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**Calculus of Variations**

# Stability and instability of the 3D incompressible viscous flow in a bounded domain

**Fucai Li[1] · Ronghua Pan[2] · Zhipeng Zhang[1]**

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**Abstract**

In this paper, we investigate the stability and instability of the steady state $(\mathbf{0}, p_s)$ ($p_s$ is a constant) for the 3D homogeneous incompressible viscous flow in a bounded simply connected domain with a smooth boundary where the velocity satisfies the Navier boundary conditions. It is shown that there exists a critical slip length $-\mathcal{C}_r\mu$, where $\mathcal{C}_r > 0$ is an explicit generic constant depending only on the domain (given in (1.7)) and $\mu > 0$ is the viscosity coefficient, such that when the slip length $\zeta$ is less than $-\mathcal{C}_r\mu$, the steady state $(\mathbf{0}, p_s)$ is linearly and nonlinearly unstable; and conversely, the steady state $(\mathbf{0}, p_s)$ is linearly and nonlinearly stable when $\zeta > -\mathcal{C}_r\mu$.

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## 1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded simply connected domain with smooth boundary $\partial\Omega$ along with unit outward normal vector $n$. The motion of a 3D homogeneous incompressible viscous fluid in $\Omega$ is governed by the following Navier-Stokes equations [24]:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \mu \Delta \mathbf{u} = 0, \\ \operatorname{div} \mathbf{u} = 0, \quad x \in \Omega, \end{cases} \tag{1.1}$$

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Dedicated to Professor Ling Hsiao on the occasion of her 80th birthday.

✉ Zhipeng Zhang
zhangzhipeng@nju.edu.cn

Fucai Li
fli@nju.edu.cn

Ronghua Pan
panrh@math.gatech.edu

1 Department of Mathematics, Nanjing University, Nanjing 210093, P.R. China

2 School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA

where the unknowns $\mathbf{u}(t,x)$ and $p(t,x)$ denote the velocity and the pressure of the fluid respectively, and $\mu > 0$ is the viscosity coefficient. We add to $\mathbf{u}(t,x)$ the Navier boundary conditions on $\partial\Omega$:

$$\begin{cases} \mathbf{u} \cdot n = 0, \\ \mu(S(\mathbf{u}) \cdot n)_\tau = -\zeta \mathbf{u}_\tau, \end{cases} \tag{1.2}$$

where $S(\cdot)$ is the strain tensor,

$$S(\mathbf{u}) = \nabla\mathbf{u} + (\nabla\mathbf{u})^\top, \tag{1.3}$$

$\zeta \in \mathbb{R}$ is slip length measuring the tendency of the fluid to slip on the boundary, and $(\cdot)_\tau$ represents the tangential part of the vector $(\cdot)$ on $\partial\Omega$. The boundary condition (1.2) was first introduced by Navier [27], allowing the fluid to slip along the boundary, and is often used to model rough boundaries [2, 12]. In this setting, the boundary $\partial\Omega$ is said to be dissipative if $\zeta \geq 0$.

As one of the important boundary conditions, there is an extensive literature on the existence, uniqueness, and regularity theory of the solutions to the incompressible Navier-Stokes equations with the Navier boundary conditions (1.2). In particular, since the work of Solonnikov and Ščadilov [29] on the existence and regularity of weak solutions for the Navier-Stokes equations with Navier boundary conditions, many significant results have been obtained by many experts, see for an incomplete list [1, 3, 11, 13, 19, 26] and the references cited therein.

