Combinatorial Conditions for Directed Collapsing

Robin Belton, Robyn Brooks, Stefania Ebli, Lisbeth Fajstrup, Brittany Terese Fasy, Nicole Sanderson, and Elizabeth Vidaurre

Abstract While collapsibility of CW complexes dates back to the 1930s, collapsibility of *directed* Euclidean cubical complexes has not been well studied to date. The classical definition of collapsibility involves certain conditions on pairs of cells of the complex. The direction of the space can be taken into account by requiring that the past links of vertices remain homotopy equivalent after collapsing. We call this type of collapse a *link-preserving directed collapse*. In the undirected setting, pairs of cells are removed that create a deformation retract. In the directed setting, topological properties—in particular, properties of spaces of directed paths—are not always preserved. In this paper, we give computationally simple conditions for preserving the topology of past links. Furthermore, we give conditions for when link-preserving directed collapses preserve the contractability and connectedness of spaces of directed paths. Throughout, we provide illustrative examples.

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1 Introduction

A directed Euclidean cubical complex is a subset of $\mathbb{R}^n$ comprising a finite union of directed unit cubes. Directed paths (i.e., paths that are nondecreasing in all coordinates) and spaces of directed paths are the objects of study in this paper. In particular, we address the question of how to simplify directed Euclidean complexes without significantly changing the spaces of directed paths.

This model is motivated by several applications, where each axis of the model corresponds to a parameter of the application (e.g., time). In particular, Euclidean cubical complexes are used to model concurreny in computer programming [4–6,21], hybrid dynamical systems [20], and motion planning [7]. Consider the application to concurrency. In this example, each axis represents a sequence of actions a process completes in the program execution. The complex itself corresponds to "compatible" parameters (i.e., when the processes can execute simultaneously). Cubes missing from the complex correspond to parameters for which the processes cannot execute simultaneously for some reason, such as when they require the same resources with limited capacity; see Fig. 1. A directed path (dipath) in the complex represents a, possibly partial, program execution. Such executions are equivalent if the corresponding dipaths are *directed homotopic*. Simplifying the complexes allows for a more compact representation of the execution space, which, in turn, reduces the complexity of validating correctness of concurrent programs.

A non-trivial Euclidean cubical complex contains uncountably many dipaths and more information than we need for understanding the topology of the spaces of dipaths. The main question we ask is, *How can we simplify a directed Euclidean cubical complex while still preserving spaces of dipaths*?

Past links are local representations of a Euclidean cubical complex at vertices. They were introduced in [21] as a means to show that any finite homotopy type can be realized as a connected component of the space of execution paths for some PV-model. In [1], we found conditions for when the local information of past links preserve the global information on the homotopy type of spaces of dipaths. Because of these relationships between past links and dipath spaces, we define collapsing in terms of past links. We call this type of collapsing *link-preserving directed collapse* (LPDC). We aim to compress a Euclidean cubical complex by LPDCs before attempting to answer questions about dipath spaces.

The main result of this paper is Theorem 3.9, which provides a simple criterion for such a collapsing to be allowed: *A pair of cubes* (τ, σ) *is an LPDC pair if and only if it is a collapsing pair in the non-directed sense and* τ *does not contain the minimum vertex of* σ. This condition greatly simplifies the definition of LPDC and is easy to add to a collapsing algorithm for Euclidean cubical complexes in the undirected setting. Algorithms and implementations in this setting already exist such as in [15]. Furthermore,

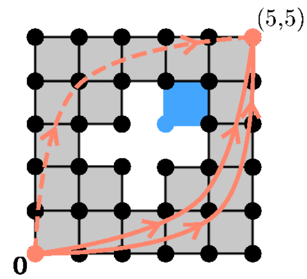


Fig. 1: The Swiss Flag and Three Directed Paths. The gray and blue squares are the two-cubes of a Euclidean cubical complex. The bi-monotone increasing paths are directed paths starting at $(0, 0)$ and ending at $(5, 5)$. This complex has a cross-shaped hole in the middle. As a consequence, the solid directed paths are directed homotopic while the dashed directed path is not directed homotopic to either of the other directed paths. Each point highlighted in blue is *unreachable*, meaning that we cannot reach any point highlighted in blue without breaking bi-monotonicity in a path starting at $(0, 0)$. This complex models the dining philsophers problem, a well-known example in concurrency, where two processes require two shared resources with limited capacity [4, 12]. The two distinct paths (solid and dashed) represent which process uses both shared resources first.[[**for camera-ready:** BTF + NS rewrote this figure caption. Can someone else take a look?]]

we provide conditions for when LPDCs preserve the contractability and connectedness of dipath spaces (Section 4) along with a discussion of some of the limitations (Section 5). This work provides a start at the mathematical foundations for developing polynomial time algorithms that collapse Euclidean cubical complexes and preserve dipath spaces.

2 Background

This paper builds on our prior work [1], as well as work by others [6, 9, 10, 16, 21]. In this section, we recall the definitions of directed Euclidean cubical complexes, which are the objects that we study in this paper. Then, we discuss the relationship between spaces of directed paths and past links in directed Euclidean cubical complexes. For additional background on directed topology (including generalizations of the definitions below), we refer the reader to [5]. We also assume the reader is familiar with the notion of homotopy equivalence of topological spaces (denoted using $\simeq$ in this paper) and homotopy between paths as presented in [11].

2.1 Directed Spaces and Euclidean Cubical Complexes

Let n be a positive integer. A *(closed) elementary cube* in $\mathbb{R}^n$ is a product of closed intervals of the following form:

$$[v_1 - j_1, v_1] \times [v_2 - j_2, v_2] \times \ldots \times [v_n - j_n, v_n] \subsetneq \mathbb{R}^n, \tag{1}$$

where $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{Z}^n$ and $\mathbf{j} = (j_1, j_2, \ldots, j_n) \in \{0,1\}^n$. We often refer to elementary cubes simply as *cubes.* The dimension of the cube is the number of unit entries in the vector $\mathbf{j}$; specifically, the dimension of the cube in Eq. (1) is the sum: $\sum_{i=1}^n j_i$. In particular, when $\mathbf{j} = \mathbf{0} := (0, 0, \ldots, 0)$, the elementary cube is a single point and often denoted using just $\mathbf{v}$. If τ and σ are elementary cubes such that $\tau \subseteq \sigma$, we say that τ is a face of σ and that σ is a coface of τ. Cubical sets were first introduced in the 1950s by Serre [17] in a more general setting; see also [2, 8, 13].

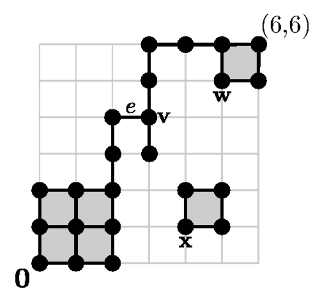


Fig. 2: Euclidean cubical complex in $\mathbb{R}^2$ with 24 zero-cubes (vertices), 28 one-cubes (edges), and six two-cubes (squares). By construction, all elementary cubes in a directed Euclidean cubical complex are axis aligned. Consider the vertex $\mathbf{v} = (3, 4)$. The edge $e = [(2, 4), (3, 4)]$ (written $e = [2, 3] \times [4, 4]$ in the notation of Eq. (1)) is one of the two lower cofaces of $\mathbf{v}$. Since e is not a face of any two-cube, e is a maximal cube (since it is not a face of a higher-dimensional cube).

Elementary cubes stratify $\mathbb{R}^n$, where two points $x, y \in \mathbb{R}^n$ are in the same stratum if and only if they are members of the same set of elementary cubes; we call this the *cubical stratification of* $\mathbb{R}^n$. Each stratum in the stratification is either an open cube or a single point. A *Euclidean cubical complex* $(K, \mathcal{K})$ is a subspace $K \subsetneq \mathbb{R}^n$ that is equal to the union of a finite set of elementary cubes, together with the stratification $\mathcal{K}$ induced by the cubical stratification of $\mathbb{R}^n$; see Fig. 2. We topologize K using the subspace topology with the standard topology on $\mathbb{R}^n$. By construction, if $\sigma \in \mathcal{K}$, then all of its faces are necessarily in $\mathcal{K}$ as well. If $\sigma \in \mathcal{K}$ with no proper cofaces, then we say that σ is a *maximal cube* in K. We denote the set of closed cubes in $(K, \mathcal{K})$ by $\overline{\mathcal{K}}$; the set of closed cubes in $\overline{\mathcal{K}}$ is in one-to-one correspondence with the open cubes in $\mathcal{K}$. Specifically, vertices in $\overline{\mathcal{K}}$ correspond to vertices in $\mathcal{K}$ and all other elementary cubes in $\overline{\mathcal{K}}$ correspond to their interiors in $\mathcal{K}$. Throughout this paper, we denote the set of zero-cubes in $\mathcal{K}$ by $\text{verts}(K)$ and note that $\text{verts}(K) \subsetneq \mathbb{Z}^n$, since all cubes in $(K, \mathcal{K})$ are elementary cubes.

The *product order on* $\mathbb{R}^n$, denoted $\preceq$, is the partial order such that for two points $\mathbf{p} = (p_1, p_2, \dots, p_n)$ and $\mathbf{q} = (q_1, q_2, \dots, q_n)$ in $\mathbb{R}^n$, we have $\mathbf{p} \preceq \mathbf{q}$ if and only if $p_i \leq q_i$ for each coordinate i. Using this partial order, we define the interval of points in $\mathbb{R}^n$ between $\mathbf{p}$ and $\mathbf{q}$ as

$$[\mathbf{p}, \mathbf{q}] := \{\mathbf{x} \mid \mathbf{p} \preceq \mathbf{x} \preceq \mathbf{q}\}.$$

The point $\mathbf{p}$ is the minimum vertex of the interval and $\mathbf{q}$ is the maximum vertex of the interval, with respect to $\preceq$. Notationally, we write this as $\min([\mathbf{p}, \mathbf{q}]) := \mathbf{p}$ and $\max([\mathbf{p}, \mathbf{q}]) := \mathbf{q}$. When $\mathbf{q} \in \mathbb{Z}^n$ and $\mathbf{p} = \mathbf{q} + \mathbf{j}$, for some $\mathbf{j} \in \{0, 1\}^n$, the interval $[\mathbf{p}, \mathbf{q}]$ is an elementary cube as defined in Eq. (1). If, in addition, $\mathbf{j}$ is not the zero vector, then we say that $[\mathbf{v} - \mathbf{j}, \mathbf{v}]$ is a *lower coface* of $\mathbf{v}$.

