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Analysis of Chorin-type projection methods for the stochastic Stokes equations with general multiplicative noise

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Abstract

This paper is concerned with numerical analysis of two fully discrete Chorin-type projection methods for the stochastic Stokes equations with general non-solenoidal multiplicative noise. The first scheme is the standard Chorin scheme and the second one is a modified Chorin scheme which is designed by employing the Helmholtz decomposition on the noise function at each time step to produce a projected divergence-free noise and a "pseudo pressure" after combining the original pressure and the curl-free part of the decomposition. An $O(k^{\frac{1}{4}})$ rate of convergence is proved for the standard Chorin scheme, which is sharp but not optimal due to the use of non-solenoidal noise, where k denotes the time mesh size. On the other hand, an optimal convergence rate $O(k^{\frac{1}{2}})$ is established for the modified Chorin scheme. The fully discrete finite element methods are formulated by discretizing both semi-discrete Chorin schemes in space by the standard finite element method. Suboptimal order error estimates are derived for both fully discrete methods. It is proved that all spatial error constants contain a growth factor $k^{-\frac{1}{2}}$, where k denotes the time step size, which explains the deteriorating performance of the standard Chorin scheme when $k \to 0$ and the space mesh size is fixed as observed earlier in the numerical tests of Carelli et al. (SIAM J Numer Anal 50(6):2917–2939, 2012). Numerical results are also provided to guage the performance of the proposed numerical methods and to validate the sharpness of the theoretical error estimates.

Keywords Stochastic Stokes equations · Multiplicative noise · Wiener process · Itô stochastic integral · Chorin projection scheme · Inf-sup condition · Error estimates.

Mathematics Subject Classification 65N12 · 65N15 · 65N30

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1 Introduction

This paper is concerned with developing and analyzing Chorin-type projection finite element methods for the following time-dependent stochastic Stokes problem:

$$d\mathbf{u} = \big[\nu\Delta\mathbf{u} - \nabla p + \mathbf{f}\big]dt + \mathbf{B}(\mathbf{u})d\mathbf{W}(t) \qquad \text{a.s. in } D_T := (0,T)\times D, \tag{1.1a}$$

$$\operatorname{div}\mathbf{u} = 0 \qquad \text{a.s. in } D_T, \tag{1.1b}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \qquad \text{a.s. in } D, \tag{1.1c}$$

where $D = (0, L)^2 \subset \mathbb{R}^d$ ($d = 2, 3$) represents a period of the periodic domain in $\mathbb{R}^d$, $\mathbf{u}$ and p stand for respectively the velocity field and the pressure of the fluid, $\mathbf{B}$ is an operator-valued random field, $\{\mathbf{W}(t); t \geq 0\}$ denotes an $\mathbf{L}^2(D)$-valued Q-Wiener process, and $\mathbf{f}$ is a body force function (see Sect. 2 for their precise definitions). Here we seek periodic-in-space solutions $(\mathbf{u}, p)$ with period L, that is, $\mathbf{u}(t, \mathbf{x} + L\mathbf{e}_i) = \mathbf{u}(t, \mathbf{x})$ and $p(t, \mathbf{x} + L\mathbf{e}_i) = p(t, \mathbf{x})$ almost surely and for any $(t, \mathbf{x}) \in (0, T) \times \mathbb{R}^d$ and $1 \leq i \leq d$, where $\{\mathbf{e_i}\}_{i=1}^d$ denotes the canonical basis of $\mathbb{R}^d$.

The system (1.1a) is a stochastic perturbation of the deterministic Stokes system by introducing a multiplicative noise force term $\mathbf{B}(\cdot)d\mathbf{W}(s)$ and it has been used to model turbulent fluids (cf. [1,2,18,22]). The stochastic Stokes system is a simplified model of the full stochastic Navier-Stokes equations by omitting the nonlinear term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ in the drift part of the stochastic Navier-Stokes equations. Although the deterministic Stokes equations is a linear PDE system which has been well studied in the literature (cf. [15,22] and the references therein), the stochastic Stokes system (1.1a) is intrinsically nonlinear because the diffusion coefficient $\mathbf{B}$ is nonlinear in the velocity $\mathbf{u}$. Due to the introduction of random forces it has been well known that the solution of problem (1.1) has very low regularities in time. We refer the reader to [1,11,19] and the references therein for a detailed account about the well-posedness and regularities of the solution for system (1.1).

Besides their mathematical and practical importance, the stochastic Stokes (and Navier-Stokes) equations have been used as prototypical stochastic PDEs for developing efficient numerical methods and general numerical analysis techniques for analyzing numerical methods for stochastic PDEs. In that regard several works have been reported in the literature [3,5,8,9,12,13]. Euler-Maruyama time discretization and divergence-free finite element space discretization was proposed and analyzed in [9] in the case of divergence-free noises (i.e., $\mathbf{B}(\mathbf{u})$ is divergence-free). Optimal order error estimates in strong norm for the velocity approximation were obtained. In [12,13] the authors considered the general noise and analyzed the standard and a modified mixed finite element methods as well as pressure stabilized methods for space discretization, suboptimal order error estimates were proved in [12] for the velocity approximation in strong norm and for the pressure approximation in a time-averaged norm, all these suboptimal order error estimates were improved to optimal order for a Helmholtz projection-enhanced mixed finite element in [13] (also see [5] for a similar approach). It should be noted that the reason for measuring the pressure errors in a time-averaged norm is because the low regularity of the pressure field which is only a distribution in general and the numerical tests of [12,13] suggest that these error estimates are sharp.

In [8] the authors proposed a Chorin time-splitting finite element method for problem (1.1) and proved a suboptimal convergence rate in strong norm for the velocity approximation in the case of divergence-free noises. In [3] the authors proposed an iterative splitting scheme for stochastic Navier-Stokes equations and a strong convergence in probability was established in the 2-D case for the velocity approximation. In a recent work [4], the authors proposed another time-splitting scheme and proved its strong L^2 convergence for the velocity approximation.

Compared to the recent advances on mixed finite element methods [9,12,13], the numerical analysis of the well-known Chorin projection/splitting scheme for the stochastic Stokes equations lags behind. To the best of our knowledge, the only analysis result obtained in [8] is the optimal convergence in the energy norm for the velocity approximation in the case of divergence-free noises (i.e., $\mathbf{B}(\mathbf{u})$ is divergence-free). Several natural and important questions arise and must be addressed for a better understanding of the Chorin projection scheme for problem (1.1). Among them are *(i) Does the pressure approximation converge even when the noise is divergence-free? If so, in what sense and what rate? (ii) Does the Chorin projection scheme converge (for both the velocity and pressure approximations) for general noises? If so, in what sense and what rate? (iii) Could the performance of the standard Chorin projection scheme be improved one way or another in the case of general noises?* The primary objective this paper is to provide a positive answer to each of the above questions.

As it was shown in [8], the adaptation of the standard deterministic Chorin projection scheme to problem (1.1) is straightforward (see Algorithm 1 of Sect. 3). The idea of the Chorin scheme is to separate the computation of the velocity and pressure at each time step which is done by solving two decoupled Poisson problems and the divergence-free constraint for the velocity approximation is enforced by a Helmholtz projection technique which can be easily obtained using the solutions of the two Poisson problems. The Chorin scheme also can be compactly rewritten as a pressure stabilization scheme at each time step as follows (cf. [8]):

$$\tilde{\mathbf{u}}^{n+1} - \tilde{\mathbf{u}}^n - k\nu\Delta\tilde{\mathbf{u}}^{n+1} + k\nabla p^n = k\mathbf{f}^{n+1} + \mathbf{B}(\tilde{\mathbf{u}}^n)\Delta W_{n+1} \qquad \text{a.s. in } D_T, \quad (1.2a)$$

$$\operatorname{div}\tilde{\mathbf{u}}^{n+1} - k\Delta p^{n+1} = 0 \qquad \text{a.s. in } D_T, \quad (1.2b)$$

$$\partial_{\mathbf{n}} p^{n+1} = 0 \qquad \text{a.s. on } \partial D_T, \quad (1.2c)$$

where $\partial_{\mathbf{n}} p^{n+1}$ denotes the normal derivative of p^{n+1} and k is the time step size.

One of advantages of the above Chorin scheme is that the spatial approximation spaces for $\tilde{\mathbf{u}}^{n+1}$ and p^{n+1} can be chosen independently, so unlike in the mixed finite element method, they are not required to satisfy an *inf-sup* condition. Notice that a time lag on pressure appears in equation (1.2a) which causes most of difficulties in the convergence analysis (cf. [8,16,20,21]). We also note that the term $-k\Delta p^{n+1}$ in equation (1.2b) is known as a pressure stabilization term.

To improve the convergence of the standard Chorin scheme, we adopt a Helmholtz projection technique as used in [13] (also see [5]). At each time step we first perform the Helmholtz decomposition $\mathbf{B}(\tilde{\mathbf{u}}^n) = \boldsymbol{\eta}^n + \nabla\xi^n$ and then rewrite (1.2a) as

$$\tilde{\mathbf{u}}^{n+1} - \tilde{\mathbf{u}}^{n} - k\nu\Delta\tilde{\mathbf{u}}^{n+1} + k\nabla r^{n} = k\mathbf{f}^{n+1} + \boldsymbol{\eta}^{n}\Delta W_{n+1} \qquad \text{a.s. in } D_T, \tag{1.3}$$

where $r^n = p^n - k^{-1}\xi^n\Delta W_{n+1}$. Our modified Chorin scheme consists of (1.3), (1.2b)–(1.2c) and the Helmholtz decomposition $\mathbf{B}(\tilde{\mathbf{u}}^n) = \boldsymbol{\eta}^n + \nabla\xi^n$. Since $\boldsymbol{\eta}^n$ is divergence-free, it turns out that the finite element approximation of the modified Chorin scheme has better convergence properties. Notice that p^n can be recovered from r^n via the simple algebraic relation $p^n = r^n + k^{-1}\xi^n\Delta W_{n+1}$.

The main contributions of this paper are summarized below.

- We proved the following error estimates in strong norms for the Chorin-P_1 finite element method (see Algorithm 3) for problem (1.1) with general multiplicative noises:

$$\Big(\mathbb{E}\Big[k\sum_{m=0}^{M}\|\mathbf{u}(t_m) - \tilde{\mathbf{u}}_h^m\|^2\Big]\Big)^{\frac{1}{2}} + \max_{0\le\ell\le M}\Big(\mathbb{E}\Big[\Big\|k\sum_{m=0}^{\ell}\nabla(\mathbf{u}(t_m) - \tilde{\mathbf{u}}_h^m)\Big\|^2\Big]\Big)^{\frac{1}{2}} \le C\Big(k^{\frac{1}{4}} + hk^{-\frac{1}{2}}\Big),$$

$$\Big(\mathbb{E}\Big[k\sum_{m=0}^{M}\Big\|P(t_m) - k\sum_{n=0}^{m}p_h^n\Big\|^2\Big]\Big)^{\frac{1}{2}} \le C\Big(k^{\frac{1}{4}} + hk^{-\frac{1}{2}}\Big),$$

  where $(\mathbf{u}(t_m), P(t_m))$ are the solution to problem (1.1) while $(\tilde{\mathbf{u}}_h^m, p_h^m)$ are the discrete solution of Algorithm 3, see Sects. 2 and 4 for their precise definitions.
- We proposed a modified Chorin-P_1 finite element method (see Algorithm 4) and proved the following error estimates in strong norms for problem (1.1) with general multiplicative noises:

$$\max_{1\le m\le M}\Big(\mathbb{E}\big[\big\|\mathbf{u}(t_m) - \tilde{\mathbf{u}}_h^m\big\|^2\big]\Big)^{\frac{1}{2}} + \Big(\mathbb{E}\Big[k\sum_{m=1}^{M}\big\|\nabla(u(t_m), u_h^m)\big\|^2\Big]\Big)^{\frac{1}{2}} \le C\Big(k^{\frac{1}{2}} + h + k^{-\frac{1}{2}}h^2\Big),$$

$$\Big(\mathbb{E}\Big[\Big\|R(t_m) - k\sum_{n=1}^{m}r_h^n\Big\|^2\Big]\Big)^{\frac{1}{2}} + \Big(\mathbb{E}\Big[\Big\|P(t_m) - k\sum_{n=1}^{m}p_h^n\Big\|^2\Big]\Big)^{\frac{1}{2}} \le C\Big(k^{\frac{1}{2}} + h + k^{-\frac{1}{2}}h^2\Big).$$

  where $(\mathbf{u}(t_m), P(t_m))$ is the solution to problem (1.1) and $R(t)$ is defined as the time-average of the pseudo pressure $r(t)$ while $(\mathbf{u}_h^m, r_h^m, p_h^m)$ is the solution of Algorithm 4, see Sects. 2 and 4 for their precise definitions.

We note that all spatial error constants contain a growth factor $k^{-\frac{1}{2}}$, which explains the deteriorating performance of the standard (and modified) Chorin scheme when $k \to 0$ and the mesh size h is fixed as observed in the numerical tests of [8]. The numerical experiments to be given in Sect. 5 indicate that the dependence on factor $k^{-\frac{1}{2}}$ is sharp.

The remainder of this paper is organized as follows. In Sect. 2, we first introduce some space notations and state the assumptions on the initial data and on **B** as well as recall the definition of solutions to (1.1). We then state and prove a Hölder continuity property for the pressure p in a time-averaged norm. In Sect. 3, we define the standard Chorin projection scheme as Algorithm 1 for problem (1.1) in Sect. 3.1 and the modified Chorin scheme as Algorithm 2 in Sect. 3.2. The highlights of this section are to prove some uniform (in k) stability estimates which are very useful for error analysis later. In Sect. 4, we formulate the finite element spatial discretization for both the standard Chorin and modified Chorin schemes in Algorithm 3 and 4, respectively and prove the quasi-optimal error estimates for both algorithms as summarized above. In Sect. 5, we present several numerical experiments to gauge the performance of the proposed numerical methods and to validate the sharpness of the proved error estimates.

This paper is a significantly shortened version of [14] where some omitted proofs and additional remarks and explanations can be found.

2 Preliminaries

Standard function and space notation will be adopted in this paper. Let $\mathbf{H}_0^1(D)$ denote the subspace of $\mathbf{H}^1(D)$ whose $\mathbb{R}^d$-valued functions have zero trace on ∂D, and $(\cdot,\cdot) := (\cdot,\cdot)_D$ denote the standard L^2-inner product, with induced norm $\|\cdot\|$. We also denote $\mathbf{L}^p_{per}(D)$ and $\mathbf{H}^k_{per}(D)$ as the Lebesgue and Sobolev spaces of the functions that are periodic and have vanishing mean, respectively. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a filtered probability space with the probability measure $\mathbb{P}$, the σ-algebra $\mathcal{F}$ and the continuous filtration $\{\mathcal{F}_t\} \subset \mathcal{F}$. For a random variable v defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, $\mathbb{E}[v]$ denotes the expected value of v. For a vector space X with norm $\|\cdot\|_X$, and $1 \leq p < \infty$, we define the Bochner space $\big(L^p(\Omega, X); \|v\|_{L^p(\Omega,X)}\big)$, where $\|v\|_{L^p(\Omega,X)} := \big(\mathbb{E}[\|v\|_X^p]\big)^{\frac{1}{p}}$. We also define

$$\mathbb{H} := \left\{\mathbf{v} \in \mathbf{L}^2_{per}(D);\ \operatorname{div}\mathbf{v} = 0 \text{ in } D\right\},$$
$$\mathbb{V} := \left\{\mathbf{v} \in \mathbf{H}^1_{per}(D);\ \operatorname{div}\mathbf{v} = 0 \text{ in } D\right\}.$$

We recall from [15] that the (orthogonal) Helmholtz projection $\mathbf{P}_{\mathbb{H}} : \mathbf{L}^2_{per}(D) \to \mathbb{H}$ is defined by $\mathbf{P}_{\mathbb{H}}\mathbf{v} = \boldsymbol{\eta}$ for every $\mathbf{v} \in \mathbf{L}^2_{per}(D)$, where $(\boldsymbol{\eta}, \xi) \in \mathbb{H} \times H^1_{per}(D)/\mathbb{R}$ is a unique tuple such that

$$\mathbf{v} = \boldsymbol{\eta} + \nabla\xi\,,$$

and $\xi \in H^1_{per}(D)/\mathbb{R}$ solves the following Poisson problem with the homogeneous Neumann boundary condition:

$$\Delta\xi = \operatorname{div}\mathbf{v}. \tag{2.1}$$

We also define the Stokes operator $\mathbf{A} := -\mathbf{P}_{\mathbb{H}}\Delta : \mathbb{V} \cap \mathbf{H}^2_{per}(D) \to \mathbb{H}$.