Beyond the theory on existence, uniqueness and regularity of the solutions, the stability and instability of viscous fluids governed by the Navier-Stokes equations is also a classical subject [5, 10], and was investigated by many mathematicians for various boundary conditions, especially for the no-slip and Navier boundary conditions. In recent years, this subject attracts more and more attentions. For the case of the no-slip boundary condition, Guo and Tice [15] studied the linear instability for a steady state profile of a 3D compressible viscous flow in an infinite slab, in which a heavier fluid lies on a lighter fluid along a planar interface, i.e., the Rayleigh-Taylor (RT) steady state. Jiang and Jiang [20] investigated the RT instability for a 3D nonhomogeneous incompressible viscous fluid driven by gravity in a bounded domain, where the steady density is heavier with increasing height, see also [16, 18, 21] for more results on the RT instability. Kagei [22] proved that if the Reynolds and Mach numbers are sufficiently small, the planar Couette flow is asymptotically stable under sufficiently small initial disturbances in viscous compressible fluid. On the other hand, some important progresses have also been made on the situation where the velocity satisfies the Navier boundary conditions. In 2016, Li and Zhang [23] considered the nonlinear stability of the planar Couette flow for the 3D compressible Navier-Stokes equations with the no-slip boundary condition on the upper flat boundary and the Navier boundary conditions on the lower flat boundary where $\zeta > 0$. Ding and Lin [7] studied the stability of the planar Couette flow for viscous incompressible fluid in a two dimensional slab domain, where the Navier boundary conditions with $\zeta \geq 0$ is imposed on both the upper and lower flat boundaries. In both articles mentioned above, the condition $\zeta \geq 0$ plays an important role.

In 1959, Serrin addressed the problem on the sign of $\zeta$ and pointed out that $\zeta$ does not need to have a defined sign (see [28], p. 240). Actually, the situation of $\zeta < 0$ does exist in the real world. For the curved gas-liquid interfaces, Haase et al. [17] investigated the evolution of the slip length with the bubble's protrusion angle. They found that the slip length is maximum for a small but finite nonzero protrusion angle, however, when the angle exceeds a critical value, the slip length becomes negative, i.e., $\zeta < 0$. The relationship between the bubble's protrusion angle and the slip length is clearly shown in figure (a) on p. 5 of [17], see also the references cited therein for more similar results. The Navier boundary conditions (1.2) with $\zeta < 0$ is also applied widely in the numerical simulations of flows, see [6, 25] and the references cited therein. Therefore, one significant problem is to analyze the stability of viscous fluids governed by Navier boundary conditions with $\zeta < 0$, and it is very interesting to perform stability analysis on the steady state $(\mathbf{0}, p_s)$ ($p_s$ is a constant) to the Navier-Stokes equations with Navier boundary conditions for all $\zeta \in \mathbb{R}$. Comparing with the situation $\zeta \geq 0$, the research on the Navier boundary value problem with $\zeta < 0$ is very limited, due to the challenges arising from the possible lack of dissipation. In 2018, Ding, Li and Xin [8] investigated the stability and instability of the trivial steady state of the 2D incompressible Navier-Stokes equations with Navier boundary conditions in a slab domain $\mathbb{R} \times [0, 1]$. They have shown that when all boundaries are dissipative, i.e., $\zeta \geq 0$, the nonlinear asymptotic stability holds true. Otherwise, there is a sharp critical viscosity, which distinguishes the linear/nonlinear stability from instability. We remark that, from the physical point of view, it is more natural to study the Navier boundary value problem for any $\zeta \in \mathbb{R}$ in a bounded domain, which will be addressed in current paper.

The purpose of this paper is to analyze the stability and instability of the steady state $(\mathbf{0}, p_s)$ ($p_s$ is a constant) to the problem (1.1)–(1.2) in $\Omega \subset \mathbb{R}^3$ for any $\zeta \in \mathbb{R}$. To this end, we denote the perturbation around the steady state $(\mathbf{0}, p_s)$ by

$$\mathbf{v}(t, x) = \mathbf{u}(t, x) - \mathbf{0}, \quad q(t, x) = p(t, x) - p_s,$$

then the equations (1.1) can be rewritten as the following perturbed form:

$$\begin{cases} \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla q - \mu \Delta \mathbf{v} = 0, \\ \operatorname{div} \mathbf{v} = 0, \quad x \in \Omega. \end{cases} \tag{1.4}$$

In addition, we shall impose the Navier boundary conditions:

$$\begin{cases} \mathbf{v} \cdot n = 0, \quad x \in \partial\Omega, \\ \mu\big(S(\mathbf{v}) \cdot n\big)_\tau = -\zeta \mathbf{v}_\tau. \end{cases} \tag{1.5}$$

Upon the linearization around the steady-state $(\mathbf{0}, p_s)$, we obtained from the equations (1.4) the following linearized equations:

$$\begin{cases} \partial_t \mathbf{v} + \nabla q - \mu \Delta \mathbf{v} = 0, \\ \operatorname{div} \mathbf{v} = 0, \quad x \in \Omega. \end{cases} \tag{1.6}$$