Using the fact that the partial order $(\mathbb{R}^n, \preceq)$ induces a partial order on the points in K, we define directed paths in K as the set of nondecreasing paths in K: A *path* in K is a continuous map from the unit interval $I = [0, 1]$ to K. We say that a path $\gamma \colon I \to K$ goes from $\gamma(0)$ to $\gamma(1)$. Letting K^I denote the set of all paths in K, the set of *directed paths* (or *dipaths* for short) is

$$\overrightarrow{P}(K) := \{\gamma \in K^I \mid \forall i, j \text{ s.t. } 0 \leq i \leq j \leq 1, \gamma(i) \preceq \gamma(j)\}.$$

We topologize $\overrightarrow{P}(K)$ using the compact-open topology. For $\mathbf{p}, \mathbf{q} \in K$, we denote the subspace of dipaths from $\mathbf{p}$ to $\mathbf{q}$ by $\overrightarrow{P}_{\mathbf{p}}^{\mathbf{q}}(K)$. We refer to $(K, \overrightarrow{P}(K))$ as a *directed Euclidean cubical complex*.[1] The connected components of $\overrightarrow{P}_{\mathbf{p}}^{\mathbf{q}}(K)$ are exactly the equivalence classes of dipaths, up to dihomotopy. If two dipaths, f and g are homotopic through a continuous family of dipaths, then f and g are called *dihomotopic*.

Given a directed complex, certain subcomplexes are of interest:

Definition 2.1 (Special Complexes)

Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$. Let $\mathbf{p} \in \text{verts}(K)$ and let σ be an elementary cube (that need not be in $\mathcal{K}$).

1. *The complex above* $\mathbf{p}$ *is* $K_{\mathbf{p}\preceq} := \{\mathbf{q} \in K \mid \mathbf{p} \preceq \mathbf{q}\}$.
2. *The complex below* $\mathbf{p}$ *is* $K_{\preceq\mathbf{p}} := \{\mathbf{q} \in K \mid \mathbf{q} \preceq \mathbf{p}\}$.

[1] Directed Euclidean cubical complexes are an example of a more general concept known as *directed space* (d-spaces). To define a d-space, we have a topological space X and we define a set of dipaths $P'(X) \subseteq X^I$ that contains all constant paths, and is closed under taking nondecreasing reparameterizations, concatenations, and subpaths. Indeed, $\overrightarrow{P}(K)$ satisfies these properties.

3. The reachable complex from $\mathbf{p}$ *is* $\text{reach}(K, \mathbf{p}) := \{\mathbf{q} \in K \mid \vec{P}_{\mathbf{p}}^{\mathbf{q}}(K) \neq \emptyset\}$.
4. The complex restricted to σ *is*

$$K|_{\sigma} := \bigcup \{\tau \in \mathcal{K} \mid \min \sigma \preceq \min \tau \preceq \max \tau \preceq \max \sigma\}.$$

5. If $K = I^n$*, then we call* $(K, \mathcal{K})$ *the* standard unit cubical complex *and often denote it by* $(I^n, \mathcal{I})$*. If* $K = I^n + \mathbf{x}$ *for some* $\mathbf{x} \in \mathbb{Z}^n$*, then* K *is a full-dimensional unit cubical complex.*

2.2 Past Links of Directed Cubical Compelxes

An *abstract simplicial complex* is a finite collection $\mathcal{S}$ of sets that is closed under the subset relation, i.e., if $A \in \mathcal{S}$ and B is a set such that $\emptyset \neq B \subseteq A$, then $B \in \mathcal{S}$. The sets in $\mathcal{S}$ are called *simplices*. If the simplex A has $k+1$ elements, then we say that the dimension of A is $\dim(A) := k$, and we say A is a k-simplex. For example, the zero-simplices are the singleton sets and are often referred to as vertices. Since every element of a set $A \in \mathcal{S}$ gives rise to a singleton set in the finite set $\mathcal{S}$, A must be finite.

In a topological space embedded in $\mathbb{R}^n$, the link of a point $\mathbf{v}$ is constructed by intersecting an arbitrarily small $(n-1)$-sphere around $\mathbf{v}$ with the space itself. In $\mathbb{R}^n$, the link of a point is an $(n-1)$-sphere. Moreover, if $\mathbf{v} \in \mathbb{Z}^n$, the link inherits the stratification as a subcomplex of $\mathbb{R}^n$, and can be represented as a simplicial complex whose i-simplices are in one-to-one correspondence with the $(i+1)$-dimensional cofaces of $\mathbf{v}$. The past link of $\mathbf{v}$ is the restriction of the link using the set of lower cofaces of $\mathbf{v}$ instead of all cofaces. Thus, we can represent each simplex in the past link as a vector in $\{0,1\}^n \setminus \{\mathbf{0}\}$, where the vector $\mathbf{j} \in \{0,1\}^n \setminus \{\mathbf{0}\}$ represents the cube $[\mathbf{v}-\mathbf{j}, \mathbf{v}]$ in the simplex-cube correspondence. As a simplicial complex, the past link of $\mathbf{v}$ in $\mathbb{R}^n$ has n vertices $\{x_i\}_{1 \leq i \leq n}$, and $\mathbf{j}$ represents the simplex $\{x_i | 1 \leq i \leq n, j_i = 1\}$ of dimension $||\mathbf{j}||_1 - 1$; for example, $(1,0,0)$ represents a vertex and $(1,0,1)$ represents an edge. We are now ready to define the past link of a vertex in a Euclidean cubical complex:

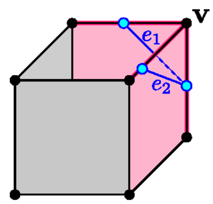
Definition 2.2 (Past Link)

Let $(K, \mathcal{K})$ *be a directed Euclidean cubical complex in* $\mathbb{R}^n$*. Let* $\mathbf{v} \in \mathbb{Z}^n$*. The* past link *of* $\mathbf{v}$ *is the following simplicial complex:*

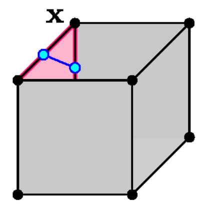
$$\text{lk}_K^-(\mathbf{v}) := \{\mathbf{j} \in \{0,1\}^n \setminus \{\mathbf{0}\} \mid [\mathbf{v} - \mathbf{j}, \mathbf{v}] \subseteq K\}.$$

As a set, the past link represents all elementary cubes in K for which $\mathbf{v}$ is the maximum vertex. As a simplicial complex, it describes (locally) the different types of dipaths to or through $\mathbf{v}$ in K; see Fig. 3.

We conclude this section with a lemma summarizing properties of the past link, most of which follow directly from definitions:



(a) Past Link of $\mathbf{v}$



(b) Past Link of $\mathbf{x}$

Fig. 3: Past link in the Open Top Box. (a) The maximum vertex of this complex is $\mathbf{v} = (v_1, v_2, v_3)$. The past link $\mathrm{lk}^-_K(\mathbf{v})$ is the simplicial complex comprising three vertices and two edges (shown in blue/cyan). These simplices are in one-to-one correspondence with the set of lower cofaces of $\mathbf{v}$ (highlighted in pink). For example, the edges of $\mathrm{lk}^-_K(\mathbf{v})$, which are labeled e_1 and e_2, are in one-to-one correspondence with the elementary two-cubes that are lower cofaces of $\mathbf{v}$ ($\sigma_1 = [(v_1-1, v_2, v_3-1), \mathbf{v}]$ and $\sigma_2 = [(v_1, v_2-1, v_3-1), \mathbf{v}]$, respectively). In the vector notation for simplices of $\mathrm{lk}^-_K(\mathbf{v})$, we write $e_1 = (1, 0, 1)$ and $e_2 = (0, 1, 1)$. (b) The past link of a vertex $\mathbf{x}$ that is neither the minimum nor the maximum vertex in the complex.

Lemma 2.3 (Properties of Past Links)

Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$. Then, the following statements hold for all $\mathbf{v} \in \mathbb{Z}^n$:

1. $\mathrm{lk}^-_K(\mathbf{v}) = \bigcup_{\mathbf{p} \in \mathbb{R}^n} \mathrm{lk}^-_{K_{\mathbf{p}\preceq}}(\mathbf{v})$.
2. *If $(K', \mathcal{K}')$ is a subcomplex of $(K, \mathcal{K})$, then $\mathrm{lk}^-_{K'}(\mathbf{v}) \subseteq \mathrm{lk}^-_K(\mathbf{v})$.*
3. $\mathrm{lk}^-_K(\mathbf{v}) = \mathrm{lk}^-_{K_{\preceq \mathbf{v}}}(\mathbf{v})$.
4. *If there exists $\mathbf{w} \in \mathbb{Z}^n$ such that $K = [\mathbf{w} - \mathbf{1}, \mathbf{w}]$, then $\mathrm{lk}^-_K(\mathbf{w})$ is the complete simplicial complex on n vertices.*
5. *$\mathrm{lk}^-_K(\mathbf{v})$ is a subcomplex of the complete simplicial complex on n vertices.[[**for camera-ready:** this is not cited in the proofs, but I think we do use it. can we identify where so that it can be cited?]]*

Proof Statement *1*: If $K = \emptyset$, then all past links are empty and the equality trivially holds. If $K \neq \emptyset$, then $\mathrm{verts}(K)$ is a finite non empty set. Thus, there exists $\mathbf{q} \in \mathbb{R}^n$ such that for all $\mathbf{w} \in \mathrm{verts}(K)$, $\mathbf{q} \preceq \mathbf{w}$. Let $\mathbf{j} \in \mathrm{lk}^-_K(\mathbf{v})$. Then, $[\mathbf{v} - \mathbf{j}, \mathbf{v}] \subseteq K$ and so $\mathbf{v} - \mathbf{j} \in \mathrm{verts}(K)$. Hence, $\mathbf{q} \preceq \mathbf{v} - \mathbf{j}$, which means that $\mathbf{j} \in \mathrm{lk}^-_{K_{\mathbf{q}\preceq}}(\mathbf{v}) \subseteq \bigcup_{\mathbf{p} \in \mathbb{R}^n} \mathrm{lk}^-_{K_{\mathbf{p}\preceq}}(\mathbf{v})$. The reverse inclusion follows from the fact that each of these statements holds if and only if.

