

STABLE NUMERICAL METHODS FOR A STOCHASTIC NONLINEAR SCHRÖDINGER EQUATION WITH LINEAR MULTIPLICATIVE NOISE

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ABSTRACT. This paper is concerned with fully discrete finite element approximations of a stochastic nonlinear Schrödinger (sNLS) equation with linear multiplicative noise of the Stratonovich type. The goal of studying the sNLS equation is to understand the role played by the noises for a possible delay or prevention of the collapsing and/or blow-up of the solution to the sNLS equation. In the paper we first carry out a detailed analysis of the properties of the solution which lays down a theoretical foundation and guidance for numerical analysis, we then present a family of three-parameters fully discrete finite element methods which differ mainly in their time discretizations and contains many well-known schemes (such as the explicit and implicit Euler schemes and the Crank-Nicolson scheme) with different combinations of time discretization strategies. The prototypical θ -schemes are analyzed in detail and various stability properties are established for its numerical solution. An extensive numerical study and performance comparison are also presented for the proposed fully discrete finite element schemes.

1. Introduction. This paper is concerned with numerical approximations of the following initial-boundary value problem for a stochastic nonlinear Schrödinger (sNLS) equation with linear multiplicative noise of Stratonovich type:

$$idu + \Delta u dt + \lambda |u|^2 u dt = u \circ dW(t) \quad \text{in } \mathcal{D}_T := \mathcal{D} \times (0, T), \quad (1)$$

$$u = 0 \quad \text{on } \partial \mathcal{D} \times (0, T), \quad (2)$$

$$u(0) = u_0 \quad \text{in } \mathcal{D}, \quad (3)$$

where $W(t)$ denotes a standard $\mathbb{R}$ -valued Brownian motion (i.e., Wiener process) on a given filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})$. Moreover, $\mathcal{D} \subset \mathbb{R}^d$ ($d \geq 1$) is a bounded domain, $i := \sqrt{-1}$ denotes the imaginary unit. $\lambda \in \mathbb{R}$ is a

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given parameter. When $\lambda = 0$, equation (1) is linear, and it is called respectively the *defocusing* and *focusing* case when $\lambda > 0$ and $\lambda < 0$. The equation is obviously nonlinear in both cases. $u_0 : \mathcal{D} \times \Omega \rightarrow \mathbb{C}$ is a given complex-valued initial function. Hence $u : \mathcal{D} \times (0, T) \times \Omega \rightarrow \mathbb{C}$ is a complex-valued function. By the Stratonovich to Itô conversion formula, it is easy to verify that (1) has the following equivalent Itô form:

$$idu + \Delta u dt + \lambda |u|^2 u dt + \frac{i}{2} u dt = u dW(t). \quad (4)$$

Equation (1) without the right-hand noise term is known as the (classical) nonlinear (deterministic) Schrödinger (NLS) equation which arises as the governing equation for light propagation in nonlinear optical fibers and planar waveguides and for Bose-Einstein condensates confined to highly anisotropic cigar-shaped traps in the mean-field regime. Moreover, the NLS equation appears in the studies of small-amplitude gravity waves on the surface of deep inviscid (zero-viscosity) water; the Langmuir waves in hot plasmas; the propagation of plane-diffracted wave beams in the focusing regions of the ionosphere; the propagation of Davydov's alpha-helix solitons, which are responsible for energy transport along molecular chains; and many others. Furthermore, the NLS equation appears as one of universal equations that describe the evolution of slowly varying packets of quasi-monochromatic waves in weakly nonlinear media that have dispersion. Mathematically, the NLS equation is a prototypical dispersive wave equation, its solutions exhibit some intriguing properties such as energy conservation, soliton wave, and possible blow-ups [6, 17]. In particular, we mention the following two conserved quantities (in time):

$$\mathcal{M}(u)(t) := \|u(t)\|_{L^2}^2 = \int_{\mathcal{D}} |u(t)|^2 dx, \quad (5)$$

$$\mathcal{H}(u)(t) := \int_{\mathcal{D}} \left(\frac{1}{2} |\nabla u(t)|^2 - \frac{\lambda}{4} |u(t)|^4 \right) dx. \quad (6)$$

In other words, $\mathcal{M}(u)$ and $\mathcal{H}(u)$ are constant functions in t along the solution u of the deterministic NLS problem.

One of the primary motivations for considering the stochastic counterpart of the NLS equation is to examine the possible role and mechanism of noise in preventing or delaying the collapse phenomenon. The special choice of the multiplicative noise is to maintain the above conserved quantities along the solution of problem (1)–(3) either $\mathbb{P}$ -a.s. or in mean, see Lemma 2.2. Besides developing accurate numerical methods, another important goal of numerical approximations is to design efficient numerical methods which are energy-conserved in the sense that numerical solutions also satisfy the same energy conservation properties as the PDE solution does. It turns out that it is quite challenging to construct such desired numerical methods for both deterministic and stochastic problems. As expected, the situation for the latter case is more difficult.

Numerical analysis for the deterministic NLS equation has been carried out by many people, various numerical PDE methods such as finite difference and Galerkin-type methods (including finite element, discontinuous Galerkin and spectral methods) have been developed and analyzed in the past twenty years, we refer the reader to [7, 18, 12, 11] and the references therein for a detailed discussion about the recent developments. Numerical approximations of the stochastic NLS equation has also garnered some attention, several works have been reported in the literature [4, 5, 8, 9, 10, 13, 14]. In order to make the energy $\mathcal{M}(u_h^n)$ is constant in n for

the numerical solution u_h^n , all these works use the Crank-Nicolson time-stepping scheme, moreover, the noise term is approximated *implicitly* by $u_h^{n+1}\Delta W_{n+1}$. Computationally, this is a very expensive and inefficient strategy because the resulting stiffness matrix changes not only at each time step but also varies for each Monte Carlo sampling.

To overcome this difficulty, we propose to approximate the noise term *explicitly* by $u_h^n\Delta W_{n+1}$, which becomes a source term in the equation for u_h^{n+1} so it does not contribute to the stiffness matrix at the time step t_{n+1} , we then combine this noise treatment with the explicit discretization and various implicit discretizations of the drift term to obtain our overall time-stepping schemes. For the spatial discretization, we focus on the finite element discretization although it can be replaced by other Galerkin-type spatial discretization methodologies. It should be noted that the price for the advantage gained by the explicit treatment of the noise term is that the energy $\mathcal{M}$ and $\mathcal{H}$ may not be constant along our numerical solutions, instead, they may only conserve both energies approximately. Another goal of this paper is to examine numerically the impact of this relaxed energy conservation on the quality of the resulted numerical solutions.

The remainder of the paper is organized as follows. In Section 2 we first introduce some notation and then present a detailed analysis about the weak solution of problem (1)–(3) which include the derivations of the conserved quantities, high order stability estimates and Hölder continuity in time in various norms. We note that most of these results have already been reported in the literature, however, our derivations and proofs are somewhat different and are easier to follow. Section 3, which is the main section of this paper, is devoted to the construction and analysis of our semi-discrete (in time) discretizations for problem (1)–(3). We use the θ -time schemes to present a detailed stability analysis for their solutions. In Section 4 we briefly describe the finite element spatial discretization for our semi-discrete (in time) scheme to obtain our fully discrete finite element methods. In Section 5 we present extensive numerical experiment results to illustrate the performance of the proposed numerical methods and also present a performance comparison for the proposed fully discrete finite element schemes using different time-stepping strategies including the explicit and implicit Euler schemes and the Crank-Nicolson scheme. Finally, we conclude the paper with a short summary given in Section 6.

2. Preliminaries and PDE analysis.

2.1. Notation. The standard function and space notations will be adopted in this paper, in particular, $\mathbb{H}^k(\mathcal{D})$ for $k \geq 0$ denotes the complex-valued Sobolev space of order k and $(\cdot, \cdot) := (\cdot, \cdot)_{\mathcal{D}}$ denotes the standard inner product of the complex-valued $\mathbb{L}^2(\mathcal{D})$ space, namely,

$$(u, v) := \int_{\mathcal{D}} u(x) \bar{v}(x) dx,$$

where $\bar{v}$ denotes the complex conjugate of v . For notation brevity, we also set $\|\cdot\|_{H^k} := \|\cdot\|_{H^k(\mathcal{D})}$. Throughout the paper, unless stated otherwise, C, K will be used to denote generic positive constants independent of the space and time mesh sizes h and τ .

2.2. Weak formulation and properties of weak solutions. In this subsection we define a variational weak solution concept for problem (1)–(3), and establish a certain properties of the weak solution u of the stochastic Schrödinger problem,

such as conserved quantities $\mathcal{M}(u)$ and $\mathcal{H}(u)$, a priori estimates of u in various norms and Hölder continuity in time with respect to the $L^2(\mathcal{D})$ -norm.

Definition 2.1. For every $t \in [0, T]$, there holds $\mathbb{P}$ -a.s.

$$\begin{aligned} i(u(t), v) - \int_0^t (\nabla u(s), \nabla v) ds + \lambda \int_0^t (|u(s)|^2 u(s), v) ds + \frac{i}{2} \int_0^t (u(s), v) ds \\ = i(u_0, v) + \int_0^t (u(s), v) dW(s), \quad \forall v \in \mathbb{H}_0^1(\mathcal{D}). \end{aligned} \quad (7)$$

We start with establishing the following conservation results for the weak solution.

Lemma 2.2. Let u be a weak solution to problem (7), then there hold

$$\mathcal{M}(u)(t) = \mathcal{M}(u_0) \quad \forall t \in [0, T], \mathbb{P} - a.s. \quad (8)$$

$$\mathbb{E}[\mathcal{H}(u)^p(s)] = \mathbb{E}[\mathcal{H}(u_0)^p] \quad \forall t \in [0, T], p \geq 1. \quad (9)$$

Proof. Applying Itô's formula to $\Phi(u(t)) := \|u(t)\|_{L^2}^2$ to get

$$\begin{aligned} \|u(t)\|_{L^2}^2 - \|u_0\|_{L^2}^2 = 2\Re \int_0^t (i\Delta u(s) + i\lambda|u(s)|^2 u(s) - \frac{1}{2}u(s), u(s)) ds \\ + \Re \int_0^t (u(s), u(s)) ds - 2\Re \int_0^t (iu(s)dW(s), u(s)) = 0, \end{aligned} \quad (10)$$

which immediately implies $\|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2$. Hence, (8) holds.

To show (9), by direct calculations we get

$$D(\mathcal{H}(u)^p)(v) = p\mathcal{H}(u)^{p-1}(\Re(\nabla u, \nabla v) - \lambda\Re(|u|^2 u, v)), \quad (11)$$

$$\begin{aligned} D^2(\mathcal{H}(u)^p)(v, w) = p\mathcal{H}(u)^{p-1}(\Re(\nabla v, \nabla w) - 2\lambda\Re(|u|^2 v, w) - \lambda\Re(u^2, vw)) \\ + p(p-1)\mathcal{H}(u)^{p-2}(\Re(\nabla u, \nabla v) - \lambda\Re(|u|^2 u, v)) \\ \otimes (\Re(\nabla u, \nabla w) - \lambda\Re(|u|^2 u, w)). \end{aligned} \quad (12)$$

Applying Itô's formula to $\Phi(u(\cdot)) := \mathcal{H}(u)^p(\cdot)$ we have

$$\mathcal{H}(u)^p(t) = \mathcal{H}(u_0)^p + M_t, \quad (13)$$

where M_t is the martingale given by

$$M_t = p \int_0^t \mathcal{H}(u)^{p-1}(s) \Re(\nabla u, -iud\nabla W(s)) ds. \quad (14)$$

Taking expectation on (13), it follows from the martingale property $\mathbb{E}[M_t] = 0$ that

$$\mathbb{E}[\mathcal{H}(u)^p(t)] = \mathbb{E}[\mathcal{H}(u_0)^p]. \quad (15)$$

Hence, (9) holds. The proof is complete. $\square$

Corollary 1. Let u be a weak solution to problem (7) and $p \geq 1$ be an integer. Suppose u_0 satisfies $\mathbb{E}[\mathcal{H}(u_0)^p] < \infty$ and $u_0 = 0$ on $\partial\mathcal{D}$. Then there hold

$$\sup_{t \in [0, T]} \left(\mathbb{E}[\|\nabla u(t)\|_{L^2}^{2p}] - \lambda \mathbb{E}[\|u(t)\|_{L^4}^{4p}] \right) \leq K_1 := 4^p \mathbb{E}[\mathcal{H}(u_0)^p], \quad (16)$$

$$\sup_{t \in [0, T]} \mathbb{E}[\|u(t)\|_{H^1}^{2p}] \leq K_2 := (4^p + 1) \mathbb{E}[\mathcal{H}(u_0)^p] + K_E^3 |\lambda| \mathbb{E}[\mathcal{M}(u_0)^{4p}]. \quad (17)$$

where k_E^3 is the constant in Sobolev embedding $L^2 \hookrightarrow L^4$,

The next lemma establishes an a priori estimate for u in the H^2 -norm.