Throughout this paper we assume that $\mathbf{B} : \mathbf{L}^2_{per}(D) \to \mathbf{L}^2_{per}(D)$ is a Lipschitz continuous mapping and has linear growth, that is, there exists a constant $C > 0$ such that for all $\mathbf{v}, \mathbf{w} \in \mathbf{L}^2_{per}(D)$

$$\|\mathbf{B}(\mathbf{v}) - \mathbf{B}(\mathbf{w})\| \leq C\|\mathbf{v} - \mathbf{w}\|\,, \tag{2.2a}$$

$$\|\mathbf{B}(\mathbf{v})\| \leq C\big(1 + \|\mathbf{v}\|\big)\,, \tag{2.2b}$$

Since we shall not explicitly track the dependence of all constants on ν, for ease of the presentation, unless it is stated otherwise, we shall set $\nu = 1$ in the rest of the paper and assume that $\mathbf{f} \in L^2(\Omega; L^2_{per}(D))$. In addition, we shall use C to denote a generic positive constant which may depend on T, the datum functions $\mathbf{u}_0$ and $\mathbf{f}$, and the domain D but is independent of the mesh parameters h and k.

2.1 Variational formulation of the stochastic Stokes equations

We first define the variational solution concept for (1.1) and refer the reader to [10,11] for a proof of its existence and uniqueness.

Definition 2.1 Given $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, let W be an $\mathbb{R}$-valued Wiener process on it. Suppose $\mathbf{u}_0 \in L^2(\Omega, \mathbb{V})$ and $\mathbf{f} \in L^2(\Omega; L^2((0,T); L^2_{per}(D)))$. An $\{\mathcal{F}_t\}$-adapted stochastic process $\{\mathbf{u}(t); 0 \leq t \leq T\}$ is called a variational solution of (1.1) if $\mathbf{u} \in L^2\big(\Omega; C([0,T]; \mathbb{V})\big) \cap L^2\big(\Omega; 0, T; \mathbf{H}^2_{per}(D)\big)$, and satisfies $\mathbb{P}$-a.s. for all $t \in (0, T]$

$$\begin{aligned}\big(\mathbf{u}(t), \mathbf{v}\big) + \int_0^t \big(\nabla \mathbf{u}(s), \nabla \mathbf{v}\big)\, ds &= (\mathbf{u}_0, \mathbf{v}) + \int_0^t \big(\mathbf{f}(s), \mathbf{v}\big)\, ds \\ &\quad + \int_0^t \Big(\mathbf{B}\big(\mathbf{u}(s)\big), \mathbf{v}\Big)\, dW(s) \qquad \forall\, \mathbf{v} \in \mathbb{V}\,. \end{aligned} \tag{2.3}$$

We cite the following Hölder continuity estimates for the variational solution whose proofs can be found in [8,12].

Lemma 2.1 *Suppose* $\mathbf{u}_0 \in L^2\big(\Omega; \mathbb{V} \cap \mathbf{H}^2_{per}(D)\big)$ *and* $\mathbf{f} \in L^2(\Omega; C^{\frac{1}{2}}([0,T]); H^1_{per}(D))$. *Then there exists a constant* $C \equiv C(D_T, \mathbf{u}_0, \mathbf{f}) > 0$, *such that the variational solution to problem (1.1) satisfies for* $s, t \in [0, T]$

$$\mathbb{E}\big[\|\mathbf{u}(t) - \mathbf{u}(s)\|^2\big] + \mathbb{E}\Big[\int_s^t \|\nabla\big(\mathbf{u}(\tau) - \mathbf{u}(s)\big)\|^2\, d\tau\Big] \leq C|t - s|\,, \tag{2.4a}$$

$$\mathbb{E}\big[\|\nabla\big(\mathbf{u}(t) - \mathbf{u}(s)\big)\|^2\big] + \mathbb{E}\Big[\int_s^t \|\mathbf{A}\big(\mathbf{u}(\tau) - \mathbf{u}(s)\big)\|^2\, d\tau\Big] \leq C|t - s|\,. \tag{2.4b}$$

Definition 2.1 only defines the velocity $\mathbf{u}$ for (1.1), its associated pressure p is subtle to define. In that regard we quote the following theorem from [13].

Theorem 2.2 *Let* $\{\mathbf{u}(t); 0 \leq t \leq T\}$ *be the variational solution of (1.1). There exists a unique adapted process* $P \in L^2\big(\Omega, L^2(0, T; H^1_{per}(D)/\mathbb{R})\big)$ *such that* $(\mathbf{u}, P)$ *satisfies* $\mathbb{P}$*-a.s. for all* $t \in (0, T]$

$$\big(\mathbf{u}(t), \mathbf{v}\big) + \int_0^t \big(\nabla \mathbf{u}(s), \nabla \mathbf{v}\big)\, ds - \big(div\, \mathbf{v}, P(t)\big) \tag{2.5a}$$

$$= (\mathbf{u}_0, \mathbf{v}) + \int_0^t \big(\mathbf{f}(s), \mathbf{v}\big)\, ds + \int_0^t \big(\mathbf{B}\big(\mathbf{u}(s)\big), \mathbf{v}\big)\, dW(s) \quad \forall\, \mathbf{v} \in \mathbf{H}^1_{per}(D; \mathbb{R}^d)\,,$$

$$\big(div\, \mathbf{u}, q\big) = 0 \qquad \forall\, q \in L^2_0(D) := \{q \in L^2_{per}(D) :\ (q, 1) = 0\}\,. \tag{2.5b}$$

System (2.5) is a mixed formulation for the stochastic Stokes system (1.1), where the (time-averaged) pressure P is defined. The distributional derivative $p := \frac{\partial P}{\partial t}$, which was shown to belong to $L^1\big(\Omega; W^{-1,\infty}(0, T; H^1_{per}(D)/\mathbb{R})\big)$, was defined as the pressure. Below, we also define another time-averaged "pressure"

$$R(t) := P(t) - \int_0^t \xi(s)\, dW(s), \tag{2.6}$$

using the Helmholtz decomposition $\mathbf{B}(\mathbf{u}(t)) = \boldsymbol{\eta}(t) + \nabla \xi(t)$, where $\xi \in H^1_{per}(D)/\mathbb{R}$ $\mathbb{P}$-a.s. such that

$$\big(\nabla \xi(t), \nabla \phi\big) = \big(\mathbf{B}(\mathbf{u}(t)), \nabla \phi\big) \qquad \forall\, \phi \in H^1_{per}(D)\,. \tag{2.7}$$

Then, it is easy to check that $\mathbb{P}$-a.s.

$$\nabla R(t) = -\mathbf{u}(t) + \int_0^t \mathbf{u}(s)\, ds + \mathbf{u}_0 + \int_0^t \mathbf{f}(s)\, ds + \int_0^t \boldsymbol{\eta}(s)\, dW(s) \quad \forall\, t \in (0, T). \tag{2.8}$$

The process $\{R(t); 0 \le t \le T\}$ will also be approximated in our numerical methods.

Next, we establish some stability estimates for the velocity $\mathbf{u}$ and the pressure P in the following lemma, its proof can be found in [14, Lemma 2.4].

Lemma 2.3 *Suppose that $\mathbf{u}_0 \in L^2(\Omega; \mathbb{V})$. Let $(\mathbf{u}, P)$ solve (2.5). Then there exists a constant $C \equiv C(D_T, \mathbf{u}_0, \mathbf{f})$ such that*

$$\mathbb{E}\Big[\sup_{0\le t\le T} \|\nabla \mathbf{u}(t)\|^2\Big] + \mathbb{E}\Big[\int_0^T \|\mathbf{A}\mathbf{u}(s)\|^2\, ds\Big] \le C, \tag{2.9}$$

$$\sup_{0\le t\le T} \mathbb{E}\Big[\|\nabla P(t)\|^2\Big] \le C. \tag{2.10}$$

We finish this section by establishing the following Hölder continuity result for P. We also omit its proof here and refer the reader to [14, Lemma 2.5]

Lemma 2.4 *Suppose that $\mathbf{u} \in L^2(\Omega; C([0, T]; \mathbb{V})) \cap L^2(\Omega; 0, T; \mathbf{H}^2_{per}(D))$, $\mathbf{f} \in L^2(\Omega; 0, T; \mathbf{L}^2_{per}(D))$ and $P \in L^2(\Omega; L^2(0, T; H^1_{per}(D/\mathbb{R})))$. Then, there holds*

$$\mathbb{E}\big[\|\nabla\big(P(s) - P(t)\big)\|^2\big] \le C|s - t| \qquad \forall s, t \in [0, T], \tag{2.11}$$

where the constant $C > 0$ depends on D_T, $\mathbf{u}_0$ and $\mathbf{f}$.

3 Two Chorin-type time-stepping schemes

In this section, we first formulate two Chorin-type semi-discrete-in-time schemes for problem (1.1). The first scheme is the standard Chorin scheme and the second one is a Helmholtz decomposition enhanced nonstandard Chorin scheme. We then present a complete convergence analysis for each scheme which includes establishing their stability and error estimates in strong norms for both velocity and pressure approximations.

3.1 Standard Chorin projection scheme

We first consider the standard Chorin scheme for (1.1), its formulation is a straightforward adaptation of the well-known scheme for the deterministic Stokes problem and is given in Algorithm 1 below. As mentioned earlier, the standard Chorin scheme for (1.1) was already studied in [8] in the special case when the noise is divergence-free and error estimates were only obtained for the velocity approximation. In contrast, here we consider the Chorin scheme for general multiplicative noise and to derive error estimates in strong norms not only for the velocity but also for pressure approximations and to achieve a full understanding about the scheme.

3.1.1 Formulation of the standard Chorin scheme

Let M be a (large) positive integer and $k := T/M$ be the time step size. Set $t_j = jk$ for $j = 0, 1, 2, \cdots, M$, then $\{t_j\}_{j=0}^M$ forms a uniform mesh for $(0, T)$. The standard Chorin projection scheme is given as follows (cf. [8,15,22]):

Algorithm 1 Let $\tilde{\mathbf{u}}^0 = \mathbf{u}^0 = \mathbf{u}_0$. For $n = 0, 1, 2, \cdots, M-1$, do the following three steps.

Step 1: Given $\mathbf{u}^n \in L^2(\Omega; \mathbb{H})$ and $\tilde{\mathbf{u}}^n \in L^2(\Omega; \mathbf{H}^1_{per}(D))$, find $\tilde{\mathbf{u}}^{n+1} \in L^2(\Omega; \mathbf{H}^1_{per}(D))$ such that $\mathbb{P}$-a.s.

$$\tilde{\mathbf{u}}^{n+1} - k\Delta\tilde{\mathbf{u}}^{n+1} = \mathbf{u}^n + k\mathbf{f}^{n+1} + \mathbf{B}(\tilde{\mathbf{u}}^n)\Delta W_{n+1} \qquad \text{in } D. \tag{3.1}$$

Step 2: Find $p^{n+1} \in L^2(\Omega; H^1_{per}(D)/\mathbb{R})$ such that $\mathbb{P}$-a.s.

$$-\Delta p^{n+1} = -\frac{1}{k}\operatorname{div}\tilde{\mathbf{u}}^{n+1} \qquad \text{in } D. \tag{3.2}$$

Step 3: Define $\mathbf{u}^{n+1} \in L^2(\Omega; \mathbb{H})$ by

$$\mathbf{u}^{n+1} = \tilde{\mathbf{u}}^{n+1} - k\nabla p^{n+1}. \tag{3.3}$$

Remark 3.1 (a) The above formulation is written in the way in which the scheme is implemented, it is slightly different from the traditional writing which combines Step 2 and 3 together.

(b) It is easy to check $(\tilde{\mathbf{u}}^{n+1}, \mathbf{u}^{n+1}, p^{n+1})$ satisfies the following system:

$$\tilde{\mathbf{u}}^{n+1} - \tilde{\mathbf{u}}^{n} - k\Delta\tilde{\mathbf{u}}^{n+1} + k\nabla p^{n} = k\mathbf{f}^{n+1} + \mathbf{B}(\tilde{\mathbf{u}}^{n})\Delta W_{n+1} \qquad \text{in } D, \tag{3.4a}$$

$$\operatorname{div}\tilde{\mathbf{u}}^{n+1} - k\Delta p^{n+1} = 0 \qquad \text{in } D, \tag{3.4b}$$

where $\tilde{u}^0 = u_0$.

3.1.2 Stability estimates for the standard Chorin method

The goal of this subsection is to establish some stability estimates for Algorithm 1 in strong norms. These estimates will play an important role in the error estimations for the fully discrete finite element Chorin scheme to be given in the next section.

Lemma 3.1 *The discrete processes $\{(\tilde{\mathbf{u}}^n, p^n)\}_{n=0}^M$ defined in (3.4) satisfy*

$$\max_{0\le n\le M}\mathbb{E}[\|\tilde{\mathbf{u}}^n\|^2] + \mathbb{E}\left[\sum_{n=1}^{M}\|\tilde{\mathbf{u}}^n - \tilde{\mathbf{u}}^{n-1}\|^2\right] + \mathbb{E}\left[k\sum_{n=0}^{M}\|\nabla\tilde{\mathbf{u}}^n\|^2\right] \le C, \tag{3.5a}$$

$$\mathbb{E}\left[k\sum_{n=0}^{M}\|\nabla p^n\|^2\right] \le \frac{C}{k}, \tag{3.5b}$$

$$\max_{0\le n\le M}\mathbb{E}[\|\nabla\tilde{\mathbf{u}}^n\|^2] + \mathbb{E}\left[k\sum_{n=0}^{M}\|\Delta\tilde{\mathbf{u}}^n\|^2\right] \le \frac{C}{k}, \tag{3.5c}$$

where $C > 0$ depends only on $D_T, u_0, \mathbf{f}$.

We refer the reader to [14, Lemma 3.1] for a complete proof of this lemma.

3.1.3 Error estimates for the standard Chorin scheme

In this subsection we shall derive some error estimates for the time-discrete processes generated by Algorithm 1. To the best of our knowledge, these are the first error estimate results for the standard Chorin scheme in the case general multiplicative noises. For the sake of brevity, but without loss of the generality, we set $\mathbf{f} = 0$ in this subsection.

First, we state the following error estimate result for the velocity.

Theorem 3.2 *Let $\{(\tilde{\mathbf{u}}^n, p^n)\}_{n=0}^M$ be generated by Algorithm 1, then there exists a positive constant C which depends on D_T, $\mathbf{u}_0$, and $\mathbf{f}$ such that*

$$\left(\mathbb{E}\left[k\sum_{n=0}^{M}\|\mathbf{u}(t_n) - \tilde{\mathbf{u}}^n\|^2\right]\right)^{\frac{1}{2}} + \max_{0\le \ell\le M}\left(\mathbb{E}\left[\left\|k\sum_{n=0}^{\ell}\nabla(\mathbf{u}(t_n) - \tilde{\mathbf{u}}^n)\right\|^2\right]\right)^{\frac{1}{2}} \le Ck^{\frac{1}{4}}. \tag{3.6}$$

Proof Let $\mathbf{e}_{\tilde{\mathbf{u}}}^m = \mathbf{u}(t_m) - \tilde{\mathbf{u}}^m$ and $\mathcal{E}_p^m = P(t_m) - k\sum_{n=0}^{m} p^n$. Obviously, $\mathbf{e}_{\tilde{\mathbf{u}}}^m \in L^2\big(\Omega; \mathbf{H}_{per}^1(D)\big)$ and $\mathcal{E}_p^m \in L^2(\Omega; H_{per}^1(D)/\mathbb{R})$. In addition, from (2.5a), we have

$$\begin{aligned}&\big(\mathbf{u}(t_{m+1}), \mathbf{v}\big) + \sum_{n=0}^{m}\int_{t_n}^{t_{n+1}}\big(\nabla\mathbf{u}(s), \nabla\mathbf{v}\big)\,ds + \big(\mathbf{v}, \nabla P(t_{m+1})\big) \\ &\quad= \big(\mathbf{u}_0, \mathbf{v}\big) + \Big(\sum_{n=0}^{m}\int_{t_n}^{t_{n+1}}\big(\mathbf{B}(\mathbf{u}(s))\,dW(s), \mathbf{v}\Big) \quad \forall \mathbf{v} \in \mathbf{H}_{per}^1(D)\ a.s.\end{aligned} \tag{3.7}$$

Applying the summation operator $\sum_{n=0}^{m}$ to (3.4a) yields

$$\begin{aligned}&\big(\tilde{\mathbf{u}}^{m+1}, \mathbf{v}\big) + k\sum_{n=0}^{m}\big(\nabla\tilde{\mathbf{u}}^{n+1}, \nabla\mathbf{v}\big) + \Big(k\sum_{n=0}^{m}\nabla p^n, \mathbf{v}\Big) \\ &\quad= \big(\tilde{\mathbf{u}}_0, \mathbf{v}\big) + \Big(\sum_{n=0}^{m}\int_{t_n}^{t_{n+1}}\mathbf{B}(\tilde{\mathbf{u}}^n)\,dW(s), \mathbf{v}\Big).\end{aligned} \tag{3.8}$$