Before stating our main results, we now clarify the notations used throughout this paper. For convenience, we will drop the domain $\Omega$ in Sobolev spaces and their corresponding norms as well as in the integrals over $\Omega$, for example,

$$\begin{aligned}
&L^p := L^p(\Omega), \quad H^k := W^{k,2}(\Omega), \quad \int := \int_\Omega \ , \\
&L^2_\sigma := \big\{\mathbf{u} \in L^2 \,\big|\, \operatorname{div} \mathbf{u} = 0,\ \mathbf{u} \cdot n = 0 \text{ on } \partial\Omega\big\}, \\
&H^1_\sigma := \big\{\mathbf{u} \in H^1 \,\big|\, \operatorname{div} \mathbf{u} = 0,\ \mathbf{u} \cdot n = 0 \text{ on } \partial\Omega\big\}, \\
&\mathcal{W} := \big\{\mathbf{u} \in H^1_\sigma \cap H^2 \,\big|\, \mu\big(S(\mathbf{u}) \cdot n\big)_\tau = -\zeta \mathbf{u}_\tau \text{ on } \partial\Omega\big\}, \\
&V^1_\sigma := \Big\{\mathbf{u} \in H^1_\sigma \,\Big|\, \int_{\partial\Omega} |\mathbf{u}|^2 d\sigma \neq 0\Big\}, \quad \mathcal{D}_\sigma := \big\{\mathbf{u} \in C_0^\infty \,\big|\, \operatorname{div} \mathbf{u} = 0\big\}.
\end{aligned}$$

In addition, a product space $(X)^3$ of vector functions is still denoted by $X$ for presentation simplicity, for examples, the vector function $\mathbf{u} \in (H^2)^3$ is denoted by $\mathbf{u} \in H^2$.

The constant $\mathcal{C}_r$, defined by

$$\mathcal{C}_r := \inf_{\mathbf{u}\in V_\sigma^1} \frac{\displaystyle\int |S(\mathbf{u})|^2 \mathrm{d}x}{2\displaystyle\int_{\partial\Omega} |\mathbf{u}|^2 \mathrm{d}\sigma}, \tag{1.7}$$

plays an important role in our results and analysis. Clearly, $\mathcal{C}_r$ is well defined and $\mathcal{C}_r \geq 0$. In the case of slab domain, see for instance [8] on $\mathbb{R}\times[0,1]$, $\mathcal{C}_r$ could take zero value. One of the distinct features in our situation is that the bounded simply connected domain $\Omega$ guarantees the positivity of $\mathcal{C}_r$. Indeed, from Theorem 2 in [9], we have the following Korn's inequality for functions with vanishing normal trace on the boundary: there exists a constant $C_{1\Omega} > 0$ such that

$$\|\mathbf{u}\|_{H^1} \leq C_{1\Omega}\|S(\mathbf{u})\|_{L^2} \text{ for any } \mathbf{u}\in H^1 \text{ with } \mathbf{u}\cdot n = 0. \tag{1.8}$$

On the other hand, we know from the standard trace theorem that, there is a constant $C_{2\Omega} > 0$ such that

$$\|\mathbf{u}\|_{L^2(\partial\Omega)} \leq C_{2\Omega}\|\mathbf{u}\|_{H^1} \text{ for any } \mathbf{u}\in H^1. \tag{1.9}$$

Therefore, one finds that

$$\mathcal{C}_r \geq \frac{1}{2C_{1\Omega}^2 C_{2\Omega}^2} > 0.$$

We note that, both $C_{1\Omega}$ and $C_{2\Omega}$ only depend on the domain $\Omega$. In what follows, we denote by $C$ a generic positive constant which may depend on $\Omega$, $\mu$ and $\zeta$.

Our first result is on the instability to the problem (1.5)–(1.6).

