

# Proportionate Adaptive Filters Based on Minimizing Diversity Measures for Promoting Sparsity

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**Abstract**—In this paper, a novel way of deriving proportionate adaptive filters is proposed based on diversity measure minimization using the iterative reweighting techniques well-known in the sparse signal recovery (SSR) area. The resulting least mean square (LMS)-type and normalized LMS (NLMS)-type sparse adaptive filtering algorithms can incorporate various diversity measures that have proved effective in SSR. Furthermore, by setting the regularization coefficient of the diversity measure term to zero in the resulting algorithms, Sparsity promoting LMS (SLMS) and Sparsity promoting NLMS (SNLMS) are introduced, which exploit but do not strictly enforce the sparsity of the system response if it already exists. Moreover, unlike most existing proportionate algorithms that design the step-size control factors based on heuristics, our SSR-based framework leads to designing the factors in a more systematic way. Simulation results are presented to demonstrate the convergence behavior of the derived algorithms for systems with different sparsity levels.

**Index Terms**—adaptive filtering, proportionate adaptation, sparse system identification, diversity measure minimization, iterative reweighting

## I. INTRODUCTION

In many applications of adaptive filters, the impulse responses (IRs) to be identified are often sparse or compressible (quasi-sparse), i.e., only a small percentage of the IR components have a significant magnitude while the rest are zero or small. Examples include network and acoustic echo cancellation [1]–[3], acoustic feedback control in hearing aids [4], [5], etc. Therefore, designing adaptive filters that can exploit the structural sparsity for performance improvement over the conventional approaches such as the least mean square (LMS) and the normalized LMS (NLMS) [6]–[9] has been an area of great interest over the past few decades.

An early and influential work on identifying sparse IRs is the proportionate NLMS (PNLMS) algorithm proposed by Duttweiler [1]. The PNLMS algorithm was developed in an intuitive way, i.e., the equations used to calculate the step-size control factors were not based on any optimization criterion but were based on good heuristics. The main idea behind the approach is to update each filter coefficient using a step size proportional to the magnitude of the estimated coefficient. Variants that also utilize this concept of proportionate adaptation were later proposed and [10] provides a good summary.

The recent progress on sparse signal recovery (SSR) has led to a number of computational algorithms [11], [12]. Inspired by the batch estimation techniques in SSR, methods have been

proposed for theoretically justifying the formulation of the proportionate adaptation scheme in adaptive filters [13], [14], or even obtaining a general adaptive filtering framework that incorporates sparsity [15]–[17]. Another class of algorithms, i.e., the family of sparsity regularization-based approaches [18]–[22], has also been proposed by adding a sparsity-inducing penalty to the ordinary LMS objective function.

In this paper, we build on this trend of drawing inspiration from SSR, and propose a novel way of deriving proportionate adaptive filters based on minimizing diversity measures using the well-known iterative reweighting methods [23]. The resulting algorithms can incorporate flexible diversity measures that have proved effective in SSR. Furthermore, by setting the regularization coefficient of the diversity measure term to zero, we introduce the Sparsity promoting LMS (SLMS) and Sparsity promoting NLMS (SNLMS) that exploit but do not strictly enforce the sparsity of the system response if it already exists. The proposed SSR-based framework leads to designing the step-size control factors for proportionate adaptation in a more systematic way, as opposed to most existing proportionate algorithms that design the factors based on heuristics [1], [10]. Simulation results are presented to demonstrate the convergence behavior of the derived algorithms for systems with different levels of sparsity.

## II. BACKGROUND

### A. Adaptive Filters for System Identification

Let  $\mathbf{h}_n = [h_{0,n}, h_{1,n}, \dots, h_{M-1,n}]^T$  denote the adaptive filter of length  $M$  at discrete time instant  $n$ . Assume the IR of the underlying system is  $\mathbf{h}^o = [h_0^o, h_1^o, \dots, h_{M-1}^o]^T$ , and the model for the observed or desired signal is  $d_n = \mathbf{u}_n^T \mathbf{h}^o + v_n$ , where  $\mathbf{u}_n = [u_n, u_{n-1}, \dots, u_{n-M+1}]^T$  is the vector containing the  $M$  most recent samples of the input signal  $u_n$  and  $v_n$  is an additive noise signal. The output of the adaptive filter  $\mathbf{u}_n^T \mathbf{h}_n$  is subtracted from  $d_n$  to obtain the error signal  $e_n = d_n - \mathbf{u}_n^T \mathbf{h}_n$ . The goal in general is to continuously adjust the coefficients of  $\mathbf{h}_n$  such that eventually  $\mathbf{h}_n = \mathbf{h}^o$ ; i.e., to identify the unknown system.