Lemma 2.3. *Let u be the weak solution to problem (7) with $d = 1$, assume that $u \in L^\infty(0, T; L^q(\Omega, H^2(\mathcal{D})))$ for any $1 < q < \infty$. Then there holds for any integer $p \geq 1$*

$$\sup_{t \in [0, T]} \mathbb{E} \left[\|u(t)\|_{H^2}^{2p} \right] \leq K_3, \quad (18)$$

where

$$\begin{aligned} K_3 &:= K_p (L(u_0)^p + 15K_{E,p}K_2T) e^{26K_{E,p}K_2T} + (K_p(K_E^1)^p + 1)K_2, \\ L(u_0) &:= \|\Delta u_0\|_{L^2}^2 - \Re(\Delta u_0, |u_0|^2 u_0), \\ K_{E,p} &:= (1 + K_E^1 + (K_E^2)^4)p(p-1)|\lambda|, \end{aligned}$$

and k_E^1 is the constant in Sobolev embedding $H^1(\mathcal{D}) \hookrightarrow L^6(\mathcal{D})$, k_E^2 is the constant in Sobolev embedding $H^1 \hookrightarrow L^\infty$, K_p is the constant in inequality $(a+b)^p \leq K_p(a^p + b^p)$,

Proof. To control the nonlinear term $|u|^2 u$ in (1), we introduce the Lyapunov functional:

$$L(v) = \|\Delta v\|_{L^2}^2 + \lambda \Re(\Delta v, |v|^2 v), \quad \forall v \in H^2(\mathcal{D}). \quad (19)$$

By Young's inequality and Sobolev embedding $H^1(\mathcal{D}) \hookrightarrow L^6(\mathcal{D})$, we have

$$\|\Delta v\|_{L^2}^{2p} \leq K_p \left(L(v)^p + (K_E^1)^p |\lambda|^{2p} \|v\|_{H^1}^{6p} \right), \quad (20)$$

where K_p is the constant in inequality $(2a+b)^p \leq K_p(a^p + b^p)$. For example, it's easy to conclude $\|\Delta v\|_{L^2}^2 \leq 2L(v) + \frac{K_E^1}{2} |\lambda|^2 \|v\|_{H^1}^6$ when $p = 1$.

We formally apply Itô's formula to $\Phi(u(\cdot)) := L(u(\cdot))^p$ to get

$$\begin{aligned} L(u(t))^p &= L(u_0)^p + \int_0^t D\Phi(u(s)) \left(i\Delta u dt + i\lambda |u|^2 u dt - \frac{1}{2} u \right) ds \\ &\quad + \frac{1}{2} \int_0^t \text{Tr} \left[D^2\Phi(u(s)) (-iu) (-iu)^* \right] ds \\ &\quad + \int_0^t D\Phi(u(s)) (-iudW(s)) \\ &=: L(u_0)^p + I_1(t) + I_2(t) + I_3(t). \end{aligned} \quad (21)$$

The first and second order derivatives in (21) are given by

$$D\Phi(u)(v) = DL(u(t))^p(v) = pL(u)^{p-1} [DL(u)(v)], \quad (22)$$

$$D^2\Phi(u)(v, w) = p(p-1)L(u)^{p-2} \left([DL(u)(v)] \otimes [DL(u)(w)] \right) \quad (23)$$

$$\begin{aligned} &+ pL(u)^{p-1} \left(2\Re(\Delta v, \Delta w) + 2\lambda \Re(\Delta u, w\Re[\bar{u}v]) + 2\lambda \Re(\Delta w, u\Re[\bar{u}v]) \right. \\ &+ 2\lambda \Re(\Delta u, u\Re[\bar{v}w]) + 2\lambda \Re(\Delta u, v\Re[\bar{u}w]) + \lambda(\Delta w, |u|^2 v) \\ &\left. + 2\lambda \Re(\Delta v, u\Re[\bar{u}w]) + \lambda \Re(\Delta v, |u|^2 w) \right) \end{aligned}$$

for any $v, w \in C_0^\infty$, and $DL(u)(\vartheta) = 2\Re(\Delta u, \Delta \vartheta) + 2\lambda \Re(\Delta u, u\Re[\bar{u}\vartheta]) + \lambda \Re(\Delta u, |u|^2 \vartheta) + \lambda(\Delta \vartheta, |u|^2 u)$.

For notation brevity we use L^p to denote $L(u(s))^p$. Substituting the expressions (22) and (23) of $D\Phi(u)$ and $D^2\Phi(u)$ into $I_1(t)$ and $I_2(t)$, respectively, and performing integration by parts, we get the following equality

$$\begin{aligned}
I_1(t) &= p(p-1) \int_0^t L^{p-1} \left[\lambda \Im(\Delta u(s), \Delta(|u(s)|^2 u(s))) \right. \\
&\quad + \lambda \Im(\Delta u(s), u(s)^2 \Delta \bar{u}(s)) - \|\Delta u(s)\|_{L^2}^2 \\
&\quad \left. + \lambda \Im(\Delta u(s), |u(s)|^4 u(s)) - \lambda \Re(\Delta u(s), |u(s)|^2 u(s)) \right] ds \\
&= p(p-1) \lambda \int_0^t L^{p-1} 4 \Im(\Delta u(s), u(s) |\nabla u(s)|^2) \\
&\quad + 2 \Im(\Delta u(s), \bar{u}(s) (\nabla u(s))^2) ds - p(p-1) \int_0^t L^{p-1} \|\Delta u(s)\|_{L^2}^2 ds \\
&\quad + p(p-1) \lambda \int_0^t L^{p-1} \left(\Im(\Delta u(s), |u(s)|^4 u(s)) \right. \\
&\quad \left. - \Re(\Delta u(s), |u(s)|^2 u(s)) \right) ds \\
&=: I_{11}(t) + I_{12}(t) + I_{13}(t),
\end{aligned} \tag{24}$$

where we use the identities $\Delta(|u|^2 u) = 2|u|^2 \Delta u + 4u |\nabla u|^2 + 2\bar{u} (\nabla u)^2 + u^2 \Delta \bar{u}$ and the facts that $\Re(\Delta u, \Delta(i\Delta u)) = 0$, $\Re(\Delta u, i|u|^2 \Delta u) = 0$, $\Re(\Delta(-i|u|^2 u), i|u|^2 u) = 0$, and $\Re[i\bar{u}|u|^2 u] = 0$.

Taking expectation on both sides of (24) and using the Sobolev embedding $H^1 \hookrightarrow L^\infty$, Gagliardo-Nirenberg inequality $\|f\|_{L^\infty}^2 \leq 2\|f\|_{L^2} \|\nabla f\|_{L^2}$, Young's inequality and equation (20), we get

$$\begin{aligned}
I_{11}(t) &\leq 6p(p-1) |\lambda| \int_0^t L^{p-1} \left(\|\Delta u\|_{L^2} \|u\|_{L^\infty} \|\nabla u\|_{L^\infty} \|\nabla u\|_{L^2} \right) ds \\
&\leq 6p(p-1) |\lambda| \int_0^t L^{p-1} \left(\|\Delta u\|_{L^2}^{\frac{3}{2}} \cdot \sqrt{2} K_E^2 \|\nabla u\|_{L^2}^{\frac{5}{2}} \right) ds \\
&\leq \frac{9}{2} p(p-1) |\lambda| \int_0^t L^{p-1} \|\Delta u\|_{L^2}^2 ds \\
&\quad + 6(K_E^2)^4 p(p-1) |\lambda| \int_0^t L^{p-1} \|u\|_{H^1}^{10} ds \\
&\leq \left(9p(p-1) + \left(\frac{9}{2} K_E^1 + 6(K_E^2)^4 \right) (p-1)^2 \right) |\lambda| \int_0^t L^p ds \\
&\quad + 6(K_E^2)^4 (p-1) |\lambda| \int_0^t \|u\|_{H^1}^{10p} ds + \frac{9}{2} K_E^1 (p-1) |\lambda| \int_0^t \|u\|_{H^1}^{6p} ds.
\end{aligned} \tag{25}$$

$$\begin{aligned}
\mathbb{E}[I_{11}(t)] &\leq 9(1 + K_E^1 + (K_E^2)^4) p(p-1) |\lambda| \int_0^t \mathbb{E}[L(u(s))^p] ds \\
&\quad + 6(K_E^2)^4 (p-1) |\lambda| \int_0^t \mathbb{E}[\|u\|_{H^1}^{10p}] ds \\
&\quad + \frac{9}{2} K_E^1 (p-1) |\lambda| \int_0^t \mathbb{E}[\|u\|_{H^1}^{6p}] ds.
\end{aligned} \tag{26}$$

By Young's inequality and equation (20), we can bound $I_{12}(t)$ as follows:

$$\mathbb{E}[I_{12}(t)] \leq 2(1 + K_E^1) p^2 \int_0^t \mathbb{E}[L(u(s))^p] ds + K_E^1 p \int_0^t \mathbb{E}[\|u\|_{H^1}^{6p}] ds. \quad (27)$$

Similar to the estimation of $I_{11}(t)$, we can control $I_{13}(t)$ by

$$I_{13}(t) \leq 6p(p-1)|\lambda| \int_0^t L(u(s))^{p-1} \left(\|\Delta u\|_{L^2} \|u\|_{L^{10}}^5 + \|\Delta u\|_{L^2} \|u\|_{L^6}^3 \right) ds \quad (28)$$

$$\begin{aligned} &\leq (12p(p-1) + 6K_E^1(p-1)^2)|\lambda| \int_0^t L(u(s))^p ds \\ &\quad + 3K_E^1(p-1)|\lambda| \int_0^t \|u\|_{H^1}^{10p} ds + 9K_E^1(p-1)|\lambda| \int_0^t \|u\|_{H^1}^{6p} ds. \end{aligned}$$

$$\mathbb{E}[I_{13}(t)] \leq 12(1 + K_E^1)p(p-1)|\lambda| \int_0^t \mathbb{E}[L(u(s))^p] ds \quad (29)$$

$$+ 3K_E^1(p-1)|\lambda| \int_0^t \mathbb{E}[\|u\|_{H^1}^{10p}] ds + 9K_E^1(p-1)|\lambda| \int_0^t \mathbb{E}[\|u\|_{H^1}^{6p}] ds.$$

Now we turn to the term $I_2(t)$ in (21). Using Cauchy-Schwarz inequality we get

$$I_2(t) = \frac{p}{2} \int_0^t L(u(s))^{p-1} \left(2\|\Delta u(s)\|_{L^2}^2 + 3\lambda \Re(\Delta u(s), |u(s)|^2 u(s)) \right) ds \quad (30)$$

$$\begin{aligned} &\leq \frac{p}{4} |\lambda| \int_0^t L(u(s))^{p-1} \left(\|\Delta u(s)\|_{L^2}^2 + K_E^1 \|u(s)\|_{H^1}^6 \right) ds \\ &\quad + \frac{p}{2} |\lambda| \int_0^t L(u(s))^p ds, \end{aligned}$$

$$\mathbb{E}[I_{12}(t)] \leq (1 + K_E^1)p(p-1)|\lambda| \int_0^t \mathbb{E}[L(u(s))^p] ds + \frac{|\lambda|}{2} K_E^1 \int_0^t \mathbb{E}[\|u\|_{H^1}^{6p}] ds. \quad (31)$$

On the other hand, by the property of Itô integral, we have $\mathbb{E}[I_3(t)] = 0$. Combining above estimates, we get the following inequality:

$$\begin{aligned} \mathbb{E}[L(u(t))^p] &\leq L(u_0)^p + 26K_{E,p} \int_0^t \mathbb{E}[L(u(s))^p] ds \\ &\quad + 15K_{E,p} \left(\int_0^t \mathbb{E}[\|u\|_{H^1}^{10p}] ds + \int_0^t \mathbb{E}[\|u\|_{H^1}^{6p}] ds \right) \\ &\leq L(u_0)^p + 26K_{E,p} \int_0^t \mathbb{E}[L(u(s))^p] ds + 15K_{E,p} K_2 T. \end{aligned} \quad (32)$$

where $K_{E,p} = (1 + K_E^1 + (K_E^2)^4)p(p-1)|\lambda|$.