**Theorem 1.1** *Let $\Omega$ be a bounded simply connected subset in $\mathbb{R}^3$ with $\mathcal{C}^2$ boundary $\partial\Omega$ and $n$ the unit outward normal. If*

$$\zeta < -\mathcal{C}_r\mu, \tag{1.10}$$

*holds for $\mathcal{C}_r$ defined in (1.7), then the linearized problem (1.5)–(1.6) is unstable. That is, there exists an unstable solution*

$$(\mathbf{v}, q) := e^{\Lambda t}\big(\tilde{\mathbf{v}}(x), \tilde{q}(x)\big) \tag{1.11}$$

*to the linearized problem (1.5)–(1.6), where $(\tilde{\mathbf{v}}, \tilde{q}) \in H^2\times H^1$ solves the following equations*

$$\begin{cases} \Lambda\tilde{\mathbf{v}} + \nabla\tilde{q} - \mu\Delta\tilde{\mathbf{v}} = 0, \\ \operatorname{div}\tilde{\mathbf{v}} = 0, \quad x\in\Omega, \end{cases} \tag{1.12}$$

*with the Navier boundary conditions*

$$\begin{cases} \tilde{\mathbf{v}}\cdot n = 0, \quad x\in\partial\Omega, \\ \mu\big(S(\tilde{\mathbf{v}})\cdot n\big)_\tau = -\zeta\tilde{\mathbf{v}}_\tau, \end{cases} \tag{1.13}$$

*and the constant growth rate $\Lambda > 0$ is defined by*

$$\Lambda := \sup_{\tilde{\mathbf{v}}\in H_\sigma^1} \frac{-\frac{1}{2}\mu\displaystyle\int |S(\tilde{\mathbf{v}})|^2\mathrm{d}x - \zeta\int_{\partial\Omega}|\tilde{\mathbf{v}}|^2\mathrm{d}\sigma}{\displaystyle\int|\tilde{\mathbf{v}}|^2\mathrm{d}x}. \tag{1.14}$$

Based on Theorem 1.1, we can establish the following nonlinear instability result.

**Theorem 1.2** *Let $\Omega$ be a bounded simply connected subset in $\mathbb{R}^3$ with $\mathcal{C}^2$ boundary $\partial\Omega$ and $n$ the unit outward normal. Under the condition (1.10), the nonlinear problem (1.4)–(1.5) is unstable in the Hadamard sense. That is, there are positive constants $\epsilon$, $\delta_0$ and a function $\bar{\mathbf{v}}_0$, $\|\bar{\mathbf{v}}_0\|_{H^2} = 1$, such that, for any $\delta \in (0, \delta_0)$ and the initial datum $\bar{\mathbf{v}}_0^\delta = \delta\bar{\mathbf{v}}_0$, there exists a unique strong solution $\mathbf{v}^\delta \in \mathcal{C}([0, T^{\max}], H^2)$ with an associated pressure $p^\delta$ of the nonlinear problem (1.4)–(1.5) satisfying*

$$\|\mathbf{v}^\delta(T^\delta)\|_{L^2} \geq \epsilon\|\bar{\mathbf{v}}_0\|_{L^2}, \tag{1.15}$$

*for the escape time $T^\delta = \frac{1}{\Lambda}\ln\frac{2\epsilon}{\delta}$.*

It remains an interesting problem on the stability of the linear and nonlinear systems for the case $\zeta \geq -\mathcal{C}_r\mu$. In the following Theorem, we prove the exponentially asymptotic stability when $\zeta > -\mathcal{C}_r\mu$.

**Theorem 1.3** *Let $\Omega$ be a bounded simply connected subset in $\mathbb{R}^3$ with $\mathcal{C}^2$ boundary $\partial\Omega$ and $n$ the unit outward normal. Under the condition*

$$\zeta > -\mathcal{C}_r\mu, \tag{1.16}$$

*the linearized problem (1.5)–(1.6) is globally stable. Indeed, for any $\mathbf{v}_0 \in H^2$ satisfying $\operatorname{div}\mathbf{v}_0 = 0$ and the boundary compatibility conditions, there exists a unique global strong solution $(\mathbf{v}, q) \in H^2 \times H^1$ to the linearized problem (1.5)–(1.6) with the initial datum $\mathbf{v}_0$. Furthermore, there are positive constants $\alpha$ and $C$ depending only on $\Omega$, $\mu$, $\zeta$ and $\mathcal{C}_r$, such that for any $t > 0$ it holds*

$$\|\mathbf{v}(t)\|_{H^2} + \|q(t)\|_{H^1} + \|\mathbf{v}_t(t)\|_{L^2} \leq Ce^{-\frac{\alpha}{2}t}\|\mathbf{v}_0\|_{H^2}. \tag{1.17}$$