The classic LMS and NLMS algorithms can be derived from the following unconstrained optimization problem using an

objective function  $J_n(\mathbf{h})$  of the instantaneous error<sup>1</sup> [6]:

$$\min_{\mathbf{h}} J_n(\mathbf{h}) \triangleq e_n^2 = \left(d_n - \mathbf{u}_n^T \mathbf{h}\right)^2. \quad (1)$$

Applying the stochastic gradient descent which utilizes a first order approximation results in the LMS [7]:

$$\mathbf{h}_{n+1} = \mathbf{h}_n - \frac{\mu}{2} \nabla_{\mathbf{h}} J_n(\mathbf{h}_n) = \mathbf{h}_n + \mu \mathbf{u}_n e_n, \quad (2)$$

where  $\nabla_{\mathbf{h}}$  denotes the gradient operator with respect to  $\mathbf{h}$  and  $\mu > 0$  is the step size.

Applying the stochastic regularized Newton's method which utilizes a second order approximation gives the NLMS [8]:

$$\begin{aligned} \mathbf{h}_{n+1} &= \mathbf{h}_n - \mu \left( \nabla_{\mathbf{h}}^2 J_n(\mathbf{h}_n) + 2\delta \mathbf{I} \right)^{-1} \nabla_{\mathbf{h}} J_n(\mathbf{h}_n) \\ &= \mathbf{h}_n + \frac{\mu \mathbf{u}_n e_n}{\mathbf{u}_n^T \mathbf{u}_n + \delta}, \end{aligned} \quad (3)$$

where  $\nabla_{\mathbf{h}}^2$  denotes the Hessian operator with respect to  $\mathbf{h}$ ,  $\mathbf{I}$  denotes the identity matrix, and  $\delta > 0$  is a small regularization constant for preventing singularity.

### B. Diversity Measure Minimization for SSR

The concept of SSR is to search for sparse solutions to an underdetermined system of linear equations  $\mathbf{y} = \mathbf{A}\mathbf{x}$ , where  $\mathbf{A} \in \mathbb{R}^{N \times M}$  represents an overcomplete dictionary with  $\text{rank}(\mathbf{A}) = N$  and  $N < M$ ,  $\mathbf{x} \in \mathbb{R}^M$  is the underlying sparse representation, and  $\mathbf{y} \in \mathbb{R}^N$  is the measurement vector. A popular approach is to consider the following diversity measure minimization problem:

$$\min_{\mathbf{x}} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + \lambda G(\mathbf{x}), \quad \lambda > 0, \quad (4)$$

where  $G(\cdot)$  is the *general diversity measure* weighted by  $\lambda$  that induces sparsity in its argument. We further define a *separable* diversity measure that has the form  $G(\mathbf{x}) = \sum_{i=0}^{M-1} g(x_i)$ , where  $g(\cdot)$  has the following properties:

**Property 1:**  $g(t)$  is symmetric, i.e.,  $g(t) = g(-t) = g(|t|)$ ;

**Property 2:**  $g(|t|)$  is monotonically increasing with  $|t|$ ;

**Property 3:**  $g(0)$  is finite;

**Property 4:**  $g(t)$  is strictly concave in  $|t|$  or  $t^2$ .

The iterative reweighting methods [23] are popular techniques for solving (4). By introducing a weighted  $\ell_2$  [24], [25] or  $\ell_1$  [26] norm term as an upper bound for  $G(\mathbf{x})$  in each iteration, they form and solve for a new optimization problem accordingly to approach the optimal solution [23]. We briefly review the reweighted  $\ell_2$  method here as we will be using it for deriving adaptive filtering algorithms that incorporate sparsity.

To apply the iterative reweighted  $\ell_2$  approach, first note that the function  $g(t)$  has to be concave in  $t^2$  for Property 4; i.e., it satisfies  $g(t) = f(t^2)$ , where  $f(z)$  is concave for  $z \in \mathbb{R}_+$ .