Statement *2*: Observe that if $\mathbf{j} \in \mathrm{lk}^-_{K'}(\mathbf{v})$, then, by definition of the past link, $[\mathbf{v} - \mathbf{j}, \mathbf{v}] \subseteq K'$. Since $K' \subseteq K$, we have $[\mathbf{v} - \mathbf{j}, \mathbf{v}] \subseteq K' \subseteq K$. Therefore, we can conclude that $\mathbf{j} \in \mathrm{lk}^-_K(\mathbf{v})$.

Statement *3*: By Statement *2* (which we just proved), we have the following inclusion $\mathrm{lk}^-_{K_{\preceq \mathbf{v}}}(\mathbf{v}) \subseteq \mathrm{lk}^-_K(\mathbf{v})$. To prove the inclusion in the other direction, let $\mathbf{j} \in \mathrm{lk}^-_K(\mathbf{v})$. Since $\mathbf{v}-\mathbf{j} \preceq \mathbf{v}$, then $[\mathbf{v}-\mathbf{j}, \mathbf{v}] \subseteq K_{\preceq \mathbf{v}}$. Therefore, we conclude that $\mathrm{lk}^-_K(\mathbf{v}) \subseteq \mathrm{lk}^-_{K_{\preceq \mathbf{v}}}(\mathbf{v})$.

Statement *4*: Since $K = [\mathbf{w}-\mathbf{1}, \mathbf{w}]$, we know that K is full-dimensional, and so for all $\mathbf{j} \in \{0,1\}^n$, $[\mathbf{w}-\mathbf{j}, \mathbf{w}] \subseteq K$. Thus, by definition of past link, we have that the past link of $\mathbf{w}$ is: $\mathrm{lk}^-_K(\mathbf{w}) := \{0,1\}^n \setminus \{\mathbf{0}\}$, which is the complete simplicial complex on n vertices.

Statement *5*: Let $L = K \cap [\mathbf{v}-\mathbf{1}, \mathbf{v}]$. By definition of past link, we know $\mathrm{lk}^-_L(\mathbf{v}) = \mathrm{lk}^-_K(\mathbf{v})$. By Statement *2*, since L is a subcomplex of $[\mathbf{v}-\mathbf{1}, \mathbf{v}]$, we know $\mathrm{lk}^-_L(\mathbf{v}) \subseteq \mathrm{lk}^-_{[\mathbf{v}-\mathbf{1},\mathbf{v}]}(\mathbf{v})$. By Statement *4*, $\mathrm{lk}^-_{[\mathbf{v}-\mathbf{1},\mathbf{v}]}(\mathbf{v})$ is the complete simplicial complex on n vertices. Therefore, $\mathrm{lk}^-_K(\mathbf{w})$ is the complete simplicial complex on n vertices. □

2.3 Relationship Between Past Links and Path Spaces

The topology of the past links is intrinsically related to spaces of dipaths. Specifically, in [1] we prove that the contractability and/or connectedness of past links of vertices in directed Euclidean cubical complexes with a minimum vertex[2] implies that all spaces of dipaths with $\mathbf{w}$ as initial point are also contractible and/or connected.

Theorem 2.4 (Contractability [1, Theorem 1])

Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$ that has a minimum vertex $\mathbf{w}$. If, for all vertices $\mathbf{v} \in \mathrm{verts}(K)$, the past link $\mathrm{lk}^-_K(\mathbf{v})$ is contractible, then the space $\vec{P}^{\mathbf{k}}_{\mathbf{w}}(K)$ is contractible for all $\mathbf{k} \in \mathrm{verts}(K)$.

An analogous theorem for connectedness also holds.

Theorem 2.5 (Connectedness [1, Theorem 2])

Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$ that has a minimum vertex $\mathbf{w}$. Suppose that, for all $\mathbf{v} \in \mathrm{verts}(K)$, the past link $\mathrm{lk}^-_K(\mathbf{v})$ is connected. Then, for all $\mathbf{k} \in \mathrm{verts}(K)$, the space $\vec{P}^{\mathbf{k}}_{\mathbf{w}}(K)$ is connected.

Furthermore, we proved a partial converse to Theorem 2.5. Specifically, the converse holds only if K is a reachable directed Euclidean cubical complex as defined in Statement *3* of Definition 2.1. This is expected: Properties of parts of the directed Euclidean complex which are not reachable from $\mathbf{w}$, do not influence the dipath spaces from $\mathbf{w}$.

Theorem 2.6 (Realizing Obstructions [1, Theorem 3])

Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$. Let $\mathbf{w} \in \mathrm{verts}(K)$, and let $L = \mathrm{reach}(K, \mathbf{w})$. Let $\mathbf{v} \in \mathrm{verts}(L)$. If the past link $\mathrm{lk}^-_L(\mathbf{v})$ is disconnected, then the space $\vec{P}^{\mathbf{v}}_{\mathbf{w}}(K)$ is disconnected.

[2] In [1], the minimum (initial) vertex was often assumed to be $\mathbf{0}$ for ease of exposition. We restate the lemmas and theorems here using more general notation, where K has a minimum vertex $\mathbf{w}$.

3 Directed Collapsing Pairs

Although simplicial collapses preserve the homotopy type of the underlying space [14, Proposition 6.14] and hence of all path spaces, this type of collapsing in directed Euclidean cubical complexes may not preserve topological properties of spaces of dipaths. In this section, we study a specific type of collapsing called a link-preserving directed collapse. We define link-preserving directed collapses in Section 3.1 and give properties of link-preserving directed collapses in Section 3.2.

3.1 Link-Preserving Directed Collapses

Since we are interested in preserving the dipath spaces through collapses, the results from Section 2.3 motivate us to study a type of directed collapse (DC) via past links, introduced in [1]. However, we call it a *link-preserving directed collapse* (LPDC) (as opposed to a *directed collapse*) since we show in the last sections of this paper that when the spaces of dipaths starting from the minimum vertex are not connected, the following definition of collapse does not preserve the number of components.

Definition 3.1 (Link Preserving Directed Collapse)

Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$. Let $\sigma \in \mathcal{K}$ be a maximal cube, and let τ be a proper face of σ such that no other maximal cube contains τ (in this case, we say that τ is a free face *of σ). Then, we define the (τ, σ)-collapse of K as the subcomplex obtained by removing everything in between τ and σ:*

$$K' = K \setminus \{\gamma \in \mathcal{K} \mid \overline{\tau} \subseteq \overline{\gamma} \subseteq \overline{\sigma}\}, \tag{2}$$

and let $\mathcal{K}'$ denote the stratification of the set K' induced by the cubical stratification of $\mathbb{R}^n$ (thus, $\mathcal{K}' \subsetneq \mathcal{K}$).

We call the directed Euclidean cubical complex $(K', \mathcal{K}')$ a link-preserving directed collapse (LPDC) *of $(K, \mathcal{K})$ if, for all $\mathbf{v} \in \text{verts}(K')$, the past link $\text{lk}^-_K(\mathbf{v})$ is homotopy equivalent to $\text{lk}^-_{K'}(\mathbf{v})$ (denoted $\text{lk}^-_K(\mathbf{v}) \simeq \text{lk}^-_{K'}(\mathbf{v})$). The pair (τ, σ) is then called an* LPDC pair.

Remark 3.2 (Simplicial Collapses) *The study of simplicial collapses is known as* simple homotopy theory *[3, 19], and traces back to the work of Whitehead in the 1930s [18]. The idea is very similar: If C is an abstract simplicial complex and $\alpha \in C$ such that α is a proper face of exactly one maximal simplex β, then the following complex is the α-*collapse of C in the simplicial setting*:*

$$C' = C \setminus \{\gamma \in C \mid \alpha \subseteq \gamma \subseteq \beta\}.$$

Note that we use only the free face (α) when defining a simplicial collapse, as doing so helps to distinguish between discussing a simplicial collapse and a directed Euclidean cubical collapse. In addition, we always explicitly state "in the simplicial setting" when talking about a simplicial collapse.

Applying a sequence of LPDCs to a directed Euclidean cubical complex can reduce the number of cubes, and hence can more clearly illustrate the number of dihomotopy classes of dipaths within the directed Euclidean cubical complex. For an example, see Fig. 4. However, it is not necessarily true that LPDCs preserve dipath spaces. We discuss the relationship between dipath spaces and LPDCs in Section 4.

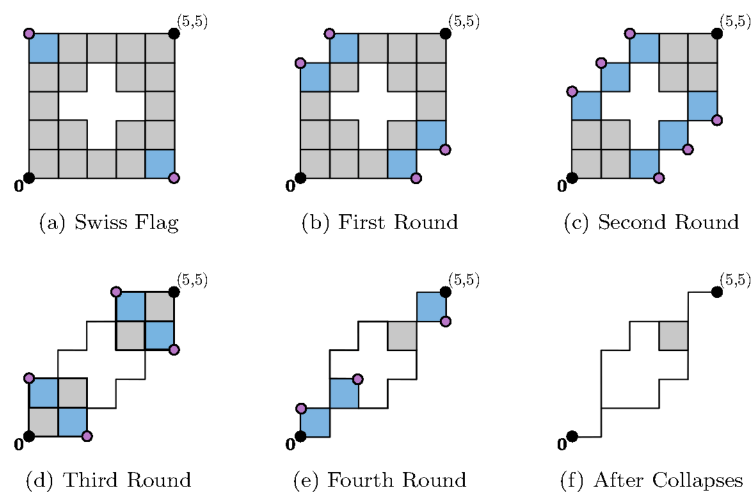


Fig. 4: Collapsing the Swiss Flag. A sequence of vertex collapses is presented from the top left to bottom right. At each stage, the faces and vertices shaded in blue and purple represent the vertex collapsing pairs with the blue Euclidean cube being σ and the purple vertex being τ. The result of the sequence of LPDCs is shown in (f) is a one-dimensional directed Euclidean cubical complex and one two-cube. Observe that this directed Euclidean cubical complex clearly illustrates the two dihomotopy classes of $\vec{P}_{\mathbf{0}}^{(5,5)}(K)$.

3.2 Properties of LPDCs

We give a combinatorial condition for a collapsing pair (τ, σ) to be an LPDC pair; namely, the condition is that τ does not contain the vertex $\min(\sigma)$.

