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Initial degenerations of Grassmannians

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Abstract

We construct closed immersions from initial degenerations of $\mathrm{Gr}_0(d, n)$—the open cell in the Grassmannian $\mathrm{Gr}(d, n)$ given by the nonvanishing of all Plücker coordinates—to limits of thin Schubert cells associated to diagrams induced by the face poset of the corresponding tropical linear space. These are isomorphisms when (d, n) equals $(2, n)$, $(3, 6)$ and $(3, 7)$. As an application we prove $\mathrm{Gr}_0(3, 7)$ is schön, and the Chow quotient of $\mathrm{Gr}(3, 7)$ by the maximal torus in $\mathrm{PGL}(7)$ is the log canonical compactification of the moduli space of 7 points in $\mathbb{P}^2$ in linear general position, making progress on a conjecture of Hacking, Keel, and Tevelev.

Keywords Grassmannians · Initial degenerations · Thin Schubert cells · Matroids · Chow quotient · Log canonical compactification

Mathematics Subject Classification Primary 14T05; Secondary 14M15 · 14E25

1 Introduction

Let $\mathrm{Gr}(d, n)$ be the Grassmannian of d-dimensional subspaces of $\mathbf{k}^n$, for an algebraically closed field $\mathbf{k}$, and $\mathrm{Gr}_0(d, n)$ the open cell given by the nonvanishing of all Plücker coordinates. We consider its tropicalization $\mathrm{TGr}_0(d, n)$ through two frameworks. Via Gröbner theory, the set $\mathrm{TGr}_0(d, n)$ consists of those $w \in \wedge^d \mathbb{R}^n$ such that the initial degeneration $\mathrm{in}_w \mathrm{Gr}_0(d, n)$ is nonempty. Alternatively, the set $\mathrm{TGr}_0(d, n)$ has a modular interpretation as the space of d-dimensional tropical linear subspaces of $\mathbb{R}^n$ that are realizable over valued extensions of $\mathbf{k}$ [37]. The goal of this paper is to study initial degenerations of $\mathrm{Gr}_0(d, n)$ via their relation to tropical linear spaces.

Suppose $w \in \mathrm{TGr}_0(d, n)$ and L_w is the corresponding tropical linear space. Then w induces a regular subdivision Δ_w of the hypersimplex $\Delta(d, n) \subset \mathbb{R}^n$ into matroid polytopes, and there is a bijection between the bounded cells of L_w and the internal

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cells of Δ_w, reversing the face order [36]. Equipping $\mathrm{TGr}_0(d,n)$ with its Gröbner fan structure, the initial degeneration $\mathrm{in}_w \mathrm{Gr}_0(d,n)$ and the matroid subdivision Δ_w depend only on the cone of $\mathrm{TGr}_0(d,n)$ that contains w in its relative interior [40].

The collection of all subspaces realizing a matroid M defines a locally closed subscheme $\mathrm{Gr}_M \subset \mathrm{Gr}(d,n)$ called the *thin Schubert cell* of M. Let us describe an inverse system of thin Schubert cells associated to the matroid subdivision Δ_w. Given a matroid polytope $Q \subset \Delta(d,n)$, write M_Q for its matroid and ρ_{M_Q} for the rank function. Any facet of Q has the form $Q' = Q \cap \{\sum_{i\in\eta} x_i = \rho_{M_Q}(\eta)\}$ for some $\eta \subset [n] := \{0,1,\ldots,n-1\}$, and $M_{Q'}$ decomposes as a direct sum of the contraction M_Q/η and restriction $M_Q|\eta$ [11]. If $F \in \mathrm{Gr}(d,n)$ realizes M_Q and $\mu = [n]\backslash\eta$, then $F \cap \mathbf{k}^\mu$ and $F/(F\cap \mathbf{k}^\mu)$ realize M_Q/η and $M_Q|\eta$, respectively. We have a morphism

$$\mathrm{Gr}_{M_Q} \to \mathrm{Gr}_{M_{Q'}} \qquad F \mapsto (F\cap \mathbf{k}^\mu) \oplus F/(F\cap \mathbf{k}^\mu).$$

Thus $\{\mathrm{Gr}_{M_Q} \mid Q \in \Delta_w\}$ defines an inverse system, and we may form the limit $\varprojlim_{Q\in\Delta_w} \mathrm{Gr}_{M_Q}$.

Theorem 1.1 *For $w \in \mathrm{TGr}_0(d,n)$, there is a closed immersion*

$$\psi_w : \mathrm{in}_w \mathrm{Gr}_0(d,n) \hookrightarrow \varprojlim_{Q\in\Delta_w} \mathrm{Gr}_{M_Q}.$$

The morphisms $\mathrm{Gr}_{M_Q} \to \mathrm{Gr}_{M_{Q'}}$ and limit $\varprojlim_{Q\in\Delta_w} \mathrm{Gr}_{M_Q}$ originally appear in [27]. This limit parameterizes collections of subspaces $\{F_Q \in \mathrm{Gr}_{M_Q} \mid Q \in \Delta_w\}$ such that, if Q_1 and Q_2 share a common face defined by $\sum_{i\in\eta} x_i = \rho_{M_{Q_1}}(\eta) = d - \rho_{M_{Q_2}}(\mu)$ with $\mu = [n]\backslash\eta$, then

$$F_{Q_1}/(F_{Q_1}\cap \mathbf{k}^\mu) = F_{Q_2}\cap \mathbf{k}^\eta \qquad \text{and} \qquad F_{Q_2}/(F_{Q_2}\cap \mathbf{k}^\mu) = F_{Q_1}\cap \mathbf{k}^\eta$$

under the identifications $\mathbf{k}^n/\mathbf{k}^\mu = \mathbf{k}^\eta$ and $\mathbf{k}^n/\mathbf{k}^\eta = \mathbf{k}^\mu$.

In the construction of a morphism to $\varprojlim_{Q\in\Delta_w} \mathrm{Gr}_{M_Q}$, it suffices to construct compatible morphisms $\mathrm{in}_w \mathrm{Gr}_0(d,n) \to \mathrm{Gr}_{M_Q}$ whenever Q is an internal cell of Δ_w. Let us sketch a geometric characterization of these morphisms. Choose a $\mathbf{k}$-point x of $\mathrm{in}_w \mathrm{Gr}_0(d,n)$ and set $\mathbb{K} = \mathbf{k}(\!(t^\mathbb{R})\!)$. By surjectivity of exploded tropicalization [33], there is a $\mathbb{K}$-point p of $\mathrm{Gr}_0(d,n)$ such that $\mathfrak{Trop}(p) = x$; because $\mathbb{K}$ is a generalized power series field, the exploded tropicalization $\mathfrak{Trop}(p)$ is simply the vector of lead coefficients. The Plücker vector p defines a linear subspace F_p of $\mathbb{K}^n$ such that $\mathrm{Trop}(F_p^\circ) = L_w$, where $F_p^\circ = F_p \cap (\mathbb{K}^*)^n$. For any v with $-v$ in the bounded cell of L_w dual to Q, the closure of $\mathrm{in}_{-v}(F_p^\circ)$ in $\mathbf{k}^n$, denoted by $\overline{\mathrm{in}_{-v}(F_p^\circ)}$, is a linear realization of M_Q. The morphism $\mathrm{in}_w \mathrm{Gr}_0(d,n) \to \mathrm{Gr}_{M_Q}$ sends x to $\overline{\mathrm{in}_{-v}(F_p^\circ)}$. We will produce a scheme-theoretic construction of these morphisms in Sect. 3, and provide compatibility with this geometric description in Remark 3.6.

Our main application of Theorem 1.1 is to determine smoothness and irreducibility of initial degenerations of Grassmannians, especially for $\mathrm{Gr}_0(3,7)$. Because $\mathrm{TGr}_0(d,n)$ is sensitive to the characteristic of the underlying field, we assume that $\mathrm{char}\,\mathbf{k} = 0$.

Theorem 1.2 *The initial degenerations of* $\mathrm{Gr}_0(3,7)$ *are smooth and irreducible.*

The computation of $\mathrm{TGr}_0(3,7)$ in [19] allows us to compute the initial degenerations of $\mathrm{Gr}_0(3,7)$ and matroid subdivisions of $\Delta(3,7)$. Given the size of the initial ideals, determining smoothness and irreducibility of the $\mathrm{in}_w\,\mathrm{Gr}_0(3,7)$ directly is impractical, even with computer assistance. In comparison, thin Schubert cells and the morphisms between them are easier to describe, as we do in Sects. 4 and 5. Each $\mathrm{Gr}_M \subset \mathrm{Gr}(3,7)$ is smooth and irreducible and the morphisms $\mathrm{Gr}_M \to \mathrm{Gr}_{M_Q}$ are smooth and dominant with connected fibers, provided Q is not a face of $\Delta(3,7)$. This allows us to determine smoothness and irreducibility of $\varprojlim_{Q\in\Delta_w} \mathrm{Gr}_{M_Q}$; see Examples 6.2 and 6.3 for an illustration of this analysis. Being a closed immersion of affine schemes, the map ψ_w : $\mathrm{in}_w\,\mathrm{Gr}_0(d,n) \hookrightarrow \varprojlim_{Q\in\Delta_w} \mathrm{Gr}_{M_Q}$ is an isomorphism whenever $\varprojlim_{Q\in\Delta_w} \mathrm{Gr}_{M_Q}$ is an integral scheme of dimension $d(n-d)$. While the inequality $\dim(\varprojlim_{Q\in\Delta_w} \mathrm{Gr}_{M_Q}) \geq d(n-d)$ may be strict when $d=3$ and $n \geq 9$, as demonstrated in Example 8.2, it is an equality for all w in the $(3,7)$ case. This will yield a proof of Theorem 1.2, and our techniques will allow us to prove an analog of this theorem for any $\mathrm{Gr}_M \subset \mathrm{Gr}(d,n)$ for $(2,n)$, $(3,6)$, and $(3,7)$, see Theorem 6.1.

As a consequence of Theorem 1.2, the variety $\mathrm{Gr}_0(3,7)$ is schön in the sense of Tevelev [40]. This is important because, when X_0 is a schön subvariety of a torus, we may use tropical geometry to construct compactifications of X_0 with desirable properties. Indeed, the closure X of X_0 in any toric variety whose fan has support $\mathrm{Trop}\,X_0$ is a schön compactification [29]. The strata of X are schön, and $(X, B := X\backslash X_0)$ has toroidal singularities. Hacking, Keel, and Tevelev prove that $K_X + B$ is ample if and only if each irreducible stratum of X is log minimal, and a schön subvariety of a torus is log minimal if and only if its tropicalization is not invariant under translation by a rational subspace [16]. They apply this to Y^n, the moduli space of smooth marked del Pezzo surfaces of degree $9-n$ for $n \leq 7$, demonstrating that the Sekiguchi cross-ratio variety $\overline{Y}^n$ [34,35], introduced by Naruki when $n=6$ [31], is a schön and log canonical compactification of Y^n.

While $\mathrm{Gr}_0(d,n)$ is not log minimal, its quotient $X_0(d,n)$ by the free action of the maximal torus $H \subset \mathrm{PGL}(d)$ does have this property [26, Proposition 2.20]. Via the Gelfand–MacPherson correspondence, we interpret $X_0(d,n)$ as the moduli space of d marked points in $\mathbb{P}^{n-1}$ in linear general position up to the $\mathrm{PGL}(d)$-action. The Chow quotient $\mathrm{Gr}(d,n)//H$ compactifies $X_0(d,n)$. Let $X(d,n)$ be its normalization. Kapranov [21] proves $X(2,n) \cong \overline{M}_{0,n}$, the Grothendieck–Knudsen moduli space of genus 0 stable n-marked curves. This compactification of $X_0(2,n)$ is schön [40] and log canonical [25]. Keel and Tevelev prove $X(3,n)$ is not log canonical when $n \geq 9$, and together with Hacking they conjecture $X(3,n)$ is a schön and log canonical compactification for $X_0(3,n)$ when $n = 6, 7$, and 8 [40, Theorem 5.7], [26, Conjecture 1.6]. Luxton handles the $n=6$ case by investigating the relationship between $X_0(3,6)$ to Y^6 [28]. He proves that $X_0(3,6)$ is schön by showing that the toric strata of $X(3,6)$ are smooth via a delicate analysis of how the log canonical fan of Y^6 maps onto $\mathrm{Trop}\,X_0(3,6)$. A direct adaptation of Luxton's strategy does not carry over to this setting; see Remark 7.4. Instead, we determine that $X_0(3,7)$ is schön directly from Theorem 1.2. We use this to verify the above conjecture when $n=7$.

Theorem 1.3 *The variety $X(3,7)$ is a schön and log canonical compactification of $X_0(3,7)$.*

In Sect. 8, we investigate the behavior of initial degenerations of $\mathrm{Gr}_0(3,n)$ for larger values of n. Given the relationship between thin Schubert cells and initial degenerations of $\mathrm{Gr}_0(d,n)$, it is reasonable to expect that in general $\mathrm{Gr}_0(d,n)$ will have initial degenerations that are not smooth or reducible. Indeed, the Perles matroid (see Fig. 2) is a rank 3 matroid P on [9] such that Gr_P is reducible. We use this to find an initial degeneration of $\mathrm{Gr}_0(3,9)$ with the same property.

Theorem 1.4 *The Grassmannian $\mathrm{Gr}_0(3,9)$ has an initial degeneration with two connected components.*

We conclude with three appendices. Appendix A gathers various properties of morphisms that are smooth and dominant with connected fibers. It also includes a discussion on finite limits of **k**-schemes. Appendix B contains a table with the data necessary for the proof of Lemma 6.4. Appendix C, written by María Angélica Cueto, includes various arguments that reduce the study of thin Schubert cells Gr_M, and the morphisms $\mathrm{Gr}_M \to \mathrm{Gr}_{M'}$ to the case where M is a simple and connected matroid, and a treatment of the rank 2 case. It also includes an argument that the limits of thin Schubert cells over Δ_w may be computed on the smaller poset consisting of cells that have codimension at most one.

Conventions

The field **k** is algebraically closed of characteristic 0. We fix the assumption on the characteristic because of the dependence on computer calculations. However, the proof of Theorem 1.1 works for all characteristics, and we expect that Theorems 1.2 and 1.4 remain true provided char $\mathbf{k} \neq 2$ or 5 respectively.

Computations

The software packages `gfan` [20], `Macaulay2` [13], `polymake` [8], and `sage` [41] in the proofs of Proposition 3.9, Lemma 6.4, and Lemma 7.3. The matroid subdivisions in Examples 6.2, 6.3, 8.1, 8.2 and Proposition 8.6 are computed using `polymake`. No computation takes longer than a few hours on a standard desktop computer. The code may be found at the following website.

https://github.com/dcorey2814/initialDegenerationsOfGrassmannians

2 Preliminaries

2.1 Initial degenerations

We recall some facts about initial degenerations and tropicalization of varieties defined over a trivially valued field from the Gröbner-theoretic perspective, see [30, Chapters 2, 3] for a comprehensive treatment, including the non-trivially valued case.

Let X be the closed subvariety of $\mathbb{P}^a = \mathrm{Proj}(\mathbf{k}[t_0, \dots, t_a])$ with homogeneous ideal $I \subset \mathbf{k}[t_0, \dots, t_a]$. Assume that X meets the dense torus T. Set $X_0 = X \cap T$ and $I_0 = I \cdot \mathbf{k}[t_0^{\pm}, \dots, t_a^{\pm}]$. We will often find it easier to work with $\mathrm{Spec}(\mathbf{k}[t_0^{\pm}, \dots, t_a^{\pm}]/I_0)$. This space is $\pi^{-1}(X_0)$, where $\pi : \mathbb{A}^{a+1} \backslash \{0\} \to \mathbb{P}^a$ is the natural projection. Note that $\pi^{-1}(X_0) \cong X_0 \times \mathbb{G}_m$.

Let $N_T = \mathbb{Z}^{a+1}/\mathbb{Z}\cdot\mathbf{1}$ denote the cocharacter lattice of T, where $\mathbf{1} = (1, \dots, 1)$. For $z = (z_0, \dots, z_a)$, we write $t^z = t^{z_0} \dots t^{z_a}$. The *initial form* of $f \in \mathbf{k}[t_0^{\pm}, \dots, t_a^{\pm}]$ with respect to $w \in (N_T)_{\mathbb{R}} := (N_T) \otimes_{\mathbb{Z}} \mathbb{R}$ is

$$\mathrm{in}_w f = \sum_{z:\langle w,z\rangle \text{ minimal}} a_z t^z \quad \text{where} \quad f = \sum a_z t^z.$$

That is, the (Laurent) polynomial $\mathrm{in}_w f$ is the sum of all monomials $a_z t^z$ of f with minimal w-weight. The *initial ideals* of I_0 and I with respect to w are

$$\mathrm{in}_w I_0 = \langle \mathrm{in}_w f \mid f \in I_0 \rangle \quad \text{and} \quad \mathrm{in}_w I = \langle \mathrm{in}_w f \mid f \in I \rangle,$$

respectively. The *initial degeneration* of X_0 with respect to w is

$$\mathrm{in}_w X_0 = T \cap \mathrm{Proj}(\mathbf{k}[t_0, \dots, t_a]/ \mathrm{in}_w I).$$

There is a complete polyhedral fan $\Sigma_{\mathrm{G}}(X_0)$ in $N_{\mathbb{R}}$, called the *Gröbner fan*, where w and w' belong to the relative interior of the same cone in $\Sigma_{\mathrm{G}}(X_0)$ if and only if $\mathrm{in}_w I = \mathrm{in}_{w'} I$ [39, Theorem 1.2]. The *tropicalization* of X_0 is

$$\mathrm{Trop}\, X_0 = \{w \in (N_T)_{\mathbb{R}} \mid \mathrm{in}_w I_0 \neq \langle 1 \rangle\}.$$

When X_0 is irreducible, its tropicalization $\mathrm{Trop}\, X_0$ is the support of a pure $\dim(X_0)$-dimensional subfan of $\Sigma_{\mathrm{G}}(X_0)$. Denote the restriction of $\Sigma_{\mathrm{G}}(X_0)$ to $\mathrm{Trop}\, X_0$ by $\mathcal{G}_{X_0}$. While $\mathrm{in}_w X_0$ depends only on the cone of $\mathcal{G}_{X_0}$ containing w in its relative interior, it is possible that $\mathrm{in}_w X_0 = \mathrm{in}_{w'} X_0$ when w and w' belong to distinct locally closed cones. In this case, the ideals $\mathrm{in}_w I$ and $\mathrm{in}_{w'} I$ differ by primary components contained in $\langle t_0, \dots, t_a \rangle$.

2.2 Matroid polytopes

We assume familiarity with basic notions of matroids and refer the reader to [32] for a detailed treatment. For brevity, we say that a rank d matroid on $[n]$ is a (d, n)-matroid. Given a matroid M, we write $\mathcal{B}(M)$ for its set of bases and ρ_M for its rank function. The uniform (d, n)-matroid is denoted by $U(d, n)$. For $\eta \subset [n]$, the matroid M/η denotes the contraction of M by η and $M|\eta$ denotes the restriction of M to η.

Let $e_0, \dots, e_{n-1}$ denote the standard basis of $\mathbb{R}^n$, and for a subset $\lambda = \{\lambda_0, \dots, \lambda_k\}$ of $[n]$, let $e_\lambda = e_{\lambda_0} + \dots + e_{\lambda_k}$. The *hypersimplex* $\Delta(d, n)$ is the polytope in $\mathbb{R}^n$ defined by

$$\Delta(d,n)=\left\{(x_0,\ldots,x_{n-1})\in\mathbb{R}^n\;\middle|\;\sum_{i\in[n]}x_i=d,\;0\le x_i\le 1\right\}. \tag{2.1}$$

The vertices of $\Delta(d,n)$ are the points e_λ for $\lambda\in\binom{[n]}{d}:=\{\sigma\subset[n]\,|\,|\sigma|=d\}$. The *matroid polytope* of M is

$$Q_M=\left\{(x_0,\ldots,x_{n-1})\in\mathbb{R}^n\;\middle|\;\sum_{i\in[n]}x_i=d,\;\sum_{i\in\eta}x_i\le\rho_M(\eta),\;\eta\subset[n]\right\}. \tag{2.2}$$

The vertices of Q_M are the points e_β for $\beta\in\mathcal{B}(M)$. Given a collection of vertices of $\Delta(d,n)$, its convex hull Q is a matroid polytope if and only if every edge of Q is parallel to some e_i-e_j [9, Theorem 4.1]; we write M_Q for the corresponding matroid. In particular, any face of a matroid polytope is a matroid polytope.

