A Code and Rate Equivalence Between Secure Network and Index Coding

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***Abstract*—Establishing code equivalences between index coding and network coding provides important insights for code design. Previous works showed an equivalence relation between any index-coding instance and a network-coding instance, for which a code for one instance can be translated to a code for the other instance with the same decoding-error performance. The equivalence also showed a surprising result that any network-coding instance can be mapped to an index-coding instance with a properly designed code translation. In this article, we extend the existing equivalence (instance map and code translation) to one between secure index coding and secure network coding, where eavesdroppers are present in the network. In the secure setting, any code construction needs to guarantee security constraints in addition to decoding-error performance. A rate equivalence between these two problems is also established.**

***Index Terms*—Code equivalence, Index coding, network coding, secure communications, wiretap.**

I. Introduction

EQUIVALENCE results in information theory and network coding are of significant interest because such results uniquely reduce one communication problem to another equivalent problem that is potentially easier to study. Some equivalence results already established include those between instances of multiple-unicast network coding and those of (i) multiple-multicast network coding [1], (ii) secure network coding [2], [3], (iii) index coding [4], [5], (iv) and with respect to a capacity equivalence for networks with adversarial state [6]. This article focuses on the equivalence between index coding and network coding.

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Prima facie, the two problems of index coding and network coding appear different. Index coding [7] considers a one-hop network where a sender conveys multiple messages to multiple receivers through a noiseless broadcast medium, where each receiver wants some messages from the sender, but already knows some other messages. On the other hand, network coding [8] considers a network of interconnected links with fixed capacities, where multiple senders send multiple messages to multiple receivers through these links.

Despite the differences, the following equivalence between them has been demonstrated [4], [5]. For any index-coding instance $\mathbb{I}$, an *instance map* constructs an equivalent network-coding instance $\mathbb{N}$. This instance pair $(\mathbb{I}, \mathbb{N})$ has the following properties: Any index code for $\mathbb{I}$ can be translated to a network code for $\mathbb{N}$, and vice versa. This *code translation* preserves the message length, code length, and probability of decoding error. Similarly, for any network-coding instance $\mathbb{N}'$, an instance map constructs an equivalent index-coding instance $\mathbb{I}'$, and a code translation can translate codes between the pair $(\mathbb{N}', \mathbb{I}')$.

In this article, we investigate the equivalence when we impose *security constraints* in addition to decodability constraints (the probability of decoding error). Separately, the secure version of index coding [9]–[11] and that of network coding [12]–[15] have been studied, in which there are additional passive eavesdroppers that attempt to obtain some information on the communicated messages. Codes for the secure version of these problem must prevent eavesdroppers from knowing the messages they attempt to decode (where knowing is quantified by the information-theoretic security measure[1] [18, Ch. 22]) in addition to guaranteeing that all receivers can obtain their requested messages (by bounding the probability of decoding error).

The *non-secure*[2] instance maps and code translations [4], [5] do not trivially apply to the secure version of the problems. In particular, we pointed out [19] that mapping an eavesdropper in secure network coding to secure index coding is not straightforward, as the eavesdroppers in the two problems have different characteristics. Eavesdroppers in network coding listen to transmission on certain links, while those in index

[1]This is a common criteria to protect classified data. Other security measures include preventing eavesdroppers from knowing what messages the receivers request (private information retrieval [16]) or preventing eavesdroppers from detecting whether communications occur (covert communications [17]).

[2]In this article, we use the term "non-secure" to denote existing instance maps and code translations in the absence of security constraints.

coding listen to the common broadcast and have access to some subset of messages.

Also, the non-secure code translation was designed for deterministic codes. But randomized encoding is inevitable in some secure network-coding instances [15], and we have shown that the non-secure code translation breaks down when randomized encoding is allowed [19].

In this article, we establish an equivalence between secure network coding and secure index coding. Similar to the non-secure equivalence, we construct instance maps for two directions (from secure index coding to secure network coding, and vice versa). For each instance map, we construct two code translations (from an index code to a network code, and back).

This equivalence carries the practical significance of comparing secure communication against eavesdropping in wired networks with that in wireless networks. Our equivalence results reveal that a passive eavesdropper that listens to the common wireless broadcast is not advantageous—that is, more difficult to deal with—compared to a passive eavesdropper in a wired networks that taps only certain wired links. In fact, our code translation results guarantee that the same approach can be implemented in both networks with help of side information (receivers knowing some other messages a priori) in the wireless case.

A. Contributions and Approaches

Our approach to establish an equivalence is summarized as follows:

- First, we construct the maps for translating problem instances. We build on the existing non-secure instance maps that map legitimate receivers. This involves mapping the eavesdroppers in any secure index-coding instance (having certain messages) to those in the corresponding secure network-coding instance (listening to certain links), and vice versa.
- Second, we construct code translations for the problem-instance pairs. Again, we build on the existing non-secure code translations, which have been shown to preserve the decoding criteria. As mentioned earlier, the non-secure code translations were designed for deterministic codes, but randomized network codes are necessary for secure network coding [15]. To deal with this issue, we (i) construct a two-step code translation to convert randomized codes to deterministic codes, and (ii) restrict the randomized encoding functions to certain nodes in the mapped problem instance.
- Third, we show that although eavesdroppers in the two problems instances observe different types of messages, the code translations output codes with comparable message size, code length, probability of decoding error, and information leakage to the eavesdroppers.
- Lastly, using the code translations, we show a rate equivalence between the two problems, that is, if a rate tuple is achievable[3] for index coding, an appropriately scaled rate tuple is also achievable for the mapped network coding, and vice versa.

[3]Achievability is defined in the Shannon sense, that is, both probability of decoding error and information leakage diminish, as the code length increases.

II. Problem Definition and Notation

Let X be a set whose elements are indexed by a strictly ordered set $S = \{s_1, s_2, \dots s_{|S|}\}$, with a total order $<$, i.e., $s_1 < s_2 < \cdots < s_{|S|}$, and let for any $A \subseteq S$, $X_A \coloneqq (X_s)_{s\in A}$. Thus, for instance, $X_S = (X_{s_1}, X_{s_2}, \dots, X_{s_{|S|}})$. Consider a directed graph $G = (V, E)$ with a node set V and an link set E. For a link $e = (u \to v) \in E$, where $u, v \in V$, its tail is $\texttt{tail}(e) \coloneqq u$, and its head is $\texttt{head}(e) \coloneqq v$. For any node $v \in V$, the set of incoming links is denoted by $\texttt{in}(v) \coloneqq \{e \in E : \texttt{head}(e) = v\}$, and the set of outgoing links by $\texttt{out}(v) \coloneqq \{e \in E : \texttt{tail}(e) = v\}$. For any $a \in \mathbb{Z}^+ \coloneqq \{1, 2, \dots\}$, denote $[a] \coloneqq \{1, 2, \dots, a\}$. $\mathbb{R}^+ \coloneqq (0, \infty)$ and $\mathbb{R}_0^+ \coloneqq [0, \infty)$.

A. Secure Network Coding

1) Problem Instance: Denote a secure network-coding instance [13] by $\mathbb{N} = (G, C, P)$, where

- $G = (V, E)$ is an acyclic directed graph with a node set V and a link set E. Each link $e\in E$ has a capacity $c_e\in\mathbb{R}^+$, where $\texttt{tail}(e)$ can send a message $X_e \in [2^{\lfloor c_e n\rfloor}]$ to $\texttt{head}(e)$ noiselessly over $n \in \mathbb{Z}^+$ link uses.
- $C = (S, O, D)$ is the connection requirement. S contains the message indices, where the messages are $\{X_s : s\in S\}$. $O(s) \in V$ is the originating node[4] for message X_s. $D(s)\subsetneq V$ is the set of nodes that require message X_s.
- $P = ((A_z, B_z) : z \in Z)$ defines the eavesdroppers indexed by elements of Z. Each eavesdropper $z \in Z$ observes messages X_{B_z} on links $B_z \subseteq E$ and tries to reconstruct messages X_{A_z} for some $A_z \subseteq S$.

We assume that vertices with no incoming links are originating nodes for some source messages, and vertices with no outgoing links are destinations for some source messages. Otherwise, they can be deleted without any consequence. Similarly, each message is requested by at least one node.

2) Deterministic Codes: Consider n uses of each link. Let the messages $\{X_s : s \in S\}$ be mutually independent, and let each X_s be uniformly distributed over $[M_s]$ for some $M_s \in \mathbb{Z}^+$. A deterministic (M_S, n)-network code consists of the following:

- A deterministic encoding function e_e for each link $e \in E$ that takes in encoded messages that are conveyed on incoming links of $\texttt{tail}(e)$ and those messages originating from $\texttt{tail}(e)$, i.e., $(X_{\texttt{in}(\texttt{tail}(e))}, X_{O^{-1}(\texttt{tail}(e))})$, and outputs message $X_e \coloneqq \mathsf{e}_e(X_{\texttt{in}(\texttt{tail}(e))}, X_{O^{-1}(\texttt{tail}(e))})$ to be conveyed on link e taking values in $[2^{\lfloor c_e n\rfloor}]$. Here, $O^{-1}(v)$ denotes the indices of the messages originating from node v.
- A decoding function d_v for each node $v \in V$ that takes in $(X_{\texttt{in}(v)}, X_{O^{-1}(v)})$ and outputs an estimate $X^{(v)} = \mathsf{d}_v(X_{\texttt{in}(v)}, X_{O^{-1}(v)})$ of the messages $X_{D^{-1}(v)}$ that v requires. Here $D^{-1}(v)$ is the set of indices of messages whose destinations include v.

[4]Without loss of generality, each message is available precisely at one node. Otherwise, if message X is available at nodes a and b, we can always construct an equivalent instance with an additional node c that is the sole originating node for message X and has links with large capacities to nodes a and b.

Here, n is referred to as the block length of the code. It is the number of times each link is used. We assume that $c_e n \geq 1$ for every link e, such that we can transmit at least one bit.

3) Randomized Codes: In this article, we consider randomized network codes with the use of random keys. Each node v generates an independent random key Y_v that is uniformly distributed over $[K_v]$ for some $K_v \in \mathbb{Z}^+$. The key Y_v is known only to the generating node v.

A randomized network code is similar to a deterministic one, except that each link encoding function e_e deterministically maps $(X_{\mathtt{in}(\mathtt{tail}(e))}, X_{O^{-1}(\mathtt{tail}(e))}, Y_{\mathtt{tail}(e)})$ to X_e.

Since the graph G is acyclic, any encoding function $\mathsf{e}_e(X_{\mathtt{in}(\mathtt{tail}(e))}, X_{O^{-1}(\mathtt{tail}(e))}, Y_{\mathtt{tail}(e)}))$ can be replaced by a suitable *global* encoding function $\mathsf{g}_e(X_S, Y_{\mathtt{tail}(e)})$ if all upstream links of e have deterministic encoding functions.

4) Decodability: A network code has a probability of decoding error of at most $\epsilon \in \mathbb{R}_0^+$ iff

$$P_{\mathrm{e}} = 1 - \Pr\left\{X^{(v)}_{D^{-1}(v)} = X_{D^{-1}(v)} \text{ for all } v \in V\right\} \leq \epsilon, \quad (1)$$

where $X^{(v)}$ is the set of all decoded messages whose destination is v (see Section II-A2). Note that when $\epsilon = 0$, the code guarantees perfect decoding.

5) Leakage: A network code has a leakage of at most $\eta \in \mathbb{R}_0^+$ iff

$$\frac{1}{n} I\left(X_{A_z}; X_{B_z}\right) \leq \eta, \qquad \text{for all } z \in Z. \quad (2)$$

With the normalization factor of $\frac{1}{n}$, (2) is commonly referred to as weak security in the literature. When $\eta = 0$, we say that the code is perfectly secure.

6) Feasibility: A secure network-coding instance $\mathbb{N}$ is said to be (M_S, n, ϵ, η)-feasible iff there exists an (M_S, n)-network code that has a probability of decoding error of at most ϵ and a leakage of at most η.

A message rate tuple of an (M_S, n)-network code is $R_S := (\frac{\log_2 M_s}{n} : s \in S)$. A rate tuple R_S is said to be achievable iff there exists a sequence of $((2^{\lceil nR_s \rceil} : s \in S), n)$-network codes, for $n \in \ell\mathbb{Z}^+$ for some $\ell \in \mathbb{Z}^+$, such that $\epsilon \to 0$ and $\eta \to 0$ as $n \to \infty$.[5] For such a sequence of network codes, R_V^{key} are called the key rates of the sequence, if for each code $((2^{\lceil nR_s \rceil} : s \in S), n)$ in the sequence, the alphabet size of random key Y_v is $K_v = 2^{\lceil nR_v^{\mathrm{key}} \rceil}$, for all $v \in V$.

B. Secure Index Coding

1) Problem Instance: Denote a secure index-coding instance [9] by $\mathbb{I} = (\widehat{S}, \widehat{T}, \{(\widehat{W}_t, \widehat{H}_t) : t \in \widehat{T}\}, \widehat{P})$.

- $\widehat{S}$ is the (ordered) message index set.
- $\widehat{T}$ is the receiver index set.
- $\widehat{W}_t \subseteq \widehat{S}$ contains the indices of messages required by receiver $t \in \widehat{T}$.
- $\widehat{H}_t \subsetneq \widehat{S}$ contains the indices of messages known a priori (known as side information) to receiver $t \in \widehat{T}$.
- $\widehat{P} = ((\widehat{A}_z, \widehat{B}_z) : z \in \widehat{Z})$ defines the eavesdroppers with indices in $\widehat{Z}$. Each eavesdropper $z \in \widehat{Z}$ has access to the code word broadcast by the sender and messages indexed by $\widehat{B}_z \subsetneq \widehat{S}$, and attempts to reconstruct messages indexed by $\widehat{A}_z \subseteq \widehat{S}$. Note that $\widehat{A}_z \cap \widehat{B}_z = \emptyset$.

2) Deterministic Codes: Let the messages $\{\widehat{X}_s : s \in \widehat{S}\}$ be mutually independent, and each $\widehat{X}_s$ be uniformly distributed over $[\widehat{M}_s]$ for some $\widehat{M}_s \in \mathbb{Z}^+$. A deterministic $(\widehat{M}_{\widehat{S}}, \widehat{n})$-index code consists of the following:

- A deterministic encoding (or broadcast) function for the sender: $\widehat{X}_{\mathrm{b}} = \widehat{\mathsf{e}}(\widehat{X}_{\widehat{S}}) \in [2^{\widehat{n}}]$, for some $\widehat{n} \in \mathbb{Z}^+$.
- A decoding function $\widehat{\mathsf{d}}_t$ for each receiver $t \in \widehat{T}$ that takes in $(\widehat{X}_{\mathrm{b}}, \widehat{X}_{\widehat{H}_t})$, and outputs an estimate $\widehat{X}^{(t)} = \widehat{\mathsf{d}}_t(\widehat{X}_{\mathrm{b}}, \widehat{X}_{\widehat{H}_t})$ of the messages $\widehat{X}_{\widehat{W}_t}$ that receiver t requires.

Here, the number of binary bits $\widehat{n}$ transmitted by the sender is referred to as the block length.