From the definition of an LPDC, we see that finding an LPDC pair requires computing the past link of *all* vertices in $\text{verts}(K')$. In [1], we discussed how we can reduce the check down to only the vertices in σ since no other vertices have their past links affected. In this paper, we prove we need to only check *one* condition to determine if we have an LPDC pair. The one simple condition dramatically reduces the number of computations we need to perform in order to verify we have an LPDC. This result given in Theorem 3.9 depends on the following lemmas about the properties of past links on vertices.

Lemma 3.3 (Properties of Past Links in a Vertex Collapse)

Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$. Let $\sigma \in \mathcal{K}$ and $\tau, \mathbf{v} \in \text{verts}(\sigma)$ such that $\tau \preceq \mathbf{v}$. If τ is a free face of σ and K' is the (τ, σ)-collapse, then the following two statements hold:

1. $\text{lk}^-_{K|_\sigma}(\mathbf{v}) = \{\mathbf{j} \in \{0,1\}^n \setminus \{\mathbf{0}\} \mid \min(\sigma) \preceq \mathbf{v} - \mathbf{j}\}$.
2. $\text{lk}^-_{K'|_\sigma}(\mathbf{v}) = \text{lk}^-_{K|_\sigma}(\mathbf{v}) \setminus \{\mathbf{j} \in \{0,1\}^n \setminus \{\mathbf{0}\} \mid \mathbf{v} - \mathbf{j} \preceq \tau\}$.

Proof To ease notation, we define the following two sets:

$$J := \{\mathbf{j} \in \{0,1\}^n \setminus \{\mathbf{0}\} \mid \min(\sigma) \preceq \mathbf{v} - \mathbf{j}\}$$
$$I := \{\mathbf{j} \in \{0,1\}^n \setminus \{\mathbf{0}\} \mid \mathbf{v} - \mathbf{j} \preceq \tau\}.$$

First, we prove Statement *1* (that $\text{lk}^-_{K|_\sigma}(\mathbf{v}) = J$). We start with the forward inclusion. Let $\mathbf{j} \in \text{lk}^-_{K|_\sigma}(\mathbf{v})$. By the definition of past links (see Definition 2.2), we know that $[\mathbf{v} - \mathbf{j}, \mathbf{v}] \subseteq K|_\sigma$. By the definition of $K|_\sigma$ (see Definition 2.1), we know that $\min(\sigma) \preceq \min([\mathbf{v} - \mathbf{j}, \mathbf{v}]) = \mathbf{v} - \mathbf{j}$. This implies $\mathbf{j} \in J$. Therefore, $\text{lk}^-_{K|_\sigma}(\mathbf{v}) \subseteq J$. For the backward inclusion, let $\mathbf{j} \in J$. Then, since $\mathbf{v} \in \text{verts}(\sigma)$ and σ is an elementary cube by assumption, and $\min(\sigma) \preceq \mathbf{v} - \mathbf{j}$ by definition of J, we have $\mathbf{v} - \mathbf{j} \in \text{verts}(\sigma)$. Since $\sigma \in \mathcal{K}$, all faces must be in $\mathcal{K}$; hence, $[\mathbf{v} - \mathbf{j}, \mathbf{v}] \subseteq K|_\sigma$. Therefore, $\mathbf{j} \in \text{lk}^-_{K|_\sigma}(\mathbf{v})$, and so $\text{lk}^-_{K|_\sigma}(\mathbf{v}) \supseteq J$. Since we have both inclusions, then Statement *1* holds.

Now, we prove Statement *2* (that $\text{lk}^-_{K'|_\sigma}(\mathbf{v}) = J \setminus I$). Again, we prove the inclusions in both directions. For the forward inclusion, let $\mathbf{j} \in \text{lk}^-_{K'|_\sigma}(\mathbf{v})$. By Statement *2* of Lemma 2.3, we have $\text{lk}^-_{K'|_\sigma}(\mathbf{v}) \subseteq \text{lk}^-_{K|_\sigma}(\mathbf{v})$, and so, we obtain $\mathbf{j} \in \text{lk}^-_{K|_\sigma}(\mathbf{v}) = J$. Next, we must show that $\mathbf{j} \notin I$. Assume, for a contradiction, that $\mathbf{j} \in I$. Then, by definition of I, $\mathbf{v} - \mathbf{j} \preceq \tau$. Since $\tau \preceq \mathbf{v}$, we obtain the partial order $\mathbf{v} - \mathbf{j} \preceq \tau \preceq \mathbf{v}$. This implies that $[\tau, \mathbf{v}] \subseteq [\mathbf{v} - \mathbf{j}, \mathbf{v}]$. Since $[\mathbf{v} - \mathbf{j}, \mathbf{v}]$ is an elementary cube in $K'|_\sigma$, then its face $[\tau, \mathbf{v}]$ must also be an elementary cube in $K'|_\sigma$. Setting $\overline{\gamma} = [\tau, \mathbf{v}]$ and observing $\tau = \overline{\tau} \subseteq \overline{\gamma} \subseteq \overline{\sigma}$, we observe that γ is not an elementary cube in K' by Eq. (2). This gives us a contradiction and so $\mathbf{j} \notin I$. Therefore, $\text{lk}^-_{K'|_\sigma}(\mathbf{v}) \subseteq J \setminus I$.

Finally, we prove the backward inclusion of Statement *2*. Let $\mathbf{j} \in J \setminus I$. Then, by Statement *1*, $\mathbf{j} \in \text{lk}^-_{K|_\sigma}(\mathbf{v})$ and either $\tau \prec \mathbf{v} - \mathbf{j}$ or τ is not comparable to $\mathbf{v} - \mathbf{j}$ under $\preceq$. Thus, by Eq. (2), $[\mathbf{v} - \mathbf{j}, \mathbf{v}]$ is an elementary cube of $K'|_\sigma$.

Thus, by Definition 2.2, we have that $\mathbf{j} \in \mathrm{lk}^-_{K'|_\sigma}(\mathbf{v})$. Hence, $J \setminus I \subseteq \mathrm{lk}^-_{K'|_\sigma}(\mathbf{v})$, and so Statement *2* holds. □

Using Lemma 3.3, we see why τ cannot be the vertex $\min(\sigma)$ when performing an LPDC. If $\tau = \min(\sigma)$, then

$$\begin{aligned}
\mathrm{lk}^-_{K'|_\sigma}(\mathbf{v}) = \{&\mathbf{j} \in \{0,1\}^n \setminus \{\mathbf{0}\} \mid \min(\sigma) \preceq \mathbf{v} - \mathbf{j}\} \\
&\setminus \{\mathbf{j} \in \{0,1\}^n \setminus \{\mathbf{0}\} \mid \mathbf{v} - \mathbf{j} \preceq \min(\sigma)\} \\
&= \{\mathbf{j} \in \{0,1\}^n \setminus \{\mathbf{0}\} \mid \min(\sigma) \preceq \mathbf{v} - \mathbf{j} \text{ and } \mathbf{v} - \mathbf{j} \succ \min(\sigma)\} \\
&= \{\mathbf{j} \in \{0,1\}^n \setminus \{\mathbf{0}\} \mid \min(\sigma) \prec \mathbf{v} - \mathbf{j}\} \\
&= \{\mathbf{j} \in \{0,1\}^n \setminus \{\mathbf{0}\} \mid \mathbf{j} \prec \mathbf{v} - \min(\sigma)\}.
\end{aligned}$$

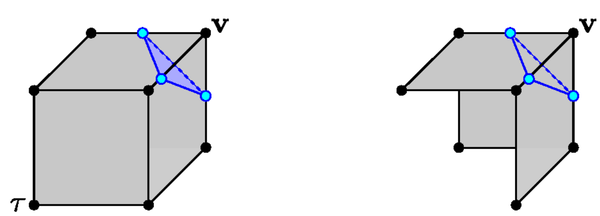
If $\mathbf{v}$ is the maximum vertex of σ, then we obtain $\mathrm{lk}^-_{K'|_\sigma}(\mathbf{v}) = \{0,1\}^n \setminus \{\mathbf{0}, \mathbf{v} - \min(\sigma)\}$. This computation gives us the following corollary, which we illustrate in Fig. 5 when K is a single closed three-cube.

Corollary 3.4 (Caution for a $(\min(\sigma), \sigma)$-Collapse)

Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$. Let $\sigma \in \mathcal{K}$, $\tau = \min(\sigma)$, and $\mathbf{v} \in \mathrm{verts}(\sigma)$. If τ is a free face and K' is the (τ, σ)-collapse, then the past link of $\mathbf{v}$ in $K'|_\sigma$ is:

$$\{\mathbf{j} \in \{0,1\}^n \setminus \{\mathbf{0}\} \mid \mathbf{j} \prec \mathbf{v} - \min(\sigma)\}$$

In particular, if $\mathbf{v} = \max(\sigma)$ and $k = \dim(\sigma)$, then the past link is the complete complex on k elements before the collapse, and, after the collapse, it is homeomorphic to $\mathbb{S}^{k-2}$. Thus, (τ, σ) is not an LPDC pair.



(a) Initial Complex (b) After Collapse

Fig. 5: Removing the minimum vertex of a cube. Consider the directed Euclidean cubical complex in (a), which as a subset of $\mathbb{R}^3$ is a single closed three-cube; call this three-cube σ. Letting $\tau = \min(\sigma)$, we observe that the past link of $\mathbf{v} = \max(\sigma)$ is contractible before the (τ, σ)-collapse and is homeomorphic to $\mathbb{S}^1$ after the collapse. Thus, the past links before and after the collapse are not homotopy equivalent, and so this collapse is not an LPDC.

The following lemma shows under which condition a directed Euclidean cubical collapse induces a simplicial collapse in the past link.