Subtracting (3.8) from (3.7) we get

$$\begin{aligned}&\big(\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}, \mathbf{v}\big) + k\Big(\sum_{n=0}^{m}\nabla\mathbf{e}_{\tilde{\mathbf{u}}}^{n+1}, \nabla\mathbf{v}\Big) + \big(\nabla\mathcal{E}_p^m, \mathbf{v}\big) \\ &= \sum_{n=0}^{m}\int_{t_n}^{t_{n+1}}\big(\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s)), \nabla\mathbf{v}\big)\,ds - \big(\nabla(P(t_{m+1}) - P(t_m)), \mathbf{v}\big) \\ &\quad+ \Big(\sum_{n=0}^{m}\int_{t_n}^{t_{n+1}}(\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\tilde{\mathbf{u}}^n))\,dW(s), \mathbf{v}\Big) \quad \forall \mathbf{v} \in \mathbf{H}_{per}^1(D).\end{aligned} \tag{3.9}$$

Setting $\mathbf{v} = \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}$ in (3.9) we obtain

$$\begin{aligned}&\|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|^2 + k\Big(\sum_{n=0}^{m+1}\nabla\mathbf{e}_{\tilde{\mathbf{u}}}^{n}, \nabla\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\Big) + \big(\nabla\mathcal{E}_p^m, \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\big) \\ &\quad= \sum_{n=0}^{m}\int_{t_n}^{t_{n+1}}\big(\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s)), \nabla\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\big)\,ds \\ &\quad\quad- \big(\nabla(P(t_{m+1}) - P(t_m)), \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\big) \\ &\quad\quad+ \Big(\sum_{n=0}^{m}\int_{t_n}^{t_{n+1}}(\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\tilde{\mathbf{u}}^n))\,dW(s), \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\Big).\end{aligned} \tag{3.10}$$

Similarly, by (2.5b) and (3.4b) we get

$$\operatorname{div} \mathbf{e}_{\tilde{\mathbf{u}}}^n + k\Delta p^n = 0. \tag{3.11}$$

Applying the summation $\sum_{n=0}^{m+1}$ to (3.11) and then adding $\pm\Delta P(t_{m+1})$ yield

$$\operatorname{div}\left(\sum_{n=0}^{m+1} \mathbf{e}_{\tilde{\mathbf{u}}}^n\right) - \Delta\mathcal{E}_p^{m+1} = -\Delta P(t_{m+1}).$$

Therefore,

$$\operatorname{div} \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1} - \Delta(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m) = -\Delta(P(t_{m+1}) - P(t_m)). \tag{3.12}$$

Testing (3.12) by any $q \in L^2(\Omega; H^1_{per}(D)/\mathbb{R})$, we have

$$\left(\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}, \nabla q\right) = \left(\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m), \nabla q\right) - \left(\nabla(P(t_{m+1}) - P(t_m)), \nabla q\right). \tag{3.13}$$

Choosing $q = \mathcal{E}_p^m$ in (3.13) gives

$$\begin{aligned}\left(\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}, \nabla\mathcal{E}_p^m\right) &= \left(\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m), \nabla\mathcal{E}_p^m\right) - \left(\nabla(P(t_{m+1}) - P(t_m)), \nabla\mathcal{E}_p^m\right) \\ &= \left(\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m), \nabla\mathcal{E}_p^{m+1}\right) - \|\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m)\|^2 \\ &\quad - \left(\nabla(P(t_{m+1}) - P(t_m)), \nabla\mathcal{E}_p^m\right).\end{aligned} \tag{3.14}$$

Substituting (3.14) into (3.10) we obtain

$$\begin{aligned}&\|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|^2 + k\left(\sum_{n=0}^{m+1} \nabla\mathbf{e}_{\tilde{\mathbf{u}}}^n, \nabla\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\right) + \left(\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m), \nabla\mathcal{E}_p^{m+1}\right) \\ &\leq \sum_{n=0}^{m} \int_{t_n}^{t_{n+1}} \left(\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s)), \nabla\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\right) ds \\ &\quad + \|\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m)\|^2 + \left(\nabla(P(t_{m+1}) - P(t_m)), \nabla\mathcal{E}_p^m\right) \\ &\quad + \|\nabla(P(t_{m+1}) - P(t_m))\|^2 + \frac{1}{4}\|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|^2 \\ &\quad + \left\|\sum_{n=0}^{m} \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\tilde{\mathbf{u}}^n))\, dW(s)\right\|^2 + \frac{1}{4}\|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|^2.\end{aligned}$$

Therefore,

$$\frac{1}{2}\|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|^2 + k\Big(\sum_{n=0}^{m+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{n}, \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\Big) + \big(\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^{m}), \nabla \mathcal{E}_p^{m+1}\big) \tag{3.15}$$
$$\leq \sum_{n=0}^{m} \int_{t_n}^{t_{n+1}} \big(\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s)), \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\big)\, ds$$
$$+ \|\nabla(P(t_{m+1}) - P(t_m))\|^2 + \|\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^{m})\|^2$$
$$+ \big(\nabla(P(t_{m+1}) - P(t_m)), \nabla \mathcal{E}_p^{m}\big)$$
$$+ \Big\|\sum_{n=0}^{m} \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\tilde{\mathbf{u}}^n))\, dW(s)\Big\|^2.$$

Using the identity $2a(a-b) = a^2 - b^2 + (a-b)^2$ in (3.15) yields

$$\frac{1}{2}\|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|^2 + \frac{k}{2}\Big[\Big\|\sum_{n=0}^{m+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{n}\Big\|^2 - \Big\|\sum_{n=0}^{m} \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{n}\Big\|^2 + \|\nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|^2\Big] \tag{3.16}$$
$$+ \frac{1}{2}\big[\|\nabla \mathcal{E}_p^{m+1}\|^2 - \|\nabla \mathcal{E}_p^{m}\|^2 + \|\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^{m})\|^2\big]$$
$$\leq \sum_{n=0}^{m} \int_{t_n}^{t_{n+1}} \big(\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s)), \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\big)\, ds$$
$$+ \|\nabla(P(t_{m+1}) - P(t_m))\|^2 + \|\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^{m})\|^2$$
$$+ \big(\nabla(P(t_{m+1}) - P(t_m)), \nabla \mathcal{E}_p^{m}\big)$$
$$+ \Big\|\sum_{n=0}^{m} \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\tilde{\mathbf{u}}^n))\, dW(s)\Big\|^2.$$

Next, we apply the summation operator $k\sum_{m=0}^{\ell}$ for $0 \leq \ell \leq M-1$, followed by applying the expectation operator $\mathbb{E}[\cdot]$, to (3.16) to obtain

$$k\sum_{m=0}^{\ell} \mathbb{E}\big[\|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|^2\big] + \mathbb{E}\Big[\Big\|k\sum_{m=0}^{\ell+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{m}\Big\|^2\Big] + k^2\sum_{m=0}^{\ell} \mathbb{E}\big[\|\nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|^2\big] \tag{3.17}$$
$$+ k\mathbb{E}\big[\|\nabla \mathcal{E}_p^{\ell+1}\|^2\big] + k\sum_{m=0}^{\ell} \mathbb{E}\big[\|\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^{m})\|^2\big]$$
$$\leq 2\mathbb{E}\Big[k\sum_{m=0}^{\ell}\sum_{n=0}^{m} \int_{t_n}^{t_{n+1}} \big(\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s)), \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\big)\, ds\Big]$$
$$+ 2\mathbb{E}\Big[k\sum_{m=0}^{\ell} \|\nabla(P(t_{m+1}) - P(t_m))\|^2\Big] + 2\mathbb{E}\Big[k\sum_{m=0}^{\ell} \|\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^{m})\|^2\Big]$$

$$
\begin{aligned}
&+ 2\mathbb{E}\Big[k\sum_{m=0}^{\ell}\big(\nabla(P(t_{m+1})-P(t_m)),\nabla\mathcal{E}_p^m\big)\Big]\\
&+ 2\mathbb{E}\Big[k\sum_{m=0}^{\ell}\Big\|\sum_{n=0}^{m}\int_{t_n}^{t_{n+1}}(\mathbf{B}(\mathbf{u}(s))-\mathbf{B}(\tilde{\mathbf{u}}^n))\,dW(s)\Big\|^2\Big]\\
:= {}& \mathtt{I}+\mathtt{II}+\mathtt{III}+\mathtt{IV}+\mathtt{V}.
\end{aligned}
$$

Now we estimate each term on the right-hand side of (3.17) as follows. We refer the reader to see [14, Theorem 3.2] for the details of the following estimations.

By using the discrete and continuous Hölder inequality estimates (2.4b), (2.9) and (3.5a), we obtain

$$
\begin{aligned}
\mathtt{I} &= 2\mathbb{E}\Big[k\sum_{m=0}^{\ell}\Big(\sum_{n=0}^{m}\int_{t_n}^{t_{n+1}}\nabla(\mathbf{u}(t_{n+1})-\mathbf{u}(s))\,ds,\nabla\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\Big)\Big]\\
&\le C\Big(\mathbb{E}\Big[k\sum_{m=0}^{\ell}\sum_{n=0}^{m}\int_{t_n}^{t_{n+1}}\|\nabla(\mathbf{u}(t_{n+1})-\mathbf{u}(s))\|^2\,ds\Big]\Big)^{\frac12}\Big(\mathbb{E}\Big[k\sum_{m=0}^{\ell}\|\nabla\mathbf{e}_{\tilde{\mathbf{u}}}^n\|^2\Big]\Big)\\
&\le Ck^{\frac12}.
\end{aligned}
\tag{3.18}
$$

Next, by using (2.11) we have

$$
\mathtt{II} = 2\mathbb{E}\Big[k\sum_{m=0}^{\ell}\|\nabla(P(t_{m+1})-P(t_m))\|^2\Big]\le Ck. \tag{3.19}
$$

By using (2.11) and the stability estimate (3.5b) we obtain

$$
\mathtt{III}\le C\mathbb{E}\Big[k\sum_{m=0}^{\ell}\|\nabla(P(t_{m+1})-P(t_m))\|^2 + Ck^3\sum_{m=0}^{\ell}\|\nabla p^{m+1}\|^2\Big]\le Ck. \tag{3.20}
$$

It follows from the Itô isometry and (2.4a) that

$$
\begin{aligned}
\mathtt{V} &= 2k\sum_{m=0}^{\ell}\mathbb{E}\Big[\sum_{n=0}^{m}\int_{t_n}^{t_{n+1}}\|\mathbf{B}(\mathbf{u}(s))-\mathbf{B}(\tilde{\mathbf{u}}^n)\|^2\,ds\Big]\\
&\le Ck\sum_{m=0}^{\ell}k\sum_{n=0}^{m}\mathbb{E}[\|\mathbf{e}_{\mathbf{u}}^n\|^2] + Ck\sum_{m=0}^{\ell}\sum_{n=0}^{m}\int_{t_n}^{t_{n+1}}\mathbb{E}[\|\mathbf{u}(s)-\mathbf{u}(t_n)\|^2]\,ds\\
&\le Ck\sum_{m=0}^{\ell}k\sum_{n=0}^{m}\mathbb{E}[\|\mathbf{e}_{\mathbf{u}}^n\|^2] + Ck.
\end{aligned}
\tag{3.21}
$$

To bound term $\mathtt{IV}$, we first derive its rewriting as follows:

$$\begin{aligned}\mathtt{IV} = 2\mathbb{E}\Big[k\sum_{m=0}^{\ell}\big(\nabla(P(t_{m+1}) - P(t_m)), \nabla\mathcal{E}_p^{m+1}\big)\Big]\\ + 2\mathbb{E}\Big[k\sum_{m=0}^{\ell}\big(\nabla(P(t_{m+1}) - P(t_m)), \nabla(\mathcal{E}_p^{m} - \mathcal{E}_p^{m+1})\big)\Big].\end{aligned} \tag{3.22}$$

By using the summation by parts, the first term above can be rewritten as

$$\begin{aligned}&2\mathbb{E}\Big[k\sum_{m=0}^{\ell}\big(\nabla(P(t_{m+1}) - P(t_m)), \nabla\mathcal{E}_p^{m+1}\big)\Big]\\ &= 2k\mathbb{E}\big[\big(\nabla P(t_{\ell+1}), \nabla\mathcal{E}_p^{\ell+1}\big)\big] - 2k\mathbb{E}\Big[\sum_{m=0}^{\ell}\big(\nabla P(t_m), \nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^{m})\big)\Big].\end{aligned} \tag{3.23}$$

Substituting (3.23) into (3.22) yields

$$\begin{aligned}\mathtt{IV} &= 2k\mathbb{E}\big[\big(\nabla P(t_{\ell+1}), \nabla\mathcal{E}_p^{\ell+1}\big)\big] - 2k\mathbb{E}\Big[\sum_{m=0}^{\ell}\big(\nabla P(t_m), \nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^{m})\big)\Big]\\ &\quad + 2\mathbb{E}\Big[k\sum_{m=0}^{\ell}\big(\nabla(P(t_{m+1}) - P(t_m)), \nabla(\mathcal{E}_p^{m} - \mathcal{E}_p^{m+1})\big)\Big]\\ &:= \mathtt{IV}_1 + \mathtt{IV}_2 + \mathtt{IV}_3.\end{aligned} \tag{3.24}$$

We now bound each term above. Using the stability (2.10) we get

$$\mathtt{IV}_1 \le Ck\mathbb{E}\big[\|\nabla P(t_{\ell+1})\|^2\big] + \frac{k}{4}\mathbb{E}\big[\|\nabla\mathcal{E}_p^{\ell+1}\|^2\big] \le Ck + \frac{k}{4}\mathbb{E}\big[\|\nabla\mathcal{E}_p^{\ell+1}\|^2\big]. \tag{3.25}$$

Expectedly, the term $\frac{k}{4}\mathbb{E}\big[\|\nabla\mathcal{E}_p^{\ell+1}\|^2\big]$ will be absorbed to the left side of (3.17) later.

To bound term $\mathtt{IV}_2$, we reuse the estimation from $\mathtt{III}$ in (3.20) together with the stability of P given in (2.10) to get

$$\mathtt{IV}_2 \le C\Big(\mathbb{E}\Big[k\sum_{m=0}^{\ell}\|\nabla P(t_m)\|^2\Big]\Big)^{\frac{1}{2}}\Big(\mathbb{E}\Big[k\sum_{m=0}^{\ell}\|\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^{m})\|^2\Big]\Big)^{\frac{1}{2}} \le Ck^{\frac{1}{2}}. \tag{3.26}$$

Using again (3.20) and (2.11) we have

$$\mathtt{IV}_3 \le C\mathbb{E}\Big[k\textstyle\sum_{m=0}^{\ell}\|\nabla(P(t_{m+1}) - P(t_m))\|^2 + k\sum_{m=0}^{\ell}\|\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^{m})\|^2\Big] \le Ck. \tag{3.27}$$

Substituting estimates (3.25)–(3.27) into (3.24) yields

$$\mathtt{IV} \le Ck^{\frac{1}{2}} + \frac{k}{4}\mathbb{E}\Big[\|\nabla \mathcal{E}_p^{\ell+1}\|^2\Big]. \tag{3.28}$$

Now, substituting the estimates for $\mathtt{I},\ \mathtt{II},\ \mathtt{III},\ \mathtt{IV},\ \mathtt{V}$ into (3.17) and using the notation $X^\ell = k\sum_{m=0}^{\ell}\mathbb{E}[\|\mathbf{e}_{\tilde{\mathbf{u}}}^m\|^2]$ we obtain

$$\begin{aligned} X^{\ell+1} &+ \mathbb{E}\Big[\Big\|k\sum_{m=0}^{\ell+1}\nabla \mathbf{e}_{\tilde{\mathbf{u}}}^m\Big\|^2 + k^2\sum_{m=0}^{\ell}\|\nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|^2 \\ &+ \frac{3k}{4}\|\nabla \mathcal{E}_p^{\ell+1}\|^2 + k\sum_{m=0}^{\ell}\|\nabla(\mathcal{E}_p^{m+1} - \mathcal{E}_p^m)\|^2\Big] \le Ck^{\frac{1}{2}} + Ck\sum_{m=0}^{\ell}X^m. \end{aligned}$$

Thus, it follows from the discrete Gronwall inequality that

$$X^{\ell+1} + \mathbb{E}\Big[\Big\|k\sum_{m=0}^{\ell+1}\nabla \mathbf{e}_{\tilde{\mathbf{u}}}^m\Big\|^2\Big] \le Ck^{\frac{1}{2}}\exp(Ct_\ell),$$

which yields the desired error estimate for the velocity approximation. □

Next, we derive an error estimate for the pressure approximation.

