known quasisimple 2-fusion systems, and fix a fully centralized involution z of $\mathcal{F}$. Then all members of $\mathfrak{C}(C_{\mathcal{F}}(z))$ are known.*

Proof. Given a fully centralized involution t in $C_{\mathcal{F}}(z)$ and a component $\mathcal{K}$ of $C_{C_{\mathcal{F}}(z)}(t)$, it follows from [Asc20, 1.3] that there is a component $\mathcal{L}$ of $C_{\mathcal{F}}(z)$ such that either $\mathcal{K}$ is a homomorphic image of $\mathcal{L}$, or $\mathcal{L}$ is t-invariant, t does not centralize $\mathcal{L}$, and $\mathcal{K} \in \mathrm{Comp}(C_{\mathcal{L}\langle t\rangle}(t))$. In the former case, $\mathcal{K}$ is known, so assume the latter case. Then $\mathcal{K}$ is a component in the centralizer of some involution in an almost quasisimple extension $\mathcal{L}\langle t\rangle$ of $\mathcal{L} \in \mathfrak{C}(\mathcal{F})$. By assumption and Theorem 2.29 of [AO16], either $\mathcal{L}$ is a Benson-Solomon system, or $\mathcal{L}$ is tamely realized by a finite quasisimple group. If $\mathcal{L}$ is a quasisimple extension of $\mathcal{F}_{\mathrm{Sol}}(q)$, then $\mathcal{L} = \mathcal{F}_{\mathrm{Sol}}(q)$ by a result of Linckelmann [HL18, Theorem 4.2]. Thus Lemma 2.39 and Proposition 2.45 yield that $\mathcal{K}$ is either the fusion system of $\mathrm{Spin}_7(q)$ (if t is inner) or a Benson-Solomon system (if t is not inner). Hence, $\mathcal{K}$ is known in this case. On the other hand, if $\mathcal{L}$ is tamely realized by a finite quasisimple group L, then $\mathcal{L}\langle t\rangle$ is tamely realized by an extension $L\langle t\rangle$ of L by Theorem 2.13. Hence, components in centralizers of involutions in $L\langle t\rangle$ are known by Lemma 2.53, and so $\mathcal{K}$ is known by Lemma 2.54. $\square$

Lemma 7.4. *If $\mathcal{M}$ is a saturated subsystem of $\mathcal{F}$ such that $O^2(\mathcal{M})$ is a component of $\mathcal{F}$, then $\mathfrak{C}(\mathcal{M}) \subseteq \mathfrak{C}(\mathcal{F})$.*

Proof. Let $\mathcal{M}$ be a saturated subsystem of $\mathcal{F}$ over the subgroup $M \leqslant S$ such that $\mathcal{D} := O^2(\mathcal{M})$ is a component of $\mathcal{F}$. Write D for the Sylow subgroup of $\mathcal{D}$. Fix $\mathcal{C} \in \mathfrak{C}(\mathcal{M})$ and $t \in \mathcal{I}_{\mathcal{M}}(\mathcal{C})$. We will show that $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$. By definition of $\mathcal{I}_{\mathcal{M}}(\mathcal{C})$ and by Lemma 2.17(b), replacing $(t, \mathcal{C})$ by $(t^\alpha, \mathcal{C}^\alpha)$ for some suitable $\alpha \in \mathfrak{A}_{\mathcal{M}}(t)$, we may assume that $\langle t\rangle$ is fully $\mathcal{M}$-normalized and $\mathcal{C}$ is a component of $C_{\mathcal{M}}(t)$.

As $\mathcal{D}$ is a component of $\mathcal{F}$, the normalizer $N_{\mathcal{F}}(\mathcal{D})$ is defined in [Asc19, Definition 2.2.1]. By construction, this is a subsystem of $\mathcal{F}$ over $N_S(D)$. Moreover, by [Asc19, Theorem 2.1.14, 2.1.15, 2.1.16], $N_{\mathcal{F}}(\mathcal{D})$ is a saturated, $\mathcal{D}$ is normal in $N_{\mathcal{F}}(\mathcal{D})$, and every saturated subsystem of $\mathcal{F}$ in which $\mathcal{D}$ is normal is contained in $N_{\mathcal{F}}(\mathcal{D})$. In particular, $\mathcal{M} \leqslant N_{\mathcal{F}}(\mathcal{D})$. Observe that $\mathcal{D}$ is also normal in $E(\mathcal{F})\langle t\rangle$ and thus $E(\mathcal{F})\langle t\rangle \leqslant N_{\mathcal{F}}(\mathcal{D})$. By [Hen13, Theorem 1], $(\mathcal{D}\langle t\rangle)_{N_{\mathcal{F}}(\mathcal{D})}$ is the unique saturated subsystem $\mathcal{Y}$ of $N_{\mathcal{F}}(\mathcal{D})$ over $D\langle t\rangle$ such that $O^2(\mathcal{Y}) = O^2(\mathcal{D}) = \mathcal{D}$. Thus, $(\mathcal{D}\langle t\rangle)_{\mathcal{M}} = (\mathcal{D}\langle t\rangle)_{N_{\mathcal{F}}(\mathcal{D})} = (\mathcal{D}\langle t\rangle)_{E(\mathcal{F})\langle t\rangle}$ and we will denote this subsystem by $\mathcal{D}\langle t\rangle$. As a consequence, $C_{\mathcal{D}}(t)_{\mathcal{M}} = C_{\mathcal{D}}(t)_{E(\mathcal{F})\langle t\rangle}$ and again we will denote this subsystem just by $C_{\mathcal{D}}(t)$.