Remark 1: This index-code definition is consistent with the index-coding literature [20]–[23], but is different from that by Effros *et al.*, where the sender transmits $\widehat{X}_{\mathrm{b}} \in [2^{\widehat{c}_{\mathrm{b}}\widehat{n}}]$, and $\widehat{c}_{\mathrm{b}}$ is then chosen to be a function of the link capacities of the equivalent network-coding instance. The difference results in a scaling factor in our rate equivalence.

3) Randomised Codes: A randomized index code is similar to deterministic index codes except that the sender's encoding function takes in an independent random key $\widehat{Y} \in [\widehat{K}]$ in addition to $\widehat{X}_{\widehat{S}}$, for some $\widehat{K} \in \mathbb{Z}^+$.

4) Decodability: As with network coding, an index code has a probability of decoding error of at most $\epsilon \in \mathbb{R}_0^+$ iff

$$\widehat{P}_{\mathrm{e}} := 1 - \Pr\left\{\widehat{X}^{(t)} = \widehat{X}_{\widehat{W}_t} \text{ for all } t \in \widehat{T}\right\} \leq \epsilon. \quad (3)$$

5) Leakage: An index code has a leakage of at most $\eta \in \mathbb{R}_0^+$ iff

$$\frac{1}{\widehat{n}} I\left(\widehat{X}_{\widehat{A}_z}; \widehat{X}_{\mathrm{b}}, \widehat{X}_{\widehat{B}_z}\right) \leq \eta, \qquad \text{for all } z \in \widehat{Z}. \quad (4)$$

6) Feasibility: A secure index-coding instance $\mathbb{I}$ is said to be $(\widehat{M}_{\widehat{S}}, \widehat{n}, \epsilon, \eta)$-feasible iff there exists an $(\widehat{M}_{\widehat{S}}, \widehat{n})$-index code that has a probability of decoding error of at most ϵ and a leakage of at most η. A message rate tuple of an $(\widehat{M}_{\widehat{S}}, \widehat{n})$-index code is $\widehat{R}_{\widehat{S}} := (\frac{\log_2 \widehat{M}_s}{\widehat{n}} : s \in \widehat{S})$. A message rate tuple $\widehat{R}_{\widehat{S}}$ is said to be achievable iff there exists a sequence of $((2^{\lceil \widehat{n} R_s \rceil} : s \in \widehat{S}), \widehat{n})$-index codes, for $\widehat{n} \in \ell\mathbb{Z}^+$ for some $\ell \in \mathbb{Z}^+$, such that $\epsilon \to 0$ and $\eta \to 0$ as $\widehat{n} \to \infty$.

Table I summaries the notation used in network coding and index coding.

C. A Note on Achievable Rates

For secure network coding, if R_S is achievable, then R'_S, where $0 \leq R'_s \leq R_s$ for all $s \in S$, is also achievable using randomized codes. This can be achieved by replacing each message of $\lceil nR_s \rceil$ bits with a new message of $\lceil nR'_s \rceil$ bits and a random key of $\lceil nR_s \rceil - \lceil nR'_s \rceil$ bits. Doing so will not affect security, as both the shorter new message and the random key are now secure.

The above observation is not true in general for secure index coding. This is because the replacement step for a particular

[5]The condition of non-consecutive code lengths is introduced to match the translation of network codes of length n to index codes of length $\sum_{e \in E} \lfloor c_e n \rfloor$. While it preserves the spirit of having infinitely many codes with sufficiently large n, it does not require codes with every code length to exist or to satisfy the criteria.

TABLE I
LIST OF SYMBOLS USED FOR SECURE NETWORK CODING AND SECURE INDEX CODING

	Secure network coding	Secure index coding
Nodes		
The set of nodes	V	Sender and $\widehat{T}$
Node j wants these messages	$X_{D^{-1}(j)}$	$\widehat{X}_{\widehat{W}_j}$
Node j has these messages	$X_{O^{-1}(j)}$	Sender has $\widehat{X}_{\widehat{S}}$ Node $j \in \widehat{T}$ has $\widehat{X}_{\widehat{H}_j}$
Node j listens to these transmissions	$X_{\mathtt{in}(j)}$	$\widehat{X}_{\mathrm{b}}$
Source messages		
The set of messages	X_S	$\widehat{X}_{\widehat{S}}$
Message $X_i/\widehat{X}_i$ is known to these nodes	$O(i)$	Sender and $\{j : i \in \widehat{H}_j\}$
Message $X_i/\widehat{X}_i$ is wanted by these nodes	$D(i)$	$\{j : i \in \widehat{W}_j\}$
Links		
The set of links	E	One broadcast link
Transmission on link $e \in E$ or broadcast link	X_e	$\widehat{X}_{\mathrm{b}}$
Capacity of link $e \in E$ or broadcast link	c_e	1
Eavesdroppers		
The index set of eavesdroppers	Z	$\widehat{Z}$
Eavesdropper z wants these messages	$X_{A_z}, A_z \subseteq S$	$\widehat{X}_{\widehat{A}_z}, \widehat{A}_z \subseteq \widehat{S}$
Eavesdropper z has these messages	$\emptyset$	$\widehat{X}_{\widehat{B}_z}, \widehat{B}_z \subseteq \widehat{S}$
Eavesdropper z listens to these transmissions	$X_{B_z}, B_z \subseteq E$	$\widehat{X}_{\mathrm{b}}$
Codes		
Encoding function on link e or broadcast link	$X_e = \mathsf{e}_e\left(X_{\mathtt{in}(\mathtt{tail}(e))}, X_{O^{-1}(\mathtt{tail}(e))}, Y_{\mathtt{tail}(e)}\right)$	$\widehat{X}_{\mathrm{b}} = \widehat{\mathsf{e}}(\widehat{X}_{\widehat{S}}, \widehat{Y})$
Decoding function at node j	$\mathsf{d}_j(X_{\mathtt{in}(j)}, X_{O^{-1}(j)})$	$\widehat{\mathsf{d}}_j(\widehat{X}_{\mathrm{b}}, \widehat{X}_{\widehat{H}_j})$

message cannot be replicated at the receivers having that particular message as side information. The following example illustrates this point:

Example 1: Consider a secure index coding problem with two receivers {1,2} and an eavesdropper. Receiver 1 wants X_1 and knows X_2; receiver 2 wants X_2 and knows X_1; the eavesdropper wants X_1 and knows nothing. The rate (R, R) for all $0 \le R \le 1$ is achievable by sending $X_1 + X_2 \mod 2^{\lceil \widehat{n}R \rceil}$. However, the rate $(R, R-\delta)$ is not achievable. To secure X_1, the sender needs to pad X_2 with $\lceil \widehat{n}R \rceil - \lceil \widehat{n}(R-\delta) \rceil$ random bits. But by doing so, receiver 1 cannot decode X_1 as it does not know these random bits.

III. SUMMARY OF RESULTS

A. Code Feasibility Equivalence

Given any secure index-coding instance $\mathbb{I}$, Section IV defines a map to obtain a corresponding secure network-coding instance $\mathbb{N}$, with the following code feasibility equivalence:

Theorem 1 (A brief version): $\mathbb{I}$ is $(\widehat{M}_{\widehat{S}}, \widehat{n}, \epsilon, \eta)$-feasible iff $\mathbb{N}$ is $(\widehat{M}_{\widehat{S}}, \widehat{n}, \epsilon, \eta)$-feasible.

We will prove the forward assertion in Section V and the backward assertion in Section VI.

In the other direction, given any secure network-coding instance $\mathbb{N}$, Section VII defines a map to a corresponding secure index-coding instance $\mathbb{I}$, with the following code feasibility equivalence:

Theorem 2 (A brief version): Let $\widehat{n} = \sum_{e \in E} \lfloor c_e n \rfloor$ and $2^{\lfloor c_E n \rfloor} := (2^{\lfloor c_e n \rfloor} : e \in E)$.

1) If $\mathbb{N}$ is (M_S, n, ϵ, η)-feasible, then $\mathbb{I}$ is $((M_S, K_V, 2^{\lfloor c_E n \rfloor}), \widehat{n}, \epsilon, \theta_1)$-feasible, where
$$\theta_1 := \frac{\eta}{\left(\sum_{e \in E} c_e - \frac{|E|}{n}\right)}.$$
Note that $\mathbb{I}$ has $|S| + |V| + |E|$ messages with sizes $\widehat{M}_{\widehat{S}} = (M_S, K_V, 2^{\lfloor c_E n \rfloor})$.
2) If $\mathbb{I}$ is $((M_S, K_V, 2^{\lfloor c_E n \rfloor}), \widehat{n}, \epsilon, \eta)$-feasible, then $\mathbb{N}$ is $(M_S, n, (|Z|+1)\epsilon, \theta_2)$-feasible, where
$$\theta_2 := (|Z|+1)\left[\left(\frac{\eta}{1-\epsilon} + \epsilon\right)\sum_{e \in E} c_e + \frac{1}{n}\left(\frac{H_{\mathrm{b}}(\epsilon)}{1-\epsilon} + H_{\mathrm{b}}(\epsilon)\right)\right]$$
and $H_{\mathrm{b}}(\cdot)$ denotes the binary entropy function.

We will prove Part 1 in Section VIII and Part 2 in Section IX.

Remark 2:

1) For perfect decoding, that is, $\epsilon = 0$, we have that $\theta_1 = \eta \frac{1}{\left(\sum_{e \in E} c_e - \frac{|E|}{n}\right)} \le \psi_1 \eta$ and $\theta_2 = \eta(|Z|+1)\sum_{e \in E} c_e = \psi_2 \eta$, for some constants ψ_1, ψ_2. The term $\sum_{e \in E} c_e$ in the scaling factor is a result of the way leakage is normalized: $\frac{1}{n}$ for network coding; $\frac{1}{\widehat{n}} = \frac{1}{\sum_{e \in E} \lfloor c_e n \rfloor}$ for index coding.
2) For perfect decoding and zero leakage ($\epsilon = \eta = 0$), we have $\theta_1 = \theta_2 = 0$, regardless of n.
3) If $n \to \infty$, $\epsilon \to 0$, and $\eta \to 0$, then $(|Z|+1)\epsilon \to 0$, $\theta_1 \to 0$, and $\theta_2 \to 0$.

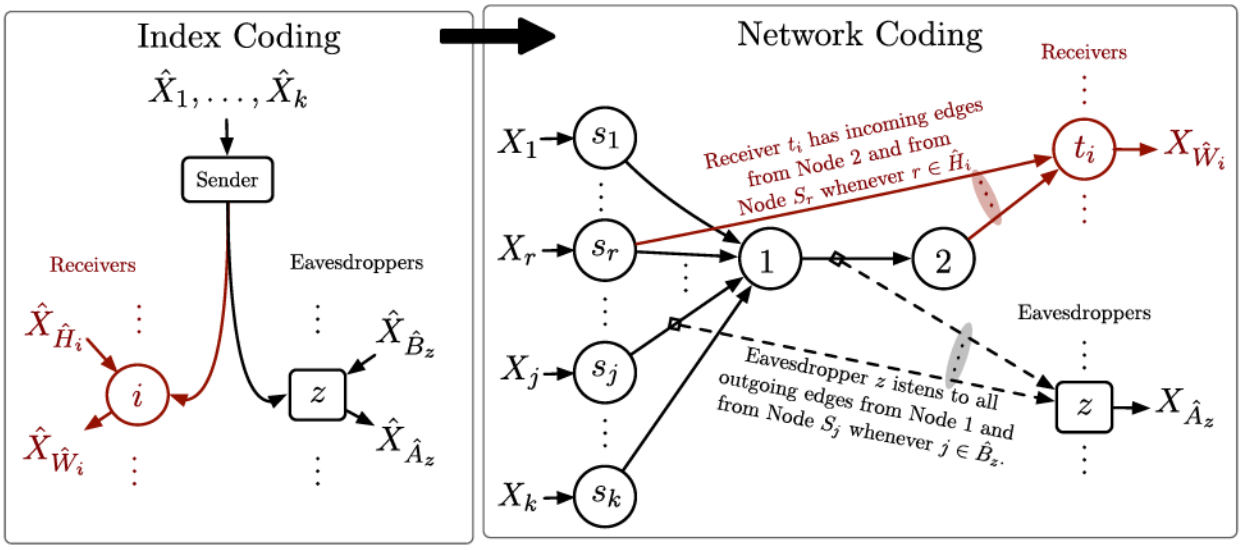


Fig. 1. The map from secure index coding to secure network coding: Circular nodes represent receivers and square nodes represent eavesdroppers.

4) A network-coding instance $\mathbb{N}$ with $|S|$ sources is mapped to an index-coding instance $\mathbb{I}$ with $|S|+|V|+|E|$ sources.
5) For Theorem 2, deterministic index codes suffice.

B. Rate Equivalence

From the above code translations, we obtain the following rate-equivalence. Given any secure index-coding instance $\mathbb{I}$ and its corresponding secure network-coding instance $\mathbb{N}$ (via the instance map defined in Section IV), we have the following:

Corollary 1 (A Brief Version): The rate tuple $\widehat{R}_{\widehat{S}}$ is achievable for $\mathbb{I}$ iff it is achievable for $\mathbb{N}$.

For the other direction, we consider any secure network-coding instance $\mathbb{N}$, where all link capacities c_e are integers, and the corresponding secure index-coding instance $\mathbb{I}$ (via the instance map defined in Section VII).

Corollary 2 (A Brief Version): The rate tuple R_S is achievable for $\mathbb{N}$ (with integer link capacities), using random key with rates R_V^{key}, iff the rate tuple $\frac{1}{\sum_{e\in E} c_e}(R_S, R_V^{\text{key}}, c_E)$ is achievable for $\mathbb{I}$.

Remark 3: Corollary 2 can be extended to any secure network-coding instance $\mathbb{N}$ with rational link capacities. To this end, we consider $\lambda \in \mathbb{Z}^+$ uses of the links as a group, where λ is the least common multiple of the denominators of the link capacities. Each group of λ uses of the links is equivalent to another network coding instance $\mathbb{N}'$ with link capacities $c'_e = \lambda c_e \in \mathbb{Z}^+$. We apply Corollary 2 to $\mathbb{N}'$ to get a rate equivalence between $R'_S = \lambda R_S$ for $\mathbb{N}'$ (which is R_S for $\mathbb{N}$) and $\frac{1}{\sum_{e\in E} c'_e}(R'_S, R'^{\text{key}}_V, c'_E) = \frac{1}{\sum_{e\in E} c_e}(R_S, R_V^{\text{key}}, c_E)$ for $\mathbb{I}$.

IV. From Secure Index to Secure Network Coding

Given a secure index-coding instance $\mathbb{I}=(\widehat{S}, \widehat{T}, \{(\widehat{W}_t, \widehat{H}_t) : t \in \widehat{T}\}, \widehat{P})$, let $\widehat{S} = [k]$ and $\widehat{T} = [\ell]$ for some positive integers k and ℓ. We will first propose a map to a secure network-coding instance $\mathbb{N} = (G, C, P)$.