Theorem 3.3 *Let $\{(\tilde{\mathbf{u}}^m, p^m)\}_{m=0}^M$ be generated by Algorithm 1. Then, there exists a positive constant C which depends on $D_T, \mathbf{u}_0, \mathbf{f}$, and β such that*

$$\left(\mathbb{E}\Big[k\sum_{m=0}^{M}\Big\|P(t_m) - k\sum_{n=0}^{m}p^n\Big\|^2\Big]\right)^{\frac{1}{2}} \le Ck^{\frac{1}{4}}, \tag{3.29}$$

where β denotes the stochastic inf-sup constant (see below).

Proof We first recall the following inf-sup condition (cf. [6]):

$$\sup_{\mathbf{v}\in \mathbf{H}_{per}^1(D)} \frac{\big(q, \operatorname{div}\mathbf{v}\big)}{\|\nabla \mathbf{v}\|} \ge \beta\,\|q\| \qquad \forall q \in L_{per}^2(D)/\mathbb{R}, \tag{3.30}$$

where β is a positive constant.

Below we reuse all the notations from Theorem 3.2. From the error equation (3.9) we obtain for all $\mathbf{v} \in \mathbf{H}_{per}^1(D)$

$$
\begin{aligned}
\left(\mathcal{E}_p^m, \operatorname{div} \mathbf{v}\right) &= \left(\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}, \mathbf{v}\right) + \left(k \sum_{n=0}^{m} \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{n+1}, \nabla \mathbf{v}\right) + \left(\nabla(P(t_{m+1}) - P(t_m)), \mathbf{v}\right) \\
&\quad - \left(\sum_{n=0}^{m} \int_{t_n}^{t_{n+1}} \nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s))\, ds, \nabla \mathbf{v}\right) \\
&\quad - \left(\sum_{n=0}^{m} \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\tilde{\mathbf{u}}^n))\, dW(s), \mathbf{v}\right).
\end{aligned} \tag{3.31}
$$

By using Schwarz and Poincaré inequalities on the right side of (3.31), we get

$$
\begin{aligned}
\left(\mathcal{E}_p^m, \operatorname{div} \mathbf{v}\right) &\le C \|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\| \|\nabla \mathbf{v}\| + \left\| k \sum_{n=0}^{m} \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{n+1} \right\| \|\nabla \mathbf{v}\| \\
&\quad + \left\| \sum_{n=0}^{m} \int_{t_n}^{t_{n+1}} \nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s))\, ds \right\| \|\nabla \mathbf{v}\| \\
&\quad + C \|\nabla(P(t_{m+1}) - P(t_m))\| \|\nabla \mathbf{v}\| \\
&\quad + C \left\| \sum_{n=0}^{m} \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\tilde{\mathbf{u}}^n))\, dW(s) \right\| \|\nabla \mathbf{v}\|
\end{aligned} \tag{3.32}
$$

Applying (3.30) yields

$$
\begin{aligned}
\beta \|\mathcal{E}_p^m\| &\le \sup_{\mathbf{v} \in \mathbf{H}_{per}^1(D)} \frac{\left(\mathcal{E}_p^m, \operatorname{div} \mathbf{v}\right)}{\|\nabla \mathbf{v}\|} \\
&\le C \|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\| + \left\| k \sum_{n=0}^{m} \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{n+1} \right\| + \left\| \sum_{n=0}^{m} \int_{t_n}^{t_{n+1}} \nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s))\, ds \right\| \\
&\quad + C \|\nabla(P(t_{m+1}) - P(t_m))\| \\
&\quad + C \left\| \sum_{n=0}^{m} \int_{t_n}^{t_{n+1}} (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\tilde{\mathbf{u}}^n))\, dW(s) \right\|.
\end{aligned} \tag{3.33}
$$

Next, squaring both sides of (3.33) followed by applying operators $k \sum_{m=0}^{\ell}$ and $\mathbb{E}[\cdot]$, we obtain

$$
\begin{aligned}
\beta^2 \mathbb{E}\left[k \sum_{m=0}^{\ell} \|\mathcal{E}_p^m\|^2\right] &\le C \mathbb{E}\left[k \sum_{m=0}^{\ell} \|\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}\|^2\right] + Ck \sum_{m=0}^{\ell} \mathbb{E}\left[\left\| k \sum_{n=0}^{m} \nabla \mathbf{e}_{\tilde{\mathbf{u}}}^{n+1} \right\|^2\right] \\
&\quad + Ck \sum_{m=0}^{\ell} \mathbb{E}\left[\left\| \sum_{n=0}^{m} \int_{t_n}^{t_{n+1}} \nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(s))\, ds \right\|^2\right]
\end{aligned} \tag{3.34}
$$

$$
\begin{aligned}
&+ Ck\sum_{m=0}^{\ell}\mathbb{E}\big[\|\nabla(P(t_{m+1})-P(t_m))\|^2\big]\\
&+ Ck\sum_{m=0}^{\ell}\mathbb{E}\Bigg[\Bigg\|\sum_{n=0}^{m}\int_{t_n}^{t_{n+1}}(\mathbf{B}(\mathbf{u}(s))-\mathbf{B}(\tilde{\mathbf{u}}^n))\,dW(s)\Bigg\|^2\Bigg]\\
&:= \mathtt{I}+\mathtt{II}+\mathtt{III}+\mathtt{IV}+\mathtt{V}.
\end{aligned}
$$

It remains to bound each term on the right side above. We refer the reader to see the details of the proof in [14, Theorem 3.3] using Theorem 3.2. We only state the final results below.

$$
\beta^2\mathbb{E}\Bigg[k\sum_{m=0}^{\ell}\|\mathcal{E}_p^m\|^2\Bigg] \le Ck^{\frac{1}{2}} \qquad \text{for } 1\le \ell\le M. \tag{3.35}
$$

The proof is complete. □

Remark 3.2 It is interesting to point out that the above proof uses the technique from the (non-splitting) mixed method error analysis although Chorin scheme is a splitting scheme.

We conclude this subsection by stating two stability estimates for $(\tilde{\mathbf{u}}^m, p^m)$ in high norms as immediate corollaries of the above error estimates, they will be used in the next section in deriving error estimates for a fully discrete finite element Chorin method. We note that these stability estimates improve those given in Lemma 3.1 and may not be obtained directly without using the above error estimates.

Corollary 3.4 *Under the assumptions of Theorem 3.2, there exists a positive constant C which depends on D_T, $\mathbf{u}_0$ and $\mathbf{f}$ such that*

$$
\mathbb{E}\Bigg[k\sum_{m=0}^{M}\Bigg\|k\sum_{n=0}^{m}\nabla p^n\Bigg\|^2\Bigg] \le C, \tag{3.36}
$$

$$
\max_{0\le m\le M}\mathbb{E}\Bigg[\Bigg\|k\sum_{n=0}^{m}\nabla\tilde{\mathbf{u}}^n\Bigg\|^2\Bigg] + \mathbb{E}\Bigg[k\sum_{m=0}^{M}\Bigg\|k\sum_{n=0}^{m}\Delta\tilde{\mathbf{u}}^n\Bigg\|^2\Bigg] \le C. \tag{3.37}
$$

We refer the reader to [14, Corollary 3.4] for a detailed proof.

3.2 A modified Chorin projection scheme

In this subsection, we consider a modification of Algorithm 1 which was already pointed out in [8] but did not analyze there. The modification is to perform a Helmholtz decomposition of $\mathbf{B}(\tilde{\mathbf{u}}^m)$ at each time step which allows us to eliminate the curl-free part in Step 1 of Algorithm 1, this then results in a divergent-free Helmholtz projected noise. The goal of this subsection is to present a complete convergence analysis for the modified Chorin scheme which includes deriving stronger error estimates for both velocity and pressure approximations than those for the standard Chorin scheme. We note that this Helmholtz decomposition enhancing technique was also used in [13] to improve the standard mixed finite element method for (1.1).

3.2.1 Formulation of the modified Chorin scheme

For ease of the presentation, we assume $W(t)$ is a real-valued Wiener process and independent of the spatial variable. The case of more general $W(t)$ can be dealt with similarly. The modified Chorin scheme is given as follows.

Algorithm 2 Set $\tilde{\mathbf{u}}^0 = \mathbf{u}^0 = \mathbf{u}_0$. For $m = 0, 1, \cdots, M-1$, do the following five steps.

Step 1: Given $\tilde{\mathbf{u}}^m \in L^2(\Omega, \mathbf{H}^1_{per}(D))$, find $\xi^m \in L^2(\Omega, H^1_{per}(D)/\mathbb{R})$ such that $\mathbb{P}$-a.s.

$$\Delta\xi^m = \operatorname{div} \mathbf{B}(\tilde{\mathbf{u}}^m) \qquad \text{in } D. \tag{3.38}$$

Step 2: Set $\boldsymbol{\eta}^m_{\tilde{\mathbf{u}}} = \mathbf{B}(\tilde{\mathbf{u}}^m) - \nabla\xi^m$. Given $\mathbf{u}^m \in L^2(\Omega, \mathbb{H})$ and $\tilde{\mathbf{u}}^m \in L^2(\Omega, \mathbf{H}^1_{per}(D))$, find $\tilde{\mathbf{u}}^{m+1} \in L^2(\Omega, \mathbf{H}^1_{per}(D))$ such that $\mathbb{P}$-a.s.

$$\tilde{\mathbf{u}}^{m+1} - k\Delta\tilde{\mathbf{u}}^{m+1} = \mathbf{u}^m + k\mathbf{f}^{m+1} + \boldsymbol{\eta}^m_{\tilde{\mathbf{u}}}\Delta W_{m+1} \qquad \text{in } D. \tag{3.39}$$

Step 3: Find $r^{m+1} \in L^2(\Omega, H^1_{per}(D)/\mathbb{R})$ such that $\mathbb{P}$-a.s.

$$-\Delta r^{m+1} = -\frac{1}{k}\operatorname{div} \tilde{\mathbf{u}}^{m+1} \qquad \text{in } D. \tag{3.40a}$$

Step 4: Define $\mathbf{u}^{m+1} \in L^2(\Omega, \mathbb{H})$ as

$$\mathbf{u}^{m+1} = \tilde{\mathbf{u}}^{m+1} - k\nabla r^{m+1} \qquad \text{in } D. \tag{3.41}$$

Step 5: Define the pressure approximation p^{m+1} as

$$p^{m+1} = r^{m+1} + \frac{1}{k}\xi^m \Delta W_{m+1} \qquad \text{in } D. \tag{3.42}$$

Remark 3.3 It follows from (2.2b) and (3.38) that the Helmholtz projection $\boldsymbol{\eta}^m_{\tilde{\mathbf{u}}}$ can be bounded in terms of $\tilde{\mathbf{u}}^m$ as follows:

$$\|\boldsymbol{\eta}^m_{\tilde{\mathbf{u}}}\|_{L^2} \leq \|\mathbf{B}(\tilde{\mathbf{u}}^m)\|_{L^2} + \|\nabla\xi^m_{\tilde{\mathbf{u}}}\|_{L^2} \leq 2\|\mathbf{B}(\tilde{\mathbf{u}}^m)\|_{L^2} \leq C\|\tilde{\mathbf{u}}^m\|_{L^2} \tag{3.43}$$

3.2.2 Stability estimates for the modified Chorin scheme

In this subsection we first state some stability estimates for Algorithm 2. We then recall the Euler-Maruyama scheme for (1.1) and its stability and error estimates from [13], which will be utilized as a tool in the stability and error analysis of the modified Chorin scheme in the next subsection.

Lemma 3.5 *The discrete processes* $\{(\tilde{\mathbf{u}}^m, r^m)\}_{m=0}^M$ *defined by Algorithm 2 satisfy*

$$\max_{0\le m\le M} \mathbb{E}[\|\tilde{\mathbf{u}}^m\|^2] + \mathbb{E}\Big[\sum_{m=1}^M \|\tilde{\mathbf{u}}^m - \tilde{\mathbf{u}}^{m-1}\|^2\Big] + \mathbb{E}\Big[k\sum_{m=0}^M \|\nabla\tilde{\mathbf{u}}^m\|^2\Big] \le C, \quad (3.44a)$$

$$\mathbb{E}\Big[k^2\sum_{m=0}^M \|\nabla r^m\|^2\Big] \le C, \quad (3.44b)$$

where C *is a positive constant which depends on* D_T, $\mathbf{u}_0$ *and* $\mathbf{f}$.

Since the proof of this lemma follows the same lines as those of Lemma 3.1. We omit the proof to save space.

Next, we recall the Helmholtz enhanced Euler-Maruyama scheme for (1.1) which was proposed and analyzed in [13]. This scheme will be used as an auxiliary scheme in our error estimates for the velocity and pressure approximations of Algorithm 2 in the next subsection. The Euler-Maruyama scheme reads as

$$(\mathbf{v}^{m+1} - \mathbf{v}^m) - k\Delta\mathbf{v}^{m+1} + k\nabla q^{m+1} = k\mathbf{f}^{m+1} + \boldsymbol{\eta}_{\mathbf{v}}^m \Delta W_{m+1} \qquad \text{in } D, \quad (3.45a)$$

$$\operatorname{div} \mathbf{v}^{m+1} = 0 \qquad \text{in } D, \quad (3.45b)$$

where $\boldsymbol{\eta}_{\mathbf{v}}^m = \mathbf{B}(\mathbf{v}^m) - \nabla\xi_{\mathbf{v}}^m$ denotes the Helmholtz projection of $\mathbf{B}(\mathbf{v}^m)$.

It was proved in [13] that the following stability and error estimates hold for the solution of the above Euler-Maruyama scheme.

Lemma 3.6 *The discrete processes* $\{(\mathbf{v}^m, q^m)\}_{m=0}^M$ *defined by (3.45) satisfy*

$$\max_{0\le m\le M} \mathbb{E}\big[\|\mathbf{v}^m\|^2\big] + \mathbb{E}\Big[\sum_{m=1}^M \|\mathbf{v}^m - \mathbf{v}^{m-1}\|^2\Big] + \mathbb{E}\Big[k\sum_{m=0}^M \|\nabla\mathbf{v}^m\|^2\Big] \le C, \quad (3.46a)$$

$$\max_{0\le m\le M} \mathbb{E}\big[\|\nabla\mathbf{v}^m\|^2\big] + \mathbb{E}\Big[\sum_{m=1}^M \Big(\|\nabla(\mathbf{v}^m - \mathbf{v}^{m-1})\|^2 + k\|\mathbf{A}\mathbf{v}^m\|^2\Big)\Big] \le C, \quad (3.46b)$$

$$\mathbb{E}\Big[k\sum_{m=0}^M \|\nabla q^m\|^2\Big] \le C, \quad (3.46c)$$

where C *is a positive constant which depends on* D_T, $\mathbf{u}_0$ *and* $\mathbf{f}$.

Remark 3.4 We note that to ensure the stability estimate (3.46b) is the only reason for restricting to the periodic boundary condition in this paper.

Lemma 3.7 *There hold the the following error estimates for the discrete processes* $\{(\mathbf{v}^m, q^m)\}_{m=0}^M$*:*

$$\max_{0\le m\le M} \big(\mathbb{E}\big[\|\mathbf{u}(t_m) - \mathbf{v}^m\|^2\big]\big)^{\frac{1}{2}} + \left(\mathbb{E}\left[k\sum_{m=0}^{M}\|\nabla(\mathbf{u}(t_m) - \mathbf{v}^m)\|^2\right]\right)^{\frac{1}{2}} \le C\sqrt{k}, \tag{3.47a}$$

$$\left(\mathbb{E}\left[\left\|R(t_\ell) - k\sum_{m=0}^{\ell} q^m\right\|^2\right]\right)^{\frac{1}{2}} \le C\sqrt{k} \tag{3.47b}$$

for $0 \le \ell \le M$. *Where* C *is a positive constant which depends on* D_T, $\mathbf{u}_0$ *and* $\mathbf{f}$.

3.2.3 Error estimates for the modified Chorin scheme

The goal of this subsection is to derive error estimates for both the velocity and pressure approximations generated by Algorithm 2. The anticipated error estimates are optimal to compare with those for the standard Chorin scheme proved in the previous subsection. We note that our error estimate for the velocity approximation recovers the same estimate obtained in [8, Theorem 3.1] although the analysis given here is a lot simpler. On the other hand, the error estimate for the pressure approximation is apparently new. The main idea of the proofs of this subsection is to use the Euler-Maruyama scheme analyzed in [13] as an auxiliary scheme which bridges the exact solution of (1.1) and the discrete solution of Algorithm 2.

The follow theorem gives an error estimate in strong norms for the velocity approximation.