<sup>1</sup>One can also derive the algorithms by using the expected value of the error function and then replacing the gradient/Hessian by the instantaneous gradient/Hessian as an estimate [6]–[8]. We utilize the direct approach because of the simplicity and to shorten the derivation.

Assume at the  $k$ -th iteration we have an estimate  $\mathbf{x}^{(k)}$ . Then the estimate of the next iteration  $k+1$  is given as:

$$\mathbf{x}^{(k+1)} = \arg \min_{\mathbf{x}} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + \lambda \left\| (\mathbf{W}^{(k)})^{-1} \mathbf{x} \right\|_2^2, \quad (5)$$

where  $\mathbf{W}^{(k)} = \text{diag}\{w_i^{(k)}\}$  with

$$w_i^{(k)} = \left( \left. \frac{df(z)}{dz} \right|_{z=(x_i^{(k)})^2} \right)^{-\frac{1}{2}}, \quad (6)$$

and  $d$  denotes the differential operator. In each iteration  $k$ , the matrix  $\mathbf{W}^{(k)}$  provides a surrogate function as an upper bound for the objective function in (4).<sup>2</sup> Sequentially minimizing the surrogate functions allows the algorithm to produce more focal estimates as optimization progresses [23].

### III. INCORPORATING SPARSITY INTO ADAPTIVE FILTERS

To incorporate sparsity into the adaptive filtering framework, we propose to add the general diversity measure  $G(\mathbf{h}) = \sum_{i=0}^{M-1} g(h_i)$  to the ordinary objective function  $J_n(\mathbf{h})$  in (1) as:

$$\min_{\mathbf{h}} J_n(\mathbf{h}) + \lambda G(\mathbf{h}), \quad (7)$$

where  $\lambda$  is the regularization coefficient. Inspired by the conceptual similarity with SSR, we show that by upper bounding  $G(\mathbf{h})$  in (7) with a weighted  $\ell_2$  norm term, both LMS-type and NLMS-type sparse adaptive filters can be derived. Thus, instead of (7), we consider the following problem:

$$\min_{\mathbf{h}} J_n(\mathbf{h}) + \lambda \left\| \mathbf{W}_n^{-1} \mathbf{h} \right\|_2^2, \quad (8)$$

where  $\mathbf{W}_n = \text{diag}\{w_{i,n}\}$ <sup>3</sup> and each  $w_{i,n}$  is computed based on the current estimate  $h_{i,n}$ , depending on the choice of the diversity measure  $G(\cdot)$ . Observing the similarity between (5) and (8), we make the following correspondences:  $\mathbf{x}^{(k)} \leftrightarrow \mathbf{h}_n$  and  $\mathbf{W}^{(k)} \leftrightarrow \mathbf{W}_n$ . Then the relationship between (5) and (6) suggests the following update rule for  $\mathbf{W}_n$  in (8):

$$w_{i,n} = \left( \left. \frac{df(z)}{dz} \right|_{z=h_{i,n}^2} \right)^{-\frac{1}{2}}, \quad (9)$$

where  $f(\cdot)$  is a function depending on the  $g(\cdot)$  used.

Recall that in (5) for SSR, in the  $k$ -th iteration the matrix  $\mathbf{W}^{(k)}$  is updated as a function of  $\mathbf{x}^{(k)}$  to create a new upper bound for  $G(\mathbf{x})$ . Similarly, in the adaptive filtering case here, we propose to utilize  $\mathbf{h}_n$  at time  $n$  for computing the matrix  $\mathbf{W}_n$  to form a new upper bound for  $G(\mathbf{h})$  accordingly. The whole concept, similar to the reweighting techniques in SSR, has now been applied to adaptive filtering where the upper bound evolves and adapts over time.

Before proceeding, we reparameterize the problem in terms of the scaled variable  $\mathbf{q}$ :

$$\mathbf{q} \triangleq \mathbf{W}_n^{-1} \mathbf{h}, \quad (10)$$

<sup>2</sup>Note that we will use a practical assumption that the diagonal matrix  $\mathbf{W}^{(k)}$  is positive definite at each iteration. This can be shown to hold for a wide variety of diversity measures used in SSR.