*If one further assumes that the initial datum $\mathbf{v}_0 \in H^2$ is sufficiently small, then the nonlinear problem (1.4)–(1.5) is globally stable. More precisely, there is a positive constant $\varepsilon_1 > 0$, such that if $\|\mathbf{v}_0\|_{H^2} \leq \varepsilon_1$, then the nonlinear problem (1.4)–(1.5) admits a unique global strong solution $(\mathbf{v}, q) \in H^2 \times H^1$. Furthermore, there are positive constants $\gamma$ and $C$ depending only on $\Omega$, $\mu$, $\zeta$ and $\mathcal{C}_r$, such that for any $t > 0$ it holds*

$$\|\mathbf{v}(t)\|_{H^2} + \|q(t)\|_{H^1} + \|\mathbf{v}_t(t)\|_{L^2} \leq Ce^{-\frac{\gamma}{2}t}\|\mathbf{v}_0\|_{H^2}, \tag{1.18}$$

*for any $t > 0$.*

***Remark 1.1*** For the critical case $\zeta = -\mathcal{C}_r\mu$, the linearized problem (1.5)–(1.6) also possesses a unique global strong solution $(\mathbf{v}, q) \in H^2 \times H^1$ and $\|\mathbf{v}\|_{H^2} \leq C\|\mathbf{v}_0\|_{H^2}$, whose proof is similar to that of Theorem 1.3. However, for the nonlinear problem (1.4)–(1.5), due to the absence of dissipation, it is difficult to prove whether it is stable or not.

***Remark 1.2*** It is noticed that we use the critical value of the slip length $\zeta$, instead of the viscosity coefficient $\mu$ to determine the stability or instability of the solution, which is different from [8], where they used the critical viscosity.

***Remark 1.3*** For the slab domain $\mathbb{T}^2 \times [0, 1]$, which is not simply connected, the constant $\mathcal{C}_r = 0$ and the conclusions of the above Theorems also hold. One also finds that the value of $\mathcal{C}_r$ for our domain $\Omega$ is different from that for $\mathbb{T}^2 \times [0, 1]$, which reveals that the geometric structure of the domain has an effect on the stability and instability of $(\mathbf{0}, p_s)$.

***Remark 1.4*** From the above Theorems and Remarks, one finds that when the boundary is dissipative, the steady state $(\mathbf{0}, p_s)$ is always linearly and nonlinearly stable. This is in accordance with the results in [8] on $\mathbb{R} \times [0, 1]$ with dissipative boundary condition for the upper and lower boundaries.

We now make some brief comments on the proofs of Theorems 1.1–1.3. In order to construct an unstable solution to the linearized problem (1.5)–(1.6), we start with a growing mode solution to the linearized problem (1.5)-(1.6) in the form of (1.11). Inserting this ansatz into (1.6) yields (1.12) with the boundary condition (1.13). In [8], the boundaries of $\mathbb{R} \times [0, 1]$ are flat, the authors took this advantage to transfer the boundary problem (1.6) into ordinary differential equations (ODEs) by employing the horizontal Fourier transform. In our case, the domain $\Omega$ is any bounded smooth simply connected region in $\mathbb{R}^3$, thus $\partial\Omega$ could be smooth surface of various shape. Therefore, we can not directly follow the idea of [8]. In order to overcome the difficulties caused by the boundary, we adopt the variational method [20] to construct a solution of the problem (1.12)–(1.13).

In order to prove Theorem 1.2, we first need to establish the Stokes estimates with the Navier boundary conditions for any $\zeta \in \mathbb{R}$. However, when $\zeta < -\mathcal{C}_r\mu$, the existence of weak solutions to the corresponding Stokes problem is not known. Instead, we will study a modified Stokes problem (3.4)–(3.5), which will help us to prove the local well-posedness of the problem (1.4)–(1.5) and derive some important estimates. Finally, based on the constructed unstable solution in Theorem 1.1 and the local well-posedness of the problem (1.4)–(1.5), we can prove the nonlinear instability by employing some ideas in [14].