Theorem 3.8 *Let* $\{(\tilde{\mathbf{u}}^m, p^m)\}_{m=0}^M$ *be the solution of Algorithm 2 and* $\{(\mathbf{u}(t), P(t));\ 0 \le t \le T\}$ *be the solution of (1.1). Then there holds the following estimate:*

$$\max_{0\le m\le M} \left(\mathbb{E}\left[\|\mathbf{u}(t_m) - \tilde{\mathbf{u}}^m\|^2\right]\right)^{\frac{1}{2}} + \left(\mathbb{E}\left[k\sum_{m=0}^{M}\|\nabla(\mathbf{u}(t_m) - \tilde{\mathbf{u}}^m)\|^2\right]\right)^{\frac{1}{2}} \le C\sqrt{k}, \tag{3.48}$$

where C is a positive constant which depends on D_T, $\mathbf{u}_0$ and $\mathbf{f}$.

We refer the reader to [14, Theorem 3.8] for a detailed proof. We also note that in that proof, only one auxiliary scheme (i.e., (3.45)) was used, while the proof given in [8, Theorem 3.1] required to use two auxiliary schemes to carry out the proof.

An immediate corollary of the above error estimate is the following stronger stability estimates for $\{(\tilde{\mathbf{u}}^m, r^m)\}$, which may not be obtainable directly and will play an important role in the error analysis of fully discrete counterpart of Algorithm 2 in the next section. We skip its proof and refer the reader to [14, Corollary 3.9] for a complete proof.

Corollary 3.9 *There exists* $C > 0$ *which depends on* D_T, $\mathbf{u}_0$ *and* $\mathbf{f}$ *such that*

$$\max_{0\le m\le M} \mathbb{E}[\|\nabla\tilde{\mathbf{u}}^m\|^2] + \mathbb{E}\Big[k\sum_{m=0}^{M}\|\Delta\tilde{\mathbf{u}}^m\|^2\Big] \le C, \tag{3.49a}$$

$$\mathbb{E}\Big[k\sum_{m=0}^{M}\|\nabla r^m\|^2\Big] \le C. \tag{3.49b}$$

Similarly, the following estimate holds for $\{\mathbf{u}^m\}$.

Corollary 3.10 *There exists* $C > 0$ *which depends on* D_T, $\mathbf{u}_0$ *and* $\mathbf{f}$ *such that*

$$\max_{0\le m\le M}\big(\mathbb{E}[\|\mathbf{u}(t_m)-\mathbf{u}^m\|^2]\big)^{\frac{1}{2}} + \left(\mathbb{E}\Big[k\sum_{m=0}^{M}\|\mathbf{u}(t_m)-\mathbf{u}^m\|^2\Big]\right)^{\frac{1}{2}} \le C\sqrt{k}. \tag{3.50}$$

The proof of (3.50) readily follows from (3.41) and Theorem (3.8) as well as the estimate (3.49b).

Next, we derive error estimates for the pressure approximations r^m and p^m generated by Algorithm 2. First, we state the following lemma.

Lemma 3.11 *Let* $\{r^m\}_{m=0}^M$ *be generated by Algorithm 2. Then, there exists a constant* $C > 0$ *depending on* D_T, $\mathbf{u}_0$, $\mathbf{f}$ *and* β *such that for* $0 \le \ell \le M$

$$\left(\mathbb{E}\left[\left\|k\sum_{m=1}^{\ell}(r^m - r^{m-1})\right\|^2\right]\right)^{\frac{1}{2}} \le C\sqrt{k}. \tag{3.51}$$

Proof The idea of the proof is to utilize the inf-sup condition (3.30). Testing (3.41) by any $\mathbf{v} \in L^2(\Omega; \mathbf{H}^1_{per}(D))$, we obtain

$$k\big(r^{m+1}, \operatorname{div}\mathbf{v}\big) = \big(\mathbf{u}^{m+1} - \tilde{\mathbf{u}}^{m+1}, \mathbf{v}\big),$$
$$k\big(r^{m}, \operatorname{div}\mathbf{v}\big) = \big(\mathbf{u}^{m} - \tilde{\mathbf{u}}^{m}, \mathbf{v}\big).$$

Then, subtracting the above equations yields

$$k\big(r^{m+1} - r^m, \operatorname{div}\mathbf{v}\big) = \big((\mathbf{u}^{m+1} - \mathbf{u}^m) - (\tilde{\mathbf{u}}^{m+1} - \tilde{\mathbf{u}}^m), \mathbf{v}\big). \tag{3.52}$$

Applying the summation operator $\sum_{m=0}^{\ell}$ for $0 \le \ell \le M-1$ to (3.52), we get

$$\left(k\sum_{m=0}^{\ell}(r^{m+1}-r^{m}),\operatorname{div}\mathbf{v}\right)=\big((\mathbf{u}^{\ell+1}-\mathbf{u}^{0})-(\tilde{\mathbf{u}}^{\ell+1}-\tilde{\mathbf{u}}^{0}),\mathbf{v}\big) \tag{3.53}$$
$$=\big(\mathbf{e}_{\mathbf{u}}^{\ell+1}-\mathbf{e}_{\tilde{\mathbf{u}}}^{\ell+1},\mathbf{v}\big)$$
$$\leq C\big(\|\mathbf{e}_{\mathbf{u}}^{\ell+1}\|+\|\mathbf{e}_{\tilde{\mathbf{u}}}^{\ell+1}\|\big)\|\nabla\mathbf{v}\|,$$

where $\mathbf{e}_{\mathbf{u}}^{m}$ and $\mathbf{e}_{\tilde{\mathbf{u}}}^{m}$ are the same as defined in the preceding subsection and we have used the fact that $\mathbf{u}^0-\tilde{\mathbf{u}}^0=0$.

Finally, by using the inf-sup condition (3.30) and then taking the expectation we get

$$\beta^2\mathbb{E}\left[\left\|k\sum_{m=0}^{\ell}(r^{m+1}-r^{m})\right\|^2\right]\leq C\Big(\mathbb{E}\big[\|\mathbf{e}_{\mathbf{u}}^{\ell+1}\|^2\big]+\mathbb{E}\big[\|\mathbf{e}_{\tilde{\mathbf{u}}}^{\ell+1}\|^2\big]\Big),$$

which and the estimates for $\mathbf{e}_{\mathbf{u}}^{\ell+1}$ and $\mathbf{e}_{\tilde{\mathbf{u}}}^{\ell+1}$ infer the desired estimate (3.51). The proof is complete. □

We then are ready to state the following error estimate result for r^m.

Theorem 3.12 *Let $\{r^m\}_{m=0}^{M}$ be generated by Algorithm 2 and $R(t)$ be defined in (2.6). Then there exists a constant $C>0$ depending on $D_T,\mathbf{u}_0,\mathbf{f}$ and β such hat for $0\leq\ell\leq M$*

$$\left(\mathbb{E}\left[\left\|R(t_\ell)-k\sum_{m=0}^{\ell}r^m\right\|^2\right]\right)^{\frac{1}{2}}\leq C\sqrt{k}. \tag{3.54}$$

Proof Let $\mathbf{e}_{\tilde{\mathbf{u}}}^{m}=\mathbf{v}^m-\tilde{\mathbf{u}}^m$. Subtracting (3.45) from (3.4a) and then testing the resulting equation by $\mathbf{v}\in L^2(\Omega;\mathbf{H}^1_{per}(D))$, we obtain

$$\big(\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1}-\mathbf{e}_{\tilde{\mathbf{u}}}^{m},\mathbf{v}\big)+k\big(\nabla\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1},\nabla\mathbf{v}\big)-k\big(q^{m+1}-r^m,\operatorname{div}\mathbf{v}\big) \tag{3.55}$$
$$=\big((\boldsymbol{\eta}_{\mathbf{v}}^{m}-\boldsymbol{\eta}_{\tilde{\mathbf{u}}}^{m})\Delta W_{m+1},\mathbf{v}\big).$$

Applying the summation operator $\sum_{m=0}^{\ell}$ to (3.55) for $0\leq\ell\leq M-1$ yields

$$\left(k\sum_{m=0}^{\ell}(q^{m+1}-r^{m}),\operatorname{div}\mathbf{v}\right)=\big(\mathbf{e}_{\tilde{\mathbf{u}}}^{\ell+1}-\mathbf{e}_{\tilde{\mathbf{u}}}^{0},\mathbf{v}\big)+\left(k\sum_{m=0}^{\ell}\nabla\mathbf{e}_{\tilde{\mathbf{u}}}^{m+1},\nabla\mathbf{v}\right) \tag{3.56}$$
$$-\left(\sum_{m=0}^{\ell}\big(\boldsymbol{\eta}_{\mathbf{v}}^{m}-\boldsymbol{\eta}_{\tilde{\mathbf{u}}}^{m}\big)\Delta W_{m+1},\mathbf{v}\right)$$
$$=\mathtt{I}+\mathtt{II}+\mathtt{III}.$$

To the end, we bound each term on the right-hand side of (3.56). We suggest the reader to see the details of these estimations in [14, Theorem 3.12]. Here, we just state the final result below.

$$\left(\mathbb{E}\left[\left\|k\sum_{m=0}^{\ell}(q^{m+1}-r^m)\right\|^2\right]\right)^{\frac{1}{2}} \leq C\sqrt{k}. \tag{3.57}$$

The proof is completed by applying the triangular inequality, Lemma 3.11 and (3.47b). □

Corollary 3.13 *Let $\{p^m\}_{m=0}^M$ be generated by Algorithm 2. Then, there exists a constant $C > 0$ which depends on D_T, $\mathbf{u}_0$, $\mathbf{f}$ and β such that for $0 \leq \ell \leq M$*

$$\left(\mathbb{E}\left[\left\|P(t_\ell)-k\sum_{m=0}^{\ell}p^m\right\|^2\right]\right)^{\frac{1}{2}} \leq C\sqrt{k}. \tag{3.58}$$

The proof follows readily from Theorem 3.12 and the relationship between $P(t)$ and $R(t)$ and between p^m and r^m, see [14, Corollary 3.13] for the details.

4 Fully discrete finite element methods

In this section, we formulate and analyze finite element spatial discretization for Algorithm 1 and 2. To the end, let $\mathcal{T}_h$ be a quasi-uniform triangulation of the polygonal ($d = 2$) or polyhedral ($d = 3$) bounded domain D. We introduce the following two basic Lagrangian finite element spaces:

$$V_h = \{\phi \in C(\overline{D});\quad \phi_{|K} \in P_\ell(K)\quad \forall K \in \mathcal{T}_h\}, \tag{4.1}$$

$$X_h = \{\phi \in C(\overline{D});\quad \phi_{|K} \in P_\ell(K)\quad \forall K \in \mathcal{T}_h\}, \tag{4.2}$$

where $P_\ell(K)$ ($\ell \geq 1$) denotes the set of polynomials of degree less than or equal to ℓ over the element $K \in \mathcal{T}_h$. The finite element spaces to be used to formulate our finite element methods are defined as follows:

$$\mathbf{H}_h = [V_h \cap H^1_{per}(D)]^d, \quad L_h = V_h \cap L^2_{per}(D)/\mathbb{R}, \quad S_h = X_h \cap L^2_{per}(D)/\mathbb{R}. \tag{4.3}$$

In addition, we introduce spaces

$$\mathbb{V}_h = L^2(\Omega, \mathbf{H}_h), \qquad \mathbb{W}_h = L^2(\Omega, L_h). \tag{4.4}$$

Recall that the L^2-projection $\mathcal{P}_h^0 : [L^2_{per}(D)]^d \to \mathbf{H}_h$ is defined by

$$(\phi - \mathcal{P}_h^0\phi,\, \xi) = 0 \qquad \forall \xi \in \mathbf{H}_h \tag{4.5}$$

and the H^1-projection $\mathcal{P}_h^1 : H^1_{per}(D)/\mathbb{R} \to L_h$ is defined by

$$\big((\nabla(\chi - \mathcal{P}_h^1\chi), \nabla\eta\big) = 0 \qquad \forall \eta \in L_h. \tag{4.6}$$

It is well known [6] that $\mathcal{P}_h^0$ and $\mathcal{P}_h^1$ satisfy following estimates:

$$\|\phi - \mathcal{P}_h^0\phi\| + h\|\nabla(\phi - \mathcal{P}_h^0\phi)\| \le C\,h^2\|\phi\|_{H^2} \qquad \forall \phi \in \mathbf{H}^2_{per}(D), \tag{4.7}$$

$$\|\chi - \mathcal{P}_h^1\chi\| + h\|\nabla(\chi - \mathcal{P}_h^1\chi)\| \le C\,h^2\|\chi\|_{H^2} \qquad \forall \chi \in H^1_{per}/\mathbb{R} \cap H^2_{per}(D). \tag{4.8}$$

For the clarity we only consider P_1-finite element space in this section (i.e., $\ell = 1$), the results of this section can be easily extended to high order finite elements.

4.1 Finite element methods for the standard Chorin scheme

Approximating the velocity space and pressure space respectively by the finite element spaces $\mathbf{H}_h$ and L_h in Algorithm 1, we then obtain the fully discrete finite element version of the standard Chorin scheme given below as Algorithm 3. We also note that a similar algorithm was proposed in [8].

Algorithm 3 Let $n \ge 0$. Set $\tilde{\mathbf{u}}_h^0 = \mathcal{P}_h^0\mathbf{u}_0$. For $n = 0, 1, 2, \cdots$ do the following steps:

Step 1: Given $\mathbf{u}_h^n \in L^2(\Omega, \mathbf{H}_h)$ and $\tilde{\mathbf{u}}_h^n \in L^2(\Omega, \mathbf{H}_h)$, find $\tilde{\mathbf{u}}_h^{n+1} \in L^2(\Omega, \mathbf{H}_h)$ such that $\mathbb{P}$-a.s.

$$\begin{aligned}&\big(\tilde{\mathbf{u}}_h^{n+1}, \mathbf{v}_h\big) + k\big(\nabla\tilde{\mathbf{u}}_h^{n+1}, \nabla\mathbf{v}_h\big) \\ &\quad= \big(\mathbf{u}_h^n, \mathbf{v}_h\big) + k\big(\mathbf{f}^{n+1}, \mathbf{v}_h\big) + \big(\mathbf{B}(\tilde{\mathbf{u}}_h^n)\Delta W_{n+1}, \nabla\mathbf{v}_h\big) \qquad \forall \mathbf{v}_h \in \mathbf{H}_h.\end{aligned} \tag{4.9}$$

Step 2: Find $p_h^{n+1} \in L^2(\Omega, L_h)$ such that $\mathbb{P}$-a.s.

$$\big(\nabla p_h^{n+1}, \nabla\phi_h\big) = \frac{1}{k}\big(\tilde{\mathbf{u}}_h^{n+1}, \nabla\phi_h\big) \qquad \forall \phi_h \in L_h. \tag{4.10}$$

Step 3: Define $\mathbf{u}_h^{n+1} \in L^2(\Omega, \mathbf{H}_h)$ by

$$\mathbf{u}_h^{n+1} = \tilde{\mathbf{u}}_h^{n+1} - k\nabla p_h^{n+1}. \tag{4.11}$$

As mentioned in Sect. 1, eliminating $\mathbf{u}^n$ in (4.9) using (4.10), we obtain

$$\begin{aligned}&(\tilde{\mathbf{u}}_h^{n+1} - \tilde{\mathbf{u}}_h^n, \mathbf{v}_h) + k(\nabla\tilde{\mathbf{u}}_h^{n+1}, \nabla\mathbf{v}_h) \\ &\qquad + k(\nabla p_h^n, \mathbf{v}_h) = k\big(\mathbf{f}^{n+1}, \mathbf{v}_h\big) + (\mathbf{B}(\tilde{\mathbf{u}}_h^n)\Delta W_{n+1}, \mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \mathbf{H}_h,\end{aligned} \tag{4.12a}$$

$$(\tilde{\mathbf{u}}_h^{n+1}, \nabla\phi_h) = k(\nabla p_h^{n+1}, \nabla\phi_h) \qquad \forall \phi_h \in L_h. \tag{4.12b}$$

Next, we state the stability estimates for $\{(\tilde{\mathbf{u}}_h^n, p_h^n)\}_{n=0}^M$ in the following lemma, which will be used in the fully discrete error analysis later. Since its proof follows from the same lines of that for Lemma 3.1, we omit it to save space.