<sup>3</sup>Again, the positive definiteness of the diagonal matrix  $\mathbf{W}_n$  is assumed.

in which  $\mathbf{W}_n$  is used as the *scaling matrix*. This step, similarly utilized in [4], [13], [16], can be interpreted as performing the affine scaling transformation (AST) commonly employed by the interior point approach to solving linear and nonlinear programming problems [27]. In the optimization literature, AST-based methods transform the original problem into an equivalent one, in which the current point is favorably positioned at the center of the feasible region [28], thus expediting the optimization process.

Using (10) for the objective function in (8) and performing minimization with respect to  $\mathbf{q}$ , that is:

$$\min_{\mathbf{q}} J_n^{\ell_2}(\mathbf{q}) \triangleq J_n(\mathbf{W}_n \mathbf{q}) + \lambda \|\mathbf{q}\|_2^2, \quad (11)$$

then we transform the problem into a new form. To proceed, we define the *a posteriori* AST variable at time  $n$ :

$$\mathbf{q}_{n|n} \triangleq \mathbf{W}_n^{-1} \mathbf{h}_n \quad (12)$$

and the *a priori* AST variable at time  $n$ :

$$\mathbf{q}_{n+1|n} \triangleq \mathbf{W}_n^{-1} \mathbf{h}_{n+1}. \quad (13)$$

In the following, we show that by applying the stochastic gradient descent and regularized Newton's method to (11), along with using (12) and (13), both LMS-type and NLMS-type sparse adaptive filtering algorithms can be derived.

#### A. LMS-Type Sparse Adaptive Filtering Algorithm

We formulate a recursive update by using the stochastic gradient descent in the  $\mathbf{q}$  domain:

$$\mathbf{q}_{n+1|n} = \mathbf{q}_{n|n} - \frac{\mu}{2} \nabla_{\mathbf{q}} J_n^{\ell_2}(\mathbf{q}_{n|n}). \quad (14)$$

Using the chain rule, (10), and (12), we can show the gradient term:

$$\begin{aligned} \nabla_{\mathbf{q}} J_n^{\ell_2}(\mathbf{q}_{n|n}) &= \mathbf{W}_n \nabla_{\mathbf{h}} J_n(\mathbf{h}_n) + 2\lambda \mathbf{q}_{n|n} \\ &= -2\mathbf{W}_n \mathbf{u}_n e_n + 2\lambda \mathbf{q}_{n|n}. \end{aligned} \quad (15)$$

Substituting (15) into (14) leads to:

$$\mathbf{q}_{n+1|n} = (1 - \mu\lambda) \mathbf{q}_{n|n} + \mu \mathbf{W}_n \mathbf{u}_n e_n. \quad (16)$$

Multiplying both sides of (16) by  $\mathbf{W}_n$  and using the relationships (12) and (13), we will get back to the  $\mathbf{h}$  domain:

$$\boxed{\mathbf{h}_{n+1} = (1 - \mu\lambda) \mathbf{h}_n + \mu \mathbf{W}_n^2 \mathbf{u}_n e_n.} \quad (17)$$

This is the update rule of the *generalized LMS-type sparse adaptive filtering algorithm using reweighted  $\ell_2$* .

#### B. NLMS-Type Sparse Adaptive Filtering Algorithm

For the reweighted  $\ell_2$  problem (11) we can also consider the stochastic regularized Newton's method option to formulate the recursive update in the  $\mathbf{q}$  domain:

$$\mathbf{q}_{n+1|n} = \mathbf{q}_{n|n} - \mu \left( \nabla_{\mathbf{q}}^2 J_n^{\ell_2}(\mathbf{q}_{n|n}) + 2\delta \mathbf{I} \right)^{-1} \nabla_{\mathbf{q}} J_n^{\ell_2}(\mathbf{q}_{n|n}). \quad (18)$$

Using the chain rule, (10), and (12), we can show the Hessian term:

$$\begin{aligned} \nabla_{\mathbf{q}}^2 J_n^{\ell_2}(\mathbf{q}_{n|n}) &= \mathbf{W}_n \nabla_{\mathbf{h}}^2 J_n(\mathbf{h}_n) \mathbf{W}_n + 2\lambda \mathbf{I} \\ &= 2\mathbf{W}_n \mathbf{u}_n \mathbf{u}_n^T \mathbf{W}_n + 2\lambda \mathbf{I}. \end{aligned} \quad (19)$$