As $\langle t\rangle$ is fully $\mathcal{M}$-normalized, it follows from Lemma 2.8 (applied with $\mathcal{M}$ and $\mathcal{D}$ in place of $\mathcal{F}$ and $\mathcal{E}$) that $\langle t\rangle$ is fully $\mathcal{D}\langle t\rangle$-normalized and $C_{\mathcal{D}}(t)$ is a normal subsystem of $C_{\mathcal{M}}(t)$. Recall that $\mathcal{C}$ is a component of $C_{\mathcal{M}}(t)$ and write T for the Sylow of $\mathcal{C}$. Observe that $\mathcal{C} = O^2(\mathcal{C}) \leqslant O^2(C_{\mathcal{M}}(t)) \leqslant \mathcal{D}$ and thus $T \leqslant C_D(t)$. By [Asc11, 9.1.2], T is nonabelian and thus $[T, C_D(t)] \neq 1$. Hence, [Asc11, 9.6] gives that $\mathcal{C}$ is a component of $C_{\mathcal{D}}(t)$.

Let $\varphi \in \mathfrak{A}_{E(\mathcal{F})\langle t\rangle}(t)$ so that $t^\varphi \in (E(\mathcal{F})\langle t\rangle)^f$. Note that $E(\mathcal{F})\langle t\rangle = E(\mathcal{F})\langle t^\varphi\rangle$. By Lemma 2.9 applied with $E(\mathcal{F})\langle t\rangle$ and $\mathcal{D}$ in place of $\mathcal{F}$ and $\mathcal{E}$, we have $\langle t^\varphi\rangle \in (\mathcal{D}\langle t^\varphi\rangle)^f$, $C_D(t)^\varphi = C_D(t^\varphi)$, and $\varphi|_{C_D(t)}$ induces an isomorphism from $C_{\mathcal{D}}(t)$ to $C_{\mathcal{D}}(t^\varphi)$. Thus $\mathcal{C}^\varphi$ is a component of $C_{\mathcal{D}}(t^\varphi)$.

By Lemma 2.8 applied with $E(\mathcal{F})\langle t\rangle$, $\mathcal{D}$ and $\langle t^\varphi\rangle$ in place of $\mathcal{F}$, $\mathcal{E}$ and P, we have $C_{\mathcal{D}}(t^\varphi) \trianglelefteq C_{E(\mathcal{F})\langle t^\varphi\rangle}(t^\varphi)$. So $\mathcal{C}^\varphi$ is subnormal in $C_{E(\mathcal{F})\langle t^\varphi\rangle}(t^\varphi)$ and thus a component of $C_{E(\mathcal{F})\langle t^\varphi\rangle}(t^\varphi)$. As $t^\varphi \in (E(\mathcal{F})\langle t\rangle)^f$, it follows from Lemma 2.8 applied with $E(\mathcal{F})\langle t^\varphi\rangle$ and $E(\mathcal{F})$ in place of $\mathcal{F}$ and $\mathcal{E}$ that $C_{E(\mathcal{F})}(t^\varphi) \trianglelefteq C_{E(\mathcal{F})\langle t^\varphi\rangle}(t^\varphi)$. Writing E for the Sylow subgroup of $E(\mathcal{F})$, we have $T^\varphi \leqslant C_D(t^\varphi) \leqslant C_E(t^\varphi)$. As T^φ is nonabelian, it follows thus from [Asc11, 9.6] that $\mathcal{C}^\varphi$ is a component of $C_{E(\mathcal{F})}(t^\varphi)$.

Let now $\alpha \in \mathfrak{A}_{\mathcal{F}}(t^\varphi)$. By Lemma 2.9, $t^{\varphi\alpha}$ is fully $E(\mathcal{F})\langle t^{\varphi\alpha}\rangle$-centralized, $C_E(t^\varphi)^\alpha = C_E(t^{\varphi\alpha})$, and $\alpha|_{C_E(t^\varphi)}$ induces an isomorphism from $C_{E(\mathcal{F})}(t^\varphi)$ to $C_{E(\mathcal{F})}(t^{\varphi\alpha})$. So $\mathcal{C}^{\varphi\alpha}$ is a component of $C_{E(\mathcal{F})}(t^{\varphi\alpha})$. By Lemma 2.8, $C_{E(\mathcal{F})}(t^{\varphi\alpha}) \trianglelefteq C_{\mathcal{F}}(t^{\varphi\alpha})$ and thus $\mathcal{C}^{\varphi\alpha}$ is a component of $C_{\mathcal{F}}(t^{\varphi\alpha})$. In particular, $t^{\varphi\alpha} \in \tilde{\mathcal{X}}(\mathcal{C}^{\varphi\alpha})$ and so $t \in \tilde{\mathcal{X}}(\mathcal{C})$ by Lemma 2.17(b) applied with $t^{\varphi\alpha}$, $\mathcal{C}^{\varphi\alpha}$ and $(\varphi\alpha|_{T\langle t\rangle})^{-1}$ in place of X, $\mathcal{C}$ and φ. This implies $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ as required. $\square$