Applying Gronwall's inequality to (32), we obtain

$$\mathbb{E}[L(u(t))^p] \leq (L(u_0)^p + 15K_{E,p} K_2 T) e^{26K_{E,p} K_2 T}. \quad (33)$$

Now recall (20) and Young's equality, we have

$$\begin{aligned} \mathbb{E}[\|u\|_{H^2}^{2p}] &= \mathbb{E}[\|\Delta u\|_{L^2}^{2p}] + \mathbb{E}[\|u\|_{H^1}^{2p}] \\ &\leq K_p \mathbb{E}[L(u)^p] + K_p (K_E^1)^p \mathbb{E}[\|u\|_{H^1}^{6p}] + \mathbb{E}[\|u\|_{H^1}^{2p}] \\ &\leq K_p (L(u_0)^p + 15K_{E,p} K_2 T) e^{26K_{E,p} K_2 T} + (K_p (K_E^1)^p + 1) K_2. \end{aligned} \quad (34)$$

Hence, we obtain the desired estimate (18). The proof is complete. $\square$

Theorem 2.4. *Let u be the weak solution to problem (7) with $d = 1$, then there exists a constant $K_4 \equiv K_4(K_2, K_3, K_E^1, T)$ such that*

$$\mathbb{E} \left[\|u(t) - u(s)\|_{L^2}^{2p} \right] \leq K_4 |t - s|^p \quad \forall 0 \leq s, t \leq T, \quad (35)$$

where $p \geq 1$ is an integer number. k_E^1 is the constant in Sobolev embedding $H^1(\mathcal{D}) \hookrightarrow L^6(\mathcal{D})$, and more especially, $K_4 = ((2 + |\lambda|^2 K_E^1) K_2 + 3(K_2 + K_3 + |\lambda|)|t - s|)e^{2T}$ when $p = 1$.

Proof. For a fix $s \in (0, T]$, by the definition of weak solution we obtain

$$\begin{aligned} i(u(t) - u(s), v) - \int_s^t (\nabla u(\xi), \nabla v) d\xi + \lambda \int_s^t (|u(\xi)|^2 u(\xi), v) d\xi \\ + \frac{i}{2} \int_s^t (u(\xi), v) d\xi = \int_s^t (u(\xi), v) dW(\xi) \quad \forall v \in \mathbb{H}_0^1(\mathcal{D}). \end{aligned} \quad (36)$$

Now applying Itô's formula to $\Phi(u(t)) := \|u(t) - u(s)\|_{L^2}^2$ to get

$$\begin{aligned} \|u(t) - u(s)\|_{L^2}^2 + \int_s^t \|u(\xi)\|_{L^2}^2 d\xi \\ = 2\Im \int_s^t (\nabla u(\xi) - \nabla u(s), \nabla u(\xi)) d\xi + 2\lambda \Im \int_s^t (u(\xi) - u(s), |u(\xi)|^2 u(\xi)) d\xi \\ - \Re \int_s^t (u(\xi) - u(s), u(\xi)) d\xi + 2\Im \int_s^t (u(\xi) - u(s), u(\xi) dW(\xi)) \\ =: I + II + III + IV. \end{aligned} \quad (37)$$

We then bound each item on the right-hand side of (37) separately. To bound I , we use integration by parts and equation (4) to obtain

$$\begin{aligned} I &= 2\Im \int_s^t (\nabla u(\xi) - \nabla u(s), \nabla u(s)) d\xi \\ &= 2\Im \int_s^t \int_{\mathcal{D}} \Delta \bar{u}(s) (u(\xi) - u(s)) d\xi ds \\ &= 2\Im \int_s^t \int_{\mathcal{D}} \Delta \bar{u}(s) \int_s^\xi \left(i\Delta u(r) + i\lambda |u(r)|^2 u(r) - \frac{1}{2} u(r) \right) dr dx d\xi \\ &\quad - 2\Im \int_s^t \int_{\mathcal{D}} \Delta \bar{u}(s) \int_s^\xi iu(r) dW(r) dx d\xi \\ &=: I_1 + I_2. \end{aligned} \quad (38)$$

We use the stability of solution $\{u(t); t \in [0, T]\}$ and iterates $\{u^n; n = 0, 1, \dots, N\}$ to obtain

$$\begin{aligned} I_1 &= 2\Im \int_s^t \int_{\mathcal{D}} \Delta \bar{u}(s) \int_s^\xi \left(i\Delta u(r) + i\lambda |u(r)|^2 u(r) - \frac{1}{2} u(r) \right) dr dx d\xi \\ &\leq \|\Delta u(s)\|_{L^2}^2 |t - s|^2 + \int_s^t \int_s^\xi \|\Delta u(r)\|_{L^2}^2 + |\lambda| \|u(r)\|_{L^6}^6 + \frac{1}{2} \|u(r)\|_{L^2}^2 dr d\xi. \end{aligned} \quad (39)$$

To estimate I_2 , we use integration by parts and Young's inequality to get

$$\begin{aligned} I_2 &= -2\Im \int_s^t \int_{\mathcal{D}} \Delta \bar{u}(s) \int_s^\xi iu(r) dW(r) dx d\xi \\ &= -2\Im \int_s^t \int_{\mathcal{D}} \bar{u}(s) \Delta \left(\int_s^\xi iu(r) dW(r) \right) dx d\xi \\ &\leq \|u(s)\|_{L^2}^2 |t-s| + \int_s^t \left\| \int_s^\xi u(r) dW(r) \right\|_{H^2}^2 d\xi. \end{aligned} \quad (40)$$

By using Young's inequality and Sobolev embedding $H^1(\mathcal{D}) \hookrightarrow L^6(\mathcal{D})$, term II can be bounded as follows:

$$\begin{aligned} II &= 2\lambda \Im \int_s^t (u(\xi) - u(s), |u(\xi)|^2 u(\xi)) d\xi \leq 2|\lambda| \int_s^t \|u(\xi)\|_{L^6}^3 \|u(\xi) - u(s)\|_{L^2} d\xi \\ &\leq |\lambda|^2 \int_s^t K_E^1 \|u(\xi)\|_{H^1}^6 d\xi + \int_s^t \|u(\xi) - u(s)\|_{L^2}^2 d\xi. \end{aligned} \quad (41)$$

To bound III , we have

$$\begin{aligned} III &= -\Re \int_s^t (u(\xi) - u(s), u(\xi)) d\xi \leq \int_s^t \|u(\xi)\|_{L^2} \|u(\xi) - u(s)\|_{L^2} d\xi \\ &\leq \int_s^t \|u(\xi)\|_{L^2}^2 d\xi + \int_s^t \|u(\xi) - u(s)\|_{L^2}^2 d\xi. \end{aligned} \quad (42)$$

Now combining these items together, we have

$$\begin{aligned} \|u(t) - u(s)\|_{L^2}^2 &\leq \|\Delta u(s)\|_{L^2}^2 |t-s|^2 + \int_s^t \int_s^\xi \|\Delta u(r)\|_{L^2}^2 + |\lambda| \|u(r)\|_{L^6}^6 \\ &\quad + \frac{1}{2} \|u(r)\|_{L^2}^2 dr d\xi + \|u(s)\|_{L^2}^2 |t-s| + \int_s^t \|u(\xi)\|_{L^2}^2 d\xi \\ &\quad + |\lambda|^2 \int_s^t K_E^1 \|u(\xi)\|_{H^1}^6 d\xi + \int_s^t \left\| \int_s^\xi u(r) dW(r) \right\|_{H^2}^2 d\xi \\ &\quad + 2 \int_s^t \|u(\xi) - u(s)\|_{L^2}^2 d\xi + IV. \end{aligned} \quad (43)$$

Taking the expectation and noticing that IV is a martingale (hence, $\mathbb{E}[IV] = 0$), we obtain

$$\begin{aligned} \mathbb{E} [\|u(t) - u(s)\|_{L^2}^2] &\leq (2 + |\lambda|^2 K_E^1) K_2 + 3(K_2 + K_3 + |\lambda|) |t-s| \\ &\quad + 2\mathbb{E} \left[\int_s^t \|u(\xi) - u(s)\|_{L^2}^2 d\xi \right]. \end{aligned} \quad (44)$$

Thus, an application of Gronwall's inequality infers

$$\mathbb{E} [\|u(t) - u(s)\|_{L^2}^2] \leq ((2 + |\lambda|^2 K_E^1) K_2 + 3(K_2 + K_3 + |\lambda|) |t-s|) e^{2T} |t-s|. \quad (45)$$

Hence, (35) holds for $p = 1$.

The proof for $p \geq 2$ can be carried out by the inductive method. To prove the assertion for $p = 2$, we multiply (43) by $\|u(t) - u(s)\|_{L^2}^2$, use Young's inequality and

apply expectation to get

$$\begin{aligned} \mathbb{E} [\|u(t) - u(s)\|_{L^2}^4] &\leq K|t - s|^2 \left(\mathbb{E} [\|\Delta u(s)\|_{L^2}^4] + \mathbb{E} [\|u(r)\|_{L^6}^{12}] + \mathbb{E} [\|u(r)\|_{L^2}^4] \right. \\ &\quad \left. + \mathbb{E} [\|u(\xi)\|_{H^1}^{12}] \right) + K \int_s^t \mathbb{E} [\|u(\xi) - u(s)\|_{L^2}^2] d\xi \\ &\quad + K|t - s|^3 \left(\mathbb{E} [\|u(r)\|_{H^2}^4] \right)^2. \end{aligned} \quad (46)$$

In order to verify this inequality, we may restrict ourselves to the stochastic integral in (43), since other terms can be easily estimated by Young's inequality. By the Burkholder-Davis-Gundy inequalities, we obtain

$$\mathbb{E} \left[\left\| \int_s^t \int_s^\xi u(r) dW(r) \right\|_{H^2}^2 \|u(t) - u(s)\|_{L^2}^2 d\xi \right] \quad (47)$$

$$\begin{aligned} &\leq \mathbb{E} \left[\left\| \int_s^t \int_s^\xi u(r) dW(r) \right\|_{H^2}^4 + \frac{1}{4} \|u(t) - u(s)\|_{L^2}^4 d\xi \right] \\ &\leq \frac{1}{4} |t - s| \mathbb{E} [\|u(t) - u(s)\|_{L^2}^4] + |t - s|^3 \left(\mathbb{E} [\|u(r)\|_{H^2}^4] \right)^2, \\ &\mathbb{E} \left[\Im \int_s^t (u(\xi) - u(s), u(\xi) dW(\xi)) \|u(t) - u(s)\|_{L^2}^2 \right] \\ &\leq \int_s^t \mathbb{E} [\|u(\xi) - u(s)\|_{L^2}^2 \|u(\xi)\|_{L^2}^2] d\xi + \frac{1}{4} \mathbb{E} [\|u(t) - u(s)\|_{L^2}^4] \\ &\leq \frac{1}{2} \mathbb{E} [\|u(t) - u(s)\|_{L^2}^4] + |t - s|^2 \mathbb{E} [\|u(\xi)\|_{L^2}^4], \end{aligned} \quad (48)$$

and the leading term in (47) and (48) can be absorbed by the left-hand side of (46). Therefore, we obtain the desired conclusion in the case $p = 2$ via the discrete Gronwall's inequality.

By repeating this procedure, we can show that the result holds for each $p \in \mathbb{N}$. Hence, (35) holds. The proof is complete. $\square$

Remark 1. We note that by using the generalized Hölder inequality, it is easy to show that the estimates of Corollary 1, Lemma 2.3 and Theorem 2.4 hold for all $1 \leq p < \infty$.

3. Semi-discretization in time. In this section we propose a family of three-parameters θ -time discretization schemes for the weak formulation (7) and establish some stability estimates for the semi-discrete solutions.