Substituting (19) and (15) into (18) results in:

$$\begin{aligned} \mathbf{q}_{n+1|n} &= \left( \mathbf{I} - \frac{\mu\lambda}{\lambda + \delta} \left[ \mathbf{I} - \frac{\mathbf{W}_n \mathbf{u}_n \mathbf{u}_n^T \mathbf{W}_n}{\mathbf{u}_n^T \mathbf{W}_n^2 \mathbf{u}_n + \lambda + \delta} \right] \right) \mathbf{q}_{n|n} \\ &\quad + \frac{\mu \mathbf{W}_n \mathbf{u}_n e_n}{\mathbf{u}_n^T \mathbf{W}_n^2 \mathbf{u}_n + \lambda + \delta}, \end{aligned} \quad (20)$$

where we have applied the matrix inversion lemma to simplify terms and avoid matrix inversion.

Multiplying both sides of (20) by  $\mathbf{W}_n$  and using the relationships (12) and (13), we will get back to the  $\mathbf{h}$  domain:

$$\boxed{\mathbf{h}_{n+1} = (\mathbf{I} - \mu\lambda \Phi_n) \mathbf{h}_n + \frac{\mu \mathbf{W}_n^2 \mathbf{u}_n e_n}{\mathbf{u}_n^T \mathbf{W}_n^2 \mathbf{u}_n + \lambda + \delta},} \quad (21)$$

where for simplicity we have let:

$$\Phi_n = \frac{1}{\lambda + \delta} \left[ \mathbf{I} - \frac{\mathbf{W}_n^2 \mathbf{u}_n \mathbf{u}_n^T}{\mathbf{u}_n^T \mathbf{W}_n^2 \mathbf{u}_n + \lambda + \delta} \right]. \quad (22)$$

This is the update rule of the *generalized NLMS-type sparse adaptive filtering algorithm using reweighted  $\ell_2$* .

#### C. Discussion

It is worth mentioning that there is considerable difference between the proposed algorithms derived from (8) and the existing SSR algorithms based on (5): the SSR techniques are batch estimation methods for recovering the underlying sparse representation, while the proposed algorithms are specifically tailored for the adaptive filtering scenario. We would also like to point out that it is not so straightforward to obtain the iteration schemes (17) and (21) if one does not consider the change of variables (10). This step, similar to the AST commonly employed in the optimization literature, is thus of great importance for obtaining the proposed algorithms. Note that the above procedure can be extended to other reweighting strategies, e.g., the reweighted  $\ell_1$  framework, for deriving a different class of algorithms.

### IV. SPARSITY PROMOTING ALGORITHMS

An interesting situation arises when we consider the limiting case of  $\lambda \rightarrow 0^+$ . For the algorithms (17) and (21), by setting  $\lambda = 0$  we see the terms with  $\lambda$  as a scaling factor vanish, leading to the following Sparsity promoting LMS (SLMS):

$$\boxed{\mathbf{h}_{n+1} = \mathbf{h}_n + \mu \mathbf{W}_n^2 \mathbf{u}_n e_n,} \quad (23)$$

and Sparsity promoting NLMS (SNLMS):

$$\boxed{\mathbf{h}_{n+1} = \mathbf{h}_n + \frac{\mu \mathbf{W}_n^2 \mathbf{u}_n e_n}{\mathbf{u}_n^T \mathbf{W}_n^2 \mathbf{u}_n + \delta}.} \quad (24)$$

Even with  $\lambda = 0$ , the SLMS and SNLMS still have a diagonal matrix term  $\mathbf{W}_n^2$  on the gradient to leverage sparsity. This indeed realizes proportionate adaptation similar to PNLMS-type algorithms. From the objective function perspective, (7) indicates a trade-off between estimation quality and solution sparsity as controlled by  $\lambda$ . In the limiting case of  $\lambda \rightarrow 0^+$ , the objective function exerts diminishing impact on enforcing sparsity on the solution, meaning that eventually no sparse solution is favored over other possible solutions. Interestingly, the SLMS and SNLMS, because of their proportionate nature similar to the PNLMS-type algorithms, are capable of speeding up convergence without compromising estimation quality should sparsity be present. This will be later supported by experimental results in Section V.