Lemma 7.5. *Let $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ be a subsystem over $T \leqslant S$. Fix $t \in \mathcal{I}(\mathcal{C})$ and let $\gamma \in \mathrm{Hom}_{\mathcal{F}}(\langle T, t\rangle, S)$. If there exists a component $\mathcal{D}$ of $\mathcal{F}$ such that one of the conditions (1), (2), or (3) in Theorem 7.1 holds, then there exists a component $\hat{\mathcal{D}}$ of $\mathcal{F}$ such that the same condition holds with t^γ, $\mathcal{C}^\gamma$ and $\hat{\mathcal{D}}$ in place of t, $\mathcal{C}$ and $\mathcal{D}$.*

Proof. By Lemma 2.14 the claim is clear if (1) holds, so suppose (2) or (3) holds for some component $\mathcal{D}$ of $\mathcal{F}$. Write $S_0 \leqslant S$ for the Sylow subgroup of $E(\mathcal{F})$ and D for the Sylow subgroup of $\mathcal{D}$. By [Asc19, 1.3.2], we have $\mathcal{F} = \langle E(\mathcal{F})S, N_{\mathcal{F}}(S_0)\rangle$. So it is sufficient to show the assertion when γ is a morphism in $E(\mathcal{F})S$ or in $N_{\mathcal{F}}(S_0)$. However, if γ is a morphism in $E(\mathcal{F})S$, then by Lemma 2.6 applied with $(\gamma, E(\mathcal{F}), S_0, \langle t\rangle, \langle T, t\rangle)$ in place of $(\varphi, \mathcal{E}, T, P, X)$, we can write γ as the composition of a morphism in $E(\mathcal{F})\langle t\rangle$ and a morphism in $\mathcal{F}_S(S) \leqslant N_{\mathcal{F}}(S_0)$. Thus, it is enough to show the claim if γ is a morphism in $E(\mathcal{F})\langle t\rangle$ or in $N_{\mathcal{F}}(S_0)$. Notice that our assumption implies $\mathcal{C} \leqslant E(\mathcal{F})$ and thus $\langle T, t\rangle \leqslant S_0\langle t\rangle$.

Suppose that (2) holds. Assume first that γ is a morphism in $E(\mathcal{F})\langle t\rangle$. Then $S_0\langle t\rangle = S_0\langle t^\gamma\rangle$. As S_0 normalizes every component of $\mathcal{F}$, the action of t^γ on the components of $\mathcal{F}$ coincides with the one of t. So $\mathcal{D}^t = \mathcal{D}^{t^\gamma}$. In particular, $\mathcal{D} \neq \mathcal{D}^{t^\gamma}$ and $\mathcal{D}\mathcal{D}^t = \mathcal{D}\mathcal{D}^{t^\gamma}$. Note that (2) implies in particular $T = C_{DD^t}(t)$, so γ is defined on $\langle C_{DD^t}(t), t\rangle$. Thus, Lemma 2.9 applied with $E(\mathcal{F})\langle t\rangle$ and $\mathcal{D}\mathcal{D}^t$ in place of $\mathcal{F}$ and $\mathcal{E}$ gives that t^γ is fully normalized in $\mathcal{D}\mathcal{D}^t\langle t^\gamma\rangle = \mathcal{D}\mathcal{D}^{t^\gamma}\langle t^\gamma\rangle$ and $\mathcal{C} = C_{\mathcal{D}\mathcal{D}^t}(t^\gamma) = C_{\mathcal{D}\mathcal{D}^{t^\gamma}}(t^\gamma)$. So (2) holds with t^γ and $\mathcal{C}^\gamma$ in place of t and $\mathcal{C}$, i.e. the assertion is true for $\hat{\mathcal{D}} = \mathcal{D}$.