Lemma 3.5 (Vertex Collapses that Induce Simplicial Collapse of Past Links)

Let $(K, \mathcal{K})$ *be a directed Euclidean cubical complex in* $\mathbb{R}^n$*. Let* $\sigma \in \mathcal{K}$ *and* $\tau, \mathbf{v} \in \text{verts}(\sigma)$ *such that* $\tau \preceq \mathbf{v}$ *and* $\tau \neq \min(\sigma)$*. If* τ *is a free face of* σ *and* K' *is the* (τ, σ)*-collapse, then* $\text{lk}^-_{K'}(\mathbf{v})$ *is the* $(\mathbf{v} - \tau)$*-collapse of* $\text{lk}^-_K(\mathbf{v})$ *in the simplicial setting.*

Proof Consider $K_{\preceq \mathbf{v}}$. Since $\tau, \mathbf{v} \in \text{verts}(\sigma)$ and σ is maximal in K, we know $[\min(\sigma), \mathbf{v}]$ and $[\tau, \mathbf{v}]$ are elementary cubes in $K_{\preceq \mathbf{v}}$. Since τ is a free face of σ, we further know that $[\min(\sigma), \mathbf{v}]$ is the only maximal proper coface of $[\tau, \mathbf{v}]$ in $K_{\preceq \mathbf{v}}$. By definition of past link (Definition 2.2), we then have that $\mathbf{v} - \min(\sigma)$ and $\mathbf{v} - \tau$ are simplices in $\text{lk}^-_{K_{\preceq \mathbf{v}}}(\mathbf{v})$, and $\mathbf{v} - \min(\sigma)$ is the only maximal proper coface of $\mathbf{v} - \tau$ in $\text{lk}^-_{K_{\preceq \mathbf{v}}}(\mathbf{v})$. Hence, $\mathbf{v} - \tau$ is free in $\text{lk}^-_{K_{\preceq \mathbf{v}}}(\mathbf{v})$. Moreover, $\text{lk}^-_{K'_{\preceq \mathbf{v}}}(\mathbf{v})$ is the $(\mathbf{v} - \tau)$-collapse of $\text{lk}^-_{K_{\preceq \mathbf{v}}}(\mathbf{v})$. One can see this by using Statement *2* of Lemma 3.3 by which $\text{lk}^-_{K'_{\preceq \mathbf{v}}}(\mathbf{v})$ can be characterized as the $(\mathbf{v} - \tau)$-collapse of $\text{lk}^-_{K_{\preceq \mathbf{v}}}(\mathbf{v})$.

By Statement *3* of Lemma 2.3, we know that $\text{lk}^-_K(\mathbf{v}) = \text{lk}^-_{K_{\preceq \mathbf{v}}}(\mathbf{v})$ and that $\text{lk}^-_{K'}(\mathbf{v}) = \text{lk}^-_{K'_{\preceq \mathbf{v}}}(\mathbf{v})$, which concludes this proof. □

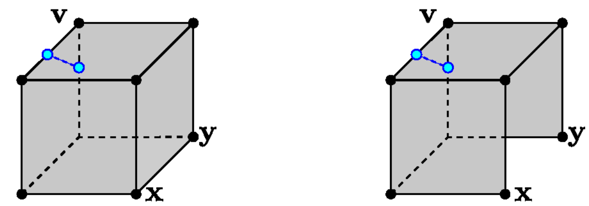
Next, we prove two lemmas concerning relationships of the past link of a vertex in the original directed Euclidean cubical complex and in the collapsed directed Euclidean cubical complex. These relationships depend on where $\mathbf{v}$ is located with respect to τ. In the first lemma, we consider the case where $\min(\tau) \not\preceq \mathbf{v}$, and we present a sufficient condition for past links in K and the (τ, σ)-collapse to be equal. See Fig. 6 for an example that illustrates the result of this lemma.

Lemma 3.6 (Condition for Past Links in K and K' to be Equal)

Let $(K, \mathcal{K})$ *be a directed Euclidean cubical complex in* $\mathbb{R}^n$*. Let* $\tau, \sigma \in \mathcal{K}$ *such that* τ *is a face of* σ*. If* τ *is a free face of* σ *and* K' *is the* (τ, σ)*-collapse, then, for all* $\mathbf{v} \in \text{verts}(K)$ *such that* $\max(\tau) \not\preceq \mathbf{v}$*, we have* $\text{lk}^-_K(\mathbf{v}) = \text{lk}^-_{K'}(\mathbf{v})$*.*

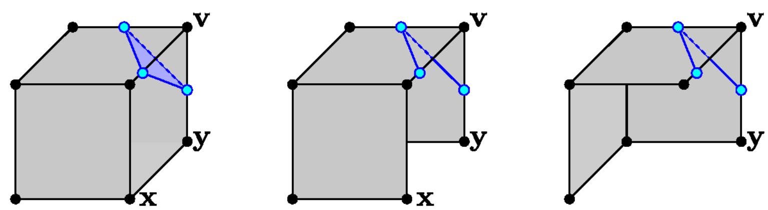
Proof By Statement *2* of Lemma 2.3, we have $\text{lk}^-_{K'}(\mathbf{v}) \subseteq \text{lk}^-_K(\mathbf{v})$. Thus, we only need to show $\text{lk}^-_K(\mathbf{v}) \subseteq \text{lk}^-_{K'}(\mathbf{v})$. Suppose $\mathbf{j} \in \text{lk}^-_K(\mathbf{v})$. By the definition of the past link (see Definition 2.2), we know that $[\mathbf{v} - \mathbf{j}, \mathbf{v}]$ is an elementary cube in K. By assumption, $\max(\tau) \not\preceq \mathbf{v}$. Thus, by Eq. (2), $[\mathbf{v} - \mathbf{j}, \mathbf{v}]$ is not removed from K and thus is an elementary cube in K'. Thus, $\mathbf{j} \in \text{lk}^-_{K'}(\mathbf{v})$.□

In the following lemma, we consider the case where $\max(\tau) \preceq \mathbf{v}$, and we present a sufficient condition for past links in the (τ, σ)-collapse and the $(\min(\tau), \sigma)$-collapse to be equal. See Fig. 7 for an example that illustrates this result.



(a) Initial Complex (b) After Collapse

Fig. 6: Past link of an "uncomparable" vertex before and after a collapse. Consider the directed Euclidean cubical complex shown, comprising a single three-cube σ and all of its faces. Let $\tau = [\mathbf{x}, \mathbf{y}]$. Since $\mathbf{v}$ and $\max(\tau) = \mathbf{y}$ are not comparable, by Lemma 3.6, the past link of $\mathbf{v}$ is the same before and after the collapse. Indeed, we see that this is the case for this example. The past link of $\mathbf{v}$ is the complete complex on two vertices, both before and after.



(a) Original Complex (b) The (τ, σ)-Collapse (c) The $(\mathbf{x}, \sigma)$-Collapse

Fig. 7: Two collapses with same past links. For example, in the directed Euclidean cubical complex K shown in (a), let σ be the three-cube, and let $\tau = [\mathbf{x}, \mathbf{y}]$. We look at the past link of the vertex $\mathbf{v}$. In the original directed Euclidean cubical complex, the past link of $\mathbf{v}$ is the complete complex on three vertices. By Lemma 3.7, the past link of $\mathbf{v}$ is the same in both the (τ, σ)-collapse and the $(\mathbf{x}, \sigma)$-collapse since $\max(\tau) = \mathbf{y} \preceq \mathbf{v}$. By Lemma 3.8, we also know that the past links of $\mathbf{v}$ in K and the $(\mathbf{x}, \sigma)$-collapse are homotopy equivalent. Indeed, we see that this is the case.

Lemma 3.7 (Comparing Past Links in a General Collapse with Past Links in a Vertex Collapse)

Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$ such that there exists cubes $\tau, \sigma \in \mathcal{K}$ with $\min(\tau)$ a free face of σ. Let K' be the (τ, σ)-collapse and let $\widehat{K}$ be the $(\min(\tau), \sigma)$-collapse. If $\mathbf{v} \in \text{verts}(K')$ and $\max(\tau) \preceq \mathbf{v}$, then $\mathbf{v} \in \text{verts}(\widehat{K})$ and $\text{lk}^-_{K'}(\mathbf{v}) = \text{lk}^-_{\widehat{K}}(\mathbf{v})$.

Proof We first show $\mathbf{v} \in \text{verts}(\widehat{K})$. If τ is a zero-cube (and hence in $\text{verts}(K)$), then $K' = \widehat{K}$, which means that $\mathbf{v} \in \text{verts}(\widehat{K})$. On the other hand, if τ is not a zero-cube, then we have $\min(\tau) \prec \max(\tau) \preceq \mathbf{v}$. In particular, $\min(\tau) \neq \mathbf{v}$.

And so, by definition of $\widehat{K}$ as a $(\min(\tau), \sigma)$-collapse and since $\mathbf{v} \in \mathcal{K}$, we conclude that $\mathbf{v} \in \widehat{K}$.

Next, we show $\mathrm{lk}^-_{K'}(\mathbf{v}) = \mathrm{lk}^-_{\widehat{K}}(\mathbf{v})$. By Statement *2* of Lemma 2.3, we have $\mathrm{lk}^-_{\widehat{K}}(\mathbf{v}) \subseteq \mathrm{lk}^-_{K'}(\mathbf{v})$. Thus, what remains to be proven is $\mathrm{lk}^-_{K'}(\mathbf{v}) \subseteq \mathrm{lk}^-_{\widehat{K}}(\mathbf{v})$. Let $\mathbf{j} \in \mathrm{lk}^-_{K'}(\mathbf{v})$. By definition of the past link (Definition 2.2), we know that $[\mathbf{v}-\mathbf{j}, \mathbf{v}] \subseteq K'$. Consider two cases: $\mathbf{v}-\mathbf{j} \preceq \min(\tau)$ and $\mathbf{v}-\mathbf{j} \not\preceq \min(\tau)$.

Case 1 ($\mathbf{v}-\mathbf{j} \preceq \min(\tau)$): Since $\mathbf{v}-\mathbf{j} \preceq \min(\tau) \preceq \max(\tau) \preceq \mathbf{v}$, we know that $\overline{\tau} \subseteq [\mathbf{v}-\mathbf{j}, \mathbf{v}]$. Thus, by Eq. (2), we have $[\mathbf{v}-\mathbf{j}, \mathbf{v}] \not\subseteq K'$, which is a contradiction. So, Case 1 cannot happen.

Case 2 ($\mathbf{v}-\mathbf{j} \not\preceq \min(\tau)$): If $\mathbf{v}-\mathbf{j} \not\preceq \min(\tau)$, then, by the definition of a $(\min(\tau), \sigma)$-collapse in Definition 3.1, we know that $[\mathbf{v}-\mathbf{j}, \mathbf{v}] \subseteq \widehat{K}$ and thus $\mathbf{j} \in \mathrm{lk}^-_{\widehat{K}}(\mathbf{v})$.