Let $\tau > 0$ be the time step and $t_n = n\tau, n = 0, 1, \dots, N$, where $t_N = T$ and denote $u^n = u(x, t_n)$. Set $U^0 = u(t_0)$ be a given $H_0^1(\mathcal{D})$ -valued random variable, then our θ -schemes for (7) are defined as seeking $\{\mathcal{F}_{t_n}\}_{0 \leq n \leq N}$ -adapted discrete process $\{U^n \in H_0^1(\mathcal{D}); 0 \leq n \leq N\}$ such that $\mathbb{P}$ -a.s.

$$\begin{aligned} i(U^{n+1}, v) - \tau(\nabla \tilde{U}^{n+\theta_1}, \nabla v) + \lambda\tau \left((\theta_2 |U^{n+1}|^2 + (1 - \theta_2) |U^n|^2) \tilde{U}^{n+\theta_2}, v \right) \\ + \frac{i}{2} \tau (\tilde{U}^{n+\theta_3}, v) = i(U^n, v) + (U^n \Delta W_{n+1}, v) \quad \forall v \in H_0^1(\mathcal{D}), \end{aligned} \quad (49)$$

where $\tilde{U}^{n+\theta_i} = \theta_i U^{n+1} + (1 - \theta_i) U^n, i = 1, 2, 3$ and $\Delta W_{n+1} := W(t_{n+1}) - W(t_n) \sim \mathcal{N}(0, \tau), 0 \leq \theta_i \leq 1$ are given parameters. The commonly used those parameters are listed in Table 1 below.

TABLE 1. The comparison between the θ -scheme and other commonly used numerical schemes ($i = 1, 2, 3$).

1	$\theta_i = 0$	Explicit Euler scheme	3	$\theta_i = 1$	Implicit Euler scheme
2	$\theta_i = \frac{1}{2}$	Crank-Nicolson scheme	4	Others	Some hybrid schemes

It is easy to see that (49) is a weak formulation for the random variable U^{n+1} . Below we shall give a detailed analysis for these θ -schemes in the remaining of this section. We begin with following uniform estimate for the expectation of the p th moment of the discrete Hamiltonian $\mathcal{H}^n$:

$$\mathcal{H}^n := \frac{1}{2} \|\nabla U^n\|_{L^2}^2 - \frac{\lambda}{4} \|U^n\|_{L^4}^4.$$

In order to establish the stability, we need the following assumption on the ranges of the parameters $\theta_i, i = 1, 2, 3$.

Assumption 1. Let $\theta_1 \in [\frac{1}{2} + c, 1]$ with $c \in [c^*, \frac{1}{2}]$, $c^* > 0$. Let $\theta_2 \in [\frac{1}{2}, 1]$ with $\lambda < 0$ or $\theta_2 \in [0, \frac{1}{2}]$ with $\lambda > 0$ or $\theta_2 \in [0, 1]$ with $\lambda = 0$. Let $\theta_3 \in [0, 1]$.

Theorem 3.1. Let $p \geq 1$ be an integer. Fix $T = t_N > 0$ and let $\theta_i (i = 1, 2, 3)$ satisfy Assumption 1. Then there exists an $H_0^1(\mathcal{D})$ -valued $\{\mathcal{F}_{t_n}\}_{0 \leq n \leq N}$ -adapted solution $\{U^n; 0, \dots, N\}$ of scheme (49) such that

$$\max_{1 \leq n \leq N} \mathbb{E}[\mathcal{H}^n] + \frac{2\theta_1 - 1}{4} \mathbb{E} \left[\sum_{l=0}^n \|\nabla U^{l+1} - \nabla U^l\|_{L^2}^2 \right] \leq C_1, \quad (50)$$

$$\max_{1 \leq n \leq N} \mathbb{E}[(\mathcal{H}^n)^{2p}] \leq C_2, \quad p \geq 2, \quad (51)$$

$$\max_{1 \leq n \leq N} \mathbb{E}[\|U^n - U^{n-1}\|_{L^2}^{2p}] \leq C_3 \tau^p, \quad p \geq 1, \quad (52)$$

where $C_1 = \tilde{C} \left(2C_E^1 T + \mathbb{E}[\mathcal{H}^0] \right) e^{6(1+|\lambda|+c^{-1})T}$ with $\tilde{C} = (1 - 3(1 + |\lambda|\tau)^{-1})$, $C_2 \equiv C_2(T, C_E^1, \lambda, c, \mathbb{E}[(\mathcal{H}^0)^{2p}])$, $C_3 \equiv C_3(\lambda, C_2, C_E^1)$ and C_E^1 is the constant in Sobolev embedding $L^4(\mathcal{D}) \hookrightarrow L^2(\mathcal{D})$.

Proof. Setting $v = U^{n+1} - U^n$ in (49) and taking the real part that yield

$$\begin{aligned} & \Re(U^{n+1} - U^n, U^{n+1} - U^n) - \tau \Re(\nabla \tilde{U}^{n+\theta_1}, \nabla U^{n+1} - \nabla U^n) \\ & \quad + \lambda \tau \Re \left((\theta_2 |U^{n+1}|^2 + (1 - \theta_2) |U^n|^2) \tilde{U}^{n+\theta_2}, U^{n+1} - U^n \right) \\ & = \frac{1}{2} \Im \tau (\tilde{U}^{n+\theta_3}, U^{n+1} - U^n) + \Re(U^n \Delta W_{n+1}, U^{n+1} - U^n). \end{aligned} \quad (53)$$

By applying identity $\Re(ia, a) = 0$, $\Re(a(\bar{a} - \bar{b})) = \frac{1}{2}(|a|^2 - |b|^2 + |a - b|^2)$ for $a, b \in \mathbb{C}$ and Young's inequality, the terms on the left-hand side of (53) can be bounded as follows:

$$\begin{aligned} \tau \Re(\nabla \tilde{U}^{n+\theta_1}, \nabla U^{n+1} - \nabla U^n) & = \frac{\tau}{2} \|\nabla U^{n+1}\|_{L^2}^2 - \frac{\tau}{2} \|\nabla U^n\|_{L^2}^2 \\ & \quad + \frac{(2\theta_1 - 1)}{2} \tau \|\nabla U^{n+1} - \nabla U^n\|_{L^2}^2, \end{aligned} \quad (54)$$

$$\begin{aligned}
& -\lambda\tau\Re\left((\theta_2|U^{n+1}|^2 + (1-\theta_2)|U^n|^2)\tilde{U}^{n+\theta_2}, U^{n+1} - U^n\right) \\
& = -\frac{\lambda}{4}\tau\|U^{n+1}\|_{L^4}^4 + \frac{\lambda}{4}\tau\|U^n\|_{L^4}^4 - \frac{(2\theta_2-1)}{4}\lambda\tau\int_{\mathcal{D}}(|U^{n+1}|^2 - |U^n|^2)dx \\
& \quad - \frac{(2\theta_2-1)}{2}\lambda\tau\int_{\mathcal{D}}(\theta_2|U^{n+1}|^2 + (\theta_2-1)|U^n|^2)|U^{n+1} - U^n|^2dx \\
& \geq -\frac{\lambda}{4}\tau\|U^{n+1}\|_{L^4}^4 + \frac{\lambda}{4}\tau\|U^n\|_{L^4}^4.
\end{aligned} \tag{55}$$

Using (54) and (55), equation (53) can be written as follows:

$$\begin{aligned}
& \left(\frac{1}{2}\|\nabla U^{n+1}\|_{L^2}^2 - \frac{1}{4}\lambda\|U^{n+1}\|_{L^4}^4\right) - \left(\frac{1}{2}\|\nabla U^n\|_{L^2}^2 - \frac{1}{4}\lambda\|U^n\|_{L^4}^4\right) \\
& \quad + \frac{(2\theta_1-1)}{2}\|\nabla U^{n+1} - \nabla U^n\|_{L^2}^2 \\
& \leq -\frac{1}{2}\Im\tau(\tilde{U}^{n+\theta_3}, U^{n+1} - U^n) - \Re(U^n\Delta W_{n+1}, U^{n+1} - U^n) \\
& =: II + III.
\end{aligned} \tag{56}$$

By using $\Im(ab) = -\Im(b\bar{a})$ for $a, b \in \mathbb{C}$ and $\tilde{U}^{n+\theta_i} = \frac{1}{2}(U^{n+1} + U^n) + \frac{2\theta_i-1}{2}(U^{n+1} - U^n)$ in (49), the first term on the right-hand side of (56) can be bounded by

$$\begin{aligned}
II & = \frac{1}{2}\Im(U^{n+1} - U^n, \frac{1}{2}(U^{n+1} + U^n) + \frac{2\theta_3-1}{2}(U^{n+1} - U^n)) \\
& = \frac{\tau}{4}\Re(\Delta\tilde{U}^{n+\theta_1}, U^{n+1} + U^n) - \frac{\tau}{8}\Im(\tilde{U}^{n+\theta_3}, U^{n+1} + U^n) \\
& \quad + \frac{\tau}{4}\lambda\Re((\theta_2|U^{n+1}|^2 + (1-\theta_2)|U^n|^2)\tilde{U}^{n+\theta_2}, U^{n+1} + U^n) \\
& \quad - \frac{1}{2}\Re(U^n\Delta W_{n+1}, U^{n+1} + U^n) \\
& =: II_1 + II_2 + II_3 + II_4.
\end{aligned} \tag{57}$$

We now need to bound the right-hand side of (57). For the first term II_1 , it follows from integration by parts, Schwarz inequality, Young's inequality that

$$\begin{aligned}
II_1 & = -\frac{\tau}{8}\Re(\nabla(U^{n+1} + U^n) + (2\theta_1-1)\nabla(U^{n+1} - U^n), \nabla U^{n+1} + \nabla U^n) \\
& \leq \frac{(2\theta_1-1)}{8}\tau\|\nabla U^n\|_{L^2}^2 \leq \frac{\tau}{8}\|\nabla U^n\|_{L^2}^2.
\end{aligned} \tag{58}$$

For the second term II_2 , by Young inequality we have

$$\begin{aligned}
II_2 & = -\frac{\tau}{16}\Im((U^{n+1} + U^n) + (2\theta_3-1)(U^{n+1} - U^n), U^{n+1} + U^n) \\
& \leq \frac{(2\theta_3-1)}{16}\tau(\|\nabla U^{n+1}\|_{L^2}^2 + \|\nabla U^n\|_{L^2}^2) \\
& \leq \frac{\tau}{16}(\|\nabla U^{n+1}\|_{L^2}^2 + \|\nabla U^n\|_{L^2}^2).
\end{aligned} \tag{59}$$

Similar to (55), it follows from Young's inequality that

$$\begin{aligned}
 II_3 &= \frac{\tau}{4} \lambda \Re((\theta_2 |U^{n+1}|^2 + (1 - \theta_2) |U^n|^2) \tilde{U}^{n+\theta_2}, U^{n+1} + U^n) \\
 &= \frac{\lambda}{16} \tau \|U^{n+1}\|_{L^4}^4 - \frac{\lambda}{16} \tau \|U^n\|_{L^4}^4 + \frac{(2\theta_2 - 1)}{16} \lambda \tau \int_{\mathcal{D}} (|U^{n+1}|^2 - |U^n|^2) dx \\
 &\quad + \frac{(2\theta_2 - 1)}{8} \lambda \tau \int_{\mathcal{D}} (\theta_2 |U^{n+1}|^2 + (\theta_2 - 1) |U^n|^2) |U^{n+1} - U^n|^2 dx \\
 &\leq \frac{\lambda}{16} \tau \|U^{n+1}\|_{L^4}^4.
 \end{aligned} \tag{60}$$

By using $\Re(a\bar{b}) = \Re(b\bar{a})$ for $a, b \in \mathbb{C}$, Young's inequality and Sobolev embedding $L^4(\mathcal{D}) \hookrightarrow L^2(\mathcal{D})$, II_4 can be estimated by

$$\begin{aligned}
 II_4 &\leq -\frac{1}{2} \Re(U^n \Delta W_{n+1}, U^{n+1} - U^n) \\
 &\leq \frac{1}{8} \|U^{n+1} - U^n\|_{L^2}^2 + \frac{1}{8} \tau \|U^n\|_{L^4}^4 + \frac{1}{2\tau} C_E^1 \|\Delta W_{n+1}\|_{L^\infty}^4.
 \end{aligned} \tag{61}$$

where C_E^1 is constant from $\|U^n\|_{L^2}^4 \leq C_E^1 \|U^n\|_{L^4}^4$.