Assume now that γ is a morphism in $N_{\mathcal{F}}(S_0)$. Then γ extends to $\alpha \in \mathrm{Hom}_{\mathcal{F}}(\langle S_0, t\rangle, S)$. So by Lemma 2.14(b), $\hat{\mathcal{D}} := \mathcal{D}^\alpha$ is a component of $\mathcal{F}$. Clearly, $\hat{\mathcal{D}} \cong \mathcal{D} \cong \mathcal{C} \cong \mathcal{C}^\gamma$. Notice that $\hat{\mathcal{D}}^{t^\gamma} = (\mathcal{D}^\alpha)^{t^\alpha} = (\mathcal{D}^t)^\alpha$. In particular, $\hat{\mathcal{D}}^{t^\gamma} \neq \hat{\mathcal{D}}$ and $\mathcal{C}^\gamma = \mathcal{C}^\alpha \leqslant (\mathcal{D}\mathcal{D}^t)^\alpha = \hat{\mathcal{D}}\hat{\mathcal{D}}^{t^\gamma}$. Moreover, α induces an isomorphism from $E(\mathcal{F})\langle t\rangle$ to $E(\mathcal{F})\langle t^\gamma\rangle$ which takes t to t^γ and $\mathcal{D}\mathcal{D}^t$ to $\hat{\mathcal{D}}\hat{\mathcal{D}}^{t^\gamma}$. Thus, α induces an isomorphism from $\mathcal{D}\mathcal{D}^t\langle t\rangle$ to $\hat{\mathcal{D}}\hat{\mathcal{D}}^{t^\gamma}\langle t^\gamma\rangle$. This implies that (2) holds with t^γ, $\mathcal{C}^\gamma$ and $\hat{\mathcal{D}}$ in place of t, $\mathcal{C}$ and $\mathcal{D}$.

Suppose now that (3) holds and assume again first that γ is a morphism in $E(\mathcal{F})\langle t\rangle$. As observed above, $\mathcal{D}$ is normal in $E(\mathcal{F})\langle t\rangle$. As $t \notin D$, it follows that $t^\gamma \notin D$. Note moreover that $S_0\langle t\rangle = S_0\langle t^\gamma\rangle$. In particular, as $S_0\langle t\rangle$ normalizes $\mathcal{D}$, we have $\mathcal{D}^{t^\gamma} = \mathcal{D}$. As $\mathcal{C} = C_{\mathcal{D}}(t)$ by assumption, we have $C_D(t) = T$ and thus $\gamma \in \mathrm{Hom}_{E(\mathcal{F})\langle t\rangle}(\langle C_D(t), t\rangle, S_0\langle t\rangle)$. Hence, by Lemma 2.9 applied with $E(\mathcal{F})\langle t\rangle$ and $\mathcal{D}$ in place of $\mathcal{F}$ and $\mathcal{E}$, we get that $\langle t^\gamma\rangle \in (\mathcal{D}\langle t^\gamma\rangle)^f$ and $\mathcal{C}^\gamma = C_{\mathcal{D}}(t)^\gamma = C_{\mathcal{D}}(t^\gamma)$. So the assertion holds in this case for $\hat{\mathcal{D}} = \mathcal{D}$.

Assume now that γ is a morphism in $N_{\mathcal{F}}(S_0)$ and choose a morphism $\alpha \in \mathrm{Hom}_{\mathcal{F}}(S_0\langle t\rangle, S)$ which extends γ. Then $\hat{\mathcal{D}} := \mathcal{D}^\alpha$ is a component of $\mathcal{F}$ over $\hat{D} := D^\alpha$, and α induces an isomorphism from $E(\mathcal{F})\langle t\rangle$ to $E(\mathcal{F})\langle t^\gamma\rangle$ which takes $\mathcal{D}$ to $\hat{\mathcal{D}}$ and t to t^γ. Hence, $t^\gamma \notin \hat{D}$ and $\hat{\mathcal{D}}^{t^\gamma} = \hat{\mathcal{D}}$. Moreover, α induces also an isomorphism from $\mathcal{D}\langle t\rangle = (\mathcal{D}\langle t\rangle)_{E(\mathcal{F})\langle t\rangle}$ to $\hat{\mathcal{D}}\langle t^\gamma\rangle = (\hat{\mathcal{D}}\langle t^\gamma\rangle)_{E(\mathcal{F})\langle t^\gamma\rangle}$. As t is fully $\mathcal{D}\langle t\rangle$-normalized, it follows that $t^\gamma = t^\alpha$ is fully $\hat{\mathcal{D}}\langle t^\gamma\rangle$-normalized. Moreover, α induces an isomorphism from $C_{\mathcal{D}\langle t\rangle}(t)$ to $C_{\hat{\mathcal{D}}\langle t^\gamma\rangle}(t^\gamma)$. Observe also that $C_D(t)^\alpha = C_{D^\alpha}(t^\alpha) = C_{\hat{D}}(t^\gamma)$. So α takes the unique normal subsystem of $C_{\mathcal{D}\langle t\rangle}(t)$ over $C_D(t)$ of 2-power index to the unique normal subsystem of $C_{\hat{\mathcal{D}}\langle t^\gamma\rangle}(t^\gamma)$ over $C_{\hat{D}}(t^\gamma)$ of 2-power index. In other words, we have $C_{\mathcal{D}}(t)^\alpha = C_{\hat{\mathcal{D}}}(t^\gamma)$ and thus $\mathcal{C}^\gamma = \mathcal{C}^\alpha = C_{\mathcal{D}}(t)^\alpha = C_{\hat{\mathcal{D}}}(t^\gamma)$. So (2) holds with t^γ, $\mathcal{C}^\gamma$ and $\hat{\mathcal{D}}$ in place of t, $\mathcal{C}$, and $\mathcal{D}$. $\square$