Lemma 4.1 *Let* $\{(\tilde{\mathbf{u}}_h^n, p_h^n)\}_{n=0}^M$ *be generated by Algorithm 3, then there holds*

$$\max_{0\le n\le M} \mathbb{E}[\|\tilde{\mathbf{u}}_h^n\|^2] + \mathbb{E}\Big[\sum_{n=1}^M \|\tilde{\mathbf{u}}_h^n - \tilde{\mathbf{u}}_h^{n-1}\|^2\Big] + \mathbb{E}\Big[k\sum_{n=0}^M \|\nabla \tilde{\mathbf{u}}_h^n\|^2\Big] \le C, \tag{4.13a}$$

$$\mathbb{E}\Big[k\sum_{n=0}^M \|\nabla p_h^n\|^2\Big] \le \frac{C}{k}, \tag{4.13b}$$

where C *is a positive constant depending only on* D_T, u_0, *and* $\mathbf{f}$.

The following theorem provides an error estimate in a strong norm for the finite element solution of Algorithm 3.

Theorem 4.2 *Let* $\{(\tilde{\mathbf{u}}^m, p^m)\}_{m=0}^M$ *and* $\{(\tilde{\mathbf{u}}_h^m, p_h^m)\}_{m=0}^M$ *be generated respectively by Algorithm 1 and Algorithm 3. Then under the assumptions of Lemmas 3.1, 4.1 and Corollary 3.4 there holds*

$$\begin{aligned}&\Big(\mathbb{E}\Big[k\sum_{m=0}^M \|\tilde{\mathbf{u}}^m - \tilde{\mathbf{u}}_h^m\|^2\Big]\Big)^{\frac{1}{2}} \\ &\quad + \max_{0\le \ell\le M}\Big(\mathbb{E}\Big[\Big\|k\sum_{n=1}^{\ell}\nabla(\tilde{\mathbf{u}}^n - \tilde{\mathbf{u}}_h^n)\Big\|^2\Big]\Big)^{\frac{1}{2}} \le C\Big(k^{\frac{1}{4}} + hk^{-\frac{1}{2}}\Big),\end{aligned} \tag{4.14}$$

where $C \equiv C(D_T, \mathbf{u}_0, \mathbf{f})$ *is a positive constant.*

Proof The proof is conceptually similar to that of Theorem 3.2. Setting $\mathbf{e}_{\tilde{\mathbf{u}}_h}^m =: \tilde{\mathbf{u}}^m - \tilde{\mathbf{u}}_h^m$ and $\varepsilon_{p_h}^m =: p^m - p_h^m$. Without loss of the generality, we assume $\mathbf{e}_{\tilde{\mathbf{u}}_h}^0 = 0$ and $\varepsilon_{p_h}^0 = 0$ because they are of high order accuracy, hence are negligible.

First, applying the summation operator $\sum_{n=0}^m$ to (4.12a), we obtain

$$\begin{aligned}&\big(\tilde{\mathbf{u}}_h^{m+1}, \mathbf{v}_h\big) + k\Big(\sum_{n=0}^{m+1}\nabla\tilde{\mathbf{u}}_h^n, \nabla\mathbf{v}_h\Big) + k\Big(\sum_{n=0}^{m}\nabla p_h^n, \mathbf{v}_h\Big) \\ &\quad = \big(\mathbf{u}_h^0, \mathbf{v}_h\big) + \Big(\sum_{n=0}^m \mathbf{B}(\tilde{\mathbf{u}}_h^n)\Delta W_{n+1}, \mathbf{v}_h\Big) \qquad \forall \mathbf{v}_h \in \mathbf{H}_h.\end{aligned} \tag{4.15}$$

Subtracting (3.8) from (4.15) yields the following error equations:

$$\begin{aligned}&\big(\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}, \mathbf{v}_h\big) + k\Big(\sum_{n=0}^{m+1}\nabla\mathbf{e}_{\tilde{\mathbf{u}}_h}^n, \nabla\mathbf{v}_h\Big) + k\Big(\sum_{n=0}^{m}\nabla \varepsilon_{p_h}^n, \mathbf{v}_h\Big) \\ &\quad = \Big(\sum_{n=0}^m (\mathbf{B}(\tilde{\mathbf{u}}^n) - \mathbf{B}(\tilde{\mathbf{u}}_h^n))\Delta W_{n+1}, \mathbf{v}_h\Big) \qquad \forall \mathbf{v}_h \in \mathbf{H}_h,\end{aligned} \tag{4.16}$$

$$\big(\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}, \nabla\phi_h\big) = k\big(\nabla\varepsilon_{p_h}^{m+1}, \nabla\phi_h\big) \qquad \forall \phi_h \in L_h. \tag{4.17}$$

Choosing $\mathbf{v}_h = \mathcal{P}_h^0 \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1} = \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1} - \boldsymbol{\theta}^{m+1}; \boldsymbol{\theta}^m = \tilde{\mathbf{u}}_h^m - \mathcal{P}_h^0 \tilde{\mathbf{u}}_h^m$, then (4.16) becomes

$$\|\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\|^2 + k\Big(\sum_{n=0}^{m+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^n, \nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\Big) + k\Big(\sum_{n=0}^{m} \nabla \varepsilon_{p_h}^n, \mathcal{P}_h^0 \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\Big) \tag{4.18}$$
$$= \big(\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}, \boldsymbol{\theta}^{m+1}\big) + k\Big(\sum_{n=0}^{m+1} \nabla \mathbf{e}_{\tilde{\mathbf{u}}_h}^n, \nabla \boldsymbol{\theta}^{m+1}\Big)$$
$$+ \Big(\sum_{n=0}^{m} (\mathbf{B}(\tilde{\mathbf{u}}^n) - \mathbf{B}(\tilde{\mathbf{u}}_h^n))\Delta W_{n+1}, \mathcal{P}_h^0 \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\Big).$$

Setting $\phi_h = \sum_{n=0}^m \mathcal{P}_h^1 \varepsilon_{p_h}^n = \sum_{n=0}^m \varepsilon_{p_h}^n - \sum_{n=0}^m \xi^n$ in (4.17), where $\xi^n = p^n - \mathcal{P}_h^1 p^n$, we obtain

$$\Big(\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}, \sum_{n=0}^{m} \nabla \mathcal{P}_h^1 \varepsilon_{p_h}^n\Big) = k\Big(\nabla \varepsilon_{p_h}^{m+1}, \sum_{n=0}^{m} \nabla \mathcal{P}_h^1 \varepsilon_{p_h}^n\Big) \tag{4.19}$$

In addition, by using the properties of $\mathcal{P}_h^0-$ and $\mathcal{P}_h^1$-projection we have

$$k\Big(\sum_{n=0}^{m} \nabla \varepsilon_{p_h}^n, \mathcal{P}_h^0 \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\Big) = k\Big(\sum_{n=0}^{m} \nabla \varepsilon_{p_h}^n, \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\Big) - k\Big(\sum_{n=0}^{m} \nabla \varepsilon_{p_h}^n, \boldsymbol{\theta}^{m+1}\Big) \tag{4.20}$$
$$= k\Big(\sum_{n=0}^{m} \nabla \mathcal{P}_h^1 \varepsilon_{p_h}^n, \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\Big) + k\Big(\sum_{n=0}^{m} \nabla \xi^n, \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\Big) - k\Big(\sum_{n=0}^{m} \nabla \varepsilon_{p_h}^n, \boldsymbol{\theta}^{m+1}\Big)$$
$$= k^2\Big(\nabla \varepsilon_{p_h}^{m+1}, \sum_{n=0}^{m} \nabla \mathcal{P}_h^1 \varepsilon_{p_h}^n\Big) + k\Big(\sum_{n=0}^{m} \nabla \xi^n, \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\Big) - k\Big(\sum_{n=0}^{m} \nabla \varepsilon_{p_h}^n, \boldsymbol{\theta}^{m+1}\Big)$$
$$= k^2\Big(\nabla \varepsilon_{p_h}^{m+1}, \sum_{n=0}^{m+1} \nabla \mathcal{P}_h^1 \varepsilon_{p_h}^n\Big) - k^2\big(\nabla \varepsilon_{p_h}^{m+1}, \nabla \mathcal{P}_h^1 \varepsilon_{p_h}^{m+1}\big)$$
$$+ k\Big(\sum_{n=0}^{m} \nabla \xi^n, \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\Big) - k\Big(\sum_{n=0}^{m} \nabla \varepsilon_{p_h}^n, \boldsymbol{\theta}^{m+1}\Big)$$
$$= k^2\Big(\nabla \varepsilon_{p_h}^{m+1}, \sum_{n=0}^{m+1} \nabla \varepsilon_{p_h}^n\Big) - k^2\Big(\nabla \varepsilon_{p_h}^{m+1}, \sum_{n=0}^{m+1} \nabla \xi^n\Big)$$
$$- k^2\big(\nabla \varepsilon_{p_h}^{m+1}, \nabla \mathcal{P}_h^1 \varepsilon_{p_h}^{m+1}\big) + k\Big(\sum_{n=0}^{m} \nabla \xi^n, \mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\Big) - k\Big(\sum_{n=0}^{m} \nabla \varepsilon_{p_h}^n, \boldsymbol{\theta}^{m+1}\Big).$$

Moreover, by using the orthogonality property of $\mathcal{P}_h^1$, we have

$$-k^2\big(\nabla \varepsilon_{p_h}^{m+1}, \nabla \mathcal{P}_h^1 \varepsilon_{p_h}^{m+1}\big) = -k^2\big(\nabla(\varepsilon_{p_h}^{m+1} - \mathcal{P}_h^1 \varepsilon_{p_h}), \nabla \mathcal{P}_h^1 \varepsilon_{p_h}^{m+1}\big) \tag{4.21}$$
$$+ k^2\|\nabla \mathcal{P}_h^1 \varepsilon_{p_h}^{m+1}\|^2 = k^2\|\nabla \mathcal{P}_h^1 \varepsilon_{p_h}^{m+1}\|^2,$$

which helps to reduce (4.20) into

$$k\Big(\sum_{n=0}^{m}\nabla\varepsilon_{p_h}^{n},\mathcal{P}_h^0\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\Big)=k^2\Big(\nabla\varepsilon_{p_h}^{m+1},\sum_{n=0}^{m+1}\nabla\varepsilon_{p_h}^{n}\Big)-k^2\Big(\nabla\varepsilon_{p_h}^{m+1},\sum_{n=0}^{m+1}\nabla\xi^{n}\Big)+k^2\|\nabla\mathcal{P}_h^1\varepsilon_{p_h}^{m+1}\|^2+k\Big(\sum_{n=0}^{m}\nabla\xi^{n},\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\Big)-k\Big(\sum_{n=0}^{m}\nabla\varepsilon_{p_h}^{n},\boldsymbol{\theta}^{m+1}\Big). \tag{4.22}$$

Substituting (4.22) into (4.18) and rearranging terms yield

$$\begin{aligned}&\|\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\|^2+k\Big(\sum_{n=0}^{m+1}\nabla\mathbf{e}_{\tilde{\mathbf{u}}_h}^{n},\nabla\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\Big)+k^2\Big(\nabla\varepsilon_{p_h}^{m+1},\sum_{n=0}^{m+1}\nabla\varepsilon_{p_h}^{n}\Big)\\&\quad+k^2\|\nabla\mathcal{P}_h^1\varepsilon_{p_h}^{m+1}\|^2\\&=k^2\Big(\nabla\varepsilon_{p_h}^{m+1},\sum_{n=0}^{m+1}\nabla\xi^{n}\Big)+k\Big(\operatorname{div}\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1},\sum_{n=0}^{m}\xi^{n}\Big)\\&\quad+k\Big(\sum_{n=0}^{m}\nabla\varepsilon_{p_h}^{n},\boldsymbol{\theta}^{m+1}\Big)+\big(\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1},\boldsymbol{\theta}^{m+1}\big)+k\Big(\sum_{n=0}^{m+1}\nabla\mathbf{e}_{\tilde{\mathbf{u}}_h}^{n},\nabla\boldsymbol{\theta}^{m+1}\Big)\\&\quad+\Big(\sum_{n=0}^{m}(\mathbf{B}(\tilde{\mathbf{u}}^n)-\mathbf{B}(\tilde{\mathbf{u}}_h^n))\Delta W_{n+1},\mathcal{P}_h^0\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\Big).\end{aligned} \tag{4.23}$$

Next, we use the identity $2a(a-b)=a^2-b^2+(a-b)^2$ to create telescoping sums on the left side of (4.23), then followed by taking the expectation and applying the summation $k\sum_{m=0}^{\ell}$ for $0\le\ell<M$ to get

$$\begin{aligned}&k\sum_{m=0}^{\ell}\mathbb{E}\big[\|\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\|^2\big]+\frac{1}{2}\mathbb{E}\Big[\Big\|k\sum_{n=0}^{\ell+1}\nabla\mathbf{e}_{\tilde{\mathbf{u}}_h}^{n}\Big\|^2\Big]+\frac{k^2}{2}\sum_{m=0}^{\ell}\mathbb{E}\Big[\|\nabla\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1}\|^2\Big]\\&\quad+\frac{k}{2}\mathbb{E}\Big[\Big\|k\sum_{n=0}^{\ell+1}\nabla\varepsilon_{p_h}^{n}\Big\|^2\Big]+\frac{k^3}{2}\sum_{m=0}^{\ell}\mathbb{E}\Big[\|\nabla\varepsilon_{p_h}^{m+1}\|^2+2\|\nabla\mathcal{P}_h^1\varepsilon_{p_h}^{m+1}\|^2\Big]\\&=\mathbb{E}\Big[k^3\sum_{m=0}^{\ell}\Big(\nabla\varepsilon_{p_h}^{m+1},\sum_{n=0}^{m+1}\nabla\xi^{n}\Big)\Big]+\mathbb{E}\Big[k^2\sum_{m=0}^{\ell}\Big(\operatorname{div}\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1},\sum_{n=0}^{m}\xi^{n}\Big)\Big]\\&\quad+\mathbb{E}\Big[k^2\sum_{m=0}^{\ell}\Big(\sum_{n=0}^{m}\nabla\varepsilon_{p_h}^{n},\boldsymbol{\theta}^{m+1}\Big)\Big]+\mathbb{E}\Big[k\sum_{m=0}^{\ell}\big(\mathbf{e}_{\tilde{\mathbf{u}}_h}^{m+1},\boldsymbol{\theta}^{m+1}\big)\Big]\end{aligned} \tag{4.24}$$

$$
\begin{aligned}
&+\mathbb{E}\Big[k^2\sum_{m=0}^{\ell}\Big(\sum_{n=0}^{m+1}\nabla\mathbf{e}^n_{\tilde{\mathbf{u}}_h},\nabla\boldsymbol{\theta}^{m+1}\Big)\Big]\\
&+k\sum_{m=0}^{\ell}\mathbb{E}\Big[\Big(\sum_{n=0}^{m}(\mathbf{B}(\tilde{\mathbf{u}}^n)-\mathbf{B}(\tilde{\mathbf{u}}^n_h))\Delta W_{n+1},\mathcal{P}^0_h\mathbf{e}^{m+1}_{\tilde{\mathbf{u}}_h}\Big)\Big]\\
&:=\mathtt{I}+\cdots+\mathtt{VI}.
\end{aligned}
$$

It remains to bound each term on the right side of (4.24), we refer to the proof of [14, Theorem 4.2] for the details and only quote its final estimate below.

$$
\begin{aligned}
&\frac{1}{2}X^{\ell+1}+\frac{3}{8}\mathbb{E}\Big[\Big\|k\sum_{n=0}^{\ell+1}\nabla\mathbf{e}^n_{\tilde{\mathbf{u}}_h}\Big\|^2\Big]+\frac{k^2}{4}\sum_{m=0}^{\ell}\mathbb{E}\Big[\|\nabla\mathbf{e}^{m+1}_{\tilde{\mathbf{u}}_h}\|^2\Big]\\
&\quad+\frac{3k}{8}\mathbb{E}\Big[\Big\|k\sum_{n=0}^{\ell+1}\nabla\varepsilon^n_{p_h}\Big\|^2\Big]+\frac{3k^3}{8}\sum_{m=0}^{\ell}\mathbb{E}\big[\|\nabla\varepsilon^{m+1}_{p_h}\|^2\big]\\
&\quad+k^3\sum_{m=0}^{\ell}\mathbb{E}\big[\|\nabla\mathcal{P}^1_h\varepsilon^{m+1}_{p_h}\|^2\big]\\
&\le Ck^{\frac{1}{2}}+\frac{Ch^2}{k}+Ck\sum_{m=0}^{\ell}X^m,
\end{aligned}
\tag{4.25}
$$

where $X^\ell=k\sum_{m=0}^{\ell}\mathbb{E}\big[\|\mathbf{e}^m_{\tilde{\mathbf{u}}_h}\|^2\big]$. The desired error estimate (4.14) then follows from an application of the discrete Gronwall inequality to (4.25). The proof is complete. □

Next, we state an error estimate result for the pressure approximation generated by Algorithm 3 in a time-averaged fashion. Recall that an important advantage of Chorin-type schemes is to allow the use of a pair of independent finite element spaces which are not required to satisfy a discrete inf-sup condition, a price for this advantage is to make error estimates for the pressure approximations become more complicated even in the deterministic case. The idea for circumventing the difficulty is to utilize the following perturbed inf-sup inequality (cf. [17]): there exists $\delta>0$ independent of $h>0$, such that

$$
\frac{1}{\delta^2}\|q_h\|^2\le\sup_{\mathbf{v}_h\in\mathbf{H}_h}\frac{|(q_h,\operatorname{div}\mathbf{v}_h)|^2}{\|\nabla\mathbf{v}_h\|^2}+h^2\|\nabla q_h\|^2\qquad\forall q_h\in S_h\,,
\tag{4.26}
$$

which was also used in [13] to derive an error estimate for a pressure-stabilization scheme for (1.1).