Throughout, we will consider the face order on polytopes: $Q'\le Q$ whenever Q' is a face of Q, and $Q'\lessdot Q$ when Q' is a facet of Q. This induces a partial order on the set of (d,n)-matroids: $M'\le M$ whenever $Q_{M'}\le Q_M$, and $M'\lessdot M$ if $Q_{M'}\lessdot Q_M$. Given $\eta\subset[n]$, let $M_\eta=M_{Q'}$ where $Q'=Q_M\cap\{\sum_{i\in\eta}x_i=\rho_M(\eta)\}$. The bases of M_η are

$$\mathcal{B}(M_\eta)=\{\beta\in\mathcal{B}(M)\,|\,|\beta\cap\eta|=\rho_M(\eta)\},$$

and the remaining $\beta\in\mathcal{B}(M)\backslash\mathcal{B}(M_\eta)$ satisfy $|\beta\cap\eta|<\rho_M(\eta)$. It is not hard to produce an isomorphism $M_\eta\cong M/\eta\oplus M|\eta$. When M is connected, a nonempty subset η is *nondegenerate* if M/η and $M|\eta$ are connected. The following proposition may be found in [11, Section 2.5].

Proposition 2.1 *If M be a connected matroid on $[n]$, then $\eta\mapsto Q_{M_\eta}$ is a one-to-one correspondence between nondegenerate subsets η and the facets of Q_M.*

Finally, we remark that if $w\in\mathbb{Z}^n$ and M_w is the matroid of minimal w-weight as in [1], then our M_η is just $M_{-\chi(\eta)}$ where χ is the characteristic function.

2.3 Thin Schubert cells

The Grassmannian $\mathrm{Gr}(d,n)$ of d-dimensional linear subspaces of $\mathbf{k}^n$ is a subvariety of $\mathbb{P}(\wedge^d\mathbf{k}^n)$ via the Plücker embedding. The homogeneous coordinate ring of $\mathbb{P}(\wedge^d\mathbf{k}^n)$ is denoted by $\mathbf{k}[p_\lambda]$, the Plücker ideal by $I^{d,n}\subset\mathbf{k}[p_\lambda]$, and the λth Plücker coordinate of $F\in\mathrm{Gr}(d,n)$ by $p_\lambda(F)$. As observed by [9], the variety $\mathrm{Gr}(d,n)$ decomposes into locally closed subschemes Gr_M called *thin Schubert cells* which are indexed by $\mathbf{k}$-realizable (d,n)-matroids. Set-theoretically,

$$\mathrm{Gr}_M=\{F\in\mathrm{Gr}(d,n)\,|\,p_\lambda(F)\ne 0\text{ if and only if }\lambda\in\mathcal{B}(M)\}.$$

Observe that $\mathrm{Gr}_0(d,n)=\mathrm{Gr}_{U(d,n)}$. We realize Gr_M as a scheme in the following way. Define

- $B_M = \mathbf{k}[p_\lambda \mid \lambda \in \mathcal{B}(M)] \subset \mathbf{k}[p_\lambda]$,
- $I_M = \left(I^{d,n} + \left\langle p_\lambda \mid \lambda \in \binom{[n]}{d} \backslash \mathcal{B}(M) \right\rangle\right) \cap B_M$,
- S_M the multiplicative semigroup of B_M generated by p_λ such that $\lambda \in \mathcal{B}(M)$, and
- $R_M = S_M^{-1} B_M / I_M$.

Then

$$\mathrm{Gr}_M = T(M) \cap \mathrm{Proj}(B_M/I_M)$$

where $T(M)$ is the dense torus of $\mathrm{Proj}(B_M)$. For computations, we will often find it easier to work with $\mathrm{Spec}(R_M) \cong \mathrm{Gr}_M \times \mathbb{G}_m$. The ideal I_M is generated by

$$P_M(\mu, \nu) = \sum_{i:\mu \cup i, \nu \backslash i \in \mathcal{B}(M)} \mathrm{sgn}(i; \mu, \nu) p_{\mu \cup i} p_{\nu \backslash i} \tag{2.3}$$

where $\mu \in \binom{[n]}{d-1}$ is independent and $\nu \in \binom{[n]}{d+1}$ has rank d and not contained in μ [30, Equation 4.4.1]. Here, the function $\mathrm{sgn}(i; \mu, \nu)$ equals $(-1)^\ell$ where ℓ is the number of $j \in \nu$ with $i < j$ plus the number of elements $j' \in \mu$ such that $i > j'$. The coordinate ring of Gr_M can be presented with far fewer generators and relations by using affine coordinates with respect to a fixed basis, which we now describe.

Construction 2.2 Suppose M is a $\mathbf{k}$-realizable (d, n)-matroid. Let $\beta = \{b_0 < \cdots < b_{d-1}\}$ be a basis, let $\gamma = \{c_0 < \cdots < c_{n-d-1}\}$ its complement, and let $\mathbf{k}[x_{ij}] := \mathbf{k}[x_{ij} \mid 0 \le i < d,\ 0 \le j < n-d]$. Define a matrix X in the following way. The submatrix of X formed by the columns from β is the identity matrix, and the submatrix formed by the columns from γ has (i, j)-entry equal to x_{ij}. For example, if $\beta = [d]$, then

$$X = \begin{pmatrix} 1 & 0 & \cdots & 0 & x_{00} & x_{01} & \cdots & x_{0,n-d-1} \\ 0 & 1 & \cdots & 0 & x_{10} & x_{11} & \cdots & x_{1,n-d-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & x_{d-1,0} & x_{d-1,1} & \cdots & x_{d-1,n-d-1} \end{pmatrix}.$$

Given $\lambda \in \binom{[n]}{d}$, let X_λ be $d \times d$ the minor of X formed by the columns from λ. For $i \in [d]$ and $j \in [n-d]$, define $\lambda_{ij} \in \binom{[n]}{d}$ by

$$\lambda_{ij} = (\beta \backslash \{b_i\}) \cup \{c_j\}.$$

Then $x_{ij} = (-1)^\ell X_{\lambda_{ij}}$ where ℓ is the number of elements of β strictly between b_i and c_j. We define

- $B_M^x = \mathbf{k}[x_{ij} \mid \lambda_{ij} \in \mathcal{B}(M)] \subset \mathbf{k}[x_{ij}]$,
- $I_M^x = \left\langle X_\lambda \mid \lambda \in \binom{[n]}{d} \backslash \mathcal{B}(M) \right\rangle \cap B_M^x$, and
- S_M^x the multiplicative semigroup in B_M^x generated by $\overline{X}_\lambda := \pi_M(X_\lambda)$ for $\lambda \in \mathcal{B}(M)$, where $\pi_M : \mathbf{k}[x_{ij}] \to \mathbf{k}[x_{ij}]/\langle x_{ij} \mid \lambda_{ij} \notin \mathcal{B}(M) \rangle \cong B_M^x$ is the quotient map.

Then the coordinate ring of Gr_M is isomorphic to $R_M^x := (S_M^x)^{-1} B_M^x / I_M^x$.

Thin Schubert cells behave well with respect to duality and direct sum of matroids. If M^* is the dual of M, then $\mathrm{Gr}_{M^*} \subset \mathrm{Gr}(n-d,n)$, and $\mathrm{Gr}_M \cong \mathrm{Gr}_{M^*}$ under the isomorphism $\mathrm{Gr}(d,n) \cong \mathrm{Gr}(n-d,n)$. If M decomposes as $M = M_1 \oplus M_2$, then $\mathrm{Gr}_M \cong \mathrm{Gr}_{M_1} \times \mathrm{Gr}_{M_2}$ [23, Proposition 9.4].

2.4 Matroid subdivisions and the tropical Grassmannian

Given a polytope $P \subset \mathbb{R}^n$ with vertices $u_0, \ldots, u_k$ and $w \in \mathbb{R}^{k+1}$, define

$$P_w = \mathrm{conv}\{(u_i, w_i) \mid 0 \leq i \leq k\}.$$

Any lower face of P_w is of the form

$$\mathrm{face}_{(v,1)}(P_w) = \{x \in P_w \mid \langle x, (v,1)\rangle \leq \langle y, (v,1)\rangle \text{ for all } y \in P_w\}$$

where $v \in \mathbb{R}^n$. The lower faces of P_w project onto P, forming a polyhedral complex whose support is P. This is called the *regular subdivision* of P induced by w. The *secondary fan* $\Sigma_{\mathrm{S}}(P)$ of P is the complete fan in $\mathbb{R}^{k+1}$ where w and w' belong to the relative interior of the same cone if and only if they induce the same regular subdivision on P [10]. The *adjacency graph* of this subdivision is the graph with vertex v_Q for each maximal cell Q and an edge between v_Q and $v_{Q'}$ whenever Q and Q' share a common facet.

Given a (d,n)-matroid M and $w \in \mathbb{R}^{\mathcal{B}(M)}$, we write $\Delta_{M,w}$ for the regular subdivision of Q_M induced by w. This subdivision is *matroidal*, or $\Delta_{M,w}$ is a *matroid subdivision*, if each $Q \in \Delta_{M,w}$ is a matroid polytope. The *Dressian* of M is

$$\mathrm{Dr}_M = \left\{ w \in \mathbb{R}^{\mathcal{B}(M)} \mid \Delta_{M,w} \text{ is matroidal} \right\}.$$

When $M = U(d,n)$, we write $\Delta_w = \Delta_{U(d,n),w}$ and $\mathrm{Dr}(d,n) = \mathrm{Dr}_{U(d,n)}$. If $\Delta_{M,w}$ is matroid subdivision and $Q \in \Delta_{M,w}$ is the projection of $\mathrm{face}_{(v,1)}((Q_M)_w)$, then

$$\mathcal{B}(M_Q) = \{\lambda \in \mathcal{B}(M) \mid \langle v, e_\lambda\rangle + w_\lambda \leq \langle v, e_{\lambda'}\rangle + w_{\lambda'} \text{ for all } \lambda' \in \mathcal{B}(M)\}. \quad (2.4)$$

We abbreviate $\mathrm{Trop}\,\mathrm{Gr}_M$ and $\mathcal{G}_{\mathrm{Gr}_M}$ by TGr_M and $\mathcal{G}_M$, respectively. We also abbreviate $\mathrm{Trop}\,\mathrm{Gr}_0(d,n)$ and $\mathcal{G}_{U(d,n)}$ by $\mathrm{TGr}_0(d,n)$ and $\mathcal{G}_{d,n}$, respectively. Denote by $\mathscr{S}_M$ the restriction of $\Sigma_{\mathrm{S}}(Q_M)$ to Dr_M; when $M = U(d,n)$, write $\mathscr{S}_M = \mathscr{S}_{d,n}$.

If $w \in \mathrm{TGr}_M$, then $\Delta_{M,w}$ is matroidal [30, Lemma 4.4.6], and hence TGr_M is a subset of Dr_M. The inclusion $\mathrm{TGr}_M \subset \mathrm{Dr}_M$ induces a morphism of fans $\mathcal{G}_M \to \mathscr{S}_M$ (when M is the uniform matroid, this is [40, Theorem 5.4]), thus $\mathrm{in}_w\,\mathrm{Gr}_M$ and $\Delta_{M,w}$ depend only on the cone of $\mathcal{G}_M$ containing w in its relative interior.

In general, not much is known about $\mathcal{G}_M \to \mathscr{S}_M$ outside of a few values of (d,n). We have $\mathrm{TGr}_0(2,n) = \mathrm{Dr}(2,n)$ and $\mathcal{G}_{2,n} = \mathscr{S}_{2,n}$. Next, we have $\mathrm{TGr}_0(3,6) = \mathrm{Dr}(3,6)$ and $\mathcal{G}_{3,6} \to \mathscr{S}_{3,6}$ is a refinement [37]. The set $\mathrm{TGr}_0(3,7)$ is properly contained in $\mathrm{Dr}(3,7)$, and $\mathcal{G}_{3,7} \to \mathscr{S}_{3,7}$ realizes $\mathcal{G}_{3,7}$ as a refinement of a subfan of $\mathscr{S}_{3,7}$

[19, Theorems 2.1, 2.2]; here, the char $\mathbf{k} \neq 2$ assumption is crucial. For $n = 6, 7$, denote by $\mathscr{S}'_{3,n}$ the restriction of $\mathscr{S}_{3,n}$ to $\mathrm{TGr}_0(3, n)$; this is a coarsening of $\mathcal{G}_{3,n}$.

3 Limits of thin Schubert cells

In this section, we construct closed immersions

$$\mathrm{in}_w \mathrm{Gr}_M \hookrightarrow \varprojlim_{Q \in \Delta_{M,w}} \mathrm{Gr}_{M_Q}$$

for any $\mathbf{k}$-realizable matroid M and $w \in \mathrm{TGr}_M$, proving Theorem 1.1. We begin with a discussion of the morphisms between thin Schubert cells. Let $Q \subset \Delta(d, n)$ be a matroid polytope, let Q' the face defined by the equation $\sum_{i \in \eta} x_i = \rho_M(\eta)$, and let $\mu = [n] \backslash \eta$. As discussed in the introduction, the map $\mathrm{Gr}_{M_Q} \to \mathrm{Gr}_{M_{Q'}}$ is given by $F \mapsto (F \cap \mathbf{k}^\mu) \oplus F/(F \cap \mathbf{k}^\mu)$. From the canonical isomorphism

$$\wedge^d F \cong \wedge^{d-\rho_{M_Q}(\eta)}(F \cap \mathbf{k}^\mu) \otimes \wedge^{\rho_{M_Q}(\eta)} F/(F \cap \mathbf{k}^\mu)$$

we see that $\mathrm{Gr}_{M_Q} \to \mathrm{Gr}_{M_{Q'}}$ is induced by the projection $\mathbf{k}^{\mathcal{B}(M_Q)} \to \mathbf{k}^{\mathcal{B}(M_{Q'})}$ [27, Proposition I.6]. We derive a scheme-theoretic characterizations of these morphisms.

Proposition 3.1 *Suppose $M' \leq M$ are (d, n)-matroids. The inclusion $B_{M'} \subset B_M$ induces a morphism of schemes $\varphi_{M,M'} : \mathrm{Gr}_M \to \mathrm{Gr}_{M'}$. Furthermore, these morphisms satisfy $\varphi_{M,M''} = \varphi_{M',M''}\varphi_{M,M'}$ if $M'' \leq M' \leq M$ and $\varphi_{M,M} = \mathrm{id}$.*

Proof It suffices to consider the case $M' = M_\eta$ for some $\eta \subset [n]$. We must show that I_{M_η} maps to I_M under the inclusion $B_{M_\eta} \subset B_M$. We will do this using the generators for I_{M_η} and I_M given by Eq. (2.3). Suppose $\mu \in \binom{[n]}{d-1}$ is independent in M_η, and $\nu \in \binom{[n]}{d+1}$ not containing μ such that $\rho_{M_\eta}(\nu) = d$. Note that μ is independent in M and $\rho_M(\nu) = d$ because $\mathcal{B}(M_\eta) \subset \mathcal{B}(M)$. We must show $P_{M_\eta}(\mu, \nu) = 0$ or $P_M(\mu, \nu)$. If $P_{M_\eta}(\mu, \nu) \neq 0$, then there is a $i_0 \in \nu \backslash \mu$ such that both $\mu \cup i_0$ and $\nu \backslash i_0$ are in $\mathcal{B}(M_\eta)$, thus $|(\mu \cup i_0) \cap \eta|$ and $|(\nu \backslash i_0) \cap \eta|$ both equal $r := \rho_M(\eta)$. In particular,

1. $|\mu \cap \eta| = r - 1$ and $|\nu \cap \eta| = r + 1$ if $i_0 \in \eta$, or
2. $|\mu \cap \eta| = r$ and $|\nu \cap \eta| = r$ if $i_0 \notin \eta$.

For each $i \in \nu \backslash \mu$, we must show that $\mu \cup i$ and $\nu \backslash i$ are in $\mathcal{B}(M_\eta)$ if and only if they are both in $\mathcal{B}(M)$. Since $\mathcal{B}(M_\eta) \subset \mathcal{B}(M)$, we need only show the "if" direction.

Suppose $\mu \cup i$ and $\nu \backslash i$ are bases of M. By the characterization of Q_M in Eq. (2.2),

$$|(\mu \cup i) \cap \eta| \leq r \text{ and } |(\nu \backslash i) \cap \eta| \leq r \tag{3.1}$$

We show that they both equal r by considering the possibilities of $|\mu \cap \eta|$ and $|\nu \cap \eta|$ as above. If $i_0 \in \eta$, then $|\mu \cap \eta| = r - 1$ and $|\nu \cap \eta| = r + 1$. By Eq. (3.1), we have that $|(\nu \backslash i) \cap \eta| = r$. In particular, we have $i \in \eta$, so $|(\mu \cup i) \cap \eta| = r$. If $i_0 \notin \eta$, then $|\mu \cap \eta| = r$ and $|\nu \cap \eta| = r$. By Eq. (3.1), we have that $|(\mu \cup i) \cap \eta| = r$. In particular, we have $i \notin \eta$, so $|(\nu \backslash i) \cap \eta| = r$. □

Proposition 3.2 *The induced morphism $\varphi^{\#}_{M,M'} : R^x_{M'} \to R^x_M$ is given by the inclusion $B^x_{M'} \subset B^x_M$.*

Proof Suppose $[d]$ is a basis of M and M'. Setting $\tilde{R}_M = S_M^{-1}\mathbf{k}[p_\lambda/p_{[d]} \mid \lambda \in \mathcal{B}(M)]/I_M$, we see that $\theta_M : \tilde{R}_M \to R^x_M$ given by $\theta_M(p_\lambda/p_{[d]}) = X_\lambda$ is an isomorphism (the inverse sends x_{ij} to $p_{\lambda_{ij}}/p_{[d]}$). By Proposition 3.1, the morphism $\varphi_{M,M'}$ is induced by the ring map $\psi_{M,M'} : \tilde{R}_{M'} \to \tilde{R}_M$ that sends p_λ to itself. Therefore, the composition $\theta_M \psi_{M,M'} \theta_{M'}^{-1}$ sends x_{ij} to itself (for $\lambda_{ij} \in \mathcal{B}(M)$), as required. □

Fix a (d, n)-matroid M and $w \in \mathrm{TGr}_M$. By Proposition 3.1, we have an inverse system $\{\mathrm{Gr}_{M_Q} \mid Q \in \Delta_{M,w}\}$. We may form $\varprojlim_{Q \in \Delta_{M,w}} \mathrm{Gr}_{M_Q}$, which we denote by $\mathrm{Gr}_{M,w}$, and write $\varphi_Q : \mathrm{Gr}_{M,w} \to \mathrm{Gr}_{M_Q}$ for the structure morphism. When $M = U(d, n)$ we write $\mathrm{Gr}_w = \mathrm{Gr}_{U(d,n),w}$. Finite limits exist in the category of affine schemes because this category has fiber products and a terminal object [2, Proposition 5.21].

Lemma 3.3 *Suppose $w \in \mathrm{TGr}_M$ and $Q \in \Delta_{M,w}$. The inclusion $B_{M_Q} \subset B_M$ induces a morphism $\psi_{M,M_Q,w} : \mathrm{in}_w \mathrm{Gr}_M \to \mathrm{Gr}_{M_Q}$.*

Proof Suppose Q is the projection of $\mathrm{face}_{(v,1)}((Q_M)_w)$. Equation (2.4) records the bases of M_Q. We must show that I_{M_Q} maps to $\mathrm{in}_w I_M$ under the inclusion $B_{M_Q} \subset B_M$. For this, it suffices to consider the quadratic generators $P_{M_Q}(\mu, \nu)$ from Eq. (2.3). Let $\mu \in \binom{[n]}{d+1}$ with $\rho_{M_Q}(\mu) = d$ and $\nu \in \binom{[n]}{d-1}$ independent in M_Q. If $P_{M_Q}(\mu, \nu) \neq 0$, then there is a $i_0 \in \mu \backslash \nu$ such that $\mu \backslash i_0, \nu \cup i_0 \in \mathcal{B}(M_Q)$. Because $\mathcal{B}(M_Q) \subset \mathcal{B}(M)$, we see that μ, ν, i_0 satisfy the same properties for M. We must show

$$P_{M_Q}(\mu, \nu) = \mathrm{in}_w P_M(\mu, \nu). \tag{3.2}$$

Observe that for any $i, j \in \mu \backslash \nu$,

$$u_{\mu\backslash j} + u_{\nu\cup j} - u_{\mu\backslash i} - u_{\nu\cup i} = w_{\mu\backslash j} + w_{\nu\cup j} - w_{\mu\backslash i} - w_{\nu\cup i}. \tag{3.3}$$

where $u_\lambda = \langle v, e_\lambda\rangle + w_\lambda$. Now, the term $p_{\mu\backslash i} p_{\nu\cup i}$ is a summand in $P_{M_Q}(\mu, \nu)$ if and only if $u_{\mu\backslash i} = u_{\mu\backslash i_0}$ and $u_{\nu\cup i} = u_{\nu\cup i_0}$. By Eq. (3.3), these equalities hold if and only if $w_{\mu\backslash i} + w_{\nu\cup i} = w_{\mu\backslash i_0} + w_{\nu\cup i_0}$. Since $w_{\mu\backslash i_0} + w_{\nu\cup i_0}$ is the smallest such sum, we have this equality if and only if $p_{\mu\backslash i} p_{\nu\cup i}$ is a summand in $\mathrm{in}_w P_M(\mu, \nu)$. □

Theorem 3.4 *The morphisms $\psi_{M,M_Q,w} : \mathrm{in}_w \mathrm{Gr}_M \to \mathrm{Gr}_{M_Q}$ induce a closed immersion $\psi_{M,w} : \mathrm{in}_w \mathrm{Gr}_M \hookrightarrow \mathrm{Gr}_{M,w}$.*

Proof Clearly $\varphi_{M_Q,M_{Q'}} \psi_{M,M_Q,w} = \psi_{M,M_{Q'},w}$, so $\psi_{M,w}$ is defined by the universal property of $\mathrm{Gr}_{M,w}$. It is a closed immersion because the induced morphism $\psi^{\#}_{M,w} : \varinjlim_{\Delta_{M,w}} R_{M'} \to S_M^{-1} B_M / \mathrm{in}_w I_M$ is surjective. □

The following Corollary is an immediate consequence of Theorem 3.4 and Proposition A.8.