Again, by using $\Re(a\bar{b}) = \Re(b\bar{a})$ for $a, b \in \mathbb{C}$ and $\tilde{U}^{n+\theta_i} = \frac{1}{2}(U^{n+1} + U^n) + \frac{2\theta_i - 1}{2}(U^{n+1} - U^n)$ in (49), the stochastic integral III can be estimated by

$$\begin{aligned}
 III &= -\frac{1}{\tau} \Re(U^{n+1} - U^n, U^n \Delta W_{n+1}) \\
 &= \Im(\Delta \tilde{U}^{n+\theta_1}, U^n \Delta W_{n+1}) + \frac{1}{2} \Re(\tilde{U}^{n+\theta_3}, U^n \Delta W_{n+1}) \\
 &\quad + \lambda \Im((\theta_2 |U^{n+1}|^2 + (1 - \theta_2) |U^n|^2) \tilde{U}^{n+\theta_2}, U^n \Delta W_{n+1}) \\
 &=: III_1 + III_2 + III_3.
 \end{aligned} \tag{62}$$

We now estimate the three terms in (62) separately. For the first term III_1 , by integration by parts and Young's inequality we have

$$\begin{aligned}
 III_1 &= -\theta_1 \Im(\nabla U^{n+1} - \nabla U^n, \nabla U^n \Delta W_{n+1}) \\
 &\leq \frac{(2\theta_1 - 1)}{4} \|\nabla U^{n+1} - \nabla U^n\|_{L^2}^2 + \frac{\theta_1^2}{2\theta_1 - 1} \|\nabla U^n\|_{L^2}^2 \|\Delta W_{n+1}\|_{L^\infty}^2 \\
 &\leq \frac{(2\theta_1 - 1)}{4} \|\nabla U^{n+1} - \nabla U^n\|_{L^2}^2 + \frac{1}{c} \|\nabla U^n\|_{L^2}^2 \|\Delta W_{n+1}\|_{L^\infty}^2.
 \end{aligned} \tag{63}$$

It follows from Young's inequality and Sobolev embedding $L^4(\mathcal{D}) \hookrightarrow L^2(\mathcal{D})$ that

$$\begin{aligned}
 III_2 &= \frac{\theta_3}{2} \Re(U^{n+1} - U^n, U^n \Delta W_{n+1}) + \frac{(1 - \theta_3)}{2} \Re(U^n, U^n \Delta W_{n+1}) \\
 &\leq \frac{1}{8} \|U^{n+1} - U^n\|_{L^2}^2 + \frac{1}{8} \tau \|U^n\|_{L^4}^4 + C_E^1 \frac{1}{2\tau} \|\Delta_n W\|_{L^\infty}^4 + M_t^1,
 \end{aligned} \tag{64}$$

where M_t^1 is the martingale given by

$$M_t^1 = \frac{(1 - \theta_3)}{2} \Re(U^n, U^n \Delta W_{n+1}). \tag{65}$$

For term III_3 , by Young inequality we have

$$\begin{aligned} III_3 &= \lambda \theta_2 \Im(\theta_2 |U^{n+1}|^2 U^{n+1} + (1 - \theta_2) |U^n|^2 U^n, U^n \Delta W_{n+1}) \\ &\leq \frac{\theta_2(1 + 2\theta_2)}{4} \tau \|U^{n+1}\|_{L^4}^4 + \frac{\theta_2(3 - 2\theta_2)}{4\tau} \|U^n\|_{L^4}^4 \|\Delta W_{n+1}\|_{L^\infty}^4 \\ &\leq \frac{3}{4} \tau \|U^{n+1}\|_{L^4}^4 + \frac{2}{\tau} \|U^n\|_{L^4}^4 \|\Delta W_{n+1}\|_{L^\infty}^4. \end{aligned} \quad (66)$$

Combining (53)-(66) we get

$$\begin{aligned} \mathcal{H}^{n+1} - \mathcal{H}^n &+ \frac{(2\theta_1 - 1)}{4} \|\nabla U^{n+1} - \nabla U^n\|_{L^2}^2 \\ &\leq \frac{\tau}{16} \|\nabla U^{n+1}\|_{L^2}^2 + \frac{3\tau}{16} \|\nabla U^n\|_{L^2}^2 + \frac{1}{c} \|\nabla U^n\|_{L^2}^2 \|\Delta W_{n+1}\|_{L^\infty}^2 \\ &\quad + \left(\frac{\lambda}{16} + \frac{3}{4}\right) \tau \|U^{n+1}\|_{L^4}^4 + \frac{1}{4} \tau \|U^n\|_{L^4}^4 + \frac{2}{\tau} \|U^n\|_{L^4}^4 \|\Delta W_{n+1}\|_{L^\infty}^4 \\ &\quad + C_E^1 \frac{1}{\tau} \|\Delta W_{n+1}\|_{L^\infty}^4 + \frac{1}{4} \|U^{n+1} - U^n\|_{L^2}^2 + M_t^1. \end{aligned} \quad (67)$$

Next, we need to bound $\|U^{n+1} - U^n\|_{L^2}^2$, which appears in the two estimates (61) and (67). For this purpose, we set $v = U^{n+1} - U^n$ in (49) and take the imaginary part, it follows from Young's inequality that

$$\begin{aligned} \|U^{n+1} - U^n\|_{L^2}^2 &= \tau \Im(\nabla \tilde{U}^{n+\theta_1}, \nabla U^{n+1} - \nabla U^n) \\ &\quad - \frac{1}{2} \Re(\tilde{U}^{n+\theta_3}, U^{n+1} - U^n) - \lambda \tau \Im\left((\theta_2 |U^{n+1}|^2 \right. \\ &\quad \left. + (1 - \theta_2) |U^n|^2) \tilde{U}^{n+\theta_2}, U^{n+1} - U^n\right) \\ &\quad + \Im(U^n \Delta W_{n+1}, U^{n+1} - U^n) \\ &=: IV_1 + IV_2 + IV_3 + IV_4. \end{aligned} \quad (68)$$

By using equality $\tilde{U}^{n+\theta_i} = \frac{1}{2}(U^{n+1} + U^n) + \frac{2\theta_i - 1}{2}(U^{n+1} - U^n)$, Young's inequality and Sobolev embedding $L^4(\mathcal{D}) \hookrightarrow L^2(\mathcal{D})$ and the estimates that

$$\begin{aligned} IV_1 &= \tau \Im\left(\frac{1}{2} \nabla(U^{n+1} + U^n) + \frac{2\theta_1 - 1}{2} \nabla(U^{n+1} - U^n), \nabla U^{n+1} - \nabla U^n\right) \\ &= \tau \Im(\nabla U^{n+1}, \nabla U^n) \leq \frac{\tau}{2} \|\nabla U^{n+1}\|_{L^2}^2 + \frac{\tau}{2} \|\nabla U^n\|_{L^2}^2, \\ IV_2 &= -\frac{1}{2} \Re\left(\frac{1}{2}(U^{n+1} + U^n) + \frac{2\theta_3 - 1}{2}(U^{n+1} - U^n), U^{n+1} - U^n\right) \\ &\leq \frac{\tau}{8} \|U^n\|_{L^4}^4 + \frac{\tau}{8} C_E^1, \\ IV_3 &= \lambda \tau \Im(\theta_2 |U^{n+1}|^2 U^{n+1} + (1 - \theta_2) |U^n|^2 U^n, U^n) \\ &\leq |\lambda| \tau \|U^{n+1}\|_{L^4}^4 + |\lambda| \tau \|U^n\|_{L^4}^4, \\ IV_4 &\leq \frac{1}{2} \|U^{n+1} - U^n\|_{L^2}^2 + \frac{\tau}{4} \|U^n\|_{L^4}^4 + \frac{1}{4\tau} C_E^1 \|\Delta W_{n+1}\|_{L^\infty}^4, \end{aligned}$$

we have

$$\begin{aligned} \|U^{n+1} - U^n\|_{L^2}^2 &\leq \tau (\|\nabla U^{n+1}\|_{L^2}^2 + \|\nabla U^n\|_{L^2}^2) + 2|\lambda| \tau \|U^{n+1}\|_{L^4}^4 \\ &\quad + (2|\lambda| + 1) \tau \|U^n\|_{L^4}^4 + \frac{1}{4} C_E^1 \tau + \frac{1}{2\tau} C_E^1 \|\Delta W_{n+1}\|_{L^\infty}^4. \end{aligned} \quad (69)$$

Combining estimates (67) and (69), we obtain

$$\begin{aligned}
 & \mathcal{H}^{n+1} - \mathcal{H}^n + \frac{(2\theta_1 - 1)}{4} \|\nabla U^{n+1} - \nabla U^n\|_{L^2}^2 \\
 & \leq \frac{5\tau}{16} \|\nabla U^{n+1}\|_{L^2}^2 + \frac{7\tau}{16} \|\nabla U^n\|_{L^2}^2 + \frac{1}{c} \|\nabla U^n\|_{L^2}^2 \|\Delta W_{n+1}\|_{L^\infty}^2 \\
 & \quad + \left(\frac{9}{16}|\lambda| + \frac{3}{4}\right)\tau (\|U^{n+1}\|_{L^4}^4 + \|U^n\|_{L^4}^4) + \frac{2}{\tau} \|U^n\|_{L^4}^4 \|\Delta W_{n+1}\|_{L^\infty}^4 \\
 & \quad + \frac{1}{16} C_E^1 \tau + \frac{9}{8\tau} C_E^1 \|\Delta W_{n+1}\|_{L^\infty}^4 + M_t^1.
 \end{aligned} \tag{70}$$

Summing over the index from $l = 0$ to n and applying expectations on both sides, and using the fact that $E[M_t^1] = 0$, we get

$$\begin{aligned}
 & \mathbb{E}[\mathcal{H}^{n+1}] - \mathbb{E}[\mathcal{H}^0] + \frac{2\theta_1 - 1}{4} \mathbb{E}[\|\nabla U^{n+1} - \nabla U^n\|_{L^2}^2] \\
 & \leq 3(1 + |\lambda|)\tau \sum_{l=0}^n \mathbb{E}[\mathcal{H}^{l+1}] + 3(1 + |\lambda| + c^{-1})\tau \sum_{l=0}^n \mathbb{E}[\mathcal{H}^l] + 2C_E^1 T.
 \end{aligned} \tag{71}$$

The discrete Gronwall's inequality then leads to

$$\mathbb{E}[\mathcal{H}^{n+1}] + \frac{2\theta_1 - 1}{4} \sum_{l=0}^n \mathbb{E}[\|\nabla U^{l+1} - \nabla U^l\|_{L^2}^2] \leq \tilde{C} e^{6(1+|\lambda|+c^{-1})T}. \tag{72}$$

where $\tilde{C} = (1 - 3(1 + |\lambda|)\tau)^{-1} (2C_E^1 T + \mathbb{E}[\mathcal{H}^0])$. Hence, assertion (50) holds.

To show (51), we employ an inductive argument. Multiply (70) by $\mathcal{H}^{n+1}$ and take expectations, using the identity $(a - b)a = \frac{1}{2}(a^2 - b^2 + (a - b)^2)$, where $a, b \in \mathbf{R}$, Young's inequality and the embedding $L^4(\mathcal{D}) \hookrightarrow L^2(\mathcal{D})$, we get

$$\begin{aligned}
 & \frac{1}{2} E[(\mathcal{H}^{n+1})^2] - \frac{1}{2} E[(\mathcal{H}^n)^2] + \frac{1}{4} E[(\mathcal{H}^{n+1} - \mathcal{H}^n)^2] \\
 & \leq (8 + 3|\lambda|)\tau E[(\mathcal{H}^{n+1})^2] + (11 + 3|\lambda| + c^{-1})\tau E[(\mathcal{H}^n)^2] + (C_E^1)(1 + C_E^1)\tau.
 \end{aligned} \tag{73}$$

In order to verify this inequality, we may restrict ourselves to the martingale term M_t^1 defined in (65), since other terms can be easily estimated by Young's inequality. By the independency property of increments of the Wiener process, we obtain

$$\begin{aligned}
 \mathbb{E}[M_t^1 \mathcal{H}^{n+1}] &= \frac{(1 - \theta_3)}{2} \mathbb{E}[(\mathcal{H}^{n+1} - \mathcal{H}^n) \Re \int_{\mathcal{D}} |U^n|^2 \Delta_n W ds] \\
 &\leq \frac{1}{4} \mathbb{E}[(\mathcal{H}^{n+1} - \mathcal{H}^n)^2] + \frac{1}{16} \mathbb{E}[\|U^n\|_{L^4}^4] \tau + \frac{1}{4} C_E^1 \tau
 \end{aligned} \tag{74}$$

and the leading term may be absorbed by the left-hand side of (73). Therefore, we prove (51) for the case $p = 2$ via discrete Gronwall's inequality. By repeating this procedure, we can show (51) holds for each $p \in \mathbb{N}$.