A. Index-to-Network Coding Instance Map

The secure version of the index-to-network coding instance map consists of the following:

1) An Existing Non-Secure Map [4, Fig. 1] for $G = (V, E)$ and $C = (S, O, D)$:

- The vertex set $V = \{s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_\ell, 1, 2\}$.
- The link set E contains the following links:
  - $(s_i \to 1)$ for each $i \in [k]$, with sufficiently large capacities.
  - $(s_i \to t_j)$ iff $i \in \widehat{H}_j$, with sufficiently large capacities.
  - $(1 \to 2)$ with link capacity 1.
  - $(2 \to t_j)$ for each $j \in [\ell]$, with link capacity 1.
- The message indices $S = \widehat{S} = [k]$.
- The originating node for message $X_i, i \in S$, is $O(i) = s_i$.
- The destinations for message X_i is $D(i) = \{t_j : i \in \widehat{W}_j\}$.

2) Our Proposed Map for $P = ((A_z, B_z) : z \in Z)$:

- $Z = \widehat{Z}$
- For each $z \in Z : A_z = \widehat{A}_z$.
- For each $z \in Z$, B_z comprises of links included by the following rules:
  - $(1 \to 2) \in B_z$.
  - For any $i \in \widehat{B}_z$, all outgoing links from node s_i are in B_z.

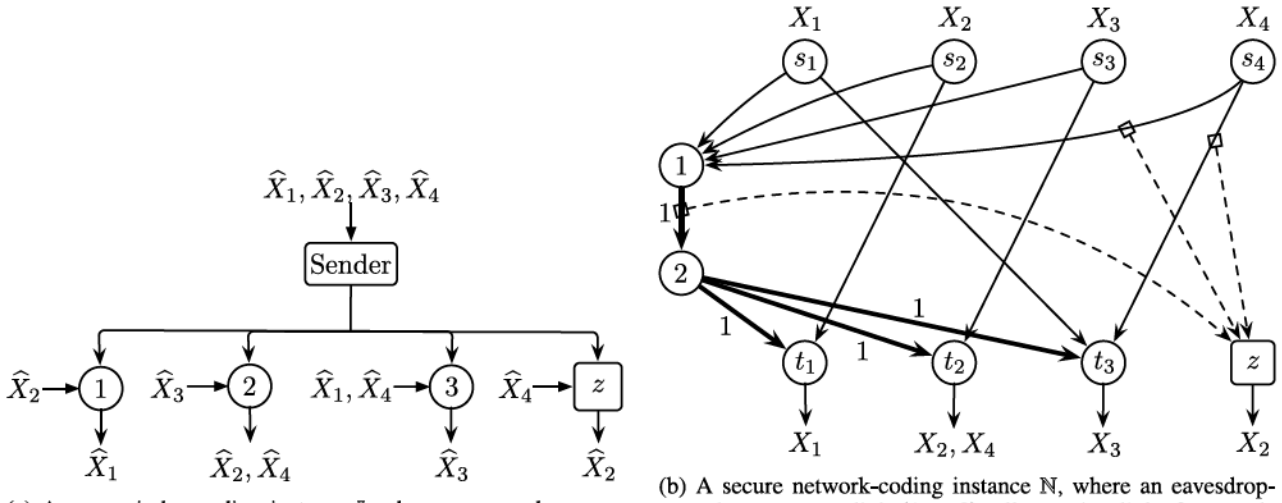
Remark 4: The number of source messages in both instances is the same. Side information in $\mathbb{I}$ manifests itself as links $(s_i \to t_j)$ in $\mathbb{N}$. In the constructed $\mathbb{N}$, sources nodes are $\{s_i : i \in \widehat{S}\}$, and destination nodes are $\{t_i : i \in \widehat{T}\}$.

The index-to-network coding instance map is summarized in Figure 1.

B. Equivalence Results

With the above map, we prove an equivalence between these two instances.

Theorem 1: Let $\mathbb{I}$ be a secure index-coding instance, and $\mathbb{N}$ be the corresponding secure network-coding instance using the index-to-network coding map. For any $\epsilon, \eta \in \mathbb{R}_0^+$ and $\widehat{n} \in \mathbb{Z}^+$, $\mathbb{I}$ is $(\widehat{M}_{\widehat{S}}, \widehat{n}, \epsilon, \eta)$-feasible iff $\mathbb{N}$ is $(\widehat{M}_{\widehat{S}}, \widehat{n}, \epsilon, \eta)$-feasible with



(a) A secure index-coding instance $\mathbb{I}$, where an eavesdropper z has access to the broadcast message $\widehat{X}_b$, side information $\widehat{X}_4$, and tries to reconstruct $\widehat{X}_2$.

(b) A secure network-coding instance $\mathbb{N}$, where an eavesdropper z has access to link $(1 \to 2)$, all outgoing links from node s_4, and tries to reconstruct X_2. The capacity of all four links given by bold arrows is 1 bit per channel use.

Fig. 2. A secure index-coding instance in (a) and its equivalent secure network-coding instance in (b).

deterministic encoding functions at vertices 2 and $\{s_i : i \in \widehat{S}\}$, and a randomized encoding function at node 1.

Proof: See Section V for the forward direction and Section VI for the backward direction. ∎

The theorem above preserves the message size, as well as the decodability and security criteria. The proof of the theorem utilisers the non-secure code translations [4], which can be easily shown to preserve the decoding error criterion ϵ when the codes are deterministic in the absence of eavesdroppers. As an equivalence for the secure instances is required here, our main contribution in the direction of secure index-to-network coding map is to show that the code translation

- works with the addition of eavesdroppers,
- works for all randomized index codes,
- preserves the security criterion η,
- still preserves the decoding criterion ϵ.

With Theorem 1, we get the following rate equivalence:

Corollary 1: Let $\mathbb{I}$ be a secure index-coding instance, and $\mathbb{N}$ be the corresponding secure network-coding instance using the index-to-network coding map. A rate tuple $\widehat{R}_{\widehat{S}}$ is achievable for $\mathbb{I}$ iff it is also achievable for $\mathbb{N}$.

Proof: If a rate tuple $\widehat{R}_{\widehat{S}}$ is achievable for $\mathbb{I}$, then there exists a sequence of $((2^{\lceil \widehat{n}\widehat{R}_s \rceil} : s \in \widehat{S}), \widehat{n})$-index codes, $\widehat{n} \in \ell\mathbb{Z}^+$, with probability of decoding error $\epsilon \to 0$ and leakage $\eta \to 0$ as $\widehat{n} \to \infty$. From Theorem 1, it follows that there exists a sequence of network codes with the same properties, and hence $\widehat{R}_{\widehat{S}}$ is also achievable for $\mathbb{N}$.

The other direction from $\mathbb{N}$ to $\mathbb{I}$ follows exactly the same argument. ∎

C. An Example

Consider the index-coding instance $\mathbb{I}$ depicted in Figure 2(a), and its mapped network-coding instance $\mathbb{N}$ in Figure 2(b). An example of index codes with $\epsilon = 0$ and $\eta = 0$ is $\widehat{\mathsf{e}}(\widehat{X}_{[4]}, \widehat{Y}) = (\widehat{X}_1 + \widehat{X}_2, \widehat{X}_2 + \widehat{X}_3, \widehat{X}_4)$. One can verify that each user can decode their intended messages, and the eavesdropper z has no information about $\widehat{X}_2$. In the translated network code, each source node s_i, $i \in [4]$, transmits X_i on every outgoing link; node 1 transmits $X_{1\to 2}{=}\mathsf{g}_{1\to 2}(X_{[4]}, Y_1) = \widehat{\mathsf{e}}(X_{[4]}, Y_1) = (X_1 + X_2, X_2 + X_3, X_4)$; node 2 forwards $X_{1\to 2}$ on every outgoing link. Clearly, each destination node in $\mathbb{N}$ can decode its required messages using the same decoding function in $\mathbb{I}$, and the eavesdropper z gains no information about X_2.

In the other direction of network-to-index code translation, consider a network code with $\epsilon{=}0$ and $\eta{=}0$ as follows: each source node s_i, $i \in [4]$, transmits X_i on every outgoing link; node 1 transmits $X_{1\to 2}{=}\mathsf{g}_{1\to 2}(X_{[4]}, Y_1) = (X_1 + X_3, X_2 + X_3, X_4)$; node 2 forwards $X_{1\to 2}$ on every outgoing link. The translated index code is $\widehat{\mathsf{e}}(\widehat{X}_{[4]}, \widehat{Y}){=}\mathsf{g}_{1\to 2}(\widehat{X}_{[4]}, \widehat{Y}) = (\widehat{X}_1 + \widehat{X}_3, \widehat{X}_2 + \widehat{X}_3, \widehat{X}_4)$. One can verify that for both instances, all users can decode their requested messages, and the eavesdropper gains no information of the messages it attempts to decode. We have used different codes for the two directions to highlight that secure network and index codes may not be unique.

V. Proof of Theorem 1 – The Forward Direction

We now prove Theorem 1 for the forward direction: if $\mathbb{I}$ is $(\widehat{M}_{\widehat{S}}, \widehat{n}, \epsilon, \eta)$-feasible, then $\mathbb{N}$ is $(\widehat{M}_{\widehat{S}}, \widehat{n}, \epsilon, \eta)$-feasible. This is achieved by showing that any index code that satisfies the feasibility condition for $\mathbb{I}$ can be translated to a network code that satisfies the feasibility condition for $\mathbb{N}$.

A. Code Translation

We start with any randomized index code that is $(\widehat{M}_{\widehat{S}}, \widehat{n}, \epsilon, \eta)$-feasible. We will show that the network code obtained by the existing non-secure code translation satisfies both the decoding and security criteria even for randomized codes. In the following, we modify the existing non-secure code translation [4, Fig. 1] to translate randomized index codes.

- For all outgoing links from s_i, $i \in [k]$: Set a deterministic link function $X_e = \mathsf{e}_e(X_{O^{-1}(s_i)}) = \mathsf{e}_e(X_i) = X_i \in [\widehat{M}_i]$, for each $e \in \mathtt{out}(s_i)$. This is possible since vertex s_i is the originating vertex for the message X_i, and the link capacity is sufficiently large.
- For link $(1 \to 2)$: Set $X_{1\to 2} = \mathsf{e}_{1\to 2}(X_{\mathtt{in}(1)}, Y_1) = \widehat{\mathsf{e}}(X_{[k]}, Y_1) \in [2^{\widehat{n}}]$. We set the cardinality of Y_1 (which is the random key used in the encoding function of vertex 1 in $\mathbb{N}$) to be the same as that of the random key $\widehat{Y}$ used in the encoding function $\widehat{\mathsf{e}}(\widehat{X}_{\widehat{S}}, \widehat{Y})$ of the sender in $\mathbb{I}$.
- For all outgoing links from 2: Set a deterministic function $X_e = \mathsf{e}_e(X_{1\to 2}) = X_{1\to 2} \in [2^{\widehat{n}}]$, for each $e \in \mathtt{out}(2)$. This is possible as every outgoing link from vertex 2 has capacity $\widehat{n}$.
- Set $\mathsf{d}_{t_i}(X_{\mathtt{in}(t_i)}) = \widehat{\mathsf{d}}_i(X_{2\to t_i}, X_{\widehat{H}_i})$ for all $i \in [\ell]$, and d_v=0 for all other vertices v.

B. Decoding Criterion

See Appendix A for the proof of the decoding criterion.

C. Security Criteria

Each eavesdropper $z \in Z$ in $\mathbb{N}$ has access to links B_z consisting of (i) link $(1 \to 2)$, which carries $X_{1\to 2} = \widehat{\mathsf{e}}(X_S, Y_1)$, and (ii) outgoing links from $\{s_i : i \in \widehat{B}_z\}$, which carry messages $X_{\widehat{B}_z}$, because each outgoing link from node s_i carries X_i by construction.

Now, we bound the leakage for the network code as follows:

$$\frac{1}{\widehat{n}} I\big(X_{A_z}; X_{B_z}\big) = \frac{1}{\widehat{n}} I\Big(X_{\widehat{A}_z}; \widehat{\mathsf{e}}(X_S, Y_1), X_{\widehat{B}_z}\Big) \tag{5a}$$

$$\overset{(a)}{=} \frac{1}{\widehat{n}} I\Big(\widehat{X}_{\widehat{A}_z}; \widehat{\mathsf{e}}\big(\widehat{X}_{\widehat{S}}, \widehat{Y}\big), \widehat{X}_{\widehat{B}_z}\Big) \overset{(b)}{\leq} \eta, \tag{5b}$$

where (a) follows from a change of random variables by noting that $(\widehat{X}_{\widehat{S}}, \widehat{Y})$ has the same distribution as (X_S, Y_1); (b) follows from the premise that the leakage for the index code is at most η. This completes the security proof for $\mathbb{N}$.

VI. Proof of Theorem 1 – The Backward Direction

We will now prove Theorem 1 for the backward direction: if $\mathbb{N}$ is $(\widehat{M}_{\widehat{S}}, \widehat{n}, \epsilon, \eta)$-feasible, then $\mathbb{I}$ is $(\widehat{M}_{\widehat{S}}, \widehat{n}, \epsilon, \eta)$-feasible.

A. Code Construction

We start with any network code for $\mathbb{N}$ that satisfies the feasibility conditions. Recall the encoding functions at node 2 and nodes $\{s_i : i \in [k]\}$ are deterministic, and that at node 1 randomized. In the following, we modify the existing non-secure code translation [4, Fig 1] to translate a network code with a random key Y_1 used at node 1.

- Set the sender's transmitted code to be $\widehat{X}_{\mathsf{b}} = \widehat{\mathsf{e}}(\widehat{X}_{\widehat{S}}, \widehat{Y}) = \mathsf{e}_{1\to 2}((\mathsf{e}_{s_i\to 1}(\widehat{X}_i) : i \in [k]), \widehat{Y}) \in [2^{\widehat{n}}]$, where $\widehat{Y}$ has the same distribution as Y_1 (the random key used in the network code).
- Set the decoding function of receiver $i \in [\ell]$ to be $\widehat{\mathsf{d}}_i(\widehat{X}_{\mathsf{b}}, \widehat{X}_{\widehat{H}_i}) = \mathsf{d}_{t_i}(\mathsf{e}_{2\to t_i}(\widehat{X}_{\mathsf{b}}), (\mathsf{e}_{s_j\to t_i}(\widehat{X}_j) : j \in \widehat{H}_i))$.

B. Restricting Network Codes

Next, we show that we only need to consider a specific class of network codes without loss of generality.[6] Specifically, we only need to consider network codes such that $\mathsf{e}_{2\to t_i}(X_{1\to 2}) = X_{1\to 2}$, for all $i \in [\ell]$. First, observe that

$$X_{\widehat{W}_i} - \big(X_{1\to 2}, \big(\mathsf{e}_{s_j\to t_i}(X_j) : j \in \widehat{H}_i\big)\big) - \big(\mathsf{e}_{2\to t_i}(X_{1\to 2}), \big(\mathsf{e}_{s_j\to t_i}(X_j) : j \in \widehat{H}_i\big)\big)$$

forms a Markov chain, where $X_{1\to 2} = \mathsf{e}_{1\to 2}((\mathsf{e}_{s_i\to 1}(X_i) : i \in [k]), Y_1)$, and $\widehat{W}_i \cap \widehat{H}_i = \emptyset$.