Corollary 3.5 *The closed immersion* $\psi_{M,w} : \mathrm{in}_w \mathrm{Gr}_M \hookrightarrow \mathrm{Gr}_{M,w}$ *is an isomorphism when* $\mathrm{Gr}_{M,w}$ *is integral and of dimension* $\dim \mathrm{Gr}_M$.

Remark 3.6 We now show that our definition of $\psi_{M,M_Q,w} : \mathrm{in}_w \mathrm{Gr}_M \to \mathrm{Gr}_{M_Q}$ agrees with the characterization discussed in the introduction. For simplicity, suppose $M = U(d,n)$. We refer the reader to [30, Chapter 4] for basic facts about circuits of linear subspaces. As before, let x be a $\mathbf{k}$-point of $\mathrm{in}_w \mathrm{Gr}_0(d,n)$, let $\mathbb{K} = \mathbf{k}(\!(t^{\mathbb{R}})\!)$, and let p a $\mathbb{K}$-point of $\mathrm{Gr}_0(d,n)$ so that $\mathfrak{Trop}(p) = x$. Let F_p be the linear subspace of $\mathbb{K}^n$ with Plücker vector p, and $I_{F_p} \subset \mathbb{K}[y_0,\ldots,y_{n-1}])$ its ideal. Given a subset $\mu = \{i_0,\ldots,i_d\} \in \binom{[n]}{d+1}$, let

$$\ell_\mu = \sum_{k=0}^{d} (-1)^k p_{\mu\setminus i_k} \cdot y_{i_k}.$$

The set $\{\ell_\mu \mid \mu \in \binom{[n]}{d+1}\}$ is a universal Gröbner basis of I_{F_p}. Let $F_p^\circ = F_p \cap (\mathbb{K}^*)^n$ and choose a vector v with $-v \in \mathrm{Trop}(F_p^\circ)$ in the cell dual to Q; the bases of M_Q are described by a Formula similar to (2.4). Then $\mathrm{in}_{-v}(F_p^\circ)$ is cut out by the linear equations

$$\mathrm{in}_{-v}\,\ell_\mu = \sum_{k:u_{\mu\setminus i_k}\ \text{minimal}} (-1)^k x_{\mu\setminus i_k} \cdot y_{i_k} \tag{3.4}$$

where $u_\lambda = \langle v, e_\lambda\rangle + w_\lambda$. Recall that the support of a linear form is $\ell = \sum a_i y_i$ is $\mathrm{supp}(\ell) = \{i \in [n] \mid a_i \neq 0\}$. The closure of $\mathrm{in}_{-v}(F_p^\circ)$ in $\mathbf{k}^n$ is a linear subspace that realizes a matroid M' whose circuits are

$$\mathcal{C}(M') = \{\mathrm{supp}(\mathrm{in}_{-v}\,\ell_\mu) \mid \mu \in \tbinom{[n]}{d+1},\ \rho_M(\mu) = d\}$$

It is easy to see that $\mathcal{B}(M') = \{\lambda \in \binom{[n]}{d} \mid u_\lambda \le u_{\lambda'} \text{ for all } \lambda' \in \binom{[n]}{d}\}$, hence $M_Q = M'$. By Eq. (3.4) and the description of $\mathcal{C}(M_Q)$, we see that the Plücker vector of the closure of $\mathrm{in}_{-v}(F_p^\circ)$ in $\mathbf{k}^n$ is the projection of $x \in \wedge^d \mathbf{k}^n$ to $\mathbf{k}^{\mathcal{B}(M_Q)}$. This is $\psi_{M,M_Q,w}(x)$, as required.

Now we show how to compute the coordinate ring of $\mathrm{Gr}_{M,w}$ in Plücker and affine coordinates. We will use this in Proposition 3.9 below to compute the dimension of $\mathrm{Gr}_{M,w}$ for any $\mathbf{k}$-realizable $(2,n)$, $(3,6)$, or $(3,7)$ matroid. Let $(\Delta_{M,w})^{\mathrm{top}}$ be the collection of top dimensional cells in $\Delta_{M,w}$, and $\Gamma_{M,w}$ the adjacency graph of $\Delta_{M,w}$, as defined in Sect. 2.4. For the uniform matroid, we write $(\Delta_w)^{\mathrm{top}} = (\Delta_{U(d,n),w})^{\mathrm{top}}$ and $\Gamma_w = \Gamma_{U(d,n),w}$. There is an inverse system over $\Gamma_{M,w}$ as in Example A.4, and $\mathrm{Gr}_{M,w} \cong \varprojlim_{\Gamma_{M,w}} \mathrm{Gr}_{M'}$ by Proposition C.12. Let

$$I_{M,w} = \langle I_{M_Q} B_M \mid Q \in (\Delta_{M,w})^{\mathrm{top}}\rangle \subset B_M$$

and $R_{M,w} = S_M^{-1} B_M / I_{M,w}$. When the polytopes in $(\Delta_{M,w})^{\mathrm{top}}$ share a common vertex, let

$$I_{M,w}^x = \langle I_{M_Q}^x B_M^x \mid Q \in (\Delta_{M,w})^{\mathrm{top}}\rangle \subset B_M^x$$

Given $f \in B^x_{M_Q}$, let $\overline{f} = \pi_{M_Q}(f)$ viewed as an element of B^x_M, where $\pi_{M_Q} : \mathbf{k}[x_{ij}] \to \mathbf{k}[x_{ij}]/\langle x_{ij} \mid \lambda_{ij} \notin \mathcal{B}(M_Q)\rangle \cong B^x_{M_Q}$ is the quotient map. Let $S^x_{M,w}$ be the multiplicative semigroup of B^x_M generated by $\overline{X}_\lambda$ for each $\lambda \in \mathcal{B}(M_Q)$ and $Q \in (\Delta_{M,w})^{\text{top}}$. Finally, set $R^x_{M,w} = (S^x_{M,w})^{-1} B^x_M / I^x_{M,w}$.

Proposition 3.7 *For any (d, n)-matroid M and $w \in \mathrm{Dr}_M$,*

$$\mathrm{Gr}_{M,w} \cong \varprojlim_{\Gamma_{M,w}} \mathrm{Gr}_{M'} \cong T(M) \cap \operatorname{Proj} R_{M,w}$$

If the polytopes in $(\Delta_{M,w})^{\text{top}}$ share a common vertex, then $\mathrm{Gr}_{M,w} \cong \operatorname{Spec} R^x_{M,w}$.

Proof The first isomorphism is established in Proposition C.12. For each $Q \in \Delta_{M,w}$ of codimension 0 or 1, we have ring maps $R_{M_Q} \to R_{M,w}$ and $R^x_{M_Q} \to R^x_{M,w}$ induced by $B_{M_Q} \subset B_M$ and $B^x_{M_Q} \subset B^x_M$ respectively. These produce morphisms

$$\Psi : \varinjlim_{\Gamma_{M,w}} R_{M_Q} \longrightarrow R_{M,w} \qquad \text{and } \Psi^x : \varinjlim_{\Gamma_{M,w}} R^x_{M_Q} \longrightarrow R^x_{M,w}$$

Now let us construct inverses Θ and Θ^x. For $\lambda \in \mathcal{B}(M)$ define $\Theta(p_\lambda) = \varphi^{\#}_{M_Q}(p_\lambda)$ where $Q \in (\Delta_{M,w})^{\text{top}}$ and $\lambda \in \mathcal{B}(M_Q)$. If Q' is another such polytope, we must show that $\varphi^{\#}_{M_Q}(p_\lambda) = \varphi^{\#}_{M_{Q'}}(p_\lambda)$. When $Q'' = Q \cap Q'$ has codimension 1,

$$\varphi^{\#}_{M_Q}(p_\lambda) = \varphi^{\#}_{M_{Q''}}(p_\lambda) = \varphi^{\#}_{M_{Q'}}(p_\lambda).$$

The general case follows from this observation and Lemma C.11. Similarly, for $\lambda_{ij} \in \mathcal{B}(M)$ define $\Theta^x(x_{ij}) = \varphi^{\#}_{M_Q}(x_{ij})$ where $Q \in (\Delta_{M,w})^{\text{top}}$ and $\lambda_{ij} \in \mathcal{B}(M_Q)$. It is easy to see that Θ and Θ^x take elements in S_M and $S^x_{M,w}$, respectively, to invertible elements.

Finally, we claim that $I_{M,w} \subset \ker(\Theta)$. It suffices to show that $\Theta(af) = 0$ for $a \in S^{-1}_{M_Q} B_{M_Q}$ and $f \in I_{M_Q}$ where $Q \in (\Delta_{M,w})^{\text{top}}$. But $\Theta(af) = \Theta(a)\varphi^{\#}_{M_Q}(f) = 0$. This shows that Θ is defined on $R_{M,w}$. A similar argument shows that $I^x_{M,w} \subset \ker(\Theta^x)$. Therefore Θ and Θ^x are defined on $R_{M,w}$ and $R^x_{M,w}$ respectively. One may verify that they are inverses to Ψ and Ψ^x respectively. □

Lemma 3.8 *If M is a rank 2 matroid and $w \in \mathrm{TGr}_M$, then $\psi_{M,w} : \mathrm{in}_w \mathrm{Gr}_M \to \mathrm{Gr}_{M,w}$ is an isomorphism.*

Proof By Theorem 3.4 and Proposition 3.7, the identity on B_M induces a surjective map $R_{M,w} \to S^{-1}_M B_M / \mathrm{in}_w I_M$, so $I_{M,w} \subset \mathrm{in}_w I_M$. The set

$$\mathcal{T} = \{P_M(\mu, \nu) \mid |\mu| = 3, |\nu| = 1, \mu \cap \nu = \emptyset\}$$

is a universal Gröbner basis for I_M (when $M = U(2, n)$, this is the set of three-term Plücker relations). Let $P_M(\mu, \nu) \in \mathcal{T}$. If $Q \in \Delta_{M,w}$ such that $P_{M_Q}(\mu, \nu) \neq 0$, then $P_{M_Q}(\mu, \nu) = \mathrm{in}_w P_M(\mu, \nu)$ by Eq. (3.2), hence $I_{M,w} = \mathrm{in}_w I_M$. □

Proposition 3.9 *Let* M *be a* $\mathbf{k}$*-realizable* $(2, n)$*,* $(3, 6)$*, or* $(3, 7)$ *matroid and* $w \in \mathrm{TGr}_M$*. Then* $\dim \mathrm{Gr}_{M,w} = \dim \mathrm{Gr}_M$.

Proof The rank 2 case follows from Lemma 3.8, so let M be a $(3, 6)$ or $(3, 7)$ matroid. By Proposition C.9 we may assume that M is simple. Once $\Delta_{M,w}$ is computed, this calculation may be done by hand, see Examples 6.2 and 6.3. Due to the large number of cases, we use a computer. The Gröbner fan structure on TGr_M is computed using `gfan`, and it catalogs all cones up to $\mathrm{Aut}(M)$-symmetry. The uniform cases were completed in [37, Theorem 5.4] for $(3, 6)$, and in [19, Theorem 2.1] for $(3, 7)$. For each cone, we choose a representative weight vector w and use `polymake` to compute $\Delta_{M,w}$. Let g be the product of all p_λ for $\lambda \in \mathcal{B}(M)$. Then $(I_{M,w} : g^\infty) \subset B_M$ is the homogeneous ideal of the closure of $\mathrm{Gr}_{M,w}$ in $\mathrm{Proj}(B_M)$. We use `Macaulay2` to show that its dimension equals $\dim \mathrm{Gr}_M$. The saturation was performed one variable at a time using the `saturate` function with the `Bayer` strategy. There are a total of 67 ideals to check among the simple $(3, 6)$-matroids, and 2815 ideals in the $(3, 7)$ case (not counting $w = 0$). The total process takes a couple of minutes for $(3, 6)$ and several hours for $(3, 7)$. □

4 Geometry of thin Schubert cells

By Mnëv universality, there exist $(3, n)$ matroids whose thin Schubert cells are singular or reducible, for sufficiently large n. Nevertheless, the thin Schubert cell Gr_M is smooth and irreducible when M is a rank 2 matroid, or a rank 3 matroid on $[6]$ or $[7]$, as we demonstrate in this section. Let M be a $\mathbf{k}$-realizable matroid. For each rank 1 flat η of M, choose a non-loop $s_\eta \in \eta$, and set $S = \{s_\eta \mid \eta \text{ is a rank 1 flat}\}$. Then $M|S$ is a simple matroid, and Gr_M is the product of $\mathrm{Gr}_{M|S}$ with an algebraic torus as discussed in Lemma C.2. Because the only simple $(2, n)$-matroid is $U(2, n)$, this leads to a straightforward proof that Gr_M is smooth and irreducible in the rank 2 case. Therefore, we will focus on rank 3 matroids.

Let M be a $\mathbf{k}$-realizable loop-free $(3, n)$-matroid for $n \geq 3$. We can represent M as a configuration of n points $p_0, \ldots, p_{n-1}$ in $\mathbb{P}^2$. The elements i, j are parallel in M if and only if p_i, p_j coincide. A subset $\beta \subset [n]$ is a basis if and only if $|\beta| = 3$ and p_i are not collinear for $i \in \beta$, and $\eta \subset [n]$ is a rank 2 flat if and only if there is a line $L \subset \mathbb{P}^2$ such that $p_i \in L$ precisely when $i \in \eta$. When drawing these pictures, we will only draw the points (labeled $0, \ldots, n-1$) and lines through at least 3 rank 1 flats, see Figs. 1 and 2. With this in mind, we say that η is a *line* of M if η is a rank 2 flat and $|\eta \cap S| \geq 3$. The set of lines of M, denoted by $\mathcal{L}(M)$, completely determines $M|S$.

All $(3, n)$-matroids for $3 \leq n \leq 7$ (up to S_n-symmetry) can be found in the online *Database of Matroids*

http://www-imai.is.s.u-tokyo.ac.jp/~ymatsu/matroid/

In showing that Gr_M is smooth and irreducible for these matroids, we start with $n = 3$ and work inductively. At each step, we need only consider simple and connected matroids by Lemmas C.1 and C.2. However, there are still 8, resp. 21, simple and connected $\mathbf{k}$-realizable $(3, 6)$, resp. $(3, 7)$-matroids. We use Lemma 4.1 to handle the remaining cases.

In this section and the next, we will need the following definitions. A morphism of schemes is said to have *connected fibers* if all of its nonempty fibers are connected. We say that $f : X \to Y$ is a *SDC-morphism* if it is smooth and dominant with connected fibers.

Lemma 4.1 *Let M be a loop-free $\mathbf{k}$-realizable $(3, n)$-matroid, let $i \in [n]$ be contained in exactly k lines where $0 \le k \le 2$, and suppose $\mathrm{Gr}_{M|[n]\setminus i}$ is integral. The composition of a dominant open immersion $\mathrm{Gr}_M \hookrightarrow \mathrm{Gr}_{M|[n]\setminus i} \times \mathbb{G}_m^{3-k}$, followed by the projection away from $\mathbb{G}_m^{3-k}$ produces a SDC-morphism $\mathrm{Gr}_M \to \mathrm{Gr}_{M|[n]\setminus i}$.*

Proof We use affine coordinates as in Construction 2.2. Assume that $\{0, 1, 2\}$ is a basis of M, let $i = n - 1$, and suppose first 3 columns of X form the identity matrix. Suppose $n - 1$ is not contained in any line. This means that $\{i, j, n - 1\} \in \mathcal{B}(M)$ for $0 \le i < j \le n - 2$, so I_M^x is generated by X_λ for suitable $\lambda \in \binom{[n-1]}{3}$. Therefore R_M^x is obtained from $R_{M|[n-1]}^x[x_{0,n-4}^{\pm}, x_{1,n-4}^{\pm}, x_{2,n-4}^{\pm}]$ by inverting X_β for $\beta \in \mathcal{B}(M)$. These ring elements are nonzero divisors since they are not 0 (by $\mathbf{k}$-realizability) and $R_{M|[n-1]}$ is an integral domain. This localization produces the open immersion $\mathrm{Gr}_M \hookrightarrow \mathrm{Gr}_{M|[n-1]} \times \mathbb{G}_m^3$.

Suppose $n-1$ is contained in exactly one line η. By applying a suitable permutation, assume $0, 1 \in \eta$. Since $\lambda_{2,j-3} \notin \mathcal{B}(M)$ when $j \in \eta$, the ideal I_M^x is generated by X_λ for suitable $\lambda \in \binom{[n-1]}{3}$, and R_M^x is obtained from $R_{M|[n-1]}^x[x_{0,n-4}^{\pm}, x_{1,n-4}^{\pm}]$ by inverting the nonzero divisors X_β for $\beta \in \mathcal{B}(M)$, producing the open immersion $\mathrm{Gr}_M \hookrightarrow \mathrm{Gr}_{M|[n-1]} \times \mathbb{G}_m^2$.

Now assume n is contained in exactly two distinct lines η_1 and η_2. We may assume $0, 1 \in \eta_1$ and $2 \in \eta_2$. Because $\lambda_{0,j-3}, \lambda_{1,j-3} \in \mathcal{B}(M)$ when $j \in \eta_2\setminus\{n - 1\}$, the corresponding $x_{0,j-3}, x_{1,j-3}$ are invertible in R_M^x. Similar to the previous case,

$$R_M^x = (S_M^x)^{-1} R_{M|[n-1]}^x[x_{0,n-4}^{\pm}, x_{1,n-4}^{\pm}]/\langle x_{0,j-3}x_{1,n-4} - x_{1,j-3}x_{0,n-4} \mid j \in \eta_2\setminus\{2\}\rangle.$$

Since $|\eta_2| \ge 3$, this ring is isomorphic to $(S_M^x)^{-1} R_{M|[n-1]}^x[x_{1,n-1}^{\pm}]$, and we have an open immersion $\mathrm{Gr}_M \hookrightarrow \mathrm{Gr}_{M|[n-1]} \times \mathbb{G}_m$. Finally, the map $\mathrm{Gr}_M \to \mathrm{Gr}_{M|[n-1]}$ is a SDC-morphism by Proposition A.3 and the fact that the projection away from $\mathbb{G}_m^{3-k}$ is SDC. □

Proposition 4.2 *For $3 \le n \le 7$, the thin Schubert cell Gr_M is smooth and irreducible for any $\mathbf{k}$-realizable $(3, n)$-matroid M.*

Proof The only $(3, 3)$-matroid is $U(3, 3)$, and its thin Schubert cell consists of a single point. Next, of the four $(3, 4)$-matroids up to S_4-symmetry, the matroid $U(3, 4)$ is the only one that is simple and connected. Since $\mathrm{Gr}_0(3, 4) \cong \mathbb{G}_m^3$, it is smooth an irreducible. That the thin Schubert cells of the remaining three are also smooth and irreducible follows from Lemmas C.1, C.2, Proposition C.3, and the $(3, 3)$-case. If M is a $(3, 5)$-matroid, then M^* is a $(2, 5)$-matroid, so Gr_M is smooth and irreducible by Proposition C.3 and the isomorphism $\mathrm{Gr}_M \cong \mathrm{Gr}_{M^*}$.