To show (52), taking expectation on (69) yields

$$\begin{aligned}
 \mathbb{E}[\|U^{n+1} - U^n\|_{L^2}^2] &\leq \tau (\mathbb{E}[\|\nabla U^{n+1}\|_{L^2}^2] + \mathbb{E}[\|\nabla U^n\|_{L^2}^2]) \\
 &\quad + (2|\lambda| + 1)\tau (\mathbb{E}[\|U^{n+1}\|_{L^4}^{4p}] + \mathbb{E}[\|U^n\|_{L^4}^{4p}]) + C_E^1 \tau \\
 &\leq ((1 + |\lambda|)C_1 + C_E^1)\tau.
 \end{aligned} \tag{75}$$

Again, we employ the inductive argument for $p \geq 2$. Multiplying (69) by $\|U^{n+1} - U^n\|_{L^2}^2$, we get

$$\begin{aligned} \|U^{n+1} - U^n\|_{L^2}^4 &\leq 8\tau^2(\|\nabla U^{n+1}\|_{L^2}^4 + \|\nabla U^n\|_{L^2}^4) + \frac{1}{\tau^2}(C_E^1)^2\|\Delta W_{n+1}\|_{L^\infty}^8 \\ &\quad + 8(|\lambda| + 1)^2\tau^2(\|U^{n+1}\|_{L^4}^8 + \|U^n\|_{L^4}^8) + \frac{1}{4}(C_E^1)^2\tau^2. \end{aligned} \quad (76)$$

Then taking expectation yields

$$\mathbb{E}[\|U^{n+1} - U^n\|_{L^2}^4] \leq (4^4(|\lambda| + 1)^2C_2 + 2(C_E^1)^2)\tau^2. \quad (77)$$

Hence, (52) holds for $p = 2$. By repeating this procedure, we can show (52) holds for each $p \in \mathbb{N}$. The proof is complete. $\square$

Next, we establish some stability estimates for the expectation of H^1 -norm of iterates $\{U^n; 0 \leq n \leq N\}$.

Theorem 3.2. *Let $p \geq 1$ be an integer. Fix $T = t_N > 0$ and let $\theta_i (i = 1, 2, 3)$ satisfy Assumption 1. Then there exists an $H_0^1(\mathcal{D})$ -valued $\{\mathcal{F}_{t_n}\}_{0 \leq n \leq N}$ -adapted solution $\{U^n; 0, \dots, N\}$ of scheme (49) such that*

$$\max_{1 \leq n \leq N} \mathbb{E}[\|U^n\|_{L^2}^{2p}] \leq C_4, \quad (78)$$

where $C_4 \equiv (T, |\lambda|, E[\|U^0\|_{L^2}^2])$.

Proof. Setting $v = U^{n+1}$ in (49) and taking the imaginary part, using identity $\Re(a(\bar{a} - \bar{b})) = \frac{1}{2}(|a|^2 - |b|^2 + |a - b|^2)$ for $a, b \in \mathbb{C}$, $\Im(U^n \Delta W_{n+1}, U^n) = 0$, Schwarz and Young's inequality, we obtain

$$\begin{aligned} &\frac{1}{2}\|U^{n+1}\|_{L^2}^2 - \frac{1}{2}\|U^n\|_{L^2}^2 + \frac{1}{2}\|U^{n+1} - U^n\|_{L^2}^2 \\ &= \tau \Im(\nabla \tilde{U}^{n+\theta_1}, \nabla U^{n+1}) - \lambda \tau \Im((\theta_2|U^{n+1}|^2 + (1 - \theta_2)|U^n|^2)\tilde{U}^{n+\theta_2}, U^{n+1}) \\ &\quad - \frac{1}{2}\tau \Re(\tilde{U}^{n+\theta_3}, U^{n+1}) + \Im(U^n \Delta W_{n+1}, U^{n+1} - U^n) \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (79)$$

Taking expectation on (79) and using the fact that

$$\begin{aligned} I_1 &= \tau(1 - \theta_1)\Im(\nabla U^n, \nabla U^{n+1}) \leq \frac{\tau}{2}\|\nabla U^{n+1}\|_{L^2}^2 + \frac{\tau}{2}\|\nabla U^n\|_{L^2}^2, \\ I_2 &= -\tau\lambda(1 - \theta_2)\Im(\theta_2|U^{n+1}|^2 U^n + (1 - \theta_2)|U^n|^2 \nabla U^{n+1}, U^{n+1}), \\ &\leq \tau|\lambda|\|U^{n+1}\|_{L^4}^4 + \tau|\lambda|\|U^n\|_{L^4}^4, \\ I_3 &= -\frac{\tau}{2}\theta_3\|U^{n+1}\|_{L^2}^2 - \frac{\tau}{2}(1 - \theta_3)\Re(U^n, U^{n+1}), \\ &\leq \frac{\tau}{4}\|U^{n+1}\|_{L^2}^2 + \frac{\tau}{4}\|U^n\|_{L^2}^2, \\ I_4 &\leq \frac{1}{4}\|U^{n+1} - U^n\|_{L^2}^2 + \|U^n\|_{L^2}^2\|\Delta W_{n+1}\|_{L^\infty}^2, \end{aligned}$$

we get

$$\begin{aligned} &\|U^{n+1}\|_{L^2}^2 - \|U^n\|_{L^2}^2 + \frac{1}{2}\|U^{n+1} - U^n\|_{L^2}^2 \\ &\leq \tau\|\nabla U^{n+1}\|_{L^2}^2 + \tau\|\nabla U^n\|_{L^2}^2 + 2\tau|\lambda|\|U^{n+1}\|_{L^4}^4 + 2\tau|\lambda|\|U^n\|_{L^4}^4 \\ &\quad + \frac{\tau}{2}\|U^{n+1}\|_{L^2}^2 + \frac{\tau}{2}\|U^n\|_{L^2}^2 + 2\|U^n\|_{L^2}^2\|\Delta W_{n+1}\|_{L^\infty}^2. \end{aligned} \quad (80)$$

After summing over the index from $l = 0$ to n and applying expectations on both sides of (80), we obtain

$$\mathbb{E}[\|U^{n+1}\|_{L^2}^2] - \mathbb{E}[\|U^0\|_{L^2}^2] \leq 8(1 + |\lambda|)C_1T + 3\tau \sum_{l=0}^n \mathbb{E}[\|U^l\|_{L^2}^2]. \quad (81)$$

The discrete Gronwall's inequality then leads to

$$\mathbb{E}[\|U^{n+1}\|_{L^2}^2] \leq (8(1 + |\lambda|)C_1T + \mathbb{E}[\|U^0\|_{L^2}^2])e^{3T}. \quad (82)$$

We again employ the inductive argument for $p \geq 2$. To obtain the result for $p = 2$, we multiply (80) by $\|U^{n+1}\|_{L^2}^2$, using the identity $(a - b)a = \frac{1}{2}(a^2 - b^2 + (a - b)^2)$, where $a, b \in \mathbf{R}$, and Young's inequality, we get the desired estimate for $p = 2$. By repeating this procedure, we can show (78) holds for each $p \in \mathbb{N}$. The proof is complete. $\square$

Corollary 2. *Let $p \geq 1$ be an integer and $d = 1$. Fix $T = t_N > 0$ and let $\theta_i (i = 1, 2, 3)$ satisfy Assumption 1. Then there exists an $H_0^1(\mathcal{D})$ -valued $\{\mathcal{F}_{t_n}\}_{0 \leq n \leq N}$ -adapted solution $\{U^n; 0, \dots, N\}$ of scheme (49) such that*

$$\max_{1 \leq n \leq N} \mathbb{E}[\|U^n\|_{H^1}^{2^p}] \leq C_5, \quad (83)$$

where $C_5 = (1 + C_E^1)C_4 + 2^p C_1$.

Remark 2. We note that by using the generalized Hölder inequality, it is easy to show that the estimates of Theorem 3.1 and 3.2, and Corollary 2 also hold when the power 2^p is replaced by any real number $1 \leq q < \infty$.

4. Fully discrete finite element methods. In order to obtain fully discrete numerical methods for problem (1), we need to discretize scheme (49) in the spatial variables. Here we briefly describe finite element methods for this purpose although other Galerkin-type methods can also be used.

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of $\mathcal{D}$. We define the finite element space

$$V_h^r := \{v_h \in H^1(\mathcal{D}) : v_h|_K \in \mathcal{P}_r(K) \quad \forall K \in \mathcal{T}_h\},$$

where $\mathcal{P}_r(K)$ denotes the space of all polynomials of degree not exceeding a given positive integer r on $K \in \mathcal{T}_h$. Then our fully discrete finite element methods for problem (1) are defined by seeking an $\mathcal{F}_{t_n}$ -adapted V_h^r -valued process $\{U_h^n\} (n = 0, 1, \dots, N)$ such that $\mathbb{P}$ -almost surely

$$i(U_h^{n+1}, v_h) - \tau(\nabla \tilde{U}_h^{n+\theta_1}, \nabla v_h) + \lambda\tau\left((\theta_2|U_h^{n+1}|^2 + (1 - \theta_2)|U_h^n|^2)\tilde{U}_h^{n+\theta_2}, v_h\right) \quad (84)$$

$$+ \frac{i}{2}\tau(\tilde{U}_h^{n+\theta_3}, v_h) = i(U_h^n, v_h) + (U_h^n \Delta W_{n+1}, v_h) \quad \forall v_h \in V_h^r,$$

where $\tilde{U}_h^{n+\theta_i} = \theta_i U_h^{n+1} + (1 - \theta_i)U_h^n (i = 1, 2, 3)$ and $\Delta W_{n+1} := W(n+1) - W(n) \sim \mathcal{N}(0, \tau)$. We choose $U_h^0 = P_h u^0$ to complement (84), where $P_h : L^2(D) \rightarrow V_h^r$ denotes the L^2 projection operator defined by

$$(P_h w, v_h) = (w, v_h) \quad \forall v_h \in V_h^r.$$

In the next section we shall provide numerical tests for all combinations of θ_i ($i = 1, 2, 3$) given in Table 1.

5. Numerical experiments. In this section, we present several one-dimensional numerical experiments to test the performance of the proposed fully discrete finite element methods with $r = 1$, i.e., V_h^r is the linear finite element space. For the nonlinear solver, Newton's method is used in all our numerical tests.

By introducing a parameter $\sigma > 0$ which measures the size of the noise, we first consider the following stochastic cubic nonlinear Schrödinger equation:

$$\begin{aligned} i du + \Delta u dt + \lambda |u|^2 u dt &= \sigma u \circ dW(t) && \text{in } (-L, L) \times (0, T], \\ u|_{t=0} &= u_0 && \text{in } (-L, L), \quad \text{with } L = 20. \end{aligned} \quad (85)$$

We choose λ and $u_0 = \text{sech}(x) \exp(2ix)$ so that the exact solution of the deterministic nonlinear Schrödinger problem ($\sigma = 0$) is given by

$$u(x, t) = \text{sech}(x - 4t) \exp(i(2x - 3t)). \quad (86)$$

This example contains a soliton wave and is often used as a benchmark for measuring the effectiveness of numerical methods for the deterministic NLS equation; see [16, 19, 15].