Recall that receiver t_i, for each $i \in [\ell]$, attempts to decode $X_{\widehat{W}_i}$ from $(\mathsf{e}_{2\to t_i}(X_{1\to 2}), (\mathsf{e}_{s_j\to t_i}(X_j) : j \in \widehat{H}_i))$. By the data-processing inequality, the probability of decoding error P_{e} cannot increase if we set $\mathsf{e}_{2\to t_i}(X_{1\to 2}) = X_{1\to 2}$ in each receiver t_i's observations. Also, by definition, none of the links $\{2 \to t_i : i \in [\ell]\}$ can be accessed by any eavesdropper. So, this choice will not affect the leakage of the code.

Consequently, for any network code for $\mathbb{N}$ with a probability of error of at most ϵ and a leakage of at most η, we can always obtain another network code by choosing the encoding functions of all outgoing links from node 2 to be $\mathsf{e}_{2\to t_i}(X_{1\to 2}) = X_{1\to 2}$, for all $i \in [\ell]$. The modified network code also has an error-decoding probability of at most ϵ and a leakage of at most η. For the subsequent subsections, we will only consider network codes of this form.

C. Decoding Criterion

See Appendix B for the proof of the decoding criterion.

D. Security Criteria

From the security criteria of $\mathbb{N}$, we have $\frac{1}{\widehat{n}} I(X_{A_z}; X_{B_z}) < \eta$ for each z. Eavesdropper z observes links B_z, which consists of all outgoing links from source nodes $\{s_i : i \in \widehat{B}_z\}$ as well as link $1 \to 2$. It attempts to decode messages indexed by $A_z = \widehat{A}_z$.

Showing that the translated index code also satisfies a similar security condition as the original network code is not trivial, as the eavesdroppers in $\mathbb{I}$ can access the messages themselves, instead of just functions of the messages as in $\mathbb{N}$. These functions may not necessarily allow one to recover the messages, as we allow non-zero decoding error probability. So, it seems that the eavesdroppers in $\mathbb{I}$ have "better" observations, which may lead to a larger leakage of the code.

We will show that this is not the case. First, note the following: (i) $\{X_S, Y_1\}$ are mutually independent; (ii) $X_{\mathtt{out}(s_i)}$, for each $i \in S$, are each a deterministic function of X_i; (iii) $\widehat{B}_z \cap A_z = \emptyset$. With these, we have the following Markov chain for every $z \in Z$:

$$X_{\widehat{B}_z} - X_{\{\mathtt{out}(s_i): i \in \widehat{B}_z\}} - \Big(Y_1, X_{A_z}, X_{S\setminus(A_z \cup \widehat{B}_z)}\Big), \tag{6}$$

which is equivalent to

$$0 = I\Big(X_{\widehat{B}_z}; Y_1, X_{A_z}, X_{S\setminus(A_z\cup\widehat{B}_z)} | X_{\{\mathtt{out}(s_i): i\in\widehat{B}_z\}}\Big) \tag{7a}$$

$$= I\Big(X_{\widehat{B}_z}; Y_1, X_{A_z}, X_{S\setminus(A_z\cup\widehat{B}_z)}, X_{\{\mathtt{out}(s_i): i\in\widehat{B}_z\}} | X_{\{\mathtt{out}(s_i): i\in\widehat{B}_z\}}\Big) \tag{7b}$$

[6] This was not shown in existing works on non-secure equivalence.

$$= I\Big(X_{\widehat{B}_z}; Y_1, X_{S\setminus\widehat{B}_z}, X_{\{\mathrm{out}(s_i):i\in\widehat{B}_z\}}, X_{\{s_i\to 1:i\in S\}} | X_{\{\mathrm{out}(s_i):i\in\widehat{B}_z\}}\Big) \tag{7c}$$

$$= I\Big(X_{\widehat{B}_z}; Y_1, X_{S\setminus\widehat{B}_z}, X_{\{\mathrm{out}(s_i):i\in\widehat{B}_z\}}, X_{\{s_i\to 1:i\in S\}}, X_{1\to 2} | X_{\{\mathrm{out}(s_i):i\in\widehat{B}_z\}}\Big) \tag{7d}$$

$$\geq I\Big(X_{\widehat{B}_z}; X_{A_z}, X_{1\to 2} | X_{\{\mathrm{out}(s_i):i\in\widehat{B}_z\}}\Big) \tag{7e}$$

$$\geq I\Big(X_{\widehat{B}_z}; X_{A_z} | X_{\{\mathrm{out}(s_i):i\in\widehat{B}_z\}}, X_{1\to 2}\Big) \tag{7f}$$

$$= I\Big(X_{\widehat{B}_z}; X_{A_z} | X_{B_z}\Big) \geq 0. \tag{7g}$$

Here, (7b) follows from $I(A|B) = I(A, B|B)$; (7c) is due to (i) $A_z \cup (S \setminus (A_z \cup \widehat{B}_z)) = S \setminus \widehat{B}_z$ (because $\widehat{B}_z \cap A_z = \emptyset$, $\widehat{B}_z \subsetneq S$, $A_z \subseteq S$), (ii) $X_{\{s_i\to 1:i\in S\}} \subseteq X_{\{\mathrm{out}(s_i):i\in\widehat{B}_z\}} \cup X_{\{\mathrm{out}(s_i):i\in S\setminus\widehat{B}_z\}}$, and (iii) $X_{\{\mathrm{out}(s_i):i\in S\setminus\widehat{B}_z\}}$ are deterministic functions of $X_{S\setminus\widehat{B}_z}$; (7d) is obtained as $X_{1\to 2}$ is a deterministic function of $X_{\{s_i\to 1:i\in S\}}$ and Y_1; (7e) follows from $I(A; B, C|D) \geq I(A; B|D)$; (7f) follows from $I(A; B, C|D) \geq I(A; B|C, D)$; the equality in (7g) follows from the definition of B_z.

This means that eavesdropper z, having observed the links X_{B_z}, does not gain any more information about $X_{\widehat{A}_z}$ even if it can also observe the source messages $X_{\widehat{B}_z}$. Now, we show that the eavesdropper cannot do better if we replace its observation of the outgoing links from the sources with the source messages:

$$\begin{aligned}\frac{1}{\widehat{n}} I\Big(X_{\widehat{B}_z}, X_{1\to 2}; X_{A_z}\Big) &= \frac{1}{\widehat{n}} I\Big(X_{\widehat{B}_z}, X_{1\to 2}, X_{\{\mathrm{out}(s_i):i\in\widehat{B}_z\}}; X_{A_z}\Big) \\ &= \frac{1}{\widehat{n}} I\Big(X_{\widehat{B}_z}, X_{B_z}; X_{A_z}\Big) \\ &= \frac{1}{\widehat{n}} I\big(X_{B_z}; X_{A_z}\big) + I\Big(X_{\widehat{B}_z}; X_{A_z} | X_{B_z}\Big) \\ &= \frac{1}{\widehat{n}} I\big(X_{B_z}; X_{A_z}\big) \leq \eta, \end{aligned} \tag{8}$$

where in the last line the equality follows from (7g), and the inequality from the premise of the network code.

Since $(\widehat{X}_{[k]}, \widehat{Y}, \widehat{X}_{\mathrm{b}})$ and $(X_{[k]}, Y_1, X_{1\to 2})$ have the same distribution, we have $\frac{1}{\widehat{n}} I(\widehat{X}_{\widehat{B}_z}, \widehat{X}_{\mathrm{b}}; \widehat{X}_{\widehat{A}_z}) \leq \eta$ for $\mathbb{I}$. ■

VII. From Secure Network to Secure Index Coding

In the other direction, consider a secure network-coding instance $\mathbb{N} = (G, C, P)$. For simplicity, let the source indices be $S = [k]$ and the vertex indices be $V = [\ell]$. Recall that each message is requested by at least one destination.

We will construct the following map to obtain an index-coding instance $\mathbb{I}$:

1) Construct an augmented secure network-coding instance $\mathbb{N}'$ from any (possibly randomized) secure network-coding instance $\mathbb{N}$. This converts any possibly randomized secure network code to a deterministic network code.
2) Map $\mathbb{N}'$ to $\mathbb{I}$:
   a) For legitimate receivers: We use the existing non-secure instance map [4, Sec. III] (which only works for deterministic codes), except that we omit one receiver in $\widehat{T}$. We will show that omitting this receiver will not affect the result.
   b) For eavesdroppers: We construct a map from the eavesdroppers in $\mathbb{N}'$ to those in $\mathbb{I}$.

For $\mathbb{I}$, we set

$$\widehat{n} = \sum_{e\in E} \lfloor c_e n \rfloor. \tag{9}$$

This means the number of bits that the sender can transmit in $\mathbb{I}$ equals the total number of bits that can be transmitted on all the links in $\mathbb{N}$.

A. Network-to-Index Coding Map

Now, we describe the instance map in detail:

1) Augmented Secure Network Coding: We construct an *augmented* secure network-coding instance $\mathbb{N}' = (G', C', P')$ as follows:

- $G' = (V, E) = G$, where each link e in G' has the same capacity c_e as that in G.
- $C' = (S', O', D')$: Here, we introduce an additional independent source $X'_{k+i} \in [K_i]$ originating at each vertex $i\in[\ell]$ that takes the role of and has the same distribution as the random key Y_i used in the randomized encoding at vertex i in $\mathbb{N}$.
  - $S' = [k + \ell]$, where the message alphabet sizes are $M'_i = M_i$ for each $i \in [k]$, and $M'_{k+i} = K_i$ for each $i \in [\ell]$.
  - $O'(i) = O(i)$ for each $i \in [k]$, and $O'(k + i) = i$ for each $i \in [\ell]$
  - $D'(i) = D(i)$ for each $i \in [k]$, and $D'(k + i) = \emptyset$ for each $i \in [\ell]$.
- $P' = ((A_z, B_z) : z \in Z) = P$.

Note that by construction, $X'_{[k+\ell]}$ and $(X_{[k]}, Y_{[\ell]})$ have the same distribution. So, there is a bijective map from a deterministic or randomized secure network code for $\mathbb{N}$ to a deterministic secure network code for $\mathbb{N}'$. Note that in $\mathbb{N}'$, the additional sources $\{X'_{k+i} : i \in [\ell]\}$ are not required to be decoded by any node. Also, they are neither known to any eavesdropper nor required to be protected.

Denote the set of vertices in $\mathbb{N}'$ that are the destinations for some source messages by $U = \{j \in [\ell] : j \in D'(i)$ for some $i\in[k]\}$.

2) Network-to-Index Coding Map: Now, we map $\mathbb{N}'$ to a secure index-coding instance $\mathbb{I}$.

- $\widehat{S} = [k+\ell]\cup E$. It consists of one message $\widehat{X}_i\in[\widehat{M}_i]=[M'_i]$ for each $i \in [k+\ell]$ and one message $\widehat{X}_e \in [\widehat{M}_e] = [2^{\lfloor c_e n\rfloor}]$ for each $e \in E$ in $\mathbb{N}'$.
- $\widehat{T} = \{\widehat{t}_i\}_{i\in U} \cup \{\widehat{t}_e\}_{e\in E}$. This means $\mathbb{I}$ has $|U| + |E|$ receivers: the first set corresponds to each destination node in $\mathbb{N}'$, and the second set corresponds to each link in $\mathbb{N}'$.
- For each $\widehat{t}_e \in \widehat{T}$ where $e \in E$, we set $\widehat{H}_{\widehat{t}_e}=\mathtt{in}(\mathtt{tail}(e))\cup O'^{-1}(\mathtt{tail}(e))$, and $\widehat{W}_{\widehat{t}_e} = \{e\}$.
- For each $\widehat{t}_i \in \widehat{T}$ where $i \in U$, we set $\widehat{H}_{\widehat{t}_i}=\mathtt{in}(i)\cup O'^{-1}(i)$, and $\widehat{W}_{\widehat{t}_i} = \{j \in [k + \ell] : i \in D'(j)\} \coloneqq D'^{-1}(i)$.
- The eavesdropper setting $\widehat{P} : \widehat{Z} = Z$. For each $z \in \widehat{Z}$, $\widehat{B}_z = B_z$, and $\widehat{A}_z = A_z$.

The two steps in the map from a network coding instance to the corresponding index-coding instance are summarized in Figures 3 and 4, respectively.

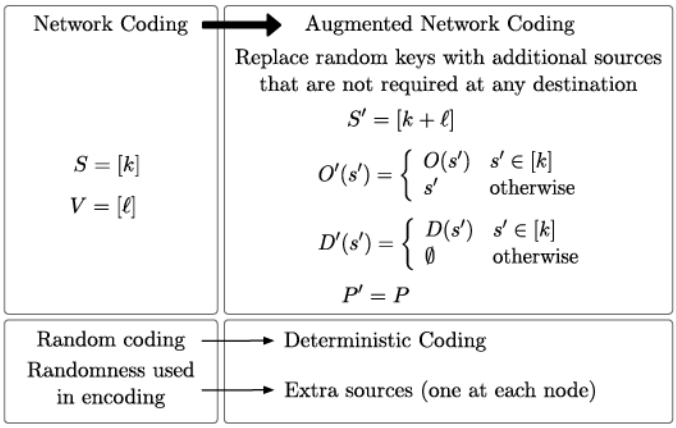


Fig. 3. From secure network coding to augmented secure network coding.

Remark 5: The mapping to the receivers in $\mathbb{I}$ from $\mathbb{N}'$ is slightly different from that in the non-secure instance map [4, Sec. III], which includes an additional receiver $\widehat{t}_{\text{all}}$ in $\mathbb{I}$. The receiver was included to guarantee a useful property, which, as we will see in Proposition 1, remains true even without receiver $\widehat{t}_{\text{all}}$.

B. Equivalence Results

With the above conversion, we state an equivalence between $\mathbb{N}$ and $\mathbb{I}$ through $\mathbb{N}'$. Recall that $\widehat{n} = \sum_{e\in E}\lfloor c_e n\rfloor$ and $2^{\lfloor c_E n\rfloor} := (2^{\lfloor c_e n\rfloor} : e \in E)$. Similarly, let $2^{\lceil R_S n\rceil} := (2^{\lceil R_s n\rceil} : s \in S)$.

Theorem 2: Let $\mathbb{N}$ be a secure network-coding instance and $\mathbb{I}$ be the corresponding secure index-coding instance. For any $\epsilon \in [0, 0.5]$, $\eta \in \mathbb{R}_0^+$, and $n \in \mathbb{Z}^+$, we have the following:

1) If $\mathbb{N}$ is (M_S, n, ϵ, η)-feasible, then $\mathbb{I}$ is $((M_S, K_V, 2^{\lfloor c_E n\rfloor}), \widehat{n}, \epsilon, \theta_1)$-feasible with a deterministic index code for some $K_V \in (\mathbb{Z}^+)^\ell$, where $\theta_1 := \frac{\eta}{(\sum_{e\in E} c_e - \frac{|E|}{n})}$.
2) If $\mathbb{I}$ is $((M_S, K_V, 2^{\lfloor c_E n\rfloor}), \widehat{n}, \epsilon, \eta)$-feasible with a deterministic index code, then $\mathbb{N}$ is $(M_S, n, (|Z| + 1)\epsilon, \theta_2)$-feasible, where $\theta_2 := (|Z| + 1)[(\frac{\eta}{1-\epsilon} + \epsilon)\sum_{e\in E} c_e + \frac{1}{n}(\frac{H_b(\epsilon)}{1-\epsilon} + H_b(\epsilon))]$.