Proof of Theorem 7.1. Let $\mathcal{F}$ be a counterexample having a minimal number of morphisms. Fix $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ and $t \in \mathcal{I}(\mathcal{C})$ such that $\mathcal{C} \cong \mathcal{F}_{\mathrm{Sol}}(q)$ and none of the conclusions (1), (2), or (3) hold for any choice of $\mathcal{D}$. Let T be the Sylow of $\mathcal{C}$, and write $Z(T) = \langle z\rangle$. Set $\mathcal{H} = C_{\mathcal{C}}(z)$.

We may and do assume that $\langle z\rangle$ is fully $\mathcal{F}$-centralized and that $\langle t\rangle$ is fully $C_{\mathcal{F}}(z)$-centralized. This follows in the standard way by choosing $\beta \in \mathfrak{A}(z)$, choosing $\gamma \in \mathfrak{A}_{C_{\mathcal{F}}(z^\beta)}(t^\beta)$, setting $\varphi = \beta\gamma$, and replacing z by z^φ, t by t^φ, and $\mathcal{C}$ by $\mathcal{C}^\varphi$. In this process, note that Lemma 2.17(b) shows that we still have $t^\varphi \in \mathcal{I}(\mathcal{C}^\varphi)$. Also by Lemma 7.5 applied with φ^{-1} in the role of γ, if there exists $\hat{\mathcal{D}}$

such that one of the conclusions (1)–(3) holds with $(t^\varphi, \mathcal{C}^\varphi, \hat{\mathcal{D}})$ in place of $(t, \mathcal{C}, \mathcal{D})$, then one of the conclusions (1)–(3) holds with respect to $t, \mathcal{C}$ and some suitable $\mathcal{D}$.

Fix $\alpha \in \mathfrak{A}(t)$. Then

$$z^\alpha \in C_{\mathcal{F}}(t^\alpha)^f \text{ and the map } \alpha\colon C_{C_{\mathcal{F}}(z)}(t) \to C_{C_{\mathcal{F}}(t^\alpha)}(z^\alpha) \text{ is an isomorphism} \tag{7.6}$$

by [Asc10, 2.2]. Moreover, Lemma 2.17(a) yields

$$\mathcal{C}^\alpha \text{ is a component of } C_{\mathcal{F}}(t^\alpha). \tag{7.7}$$

If there exists a component $\mathcal{D}$ of $\mathcal{F}$ such that one of the conclusions (1)–(3) holds with $(t^\alpha, \mathcal{C}^\alpha)$ in place of $(t, \mathcal{C})$, then it follows from Lemma 7.5 that for some (possibly different) choice of $\mathcal{D}$, one of the conclusions (1)–(3) holds. As this would contradict our assumption, it follows that

$$\begin{gathered}\text{there does not exist a component } \mathcal{D} \text{ of } \mathcal{F} \text{ such that one of the conclusions}\\ \text{(1)–(3) holds with } (t^\alpha, \mathcal{C}^\alpha) \text{ in place of } (t, \mathcal{C}).\end{gathered} \tag{7.8}$$

Since for any component $\mathcal{D}$, neither conclusion (1) nor (2) holds with $(t^\alpha, \mathcal{C}^\alpha)$ in place of $(t, \mathcal{C})$, it follows from [Asc20, 1.3] and the fact that the Benson-Solomon systems have no proper quasisimple coverings [HL18, Theorem 4.2] that there is a unique t^α-invariant component $\mathcal{D}$ of $\mathcal{F}$ containing $\mathcal{C}^\alpha$ such that $\mathcal{D} \neq \mathcal{C}^\alpha$ and $\mathcal{C}^\alpha \in \mathrm{Comp}(C_{\mathcal{D}\langle t^\alpha\rangle}(t^\alpha))$. In particular, $t^\alpha \in \mathcal{I}_{\mathcal{D}\langle t^\alpha\rangle}(\mathcal{C}^\alpha)$. All members of $\mathfrak{C}(\mathcal{D}\langle t^\alpha\rangle)$ are known by Lemma 7.4. Moreover, notice that $\mathcal{D}$ is the unique component of $\mathcal{D}\langle t^\alpha\rangle$. So if $\mathcal{D}\langle t^\alpha\rangle$ is not a counterexample, then conclusion (3) holds with $(t^\alpha, \mathcal{C}^\alpha)$ in place of $(t, \mathcal{C})$ contradicting (7.8). Hence $\mathcal{D}\langle t^\alpha\rangle$ is a counterexample, and so $\mathcal{F} = \mathcal{D}\langle t^\alpha\rangle$ by minimality of $\mathcal{F}$. This implies that

$$F^*(\mathcal{F}) = O^2(\mathcal{F}) = \mathcal{D} \text{ is quasisimple.} \tag{7.9}$$

It is possible at this point that $\mathcal{F} = \mathcal{D}$.