It is worth noting the fact that we can utilize  $\lambda = 0$  to obtain the SLMS and SNLMS could be attributed to the change of variables (10), which is similar to the concept of AST that belongs to the family of interior-point methods [28]. Due to the use of (10), optimization is performed in the  $\mathbf{q}$  domain rather than in the  $\mathbf{h}$  domain, allowing the use of the limiting case of  $\lambda \rightarrow 0^+$ . This does not apply to existing regularization-based algorithms developed in the original variable domain, e.g., [18]–[22]. Setting  $\lambda = 0$  in these algorithms reduces to the ordinary LMS without benefiting from sparsity.

For the design of the step-size control factors, many popular diversity measures in SSR can be used to instantiate the algorithms for updating  $\mathbf{W}_n$ . We present an example using the  $p$ -norm-like diversity measure [27] with  $g(h_i) = |h_i|^p$ ,  $0 < p \leq 2$ . Using (9) leads to the update rule for  $\mathbf{W}_n$ :

$$w_{i,n} = \left( \frac{2}{p} \left( |h_{i,n}| + c \right)^{2-p} \right)^{\frac{1}{2}}. \quad (25)$$

Note that we have added a small regularization constant  $c > 0$  for avoiding algorithm stagnation and instability. In (25), the parameter  $p$  controls the behavior of the adaptive filtering algorithms: using  $p \rightarrow 1$  in (25) results in a step-size control factor close to that of the PNLMS, while letting  $p = 2$  recovers the LMS/NLMS. The parameter  $p$  thus plays the role for fitting the sparsity levels of the systems.

In practice, we have found it helpful to perform normalization to the matrix term  $\mathbf{W}_n^2$  for stability purposes, i.e., to replace  $\mathbf{W}_n^2$  in (23) and (24) with  $\mathbf{S}_n$  where:

$$\mathbf{S}_n = \frac{\mathbf{W}_n^2}{\frac{1}{M} \text{tr}(\mathbf{W}_n^2)}, \quad (26)$$

and  $\text{tr}(\cdot)$  denotes the matrix trace. Similar steps of normalization can also be seen in other proportionate algorithms [10]. Note that the normalization (26) also applies to the generalized algorithms (17) and (21).

## V. SIMULATION RESULTS

The proposed algorithms are evaluated using computer simulations in MATLAB. We considered three system IRs as shown in Fig. 1 that represent different sparsity levels: quasi-sparse, sparse, and dispersive systems, respectively. Each of

the IRs has 256 taps. For demonstrating convergence behavior, experiments were conducted to obtain the mean squared error (MSE) learning curves, i.e., the ensemble average of  $e_n^2$  as a function of iteration  $n$ . The ensemble averaging was performed over 1000 independent Monte Carlo runs for obtaining each MSE curve. The adaptive filter length was 256 and the coefficients were initialized with all zeros. The input signal  $u_n$  and the noise  $v_n$  were zero mean white Gaussian processes with variance 1 and 0.001, respectively.

Fig. 2 presents the resulting MSE curves of using the SNLMS (24) as an example. The update rule (25) was used for  $\mathbf{W}_n^2$ , using  $c = 0.001$ . The normalization (26) was performed. The NLMS (3) is also compared. We used  $\mu = 0.5$  and  $\delta = 0.01$  for both SNLMS and NLMS. From the results we see that the selection of  $p$  is crucial for obtaining optimal performance for IRs with different sparsity degrees. For the quasi-sparse case in Fig. 2 (a), the fastest convergence is given by  $p = 1.5$ , which seems a reasonable value in terms of finding a balance between PNLMS ( $p \rightarrow 1$ ) and NLMS ( $p = 2$ ). For the sparse case in Fig. 2 (b),  $p = 1.2$  gives the best results, which is also intuitive since the sparsity level has increased. For the dispersive case in Fig. 2 (c),  $p = 1.8$  results in the fastest convergence and is comparable to NLMS. These results show that the algorithm exploits the underlying system structure in the way we expect.

Fig. 3 presents the mean of estimated filter coefficients to observe the converged solutions. The results were computed as the average of 500 iterations when the algorithm was in steady-state. In each case, we see that SNLMS (with the optimal  $p$  found in Fig. 2) converges to the same solution as NLMS, which agrees well with the corresponding true IR in Fig. 1. This supports the argument that the proposed strategy of utilizing  $\lambda \rightarrow 0^+$  provides the algorithms with the capability of leveraging sparsity without compromising estimation quality. In other words, the results together with Fig. 2 demonstrate that the sparsity promoting algorithms can exploit sparsity for speeding up convergence in the adaptation stage and perform equally well in steady-state should sparsity be present in the underlying system response.