Next, consider the $(3, 6)$ case. As before, we need only examine the simple and connected matroids. For every such matroid M, each $i \in [6]$ is contained in 2 or fewer lines of M. Therefore, the thin Schubert cell Gr_M is smooth and irreducible by

Lemma 4.1, Proposition A.1(2), and the previous cases. Finally, if M is any simple and connected $(3,7)$-matroid other than the Fano matroid, then M has an $i \in [n]$ contained in no more than 2 lines. Similar to the $(3,6)$ case, the thin Schubert cell Gr_M is smooth and irreducible. □

5 Morphisms between thin Schubert cells

In this section, we will consider the morphisms $\varphi_{M,M'} : \mathrm{Gr}_M \to \mathrm{Gr}_{M'}$. For arbitrary matroids M, SDC-properties of the morphisms $\varphi_{M,M'}$ are entirely determined by $\varphi_{M|S,M'|S}$ with S as in the beginning of Sect. 4, hence we need only consider simple M. These reductions are contained in Appendix C, and yield a straightforward proof that $\varphi_{M,M'}$ is a SDC-morphism when M is a $(2,n)$- matroid. Therefore, we focus on the rank 3 case. For the proof of Theorem 1.2, we will only need to verify that $\varphi_{M,M'}$ is a SDC-morphism for pairs $M' \lessdot M$ of $(3,7)$-matroids where $Q_{M'}$ is not a face of the hypersimplex. To do this, we will find it convenient to show that $\varphi_{M,M'}$ is a SDC-morphism for all pairs of $(3,m)$-matroids $M' \leq M$ where $m \leq 6$. Recall from Proposition 2.1 that the facets of Q_M correspond to the nondegenerate subsets of $[n]$, when M is connected. We begin with a test for nondegeneracy in the rank 3 setting.

Proposition 5.1 *Let M be a simple and connected $\mathbf{k}$-realizable $(3,n)$-matroid, and $\eta \subset [n]$. Then η is nondegenerate if and only if either*

1. *$|\eta| = 1$ and M/η is connected, or*
2. *$|\eta| = n-1$ and $M|\eta$ is connected, or*
3. *η is a line.*

Proof First, we claim that if η is nondegenerate, then $|\eta| = 1, n-1$ or η is a line. To that end, fix a subset η such that $1 < |\eta| < n-1$, and η is not a line (η is clearly degenerate when $|\eta| = 0$ or n). If $|\eta| = 2$, then $M|\eta \cong U(1,1) \oplus U(1,1)$, hence not connected. Otherwise, $2 < |\eta| < n-1$ and $\rho_M(\eta) = 2$ or 3. If $\rho_M(\eta) = 2$, then there is a line η' properly containing η. In this case, every element in $\eta' \backslash \eta$ becomes a loop in M/η. Because M/η has at least 2 elements, having a loop implies that it is not connected. If $\rho_M(\eta) = 3$, then every element in $[n] \backslash \eta$ is a loop in M/η. Since M is connected, we have $n - |\eta| \geq 2$, thus M/η is not connected. In all cases, the set η is degenerate, hence the claim.

If $|\eta| = 1$, then $M|\eta \cong U(1,1)$ which is connected, so η is nondegenerate if and only if M/η is connected. Similarly, if $|\eta| = n-1$, then $M/\eta \cong U(1,1)$ which is connected, so η is nondegenerate if and only if $M|\eta$ is connected. Finally, suppose η is a line. Then $M|\eta \cong U(2,k)$ $(k \geq 3)$ and $M/\eta \cong U(1,\ell)$ $(\ell \geq 2)$, both of which are connected, so η is nondegenerate. □

Lemma 4.1 and the next two lemmas will allow us to trim down the amount of $M' \lessdot M$ that we will need to check in the proofs of Proposition 5.4 and Proposition 5.5.

Lemma 5.2 *Suppose M is simple and connected and η is a line of M. If $i \in [n]$ is not contained in any line and $\mathrm{Gr}_{M|[n]\backslash i}$ is integral, then $M_\eta|([n]\backslash i) \leq M|[n]\backslash i$, and we* *have a commutative diagram*

$$\begin{array}{ccc} \mathrm{Gr}_M & \hookrightarrow & \mathrm{Gr}_{M|[n]\setminus i} \times \mathbb{G}_m^3 \\ {\scriptstyle \varphi_{M,M_\eta}}\downarrow & & \downarrow{\scriptstyle \varphi_{M|[n]\setminus i, M_\eta|[n]\setminus i}\times \pi} \\ \mathrm{Gr}_{M_\eta} & \hookrightarrow & \mathrm{Gr}_{M_\eta|[n]\setminus i} \times \mathbb{G}_m \end{array}$$

where the top and bottom arrows are dominant open immersions, and π is a coordinate projection. In particular, if $\varphi_{M|[n]\setminus i, M_\eta|[n]\setminus i}$ is a SDC-morphism, then so is φ_{M,M_η}.

Proof As usual, we use affine coordinates as in Construction 2.2, assume that $\{0, 1, 2\}$ is a basis and the first 3 columns of X form the identity matrix. We may also assume that $i = n-1$ and $0, 1 \in \eta$. As in the proof of Lemma 4.1, the dominant open immersion $\mathrm{Gr}_M \hookrightarrow \mathrm{Gr}_{M|[n-1]} \times \mathbb{G}_m^3$ is induced by the inversion of X_β in $R^x_{M|[n-1]}[x^{\pm}_{0,n-4}, x^{\pm}_{1,n-4}, x^{\pm}_{2,n-4}]$ for $\beta \in \mathcal{B}(M)$. Since $M_\eta \cong M/\eta \oplus M|\eta$ and M/η has rank 1, all elements of $[n]\setminus\eta$ become parallel to 2 in M_η, in particular $\lambda_{0,j-3} = \lambda_{1,j-3} = 0$ in $R^x_{M_\eta}$ for $j \notin \eta$. Similar to R^x_M, the ring $R^x_{M_\eta}$ is obtained from $R^x_{M_\eta|[n-1]}[x^{\pm}_{2,n-4}]$ by inverting X_β for $\beta \in \mathcal{B}(M_\eta)$. This localization induces a dominant open immersion $\mathrm{Gr}_{M_\eta} \hookrightarrow \mathrm{Gr}_{M_\eta|[n-1]} \times \mathbb{G}_m$. The morphism $\mathbb{G}_m^3 \to \mathbb{G}_m$ is induced by $\mathbf{k}[x^{\pm}_{2,n-4}] \subset \mathbf{k}[x^{\pm}_{0,n-4}, x^{\pm}_{1,n-4}, x^{\pm}_{2,n-4}]$. Commutativity of the diagram is now a simple verification at the level of rings. The last statement follows from Proposition A.3 and the fact that π is a SDC-morphism. □

Lemma 5.3 *Suppose M is simple and connected, the set η is a line of M, and $\mathrm{Gr}_{M|[n]\setminus i}$ is integral. If $i \in \eta$ is not contained in any other line, then $M_{\eta\setminus i} \leq M|[n]\setminus i$, and we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{Gr}_M & \hookrightarrow & \mathrm{Gr}_{M|[n]\setminus i} \times \mathbb{G}_m^2 \\ {\scriptstyle \varphi_{M,M_\eta}}\downarrow & & \downarrow{\scriptstyle \varphi_{M|[n]\setminus i, M_{\eta\setminus i}}\times \mathrm{id}} \\ \mathrm{Gr}_{M_\eta} & \hookrightarrow & \mathrm{Gr}_{M_{\eta\setminus i}} \times \mathbb{G}_m^2 \end{array}$$

where the top and bottom arrows are dominant open immersions. In particular, if $\varphi_{M|[n]\setminus i, M_{\eta\setminus i}}$ is a SDC-morphism, then so is φ_{M,M_η}.

Proof Similar to the proof of Lemma 5.2, we use affine coordinates as in Construction 2.2, assume that $\{0, 1, 2\}$ is a basis, the first 3 columns of X form the identity matrix, set $i = n-1$, and $0, 1, n-1 \in \eta$. As in the proof of Lemma 4.1, the dominant open immersion $\mathrm{Gr}_M \hookrightarrow \mathrm{Gr}_{M|[n-1]} \times \mathbb{G}_m^2$ is induced by the inversion of X_β in $R^x_{M|[n-1]}[x^{\pm}_{0,n-4}, x^{\pm}_{1,n-4}]$ for $\beta \in \mathcal{B}(M)$. Since $M_\eta \cong M/\eta \oplus M|\eta$ and M/η has rank 1, all elements of $[n]\setminus\eta$ become parallel to 2 in M_η. Because $\{0, 2, n-1\}$ and $\{1, 2, n-1\}$ remains bases in M_η, the terms $x_{0,n-4}$ and $x_{1,n-4}$ are still invertible in $R^x_{M_\eta}$. Similar to R^x_M, the ring $R^x_{M_\eta}$ is obtained from $R^x_{M_{\eta\setminus\{n-1\}}}[x^{\pm}_{0,n-4}, x^{\pm}_{1,n-4}]$ by inverting X_β for $\beta \in \mathcal{B}(M_\eta)$. This localization induces a dominant open immersion $\mathrm{Gr}_{M_\eta} \hookrightarrow \mathrm{Gr}_{M_{\eta\setminus\{n-1\}}} \times \mathbb{G}_m^2$. Commutativity of the diagram is now a simple verification at the level of rings. The last statement follows from Proposition A.3. □

Proposition 5.4 *Let M be a $(3, n)$ matroid for $3 \le n \le 6$ and $M' \le M$. Then $\varphi_{M,M'} : \mathrm{Gr}_M \to \mathrm{Gr}_{M'}$ is a SDC-morphism.*

Proof By Lemmas C.1 and C.2, it suffices to consider pairs of matroids of the form $M' \lessdot M$ where M is simple and connected. The only $(3, 3)$-matroid is $U(3, 3)$, and $\Delta(3, 3)$ is a point, so there is nothing to check. For $n = 4$, the only simple and connected matroid is $U(3, 4)$ so $\varphi_{M,M'}$ is a SDC-morphism by Proposition C.7. The case $n = 5$ follows from Proposition C.8 and Lemma C.6.

Finally consider $n = 6$. We may assume that M and M^* are simple by Lemma C.6, and $M' = M_\eta$ where $|\eta| = 1, n - 1$ or η is a line by Proposition 5.1. It suffices to consider pairs $M_\eta \lessdot M$ such that one of the following holds:

1. $\eta = [n]\backslash\{i\}$ and i is contained in 3 or more lines (Lemma 4.1),
2. $\eta = \{i\}$ and i is contained in 3 or more lines of M^* (Lemma C.6), or
3. $\eta \in \mathcal{L}(M)$, every $i \in [n]$ is contained in a line, (Lemma 5.2) and every $j \in \eta$ is contained in another line (Lemma 5.3).

For $(3, 6)$ matroids (1) and (2) can never happen. Up to symmetry, there is only one pair that satisfies (3):

$$\mathcal{L}(M) = \{\{0, 1, 3\}, \{0, 2, 4\}, \{1, 2, 5\}, \{3, 4, 5\}\}$$

and $\eta = \{0, 1, 3\}$. By the isomorphism $M_\eta \cong M/\eta \oplus M|\eta$, the set $\{0, 1, 3\}$ is the only line of M_η and $2, 4, 5$ are parallel to each other. We use affine coordinates as in Construction 2.2. Assume that the first 3 columns of X form the identity matrix, so $R^x_{M'} = \mathbf{k}[x^{\pm}_{00}, x^{\pm}_{10}, x^{\pm}_{11}, x^{\pm}_{22}]$ and

$$R^x_M = R^x_{M'}[x^{\pm}_{01}, x^{\pm}_{12}, x^{\pm}_{21}]/\left\langle x_{00}x_{12}x_{21} + x_{01}x_{10}x_{22}\right\rangle \cong R^x_{M'}[x^{\pm}_{12}, x^{\pm}_{21}].$$

Then $R^x_{M_\eta} \to R^x_M$ may be identified with the inclusion $R^x_{M_\eta} \subset R^x_{M_\eta}[x^{\pm}_{12}, x^{\pm}_{21}]$ and therefore φ_{M,M_η} is a SDC-morphism. □

Proposition 5.5 *Let M be a $(3, 7)$-matroid and $M' \le M$ such that $Q_{M'}$ is not a face of $\Delta(3, 7)$. Then $\varphi_{M,M'} : \mathrm{Gr}_M \to \mathrm{Gr}_{M'}$ is a SDC-morphism.*

Proof By Lemmas C.1, C.5, 5.2 and 5.3, we may assume that M is simple, connected, every element in [7] is contained in a line, and there is a line η with the property that every $i \in \eta$ is contained in another line. There are only six such matroids. We list these in Table 1, together with their nondegenerate subsets (up to symmetry) that define internal facets, i.e., those facets that are not faces of $\Delta(3, 7)$. This has the effect of excluding the subsets of size 1 or 6. The representatives of the nondegenerate subsets are chosen so that $\{0, 1, 2\}$ is a basis of both M_η and M whenever we need to perform an explicit computation in affine coordinates.

Cases 7.3(1), 7.4(2), 7.5(2), 7.6(2) follow from Lemma 5.2, and case 7.6(1) is similar to the case worked out in the proof of Proposition 5.4 (indeed, the matroid in 7.6 is obtained by adding an element to a line of this matroid). For these remaining cases, we proceed by a direct computation using affine coordinates as in Construction

Table 1 The simple connected rank 3 matroids on [7] relevant to Proposition 5.5, together with nondegenerate subsets defining internal facets

	$\mathcal{L}(M)$	Internal facets (Aut(M)-representatives)
7.1	{0, 1, 3}, {0, 2, 4}, {1, 2, 5}, {2, 3, 6}, {4, 5, 6}	(1) {0, 1, 3}, (2) {0, 2, 4}
7.2	{0, 1, 3}, {0, 2, 4}, {0, 5, 6}, {1, 2, 5}, {1, 4, 6}, {2, 3, 6}	(1) {0, 1, 3}
7.3	{0, 1, 3}, {0, 2, 4}, {1, 2, 5}, {4, 5, 6}	(1) {0, 1, 3}, (2) {0, 2, 4}
7.4	{0, 1, 3}, {0, 2, 4}, {1, 2, 5}, {2, 3, 6}	(1) {0, 1, 3}, (2) {0, 2, 4}
7.5	{0, 1, 3}, {0, 2, 4}, {1, 2, 5}, {1, 4, 6}, {2, 3, 6}	(1) {0, 1, 3}, (2) {1, 2, 5}
7.6	{0, 1, 5}, {0, 2, 3, 6}, {1, 4, 6}, {3, 4, 5}	(1) {0, 1, 5}, (2) {0, 2, 3, 6}

2.2. The first 3 columns of X will always be the identity matrix. As in the proof of Proposition 5.4, the isomorphism $M_\eta \cong M/\eta \oplus M|\eta$ gives a simple way to identify M_η.

Let $M' \lessdot M$ be the pair in Case 7.1(1). Then $R^x_{M'} = \mathbf{k}[x_{00}^{\pm}, x_{10}^{\pm}, x_{21}^{\pm}, x_{22}^{\pm}, x_{23}^{\pm}]$ and R^x_M is the quotient of $(S^x_M)^{-1}R^x_{M'}[x_{01}^{\pm}, x_{12}^{\pm}, x_{03}^{\pm}, x_{13}^{\pm}]$ by the ideal

$$\langle x_{00}x_{13} - x_{10}x_{03},\ x_{01}(x_{12}x_{23} - x_{22}x_{13}) - x_{21}x_{12}x_{03}\rangle .$$

Because $\overline{X}_{056} = x_{12}x_{23} - x_{22}x_{13}$ is in S^x_M, we have $R^x_M \cong R^x_{M'}[x_{12}^{\pm}, x_{13}^{\pm}]$. So $R^x_{M'} \to R^x_M$ may be identified with the inclusion $R^x_{M'} \subset (S^x_M)^{-1}R^x_{M'}[x_{12}^{\pm}, x_{13}^{\pm}]$. Therefore $\varphi_{M,M'}$ is a SDC-morphism.

Next consider the pair $M' \lessdot M$ in case 7.1(2). Then $R^x_{M'} = \mathbf{k}[x_{10}^{\pm}, x_{01}^{\pm}, x_{21}^{\pm}, x_{12}^{\pm}, x_{13}^{\pm}]$. By eliminating the variables x_{00} and x_{22} from R^x_M, we may identify the morphism $R^x_{M'} \to R^x_M$ with the inclusion $R^x_{M'} \subset (S^x_M)^{-1}R^x_{M'}[x_{03}^{\pm}, x_{23}^{\pm}]$. Therefore $\varphi_{M,M'}$ is SDC. Because the matroid in 7.3 is obtained from M by removing one line, case 7.3(2) is similar.

Finally consider the pair $M' \lessdot M$ in case 7.2(1). Then $R^x_{M'} = \mathbf{k}[x_{00}^{\pm}, x_{10}^{\pm}, x_{21}^{\pm}, x_{22}^{\pm}, x_{23}^{\pm}]$ and R^x_M is the quotient of $(S^x_M)^{-1}R^x_{M'}[x_{01}^{\pm}, x_{12}^{\pm}, x_{03}^{\pm}, x_{13}^{\pm}]$ by the ideal

$$\langle x_{12}x_{23} - x_{13}x_{22}, x_{01}x_{23} - x_{21}x_{03}, x_{00}x_{13} - x_{10}x_{03}\rangle.$$

By eliminating the variables x_{13}, x_{03}, and x_{12}, $R^x_{M'} \to R^x_M$ may be identified with the inclusion $R^x_{M'} \subset (S^x_M)^{-1}R^x_{M'}[x_{01}^{\pm}]$. Therefore $\varphi_{M,M'}$ is SDC. Because the matroids in 7.4 and 7.5 are obtained by removing two, resp. one, lines from M, cases 7.4(1) and 7.5(1) are similar. □

6 Smoothness and irreducibility of initial degenerations

Let M be a $\mathbf{k}$-realizable $(2, n)$, $(3, 6)$, or $(3, 7)$ matroid. We compile the results of the previous sections to prove the following more general version of Theorem 1.2.

Theorem 6.1 *The initial degenerations* $\mathrm{in}_w \mathrm{Gr}_M$ *are smooth and irreducible for all* $w \in \mathrm{TGr}_M$.

By Corollary 3.5, we must show that $\mathrm{Gr}_{M,w}$ is smooth, irreducible, and has the same dimension as Gr_M. Thanks to Proposition C.12, we may compute $\mathrm{Gr}_{M,w}$ as a limit over a diagram induced by a graph as in Example A.4. When $\Gamma_{M,w}$ is a tree, Proposition A.6 tells us that $\mathrm{Gr}_{M,w}$ is smooth and irreducible when Gr_{M_Q} is smooth and irreducible and $\varphi_{M_Q,M_{Q'}} : \mathrm{Gr}_{M_Q} \to \mathrm{Gr}_{M_{Q'}}$ is a SDC-morphism for $Q \in (\Delta_{M,w})^{\mathrm{top}}$ and $Q' \lessdot Q$ not a face of the hypersimplex. This is illustrated in Example 6.2. However, when $\Gamma_{M,w}$ is not a tree, this data is insufficient to conclude that $\mathrm{Gr}_{M,w}$ is smooth and irreducible, see Remark A.7.

Let $\eta \subset [n]$. In the examples below and in Sect. 8, we encounter the matroids $M(\eta)$ and $M(\eta)'$ defined by

$$\mathcal{B}(M(\eta)) = \left\{ \beta \in \binom{[n]}{3} \,\middle|\, |\beta \cap \eta| \geq 2 \right\}, \ \mathcal{B}(M(\eta)') = \left\{ \beta \in \binom{[n]}{3} \,\middle|\, |\beta \cap \eta| = 2 \right\}. \tag{6.1}$$

A simple computation in affine coordinates yields $\dim \mathrm{Gr}_{M(\eta)} = n + 2|\lambda| - 7$, and $\dim \mathrm{Gr}_{M(\eta)'} = n + |\lambda| - 5$. Also, we set $f_\lambda = e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_d} \in \wedge^d \mathbb{R}^n$ for $\lambda = \{\lambda_1, \ldots, \lambda_d\}$.

Example 6.2 Let

$$w = f_{013} + f_{024} + f_{056} + f_{125} + f_{146} + f_{236}$$

and C matroid 7.2 from Table 1. The adjacency graph Γ_w is a star tree with v_{Q_C} as the central vertex and a leaf vertex $v_{Q_{M(ijk)}}$ for each $\{i, j, k\} \in \mathcal{L}(C)$. The edge between C and $M(ijk)$ corresponds to the matroid $M(ijk)'$. Because Δ_w is a matroid subdivision and does not lie in the relative interior of the Fano cone as in [19, Theorem 2.1], the vector w lies in $\mathrm{TGr}_0(3, 7)$. The isomorphism from Proposition C.12 yields

$$\mathrm{Gr}_w \cong \mathrm{Gr}_C \times_{\prod \mathrm{Gr}_{M(ijk)'}} \prod \mathrm{Gr}_{M(ijk)}.$$

The thin Schubert cell Gr_C is smooth and irreducible by Proposition 4.2 and the $\mathrm{Gr}_{M(ijk)} \to \mathrm{Gr}_{M(ijk)'}$ are SDC-morphisms by Proposition 5.5. From the preceding comments, we have $\dim \mathrm{Gr}_{M(ijk)} = 6$ and $\dim \mathrm{Gr}_{M(ijk)'} = 5$. A simple computation in affine coordinates yields $\dim \mathrm{Gr}_C = 6$. By Proposition A.6, the scheme Gr_w is smooth and irreducible of dimension 12, as is $\mathrm{in}_w \mathrm{Gr}_0(3, 7)$ by Corollary 3.5.