We first prove that

$$\mathcal{H} \text{ is a component of } C_{C_{\mathcal{F}}(z)}(t). \tag{7.10}$$

Recall that $\langle z^\alpha\rangle$ is fully $C_{\mathcal{F}}(t^\alpha)$-normalized by (7.6) and that $\mathcal{C}^\alpha$ is subnormal in $C_{\mathcal{F}}(t^\alpha)$ by (7.7). Fix a subnormal series $\mathcal{C}^\alpha = \mathcal{F}_0 \trianglelefteq \cdots \trianglelefteq \mathcal{F}_n = C_{\mathcal{F}}(t^\alpha)$ for $\mathcal{C}^\alpha$. Then $\langle z^\alpha\rangle$ is fully $\mathcal{F}_i$-normalized for each i by Lemma 2.4. Also, $\mathcal{H}^\alpha = C_{\mathcal{C}^\alpha}(z^\alpha) \trianglelefteq C_{\mathcal{F}_1}(z^\alpha) \trianglelefteq \cdots \trianglelefteq C_{C_{\mathcal{F}}(t^\alpha)}(z^\alpha)$ is a subnormal series for $\mathcal{H}^\alpha$ in $C_{C_{\mathcal{F}}(t^\alpha)}(z^\alpha)$ by application of [Asc11, 8.23.2] and induction on n. Hence, $\mathcal{H}$ is a component of $C_{C_{\mathcal{F}}(z)}(t)$ by the isomorphism in (7.6).

We next apply Walter's Theorem in $C_{\mathcal{F}}(z)$. Recall that we took t to be fully $C_{\mathcal{F}}(z)$-centralized. By (7.10), $\mathcal{H} \in \mathrm{Chev}[\mathrm{large}]$ is a component of $C_{C_{\mathcal{F}}(z)}(t)$, so $t \in \mathcal{I}_{C_{\mathcal{F}}(z)}(\mathcal{H})$. All members of $\mathfrak{C}(C_{\mathcal{F}}(z))$ are known by Lemma 7.3. Thus, the hypotheses of Walter's Theorem (Theorem 7.2) are satisfied with $C_{\mathcal{F}}(z)$, $\mathcal{H}$, and t in the roles of $\mathcal{F}$, $\mathcal{L}$, and t. Let $\mathcal{M} \in \mathrm{Comp}(C_{\mathcal{F}}(z))$ be as given by Theorem 7.2 (in the role of $\mathcal{D}$). Since $\mathcal{H}$ is not the 2-fusion system of $SL_2(9)$, we are not in Theorem 7.2(4). Consider the case that (1) or (3) of Theorem 7.2 holds. Then $\mathcal{H} = O^2(\mathcal{H}) \leqslant O^2(\mathcal{M}\langle t\rangle) = \mathcal{M}$, so z is an element of the Sylow of $\mathcal{M}$. Since $\mathcal{M} \leqslant C_{\mathcal{F}}(z)$, it follows that $\langle z\rangle$ is strongly $\mathcal{M}$-closed. Thus, $z \in Z(\mathcal{M})$ in this case by [AKO11, Corollary I.4.7(a)]. As $\mathcal{F}_{\mathrm{Sol}}(q)$ has a trivial center, this shows that conclusion (3) of Walter's Theorem does not hold. Hence, in any case, Theorem 7.2(1) or (2) holds. Further, as $\mathrm{Spin}_7(q)$ has no proper quasisimple 2-coverings [GLS98, Tables 6.1.2,6.1.3], neither does $\mathcal{H}$ [BCG+07, Corollary 6.4]. So if (2) holds, then $\mathcal{H} \cong \mathcal{M}$. Therefore,

$$\mathcal{M} \in \mathrm{Chev}[\mathrm{large}],\ z \in \mathcal{I}(\mathcal{M}), \text{ and } \mathcal{H} \text{ isomorphic to a subsystem of } \mathcal{M}. \tag{7.11}$$

We next apply Walter's Theorem with $\mathcal{F}$, $\mathcal{M}$, and z in the roles of of $\mathcal{F}$, $\mathcal{L}$, and t. Recall that we took z to be fully $\mathcal{F}$-centralized. So by (7.11) and assumption on $\mathfrak{C}(\mathcal{F})$, the hypotheses of Walter's Theorem apply. By (7.9), $\mathcal{D}$ is the unique component of $\mathcal{F}$, and $\mathcal{C} = O^2(\mathcal{C}) \leqslant O^2(\mathcal{F}) = \mathcal{D}$. By (7.11), $\mathcal{H}$ is isomorphic to a subsystem of $\mathcal{M}$, so $\mathcal{M}$ is not the 2-fusion system of $SL_2(9)$. Thus, Theorem 7.2(1), (2), or (3) holds. In particular, either $\mathcal{D} \in$ Chev[large] or $\mathcal{D}$ is isomorphic to a Benson-Solomon system.