Hence, $\mathrm{lk}^-_{K'}(\mathbf{v}) \subseteq \mathrm{lk}^-_{\widehat{K}}(\mathbf{v})$. Since we have both subset inclusions, we conclude $\mathrm{lk}^-_{K'}(\mathbf{v}) = \mathrm{lk}^-_{\widehat{K}}(\mathbf{v})$. □

In general, the minimal vertex of τ is not free in K and hence, there is no vertex collapse. In the main theorem, the previous lemma is applied to a subcomplex of K; specifically, it is applied to the restriction to the unit cube corresponding to σ, where all vertices, inculding $\min \tau$ are then free. The results carry over to K.

The next result states that vertex collapses result in homotopy equivalent past links as long as we are not collapsing the minimum vertex of the directed Euclidean cubical complex.

Lemma 3.8 (Past Links in a Vertex Collapse)

Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$. Let $\sigma \in \mathcal{K}$ and let $\tau \in \mathrm{verts}(\sigma)$ such that $\tau \neq \min(\sigma)$. Let $\mathbf{v} \in \mathrm{verts}(K)$ with $\mathbf{v} \neq \tau$. If τ is a free face of σ and K' is the (τ, σ)-collapse, then $\mathrm{lk}^-_K(\mathbf{v}) \simeq \mathrm{lk}^-_{K'}(\mathbf{v})$.

Proof We consider three cases:

Case 1 ($\mathbf{v} \notin \mathrm{verts}(\sigma)$): By definition of past link (Definition 2.2), if $\mathbf{v} \notin \mathrm{verts}(\sigma)$, then the past links $\mathrm{lk}^-_K(\mathbf{v})$ and $\mathrm{lk}^-_{K'}(\mathbf{v})$ are equal.

Case 2 ($\tau \not\preceq \mathbf{v}$): By Lemma 3.6, if $\tau = \max(\tau) \not\preceq \mathbf{v}$, again we have equality of the past links $\mathrm{lk}^-_K(\mathbf{v})$ and $\mathrm{lk}^-_{K'}(\mathbf{v})$.

Case 3 ($\mathbf{v} \in \mathrm{verts}(\sigma)$ and $\tau \preceq \mathbf{v}$): By Lemma 3.5, we know that $\mathrm{lk}^-_{K'}(\mathbf{v})$ is the $\mathbf{v}-\tau$-collapse of $\mathrm{lk}^-_K(\mathbf{v})$ in the simplicial setting. Since simplicial collapses preserve the homotopy type (see e.g., [14, Proposition 6.14]), we conclude $\mathrm{lk}^-_K(\mathbf{v}) \simeq \mathrm{lk}^-_{K'}(\mathbf{v})$.

We give an example of Lemma 3.8 in Fig. 7 by showing how the LPDC induces a simplicial collapse on past links.

Lastly, we are ready to prove the main result.

Theorem 3.9 (Main Theorem)

Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$ such that there exist cubes $\tau, \sigma \in \mathcal{K}$ with τ a free face of σ. Then, (τ, σ) is an LPDC pair if and only if $\min(\sigma) \notin \mathrm{verts}(\tau)$.

Proof Let $\mathbf{v} = \max(\sigma)$ and $k = \dim(\sigma)$. Let $(K', \mathcal{K}')$ be the (τ, σ)-collapse of K. Let $(L, \mathcal{L})$ be the cubical complex such that $L = K|_\sigma$. Since $\sigma \in K$, we know $L = \overline{\sigma}$ (i.e., L is a unit cube). Since L is a single unit cube and σ is a maximal elementary cube, all proper faces of σ, including τ and $\min(\tau)$, are free faces in L. Thus, let $(L', \mathcal{L}')$ be the (τ, σ)-collapse of L, and let $(\widehat{L}, \widehat{\mathcal{L}})$ be the $(\min(\tau), \sigma)$-collapse of L.

We first prove the forward direction by contrapositive (if $\min(\sigma) \in \mathrm{verts}(\tau)$, then (τ, σ) is not an LPDC pair). Assume $\min(\sigma) \in \mathrm{verts}(\tau)$. By Corollary 3.4, we obtain $\mathrm{lk}^-_L(\mathbf{v})$ is homeomorphic to $\mathbb{B}^{k-1}$ and $\mathrm{lk}^-_{\widehat{L}}(\mathbf{v})$ is homeomorphic to $\mathbb{S}^{k-2}$. Since $\min(\sigma) \in \mathrm{verts}(\tau)$, we know that $\min(\sigma) = \min(\tau)$. Since τ is a face of σ, we know $\max(\tau) \preceq \max(\sigma) = \mathbf{v}$. Since $\min(\sigma) = \min(\tau) \in \mathrm{verts}(\tau)$ and since τ is a proper face of σ, we know that $\mathbf{v} \neq \max(\tau)$. Thus, $\mathbf{v} \in \mathrm{verts}(L')$. Applying Lemma 3.7, we obtain $\mathrm{lk}^-_{L'}(\mathbf{v}) = \mathrm{lk}^-_{\widehat{L}}(\mathbf{v})$. Putting this all together, we have:

$$\mathrm{lk}^-_L(\mathbf{v}) \simeq \mathbb{B}^{d-1} \not\simeq \mathbb{S}^{k-2} \simeq \mathrm{lk}^-_{\widehat{L}}(\mathbf{v}) = \mathrm{lk}^-_{L'}(\mathbf{v}),$$

and so $\mathrm{lk}^-_L(\mathbf{v}) \not\simeq \mathrm{lk}^-_{L'}(\mathbf{v})$.

Since no faces of σ are in $\mathcal{K} \setminus \mathcal{L}$, the past link of $\mathbf{v}$ remains the same outside of L in both K and K'. Thus, $\mathrm{lk}^-_K(\mathbf{v}) \not\simeq \mathrm{lk}^-_{K'}(\mathbf{v})$ and so we conclude that (τ, σ) is not an LPDC pair, as was to be shown.

Next, we show the backwards direction. Suppose $\min(\sigma) \notin \mathrm{verts}(\tau)$. Let $\mathbf{v} \in \mathrm{verts}(K')$, and consider two cases: $\max(\tau) \not\preceq \mathbf{v}$ and $\max(\tau) \preceq \mathbf{v}$.

Case 1 ($\max(\tau) \not\preceq \mathbf{v}$): By Lemma 3.6, we have $\mathrm{lk}^-_K(\mathbf{v}) = \mathrm{lk}^-_{K'}(\mathbf{v})$. Hence, $\mathrm{lk}^-_K(\mathbf{v}) \simeq \mathrm{lk}^-_{K'}(\mathbf{v})$. Since $\mathbf{v}$ was arbitrarily chosen, we conclude that (τ, σ) is an LPDC pair.

Case 2 ($\max(\tau) \preceq \mathbf{v}$): By Lemma 3.7, we have that $\mathrm{lk}^-_{L'}(\mathbf{v}) = \mathrm{lk}^-_{\widehat{L}}(\mathbf{v})$. Since $\min(\sigma) \notin \mathrm{verts}(\tau)$, we know that $\min(\tau) \neq \min(\sigma)$. Applying Lemma 3.8, we obtain $\mathrm{lk}^-_L(\mathbf{v}) \simeq \mathrm{lk}^-_{\widehat{L}}(\mathbf{v})$. Again, since no faces of σ are removed from $\mathcal{K}$ and $\mathcal{K}'$ to obtain $\mathcal{L}$ and $\mathcal{L}'$, the past link of $\mathbf{v}$ remains the same outside of L in both K and K'. Thus, $\mathrm{lk}^-_K(\mathbf{v}) \simeq \mathrm{lk}^-_{K'}(\mathbf{v})$. Since $\mathbf{v}$ was arbitrarily chosen, we conclude that (τ, σ) is an LPDC pair. □

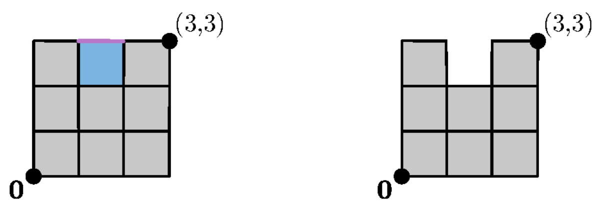
4 Preservation of Spaces of Dipaths

In [1], we proved several results on the relationships between past links and spaces of dipaths. One result, Theorem 2.4, states that for a directed Euclidean cubical complex with a minimum vertex, if all past links are contractible, then all spaces of dipaths starting at that minimum vertex are also contractible. If we start with a directed Euclidean cubical complex with a minimum vertex that has all contractible past links, then all spaces of dipaths from the minimum vertex are contractible by this theorem. We explain how those relationships extend to the LPDC setting in this section.

Applying an LPDC preserves the homotopy type of past links by definition. Hence, applying the theorem again, we see that any LPDC also has contractible dipath spaces from the minimum vertex. Notice that the minimum vertex is not removed in an LPDC, since it is a vertex and minimal in all cubes containing it (including the maximal cube). We give an example of this in Example 4.1.

Example 4.1 (3×3 filled grid)

Let K be the 3×3 filled grid. For all $\mathbf{v} \in \mathrm{verts}(K)$, $\mathrm{lk}_K^-(\mathbf{v})$ is contractible. By Theorem 2.4, this implies that all spaces of dipaths starting at $\mathbf{0}$ are contractible. Applying an LPDC such as the edge $[(1,3),(2,3)]$ results in contractible past links in K' and so all spaces of dipaths in K' are also contractible. See Fig. 8. We can generalize this example to any k^d filled grid where $k, d \in \mathbb{N}$.



(a) Initial Complex (b) After Collapse

Fig. 8: (a) The 3×3 filled grid has contractible past links and dipath spaces. The pair comprising of the purple edge $[(1,3),(2,3)]$ and the blue square $[(1,2),(2,3)]$ is an LPDC pair. (b) The result of performing the LPDC. All past links are contractible and so all dipath spaces are also contractible.

An analogous result holds for connectedness (Theorem 2.5). If we start with a directed Euclidean cubical complex such that all past links are connected, then all dipath spaces are connected. Any LPDC results in a directed Euclidean cubical complex that also has connected dipath spaces. See Example 4.2.

Example 4.2 (Outer Cubes of the $5 \times 5 \times 5$ Grid)

Let $K = [0,5]^3 \setminus [1,4]^3$, which, as an undirected complex, is homeomorphic to a thickened two-sphere. For all $\mathbf{v} \in \mathrm{verts}(K)$, $\mathrm{lk}_K^-(\mathbf{v})$ is connected. By Theorem 2.5, this implies that for all $\mathbf{v} \in \mathrm{verts}(K)$, the space of dipaths $\vec{P}_{\mathbf{0}}^{\mathbf{v}}(K)$ is connected. Applying an LPDC such as with the vertex $(5,0,0)$ in the cube $[(4,0,0),(5,1,1)]$ results in connected past links in K' and so all spaces of dipaths $\vec{P}_{\mathbf{0}}^{\mathbf{v}}(K')$ are connected. We can generalize this example to any k^d grid where $d \geq 3$ and the inner cubes of dimension d are removed.