5.1. Mass and energy conservations. We first demonstrate the differences of these numerical schemes by their performances on preserving the mass- and energy-conservation. When $\sigma = 0$, (85) reduces to a deterministic nonlinear Schrödinger equation. It is well known that the deterministic problem admits the following invariant quantities:

$$\mathcal{M}_\lambda(u) = \int_D |u(x)|^2 dx, \quad (87)$$

$$\mathcal{H}_\lambda(u) = \int_D \left(\frac{1}{2} |\nabla u(x)|^2 - \frac{\lambda}{4} |u(x)|^4 \right) dx. \quad (88)$$

In our multiplicative stochastic case, if $W(t)$ is real-valued, $\mathcal{M}_\lambda(u)$ is again conserved. The special form of the discretization of the nonlinear term ensures that in the deterministic case, the discrete mass

$$\mathcal{M}_h(t) = \int_\Omega |U_h(t)|^2 dx, \quad (89)$$

and the discrete energy

$$\mathcal{H}_h(t) = \frac{1}{2} \int_\Omega |\nabla U_h(t)|^2 dx - \frac{\lambda}{4} \int_\Omega |U_h(t)|^4 dx. \quad (90)$$

are exactly conserved. These are important physical properties for “good” numerical methods to preserve and are used as a criteria for developing such numerical methods.

The evolution of mass and energy of the numerical solutions with $\sigma = 0$ is presented in Figure 1 with $\tau = 0.05$ and $h = 0.2$. The evolution of the expectation of the mass $\mathbb{E}[\mathcal{M}_h]$ and energy $\mathbb{E}[\mathcal{H}_h]$ of the numerical solutions with $\sigma = 0.05$ is presented in Figure 2 with $\tau = 0.05$, $h = 0.2$. The classical Monte Carlo method with $M = 500$ realizations is used to compute the expectation. It is shown the effectiveness of the proposed the Crank-Nicolson scheme ($\theta_i = \frac{1}{2}, i = 1, 2, 3$) in preserving mass and energy.

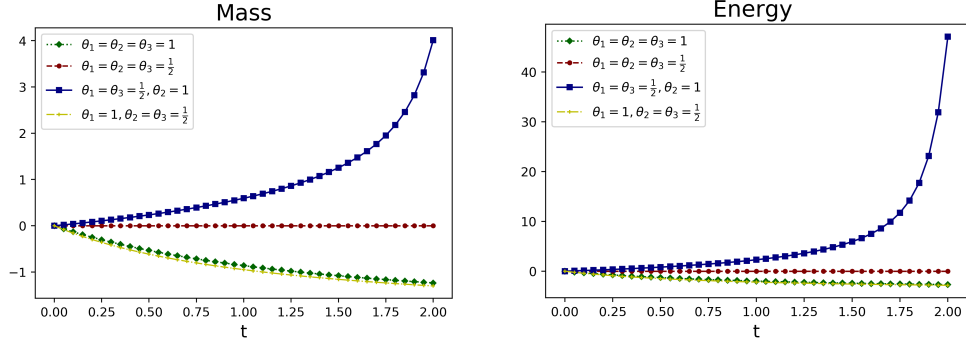


FIGURE 1. Evolution of mass $\mathcal{M}_h(t) - \mathcal{M}_h(0)$ and energy $\mathcal{H}_h(t) - \mathcal{H}_h(0)$, with $\sigma = 0$ and $\tau = 0.05, h = 0.2$.

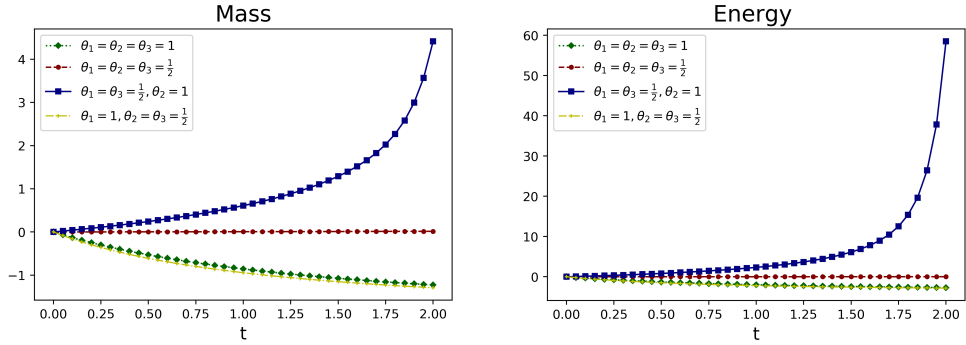


FIGURE 2. Evolution of mass $\mathbb{E}[\mathcal{M}_h(t)] - \mathbb{E}[\mathcal{M}_h(0)]$ and energy $\mathbb{E}[\mathcal{H}_h(t)] - \mathbb{E}[\mathcal{H}_h(0)]$, with $\sigma = 0.05, \tau = 0.05, h = 0.2$ and $M = 500$.

5.2. Preserving the shape of a soliton. In order to verify the validity of different θ schemes, the differences of these numerical schemes are then analyzed by their performances on preserving the shape of a soliton. In the deterministic case ($\sigma = 0$), the graph of $|u(x, t)|$ is a soliton propagating towards right. Its shape remains unchanged for all $t \geq 0$ as shown in Figure 3. The graphs of numerical solutions given by several different numerical methods with $\sigma = 0$ using the same mesh sizes are presented in Figures 4.

The numerical results show the effectiveness of the Crank-Nicolson scheme ($\theta_i = \frac{1}{2}, i = 1, 2, 3$) in preserving the shape of the soliton. We do not include the explicit method which explodes for large time, while the implicit method is heavily damped and eventually drops to zero. For other hybrid schemes, they either decay or increase, but the speed is much slower than Euler methods. It should be noted that the Crank-Nicolson scheme ($\theta_i = \frac{1}{2}, i = 1, 2, 3$) performs best in preserving mass and energy conservations. Then, we choose this scheme to analyze the effects of stochastic noises on the numerical results.

After having confirmed the validity of the Crank-Nicolson scheme ($\theta_i = \frac{1}{2}, i = 1, 2, 3$), we like to understand the effects of noise on the PDE solution. Figure 5 displays the trajectories for different values of $\sigma > 0$. The evolution of the amplitude $|U|$ is also presented. For small noise, in Figure 5(a) and (b), we see that the wave is not strongly perturbed and the noise does not prevent the propagation. On the

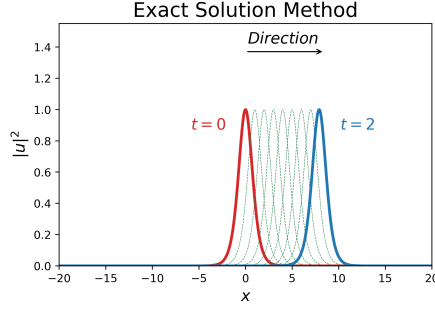


FIGURE 3. Soliton propagation when $t \in [0, 2]$: graph of the exact solution $|u(\cdot, t)|$ with $\sigma = 0$.

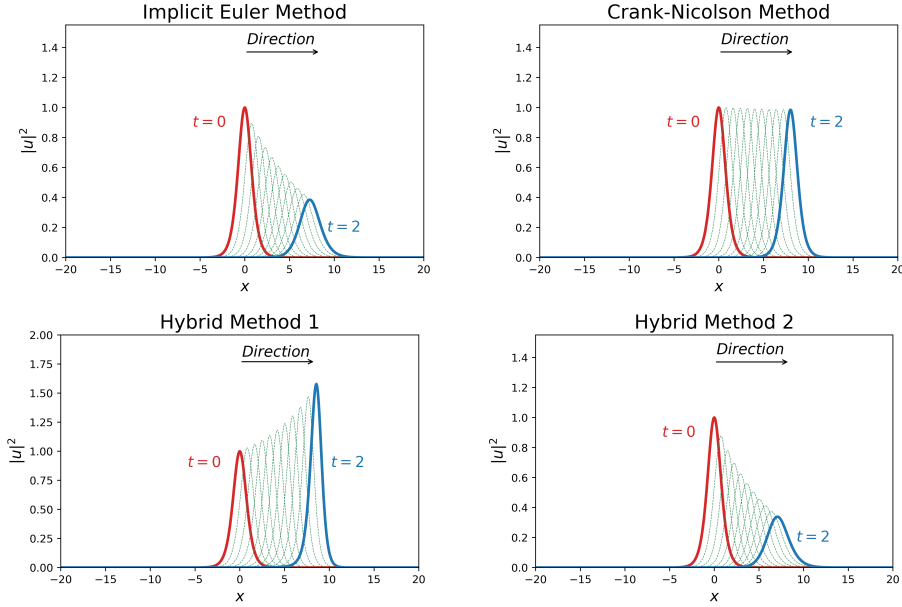


FIGURE 4. Soliton propagation when $t \in [0, 2]$: numerical solutions with $\sigma = 0$, $h = 0.2$ and $\Delta t = 0.025$.

other hand, when the noise level is high, Figure 5(c) shows that the wave profile is destroyed by noise. To have a better understanding of what happens, we use another representation of the solution and plot different level curves of the wave amplitude. The result is shown in Figure 5(d). We observe that the propagation is not stopped although the amplitude is very high. This simulation suggests that the noise has a large influence on the speed of the wave.

5.3. Convergence rates. We solve problem (85) by the proposed θ method and compare the numerical solutions with the reference solution.

We first consider the case $\sigma = 0$. The time discretization errors are presented in Figure 6, where we use a sufficiently small spatial mesh $h = 2L/2000$ so that the error from spatial discretization is negligibly small in observing the temporal

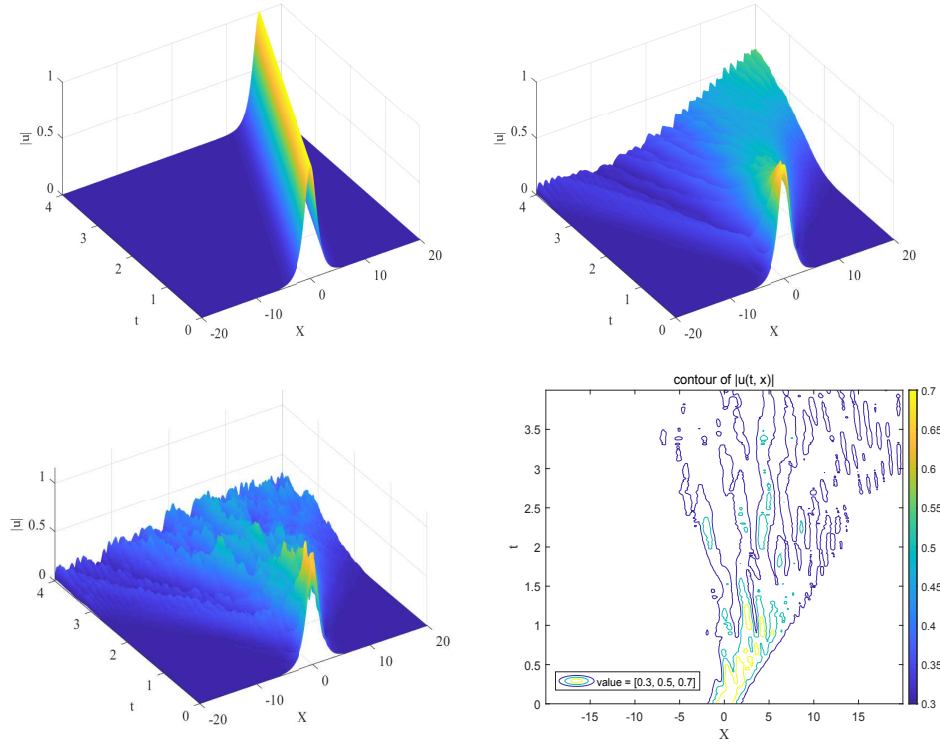


FIGURE 5. Plots in (x, t) plane of $|U|$ for one trajectory: (a) $\sigma = 0.001$, (b) $\sigma = 0.1$, (c) $\sigma = 0.5$, (d) contour plot of $|U|$ for $\sigma = 0.5$ (multiplicative noise).