Assume the former holds, namely $\mathcal{D} \in$ Chev[large]. All members of Chev[large] are tamely realized by some member of Chev$^*(p)$ for some odd prime p by [BMO19], so we may fix $D \in$ Chev$^*(p)$ tamely realizing $\mathcal{D}$. By Theorem 2.13, we may further fix a finite group G with Sylow 2-subgroup S such that $F^*(G) = D$ and $\mathcal{F} \cong \mathcal{F}_S(G)$. As $\mathcal{C}^\alpha$ is a component of $C_\mathcal{F}(t^\alpha)$, we obtain from Lemmas 2.53 and 2.54 the contradiction that $\mathcal{C}$ is not exotic.

Therefore, $\mathcal{D}$ is isomorphic to a Benson-Solomon system. Assume first that $t^\alpha \in \mathfrak{foc}(\mathcal{F})$. Then $\mathcal{F} = \mathcal{D}$. In particular, $Z(S)$ is the only fully normalized subgroup of S of order 2 by Lemma 2.39. Hence, $Z(S) = \langle t^\alpha \rangle$ as $\langle t^\alpha \rangle$ is fully $\mathcal{F}$-normalized. So $C_\mathcal{F}(t^\alpha)$ is the 2-fusion system of $\mathrm{Spin}_7(q')$ for some odd prime power q', which contradicts (7.7). Hence, $t^\alpha \notin \mathfrak{foc}(\mathcal{F})$ and so $t^\alpha \notin \mathcal{D}$. Applying Proposition 2.45, we see that $\mathcal{D} \cong \mathcal{F}_{\mathrm{Sol}}(q^2)$ and $C_\mathcal{D}(t^\alpha) = \mathcal{C}$. Therefore, (3) holds after all with $(t^\alpha, \mathcal{C}^\alpha)$ in place of $(t, \mathcal{C})$, a contradiction to (7.8). □

References

[AC10] Michael Aschbacher and Andrew Chermak, *A group-theoretic approach to a family of 2-local finite groups constructed by Levi and Oliver*, Ann. of Math. (2) **171** (2010), no. 2, 881–978.

[AKO11] Michael Aschbacher, Radha Kessar, and Bob Oliver, *Fusion systems in algebra and topology*, London Mathematical Society Lecture Note Series, vol. 391, Cambridge University Press, Cambridge, 2011. MR 2848834

[AO16] Michael Aschbacher and Bob Oliver, *Fusion systems*, Bull. Amer. Math. Soc. (N.S.) **53** (2016), no. 4, 555–615. MR 3544261

[AOV12] Kasper K. S. Andersen, Bob Oliver, and Joana Ventura, *Reduced, tame and exotic fusion systems*, Proc. Lond. Math. Soc. (3) **105** (2012), no. 1, 87–152. MR 2948790

[AOV17] ______, *Reduced fusion systems over 2-groups of small order*, J. Algebra **489** (2017), 310–372. MR 3686981

[Asc77a] Michael Aschbacher, *A characterization of Chevalley groups over fields of odd order*, Ann. of Math. (2) **106** (1977), no. 2, 353–398. MR 0498828 (58 #16865a)

[Asc77b] ______, *A characterization of Chevalley groups over fields of odd order. II*, Ann. of Math. (2) **106** (1977), no. 3, 399–468. MR 0498829 (58 #16865b)

[Asc08] ______, *Normal subsystems of fusion systems*, Proc. Lond. Math. Soc. (3) **97** (2008), no. 1, 239–271. MR 2434097 (2009e:20044)

[Asc10] ______, *Generation of fusion systems of characteristic 2-type*, Invent. Math. **180** (2010), no. 2, 225–299.

[Asc11] ______, *The generalized Fitting subsystem of a fusion system*, Mem. Amer. Math. Soc. **209** (2011), no. 986, vi+110. MR 2752788

[Asc17] ______, *Quaternion fusion packets*, preprint (2017), x+432pp.

[Asc19] ______, *On fusion systems of component type*, Mem. Amer. Math. Soc. **257** (2019), no. 1236, v+182. MR 3898993

[Asc20] ______, *Walter's theorem for fusion systems*, Proc. Lond. Math. Soc. (2020), 1–47, to appear.

[BCG+07] C. Broto, N. Castellana, J. Grodal, R. Levi, and B. Oliver, *Extensions of p-local finite groups*, Trans. Amer. Math. Soc. **359** (2007), no. 8, 3791–3858 (electronic).

[Ben98] David J. Benson, *Cohomology of sporadic groups, finite loop spaces, and the Dickson invariants*, Geometry and cohomology in group theory (Durham, 1994), London Math. Soc. Lecture Note Ser., vol. 252, Cambridge Univ. Press, Cambridge, 1998, pp. 10–23. MR 1709949 (2001i:55017)

[BLO03] Carles Broto, Ran Levi, and Bob Oliver, *The homotopy theory of fusion systems*, J. Amer. Math. Soc. **16** (2003), no. 4, 779–856 (electronic).