When the maximal cells Q of $\Delta_{M,w}$ all share a common vertex, we may determine whether $R^x_{M,w}$ defines a smooth and irreducible $\mathbf{k}$-scheme by hand, as illustrated in Example 6.3. However, many matroid subdivisions do not have this property, e.g., the subdivision $\Delta(3, 7)$ in the previous example.

Example 6.3 Let M be the following matroid

$$\mathcal{L}(M) = \{\{0, 2, 4\},\ \{0, 3, 6\},\ \{1, 2, 3\},\ \{1, 4, 6\},\ \{2, 5, 6\}\}.$$

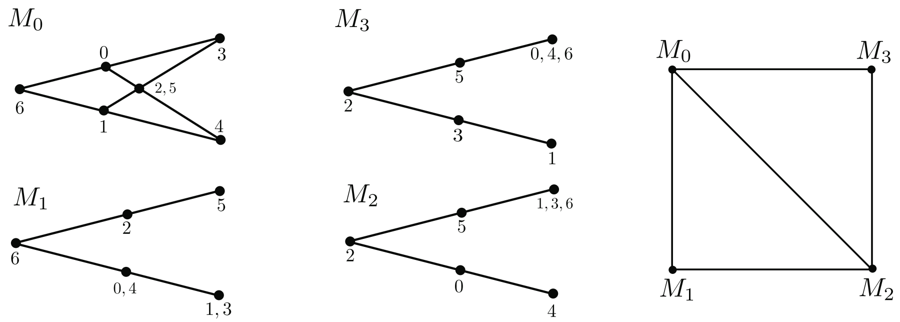

Fig. 1 The matroids and adjacency graph appearing in Example 6.3

and set

$$w = -f_{013} + f_{345} - f_{016} + f_{245} + f_{246} + f_{234} + f_{145} - f_{135} - f_{136} + f_{124} + f_{456}.$$

The subdivision $\Delta_{M,w}$ is matroidal. The matroids of maximal cells and $\Gamma_{M,w}$ are illustrated in Fig. 1. Similar to Example 6.2, we see that $w \in \mathrm{TGr}_M$. Because $\{0, 1, 2\}$ is a basis for each M_i, the ring $R^x_{M,w}$ may be computed using affine coordinates as in Construction 2.2. Assume that the first 3 columns of the matrix X form the identity. We have $I^x_{M_2} = I^x_{M_3} = \langle 0 \rangle$, and

$$I^x_{M_0} = \langle x_{10}x_{23} - x_{13}x_{20}, x_{01}x_{23} - x_{21}x_{03} \rangle, \qquad I^x_{M_1} = \langle x_{02}x_{13} - x_{03}x_{12} \rangle.$$

Therefore $I^x_{M,w} = \langle x_{10}x_{23} - x_{13}x_{20}, x_{02}x_{13} - x_{03}x_{12}, x_{01}x_{23} - x_{21}x_{03} \rangle$. Because x_{03}, x_{13} are in $S^x_{M,w}$, we may solve for these variables to produce an isomorphism

$$R^x_{M,w} \cong (S^x_{M,w})^{-1}\mathbf{k}[x^{\pm}_{01}, x^{\pm}_{02}, x^{\pm}_{10}, x^{\pm}_{12}, x^{\pm}_{20}, x^{\pm}_{21}, x^{\pm}_{22}, x^{\pm}_{23}].$$

This realizes $\mathrm{Gr}_{M,w}$ as an open subscheme of $\mathbb{G}^8_m$. Therefore $\mathrm{Gr}_{M,w}$ is smooth and irreducible of dimension 8, as is $\mathrm{in}_w \mathrm{Gr}_M$ by Corollary 3.5.

In general, we use a combination of the above techniques to show that all of the $\mathrm{Gr}_{M,w}$ are smooth and irreducible. Lemma 6.4 handles the case where $\Gamma_{M,w}$ has no leaves, showing that $\mathrm{Gr}_{M,w}$ is smooth and irreducible by a direct analysis of $R^x_{M,w}$ (for these subdivisions, all maximal cells share a common vertex). We take care of the remaining cases using this lemma together with Proposition A.2, which considers the behavior of smoothness and irreducibility under pullbacks.

Lemma 6.4 *Let M be a $\mathbf{k}$-realizable, rank 3-matroid on $[6]$ or $[7]$, and $w \in \mathrm{TGr}_M$ such that $\Gamma_{M,w}$ has no leaves. Then $\mathrm{Gr}_{M,w}$ is smooth and irreducible.*

Proof By Proposition C.9, we may assume that M is simple. We will work with affine coordinates as in Construction 2.2, and follow a strategy similar to Example 6.3. First, suppose e_β common to all $Q \in (\Delta_{M,w})^{\mathrm{top}}$ (if such a vertex exists). Let the columns of X prescribed by β be the identity matrix. Let

$$g^x_{M_j} = \prod_{\lambda \in \mathcal{B}(M_j)} \overline{X}_\lambda \quad \text{and} \quad g^x_{M,w} = \prod_{Q_{M_j} \text{ maximal}} g^x_{M_j}.$$

Finally, let $J^x_{M,w} = \left(I^x_{M,w} : (g^x_{M,w})^\infty\right) \subset B_M$ (saturation here has the effect of removing the primary components of the irrelevant ideal). By Proposition 3.7, to show that $\mathrm{Gr}_{M,w}$ is smooth an irreducible, it suffices to show that the the quotient of $\mathbf{k}[x^\pm_{ij} \mid \lambda_{ij} \in \mathcal{B}(M)]$ by the extension of $J^x_{M,w}$ is isomorphic to a Laurent polynomial ring. Due to the large number of cases that we need to check, we will find it more convenient to show that $\mathbf{k}[x^\pm_{ij}]/(J^x_{M,w} \cdot \mathbf{k}[x^\pm_{ij}])$ has this property.

We proceed by a direct computation, using computer assistance. (We emphasize that this computation may be carried out by hand for any individual w, once $\Delta_{M,w}$ is computed. We use a computer due to the large number of cases.) Representatives w of the cones in $\mathcal{G}_M$ were computed in proof of Proposition 3.9, along with the subdivisions $\Delta_{M,w}$. We use `Macaulay2` to compute the adjacency graphs and catalog those w such that $\Gamma_{M,w}$ has no leaves. There are 17 such graphs among all simple (3, 6)-matroids, and 877 for (3, 7). For each such (M, w), there is a vertex e_β common all of the $Q \in (\Delta_{M,w})^{\mathrm{top}}$. We choose such a β that is maximal with respect to the `revLex` order, compute $J^x_{M,w}$ as above, and consider its extension to $\mathbf{k}[x^\pm_{ij}]$. While this produces a large number of ideals, many end up being the same. For (3, 6), computing the ideals takes about 15 seconds, and there are 3 unique ideals:

$$\langle 0\rangle, \langle x_{02}x_{11} - x_{01}x_{12}\rangle, \langle x_{02}x_{10} - x_{00}x_{12}\rangle \subset \mathbf{k}[x^\pm_{ij}].$$

By solving for x_{12} in the last two ideals, we see that the quotients $\mathbf{k}[x^\pm_{ij}]/(J^x_{M,w} \cdot \mathbf{k}[x^\pm_{ij}])$ are all isomorphic to Laurent polynomial rings. For (3, 7), this computation takes about 50 min. We list these ideals in Appendix B, together with variables that may be eliminated to produce an isomorphism of $\mathbf{k}[x^\pm_{ij}]/(J^x_{M,w} \cdot \mathbf{k}[x^\pm_{ij}])$ with a Laurent polynomial ring. □

Let G be a connected graph. Given a leaf-vertex v, the *branch* of G containing v is the largest full subgraph of G that contains v and does not meet any cycle of G (note that this is non-standard terminology).

Theorem 6.5 *Let $w \in \mathrm{TGr}_M$ where M is a $\mathbf{k}$-realizable $(2, n)$, $(3, 6)$, or $(3, 7)$ matroid. Then $\mathrm{Gr}_{M,w}$ is smooth and irreducible.*

Proof By Proposition C.12, we know that $\mathrm{Gr}_{M,w}$ is isomorphic to a limit over the adjacency graph $\Gamma_{M,w}$. Since all relevant thin Schubert cells are smooth and irreducible, and all relevant morphisms are SDC, we may use Proposition A.6 to conclude that $\mathrm{Gr}_{M,w}$ is smooth an irreducible when $\Gamma_{M,w}$ is a tree. In particular, this completes the proof in the $d = 2$ case.

Now suppose $(d, n) = (3, 6)$ or $(3, 7)$. We need only consider those w such that $\Gamma_{M,w}$ is not a tree. We proceed by induction on the largest diameter of a branch of

$\Gamma_{M,w}$. When $\Gamma_{M,w}$ has no leaves, the scheme $\mathrm{Gr}_{M,w}$ is smooth and irreducible from Lemma 6.4, hence the base case of the induction.

Let $v_{Q_1}, \ldots, v_{Q_k}$ denote leaf vertices of $\Gamma_{M,w}$, let $L_i = M_{Q_i}$, and let L_i' for the matroid corresponding to the edge adjacent to v_{Q_i}. There is a hyperplane H_k in $\mathbb{R}^n$ such that $\Delta_M \cap H_k = \Delta_{L_k'}$, with Δ_{L_k} in one of the halfspaces of this hyperplane. The polytope given by the intersection of Δ_M with the other halfspace is also a matroid polytope: it is the convex hull of the vertices e_β such that $\beta \in (\mathcal{B}(M)\backslash\mathcal{B}(L_k)) \cup \mathcal{B}(L_k')$. The adjacency graph to this subdivision is obtained by removing the vertex and edge corresponding to Δ_{L_k} and $\Delta_{L_k'}$ respectively. Repeating this procedure for the remaining L_i's, we see that the union of the polytopes corresponding to non-leaf vertices in $\Gamma_{M,w}$ is a matroid polytope. We denote the corresponding matroid by C. By Proposition A.5 and C.12,

$$\mathrm{Gr}_{M,w} \simeq \mathrm{Gr}_{C,w} \times_{\prod \mathrm{Gr}_{L_i'}} \prod \mathrm{Gr}_{L_i} .$$

The $\mathbf{k}$-scheme $\mathrm{Gr}_{C,w}$ is smooth and irreducible by the inductive hypothesis and $\varphi_{L_i,L_i'}$ are SDC-morphisms by Proposition 5.4 and Proposition 5.5. We conclude that $\mathrm{Gr}_{M,w}$ is smooth and irreducible by Proposition A.2. □

Corollary 6.6 *If M is a $\mathbf{k}$-realizable $(3, 6)$ or $(3, 7)$ matroid and $w \in \mathrm{TGr}_M$, then $\psi_{M,w} : \mathrm{in}_w \mathrm{Gr}_M \to \mathrm{Gr}_{M,w}$ is an isomorphism.*

Proof By Theorem 3.4 and Proposition 3.9, the map $\psi_{M,w}$ is a closed immersion of affine schemes of the same dimension. Moreover $\mathrm{Gr}_{M,w}$ is integral by Theorem 6.5. Therefore $\psi_{M,w}$ is an isomorphism by Proposition A.8. □

Proofs of Theorems 1.2 and 6.1. These theorems follow from Lemma 3.8, Theorem 6.5, and Corollary 6.6. □

Remark 6.7 We are indebted to the anonymous referee for pointing out the following consequence of Theorem 6.1. Let X_0 be a closed subvariety of an algebraic torus T. Denote by X_0^{an} the Berkovich analytification of X_0 [3]. The tropicalization Trop X_0 is *faithful* if there is a continuous section to $\mathrm{Trop} : X_0^{\mathrm{an}} \to \mathrm{Trop}\, X_0$. By [14, Theorem 10.6], the tropicalization Trop X_0 is faithful when $\mathrm{in}_w X_0$ is reduced and irreducible for all $w \in \mathrm{Trop}\, X_0$. Together with Theorem 6.1, this implies that the tropicalization TGr_M is faithful for any $\mathbf{k}$-realizable $(2, n)$, $(3, 6)$, or $(3, 7)$-matroid.

7 The log canonical compactification of $X_0(3, 7)$

We now prove Theorem 1.3, that the normalization of the Chow quotient of $\mathrm{Gr}(3, 7)$ by the maximal torus $\mathrm{PGL}(7)$ is the log canonical compactification of $X_0(3, 7)$. For background on log minimality and log canonical compactifications, see the introduction of [16], for the Chow quotient of $\mathrm{Gr}(d, n)$, see [21] [26, Section 2], and for schön compactifications, see [15,29,40]. Throughout this section, we use the following notation for polyhedral fans and toric varieties that is consistent with [7]. Let N be a lattice, let T_N its torus, and let Σ a rational polyhedral fan in $N_{\mathbb{R}}$. When T is a torus, we write N_T

for its cocharacter lattice. Given a cone σ of Σ, denote by N_σ the saturated sublattice of N generated by $\sigma \cap N$. Let $N(\sigma) = N/N_\sigma$, and $\mathrm{Star}(\sigma)$ the star of σ, viewed as a fan in $N(\sigma)_\mathbb{R}$. We write $X(\Sigma)$ for the toric variety of Σ.

Let $H \subset \mathrm{PGL}(n)$ be the maximal torus, let M a loop-free matroid, and let $T(M)$ the dense torus of $\mathrm{Proj}(B_M)$. As before, we let $\{e_i \mid i \in [n]\}$ denote the standard basis of $\mathbb{Z}^n$ and $f_\lambda = e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_d}$ for $\lambda = \{\lambda_1, \ldots, \lambda_d\}$. The cocharacter lattices of H and $T(M)$ isomorphic to $\mathbb{Z}^n/\mathbb{Z}\cdot\mathbf{1}$ and $\mathbb{Z}^{\mathcal{B}(M)}/\mathbb{Z}\cdot\mathbf{1}$, respectively. The torus H embeds into $T(M)$ by

$$N_H \to N_{T(M)} \qquad e_i \mapsto \sum_{\lambda \ni i} f_\lambda. \tag{7.1}$$

Thus H acts on $\mathrm{Proj}(B_M)$ via the action of $T(M)$. This restricts to a free action on Gr_M, and we set $X_M = \mathrm{Gr}_M/H$. The quotient $\mathrm{Gr}_M \to X_M$ is induced by a monomial ring map [12, Proposition 2.1], therefore $\mathrm{Trop}\, X_M = \mathrm{TGr}_M/(N_H)_\mathbb{R}$.

Now we focus on $M = U(d,n)$; In this case we write $T = T(U(d,n))$ and $X_0(d,n) = X_{U(d,n)}$. The Plücker embedding induces an closed immersion of Chow quotients $\mathrm{Gr}(d,n)//H \hookrightarrow \mathbb{P}(\wedge^d \mathbf{k}^n)//H$. By [21,22], the normalization of $\mathbb{P}(\wedge^d \mathbf{k}^n)//H$ is the toric variety $Y_{d,n} := X(\Sigma_\mathrm{S}(d,n)/(N_H)_\mathbb{R})$. Let $X_\mathrm{S}(d,n)$ be the closure of $X_0(d,n)$ in $Y_{d,n}$. Then $Y_{d,n} \to \mathbb{P}(\wedge^d \mathbf{k}^n)//H$ induces a birational morphism $X_\mathrm{S}(d,n) \to \mathrm{Gr}(d,n)//H$, thus both have the same normalization, which we denote by $X(d,n)$. When $n = 6, 7$, the space $X(3,n)$ is also the closure of $X_0(3,n)$ in $X(\mathscr{S}''_{3,n})$ where $\mathscr{S}''_{3,n} = \mathscr{S}_{3,n}/(N_H)_\mathbb{R}$.

Lemma 7.1 *The initial degenerations of $X_0(3,7)$ are smooth and irreducible. In particular, the space $X(3,7)$ is a schön compactification of $X_0(3,7)$.*

Proof Let N_H^{sat} denote the saturation of the image of the map from Eq. (7.1). A splitting of the exact sequence $0 \to N_H^{\mathrm{sat}} \to N_T \to N_{T/H} \to 0$ induces an isomorphism $\mathrm{Gr}_0(d,n) \cong X_0(d,n) \times H$. As stated earlier, this is monomial at the level of coordinate rings. Therefore $\mathrm{in}_w \mathrm{Gr}_0(3,7) \cong \mathrm{in}_{\tilde{w}} X_0(3,7) \times H$ where $\tilde{w}$ is the projection of w to $(N_{T/H})_\mathbb{R}$. The first statement now follows from Theorem 1.2. By [29, Theorem 1.5] $X_\mathrm{S}(3,7)$ is a schön compactification, which is already normal by [40, Theorem 1.4]. □

Proof of Theorem 1.3 By Lemma 7.1, the space X is a schön compactification of $X_0(3,7)$. Let B the boundary divisor of $X_0(3,7) \subset X(3,7)$. To show that $K_{X(3,7)} + B$ is ample, we follow a strategy laid out in [28] based on [16].

For each cone $\sigma \in \mathscr{S}''_{3,7}$, let X_σ denote the locally closed stratum of $X(3,7)$ in the corresponding torus orbit of $X(\mathscr{S}''_{3,7})$. There is an isomorphism $\mathrm{in}_w X_0(3,7) \cong X_\sigma \times T_{N_\sigma}$ for any w in the relative interior of σ [18, Lemma 3.6], so each X_σ is smooth and irreducible by Lemma 7.1. Because $X(3,7)$ is a schön compactification, the pair $(X(3,7), B)$ has at worst toroidal singularities [40, Theorem 1.4]. By [16, Theorem 9.1], the divisor $K_{X(3,7)} + B$ is ample if and only if each X_σ is log minimal. We know that $X_0(3,7)$ is log minimal by [26, Proposition 2.18], so we need only consider the X_σ for $\sigma \neq 0$.

By [30, Lemma 3.3.6] $\mathrm{Trop}\, X_\sigma$ is the underlying set of $\mathrm{Star}(\sigma)$ in $N(\sigma)_\mathbb{R}$. The stratum X_σ is schön because its closure in $X(\mathrm{Star}(\sigma))$ is a schön compactification.

Therefore, the stratum X_σ is either log minimal or preserved by a nontrivial subtorus $S \subset T_{N(\sigma)}$ [16, Theorem 3.1], which occurs if and only if Trop X_σ is invariant under translation by the subspace $(N_S)_\mathbb{R} \subset N(\sigma)_\mathbb{R}$ [24, Lemma 5.2]. So it suffices to show that each Trop X_σ is not invariant under translation by any rational subspace of $N(\sigma)_\mathbb{R}$. We prove this in Lemma 7.3, using the necessary condition for such subspaces in Lemma 7.2. □

Lemma 7.2 *Suppose Σ is a rational polyhedral fan in $N_\mathbb{R}$ that is invariant under translation by the linear subspace $V \subset N_\mathbb{R}$. Then $V \subset (N_\sigma)_\mathbb{R}$ for every maximal cone σ of Σ.*

Proof If σ is a maximal cone such that $V \not\subset N_\sigma$ then $\dim(V + N_\sigma) > \dim \Sigma$, therefore V cannot preserve Σ. □

Lemma 7.3 *For each cone σ of $\mathscr{S}''_{3,7}$, the set* Trop X_σ *is not preserved under translation by any rational subspace of $N(\sigma)_\mathbb{R}$.*

Proof The case $\sigma = 0$ follows from the fact that $X_0(3, 7)$ is schön and log minimal as in the proof of Theorem 1.3, so we focus on $\sigma \neq 0$. By Lemma 7.2, to prove that $\mathrm{Trop}(X_\sigma)$ is not preserved under translation by any rational subspace of $N(\sigma)_\mathbb{R}$ it suffices to show

$$(N_\sigma)_\mathbb{R} = \bigcap (N_\tau)_\mathbb{R} \tag{7.2}$$

where the intersection is taken over all maximal cones $\tau \in \mathrm{Star}(\sigma)$. By symmetry, it suffices to show Eq. (7.2) for a collection of S_7-orbit representatives of the σ.

The Gröbner fan $\mathcal{G}_{3,7}$ was computed in `gfan` (as before), and we use `sage` to compute $\mathscr{S}''_{3,7}$ by grouping together those cones that correspond to the same matroid subdivision of $\Delta(3, 7)$. The f-vector (starting at dimension 0) for $\mathscr{S}''_{3,7}$ up to S_7-symmetry is

$$f(\mathscr{S}''_{3,7} \mod S_7) = (1, 5, 30, 107, 217, 218, 94).$$

For each representative we compute $\mathrm{Star}(\sigma)$ and the intersection in Eq. (7.2), also in `sage`. This part of the computation may be completed in under 5 min on a standard desktop computer. □

Remark 7.4 A direct adaptation of Luxton's methods for proving that $X_0(3, 6)$ is schön does not work for $X_0(3, 7)$, as we now describe. For a degree $9 - n$ del Pezzo surface S, let $e_1, \ldots, e_n, h$ denote the standard generators of Pic S, and $K = 3h - \sum e_i$ the canonical class. In this remark, we focus on the cases $n = 6, 7$. The subspace $K^\perp$ contains the root system E_n, let Λ be the $\mathbb{Z}$-lattice generated by E_n. Set $\alpha_{ij} = e_i - e_j$. For $n = 6$, let $\beta = 2h - \sum e_i$, and for $n = 7$, let $\beta_j = 2h - \sum_{i \neq j} e_i$.