Theorem 4.3 *Under the assumptions of Theorem 4.2, there exists a positive constant* $C\equiv C(D_T,\mathbf{u}_0,\mathbf{f},\delta)$ *such that*

$$
\Big(\mathbb{E}\Big[k\sum_{m=0}^{M}\Big\|k\sum_{m=1}^{m}\big(p^n-p^n_h\big)\Big\|^2\Big]\Big)^{\frac{1}{2}}\le C\Big(k^{\frac{1}{4}}+hk^{-\frac{1}{2}}\Big),
\tag{4.27}
$$

Proof We reuse all the notations from the proof of Theorem 4.2. First, from the error equations (4.16) we have

$$\Big(k\sum_{n=0}^{m}\varepsilon^n_{p_h}, \operatorname{div}\mathbf{v}_h\Big) = \big(\mathbf{e}^{m+1}_{\tilde{\mathbf{u}}_h}, \mathbf{v}_h\big) + \Big(k\sum_{n=0}^{m+1}\nabla\mathbf{e}^n_{\tilde{\mathbf{u}}_h}, \nabla\mathbf{v}_h\Big) - \Big(\sum_{n=0}^{m}(\mathbf{B}(\tilde{\mathbf{u}}^n) - \mathbf{B}(\tilde{\mathbf{u}}^n_h))\Delta W_{n+1}, \mathbf{v}_h\Big)\ \forall\mathbf{v}_h \in \mathbf{H}_h. \tag{4.28}$$

Using the Schwarz inequality on the right-hand side of (4.28) yields

$$\Big|\Big(k\sum_{n=0}^{m}\varepsilon^n_{p_h}, \operatorname{div}\mathbf{v}_h\Big)\Big| = C\|\mathbf{e}^{m+1}_{\tilde{\mathbf{u}}_h}\|\|\nabla\mathbf{v}_h\| + \Big\|k\sum_{n=0}^{m+1}\nabla\mathbf{e}^n_{\tilde{\mathbf{u}}_h}\Big\|\|\nabla\mathbf{v}_h\| + C\Big\|\sum_{n=0}^{m}(\mathbf{B}(\tilde{\mathbf{u}}^n) - \mathbf{B}(\tilde{\mathbf{u}}^n_h))\Delta W_{n+1}\Big\|\|\nabla\mathbf{v}_h\|. \tag{4.29}$$

Next, using (4.26) we conclude that

$$\begin{aligned}\frac{1}{\delta^2}\Big\|k\sum_{n=0}^{m}\varepsilon^n_{p_h}\Big\|^2 - h^2\Big\|k\sum_{n=0}^{m}\nabla\varepsilon^n_{p_h}\Big\|^2 &\le \sup_{\mathbf{v}_h\in\mathbf{H}_h}\frac{\Big|\Big(k\sum_{n=0}^{m}\varepsilon^n_{p_h}, \operatorname{div}\mathbf{v}_h\Big)\Big|^2}{\|\nabla\mathbf{v}_h\|^2} \\ &\le C\|\mathbf{e}^{m+1}_{\tilde{\mathbf{u}}_h}\|^2 + C\Big\|k\sum_{n=0}^{m+1}\nabla\mathbf{e}^n_{\tilde{\mathbf{u}}_h}\Big\|^2 \\ &\quad + C\Big\|\sum_{n=0}^{m}(\mathbf{B}(\tilde{\mathbf{u}}^n) - \mathbf{B}(\tilde{\mathbf{u}}^n_h))\Delta W_{n+1}\Big\|^2.\end{aligned} \tag{4.30}$$

Then, applying operators $k\sum_{m=0}^{\ell}$ and $\mathbb{E}[\cdot]$ on both sides we obtain

$$\begin{aligned}\frac{1}{\delta^2}\mathbb{E}\Big[k\sum_{m=0}^{\ell}\Big\|k\sum_{n=0}^{m}\varepsilon^n_{p_h}\Big\|^2\Big] &\le h^2\mathbb{E}\Big[k\sum_{m=0}^{\ell}\Big\|k\sum_{n=0}^{m}\nabla\varepsilon^n_{p_h}\Big\|^2\Big] \\ &\quad + Ck\sum_{m=0}^{\ell}\mathbb{E}\big[\|\mathbf{e}^{m+1}_{\tilde{\mathbf{u}}_h}\|^2\big] \\ &\quad + Ck\sum_{m=0}^{\ell}\mathbb{E}\Big[\Big\|k\sum_{n=0}^{m+1}\nabla\mathbf{e}^n_{\tilde{\mathbf{u}}_h}\Big\|^2\Big] \\ &\quad + Ck\sum_{m=0}^{\ell}\mathbb{E}\Big[\Big\|\sum_{n=0}^{m}(\mathbf{B}(\tilde{\mathbf{u}}^n) - \mathbf{B}(\tilde{\mathbf{u}}^n_h))\Delta W_{n+1}\Big\|^2\Big] \\ &:= \mathtt{I} + \mathtt{II} + \mathtt{III} + \mathtt{IV}.\end{aligned} \tag{4.31}$$

After bounding each terms on the right-hand side of (4.31) (see [14, Theorem 4.3] for the details) by using Theorem 4.2, we obtain the desired estimate.. □

We are now ready to state the following global error estimate theorem for Algorithm 3 which is a main result of this paper.

Theorem 4.4 *Under the assumptions of Theorems 3.2, 3.3 and Theorems 4.2 and 4.3, there hold the following error estimates:*

$$\begin{aligned}\Big(\mathbb{E}\Big[k\sum_{m=0}^{M}\|\mathbf{u}(t_m)-\tilde{\mathbf{u}}_h^m\|^2\Big]\Big)^{\frac{1}{2}}+\max_{0\le\ell\le M}\Big(\mathbb{E}\Big[\Big\|k\sum_{m=0}^{\ell}\nabla(\mathbf{u}(t_m)-\tilde{\mathbf{u}}_h^m)\Big\|^2\Big]\Big)^{\frac{1}{2}} \\ \le C\Big(k^{\frac{1}{4}}+hk^{-\frac{1}{2}}\Big),\end{aligned} \tag{4.32}$$

$$\Big(\mathbb{E}\Big[k\sum_{m=0}^{M}\Big\|P(t_m)-k\sum_{n=0}^{m}p_h^n\Big\|^2\Big]\Big)^{\frac{1}{2}}\le C\Big(k^{\frac{1}{4}}+hk^{-\frac{1}{2}}\Big), \tag{4.33}$$

where $C\equiv C(D_T,\mathbf{u}_0,\mathbf{f},\beta,\delta)$ *is positive constant independent of k and h.*

4.2 Finite element methods for the modified Chorin scheme

In this subsection, we first formulate a finite element spatial discretization for Algorithm 2 and then present a complete convergence analysis by deriving error estimates which are stronger than those obtained above for the standard Chorin scheme.

Algorithm 4 Let $m\ge 0$. Set $\tilde{\mathbf{u}}_h^0=\mathcal{P}_h^0\mathbf{u}_0$. For $m=0,1,2,\cdots$ do the following steps:
Step 1: For given $\tilde{\mathbf{u}}_h^m\in L^2(\Omega,\mathbf{H}_h)$, find $\xi_h^m\in L^2(\Omega,S_h)$ by solving the following Poisson problem: for $\mathbb{P}$-a.s.

$$\big(\nabla\xi_h^m,\nabla\phi_h\big)=\big(\mathbf{B}(\tilde{\mathbf{u}}_h^m),\nabla\phi_h\big)\qquad\forall\phi_h\in S_h. \tag{4.34}$$

Step 2: Set $\boldsymbol{\eta}_{\tilde{\mathbf{u}}_h}^m=\mathbf{B}(\tilde{\mathbf{u}}_h^m)-\nabla\xi_h^m$. For given $\mathbf{u}_h^m\in L^2(\Omega,\mathbf{H}_h)$ and $\tilde{\mathbf{u}}_h^m\in L^2(\Omega,\mathbf{H}_h)$, find $\tilde{\mathbf{u}}_h^{m+1}\in L^2(\Omega,\mathbf{H}_h)$ by solving the following problem: for $\mathbb{P}$-a.s.

$$\begin{aligned}&\big(\tilde{\mathbf{u}}_h^{m+1},\mathbf{v}_h\big)+k\big(\nabla\tilde{\mathbf{u}}_h^{m+1},\nabla\mathbf{v}_h\big)\\&\quad=\big(\mathbf{u}_h^m,\mathbf{v}_h\big)+k\big(\mathbf{f}^{m+1},\mathbf{v}_h\big)+\big(\boldsymbol{\eta}_{\tilde{\mathbf{u}}_h}^m\Delta W_{m+1},\mathbf{v}_h\big)\qquad\forall\,\mathbf{v}_h\in\mathbf{H}_h.\end{aligned} \tag{4.35}$$

Step 3: Find $r_h^{m+1}\in L^2(\Omega,L_h)$ by solving the following Poisson problem: for $\mathbb{P}$-a.s.

$$\big(\nabla r_h^{m+1},\nabla\phi_h\big)=\frac{1}{k}\big(\tilde{\mathbf{u}}_h^{m+1},\,\nabla\phi_h\big)\qquad\forall\phi_h\in L_h. \tag{4.36}$$

Step 4: Define $\mathbf{u}_h^{m+1}\in L^2(\Omega,\mathbf{H}_h)$ by

$$\mathbf{u}_h^{m+1}=\tilde{\mathbf{u}}_h^{m+1}-k\nabla r_h^{m+1}. \tag{4.37}$$

Step 5: Define $p_h^{m+1} \in L^2(\Omega, L_h)$ by

$$p_h^{m+1} = r_h^{m+1} + \frac{1}{k}\xi_h^m \Delta W_{m+1}. \tag{4.38}$$

Since each step involves a coercive problem, hence, Algorithm 4 is well defined. The next theorem establishes a convergence rate for the finite element approximation of the velocity field. Since the proof follows the same lines as those in the proof of Theorem 4.2, we omit it to save space.

Theorem 4.5 *Let $\{\tilde{\mathbf{u}}^m\}_{m=0}^M$ and $\{\tilde{\mathbf{u}}_h^m\}_{m=0}^M$ be generated respectively by Algorithm 2 and 4.*

Then, there exists a constant $C \equiv C(D_T, \mathbf{u}_0, \mathbf{f}) > 0$ such that

$$\max_{1\le m\le M}\Big(\mathbb{E}\big[\|\tilde{\mathbf{u}}^m - \tilde{\mathbf{u}}_h^m\|^2\big]\Big)^{\frac{1}{2}} + \Big(\mathbb{E}\Big[k\sum_{m=1}^{M}\|\nabla(\tilde{\mathbf{u}}^m - \tilde{\mathbf{u}}_h^m)\|^2\Big]\Big)^{\frac{1}{2}} \le C\Big(\sqrt{k} + h + h^2 k^{-\frac{1}{2}}\Big). \tag{4.39}$$

In the next theorem, we establish an error estimate for the pressure approximation of the modified Chorin finite element method given by Algorithm 4.

Theorem 4.6 *Let $\{r^m\}_{m=1}^M$ and $\{r_h^m\}_{m=1}^M$ be generated respectively by Algorithm 2 and 4. Then, there exists a constant $C \equiv C(D_T, \mathbf{u}_0, \mathbf{f}, \delta) > 0$ such that*

$$\Big(\mathbb{E}\Big[\Big\|k\sum_{m=1}^{M}(r^m - r_h^m)\Big\|^2\Big]\Big)^{\frac{1}{2}} \le C\Big(\sqrt{k} + h + h^2 k^{-\frac{1}{2}}\Big). \tag{4.40}$$

Proof Let $\boldsymbol{\varepsilon}_{\tilde{\mathbf{u}}}^m = \tilde{\mathbf{u}}^m - \tilde{\mathbf{u}}_h^m$ and $\varepsilon_r^m = r^m - r_h^m$. It is easy to check that $(\boldsymbol{\varepsilon}_{\tilde{\mathbf{u}}}^m, \varepsilon_r^m)$ satisfies the following error equation:

$$\begin{aligned}\big(\boldsymbol{\varepsilon}_{\tilde{\mathbf{u}}}^{m+1} - \boldsymbol{\varepsilon}_{\tilde{\mathbf{u}}}^m, \mathbf{v}_h\big) + k\big(\nabla\boldsymbol{\varepsilon}_{\tilde{\mathbf{u}}}^{m+1}, \nabla\mathbf{v}_h\big) + k\big(\nabla\varepsilon_r^m, \mathbf{v}_h\big) \\ = \big((\boldsymbol{\eta}_{\tilde{\mathbf{u}}}^m - \boldsymbol{\eta}_{\tilde{\mathbf{u}}_h}^m)\Delta W_{m+1}, \mathbf{v}_h\big) \qquad \forall\, \mathbf{v}_h \in \mathbf{H}_h.\end{aligned} \tag{4.41}$$

Applying the summation operator $\sum_{m=0}^{\ell}$ $(0 \le \ell \le M-1)$ to (4.41) yields

$$\begin{aligned}\Big(k\sum_{m=0}^{\ell}\varepsilon_r^m, \operatorname{div}\mathbf{v}_h\Big) &= \big(\boldsymbol{\varepsilon}_{\tilde{\mathbf{u}}}^{\ell+1} - \boldsymbol{\varepsilon}_{\tilde{\mathbf{u}}}^0, \mathbf{v}_h\big) + \Big(k\sum_{m=0}^{\ell}\nabla\boldsymbol{\varepsilon}_{\tilde{\mathbf{u}}}^{m+1}, \nabla\mathbf{v}_h\Big) \\ &\quad - \Big(\sum_{m=0}^{\ell}(\boldsymbol{\eta}_{\tilde{\mathbf{u}}}^m - \boldsymbol{\eta}_{\tilde{\mathbf{u}}_h}^m)\Delta W_{m+1}, \mathbf{v}_h\Big).\end{aligned} \tag{4.42}$$

It remains to bound terms on the right-hand side, which we refer to the proof of [14, Theorem 4.6] for the details and only quote the final estimate below.

$$\left(k\sum_{m=0}^{\ell}\varepsilon_r^m, \operatorname{div}\mathbf{v}_h\right) \le C\mathbb{E}\left[\|\boldsymbol{\varepsilon}_{\tilde{\mathbf{u}}}^{\ell+1}\|^2\right] + \mathbb{E}\left[k\sum_{m=0}^{\ell}\|\nabla\boldsymbol{\varepsilon}_{\tilde{\mathbf{u}}}^{m+1}\|^2\right]$$
$$+ C\mathbb{E}\left[k\sum_{m=0}^{\ell}\|\boldsymbol{\varepsilon}_{\tilde{\mathbf{u}}}^{m}\|^2\right] + h^2\mathbb{E}\left[k\sum_{m=0}^{\ell}\|\nabla\varepsilon_r^m\|^2\right] + Ch^2.$$

The desired estimate (4.40) follows from an application of (4.26), Theorem 4.5 and using Step 3 of Algorithm 2 and 4. The proof is complete. □

Corollary 4.7 *Let $\{p^m\}_{m=1}^M$ and $\{p_h^m\}_{m=1}^M$ be generated respectively by Algorithm 1 and 2. Then, there exists a positive constant $C \equiv C(D_T, \mathbf{u}_0, \mathbf{f}, \delta)$ such that*

$$\left(\mathbb{E}\left[\left\|k\sum_{m=1}^{M}(p^m - p_h^m)\right\|^2\right]\right)^{\frac{1}{2}} \le C\left(\sqrt{k} + h + h^2k^{-\frac{1}{2}}\right). \tag{4.43}$$

Proof Since the proof follows the same lines as those of the proof for Corollary 3.13, we only highlight the main steps. By definition of $\{p^m\}$ and $\{p_h^m\}$, we have

$$\left\|k\sum_{m=1}^{M}(p^m - p_h^m)\right\| \le \left\|k\sum_{m=1}^{M}(r^m - r_h^m)\right\| + \left\|\sum_{m=1}^{M}(\xi_{\tilde{\mathbf{u}}}^m - \xi_{\tilde{\mathbf{u}}_h}^m)\Delta W_{m+1}\right\| \tag{4.44}$$
$$=: \mathtt{I} + \mathtt{II}.$$

Term $\mathtt{I}$ can be bounded using Theorem 4.6. To bound $\mathtt{II}$, by Itô isometry, (3.38) and (4.34), Poincaré inequality, Lipschitz continuity of $\mathbf{B}$, and Theorem 4.5, we get

$$\mathbb{E}[\mathtt{II}]^2 = \mathbb{E}\left[k\sum_{m=1}^{M}\|\xi^m - \xi_h^m\|^2\right] \le C\left(k + h^2 + \frac{h^4}{k}\right).$$

The proof is complete. □

We conclude this section by stating the following global error estimate theorem for Algorithm 4, which is another main result of this paper.