Proof: See Section VIII for the proof of Part 1 and Section IX for Part 2. ∎

Unlike the index-to-network map, here $\widehat{X}_E$ and X'_E have different distributions. In $\mathbb{I}$, $\widehat{X}_E$ are the source messages, which are mutually independent; in $\mathbb{N}'$, X'_E are the link messages, which are functions of the source messages $X'_{[k+\ell]}$ and may be correlated.

Theorem 2 leads to the following rate equivalence.

Corollary 2: Let $\mathbb{N}$ be a secure network-coding instance with $c_e \in \mathbb{Z}^+$, and $\mathbb{I}$ be the corresponding secure index-coding instance obtained using the network-to-index coding map. A rate tuple R_S is achievable for $\mathbb{N}$ (with code lengths in $\ell\mathbb{Z}^+$) using random key rates R_V^{key} iff the rate tuple $\frac{1}{\sum_{e\in E} c_e}(R_S, R_V^{\text{key}}, c_E)$ is achievable for $\mathbb{I}$ (with code lengths in $\ell(\sum_{e\in E} c_e)\mathbb{Z}^+$).

Proof: Suppose that a rate tuple $R_S \in (\mathbb{R}_0^+)^{|S|}$ is achievable for $\mathbb{N}$. Then there exists a sequence of network codes $(2^{\lceil nR_S\rceil}, n)$, $n \in \ell\mathbb{Z}^+$, where each code has messages sizes $2^{\lceil nR_s\rceil}$, for $s \in S$, with probability of decoding error $\epsilon \to 0$ and leakage $\eta \to 0$ as $n \to \infty$.

From Theorem 2, it follows that there exists a sequence of index codes $([(2^{\lceil \widehat{n}\frac{n}{n}R_s\rceil}, s \in S), (2^{\lceil \widehat{n}\frac{n}{n}R_v^{\text{key}}\rceil}, v \in V), (2^{\widehat{n}\frac{\lfloor nc_e\rfloor}{n}}, e \in E)], \widehat{n})$, $\widehat{n} = \sum_{e\in E}\lfloor c_e n\rfloor = n\sum_{e\in E} c_e \in \ell(\sum_{e\in E} c_e)\mathbb{Z}^+$, with probability of decoding error $\epsilon \to 0$ and leakage $\theta_{1,n} \to 0$ as $\eta \to 0$ and $n \to \infty$. Hence, the rate $\frac{1}{\sum_{e\in E} c_e}(R_S, R_V^{\text{key}}, c_E)$ is achievable for $\mathbb{I}$.

The other direction from $\mathbb{I}$ to $\mathbb{N}$ follows a similar argument. Suppose that $\frac{1}{\sum_{e\in E} c_e}(R_S, R_V^{\text{key}}, c_E)$ is achievable for $\mathbb{I}$. Then, there exists a sequence of index codes $([(2^{\lceil \widehat{n}(\frac{R_s}{\sum_{e\in E} c_e})\rceil}, s \in S), (2^{\lceil \widehat{n}(\frac{R_v^{\text{key}}}{\sum_{e\in E} c_e})\rceil}, v \in V), (2^{\lceil \widehat{n}(\frac{c_i}{\sum_{e\in E} c_e})\rceil}, i \in E)], \widehat{n})$, $\widehat{n} \in \ell(\sum_{e\in E} c_e)\mathbb{Z}^+$, with probability of decoding error $\epsilon \to 0$ and leakage $\eta \to 0$ as $\widehat{n} \to \infty$.

From Theorem 2, it follows that there exists a sequence of network codes $(2^{\lceil nR_S\rceil}, n)$, $n = \frac{\widehat{n}}{\sum_{e\in E} c_e} \in \ell\mathbb{Z}^+$, with probability of decoding error $(|Z| + 1)\epsilon \to 0$ and leakage $\theta_2 \to 0$, $\eta \to 0$ and $n \to \infty$. Hence, the rate tuple R_S is achievable for $\mathbb{N}$. ∎

C. An Example

Consider the network-coding instance $\mathbb{N}$ depicted in Figure 5(a), its augmented version $\mathbb{N}'$ in Figure 5(b), and the mapped index-coding instance $\mathbb{I}$ in Figure 5(c).

One randomized zero-error zero-leakage network code for $\mathbb{N}$ is $X_{e_1} = \mathbf{e}_{e_1}(X_1, Y_1) = Y_1$ and $X_{e_2} = \mathbf{e}_{e_2}(X_1, Y_1) = X_1 + Y_1$, where $Y_1 \in [M_1]$ (that is, $K_1 = M_1$). Without using the random key Y_1, it is not possible to protect X_1 from the eavesdroppers.

This network code is then translated to a deterministic network code for $\mathbb{N}'$ with three messages $\{X'_1, X'_2, X'_3\}$, with $X'_3 = \alpha$ set to be a constant, as follows: $X'_{e_1} = \mathbf{g}'_{e_1}(X'_1, X'_2, X'_3) = X'_2$ and $X'_{e_2} = \mathbf{g}'_{e_2}(X'_1, X'_2, X'_3) = X'_1 + X'_2$. The augmentation from $\mathbb{N}$ to $\mathbb{N}'$ changes neither the decodability of the users nor the leakage to the eavesdroppers.

In the index-coding instance $\mathbb{I}$, two receivers $\{\widehat{t}_{e_1}, \widehat{t}_{e_2}\}$ correspond to the links in $\mathbb{N}'$ and one receiver $\widehat{t}_2$ corresponds to the destination node in $\mathbb{N}'$. The translated index code is

$$\begin{aligned}\widehat{X}_{\text{b}} &= \widehat{\mathbf{e}}(\widehat{X}_{[3]\cup E}) = (\widehat{X}_{\text{b},e_1}, \widehat{X}_{\text{b},e_2}) \\ &= (\widehat{X}_{e_1} + \mathbf{g}'_{e_1}(\widehat{X}_{[3]}), \widehat{X}_{e_2} + \mathbf{g}'_{e_2}(\widehat{X}_{[3]})) \\ &= (\widehat{X}_{e_1} + \widehat{X}_2, \widehat{X}_{e_2} + \widehat{X}_1 + \widehat{X}_2).\end{aligned}$$

All receivers can decode their required messages, and each eavesdropper gains no information about $\widehat{X}_1$.

In the other direction of secure index-to-network code translation, consider the following secure index code (where $\widehat{X}_3 = \alpha$ is a constant):

$$\begin{aligned}\widehat{X}_{\text{b}} &= \widehat{\mathbf{e}}(\widehat{X}_{[3]\cup E}) = (\widehat{\mathbf{e}}_1(\widehat{X}_{[3]\cup E}), \widehat{\mathbf{e}}_2(\widehat{X}_{[3]\cup E})) \\ &= (\widehat{X}_{e_1} + \beta_1\widehat{X}_1 + \beta_2\widehat{X}_2, \widehat{X}_{e_2} + \gamma_1\widehat{X}_1 + \gamma_2\widehat{X}_2) := (\widehat{X}_{\text{b},1}, \widehat{X}_{\text{b},2}),\end{aligned}$$

for some non-zero β_2, γ_2 such that $\beta_1/\beta_2 \neq \gamma_1/\gamma_2$. The decoding function of nodes $\widehat{t}_{e_1}, \widehat{t}_{e_2}, \widehat{t}_2$ are respectively

$$\begin{aligned}&\widehat{\mathbf{d}}_{\widehat{t}_{e_1}}(\widehat{X}_{\text{b}}, \widehat{X}_{\widehat{H}_{\widehat{t}_{e_1}}}) = \widehat{\mathbf{d}}_{\widehat{t}_{e_1}}(\widehat{X}_{\text{b}}, \widehat{X}_1, \widehat{X}_2) = \widehat{X}_{\text{b},1} - \beta_1\widehat{X}_1 - \beta_2\widehat{X}_2, \\ &\widehat{\mathbf{d}}_{\widehat{t}_{e_2}}(\widehat{X}_{\text{b}}, \widehat{X}_{\widehat{H}_{\widehat{t}_{e_2}}}) = \widehat{\mathbf{d}}_{\widehat{t}_{e_2}}(\widehat{X}_{\text{b}}, \widehat{X}_1, \widehat{X}_2) = \widehat{X}_{\text{b},2} - \gamma_1\widehat{X}_1 - \gamma_2\widehat{X}_2, \\ &\widehat{\mathbf{d}}_{\widehat{t}_2}(\widehat{X}_{\text{b}}, \widehat{X}_{\widehat{H}_{\widehat{t}_2}}) = \widehat{\mathbf{d}}_{\widehat{t}_2}(\widehat{X}_{\text{b}}, \widehat{X}_{e_1}, \widehat{X}_{e_2})\end{aligned}$$

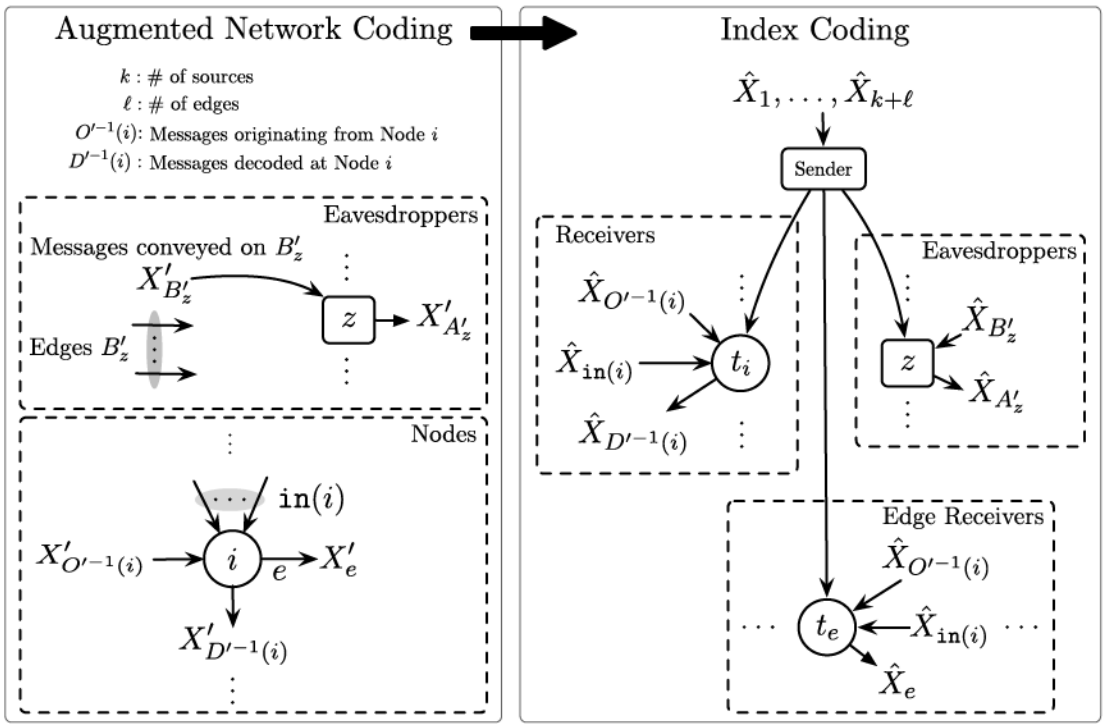
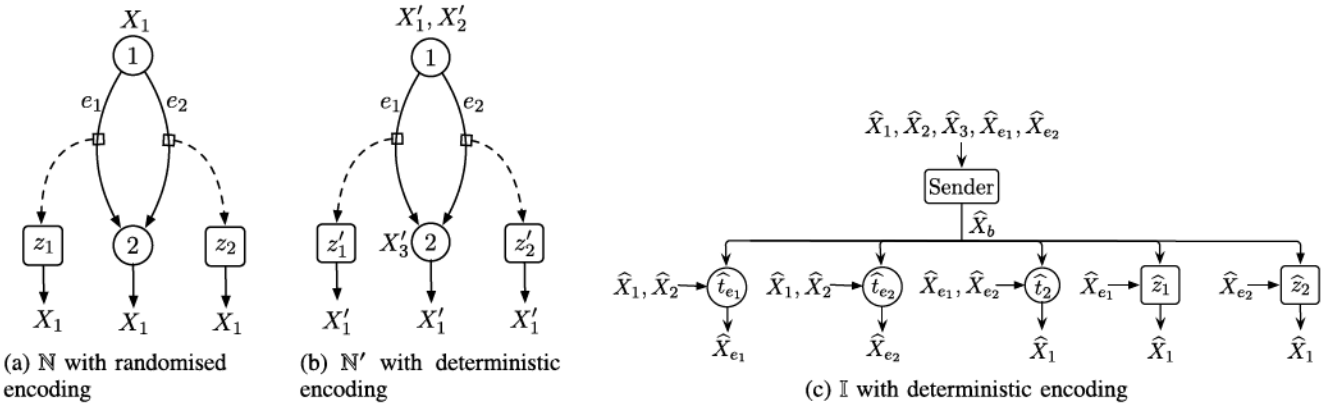


Fig. 4. From augmented secure network coding to secure index coding: Node i, edge e, and eavesdropper z in $\mathbb{N}'$ are mapped to node t_i, node t_e, and eavesdropper z respectively in $\mathbb{I}$.



(a) $\mathbb{N}$ with randomised encoding

(b) $\mathbb{N}'$ with deterministic encoding

(c) $\mathbb{I}$ with deterministic encoding

Fig. 5. A secure network-coding instance $\mathbb{N}$, its augmented version $\mathbb{N}'$ (with additional messages X'_2 and X'_3), and the corresponding secure index-coding instance $\mathbb{I}$.

$$= \left(\frac{\widehat{X}_{\mathrm{b},1} - \widehat{X}_{e1}}{\beta_2} - \frac{\widehat{X}_{\mathrm{b},2} - \widehat{X}_{e2}}{\gamma_2}\right)\left(\frac{\beta_1}{\beta_2} - \frac{\gamma_1}{\gamma_2}\right)^{-1}.$$

The translated deterministic network code for $\mathbb{N}'$ is

$$\begin{aligned} X'_{e_1} &= \mathsf{e}_{e_1}\left(X'_{\mathtt{in}(\mathtt{tail}(e_1))}, X'_{O'^{-1}(\mathtt{tail}(e_1))}\right) \\ &= \widehat{\mathsf{d}}_{\widehat{t}_{e_1}}\left(0, X'_1, X'_2\right) = -\beta_1 X'_1 - \beta_2 X'_2, \\ X'_{e_2} &= \mathsf{e}_{e_2}\left(X'_{\mathtt{in}(\mathtt{tail}(e_2))}, X'_{O'^{-1}(\mathtt{tail}(e_2))}\right) \\ &= \widehat{\mathsf{d}}_{\widehat{t}_{e_2}}\left(0, X'_1, X'_2\right) = -\gamma_1 X'_1 - \gamma_2 X'_2. \end{aligned}$$

In $\mathbb{N}'$, the destination (node 2) can recover X'_1 by choosing $\mathsf{d}_2(X'_{\mathtt{in}(2)}, X'_{O'^{-1}(2)}) = \widehat{\mathsf{d}}_{\widehat{t}_2}(0, X'_{e_1}, X'_{e_2})$. Also, each eavesdropper gains no information about X'_1 as β_2 and γ_2 are non-zero. Lastly, by replacing X'_2 with Y_1, we obtain a zero-error zero-leakage network code for $\mathbb{N}'$.