As in the introduction, denote by Y^n the moduli space of smooth marked del Pezzo surfaces of degree $9 - n$, and denote by $\mathcal{F}_n$ its log canonical fan, whose support is Trop Y^n. We recall the description of $\mathcal{F}_n$ in [16]. There is an exact sequences of free abelian groups

$$0 \to \mathrm{Sym}^2 \Lambda^\vee \xrightarrow{\phi} \mathbb{Z}^{(E_n)_+} \xrightarrow{\psi} N(E_n) \to 0$$

where $\phi(f) = \sum_{\alpha \in (E_n)_+} f(\alpha) \cdot \alpha$. Given a root subsystem Θ, let $\psi(\Theta) = \sum \psi(\alpha)$ where the sum is over the positive roots of Θ. The set of A_i root subsystems of E_n is

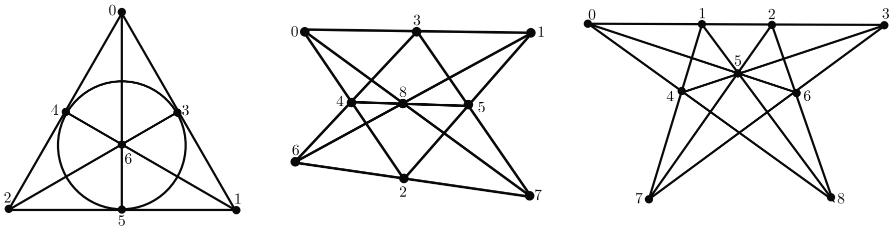

Fig. 2 From left to right: the Fano matroid, the Pappus matroid, and the Perles matroid

denoted by $\mathcal{A}_i$ and let

$$\mathcal{R}(E_6) = \mathcal{A}_1 \sqcup (\mathcal{A}_2 \times \mathcal{A}_2 \times \mathcal{A}_2), \qquad \mathcal{R}(E_7) = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup (\mathcal{A}_3 \times \mathcal{A}_3) \sqcup \mathcal{A}_7.$$

The rays of $\mathcal{F}_n$ are $\psi(\Theta)$ for any $\Theta \in \mathcal{R}(E_n)$, and $\psi(\Theta_1), \dots, \psi(\Theta_k)$ span a cone if and only if

$$\Theta_i \perp \Theta_j, \Theta_i \subset \Theta_j, \text{ or } \Theta_j \subset \Theta_i$$

When $n = 7$, exclude the "Fano" simplices spanned by 7 mutually orthogonal A_1 subsystems.

The open immersion $Y^n \hookrightarrow X_0(3,n)$ induces a surjective map $\pi : \operatorname{Trop} Y^n \to \operatorname{Trop} X_0(3,n)$ [40, Proposition 3.1]. When $n = 6$, the map π is induced by a quotient of $N(E_6)_{\mathbb{R}}$ by $\operatorname{span}_{\mathbb{R}}\{\psi(\beta)\}$. One step to prove that $X_0(3,6)$ is schön in Luxton's thesis was to show that the morphism of toric varieties $X(\mathcal{F}_6) \to X(\mathscr{S}''_{3,6})$ is smooth, see [28, Theorem 4.2.2]. When $n = 7$, the map $\pi : \operatorname{Trop} Y^n \to \operatorname{Trop} X_0(3,n)$ is induced by a quotient of $N(E_7)_{\mathbb{R}}$ by $\operatorname{span}_{\mathbb{R}}\{\psi(\beta_1), \dots, \psi(\beta_7)\}$. We claim that the morphism $X(\mathcal{F}_7) \to X(\mathscr{S}''_{3,7})$ is not smooth. Let $\Theta_1 = \{\pm\alpha_{ij}\}$ (an A_1-subsystem) and $\Theta_2 = \{\pm\alpha_{ij}, \pm\beta_i, \pm\beta_j\}$ (an A_2-subsystem). Because $\Theta_1 \subset \Theta_2$, we have that $\sigma = \operatorname{span}_{\mathbb{R}_{\geq 0}}\{\psi(\Theta_1), \psi(\Theta_2)\}$ is a cone in $\mathcal{F}_7$. Then π maps both rays $\psi(\Theta_1), \psi(\Theta_2)$ to the same ray τ in $\operatorname{Trop} X_0(3,7)$. Therefore the restriction of $X(\mathcal{F}_7) \to X(\mathscr{S}''_{3,7})$ to the toric open sets $U_\sigma \to U_\tau$ is not smooth.

8 Behavior for higher Grassmannians

The algebraic properties of both the initial degenerations and the maps between thin Schubert cells that played a central role in the proof of Theorem 1.2 fail to hold outside $(d,n) = (2,n), (3,6), (3,7)$ and their duals. In this section, we give examples for $d = 3$ and $n = 8, 9$ that show our proof-techniques will not apply beyond the cases treated earlier. We begin by showing how the analog of Proposition 3.9 does not hold when char $\mathbf{k} = 2$ or $n \geq 9$.

Example 8.1 For this example, suppose $\mathbf{k} = \overline{\mathbb{F}}_2$. The analog of Proposition 3.9 does not hold in this setting. Let F be the Fano matroid, i.e., the matroid whose set of lines is

$$\mathcal{L}(F) = \{\{0,1,3\}, \{0,2,4\}, \{0,5,6\}, \{1,2,5\}, \{1,4,6\}, \{2,3,6\}, \{3,4,5\}\}$$

as illustrated in Fig. 2. Let $w_F \in \wedge^3\mathbb{Z}^7$ be the vector

$$w_F = f_{013} + f_{024} + f_{056} + f_{125} + f_{146} + f_{236} + f_{345}.$$

This point lies in $\mathrm{TGr}_0(3, 7)$, as it is the coordinatewise t-adic valuation of the Plücker coordinates of the $\mathbb{F}_4((t))$-valued matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & t & 1 \\ 0 & 1 & 0 & 1 & t & 1 & 1+t \\ 0 & 0 & 1 & t & 1 & 1 & 1+at \end{pmatrix}$$

where $a \in \mathbb{F}_4\backslash\{0, 1\}$. The adjacency graph Γ_w is a star tree whose central node is v_{Q_F} and has a leaf vertex $v_{Q_{M(ijk)}}$ for each $\{i, j, k\} \in \mathcal{L}(F)$. The edge adjacent to $v_{Q_{M(ijk)}}$ corresponds to $M(ijk)'$. A computation in affine coordinates yields $\dim \mathrm{Gr}_F = 6$, hence $\dim \mathrm{Gr}_w = 13$ by Proposition A.8.

Example 8.2 The analog of Proposition 3.9 is also not true for other (d, n), even in characteristic 0. Consider the case $(d, n) = (3, 9)$ and let M_{Pa} be the Pappus matroid, i.e., that matroid set of lines is

$$\mathcal{L}(M_{\mathrm{Pa}}) = \{\{0, 1, 3\}, \{0, 2, 4\}, \{0, 7, 8\}, \{1, 2, 5\}, \\ \{1, 6, 8\}, \{2, 6, 7\}, \{3, 4, 6\}, \{3, 5, 7\}, \{4, 5, 8\}\}$$

as illustrated in Fig. 2. Let $w_{\mathrm{Pa}} \in \wedge^3\mathbb{Z}^9$ be the vector defined by

$$w_{\mathrm{Pa}} = f_{013} + f_{024} + f_{078} + f_{125} + f_{168} + f_{267} + f_{346} + f_{357} + f_{458}.$$

This point lies in $\mathrm{TGr}_0(3, 9)$ as it is the coordinatewise t-adic valuation of the Plücker coordinates of the $\mathbb{Q}((t))$-valued matrix

$$\begin{pmatrix} 1 & 0 & 0 & 2 & 1 & t & 1+t & -1 & 1 \\ 0 & 1 & 0 & -3 & t & 1 & -1 & 1-t^2 & 1 \\ 0 & 0 & 1 & 2t & 3 & -2 & 1 & 1 & 1+t \end{pmatrix}$$

The adjacency graph Γ_w is a star tree whose central node is $v_{Q_{M_{\mathrm{Pa}}}}$ and has a leaf vertex $v_{Q_{M(ijk)}}$ for each $\{i, j, k\} \in \mathcal{L}(M_{\mathrm{Pa}})$. The edge adjacent to $v_{Q_{M(ijk)}}$ corresponds to $M(ijk)'$. A computation in affine coordinates yields $\dim \mathrm{Gr}_{M_{\mathrm{Pa}}} = 10$, hence $\dim \mathrm{Gr}_w = 19$ by Proposition A.8.

Question 8.3 Does the analog of Proposition 3.9 hold for (3, 8)-matroids?

Next, we discuss the general behavior of the maps $\varphi_{M,M'} \colon \mathrm{Gr}_M \to \mathrm{Gr}_{M'}$ for $M' \leq M$. Recall that $\varphi_{M,M'}$ is a SDC-morphism whenever M' is a (3, 6)-matroid, or M is a (3, 7)-matroid and $\Delta_{M'}$ is not a face of $\Delta(3, 7)$. When M' is a face of $\Delta(3, 7)$, then $\varphi_{M,M'}$ may fail to be dominant.

Example 8.4 Let M be the $(3, 7)$-matroid with lines

$$\mathcal{L}(M) = \{\{0, 3, 6\}, \{1, 4, 6\}, \{2, 5, 6\}\}.$$

and $M' = M_{[5]}$. In a projective realization of M, the lines $\overline{03}$, $\overline{14}$, and $\overline{25}$ all meet at the point 6. Observe that 6 becomes a loop in M', and $M' \cong U(3, 6) \oplus U(0, 1)$. A projective realization of $U(3, 6)$ does not require that $\overline{03}$, $\overline{14}$, and $\overline{25}$ all meet at a common point. Such a condition defines a codimension 1 subscheme of $\mathrm{Gr}_0(3, 6)$. Therefore $\varphi_{M,M'}$ is not dominant. Extending the ground set by adding points in linear general position, we get non-dominant morphisms $\varphi_{M,M'}$ for any realizable $(3, n)$ matroid with $n \geq 7$. By adding elements to the ground set parallel to any of the $\{0, \ldots, 6\}$, one may produce non-dominant morphisms $\varphi_{M,M'}$ for $n \geq 8$ where $\Delta_{M'}$ is not a face of $\Delta(3, n)$.

We end with an example of an initial degeneration of $\mathrm{Gr}_0(3, 9)$ that is reducible, proving Theorem 1.4. Consider the Perles matroid P of nine points and nine lines

$$\begin{aligned}\mathcal{L}(P) = \{&\{0, 1, 2, 3\}, \{0, 4, 8\}, \{1, 4, 7\}, \{0, 5, 6\},\\ &\{1, 5, 8\}, \{3, 4, 5\}, \{2, 5, 7\}, \{2, 6, 8\}, \{3, 6, 7\}\}.\end{aligned}$$

This is depicted in Fig. 2. First, we parameterize its thin Schubert cell.

Proposition 8.5 *The thin Schubert cell* Gr_P *is isomorphic to* $X_P \times \mathbb{G}_m^8$ *and*

$$X_P \cong \mathrm{Spec}(\boldsymbol{k}[z^{\pm}]/\langle z^2 - z - 1\rangle).$$

In particular, the thin Schubert cell Gr_P *has two connected components.*

Proof As discussed in the beginning of Sect. 7, the maximal torus $H \subset \mathrm{PGL}_9(\mathbf{k})$ acts freely on Gr_P and $\mathrm{Gr}_P \cong X_P \times H$. To compute X_P, we use affine coordinates similar to Construction 2.2. Let X be the matrix with $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(1, 1, 1)$, in columns 0, 1, 4 and 5, respectively. The H-action allows us to set one nonzero entry of each remaining column to be 1. A standard calculation yields

$$X := \begin{pmatrix} 1 & 0 & -z & 1 & 0 & 1 & 1-z & 0 & 1\\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & z & 0\\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

where $z^2 - z - 1 = 0$. Therefore X_P has the required form. □

Let $w_P \in \wedge^3\mathbb{Z}^9$ be the vector

$$w_P = f_{012} + f_{013} + f_{023} + f_{123} + f_{048} + f_{147} + f_{158} + f_{257} + f_{268} + f_{367}. \quad (8.1)$$

Proposition 8.6 *The vector* w_P *from Eq.* (8.1) *induces a dominant open immersion*

$$\mathrm{Gr}_{w_P} \hookrightarrow \mathrm{Spec}(\boldsymbol{k}[z]/\langle z^2 - z - 1\rangle) \times \mathbb{G}_m^{18}.$$

In particular, the scheme Gr_{w_P} *is smooth and has two connected components.*

Proof The adjacency graph Γ_{w_P} is a star tree on nine leaves with central node v_{Q_P}. For each $\lambda \in \mathcal{L}(P)$, there is a leaf vertex $v_{Q_{M(\lambda)}}$ connected to v_{Q_P}, and $Q_P \cap Q_{M(\lambda)} = Q_{M(\lambda)'}$, where $M(\lambda)$ and $M(\lambda)'$ are defined in Eq. (6.1). By Proposition C.12,

$$\mathrm{Gr}_{w_P} \cong \mathrm{Gr}_P \times_{\prod_{\lambda\in\mathcal{L}(P)} \mathrm{Gr}_{M(\lambda)'}} \prod_{\lambda\in\mathcal{L}(P)} \mathrm{Gr}_{M(\lambda)}.$$

The embedding in the statement will be obtained by combining the associativity of fiber products with Proposition 8.5, Proposition A.5 and the following two claims.

1. $\mathrm{Gr}_P \times_{\mathrm{Gr}_{M(\lambda)'}} \mathrm{Gr}_{M(\lambda)} \cong \mathrm{Gr}_P \times \mathbb{G}_m$ for $\lambda \in \mathcal{L}(P) \setminus \{\{0,1,2,3\}\}$, and
2. $\mathrm{Gr}_P \times_{\mathrm{Gr}_{M(0123)'}} \mathrm{Gr}_{M(0123)}$ embeds into $\mathrm{Gr}_P \times \mathbb{G}_m^2$ as a dense open subscheme.

Both verification are similar. We show the second one since it is more subtle. To simplify notation, let $M = M(0123)$ and $M' = M(0123)'$. We use affine coordinates as in Construction 2.2. Let X be the matrix of variables x_{ij} such that the columns 0,1,4 form the identity. Then $R_{M'}^x \cong S'^{-1}\mathbb{Z}[x_{ij}^{\pm} \mid ij \neq 02, 12]$ and $R_M^x \cong S^{-1} R_{M'}^x[x_{20}^{\pm}, x_{21}^{\pm}]$, where S' is generated by X_{234} and S is generated by X_{023}, X_{123}. From this we conclude that

$$\mathrm{Gr}_P \times_{\mathrm{Gr}_{M'}} \mathrm{Gr}_M \cong \mathrm{Spec}(S^{-1} R_P^x[x_{20}^{\pm}, x_{21}^{\pm}]).$$

Since the x_{00}, x_{01}, x_{10}, and x_{11} are units in R_P^x, we see that X_{023} and X_{123} are not zero divisors in the above ring. Therefore the natural map $R_P^x[x_{20}^{\pm}, x_{21}^{\pm}] \to S^{-1} R_P^x[x_{20}^{\pm}, x_{21}^{\pm}]$ is injective and (ii) holds. □

Lemma 8.7 *We have an isomorphism* $\mathrm{in}_{w_P} \mathrm{Gr}_0(3,9) \cong \mathrm{Gr}_{w_P}$.

Proof Because $\psi_{w_P} : \mathrm{in}_w \mathrm{Gr}_0(3,9) \hookrightarrow \mathrm{Gr}_{w_P}$ is a closed immersion of affine schemes of the same dimension and Gr_{w_P} is reduced, it suffices to show that the image of ψ_{w_P} meets the two connected components of Gr_{w_P} by Proposition A.8. Consider the following $\mathbf{k}(\!(t)\!)$-valued matrix

$$X(a) := \begin{pmatrix} 1 & 0 & -a & 1 & 0 & 1 & 1-a & t & 1 \\ 0 & 1 & 1 & 1+t & 0 & 1 & 1 & a & 3t \\ 0 & 0 & t & 2t & 1 & 1+t & 1+2t & 1 & 1+3t \end{pmatrix}.$$

where $a = b, \bar{b}$ are the distinct solutions to $z^2 - z - 1 = 0$. Let p_a be the Plücker vector of $X(a)$. One may verify that the coordinatewise valuation of p_a is w_P. The exploded tropicalization $\mathfrak{Trop}(p_a)$ is an element of $\mathrm{in}_w \mathrm{Gr}_0(3,9)$ [33, Lemma 3.2], and $\psi_{P,w}$ maps $\mathfrak{Trop}(p_b)$ and $\mathfrak{Trop}(p_{\bar{b}})$ to different connected components of Gr_{w_P}, as required. □

Proof of Theorem 1.4 Proposition 8.6 and Lemma 8.7 yield a dominant open immersion $\mathrm{in}_{w_P} \mathrm{Gr}_0(3,9) \hookrightarrow \mathrm{Spec}(\mathbf{k}[z]/\langle z^2 - z - 1\rangle) \times \mathbb{G}_m^{18}$. Therefore $\mathrm{in}_{w_P} \mathrm{Gr}_0(3,9)$ is smooth with two connected components. □

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A Some functorial properties of SDC-morphisms

Throughout this section, all $\mathbf{k}$-schemes are of finite-type over $\mathbf{k}$. Recall from the beginning of Sect. 4 that a SDC-morphism of $\mathbf{k}$-schemes is one that is smooth and dominant with connected fibers. In this section, we will catalog properties of SDC-morphisms used throughout the paper. First, we discuss how to deduce smoothness or connectedness of a $\mathbf{k}$-scheme X from properties of a morphism $X \to Y$ and Y.

Proposition A.1 *Let X, Y be $\mathbf{k}$-schemes as above.*

1. *If $f : X \to Y$ is a dominant morphism with connected fibers and Y is irreducible, then X connected.*
2. *If $f : X \to Y$ is a SDC-morphism and Y is smooth and irreducible, then so is X.*

Proof Let V be the image of f. Since f is dominant and Y is irreducible, the scheme V is also irreducible, and therefore $f : X \to V$ is a surjective morphism with connected fibers. We conclude that X is connected by [38, Tag 0378]. Finally, (2) follows easily from (1). □

Next, we explore how SDC-morphisms behave under base change. This proposition is crucial in the proof of Theorem 1.2 as it will allow us to deduce smoothness and irreducibility of initial degenerations by studying thin Schubert cells and the morphisms between them.

Proposition A.2 *Suppose we have a pullback diagram*

$$\begin{array}{ccc} W \times_Z X & \xrightarrow{h'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ W & \xrightarrow{h} & Z, \end{array}$$

and that $W \times_Z X$ is nonempty. The following properties hold:

1. *If f is smooth and W is a smooth $\mathbf{k}$-scheme, then $W \times_Z X$ is smooth.*
2. *If f is a SDC-morphism and W is irreducible, then f' is also a SDC-morphism.*

3. *If f is a SDC-morphism and W is smooth and irreducible, then $W \times_Z X$ is smooth and irreducible.*

Proof To simplify notation, set $V := W \times_Z Y$. If f and $W \to \operatorname{Spec} \mathbf{k}$ are smooth morphisms, then so is $V \to \operatorname{Spec} \mathbf{k}$ since smoothness is preserved under composition and base-change. This proves (1).