Theorem 4.8 *Let $(\mathbf{u}, P)$ be the solution of (1.1) and $\{(\tilde{\mathbf{u}}_h^m, r_h^m, p_h^m)\}_{m=1}^M$ be the solution of Algorithm 4. Then, there exists a constant $C \equiv C(D_T, \mathbf{u}_0, \mathbf{f}, \beta, \delta) > 0$ such that*

$$\max_{1\le m\le M}\left(\mathbb{E}\left[\|\mathbf{u}(t_m) - \tilde{\mathbf{u}}_h^m\|^2\right]\right)^{\frac{1}{2}} + \left(\mathbb{E}\left[k\sum_{m=1}^{M}\|u(t_m) - \tilde{u}_h^m\|^2\right]\right)^{\frac{1}{2}}$$
$$\le C\left(\sqrt{k} + h + h^2k^{-\frac{1}{2}}\right),$$

$$\left(\mathbb{E}\left[\left\|R(t_m) - k\sum_{n=1}^{m} r_h^n\right\|^2\right]\right)^{\frac{1}{2}} + \left(\mathbb{E}\left[\left\|P(t_m) - k\sum_{n=1}^{m} p_h^n\right\|^2\right]\right)^{\frac{1}{2}} \\ \leq C\left(\sqrt{k} + h + h^2 k^{-\frac{1}{2}}\right).$$

Remark 4.1 The above error estimates are of the same nature as those obtained in [12] for the standard Euler-Maruyama mixed finite element method. On the other hand, the error estimates obtained in [13] for the Helmholtz enhanced Euler-Maruyama mixed finite element method do not have the growth term $h^2 k^{-\frac{1}{2}}$.

5 Numerical experiments

In this section, we present two 2-D numerical tests to guage the performance of the proposed numerical methods/algorithms. The first test is to verify the convergent rates proved in Theorem 4.4 for Algorithm 3 while the second test is to validate the convergent rates proved in Theorem 4.8.

In both tests the computational domain is chosen as $D = (0, 1) \times (0, 1)$, the $P_1 - P_1$ equal-order pair of finite element spaces are used for spatial discretization, the constant source function $\mathbf{f} = (1, 1)$ is applied, the terminal time is $T = 1$, the fine time and space mesh sizes $k_0 = \frac{1}{4096}$ and $h = \frac{1}{50}$ are used to compute the numerical true solution, and the number of realizations is set as $N_p = 500$ for the first test and $N_p = 800$ for the second one. Moreover, to evaluate errors in strong norms, we use the following numerical integration formulas: for any $0 \leq m \leq M$

$$\mathcal{E}_{\mathbf{u}}^m := \left(\mathbb{E}\left[\|\mathbf{u}(t_m) - \mathbf{u}_h^m(k)\|^2\right]\right)^{\frac{1}{2}} \approx \left(\frac{1}{N_p}\sum_{\ell=1}^{N_p}\left\|\mathbf{u}_h^m(k_0, \omega_\ell) - \mathbf{u}_h^m(k, \omega_\ell)\right\|^2\right)^{\frac{1}{2}},$$

$$\mathcal{E}_{\mathbf{u},av}^M := \left(\mathbb{E}\left[k\sum_{m=0}^{M}\|\mathbf{u}(t_m) - \mathbf{u}_h^m(k)\|^2\right]\right)^{\frac{1}{2}} \\ \approx \left(\frac{1}{N_p}\sum_{\ell=1}^{N_p}\left(k\sum_{m=0}^{M}\left\|\mathbf{u}_h^m(k_0, \omega_\ell) - \mathbf{u}_h^m(k, \omega_\ell)\right\|^2\right)\right)^{\frac{1}{2}},$$

$$\mathcal{E}_{p,av}^M := \left(\mathbb{E}\left[k\sum_{m=0}^{M}\left\|P(t_m) - k\sum_{n=0}^{m} p_h^n(k)\right\|^2\right]\right)^{\frac{1}{2}} \\ \approx \left(\frac{1}{N_p}\sum_{\ell=1}^{N_p}\left(k\sum_{m=0}^{M}\left\|k_0\sum_{n=1}^{\frac{t_m}{k_0}} p_h^n(k_0, \omega_\ell) - k\sum_{n=1}^{\frac{t_m}{k}} p_h^n(k, \omega_\ell)\right\|^2\right)\right)^{\frac{1}{2}}.$$

Test 1 In this test, the nonlinear multiplicative noise function $\mathbf{B}$ is chosen as $\mathbf{B}(\mathbf{u}) = 10\big((u_1^2 + 1)^{\frac{1}{2}}, (u_2^2 + 1)^{\frac{1}{2}}\big)$ and the initial value $\mathbf{u}_0 = (0, 0)^\top$. Moreover, we choose $\mathbb{R}^J$-valued Wiener process W with increments satisfying

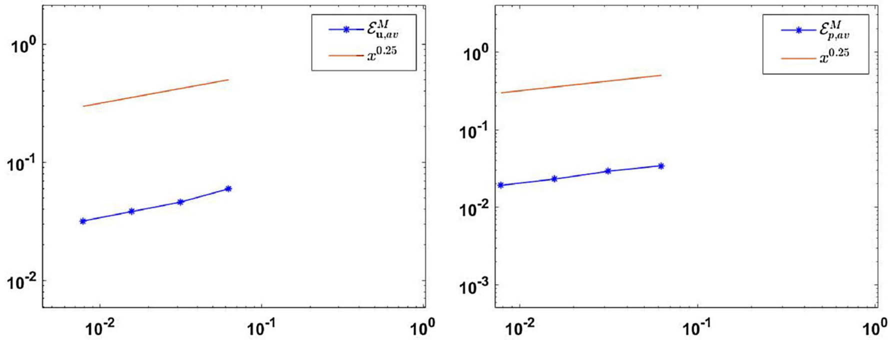

Fig. 1 Convergence rates of the time discretization for the velocity (left) and pressure (right) approximations by Algorithm 3 in the $\boldsymbol{\mathcal{E}}^{M}_{\mathbf{u},av}$ norm and $\mathcal{E}^{M}_{p,av}$ norm respectively

$$\Delta W_{m+1} = W^J(t_{m+1}, \mathbf{x}) - W^J(t_m, \mathbf{x}) = k_0 \sum_{j,\ell=0}^{J} \sqrt{\lambda_{j,\ell}}\, e_{j,\ell}(\mathbf{x}) \beta^m_{j,\ell}, \tag{5.1}$$

where $\mathbf{x} = (x_1, x_2) \in D$, $\beta^m_{j,\ell} \sim N(0, 1)$ and $\{e_{j,\ell}(\mathbf{x})\}_{j,\ell}$ are orthonormal functions defined by $e_{j,\ell}(\mathbf{x}) = g_{j,\ell}\|g_{j,\ell}\|^{-1}$ with

$$g_{j,\ell}(x_1, x_2) = \sin(j\pi x_1)\sin(\ell\pi x_2) \tag{5.2}$$

and $\lambda_{j,\ell} = \frac{1}{(j+\ell)^2}\|g_{j,\ell}\|$. In this test, we set $J = 2$, $\nu = 1$.

Figure 1 displays the convergence rates of the time discretization produced by Algorithm 3 (and Algorithm 1) using different time step size k. The left figure shows the convergence rate $O(k^{\frac{1}{4}})$ in the $\boldsymbol{\mathcal{E}}^{M}_{\mathbf{u},av}$-norm for the velocity approximation, while the right graph shows the same convergence rate in the $\mathcal{E}^{M}_{p,av}$-norm for the pressure approximation, both match the theoretical rates proved in our theoretical error estimates.

Next, we want to verify that the dependence of the error bounds on the factor $k^{-\frac{1}{2}}$ is valid. To the end, we fix $h = \frac{1}{20}$ and use again different time step size k. The numerical results in Fig. 2 shows that the errors for both the velocity and pressure approximations increase as the time step size decreases, which proves that the error bounds are indeed proportional to some negative power of k.

To verify the sharpness of the error bounds on the factor $k^{-\frac{1}{2}}$, we implement Algorithm 3 using different pairs (k, h), which satisfy the relation $h \approx k$, and display the numerical results in Fig. 3. We observe $\frac{1}{4}$ order convergence rate for both the velocity and pressure approximations as predicted in Theorem 4.4.

Test 2 We use the same test problem as in **Test** 1 to validate the theoretical error estimates for our modified Chorin scheme given by Algorithm 4. However, the nonlinear multiplicative noise functions is chosen as $\mathbf{B}(\mathbf{u}) = \big((u_1^2 + 1)^{\frac{1}{2}}, (u_2^2 + 1)^{\frac{1}{2}}\big)$. It should be noted that a similar numerical experiment was done in [8]. However, only the velocity approximation was analyzed and tested, no convergent rate for the pressure

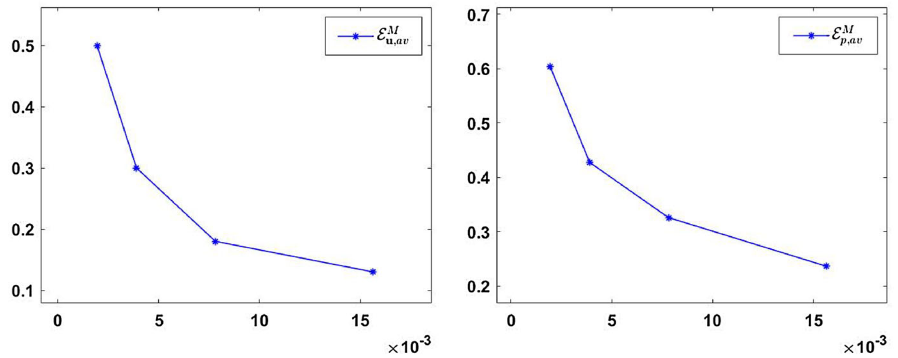

Fig. 2 Errors the velocity approximation (left) in $\mathcal{E}^M_{\mathbf{u},av}$ norm and the pressure approximation (right) in $\mathcal{E}^M_{p,av}$ norm by Algorithm 3

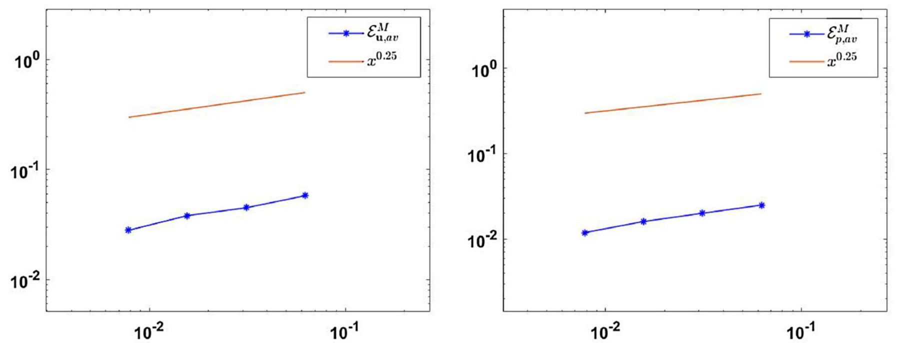

Fig. 3 Convergence rates in the $\mathcal{E}^M_{\mathbf{u},av}$ norm for the velocity (left) approximation and the $\mathcal{E}^M_{p,av}$ norm for the pressure (right) approximation by Algorithm 3 under the mesh condition $h \approx k$

approximation was proved or tested in [8]. Here we want to emphasize the optimal convergence rate for the pressure approximation in the time-averaged norm.

Figure 4 displays the $\frac{1}{2}$ order convergence rate in time for both the velocity and pressure approximations by Algorithm 4 as predicted by Theorem 4.8. We note that the velocity error is measured in the strong norm and the pressure error is measured in a time-averaged norm.

Similar to **Test** 1, we want to test whether the dependence of the error bounds on the factor $k^{-\frac{1}{2}}$ is valid and sharp. To the end, we use the same strategy as we did in **Test** 1, namely, we fix mesh size $h = \frac{1}{20}$ and decrease time step size k. As expected, we observe that the errors blow up as shown in Fig. 5.

Finally, Fig. 6 shows the $\frac{1}{2}$ order convergence rate for both the velocity and pressure approximations by Algorithm 4 when the time step size k and the space mesh size h satisfy the balancing condition $h \approx \sqrt{k}$, which verifies the sharpness of the dependence of the error bounds on on the factor $k^{-\frac{1}{2}}$ as predicted by Theorem 4.8.

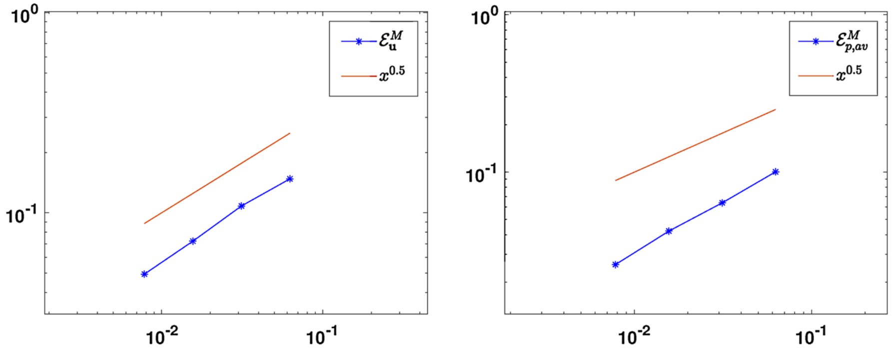

Fig. 4 Convergence rates of the time discretization for the velocity in strong norm (left) and pressure in time-averaged norm (right) by Algorithm 4

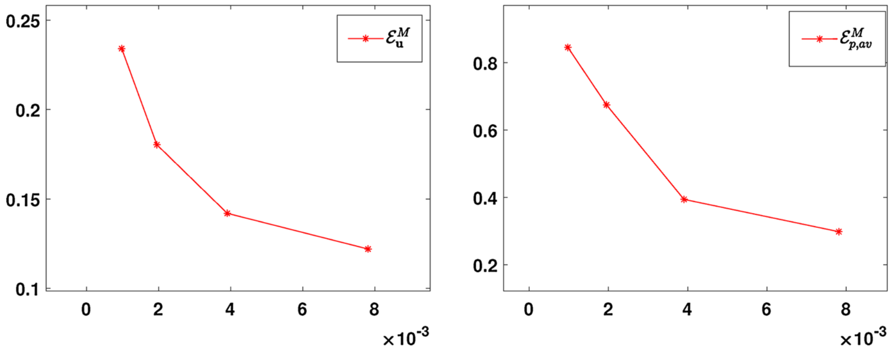

Fig. 5 Errors for the velocity approximation in strong norm (left) and pressure approximation in time-averaged norm (right) by Algorithm 4

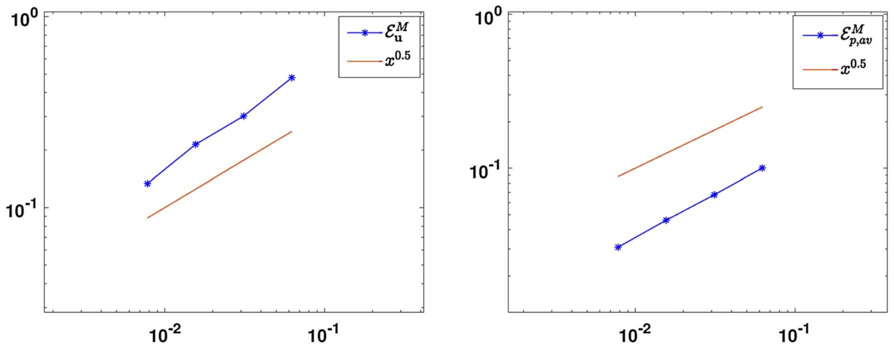

Fig. 6 Convergence rates for the velocity approximation in strong norm (left) and pressure approximation in time-averaged norm (right) under the mesh condition $h \approx \sqrt{k}$

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Author Contributions The authors contributed equally to this work.

Data Availability All datasets generated during the current study are available from the corresponding author upon reasonable request.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Code availability Available upon reasonable request.

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