The proof of Theorem 1.3 is given by the standard energy method, where the condition $\zeta > -\mathcal{C}_r\mu$ plays an essential role. This condition ensures that the boundary integrals and the terms which are caused by the nonlinear terms can be controlled by the dissipative terms $\|S(\mathbf{v})\|_{L^2}$ and $\|S(\mathbf{v}_t)\|_{L^2}$.

The rest of this paper is arranged as follows. In the next section, we prove the instability of the linearized problem and obtain Theorem 1.1. With the help of Theorem 1.1 and the a priori estimates, we show the proof of Theorem 1.2 in Sect. 3. The stability results will be presented in Sect. 4.

## 2 The linear instability

In this section, we will adopt the variational method to construct an unstable solution to the linearized problem (1.5)–(1.6).

### 2.1 Growing mode ansatz

To begin with, we assume a growing mode solution to the linearized problem (1.5)–(1.6) in the form

$$\mathbf{v}(t, x) = \tilde{\mathbf{v}}(x)e^{\lambda t}, \quad q(t, x) = \tilde{q}(x)e^{\lambda t}. \tag{2.1}$$

Inserting this ansatz into (1.6) yields that

$$\begin{cases} \lambda\tilde{\mathbf{v}} + \nabla\tilde{q} - \mu\Delta\tilde{\mathbf{v}} = 0, \\ \operatorname{div}\tilde{\mathbf{v}} = 0, \quad x \in \Omega, \end{cases} \tag{2.2}$$

and into (1.5) gives the boundary conditions

$$\begin{cases} \tilde{\mathbf{v}} \cdot n = 0, \quad x \in \partial\Omega, \\ \mu\big(S(\tilde{\mathbf{v}}) \cdot n\big)_\tau = -\zeta \tilde{\mathbf{v}}_\tau. \end{cases} \tag{2.3}$$

Obviously, the linearized problem (1.5)–(1.6) is unstable if there exists a solution $(\tilde{\mathbf{v}}, \tilde{q}, \lambda)$ to (2.2)–(2.3) with $\lambda > 0$. Now, multiplying $(2.2)_1$ by $\tilde{\mathbf{v}}$, integrating by parts and using the boundary condition (2.3) and $\operatorname{div} \tilde{\mathbf{v}} = 0$, we obtain that

$$\lambda \int \tilde{\mathbf{v}}^2 \mathrm{d}x = \mu \int \Delta\tilde{\mathbf{v}} \cdot \tilde{\mathbf{v}} \mathrm{d}x = -\frac{\mu}{2} \int |S\tilde{\mathbf{v}}|^2 \mathrm{d}x - \zeta \int_{\partial\Omega} |\tilde{\mathbf{v}}|^2 \mathrm{d}\sigma. \tag{2.4}$$

We find that the problem (2.2)–(2.3) has a natural variational structure. Therefore, we may arrive at such an aim by solving the maximization problem

$$\Lambda = \sup_{\tilde{\mathbf{v}} \in \mathcal{A}} E(\tilde{\mathbf{v}}), \tag{2.5}$$

where

$$E(\tilde{\mathbf{v}}) := -\frac{\mu}{2} \int |S(\tilde{\mathbf{v}})|^2 \mathrm{d}x - \zeta \int_{\partial\Omega} |\tilde{\mathbf{v}}|^2 \mathrm{d}\sigma \tag{2.6}$$

and the associated admissible set is defined as

$$\mathcal{A} := \left\{ \tilde{\mathbf{v}} \in V_\sigma^1 \,\Big|\, \int |\tilde{\mathbf{v}}|^2 \mathrm{d}x = 1 \right\}.$$

Next, we show that a maximizer of (2.5) exists, and that the corresponding Euler-Lagrange equations are equivalent to the problem (2.2)–(2.3).

**Proposition 2.1** *$E(\tilde{\mathbf{v}})$ can achieves its maximum on $\mathcal{A}$ and $\Lambda > 0$.*

***Proof*** Since $\zeta < -\mathcal{C}_r \mu$, it follows from the definition of $\mathcal{C}_r$ that there is a function $\tilde{\mathbf{u}} \in V_\sigma^1$ such that

$$-\frac{\zeta}{\mu} > \frac{\int |S(\tilde{\mathbf{u}})| \mathrm{d}x}{2 \int_{\partial\Omega} |\tilde{\mathbf{u}}|^2 \mathrm{d}\sigma},$$

which yields

$$E(\tilde{\mathbf{u}}) = -\frac{\mu}{2} \int |S(\tilde{\mathbf{u}})| \mathrm{d}x - \zeta \int_{\partial\Omega} |\tilde{\mathbf{u}}|^2 \mathrm{d}\sigma > 0.$$

Therefore, the condition (1.10) guarantees that $\Lambda$ is positive.