VIII. Proof of Theorem 2 – Part 1 (The Forward Direction)

We will now prove Theorem 2 for the forward direction: if $\mathbb{N}$ is (M_S, n, ϵ, η)-feasible, then $\mathbb{I}$ is $((M_S, K_V, 2^{\lfloor c_E n \rfloor}), \widehat{n}, \epsilon, \theta_1)$-feasible with a deterministic index code.

A. Code Construction

Recall that $S = [k]$ and $V = [\ell]$. Also recall that each $\widehat{X}_e$, $e \in E$, is chosen to be independently and uniformly distributed over $[\widehat{M}_e] = [2^{\lfloor c_e n \rfloor}]$.

Note that $\mathbb{N}$ is (M_S, n, ϵ, η)-feasible iff $\mathbb{N}'$ is $((M_S, K_V), n, \epsilon, \eta)$–feasible, where the random key Y_i at each node $i \in V$ in the network code for $\mathbb{N}$ is realized using an additional independent source X'_{k+i} at the same node $\mathbb{N}'$.

From a network code for $\mathbb{N}'$, we will use the non-secure code translation [4, Direction 1] to construct an index code for $\mathbb{I}$, where the sender broadcasts $\widehat{X}_{\mathrm{b}} = (\widehat{X}_{\mathrm{b},e} : e \in E)$, comprising

$$\widehat{X}_{\mathrm{b},e} = \widehat{X}_e + \mathsf{g}'_e\big(\widehat{X}_{[k+\ell]}\big) \mod 2^{\lfloor c_e n \rfloor}. \tag{10}$$

Note that each $\widehat{X}_e, \mathsf{g}'_e \in [2^{\lfloor c_e n \rfloor}]$, and therefore $\widehat{X}_{\mathrm{b}} \in [\prod_{e\in E} 2^{\lfloor c_e n \rfloor}] = [2^{\sum_{e\in E}\lfloor c_e n \rfloor}] = [2^{\widehat{n}}]$.

B. Decoding Criterion

See Appendix C for the proof of the decoding criterion, which is similar to that of the non-secure equivalence [4, Sec. V].

C. Security Criteria

Given $\frac{1}{n} I(X_{A_z}; X_{B_z}) \le \eta$ for $\mathbb{N}'$, we need to show $\frac{1}{n} I(\widehat{X}_{\widehat{A}_z}; \widehat{X}_{\mathrm{b}}, \widehat{X}_{\widehat{B}_z}) \le \eta$ for $\mathbb{I}$.

We now consider the security constraints. For each $z \in \widehat{Z}$,

$$\begin{aligned}
H\Big(\widehat{X}_{\widehat{A}_z} | \widehat{X}_{\mathrm{b}}, \widehat{X}_{\widehat{B}_z}\Big) &= H\Big(\widehat{X}_{\widehat{A}_z} | \{\widehat{X}_{\mathrm{b},e} : e \in E\}, \{\widehat{X}_{e'} : e' \in \widehat{B}_z\}\Big) \\
&= H\Big(\widehat{X}_{\widehat{A}_z} | \{\widehat{X}_{\mathrm{b},e} : e \in \widehat{B}_z\}, \{\widehat{X}_{e'} : e' \in \widehat{B}_z\}\Big) && \text{(11a)} \\
&= H\Big(\widehat{X}_{\widehat{A}_z} | \{\widehat{X}_{\mathrm{b},e}, \widehat{X}_e, \mathsf{g}'_e\big(\widehat{X}_{[k+\ell]}\big) : e \in \widehat{B}_z\}\Big) && \text{(11b)} \\
&= H\Big(\widehat{X}_{\widehat{A}_z} | \{\widehat{X}_e, \mathsf{g}'_e\big(\widehat{X}_{[k+\ell]}\big) : e \in \widehat{B}_z\}\Big) && \text{(11c)} \\
&= H\Big(\widehat{X}_{\widehat{A}_z} | \{\mathsf{g}'_e\big(\widehat{X}_{[k+\ell]}\big) : e \in \widehat{B}_z\}\Big) && \text{(11d)} \\
&= H\big(\widehat{X}_{A_z} | \{\mathsf{g}'_e\big(\widehat{X}_{[k+\ell]}\big) : e \in B_z\}\big) && \text{(11e)} \\
&= H\Big(X'_{A_z} | \{\mathsf{g}'_e\big(X'_{[k+\ell]}\big) : e \in B_z\}\Big) && \text{(11f)} \\
&= H\Big(X'_{A_z} | X'_{B_z}\Big) = H\big(X_{A_z} | X_{B_z}\big). && \text{(11g)}
\end{aligned}$$

Here, (11a) follows from the Markov chain $\widehat{X}_{\widehat{A}_z} - (\{\widehat{X}_{\mathrm{b},e} : e \in \widehat{B}_z\}, \{\widehat{X}_{e'} : e' \in \widehat{B}_z\}) - (\{\widehat{X}_{\mathrm{b},e} : e \notin \widehat{B}_z\})$, where $\{\widehat{X}_{\mathrm{b}}(e) : e \notin \widehat{B}_z\}$ are independent of $(\widehat{X}_{\widehat{A}_z}, \{\widehat{X}_{\mathrm{b},e} : e \in \widehat{B}_z\}, \{\widehat{X}_{e'} : e' \in \widehat{B}_z\})$, because the former has been randomized by independently and uniformly distributed $\{\widehat{X}_e : e \notin \widehat{B}_z\}$ (which are independent of $(\widehat{X}_{\widehat{A}_z}, \widehat{X}_{\widehat{B}_z}, \widehat{X}_{[k+\ell]})$, see (10)). (11b) follows from (10). (11c) is derived because $\widehat{X}_{\mathrm{b},e}$ is a deterministic function of $(\widehat{X}_e, \mathsf{g}'_e(\widehat{X}_{[k+\ell]}))$. (11d) follows from the Markov chain $\widehat{X}_{\widehat{A}_z} - \{\mathsf{g}'_e(\widehat{X}_{[k+\ell]}) : e \in \widehat{B}_z\} - \{\widehat{X}_e : e \in \widehat{B}_z\}$, which can be derived from noting that $\{\widehat{X}_e : e \in E\}$ are independent of $(\widehat{X}_{\widehat{A}_z}, \widehat{X}_{[k+\ell]})$. (11f) follows from a change of variables (from hatted to dashed) as $(\widehat{X}_{[k+\ell]}, \widehat{X}_{A_z})$ and $(X'_{[k+\ell]}, X'_{A_z})$ have the same distribution. (11g) is obtained from noting that $\{\mathsf{g}'_e(X'_{[k+\ell]}) : e \in B_z\} = X'_{B_z}$.

Now, for $\mathbb{N}$, if $I(X_{A_z}; X_{B_z}) \le \eta$, then

$$\begin{aligned}
\frac{1}{\widehat{n}} I\Big(\widehat{X}_{\widehat{A}_z}; \widehat{X}_{\mathrm{b}}, \widehat{X}_{\widehat{B}_z}\Big) &= \frac{1}{\widehat{n}}\Big[H\Big(\widehat{X}_{\widehat{A}_z}\Big) - H\Big(\widehat{X}_{\widehat{A}_z} | \widehat{X}_{\mathrm{b}}, \widehat{X}_{\widehat{B}_z}\Big)\Big] && \text{(12a)} \\
&= \frac{1}{\widehat{n}}\Big[H\Big(\widehat{X}_{\widehat{A}_z}\Big) - H\big(X_{A_z} | X_{B_z}\big)\Big] && \text{(12b)} \\
&= \frac{1}{\widehat{n}}\big[H\big(X_{A_z}\big) - H\big(X_{A_z} | X_{B_z}\big)\big] && \text{(12c)} \\
&= \frac{1}{\sum_{e\in E}\lfloor c_e n \rfloor} I\big(X_{A_z}; X_{B_z}\big) && \text{(12d)} \\
&\le \frac{1}{\big(n\sum_{e\in E} c_e - |E|\big)} I\big(X_{A_z}; X_{B_z}\big) && \text{(12e)} \\
&< \frac{\eta}{\Big(\sum_{e\in E} c_e - \frac{|E|}{n}\Big)} = \theta_1, && \text{(12f)}
\end{aligned}$$

where (12b) follows from (11g), and (12c) follows from $X_{[k]}$ and $\widehat{X}_{[k]}$ having the same distribution. Recall that $c_e n \ge 1$, which ensures that none of the links is degenerated (that is, cannot carry any information).

Combining the decodability and the security results, the index code is $((M_S, K_V, 2^{\lfloor c_E n \rfloor}), \widehat{n}, \epsilon, \theta_1)$-feasible.

IX. Proof of Theorem 2 – Part 2 (The Backward Direction)

We will now prove Theorem 2 for the backward direction: if $\mathbb{I}$ is $((M_S, K_V, 2^{\lfloor c_E n \rfloor}), \widehat{n}, \epsilon, \eta)$-feasible with a deterministic index code, then $\mathbb{N}$ is $(M_S, n, (|Z|+1)\epsilon, \theta_2)$-feasible.

We only need to show the result from deterministic index codes for $\mathbb{I}$ to deterministic network codes for $\mathbb{N}'$. The code translation from $\mathbb{N}'$ to $\mathbb{N}$ is straightforward. By substituting each $X'_{[k+i]} \in [K_i]$, $i \in [\ell]$, with a random key $Y_i \in [K_i]$, we conclude that if $\mathbb{N}'$ is $((M_{[k]}, K_{[\ell]}), n, \epsilon, \eta)$-feasible using a deterministic code, then $\mathbb{N}$ is $(M_{[k]}, n, \epsilon, \eta)$-feasible using a randomized code. This is possible as $X'_{[k+i]}$ originates at node i and is not required by any node or any eavesdropper.

A. Code Construction

We will start with the non-secure code translation [4, Direction 2], which translates any index code for $\mathbb{I}$ to a network code for $\mathbb{N}'$ using a parameter σ. The translated network code consists of the following:

- An encoding function for each link $e \in E$ such that $\mathsf{e}_e(x_{\widehat{H}_{t_e}}) = \widehat{\mathsf{d}}_{t_e}(\sigma, x_{\widehat{H}_{t_e}}) \in [\widehat{M}_e] = [2^{\lfloor c_e n \rfloor}]$,
- A decoding function for each destination $i \in U$ such that $\mathsf{d}_i(x_{\mathrm{in}(i)\cup O^{-1}(i)}) = \widehat{\mathsf{d}}_{\widehat{t}_i}(\sigma, x_{\mathrm{in}(i)\cup O^{-1}(i)}) \in [\widehat{M}_i]$. Recall that U is the set of destination nodes in $\mathbb{N}'$.

The block length of the network code is n.

In $\mathbb{I}$, σ is the broadcast message $\widehat{x}_{\mathrm{b}} = \widehat{\mathsf{e}}(\widehat{x}_{[k+\ell]\cup E})$, which depends on *all* messages $\widehat{x}_{[k+\ell]\cup E}$.

B. A Summary of Our Approach for the Choice of σ

This direction (the backward code translation for the secure network-to-index coding instance map) of the equivalence is the most challenging amongst the four. The difficulty here is to select a suitable $\sigma \in [2^{\widehat{n}}]$ for the network code for $\mathbb{N}'$ that satisfies both decodability and secrecy. Since σ is used in all encoding and decoding functions, and yet each node may know a different subset of messages, σ cannot be made to depend on the messages in the same way as the broadcast message does in the index code. A solution could be to either (i) choose a fixed σ for the network code, or (ii) randomly choose one σ according to some distribution—independent of the messages—but all nodes must agree on this random choice. Option (ii) is possible if all the nodes in the network observe some common randomness (for example, using the same random number generator), which need not be secure from the eavesdroppers.

Without security, a good candidate σ exists. Using this fixed σ (which is independent of the message realization), the translated network code has a probability of decoding error of at most ϵ. This particular choice was found [4, Claim 2] by considering network codes with the parameter σ randomly and uniformly over $[2^{\widehat{n}}]$.

With security, for the special case of perfect decoding and no leakage, we established [24] that the same σ found using the above method can be used to translate an index code with $\epsilon = \eta = 0$ to a network code with $\epsilon = \eta = 0$. This approach relies on an observation that if for message realizations that can be decoded correctly in $\mathbb{I}$, not only is decoding correct in $\mathbb{N}'$, but the leakage is also preserved.

For the above two scenarios, option (i) of fixing σ suffices. In general, when decoding is incorrect in $\mathbb{I}$, the leakage in $\mathbb{N}'$ may not match that in $\mathbb{I}$.[7] Consequently, for any fixed σ, even if the probability of decoding error in $\mathbb{N}'$ is small, when error occurs, the leakage can be large.

To avoid this problem, we first observe that when the broadcast message σ changes (as a function of the messages) for the index code, the probability of decoding error is bounded by ϵ, and the leakage, by η. Since the index code and the translated network code behave similarly when decoding is correct, we consider instances of correct decoding in $\mathbb{I}$ and look at the probability distribution p_Σ of the broadcast message conditioned on correct decoding.

Our approach is to design a collection of network codes, where σ for the network codes is randomly chosen based on this probability p_Σ. We show that, averaged over p_Σ, $\mathbb{N}'$ will have the desired probability of decoding error and leakage. As mentioned above, this implementation of solution option (ii) requires additional shared randomness among the nodes. Building on this solution, we will further show the existence of a good candidate for σ, thereby lifting the requirement of additional shared randomness and obtaining a solution based on option (i).

C. An Important Property of the Broadcast Message in $\mathbb{I}$

Central to the proof of the code translation from $\mathbb{I}$ to $\mathbb{N}'$ is the following property of the broadcast message in $\mathbb{I}$:

Proposition 1: Fix any broadcast message $\widehat{x}_{\mathrm{b}} \in [2^{\widehat{n}}]$ and any realization $\widehat{x}_{[k+\ell]}$. If all receivers $\widehat{T}$ can decode their requested messages correctly, then there can be at most one realization $\widehat{x}_E$ for which $\widehat{\mathsf{e}}(\widehat{x}_{[k+\ell]}, \widehat{x}_E) = \widehat{x}_{\mathrm{b}}$.

Proposition 1 was proven for a slightly different network-to-index coding map [4], which includes an additional receiver $\widehat{t}_{\mathrm{all}}$ in $\mathbb{I}$, where $\widehat{H}_{\widehat{t}_{\mathrm{all}}} = [k+\ell]$ and $\widehat{W}_{\widehat{t}_{\mathrm{all}}} = E$. We will show that the proposition remains true even without $\widehat{t}_{\mathrm{all}}$, by taking into account the fact that graph G (which was defined for $\mathbb{N}$ from which $\mathbb{I}$ has been mapped) is acyclic.