Now suppose f is a SDC-morphism and W is irreducible. So $f' : V \to W$ is smooth, in particular flat. By [17, Exercise III.9.1] f' is also open. This means that $f'(V)$ is a nonempty open subscheme of W, which is dense by the irreducibility of W. For $w \in f'(V)$, the fiber V_w is nonempty and isomorphic to $X_{h(w)}$, which is connected, hence (2). Statement (3) follows from this and Proposition A.1(2). □

Proposition A.3 *SDC-morphisms satisfy the following.*

1. *A dominant open immersion $U \hookrightarrow X$ is a SDC-morphism.*
2. *If $f : X \to Y$ and $g : Y \to Z$ are SDC-morphisms, then $gf : X \to Z$ is a SDC-morphism.*

Proof Statement (1) is clear, so consider (2). It is well known that smoothness and dominance are preserved under composition, so we need only show that gf has connected fibers. Let $z \in Z$, and X_z (resp. Y_z) be the scheme-theoretic fiber of gf (resp. g) over z. Let $f_z : X_z \to Y_z$ be the morphism obtained by pulling back f along the inclusion $Y_z \to Y$. Since Y_z is smooth and connected, it is irreducible. By Proposition A.2(2), the map f_z is a SDC-morphism. Therefore X_z is connected by Proposition A.1(1), as required. □

Many of the limits that appear in this paper come from graphs in the following way. Let $\mathcal{C}$ be a category that has finite limits, e.g, the category of $\mathbf{k}$-schemes $\mathbf{k}\text{-}\operatorname{sch}$, and G a connected graph, possibly with loops or multiple edges. We regard each edge $e \in E(G)$ as a pair of half-edges. Let us define a quiver $Q(G)$. The set of vertices of $Q(G)$ is $V(G) \cup E(G)$; we write q_v ($v \in V(G)$), resp. q_e ($e \in E(G)$), for the corresponding vertex of Q. For every half edge h of e incident to v, there is an arrow $q_h : q_v \to q_e$. In particular, if e is a loop edge, then there are two arrows from q_v to q_e. Viewing $Q(G)$ as a category in the usual way, a *diagram* of type $Q(G)$ in a category $\mathcal{C}$ is a functor $X : Q(G) \to \mathcal{C}$. We write $X_v = X(q_v)$, $X_e = X(q_e)$, $\varphi_h = X(q_h)$ and $X_G = \varprojlim_{Q(G)} X$. For example, Fig. 3 exhibits a graph and its corresponding diagram.

Example A.4 Let $\Gamma_{M,w}$ be the adjacency graph to a matroid subdivision $\Delta_{M,w}$. Let M_v, resp. M_e, denote the matroid corresponding to the vertex v, resp. edge e, of $\Gamma_{M,w}$, and $\varphi_{M_v,M_e} : \operatorname{Gr}_{M_v} \to \operatorname{Gr}_{M_e}$ whenever e is incident to v. The data of Gr_{M_v}, Gr_{M_e}, and φ_{M_v,M_e} defines a diagram of type $Q(\Gamma_{M,w})$ in $\mathbf{k}\text{-}\operatorname{sch}$.

Now, let us consider how this construction behaves with respect to contracting a connected subgraph. Let F be a connected subgraph of G, and G/F the graph obtained by contracting F to a single vertex v_F. Let $X_F = \varprojlim_{Q(F)} X$ and let $\xi_v : X_F \to X_v$ and $\xi_e : X_F \to X_e$ be the structure morphisms. Set $Y_{v_F} = X_F$, and $Y_v = X_v$ for the remaining v in $V(G/F)$. Similarly, let $Y_e = X_e$ for the edges $e \in E(G/F)$. If h is a half edge in G/F incident v_F, set $\psi_h = \varphi_h \xi_v$. Otherwise, let $\psi_h = \varphi_h$. The data (Y_v, Y_e, ψ_h) defines a diagram Y of type $Q(G/F)$.

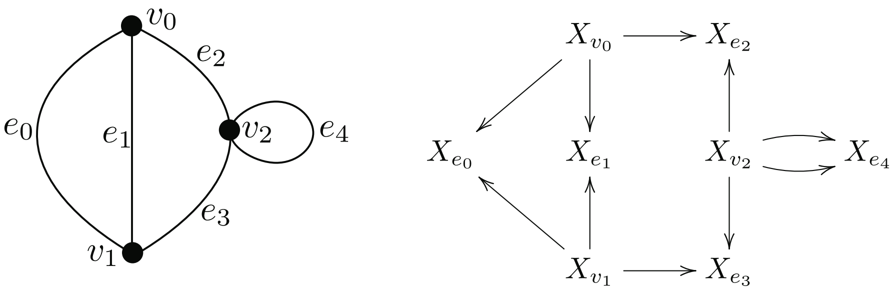


Fig. 3 A graph and its associated diagram

Proposition A.5 *We have an isomorphism*

$$\varprojlim_{G} X \cong \varprojlim_{G/F} Y.$$

Proof To simplify notation, set $Y_{G/F} = \varprojlim_{G/F} Y$. Let $\lambda_v : Y_{G/F} \to Y_v$ and $\lambda_e : Y_{G/F} \to Y_e$ denote the structure morphisms of this limit. We show that $Y_{G/F}$ satisfies the universal property for $\varprojlim_G X$. First, we must define morphisms $\alpha_v : Y_{G/F} \to X_v$ and $\alpha_e : Y_{G/F} \to X_e$ that commute with each φ_h. This is achieved by setting $\alpha_v = \xi_v \lambda_{v_F}$, (resp. $\alpha_e = \xi_e \lambda_{v_F}$) when $v \in V(F)$ (resp. $e \in E(F)$), and $\alpha_v = \lambda_v$ (resp. $\alpha_e = \lambda_e$) otherwise. One may verify that $\varphi_h \alpha_v = \alpha_e$.

Now suppose that we have a collection of morphisms $\theta_v : M \to X_v$ and $\theta_e : M \to X_e$ such that $\varphi_h \theta_v = \theta_e$ for every $q_h : q_v \to q_e$ in $Q(G)$. We will show that there is a unique morphism $\theta : M \to Y_{G/F}$ such that

$$\theta\alpha_v = \theta_v \text{ and } \theta\alpha_e = \theta_e \tag{A.1}$$

for all $v \in V(G)$ and $e \in E(G)$, respectively. By the universal property of X_F, there is a unique morphism $\theta_{v_F} : M \to Y_{v_F}$ such that $\xi_v \theta_{v_F} = \theta_v$ and $\xi_e \theta_{v_F} = \theta_e$. If h is a half edge in G/F incident to v_F, then

$$\psi_h \theta_{v_F} = \varphi_h \xi_v \theta_{v_F} = \varphi_h \theta_v = \theta_e.$$

Otherwise, we have $\psi_h \theta_v = \theta_e$ since $\psi_h = \varphi_h$. By the universal property of $Y_{G/F}$, there is a unique morphism $\theta : M \to Y_{G/F}$ satisfying $\lambda_v \theta = \theta_v$ and $\lambda_e \theta = \theta_e$.

Now we establish the equalities in Eq. (A.1). When $v \in V(F)$,

$$\alpha_v \theta = \xi_v \lambda_{v_F} \theta = \xi_v \theta_{v_F} = \theta_v.$$

A similar argument shows that $\alpha_e \theta = \theta_e$ when $e \in E(F)$. The cases where $v \in V(G) \backslash V(F)$ or $e \in E(G) \backslash E(F)$ follow from the identifications $\alpha_v = \lambda_v$ and $\alpha_e = \lambda_e$. Finally, the uniqueness of θ follows from the uniqueness of θ_F and the universal property of $Y_{G/F}$. ∎

Proposition A.6 *Suppose G is a tree. Let X be diagram of type $Q(G)$ in $\boldsymbol{k}$- sch such X_v and X_e are smooth and irreducible $\mathbf{k}$-schemes, and for each half edge h, the map $X(h) : X_v \to X_e$ is a SDC-morphism. Then X_G is smooth and irreducible. Moreover,*

$$\dim X_G = \sum_{v \in V(G)} X_v - \sum_{e \in E(G)} X_e.$$

Proof We proceed by induction on the number of vertices. When G consists of a single vertex, there is nothing to show. Now suppose that the lemma is true for all trees with fewer vertices than G. Let w be a one valent vertex of G, let e the adjacent edge, and let G' the graph consisting of the remaining vertices and edges. By Proposition A.5,

$$X_G \cong X_w \times_{X_e} X_{G'}.$$

It is smooth and irreducible by Proposition A.2 and the inductive hypothesis. Because $X_w \to X_e$ is smooth of relative dimension $\dim X_w - \dim X_e$, so is $X_G \to X'_G$, and therefore

$$\dim X_G = \dim X_w - \dim X_e + \dim X_{G'}$$

by [17, Corollary 9.6]. By the inductive hypothesis, we get the required formula for $\dim X_G$. □

Remark A.7 An arbitrary finite limit over a diagram of smooth and irreducible $\mathbf{k}$-schemes in which every morphism is SDC-need not be irreducible. Let $h = x^2 - y^2 + x + \frac{1}{4}$ and $X = \operatorname{Spec}((h)^{-1}\mathbf{k}[x, y])$. Define two morphisms $f, g : X \to \operatorname{Spec} \mathbf{k}[z]$ by:

$$f^{\#}(z) = x^2 - y^2 + x \qquad\qquad g^{\#}(z) = x.$$

One may verify that f and g are SDC-morphisms between smooth and irreducible $\mathbf{k}$-schemes. However, the equalizer of f and g is

$$\operatorname{Spec}((h)^{-1}\mathbf{k}[x, y]/\langle x^2 - y^2\rangle)$$

which is neither smooth nor irreducible.

We end with a proposition on when a closed immersion of affine schemes is an isomorphism.

Proposition A.8 *Suppose $\varphi : X \hookrightarrow Y$ is a closed immersion of affine schemes, and Y is integral. If $\dim X = \dim Y$, then φ is an isomorphism.*

Proof Let n be the Krull dimension of X and Y. Because φ is a closed immersion and Y is integral, the induced morphism on rings is of the form $\varphi^{\#} : R \to R/I$ for some integral domain R and ideal $I \subset R$. A maximal chain of prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ in R/I lifts to a maximal chain of prime ideals $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_n$ in R with $I \subset \mathfrak{q}_o$. Because R is an integral domain, we have $\mathfrak{q}_0 = \langle 0\rangle$. So $I = \langle 0\rangle$, and therefore $\varphi^{\#}$ is the identity. □

B Data for Lemma 6.4

In Table 2, we list the ideals that appear as $J^x_{M,w}$ for subdivisions of $\Delta_{M,w}$ such that M is a simple, connected $\mathbf{k}$-realizable (3, 7)-matroid and $\Gamma_{M,w}$ has no leaves, as in the proof of Lemma 6.4. We consider all of these as ideals in the ring

$$\mathbf{k}[x_{ij}^{\pm} \mid 0 \le i \le 2,\ 0 \le j \le 3].$$

We write $\mathbf{k}[x_{ij}^{\pm}]$ for short. Many of the polynomials that appear are of the form $X_{ij,k\ell} := x_{ik}x_{j\ell} - x_{i\ell}x_{jk}$. In the second column, we list variables that may be eliminated to produce an isomorphism of $\mathbf{k}[x_{ij}^{\pm}]/J^x_{M,w}$ with a Laurent polynomial ring. For example, consider the last row. In this case,

$$J^x_{M,w} = \langle X_{01,23}, X_{02,03}, X_{12,12}, X_{12,02}, X_{12,01} \rangle.$$

We use the form $X_{01,23}$ to solve for x_{02}, the form $X_{02,03}$ to solve for x_{03}, the form $X_{12,12}$ to solve for x_{11}, and finally the form $X_{12,02}$ to solve for x_{22}. This produces an isomorphism $\mathbf{k}[x_{ij}^{\pm}]/J^x_{M,w} \to \mathbf{k}[x_{ij}^{\pm} \mid ij \neq 02, 03, 11, 22]$.

C Maps between thin Schubert cells and inverse limits (written by María Angélica Cueto)

In this appendix, we discuss how to reduce the study of geometric properties of thin Schubert cells to the case of simple and connected matroids. Because the only simple rank 2 matroid is $U(2, n)$, this analysis gives us a complete understanding of Gr_M in the rank 2 case, and simplifies the study of rank 3 matroids in Sects. 4 and 5. In the following subsection, we show that the limit of thin Schubert cell $\mathrm{Gr}_{M,w}$ induced by a matroid subdivision $\Delta_{M,w}$ depends only on the adjacency graph of $\Delta_{M,w}$. This allows one to apply the results from Appendix A to study $\mathrm{Gr}_{M,w}$ as in Sect. 6.

C.1 Reduction to simple and connected matroids

The following two Lemmas demonstrate that thin Schubert cells are compatible with decomposition into connected components and removal of loops and parallel elements. Lemma C.1 appears [23, Proposition 9.4] without proof, and Lemma C.2 will appear in an upcoming paper [5]. For the reader's convenience, we sketch their proofs.

Lemma C.1 *If $M = M_1 \oplus M_2$, then $\mathrm{Gr}_M \cong \mathrm{Gr}_{M_1} \times \mathrm{Gr}_{M_2}$. In particular, we have $\mathrm{Gr}_M \cong \mathrm{Gr}_{M|T}$ where $T \subset [n]$ is the set of non-loop elements.*

Proof Suppose X_1 and X_2 are matrices giving rise to the rings $R^x_{M_1}$ and $R^x_{M_2}$ as in Construction 2.2. Let X be the block matrix with X_1 and X_2 on the diagonal. Then $R^x_M \cong R^x_{M_1} \otimes R^x_{M_2}$. The second statement follow from $M \cong M|T \oplus U(0, |T|)$. □

Given a matroid M, we define a simple matroid by removing loops and parallel elements in the following way. Let $\eta_1, \ldots, \eta_k$ be the rank 1 flats of M, choose nonloop

Table 2 Here are the unique ideals nontrivial ideals that appear in the proof of Lemma 6.4

Ideals	x_{ij} to eliminate
$X_{12,01}$	x_{21}
$X_{01,12}$	x_{12}
$X_{12,12}$	x_{22}
$X_{02,23}$	x_{23}
$X_{01,03}$	x_{13}
$X_{01,23}$	x_{13}
$X_{12,23}$	x_{23}
$X_{01,02}$	x_{12}
$x_{02}x_{10}x_{21} + x_{00}x_{11}x_{22}$	x_{22}
$x_{00}x_{12}x_{21} + x_{01}x_{10}x_{22}$	x_{22}
$x_{00}x_{13}x_{22} + x_{02}x_{10}x_{23}$	x_{23}
$x_{02}x_{10}x_{21} - x_{00}x_{12}x_{21} - x_{01}x_{10}x_{22}$	x_{22}
$x_{02}x_{11}x_{20} - x_{01}x_{12}x_{20} - x_{00}x_{11}x_{22}$	x_{22}
$x_{03}x_{10}x_{22} - x_{00}x_{13}x_{22} - x_{02}x_{10}x_{23}$	x_{23}
$x_{03}x_{10}x_{22} - x_{00}x_{13}x_{22} + x_{00}x_{12}x_{23}$	x_{23}
$x_{02}x_{10}x_{21} - x_{00}x_{12}x_{21} - x_{01}x_{10}x_{22} + x_{00}x_{11}x_{22}$	x_{11}
$X_{02,23}, X_{12,01}$	x_{21}, x_{23}
$X_{02,02}, X_{12,01}$	x_{21}, x_{22}
$X_{01,03}, X_{12,01}$	x_{13}, x_{21}
$X_{02,03}, X_{12,01}$	x_{21}, x_{23}
$X_{02,13}, X_{01,02}$	x_{12}, x_{23}
$X_{02,12}, X_{12,01}$	x_{01}, x_{22}
$X_{02,13}, X_{12,01}$	x_{10}, x_{23}
$X_{01,13}, X_{01,02}$	x_{12}, x_{13}
$X_{12,13}, x_{00}x_{12}x_{21} + x_{01}x_{10}x_{22}$	x_{22}, x_{23}
$X_{01,13}, x_{02}x_{10}x_{21} - x_{00}x_{12}x_{21} - x_{01}x_{10}x_{22}$	x_{13}, x_{22}
$X_{12,13}, x_{02}x_{10}x_{21} - x_{01}x_{10}x_{22} + x_{00}x_{11}x_{22}$	x_{02}, x_{23}
$X_{01,12}, x_{03}x_{10}x_{22} - x_{00}x_{13}x_{22} - x_{02}x_{10}x_{23}$	x_{12}, x_{23}
$X_{02,03}, X_{01,12}, X_{12,01}$	x_{12}, x_{21}, x_{23}
$X_{01,13}, X_{02,02}, X_{12,01}$	x_{13}, x_{22}, x_{21}
$X_{02,13}, X_{02,02}, X_{12,01}$	x_{10}, x_{22}, x_{23}
$X_{12,12}, X_{12,02}, X_{12,01}$	x_{22}, x_{21}
$X_{12,03}, X_{01,23}, x_{02}x_{11}x_{20} + x_{01}x_{10}x_{22} - x_{00}x_{11}x_{22}$	x_{00}, x_{12}, x_{23}
$X_{01,03}, X_{02,02}, -X_{12,01} - x_{02}x_{13}x_{21} + x_{03}x_{11}x_{22}$	x_{00}, x_{01}, x_{13}
$X_{02,03}, X_{12,12}, X_{12,02}, X_{12,01}$	x_{11}, x_{22}, x_{23}
$X_{02,13}, X_{01,12}, X_{12,01}, x_{03}x_{12}x_{20} - x_{02}x_{10}x_{23}$	x_{12}, x_{21}, x_{23}
$X_{02,02}, X_{12,01}, x_{01}x_{13}x_{22} + x_{02}x_{11}x_{23}, x_{01}x_{13}x_{20} + x_{00}x_{11}x_{23}$	x_{00}, x_{10}, x_{22}
$X_{12,23}, X_{02,13}, X_{02,02}, X_{12,01},$ $x_{00}x_{03}x_{11}x_{12} - x_{01}x_{02}x_{10}x_{13}$	$x_{00}, x_{13}, x_{20}, x_{22},$ x_{23}
$X_{01,23}, X_{02,03}, X_{12,12}, X_{12,02}, X_{12,01}$	$x_{02}, x_{03}, x_{11}, x_{22}$

elements $s_i \in \eta_i$ and set $S = \{s_1, \ldots, s_k\}$. Then $M|S$ is a simple matroid. Let ℓ be the number of loops in M.

Lemma C.2 *We have an isomorphism* $\mathrm{Gr}_M \cong \mathrm{Gr}_{M|S} \times \mathbb{G}_m^{n-k-\ell}$.

Proof By Lemma C.1, we may assume that M has no loops. Suppose i, j are parallel in M, then $\mathrm{Gr}_M \cong \mathrm{Gr}_{M_{[n]\setminus i}} \times \mathbb{G}_m$. Suppose i and j are parallel, and let $\mu_1, \mu_2 \in \binom{[n]}{d-1}$ such that $\mu_k \cup \{i\}$ and $\mu_k \cup \{j\}$ are bases of M for $k = 1, 2$. The quadratic generator from Eq. (3.1) yields

$$P_M(\mu_1 \cup \{i, j\}, \mu_2) = p_{\mu_1 \cup i} p_{\mu_2 \cup j} - p_{\mu_1 \cup j} p_{\mu_2 \cup i}$$

This means that $p_{\mu \cup i}/p_{\mu \cup j}$ is independent of μ. At the level of rings, the desired isomorphism $R_{M_{[n]\setminus i}} \otimes \mathbf{k}[t^{\pm}] \cong R_M$ is given by

$$\begin{aligned} R_{M_{[n]\setminus i}} \otimes \mathbf{k}[t^{\pm}] &\longrightarrow R_M \\ p_\lambda \otimes 1 &\mapsto p_\lambda \qquad \text{if } i \notin \lambda, \\ 1 \otimes t &\mapsto p_{\mu \cup i}/p_{\mu \cup j} \quad \text{if } \lambda = \mu \cup \{i\}. \end{aligned}$$

The Lemma now follows by induction on the number of parallel elements in M. □

Because the uniform matroid $U(2, n)$ is the only simple $(2, [n])$-matroid, and affine coordinates realize $\mathrm{Gr}_0(2, n)$ as a open subscheme of an algebraic torus, we have the following.

Proposition C.3 *If M is a rank 2 matroid then*

$$\mathrm{Gr}_M \cong \mathrm{Gr}_0(2, k) \times \mathbb{G}_m^{n-k-\ell}. \tag{C.1}$$

where k is the number of rank 1 flats and ℓ the number of loops. In particular, the thin Schubert cell Gr_M *is smooth and irreducible.*

Next, we show the morphisms $\varphi_{M,M'} : \mathrm{Gr}_M \to \mathrm{Gr}_{M'}$ are compatible with the following operations: decomposition of matroids into connected components, removal of loops and parallel elements, and duality. This will allow us to restrict our attention to pairs $M' \leq M$ where M is simple, connected, and $d = r_M([n]) \leq \lfloor n/2 \rfloor$.