[BMO12] Carles Broto, Jesper M. Møller, and Bob Oliver, *Equivalences between fusion systems of finite groups of Lie type*, J. Amer. Math. Soc. **25** (2012), no. 1, 1–20. MR 2833477

[BMO19] Carles Broto, Jesper Møller, and Bob Oliver, *Automorphisms of fusion systems of finite simple groups of Lie type*, Mem. Amer. Math. Soc. **262** (2019), no. 1267, iii+117. MR 4044462

[Che13] Andrew Chermak, *Fusion systems and localities*, Acta Math. **211** (2013), no. 1, 47–139. MR 3118305

[COS08] Andrew Chermak, Bob Oliver, and Sergey Shpectorov, *The linking systems of the Solomon 2-local finite groups are simply connected*, Proc. Lond. Math. Soc. (3) **97** (2008), no. 1, 209–238. MR 2434096 (2009g:55018)

[Cra11] David A. Craven, *Normal subsystems of fusion systems*, J. Lond. Math. Soc. (2) **84** (2011), no. 1, 137–158. MR 2819694

[GL16] George Glauberman and Justin Lynd, *Control of fixed points and existence and uniqueness of centric linking systems*, Invent. Math. **206** (2016), no. 2, 441–484. MR 3570297

[GLS98] Daniel Gorenstein, Richard Lyons, and Ronald Solomon, *The classification of the finite simple groups. Number 3. Part I. Chapter A*, Mathematical Surveys and Monographs, vol. 40, American Mathematical Society, Providence, RI, 1998, Almost simple K-groups. MR 1490581 (98j:20011)

[Gui08] *Guido's book of conjectures*, Monographies de L'Enseignement Mathématique [Monographs of L'Enseignement Mathématique], vol. 40, L'Enseignement Mathématique, Geneva, 2008, A gift to Guido Mislin on the occasion of his retirement from ETHZ June 2006, Collected by Indira Chatterji. MR 2499538

[Hen13] Ellen Henke, *Products in fusion systems*, J. Algebra **376** (2013), 300–319. MR 3003728

[Hen19] ———, *Subcentric linking systems*, Trans. Amer. Math. Soc. **371** (2019), no. 5, 3325–3373. MR 3896114

[HL18] Ellen Henke and Justin Lynd, *Extensions of the Benson-Solomon fusion systems*, Geometric and Topological Aspects of the Representation Theory of Finite Groups (J. F. Carlson, S. B. Iyengar, and J. Pevtsova, eds.), Springer Proceedings in Mathematics & Statistics, vol. 242, Springer International Publishing, 2018, pp. 1–20.

[HS15] E. Henke and J. Semeraro, *Centralizers of normal subgroups and the Z^*-theorem*, J. Algebra **439** (2015), 511–514. MR 3373382

[KS04] Hans Kurzweil and Bernd Stellmacher, *The theory of finite groups*, Universitext, Springer-Verlag, New York, 2004, An introduction, Translated from the 1998 German original. MR 2014408 (2004h:20001)

[LO02] Ran Levi and Bob Oliver, *Construction of 2-local finite groups of a type studied by Solomon and Benson*, Geom. Topol. **6** (2002), 917–990 (electronic).

[LO05] ———, *Correction to: "Construction of 2-local finite groups of a type studied by Solomon and Benson" [Geom. Topol.* **6** *(2002), 917–990 (electronic); mr1943386]*, Geom. Topol. **9** (2005), 2395–2415 (electronic).

[LS19] Justin Lynd and Jason Semararo, *Weights in a Benson-Solomon block*, arXiv:1712.02826v3 [math.GR] (2019), 33pp.

[Lyn15] Justin Lynd, *A characterization of the 2-fusion system of $L_4(q)$*, J. Algebra **428** (2015), 315–356. MR 3314296

[Oli13] Bob Oliver, *Existence and uniqueness of linking systems: Chermak's proof via obstruction theory*, Acta Math. **211** (2013), no. 1, 141–175. MR 3118306

[Oli16] ———, *Reductions to simple fusion systems*, Bull. Lond. Math. Soc. **48** (2016), no. 6, 923–934.

[Oli21] ———, *Correction to: Reductions to simple fusion systems*, arXiv:2102.00794 [math.GR] (2021), 5pp.

[Pui00] Lluis Puig, *The hyperfocal subalgebra of a block*, Invent. Math. **141** (2000), no. 2, 365–397.

[Sem15] Jason Semeraro, *Centralizers of subsystems of fusion systems*, J. Group Theory **18** (2015), no. 3, 393–405. MR 3341522

[Sol74] Ronald Solomon, *Finite groups with Sylow* 2*-subgroups of type* .3, J. Algebra **28** (1974), 182–198.

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