## VI. CONCLUSION

In this paper, we exploited the connection between sparse system identification and SSR, and utilized the iterative reweighting strategies to derive novel proportionate adaptive filters that incorporate various diversity measures for promoting sparsity. Moreover, utilizing a regularization coefficient  $\lambda \rightarrow 0^+$ , the proposed SLMS and SNLMS algorithms can take advantage of, though do not strictly enforce, the sparsity of the underlying system if it already exists. Simulation results were presented to demonstrate the effectiveness of the algorithms.

## ACKNOWLEDGMENT

This work was supported by NIH/NIDCD grants R01DC015436 and R33DC015046.

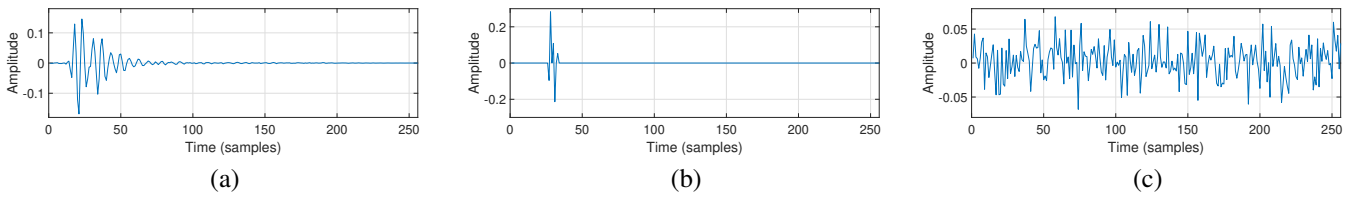


Fig. 1. IRs of (a) quasi-sparse, (b) sparse, and (c) dispersive systems.

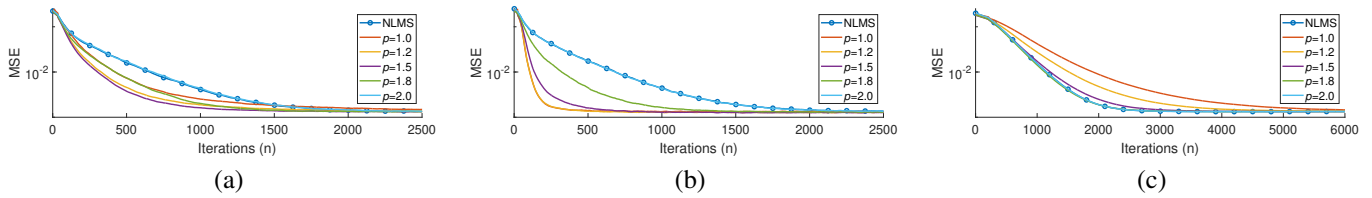


Fig. 2. MSE of SNLMS with various  $p$  for (a) quasi-sparse, (b) sparse, and (c) dispersive systems.

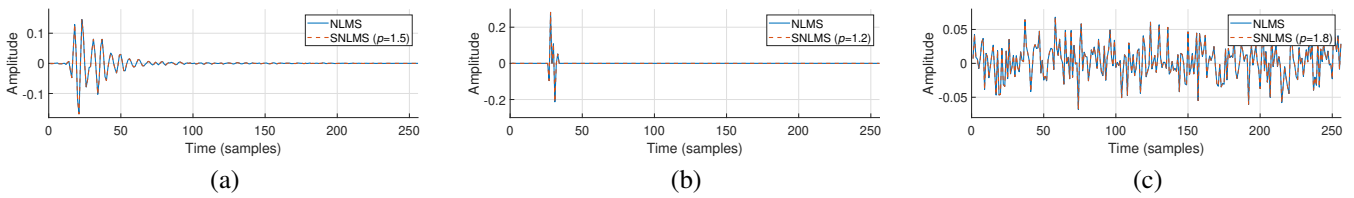


Fig. 3. Mean of estimated filter coefficients for (a) quasi-sparse, (b) sparse, and (c) dispersive systems.

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