Both Theorem 2.4 and Theorem 2.5 have assumptions on the topology of past links and results on the topology of spaces of dipaths from the minimum vertex. We may ask if the converse statements are true. Does knowing the topology of spaces of dipaths from the minimum vertex tell us anything about the topology of past links? The converse to Theorem 2.4 holds. To prove this, we first need a lemma whose proof appears in [21].

Lemma 4.3 (Homotopy Equivalence [21, Prop. 5.3])

Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$. Let $\mathbf{p}, \mathbf{q} \in \mathbb{Z}^n$. If $\vec{P}_{\mathbf{p}}^{\mathbf{q}-\mathbf{j}}(K)$ is contractible for all $\mathbf{j} \in \mathrm{lk}_K^-(\mathbf{q})$, then $\vec{P}_{\mathbf{p}}^{\mathbf{q}}(K) \simeq \mathrm{lk}_{K_{\mathbf{p}\preceq}}^-(\mathbf{q})$.

Thus, we obtain:

Theorem 4.4 (Contractability)

Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$ that has a minimum vertex $\mathbf{w}$. The following two statements are equivalent:

1. For all $\mathbf{v} \in \mathrm{verts}(K)$, the space of dipaths $\vec{P}_{\mathbf{w}}^{\mathbf{v}}(K)$ is contractible.
2. For all $\mathbf{v} \in \mathrm{verts}(K)$, the past link $\mathrm{lk}_K^-(\mathbf{v})$ is contractible.

Proof By Theorem 2.4, we obtain Statement *2* implies Statement *1*.

Next, we show that Statement *1* implies Statement *2*. Let $\mathbf{v} \in \mathrm{verts}(K)$. For all $\mathbf{j} \in \mathrm{lk}_K^-(\mathbf{v})$, the cube $[\mathbf{v}-\mathbf{j}, \mathbf{v}]$ is a subset of K, which means that $\mathbf{v}-\mathbf{j} \in \mathrm{verts}(K)$. Thus, by assumption, all dipath spaces $\vec{P}_{\mathbf{w}}^{\mathbf{v}-\mathbf{j}}(K)$ are contractible. By Lemma 4.3, we know that $\vec{P}_{\mathbf{w}}^{\mathbf{v}}(K) \simeq \mathrm{lk}_{K_{\mathbf{w}\preceq}}^-(\mathbf{v}) = \mathrm{lk}_K^-(\mathbf{v})$. Again, since $\mathbf{v} \in \mathrm{verts}(K)$, the dipath space $\vec{P}_{\mathbf{w}}^{\mathbf{v}}(K)$ is contractible. Therefore, $\mathrm{lk}_K^-(\mathbf{v})$ is contractible. □

As a consequence of this theorem, we know that if we start with a directed Euclidean cubical complex with contractible dipath spaces starting at the minimum vertex, then any LPDC also result in a directed Euclidean cubical complex with all contractible dipath spaces starting at the minimum vertex, and vice versa.

Corollary 4.5 (Preserving Directed Path Space Contractability) *Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$ that has a minimum vertex $\mathbf{w}$. Let $\tau, \sigma \in \mathcal{K}$ such that τ is a face of σ. If τ is a free face of σ, let $(K', \mathcal{K}')$ be the (τ, σ)-collapse. If K' is an LPDC of K, then the spaces of dipaths $\vec{P}_{\mathbf{w}}^{\mathbf{v}}(K)$ are contractible for all $\mathbf{v} \in \mathrm{verts}(K)$ if and only if the spaces of dipaths $\vec{P}_{\mathbf{w}}^{\mathbf{k}}(K')$ are contractible for all $\mathbf{k} \in \mathrm{verts}(K')$.*

Proof We start with the forwards direction by assuming that the spaces of dipaths $\vec{P}_{\mathbf{w}}^{\mathbf{v}}(K)$ are contractible for all $\mathbf{v} \in \mathrm{verts}(K)$. Theorem 4.4 tells us that all past links $\mathrm{lk}_K^-(\mathbf{v})$ are contractible for all $\mathbf{v} \in \mathrm{verts}(K)$. This implies that $\mathrm{lk}_{K'}^-(\mathbf{k})$ is contractible for all $\mathbf{k} \in \mathrm{verts}(K')$ because K' is an LPDC of K. Applying Theorem 4.4 again, we see that all spaces of dipaths $\overrightarrow{P}_{\mathbf{w}}^{\mathbf{k}}(K')$ are contractible for all $\mathbf{k} \in \text{verts}(K')$.

Next we prove the backwards direction by assuming that the spaces of dipaths $\overrightarrow{P}_{\mathbf{w}}^{\mathbf{k}}(K')$ are contractible for all $\mathbf{k} \in \text{verts}(K')$. Let $\mathbf{v} \in \text{verts}(K)$. Either $\mathbf{v} \in \text{verts}(K')$ or $\mathbf{v} \notin \text{verts}(K')$.

Case 1 ($\mathbf{v} \in \text{verts}(K')$): By Theorem 4.4, we know that $\text{lk}_{K'}^{-}(\mathbf{v})$ is contractible. Since K' is an LPDC of K, then $\text{lk}_{K}^{-}(\mathbf{v})$ is also contractible.

Case 2 ($\mathbf{v} \notin \text{verts}(K')$): If $\mathbf{v} \notin \text{verts}(K)$, then τ is a vertex and $\mathbf{v} = \tau$. Observe that $\text{lk}_{\overline{\sigma}}^{-}(\tau)$ is contractible since $\overline{\sigma}$ is an elementary cube and τ does not contain $\min(\sigma)$. Furthermore, notice that $\text{lk}_{K}^{-}(\tau) = \text{lk}_{\sigma}^{-}(\tau)$ because τ is a free face of σ. Hence, $\text{lk}_{K}^{-}(\tau)$ is contractible.

Therefore $\text{lk}_{K}^{-}(\mathbf{v})$ is contractible for all $\mathbf{v} \in \text{verts}(K)$. Applying Theorem 4.4, we get that $\overrightarrow{P}_{\mathbf{w}}^{\mathbf{v}}(K)$ is contractible for all $\mathbf{v} \in \text{verts}(K)$. □

Using Theorem 2.5 and the partial converse to the connectedness theorem [1, Theorem 3], we get that any LPDC of a directed Euclidean cubical complex with connected dipath spaces and reachable vertices results in a directed Euclidean cubical complex with connected dipath spaces.

Corollary 4.6 (Condition for LPDCs to Preserve Connectedness of All Directed Path Spaces)

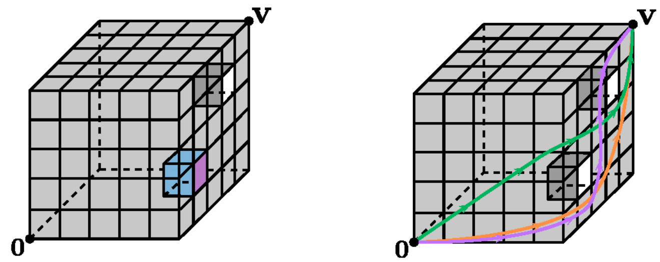
Let $(K, \mathcal{K})$ be a directed Euclidean cubical complex in $\mathbb{R}^n$ that has a minimum vertex $\mathbf{w}$. Let $(L, \mathcal{L}) = \text{reach}(K, \mathbf{w})$. Let (τ, σ) be an LPDC pair in L, and let L' be the (τ, σ)-collapse. The spaces of dipaths in $\overrightarrow{P}_{\mathbf{w}}^{\mathbf{k}}(L)$ are connected for all $\mathbf{v} \in \text{verts}(L)$ if and only if the spaces of dipaths $\overrightarrow{P}_{\mathbf{w}}^{\mathbf{v}}(L')$ are connected for all $\mathbf{v} \in \text{verts}(L')$.

We note that reachability is a necessary condition. Below we give an example of a directed Euclidean cubical complex K that has all connected dipath spaces but an LPDC yields a directed Euclidean cubical complex with a disconnected path space.

Example 4.7 (Bowling Ball)

Let K be the boundary of the $5 \times 5 \times 5$ grid union $[(4,1,1),(5,2,2)]$ and $[(4,3,3),(5,4,4)] \setminus [(5,3,3),(5,4,4)]$. See Fig. 9(a). Notice that some vertices of K are unreachable, for example, vertex $(4,1,1)$. Furthermore, all past links of vertices in K are connected and so all dipath spaces starting at $\mathbf{0}$ are also connected. After performing an LPDC with $\tau = [(5,1,1),(5,2,2)]$ and $\sigma = [(4,1,1),(5,2,2)]$, the dipath space between $\mathbf{0}$ and $(5,5,5)$ changes from having one connected component to three connected components, as shown in the figure. This example shows that the reachability condition in Corollary 4.6 is necessary for preserving connnectedness in LPDCs.

LPDCs can also preserve dihomotopy classes of dipaths starting at the minimum vertex of many directed Euclidean cubical complexes that have



(a) Original Complex (b) After Collapse

Fig. 9: The bowling ball before and after the collapse described in Example 4.7. Observe $\vec{P}_{\mathbf{0}}^{(5,5,5)}(K)$ has one connected component. Additionally, $\sigma = [(4,1,1),(5,2,2)]$ (highlighted in blue) and $\tau = [(5,1,1),(5,2,2)]$ (highlighted in purple) is an LPDC pair. After collapsing (τ,σ), $\vec{P}_{\mathbf{0}}^{(5,5,5)}(K)$ changes from having one connected component to three connected components. The three connected components are represented by the three dipaths.

disconnected past links. Recall the Swiss flag as discussed in Fig. 4. The Swiss flag has disconnected past links at $(3,4)$ and $(4,3)$, yet there exists a sequence of LPDCs that results in a directed Euclidean cubical complex that highlights the two dihomotopy classes of dipaths between $\mathbf{0}$ and (5.5). Example 4.8 gives another similar situation.