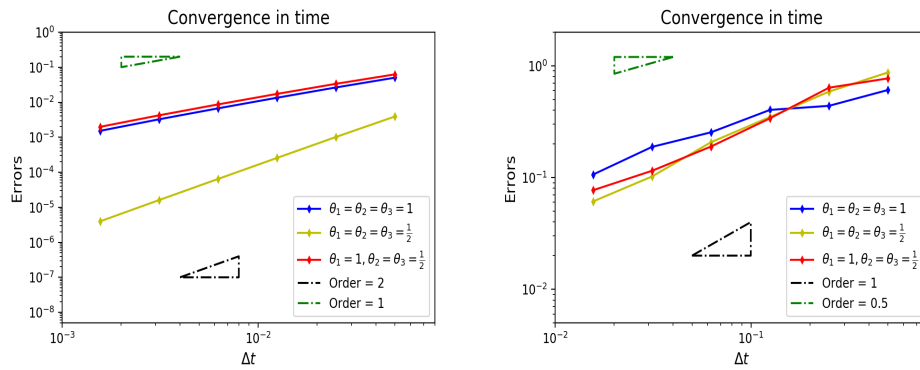


FIGURE 6. Rates of convergence with $\tau \in \{2^{-i}; 1 \leq i \leq 5\}$. left: $\sigma = 0$, $T = 0.1$, right: $\sigma = 0.05$, $T = 0.5$.

convergence rates. Figure 6 shows convergence rate 2 for the L^2 -error of the Crank-Nicolson scheme ($\theta_i = \frac{1}{2}, i = 1, 2, 3$); the order drops to 1 for the implicit Euler scheme ($\theta_i = 1, i = 1, 2, 3$) and the Hybrid scheme 2 with $\theta_1 = 1, \theta_2 = \frac{1}{2}, \theta_3 = \frac{1}{2}$.

The situation is different in the stochastic case when $\sigma = 0.05$ as shown in 6, the order of strong convergence for the above three θ schemes drops from approximately 1 to 0.5.

5.4. Sensitivity to the time step. This test is to analyze the sensitivity of different schemes to the time step τ . For this purpose, we choose $\lambda = -1$ and $\sigma = 0.05$ with the initial condition u_0 as

$$u_0 = \sin(\pi x). \quad (91)$$

We take $h = 2^{-5}$, $\tau = 2^{-i}$ ($i = 6, 7, 8$), $T = 2$ and the computational domain is $[0, 1]$. We note that the initial data results in conservation of L^2 -norm of the numerical solutions in the deterministic case. Hence, it can be used as an index to measure the differences between different numerical schemes. Since the explicit Euler scheme is unstable. Here we only consider the other four numerical schemes, and plot the variation of the solution $\mathbb{E}[\mathcal{M}_h^n]$ in Figure 7 and $\mathbb{E}[\mathcal{K}_h^n]$ in Figure 8 under different time step sizes. As expected, we use the classical Monte Carlo method with $M = 500$ realizations to compute the expectation.

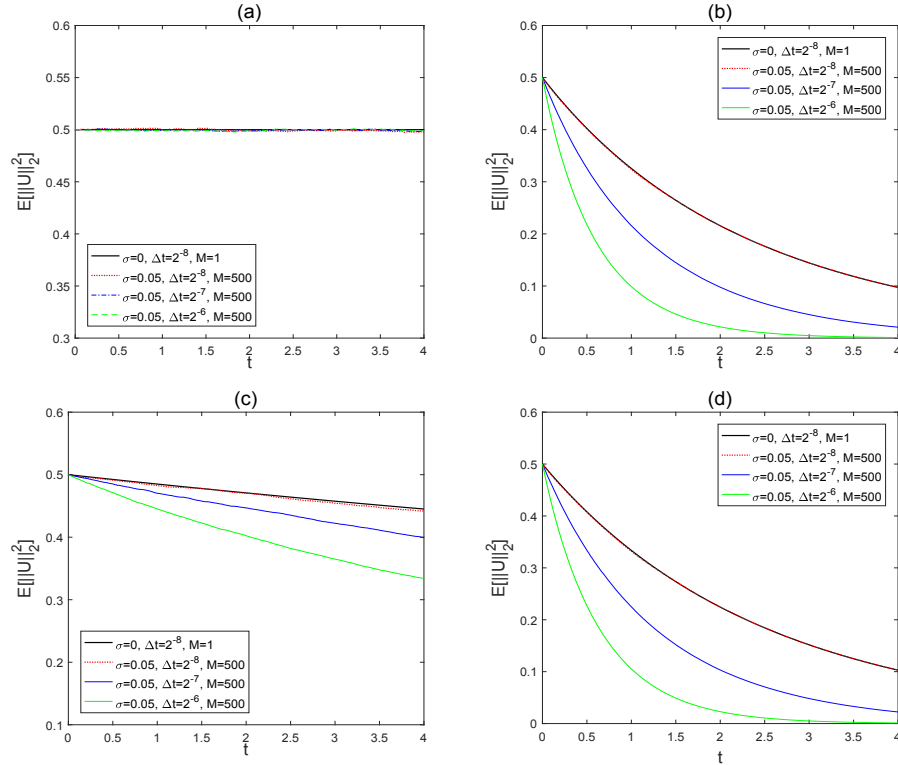


FIGURE 7. The sensitivity of $E[\mathcal{M}_h^n]$ in different subdomains using different time step sizes. (a) Crank-Nicolson scheme : $\theta_i = \frac{1}{2}, i = 1, 2, 3$; (b) Implicit Euler scheme : $\theta_i = 1, i = 1, 2, 3$; (c) Hybrid scheme 1: $\theta_1 = \frac{1}{2}, \theta_2 = 1, \theta_3 = \frac{1}{2}$; (d) Hybrid scheme 2: $\theta_1 = 1, \theta_2 = \frac{1}{2}, \theta_3 = \frac{1}{2}$.

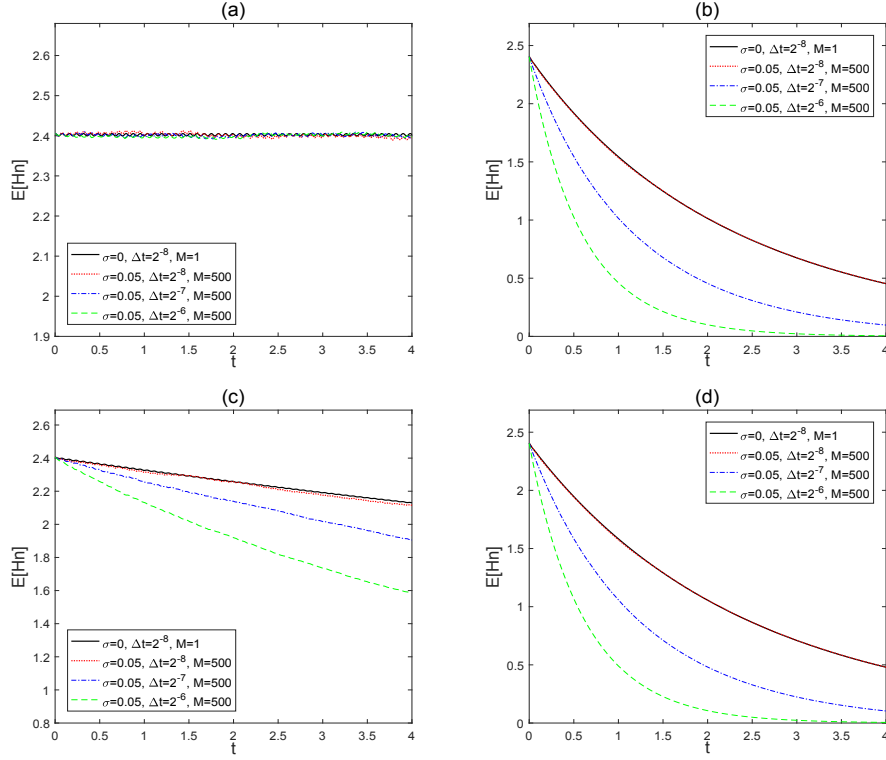


FIGURE 8. The sensitivity of $E[\mathcal{H}_h^n]$ in different subdomains using different time step sizes. (a) Crank-Nicolson scheme : $\theta_i = \frac{1}{2}, i = 1, 2, 3$; (b) Implicit Euler scheme : $\theta_i = 1, i = 1, 2, 3$; (c) Hybrid scheme 1: $\theta_1 = \frac{1}{2}, \theta_2 = 1, \theta_3 = \frac{1}{2}$; (d) Hybrid scheme 2: $\theta_1 = 1, \theta_2 = \frac{1}{2}, \theta_3 = \frac{1}{2}$.

It should be noted that because of the particularity of the chosen exact solutions, most of the numerical schemes are always decaying, except for the explicit Euler scheme ($\theta_i = 0, i = 1, 2, 3$) and the Crank-Nicolson scheme ($\theta_i = \frac{1}{2}, i = 1, 2, 3$). On the other hand, if λ or initial data changes, the hybrid schemes will either be decaying or just exploding, except for the implicit Euler scheme, which is always decaying. For example, let $\lambda = 1$ in this test, the scheme with $\theta_1 = \frac{1}{2}, \theta_2 = 1$ and $\theta_3 = \frac{1}{2}$ presents a rapid increase, but its increase speed is far less than that of the explicit Euler scheme ($\theta_i = 0, i = 1, 2, 3$), as shown in Figure 9.

Based on the properties of the stable solitons, we conclude that the Crank-Nicolson scheme ($\theta_i = \frac{1}{2}, i = 1, 2, 3$) is the only accurate numerical scheme in the family, and other numerical schemes are always decaying or unstable. For those unstable schemes, the only way to control their deterioration is to control the time step size τ . However, the deterioration cannot be avoided even if the time step is very small, because the solution will blow up when the total simulation time T becomes sufficiently large.

6. Conclusion. In this paper we propose a family of fully discrete finite element methods for a stochastic nonlinear Schrödinger (sNLS) equation with multiplicative

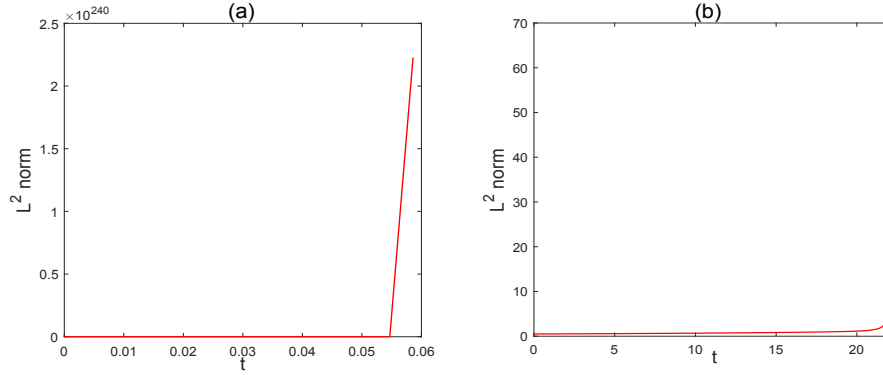


FIGURE 9. The different increasing speeds between the Euler Explicit scheme ($\theta_i = 0, i = 1, 2, 3$) and the Hybrid scheme 1 with ($\theta_1 = \frac{1}{2}, \theta_2 = 1, \theta_3 = \frac{1}{2}$).

noises of the Stratonovich type. We prove various properties of the solution to the sNLS problem such as conserved quantities and Hölder continuity in time with respect to the spatial L^2 -norm. A highlight of this paper is to establish various stability properties for the proposed numerical schemes. Better understanding and more insights about several prototypical schemes (such as the explicit and implicit Euler schemes and the Crank-Nicolson scheme and other hybrid schemes) are also obtained through the stability analysis and numerical simulations. An efficient Monte Carlo Newton nonlinear solver is also designed to solve the resulting finite element systems. Numerical results are provided to present performance comparison between different schemes and to numerically study the influence of noise on the dynamics of the numerical solutions.

It should be noted that not every commonly used scheme is useful for the stochastic nonlinear Schrödinger equation, and the Crank-Nicolson scheme is the only accurate numerical scheme in the family. The explicit Euler scheme is unstable, the implicit Euler scheme is always decaying and most of the other hybrid schemes will either be decaying or exploding, but the speed is far less than that of the explicit Euler scheme. For those unstable schemes (such as the implicit Euler scheme and other hybrid schemes with different θ_i), the only way to control their deterioration is to control the time step size τ . Also, the total simulation time T should be limited before the blow-up occurs because the deterioration cannot be avoided.

Finally, our analysis and simulations show that the multiplicative noise has different levels of influence on the solution dynamics. For small noise (i.e., when $\sigma \ll 1$), the expectation of the stochastic solution behaves the same as its deterministic counterpart, which is expected. On the other hand, for large noise (i.e., when $\sigma \geq 1$), this is no longer the case. Our numerical simulations suggest that the expectation of the stochastic solution could experience some fluctuation, shifting, and rotation for large noise. Also, our simulations show that the noise has a large influence on the speed of the solution wave.

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