[7] Take Figure 5 for example. If the decoding in $\mathbb{I}$ is correct, node $\widehat{t}_{e_2}$ outputs the same message as the side information of eavesdropper $\widehat{z}_2$. This matches $\mathbb{N}'$ where eavesdropper z_2' observes link e_2. However, when decoding in $\mathbb{I}$ is wrong at $\widehat{t}_{e_2}$, the output of $\widehat{t}_{e_2}$ and the side information of $\widehat{z}_2$ are different. So, even if the leakage in $\mathbb{I}$ is small in for this message realization, the leakage in $\mathbb{N}'$ may be large, because the observation of z_2' in $\mathbb{N}'$ does not match the side information of $\widehat{z}_2$ in $\mathbb{I}$.

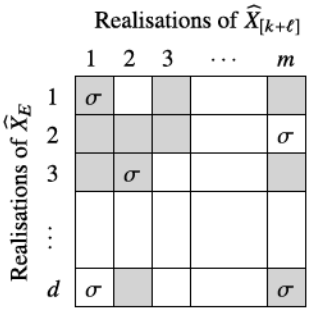


Fig. 6. A table showing the value of $\widehat{\mathsf{e}}(\widehat{x}_{[k+\ell]\cup E}) \in [2^{\widehat{n}}]$ for each message realization $(\widehat{x}_{[k+\ell]}, \widehat{x}_E)$. Shaded cells indicate message realizations that result in correct decoding for all receivers in $\mathbb{I}$.

Proof: Fix any $\widehat{x}_{[k+\ell]}$ and $\widehat{x}_{\mathrm{b}}$. The decoding function of each receiver $\widehat{t}_e$, $e \in E$, is $\widehat{\mathsf{d}}_{\widehat{t}_e}(\widehat{x}_{\mathrm{b}}, \widehat{x}_{\mathtt{in}(\mathtt{tail}(e))\cup O'^{-1}(\mathtt{tail}(e))})$, where $\mathtt{in}(\mathtt{tail}(e)) \subsetneq E$ are in the upstream of e, and $O'^{-1}(\mathtt{tail}(e)) \subseteq [k+\ell]$.

Since the decoding for all receivers are correct, for each $e \in E$, $\widehat{t}_e$ recovers $\widehat{x}_e$ by using the function $\widehat{\mathsf{d}}_{\widehat{t}_e}$. As G is acyclic, by considering decoding functions $\widehat{\mathsf{d}}_{\widehat{t}_e}$ starting from *root* nodes, that is, links e where $\mathtt{in}(\mathtt{tail}(e)) = \emptyset$, and traversing the links in the directions of the links, all *link messages* $\widehat{x}_E$ are completely determined by $\widehat{x}_{[k+\ell]}$ and $\widehat{x}_{\mathrm{b}}$. ∎

Suppose in $\mathbb{I}$ that a message realization $\widehat{x}_{[k+\ell]\cup E}$ results in correct decoding. From the proof of Proposition 1, we know that given $\widehat{x}_{[k+\ell]}$ and $\widehat{\mathsf{e}}(\widehat{x}_{[k+\ell]\cup E})$, a sequence of receiver decoding functions can collectively recover $\widehat{x}_E$. So, using the above network-code translation, we have the following observation:

Observation 1: Suppose that $\widehat{x}_{[k+\ell]\cup E}$ results in correct decoding in $\mathbb{I}$. We use the translated network code in $\mathbb{N}'$. For the message realization $x_{[k+\ell]} = \widehat{x}_{[k+\ell]}$ in $\mathbb{N}'$, if $\sigma = \widehat{\mathsf{e}}(\widehat{x}_{[k+\ell]\cup E})$ *had been chosen* for the network code, then each link $e \in E$ will send $x_e = \widehat{x}_e$ (since its encoding function is derived from the decoding function in $\mathbb{I}$), and the decoding of all receivers in $\mathbb{N}'$ will be correct (since decoding in $\mathbb{I}$ is all correct).

D. Choosing a Distribution for σ

Recall that each network code is specified by the choice of σ. We first randomly select σ, independent of the messages, according to some probability mass function p_Σ.

Remark 6: This approach requires all nodes V to know the selected σ. This can be implemented by a random public key, which the eavesdroppers may also access. However, the use of a randomized σ is only an intermediate step for us to prove the existence of a good candidate. The final result will be based on a particular pre-chosen σ.

To simplify notation, let

- $m := \prod_1^{k+\ell} M_i$ denote the total number of message realizations of $\widehat{X}_{[k+\ell]}$.
- $d := 2^{\widehat{n}} = 2^{\sum_{e\in E} \lfloor c_e n \rfloor}$ denote the total number of message realizations of $\widehat{X}_E$.

Recall that $\widehat{\mathsf{e}}(x_{[k+\ell]\cup E}) \in [2^{\widehat{n}}]$. The table in Figure 6 shows the broadcast message $\widehat{\mathsf{e}}(x_{[k+\ell]\cup E})$ for each message realization $(\widehat{x}_{[k+\ell]}, \widehat{x}_E)$. The table is constructed as follows. Each message realization in $\mathbb{I}$ is split into two parts: $\widehat{x}_{[k+\ell]}$ determines the column, and $\widehat{x}_E$, the row. Each message realization

$(\widehat{x}_{[k+\ell]}, \widehat{x}_E)$ then points to the cell in a specific column and a specific row. The value of $\widehat{\mathbf{e}}(x_{[k+\ell]\cup E})$ is placed into that cell. After that, each cell that corresponds to a message realization $(\widehat{x}_{[k+\ell]}, \widehat{x}_E)$ that results in correct decoding in $\mathbb{I}$ is shaded. Note that, due to Proposition 1, the values of $\widehat{\mathbf{e}}(x_{[k+\ell]\cup E})$ in all shaded cells in any column must be distinct.

Let N_σ be the number of realizations $\widehat{x}_{[k+\ell]\cup E}$ that results in correct decoding for all receivers and $\widehat{\mathbf{e}}(x_{[k+\ell]\cup E}) = \sigma$. In Figure 6, N_σ is the total number of shaded cells labeled as σ. Define $\bar{\epsilon}$ as the fraction of unshaded cells. As the messages are uniformly distributed, $\bar{\epsilon}$ is also the probability of decoding error $\widehat{P}_{\mathrm{e}}$ in $\mathbb{I}$. It is easy to see the following:

$$\sum_{\sigma\in[d]} N_\sigma = (1-\bar{\epsilon})md, \tag{13}$$

$$\sum_{\sigma\in[d]} \frac{1}{d}\frac{N_\sigma}{m} = 1-\bar{\epsilon}. \tag{14}$$

Define a new random variable $\widehat{C} \in \{0,1\}$ in $\mathbb{I}$, where $\widehat{C}=1$ if decoding is correct, and $\widehat{C}=0$ otherwise. Now, consider a translated network code, where σ is the realization of a random variable Σ whose distribution is given by

$$p_\Sigma(\sigma) = \frac{N_\sigma}{(1-\bar{\epsilon})md} = p_{\widehat{\mathrm{X}}_{\mathrm{b}}|\widehat{C}}(\sigma|1), \tag{15}$$

where the second equality is obtained by observing that the messages in $\mathbb{I}$ are uniformly distributed.

E. Decodability Criterion Using Randomly Chosen σ

Let $P'_{\mathrm{e},\sigma}$ be the probability of decoding error in $\mathbb{N}'$ when σ is chosen for the network code. From Observation 1, if σ is chosen, for any message realization $x_{[k+\ell]}$ such that the column $\widehat{x}_{[k+\ell]} = x_{[k+\ell]}$ in Figure 6 contains σ in a shaded cell, decoding in $\mathbb{N}'$ is correct. From Proposition 1, each σ can appear at most once over the shaded cells in each column. So, the probability of correct decoding in $\mathbb{N}'$ is

$$1-P'_{\mathrm{e},\sigma} \geq \frac{N_\sigma}{m}. \tag{16}$$

Averaged over σ, the probability of correct decoding in $\mathbb{N}'$ is

$$1-P'_{\mathrm{e}} = \sum_{\sigma\in[d]} p_\Sigma(\sigma)\left(1-P'_{\mathrm{e},\sigma}\right) \tag{17a}$$

$$\geq \sum_{\sigma\in[d]} \frac{N_\sigma}{(1-\bar{\epsilon})md}\frac{N_\sigma}{m} = \frac{1}{1-\bar{\epsilon}}\sum_{\sigma\in[d]}\frac{1}{d}\left(\frac{N_\sigma}{m}\right)^2 \tag{17b}$$

$$\overset{(a)}{\geq} \frac{1}{1-\bar{\epsilon}}\left(\sum_{\sigma\in[d]}\frac{1}{d}\frac{N_\sigma}{m}\right)^2 \overset{(b)}{=} 1-\bar{\epsilon} \overset{(c)}{\geq} 1-\epsilon, \tag{17c}$$

where (a) is obtained using Jeans's inequality; (b) follows from (14); (c) follows from the fact that the probability of decoding error in $\mathbb{I}$ is $\widehat{P}_{\mathrm{e}} = \bar{\epsilon} \leq \epsilon$.

Thus, the randomized translated network code has a probability of decoding error $P'_{\mathrm{e}} \leq \epsilon$.

F. Security Criteria Using Randomly Chosen σ

From the identity $I(\mathrm{P};\mathrm{Q}|\mathrm{X}) + I(\mathrm{P};\mathrm{R}|\mathrm{Q},\mathrm{X}) = I(\mathrm{P};\mathrm{R}|\mathrm{X}) + I(\mathrm{P};\mathrm{Q}|\mathrm{R},\mathrm{X})$, we get

$$I(\mathrm{P};\mathrm{Q}|\mathrm{X}) = I(\mathrm{P};\mathrm{Q}|\mathrm{R},\mathrm{X}) + I(\mathrm{P};\mathrm{R}|\mathrm{X}) - I(\mathrm{P};\mathrm{R}|\mathrm{Q},\mathrm{X}) \tag{18}$$

$$\geq I(\mathrm{P};\mathrm{Q}|\mathrm{R},\mathrm{X}) - I(\mathrm{P};\mathrm{R}|\mathrm{Q},\mathrm{X}) \geq I(\mathrm{P};\mathrm{Q}|\mathrm{R},\mathrm{X}) - H(\mathrm{R}). \tag{19}$$

Similarly,

$$I(\mathrm{P};\mathrm{Q}|\mathrm{R},\mathrm{X}) \geq I(\mathrm{P};\mathrm{Q}|\mathrm{X}) - H(\mathrm{R}). \tag{20}$$

Consider eavesdropper $z \in Z$. Let $B \coloneqq \widehat{B}_z = B_z$ and $A \coloneqq \widehat{A}_z = A_z$, where we drop the subscripts to ease notation. Starting with an index code that has a leakage of at most η,

$$\begin{aligned}\widehat{n}\eta &\geq I\big(\widehat{X}_A;\widehat{X}_{\mathrm{b}},\widehat{X}_B\big) \geq I\big(\widehat{X}_A;\widehat{X}_B|\widehat{X}_{\mathrm{b}}\big)\\ &\geq I\big(\widehat{X}_A;\widehat{X}_B|\widehat{X}_{\mathrm{b}},\widehat{C}\big) - H\big(\widehat{C}\big) \geq I\big(\widehat{X}_A;\widehat{X}_B|\widehat{X}_{\mathrm{b}},\widehat{C}\big) - H_{\mathrm{b}}(\epsilon)\\ &= \sum_\sigma p_{\widehat{X}_{\mathrm{b}},\widehat{C}}(\sigma,1) I\big(\widehat{X}_A;\widehat{X}_B|\widehat{X}_{\mathrm{b}}=\sigma,\widehat{C}=1\big)\\ &\quad + \sum_\sigma p_{\widehat{X}_{\mathrm{b}},\widehat{C}}(\sigma,0) I\big(\widehat{X}_A;\widehat{X}_B|\widehat{X}_{\mathrm{b}}=\sigma,\widehat{C}=0\big) - H_{\mathrm{b}}(\epsilon).\end{aligned} \tag{21}$$

To map the above to $\mathbb{N}'$, we define $C \in \{0,1\}$ as a random variable for $\mathbb{N}'$ as follows. For a specific chosen σ and message realization $x_{[k+\ell]}$, $C=1$ iff σ appears in a shaded cell in the column $\widehat{x}_{[k+\ell]} = x_{[k+\ell]}$ in Figure 6. $C=1$ implies that the decoding in $\mathbb{N}'$ is correct (but the converse is not always true). Noting that N_σ/m is the fraction of message realizations whose column contains σ in a shaded cell in Figure 6, we have

$$p_C(1) = \sum_\sigma p_\Sigma(\sigma) N_\sigma/m \geq 1-\bar{\epsilon} = p_{\widehat{C}}(1) \geq 1-\epsilon, \tag{22}$$

where (22) follows from (17a)–(17c). This means

$$\frac{p_{\widehat{C}}(1)}{1-\epsilon} \geq 1 \geq P_C(1) \quad\Rightarrow\quad p_{\widehat{C}}(1) \geq (1-\epsilon)p_C(1). \tag{23}$$

From Proposition 1, if a message realization $(\widehat{x}_{[k+\ell]}, \widehat{x}_E)$ results in correct decoding in $\mathbb{I}$ (that is, it maps to a shaded cell in Figure 6), then $\widehat{x}_E = \phi_\sigma(\widehat{x}_{[k+\ell]})$ for some deterministic function ϕ_σ, where $\sigma = \widehat{\mathbf{e}}(\widehat{x}_{[k+\ell]}, \widehat{x}_E)$. Now, suppose that $\sigma = \widehat{\mathbf{e}}(\widehat{x}_{[k+\ell]}, \widehat{x}_E)$ is chosen for the network code. From Observation 1, if $x_{[k+\ell]} = \widehat{x}_{[k+\ell]}$ is transmitted, then $x_E = \phi_\sigma(x_{[k+\ell]})$. Then, for any (a,σ) such that $p_{\widehat{X}_{[k+\ell]},\widehat{X}_{\mathrm{b}},\widehat{C}}(a,\sigma,1) > 0$, and any b, we have

$$p_{X_E|X_{[k+\ell]},\Sigma,C}(b|a,\sigma,1) = p_{\widehat{X}_E|\widehat{X}_{[k+\ell]},\widehat{X}_{\mathrm{b}},\widehat{C}}(b|a,\sigma,1). \tag{24}$$

As the messages $X_{[k+\ell]}$ in $\mathbb{I}$ and $\widehat{X}_{[k+\ell]}$ in $\mathbb{N}'$ are uniformly distributed, we see from Figure 6 that for any σ with $N_\sigma > 0$,

$$\begin{aligned}p_{X_{[k+\ell]}|\Sigma,C}(a|\sigma,1) &= p_{\widehat{X}_{[k+\ell]}|\widehat{X}_{\mathrm{b}},\widehat{C}}(a|\sigma,1)\\ &= \begin{cases} 1/N_\sigma, & \text{if a shaded cell in column } a \text{ contains } \sigma,\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Thus, for any σ with $N_\sigma > 0$, a, and b, we get the following:

$$p_{X_{[k+\ell]},X_E|\Sigma,C}(a,b|\sigma,1) = p_{\widehat{X}_{[k+\ell]},\widehat{X}_E|\widehat{X}_{\mathrm{b}},\widehat{C}}(a,b|\sigma,1), \tag{25}$$

which then necessitates that

$$I\big(\widehat{X}_A;\widehat{X}_B|\widehat{X}_{\mathrm{b}}=\sigma,\widehat{C}=1\big) = I(X_A;X_B|\Sigma=\sigma,C=1). \tag{26}$$

Substituting (26) into (21), we get

$$\begin{aligned}\widehat{n}\eta + H_{\mathrm{b}}(\epsilon) &\geq \sum_\sigma p_{\widehat{X}_{\mathrm{b}},\widehat{C}}(\sigma,1) I(X_A;X_B|\Sigma=\sigma,C=1)\\ &= p_{\widehat{C}}(1)\sum_\sigma p_{\widehat{X}_{\mathrm{b}}|\widehat{C}}(\sigma|1) I(X_A;X_B|\Sigma=\sigma,C=1)\\ &= p_{\widehat{C}}(1) I(X_A;X_B|\Sigma,C=1)\end{aligned} \tag{27a}$$

$$\geq (1-\epsilon)p_C(1)I(X_A; X_B|\Sigma, C=1), \tag{27b}$$

where (27a) is obtained noting (15); (27b) follows from (23).