Example 4.8 (Window)

Let K be the 5×5 grid with the following two-cube interiors removed: $[(1,1),(2,2)]$, $[(3,1),(4,2)]$, $[(1,3),(2,4)]$, $[(3,3),(4,4)]$. *See Fig. 10(a). K has disconnected past links at the vertices* $(2,2),(4,2),(2,4),(4,4)$ *so K does not satisfy Corollary 4.5 or Corollary 4.6. Observe that* $\vec{P}(K)_{\mathbf{0}}^{(5,5)}$ *has six connected components. We can perform a sequence of LPDCs that preserves the dihomotopy classes of dipaths between* $\mathbf{0}$ *and* $(5,5)$ *at each step. First, we apply vertex LPDCs to remove the two-cubes along the border. Then we can apply four edge LPDCs and one vertex LPDC to get a graph of vertices and edges. This graph more clearly illustrates the six dihomotopy classes of dipaths in* $\vec{P}(K)_{\mathbf{0}}^{(5,5)}$.

In summary, LPDCs preserve the connectedness and/or contractability of dipath spaces starting at the minimum vertex as long as K has all reachable vertices and all dipath spaces starting at the minimum vertex in K connected and/or contractible to begin with. If K does not have these properties, the first step could be to remove all unreachable vertices and cubes before collapsing. In the next section, it will become clear that this will not suffice, if the dipath spaces are not all connected or contractible. [[**for camera-ready:** BTF: this paragraph seems out of place, and uses future tense]]

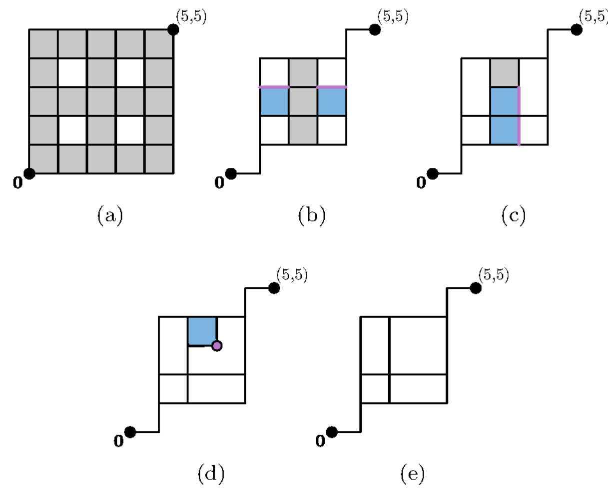


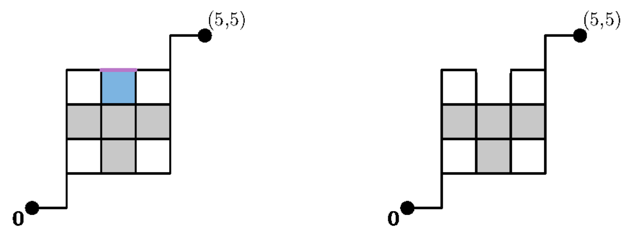
Fig. 10: Link-preserving DCs of the window. A sequence of LPDCs is presented from (a)-(e). The directed Euclidean cubical complex in (b) comes from performing several vertex LPDCs to remove the two-cubes along the border of K. In (b)-(d), the LPDC pairs (τ, σ) are highlighted in purple and blue respectively. The result of the sequence of LPDCs is a graph of vertices and edges that more clearly illustrates the dihomotopy classes of dipaths in the dipath space.

5 Discussion

LPDCs preserve spaces of dipaths in many examples (see Section 4), in particular, if they are all trivial in the sense of either all connected or all contractible and the directed Euclidean cubical complex is reachable for the minimum. However, LPDCs do not always preserve spaces of dipaths. We discuss some of those instances here. One limitation of LPDCs is that the number of components may increase after an LPDC as we saw in Example 4.7 or, as we see in Example 5.1, they may decrease.

Example 5.1 (A Sequence of LPDCs of the Window That Decreases the Number of Connected Components of the Dipath Space)

Consider K as given in Example 4.8. After applying vertex LPDCs that remove the two-cubes on the border of K, we can apply an LPDC to the edge $[(2,4),(3,4)]$. Now $\vec{P}(K')_{\mathbf{0}}^{(5,5)}$ has five connected components; whereas, the dipath space $\vec{P}(K)_{\mathbf{0}}^{(5,5)}$ has six connected components. See Fig. 11. This example shows that there are both "good" and "bad" ways to apply a sequence of LPDCs to a directed Euclidean cubical complex. As illustrated in Exam-



(a) Initial Complex (b) After Collapse

Fig. 11: Link-preserving DC of the window that changes dipath space. The LPDC of the edge $[(2,4),(3,4)]$ changes the dipath space between $\mathbf{0}$ to $(5,5)$ from having six connected components to five connected components.

ple 4.8, there exists a sequence of LPDCs that preserves the six connected components in $\overrightarrow{P}(K)_{\mathbf{0}}^{(5,5)}$. *However, if we perform a sequence of LPDCs that removes the edge* $[(2,4),(3,4)]$ *as in this example, then we get a directed Euclidean cubical complex that does not preserve the dihomotopy classes of dipaths in* $\overrightarrow{P}(K)_{\mathbf{0}}^{(5,5)}$.

Example 5.1 illustrates the need to investigate other properties if we want to preserve dipath spaces when performing an LPDC.

In Example 4.7, the problem was the existence of unreachable vertices. In Example 5.1, the vertex $(2,4)$ is a *deadlock* after the LPDC: only trivial dipaths initiate from there; whereas, before collapse, that was not the case. This seems to suggest that the introduction of new deadlocks should not be allowed; in practice, this would require an extra—but computationally easy—check on vertices of σ.

In the non-directed setting, if K' is obtained from K by collapsing a collapsing pair (τ,σ), then not only is the inclusion of K' in K a homotopy equivalence. K' is a deformation retract of K. The following example removes any hope of such a result in the directed setting:

Example 5.2 (LDPC of the Four-Cube With No Directed Retraction to the Collapsed Complex)

Let $(I^4,\mathcal{I}^4)$ *be the standard unit four-cube. Let* τ *be the vertex* $(1,1,0,0)$, *and* σ *be the cube* $[\mathbf{0},\mathbf{1}]$. *Since* τ *is free and not the minimum vertex of* σ, *the pair* (τ,σ) *is an LPDC pair. Thus, let* $(K',\mathcal{K}')$ *be the collapsed complex. Next, we show that there is no directed retration, i.e., no directed map from* I^4 *to* K' *that is the identity on* K'.

Suppose, for a contradiction, that $f: I^4 \to K'$ *is such a directed retraction. Let* $\mathbf{p}_1=(0,1,0,0)$, $\mathbf{p}_2=(1,0,0,0)$, $\mathbf{q}_1=(1,1,1,0)$, *and* $\mathbf{q}_2=(1,1,0,1)$. *By the product order on* $\mathbb{R}^4$, *we have* $\mathbf{p}_1,\mathbf{p}_2\preceq\tau$ *and* $\tau\preceq\mathbf{q}_1,\mathbf{q}_2$. *Since the points* $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{q}_1$, *and* $\mathbf{q}_2$ *are vertices of* I^4 *and are not equal to* τ, *we also* *know that* $\mathbf{p}_1$*,* $\mathbf{p}_2$*,* $\mathbf{q}_1$*, and* $\mathbf{q}_2$ *are points in* K'*. Since* f *is a directed retraction, we have that* $\mathbf{p}_1 = f(\mathbf{p}_1) \preceq f(\tau)$ *and that* $\mathbf{p}_2 = f(\mathbf{p}_2) \preceq f(\tau)$*. Similarly, we obtain that* $f(\tau) \preceq f(\mathbf{q}_1) = \mathbf{q}_1$ *and that* $f(\tau) \preceq f(\mathbf{q}_2) = \mathbf{q}_2$*.*

Let $x_1, x_2, x_3, x_4 \in I$ *such that* $f(\tau) = (x_1, x_2, x_3, x_4)$*. Then,*

$$\begin{aligned}
\mathbf{p}_1 \preceq f(\tau) &\Rightarrow x_2 \geq 1 \text{ and hence } x_2 = 1,\\
\mathbf{p}_2 \preceq f(\tau) &\Rightarrow x_1 \geq 1 \text{ and hence } x_1 = 1,\\
f(\tau) \preceq \mathbf{q}_1 &\Rightarrow x_4 \leq 0 \text{ and hence } x_4 = 0,\\
f(\tau) \preceq \mathbf{q}_2 &\Rightarrow x_3 \leq 0 \text{ and hence } x_3 = 0.
\end{aligned}$$

Thus, $f(\tau) = (1, 1, 0, 0) = \tau$*, which is not in* K' *and hence a contradiction. In fact, this argument extends to* $(I^k, \mathcal{I}^k)$ *for* $k \geq 4$*.*

As further evidence that such a (τ, σ)*-collapse does not preserve the directed topology, consider the spaces of dipaths in* $(I^4, \mathcal{I}^4)$ *and* $(K', \mathcal{K}')$*. We would need dipaths in the original space to map to dipaths in the collapsed space. However, notice that the dipath from* $\mathbf{p}_1$ *to* $\mathbf{q}_1$ *through* τ *cannot be mapped to a dipath in* $(K', \mathcal{K}')$*.*

We observe that vertex LPDCs appear to not introduce the problems of unreachability and deadlocks. These observations lead us to suspect that studying unreachability, deadlocks, and vertex LPDCs can help us better understand when LPDCs preserve and do not preserve dipath spaces between the minimum and a given vertex. We leave this as future work.

In summary, we provide an easy criterion for determining when we have an LPDC pair, as well as discuss various settings for when LPDCs preserve spaces of dipaths. Fully understanding when LPDCs preserve spaces of dipaths between two given vertices is a step towards developing algorithms that compress directed Euclidean cubical complexes and preserve directed topology.

Acknowledgements This research is a product of one of the working groups at the Women in Topology (WIT) workshop at MSRI in November 2017. This workshop was organized in partnership with MSRI and the Clay Mathematics Institute, and was partially supported by an AWM ADVANCE grant (NSF-HRD 1500481).

This material is based upon work supported by the US National Science Foundation under grant No. DGE 1649608 (Belton) and DMS 1664858 (Fasy), as well as the Swiss National Science Foundation under grant No. 200021-172636 (Ebli).

We thank the Computational Topology and Geometry (CompTaG) group at Montana State University for giving helpful feedback on drafts of this work.

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