Lemma C.4 *If $M' \leq M$ and $M = M_1 \oplus M_2$, then $M' = M_1' \oplus M_2'$ with $M_i' \leq M_i$ for $i = 1, 2$. Furthermore, we have $\varphi_{M,M'} = \varphi_{M_1,M_1'} \times \varphi_{M_2,M_2'}$.*

Proof Recall that $Q_M = Q_{M_1} \times Q_{M_2}$ if and only if $M = M_1 \oplus M_2$. Thus, a face of Q_M must be of the form $Q_{M_1'} \times Q_{M_2'}$ for $M_i' \leq M_i$ for $i = 1, 2$ so $M' = M_1' \oplus M_2'$. The statement regarding $\varphi_{M,M'}$ follows by combining this decomposition with Proposition 3.2 and Lemma C.1. □

Lemma C.5 *If $M' \leq M$, then we have $M'|S \leq M|S$ and the restrictions fit into the commutative diagram:*

$$\begin{array}{ccc} \mathrm{Gr}_M & \xrightarrow{\cong} & \mathrm{Gr}_{M|S} \times \mathbb{G}_m^{n-k-\ell} \\ \Big\downarrow \varphi_{M,M'} & & \Big\downarrow \varphi_{M|S,M'|S} \times id \\ \mathrm{Gr}_{M'} & \xrightarrow{\cong} & \mathrm{Gr}_{M'|S} \times \mathbb{G}_m^{n-k-\ell}. \end{array} \tag{C.2}$$

Proof The top horizontal map in (C.2) arises from the isomorphism $\mathrm{Gr}_M \simeq \mathrm{Gr}_{M|S} \times \mathbb{G}_m^{n-k-\ell}$ described in Lemma C.2. Since $M' \leq M$, rank-one flats in M yield rank-one flats in M' we have $M'|S \leq M|S$. The same lemma yields $\mathrm{Gr}_M \cong \mathrm{Gr}_{M'|S} \times \mathbb{G}_m^{n-k-\ell}$. This determines the bottom horizontal map. □

Now we ensure the compatibility of $M' \leq M$ with the duality operation and the isomorphism

$$\psi : \mathrm{Gr}(d,n) \to \mathrm{Gr}(n-d,n) \quad (p_\beta)_\beta \mapsto ((-1)^{\mathrm{sign}(\beta,\beta^c)} p_{\beta^c})_{\beta^c}$$

induced from $\mathbb{P}(\wedge^d \mathbf{k}^n) \cong \mathbb{P}(\wedge^{n-d} \mathbf{k}^n)$. Here, the symbol (β, β^c) is a permutation of S_n in one-line notation and $\beta^c = [n] \setminus \beta$. On affine patches, the correspondence for matrices is explicit: for example, a $d \times n$ matrix $(I_d|X)$ in $\{p_{[d]} \neq 0\}$ is identified with the $(n-d) \times n$ matrix $(-X^t|I_{n-d})$ in $\{p_{[d]^c} \neq 0\}$.

Lemma C.6 *If $M' \leq M$ then $(M')^* \leq M^*$ and $\varphi_{M^*,(M')^*} = \psi \circ \varphi_{M,M'} \circ \psi^{-1}$.*

Proof By definition, we have $Q_{M^*} = \mathrm{conv}(\{\mathbf{1} - e_\beta : \beta \in \mathcal{B}(M)\}) = \mathbf{1} - Q_M$. In particular, if $Q_{M'}$ is a face of Q_M, then $Q_{(M')^*} = (\mathbf{1} - Q_{M'}) \prec (\mathbf{1} - Q_M) = Q_{M^*}$, as required. The isomorphism ψ identifies each $p_\beta \in \mathbf{k}[\mathrm{Gr}(d,n)]$ with $p_{\beta^c} \in \mathbf{k}[\mathrm{Gr}(n-d,n)]$. The expression $\varphi_{M^*,(M')^*} = \psi \circ \varphi_{M,M'} \circ \psi^{-1}$ follows from this observation. □

As an application, we prove that $\varphi_{M,M'}$ is a SDC-morphism whenever $M = U(d,n)$ or M is a rank 2 matroid.

Proposition C.7 *For any $M' \leq M := U(d,n)$, the map $\varphi_{M,M'} \colon \mathrm{Gr}_0(d,n) \to \mathrm{Gr}_{M'}$ is a SDC-morphism.*

Proof By Proposition A.3, it suffices to show that $\varphi_{M,M'}$ is a SDC morphism when $M' \lessdot M$. The nondegenerate subsets of M are of the form $\{i\}$ or $[n] \backslash i$ for some $i \in [n]$. If $M' = M_{\{i\}}$, then $(M')^* = M^*_{[n]\backslash i}$ where $M^* \cong U(n-d,n)$. By Lemma C.6, it suffices to consider just $M' = M_{[n]\backslash\{i\}}$. In this case, we have $R^x_{M'} = (S^x_{M'})^{-1} B_{M'}$ and $R^x_{M'} = (S^x_M)^{-1} R^x_{M'}[x^{\pm}_{i,n-4} \,|\, i \in [d]]$. Therefore $\mathrm{Gr}_M \subset \mathbb{G}_m^{d \times (n-d)}$ and $\mathrm{Gr}_M \subset \mathbb{G}_m^{d \times (n-d-1)}$ are open subvarieties, and $\varphi_{M,M'}$ is induced by a coordinate projection $\mathbb{G}_m^{d \times (n-d)} \to \mathbb{G}_m^{d \times (n-d-1)}$, which is clearly a SDC-morphism. The result now follows from Proposition A.3. □

Proposition C.8 *For $(2, [n])$-matroids $M' \leq M$, the map $\varphi_{M,M'}$ is a SDC-morphism.*

Proof Because every simple rank 2 matroid is uniform, the Proposition follows from Lemmas C.4, C.5, and C.7. □

Our final result in this subsection say that the reduction to simple matroids as above is compatible taking initial degenerations and inverse limits.

Proposition C.9 *Fix $w \in \mathrm{TGr}_M$ and let $\tilde{w}$ be the projection of w to $\mathbb{R}^{\mathcal{B}(M|S)}/\mathbb{R}\cdot\boldsymbol{1}$. Then $\tilde{w} \in \mathrm{TGr}_{M|S}$ and*

$$\mathrm{in}_w \mathrm{Gr}_M \simeq \mathrm{in}_{\tilde{w}} \mathrm{Gr}_{M|S} \times \mathbb{G}_m^{n-k-\ell} \qquad \mathrm{Gr}_{M,w} \cong \mathrm{Gr}_{M|S,\tilde{w}} \times \mathbb{G}_m^{n-k-\ell}$$

Proof The assertion on initial degenerations follows from the fact that the isomorphism $\mathrm{Gr}_M \cong \mathrm{Gr}_{M|S} \times \mathbb{G}_m^{n-k-\ell}$ from Lemma C.2 is induced by a monomial map on coordinate rings. The isomorphism of limits follows from this Lemma and the description of the coordinate ring of $\mathrm{Gr}_{M,w}$ in Proposition 3.7. □

C.2 Limits of thin Schubert cells via adjacency graphs

Recall that the matroid subdivision $\Delta_{M,w}$ yields a system of maps $\varphi_{M_Q,M_{Q'}} : \mathrm{Gr}_{M_Q} \to \mathrm{Gr}_{M_{Q'}}$ whenever $Q' \leq Q$ that satisfy $\varphi_{M_Q,M_{Q''}} = \varphi_{M_{Q'},M_{Q''}}\varphi_{M_Q,M_{Q'}}$ and $\varphi_{M_Q,M_Q} = \mathrm{id}$. This allows us to form the limit

$$\mathrm{Gr}_{M,w} := \lim_{\overleftarrow{M_Q \in \Delta_{M,w}}} \mathrm{Gr}_{M_Q} \tag{C.3}$$

Rather than keeping track of the full face poset of $\Delta_{M,w}$ it is desirable to restrict ourselves to cells of codimension 0 and 1. The following construction mimics the definition of adjacency graphs for triangulations of polytopes [6, Definition 4.5.10], so we use the same name.

Definition C.10 Given w in Dr_M, let $\Gamma_{M,w}$ be the *adjacency graph* of $\Delta_{M,w}$ defined as follows. The graph $\Gamma_{M,w}$ has a vertex v_Q for each Q in $\mathrm{TC}_{M,w}$. Two vertices v_{Q_1}, v_{Q_2} are connected by an edge if $Q_1 \cap Q_2$ is a facet of both cells. Similarly, given a cell F of $\Delta_{M,w}$, we let $\Gamma_{M,w}^F$ be the full subgraph of $\Gamma_{M,w}$ generated by those vertices v_Q of $\Gamma_{M,w}$ with $F \leq Q$.

Our next lemma shows that the graphs defined above are connected. It will play a crucial role in Proposition C.12 below.

Lemma C.11 *For any $w \in \mathrm{Dr}_M$ and any cell F of $\Delta_{M,w}$, the graphs $\Gamma_{M,w}$ and $\Gamma_{M,w}^F$ are connected.*

Proof The first claim follows by convexity and is valid for the adjacency graph associated to a pure-dimensional polyhedral subdivision of any polytope. We argue for Q_M and $\Delta_{M,w}$. Indeed, given two vertices v_{Q_1}, v_{Q_2} of $\Gamma_{M,w}$, choose two points x_1, x_2, with $x_i \in \mathrm{rel\,int}(Q_i)$ so that the segment $[x_1, x_2]$ does not meet any cell whose codimension is 2 or greater. Since Q_M is convex, the $[x_1, x_2]$ lies in rel int(Q_M). All but finitely many points in $[x_1, x_2]$ lie in the relative interior of top-dimensional cells. We label the encountered cells as we move from x_1 towards x_2 by $Q_1 =: Q'_0, Q'_1, \dots, Q'_k := Q_2$. The collection $\{Q'_i\}_i$ yields a path from v_{Q_1} to v_{Q_2} in $\Gamma_{M,w}$.

A similar argument can be used to prove the statement for $\Gamma^F_{M,w}$. Let $E \subset \mathbb{R}^n$ be the affine span of Q_M. Given F in $\Delta_{M,w}$, write $s := \dim F$ and pick a point p in its relative interior. We let H be the orthogonal complement to the linear subspace $F - p$ in $E - p$, and Q a $(m - s)$-dimensional cube in H centered at the origin with diameter $0 < \varepsilon \ll 1$.

We consider the full-dimensional polytope $P' := (Q + p) \cap Q_M$ in $H + p$, and the polyhedral subdivision on P' induced by $\Delta_{M,w}$. Each cell in this subdivision equals $Q' \cap Q_M$ for some Q', and has dimension $(s - \dim Q_M + \dim Q)$. By construction, a matroid polytope Q' yields a vertex or edge of $\Gamma^F_{M,w}$ if and only if $Q' \in \Delta_{M,w}$ and $Q' \cap (Q + p) \neq \emptyset$. Thus, the graph $\Gamma_{M,w}$ agrees with the adjacency graph of the subdivision of P'. Since the latter is connected by the discussion above, the result follows. □

The adjacency graph $\Gamma_{M,w}$ encodes a subsystem of the inverse system $\mathrm{Gr}_{M,w}$ from (C.3) as in Example A.4. Our final result shows that $\mathrm{Gr}_{M,w}$ agrees with the inverse system induced by $\Gamma_{M,w}$.

Proposition C.12 *Let M be a $\boldsymbol{k}$-realizable $(d, [n])$-matroid and $w \in \mathrm{TGr}_M$. Then,*

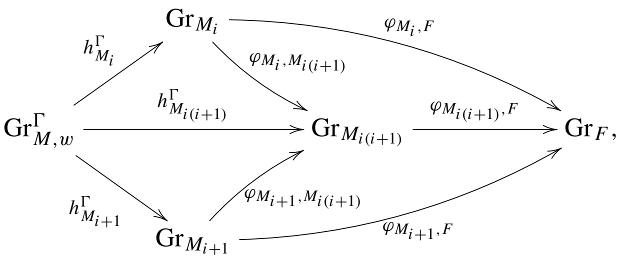
$$\mathrm{Gr}_{M,w} \cong \varprojlim_{\Gamma_{M,w}} \mathrm{Gr}_{M'} . \tag{C.4}$$

Proof We write $\mathrm{Gr}^{\Gamma}_{M,w}$ for the inverse limit on the right-hand side of Eq. (C.4). Given M' labeling a cell of $\Gamma_{M,w}$, we write $h^{\Gamma}_{M'}\colon \mathrm{Gr}^{\Gamma}_{M,w} \to \mathrm{Gr}_{M'}$ for the associated morphism. Since $\Gamma_{M,w}$ determines a subsystem of $\Delta_{M,w}$, the universal property of $\mathrm{Gr}^{\Gamma}_{M,w}$ guarantees the existence of a morphism $\psi : \mathrm{Gr}_{M,w} \to \mathrm{Gr}^{\Gamma}_{M,w}$. Next, we build a morphism $\phi : \mathrm{Gr}^{\Gamma}_{M,w} \to \mathrm{Gr}_{M,w}$.

First, we construct morphisms $g_F\colon \mathrm{Gr}^{\Gamma}_{M,w} \to \mathrm{Gr}_F$ for each cell Q_F of $\Delta_{M,w}$, satisfying $g_{M''} = \varphi_{M',M''} \circ g_{M'}$ for each pair of cells in $\Delta_{M,w}$ with $M' \leq M''$. The morphism ϕ will be unique determined once we establish the compatibility of all g_F's with the subdivision $\Delta_{M,w}$.

Let $\mathscr{V}_F$ be the set of vertices of the graph $\Gamma^F_{M,w}$. Set $g_F := \varphi_{M',F} \circ h^{\Gamma}_{M'}$ where $v_{M'} \in \mathscr{V}_F$. We must show this morphism is independent of our choice of M'. Suppose $v_{M''} \in V_F$ as well. Since $\Gamma^F_{M,w}$ is connected by Lemma C.11, we can find a collection of vertices $v_{Q_{M'}} =: v_{Q_0}, v_{Q_1}, \dots, v_{Q_k} := v_{Q_{M''}}$ where $(v_{Q_i}, v_{Q_{i+1}})$ is an edge of $\Gamma^F_{M,w}$ for each $i = 0, \dots, k-1$. We write $M_i := Q_{M_i}$ and $M_{i(i+1)} := M_{Q_i \cap Q_{i+1}}$;

note that $F \leq Q_i \cap Q_{i+1}$ for each i. The definition of inverse limit yields k diagrams



(C.5)

where all four triangles commute. It follows that $\varphi_{M',F} \circ h^{\Gamma}_{M'} = \varphi_{M'',F} \circ h^{\Gamma}_{M''}$, so g_F is well-defined.

Finally, the identity $g_{F'} = \varphi_{F,F'} \circ g_F$ for each pair $F' \leq F$ in $\Delta_{M,w}$ follows from a similar commutative diagram argument after choosing a vertex M' in $\Gamma^F_{M,w}$. These two properties determine ϕ. The universal property of both schemes $\mathrm{Gr}_{M,w}$ and $\mathrm{Gr}^{\Gamma}_{M,w}$ ensures that $\phi = \psi^{-1}$, as desired. □

References

1. Ardila, F., Klivans, C.J.: The Bergman complex of a matroid and phylogenetic trees. J. Combin. Theory Ser. B **96**(1), 38–49 (2006)
2. Awodey, S.: Category Theory, volume 52 of Oxford Logic Guides, 2nd edn. Oxford University Press, Oxford (2010)
3. Berkovich, V.G.: Spectral Theory and Analytic Geometry Over Non-Archimedean Fields. Mathematical Surveys and Monographs, vol. 33. American Mathematical Society, Providence, RI (1990)
4. Corey, D.: Initial degenerations of Grassmannians. arXiv:1708.03060 (2017)
5. Cueto, M.A.: The compact tropical Grassmannian. In preparation (2020)
6. De Loera, J.A., Rambau, J., Santos, F.: Triangulations, volume 25 of Algorithms and Computation in Mathematics. Structures for Algorithms and Applications. Springer, Berlin (2010)
7. Fulton, W.: Introduction to Toric Varieties. Princeton University Press, Princeton (1993)
8. Gawrilow, E., Joswig, M.: `polymake`: a framework for analyzing convex polytopes. In: Polytopes—Combinatorics and Computation (Oberwolfach, 1997), volume 29 of DMV Sem., pp. 43–73. Birkhäuser, Basel (2000)
9. Gelfand, I., Goresky, M., MacPherson, R., Serganova, V.: Combinatorial geometries, convex polyhedra, and Schubert cells. Adv. Math. **63**(3), 301–316 (1987)
10. Gelfand, I.M., Kapranov, M.M., Zelevinsky, A.V.: Discriminants, resultants, and multidimensional determinants. Birkhäuser, Boston (1994)
11. Gelfand, I.M., Serganova, V.V.: Combinatorial geometries and the strata of a torus on homogeneous compact manifolds. Uspekhi Mat. Nauk. **42**(2(254)):107–134, 287 (1987)
12. Gibney, A., Maclagan, D.: Equations for Chow and Hilbert quotients. Algebra Number Theory **4**(7), 855–885 (2010)
13. Grayson, D.R., Stillman, M.E.: Macaulay2, a software system for research in algebraic geometry. http://www.math.uiuc.edu/Macaulay2/
14. Gubler, W., Rabinoff, J., Werner, A.: Skeletons and tropicalizations. Adv. Math. **294**, 150–215 (2016)
15. Hacking, P.: The homology of tropical varieties. Collect. Math. **59**(3), 263–273 (2008)
16. Hacking, P., Keel, S., Tevelev, J.: Stable pair, tropical, and log canonical compactifications of moduli spaces of del Pezzo surfaces. Invent. Math. **178**(1), 173–227 (2009)
17. Hartshorne, R.: Algebraic Geometry. Springer, Berlin (1977)
18. Helm, D., Katz, E.: Monodromy filtrations and the topology of tropical varieties. Can. J. Math. **64**(4), 845–868 (2012)

19. Herrmann, S., Jensen, A., Joswig, M., Sturmfels, B.: How to draw tropical planes. Electron. J. Combin. **16**(2), R6 (2009)
20. Jensen, A.N.: Gfan, a software system for Gröbner fans and tropical varieties. http://home.imf.au.dk/jensen/software/gfan/gfan.html
21. Kapranov, M.: Chow quotients of Grassmannians. I. In I. M. Gelfand Seminar, volume 16 of Adv. Soviet Math., pp. 29–110. American Mathematical Society, Providence, RI (1993)
22. Kapranov, M.M., Sturmfels, B., Zelevinsky, A.V.: Quotients of toric varieties. Math. Ann. **290**(4), 643–655 (1991)
23. Katz, E.: Matroid theory for algebraic geometers. In: Nonarchimedean and Tropical Geometry, Simons Symposium, pp. 435–517. Springer, Cham (2016)
24. Katz, E., Payne, S.: Realization spaces for tropical fans. In: Combinatorial Aspects of Commutative Algebra and Algebraic Geometry, volume 6 of Abel Symposium, pp. 73–88. Springer, Berlin (2011)
25. Keel, S., McKernan, J.: Contractible extremal rays on $\overline{M}_{0,n}$. In: Handbook of Moduli. Vol. II, volume 25 of Adv. Lect. Math. (ALM), pp. 115–130. Int. Press, Somerville, MA (2013)
26. Keel, S., Tevelev, J.: Geometry of Chow quotients of Grassmannians. Duke Math. J. **134**(2), 259–311 (2006)
27. Lafforgue, L.: Chirurgie des grassmanniennes. CRM Monograph Series, vol. 19. American Mathematical Society, Providence, RI (2003)
28. Luxton, M.: The log canonical compactification of the moduli space of six lines in $\mathbb{P}^2$. ProQuest LLC, Ann Arbor, MI, (2008). Thesis (Ph.D.)–The University of Texas at Austin
29. Luxton, M., Qu, Z.: Some results on tropical compactifications. Trans. Am. Math. Soc. **363**(9), 4853–4876 (2011)
30. Maclagan, D., Sturmfels, B.: Introduction to Tropical Geometry. Graduate Studies in Mathematics, vol. 161. American Mathematical Society, Providence, RI (2015)
31. Naruki, I.: Cross ratio variety as a moduli space of cubic surfaces. Proc. Lond. Math. Soc. (3), 45(1):1–30 (1982) (With an appendix by Eduard Looijenga)
32. Oxley, J.: Matroid Theory. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York (1992)
33. Payne, S.: Fibers of tropicalization. Math. Z. **262**(2), 301–311 (2009)
34. Sekiguchi, J.: Cross ratio varieties for root systems. Kyushu J. Math. **48**(1), 123–168 (1994)
35. Sekiguchi, J.: Cross ratio varieties for root systems. II. The case of the root system of type E_7. Kyushu J. Math. **54**(1), 7–37 (2000)
36. Speyer, D.: Tropical linear spaces. SIAM J. Discrete Math. **22**(4), 1527–1558 (2008)
37. Speyer, D., Sturmfels, B.: The tropical Grassmannian. Adv. Geom. **4**, 389–411 (2004)
38. T. Stacks Project Authors. Stacks project. http://stacks.math.columbia.edu (2017)
39. Sturmfels, B.: Grobner Bases and Convex Polytope. Univ, vol. 8. Lectures Series. American Mathematical Society, Providence, RI (1996)
40. Tevelev, J.: Compactifications of subvarieties of tori. Am. J. Math. **129**(4), 1087–1104 (2007)
41. The Sage Developers. SageMath, the Sage Mathematics Software System (Version 9.0) (2020). https://www.sagemath.org