We are now ready to bound the leakage in $\mathbb{N}'$, where Σ is known to all nodes. First, we have

$$\begin{aligned} I(X_A; X_B, \Sigma) &= I(X_A; X_B|\Sigma) && (28a)\\ &\leq I(X_A; X_B|\Sigma, C) + H(C) && (28b)\\ &= p_C(1)I(X_A; X_B|\Sigma, C=1) + p_C(0)I(X_A; X_B|\Sigma, C=0)\\ &\quad + H(C) && (28c)\\ &\leq p_C(1)I(X_A; X_B|\Sigma, C=1) + \epsilon n\sum_{e\in E} c_e + H_{\mathrm{b}}(\epsilon) && (28d)\\ &\leq \frac{\widehat{n}\eta + H_{\mathrm{b}}(\epsilon)}{1-\epsilon} + \epsilon n\sum_{e\in E} c_e + H_{\mathrm{b}}(\epsilon), && (28e)\end{aligned}$$

where (28a) follows since Σ is chosen independent of the messages; (28b) follows from (20); (28d) follows from $p_C(0) \leq \epsilon$, $I(X_A; X_B|\Sigma, C=0) \leq H(X_B) \leq H(X_E) = \sum_{e\in E}\log_2 M_e \leq n\sum_{e\in E} c_e$, and $H(C) \leq H_{\mathrm{b}}(\epsilon)$; (28e) follows from (27b).

Thus, the randomized translated network code has a leakage of

$$\begin{aligned} \frac{1}{n}I(X_A; X_B|\Sigma) &< \frac{1}{n}\left(\frac{\widehat{n}\eta + H_{\mathrm{b}}(\epsilon)}{1-\epsilon} + H_{\mathrm{b}}(\epsilon)\right) + \epsilon\sum_{e\in E} c_e && (29a)\\ &\leq \left(\frac{\eta}{1-\epsilon} + \epsilon\right)\sum_{e\in E} c_e + \frac{1}{n}\left(\frac{H_{\mathrm{b}}(\epsilon)}{1-\epsilon} + H_{\mathrm{b}}(\epsilon)\right) := \theta, && (29b)\end{aligned}$$

where $\widehat{n} = \sum_{e\in E}\lfloor c_e n\rfloor \leq n\sum_{e\in E} c_e$, and $n \geq 1$.

G. Existence of a Candidate σ for Decodability and Security

Averaged over all realizations of Σ, inequalities (17c) and (29b) give

$$\sum_{\sigma\in[d]} p_\Sigma(\sigma)P'_{\mathrm{e},\sigma} \leq \epsilon,$$

$$\sum_{\sigma\in[d]} p_\Sigma(\sigma)\frac{1}{n}I\big(X_{A_z}; X_{B_z}|\Sigma=\sigma\big) \leq \theta, \quad \text{for each } z\in Z.$$

Invoking Markov's inequality gives the following for some $\lambda > 0$:

$$\Pr\left\{\underbrace{P'_{\mathrm{e},\sigma} \geq (|Z|+1+\lambda)\epsilon}_{:=V_0(\lambda)}\right\} \leq \frac{1}{|Z|+1+\lambda},$$

$$\Pr\left\{\underbrace{\frac{1}{n}I\big(X_{A_z}; X_{B_z}|\Sigma=\sigma\big) \geq (|Z|+1+\lambda)\theta}_{:=V_z(\lambda)}\right\} \leq \frac{1}{|Z|+1+\lambda},$$

for each $z\in Z$.

Using the union bound, we get

$$\Pr\left\{\bigcup_{z\in Z\cup\{0\}} V_z(\lambda)\right\} \leq \frac{|Z|+1}{|Z|+1+\lambda}, \tag{30}$$

or equivalently,

$$\Pr\left[\bigcap_{z\in Z\cup\{0\}} V_z^{\mathrm{c}}(\lambda)\right] \geq 1 - \frac{|Z|+1}{|Z|+1+\lambda} > 0. \tag{31}$$

The alphabet of Σ is $[d]$, which is finite. And for each $\lambda > 0$, there exists σ such that $\bigcap_{z\in Z\cup\{0\}} V_z(\lambda)^{\mathrm{c}}$ holds. Consequently, there must exist σ for which $\bigcap_{z\in Z\cup\{0\}} V_z(0)^{\mathrm{c}}$ holds, that is, there exists a network code (using one particular σ) for which

$$P'_{\mathrm{e}} < (|Z|+1)\epsilon, \tag{32}$$

$$\frac{1}{n}I\big(X_{A_z}; X_{B_z}\big) < (|Z|+1)\theta = \theta_2, \tag{33}$$

for all $z\in Z$. This proves Part 2 of Theorem 2. ■

X. Conclusion

We have established an equivalence between secure network coding and secure index coding. The equivalence includes mapping nodes from one instance to the other, as well as translating codes of the same rate between these instances.

While the equivalence has been established for networks with error-free links, whether a similar equivalence holds for networks with noisy channels is yet to be determined—even for the non-secure setting.

Also, only a specific notion of security has been investigated in this article, where each eavesdropper must not gain any information about a set of messages. Possible future research in this direction includes other security measures like covert communications and data privacy.

Appendix A
Proof of the Decoding Criterion for the Forward Direction of Theorem 1

Note that in the network-coding instance $\mathbb{N}$, only receivers $\{t_i : i \in [\ell]\}$ need to decode messages, and each t_i receives $X_{\mathrm{in}(t_i)} = (X_{\widehat{H}_i}, X_{2\to t_i})$ over its incoming links, where $X_{2\to t_i} = X_{1\to 2} = \widehat{\mathbf{e}}(X_S, Y_1)$. These are the same functions that each receiver $i \in \widehat{T}$ receives in the index-coding instance $\mathbb{I}$. Using the same decoding functions for receivers $\{t_i : i \in \widehat{T}\}$ in $\mathbb{N}$, if $\widehat{P}_{\mathrm{e}} \leq \epsilon$ for $\mathbb{I}$, we also must have $P_{\mathrm{e}} \leq \epsilon$ for $\mathbb{N}$.

Appendix B
Proof of the Decoding Criterion for the Backward Direction of Theorem 1

In $\mathbb{N}$, receiver t_i, for each $i \in [\ell]$, tries to decode $X_{\widehat{W}_i}$ using the decoding function $\mathbf{d}_{t_i}(X_{\mathrm{in}(t_i)}) = \mathbf{d}_{t_i}(X_{2\to t_i}, (X_{s_j\to t_i} : j\in\widehat{H}_i))$, where

- $X_{2\to t_i} = X_{1\to 2} = \mathbf{e}_{1\to 2}((\mathbf{e}_{s_j\to 1}(X_j) : j \in [k]), Y_1)$,
- $X_{s_j\to t_i} = \mathbf{e}_{s_j\to t_i}(X_j)$ for $j \in \widehat{H}_i$.

The premise states that $P_{\mathrm{e}} \leq \epsilon$ for $\mathbb{N}$.

In $\mathbb{I}$, according to the code translation, receiver i, for each $i \in [\ell]$, tries to decode $\widehat{X}_{\widehat{W}_i}$ using $\widehat{\mathbf{d}}_i(\widehat{X}_{\mathrm{b}}, \widehat{X}_{\widehat{H}_i})$, where

- $\widehat{X}_{\mathrm{b}} = \mathbf{e}_{1\to 2}((\mathbf{e}_{s_j\to 1}(\widehat{X}_j) : j \in [k]), \widehat{Y})$,
- $\widehat{X}_{\widehat{H}_i} = (\mathbf{e}_{s_j\to t_i}(\widehat{X}_j) : j \in \widehat{H}_i)$

Since the decoding functions in $\mathbb{I}$ exactly match those in $\mathbb{N}$, and since $(\widehat{X}_{[k]}, \widehat{Y})$, and $(X_{[k]}, Y_1)$ have the same distribution, we must have $\widehat{P}_{\mathrm{e}} \leq \epsilon$ for $\mathbb{I}$.

APPENDIX C
PROOF OF THE DECODING CRITERION FOR THE FORWARD DIRECTION OF THEOREM 2

In $\mathbb{N}$, by definition, with probability of at least $(1-\epsilon)$ (over the message realizations $x_{\mathcal{S}}$), every vertex $v \in U$ can decode all messages that it requires from the message on all incoming links and messages originating at v. Recall that only messages $X'_{[k]}$ of all messages $X'_{[k+\ell]}$ in $\mathbb{N}'$ need to be decoded. Suppose that, every node $v \in U$ can decode its required messages correctly with probability of at least $(1-\epsilon_v)$, that is,

$$\Pr\left\{X'_{D'^{-1}(v)} = \mathsf{d}'_v\left(X'_{\mathtt{in}(v)}, X'_{O'^{-1}(v)}\right)\right\} \geq 1-\epsilon_v, \tag{34}$$

or equivalently,

$$\Pr\left\{X'_{D'^{-1}(v)} = \mathsf{d}'_v\left(\left[\mathsf{g}'_e\left(X'_{[k+\ell]}\right)\right]_{e\in\mathtt{in}(v)}, X'_{O'^{-1}(v)}\right)\right\} \geq 1-\epsilon_v. \tag{35}$$

For $\mathbb{I}$, we first consider receivers $\hat{t}_i \in \widehat{T}$ where $i \in U$. While source messages $X'_{O'^{-1}(v)}$ in $\mathbb{N}'$ and $\hat{X}_{O'^{-1}(v)}$ in $\mathbb{I}$ have the same distribution, link messages $X'_{\mathtt{in}(v)}$ in $\mathbb{N}'$ and $\hat{X}_{\mathtt{in}(v)}$ in $\mathbb{I}$ may not. So, although node $\hat{t}_i \in \hat{T}$ in $\mathbb{I}$ has side information $(\hat{X}_{\mathtt{in}(i)}, \hat{X}'_{O'^{-1}(i)})$, using (34) in $\mathbb{I}$ will not work, as the pmf of $X'_{[k+\ell]\cup E}$ is different from that of $\hat{X}_{[k+\ell]\cup E}$.

To deal with this issue, we use (35), which requires $[\mathsf{g}'_e(\hat{X}_{[k+\ell]})]_{e\in\mathtt{in}(i)}$ instead. This will work because $(X'_{[k+\ell]}, [\mathsf{g}'_e(X'_{[k+\ell]})]_{e\in E})$ has the same distribution as $(\hat{X}_{[k+\ell]}, [\mathsf{g}'_e(\hat{X}_{[k+\ell]})]_{e\in E})$.

In $\mathbb{I}$, $\widehat{H}_{\hat{t}_i} = \mathtt{in}(i) \cup O'^{-1}(i)$. Receiver $\hat{t}_i$ knows $\hat{X}_{O'^{-1}(i)}$ and calculates $[\mathsf{g}'_e(\hat{X}_{[k+\ell]})]_{e\in\mathtt{in}(i)}$ from the broadcast message $\hat{X}_{\mathrm{b}}$ and side information $\hat{X}_{\mathtt{in}(i)}$ using (10). So, using (35) with a change of variables (from non-hatted to hatted), we have

$$\begin{aligned}
&\Pr\left\{\hat{X}_{\widehat{W}_{\hat{t}_i}} = \mathsf{d}'_i\left(\left[\mathsf{g}'_e\left(\hat{X}_{[k+\ell]}\right)\right]_{e\in\mathtt{in}(i))}, \hat{X}_{O'^{-1}(i)}\right)\right\} \\
&= \Pr\left\{\hat{X}_{D'^{-1}(i)} = \mathsf{d}'_i\left(\left[\mathsf{g}'_e\left(\hat{X}_{[k+\ell]}\right)\right]_{e\in\mathtt{in}(i))}, \hat{X}_{O'^{-1}(i)}\right)\right\} \\
&\geq 1-\epsilon_i.
\end{aligned}$$

This means each receiver $\hat{t}_i \in \widehat{T}$, $i \in U$, can correctly decode its required messages with probability of at least $(1-\epsilon_i)$.

Now, we consider receivers $\hat{t}_e \in \widehat{T}$ where $e \in E$. Recall that $\widehat{H}_{\hat{t}_e} = \mathtt{in}(\mathtt{tail}(e)) \cup O'^{-1}(\mathtt{tail}(e))$, and $\widehat{W}_{\hat{t}_e} = \{e\}$. Receiver $\hat{t}_e$ performs the following steps:

1) Knowing $\{\hat{X}_d : d \in \mathtt{in}(\mathtt{tail}(e))\}$, it obtains $\{\mathsf{g}'_d(\hat{X}_{[k+\ell]}) : d \in \mathtt{in}(\mathtt{tail}(e))\}$ from (10).
2) Knowing $\hat{X}_{O'^{-1}(\mathtt{tail}(e))}$ as side information, it calculates $\mathsf{g}'_e(\hat{X}_{[k+\ell]}) = \mathsf{e}'_e([\mathsf{g}'_d(\hat{X}_{[k+\ell]}) : d \in \mathtt{in}(\mathtt{tail}(e))], \hat{X}_{O'^{-1}(\mathtt{tail}(e))})$.
3) With $\mathsf{g}'_e(\hat{X}_{[k+\ell]})$ and the broadcast message $\hat{X}_{\mathrm{b},e}$, it obtains the required $\hat{X}_e$ using (10).

So, every receiver $\hat{t}_e \in \widehat{T}$, $e \in E$, must be able to correctly decode the required $\hat{X}_e$ without error.

Combining these two classes of receivers, we have shown that all receivers in $\mathbb{I}$ can correctly decode their required messages with probability of at least $(1-\epsilon)$.

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