Now, we turn to prove the first claim in Proposition 2.1. Using the trace theorem [4], we know that there is a constant $C_{3\Omega}$ depending only on $\Omega$, such that

$$\int_{\partial\Omega} |\tilde{\mathbf{v}}|^2 \mathrm{d}\sigma \le C_{3\Omega} \|\tilde{\mathbf{v}}\|_{H^1} \|\tilde{\mathbf{v}}\|_{L^2}. \tag{2.7}$$

Furthermore, combining (2.7) with Korn's inequality (1.8) and Cauchy's inequality, we derive that, for any $\tilde{\mathbf{v}} \in \mathcal{A}$, there is a positive constant $C_\Omega = C_{1\Omega} C_{3\Omega}$, depending only on $\Omega$, such that

$$\begin{aligned} E(\tilde{\mathbf{v}}) &\le -\zeta C_\Omega \|S(\tilde{\mathbf{v}})\|_{L^2} \|\tilde{\mathbf{v}}\|_{L^2} - \frac{\mu}{2} \|S(\tilde{\mathbf{v}})\|_{L^2}^2 \\ &\le -\frac{\mu}{2} \|S(\tilde{\mathbf{v}})\|_{L^2}^2 + \frac{\mu}{2} \|S(\tilde{\mathbf{v}})\|_{L^2}^2 + \frac{C_\Omega^2 \zeta^2}{2\mu} \|\tilde{\mathbf{v}}\|_{L^2}^2 \end{aligned}$$

$$\leq \frac{C_\Omega^2 \zeta^2}{2\mu} \|\tilde{\mathbf{v}}\|_{L^2}^2 \leq \frac{C_\Omega^2 \zeta^2}{2\mu}. \tag{2.8}$$

Therefore, $E(\tilde{\mathbf{v}})$ has an upper bound on $\mathcal{A}$, and $\sup_{\tilde{\mathbf{v}} \in \mathcal{A}} E(\tilde{\mathbf{v}})$ is well-defined and finite.

Letting $\tilde{\mathbf{v}}_n \in \mathcal{A}$ be a maximizing sequence, then $E(\tilde{\mathbf{v}}_n)$ is bounded. It follows from (2.8) that

$$0 \leq E(\tilde{\mathbf{v}}_n) \leq \Lambda \leq \frac{C_\Omega^2 \zeta^2}{2\mu}.$$

Similar to the arguments in (2.8), we arrive at

$$\begin{aligned} 0 \leq E(\tilde{\mathbf{v}}_n) &\leq -\zeta C_\Omega \|S(\tilde{\mathbf{v}}_n)\|_{L^2} \|\tilde{\mathbf{v}}_n\|_{L^2} - \frac{\mu}{2} \|S(\tilde{\mathbf{v}}_n)\|_{L^2}^2 \\ &\leq -\frac{\mu}{2} \|S(\tilde{\mathbf{v}}_n)\|_{L^2}^2 + \frac{\mu}{4} \|S(\tilde{\mathbf{v}}_n)\|_{L^2}^2 + \frac{C_\Omega^2 \zeta^2}{\mu} \|\tilde{\mathbf{v}}_n\|_{L^2}^2 \\ &\leq -\frac{\mu}{4} \|S(\tilde{\mathbf{v}}_n)\|_{L^2}^2 + \frac{C_\Omega^2 \zeta^2}{\mu}. \end{aligned} \tag{2.9}$$

Therefore, we obtained that

$$\|S(\tilde{\mathbf{v}}_n)\|_{L^2}^2 \leq \frac{4C_\Omega^2 \zeta^2}{\mu^2}, \tag{2.10}$$

which implies that $\tilde{\mathbf{v}}_n$ is bounded in $H^1$ due to Korn's inequality (1.8). Thus there exists a function $\tilde{\mathbf{v}}_0 \in H^1$ and a subsequence (still denoted by $\tilde{\mathbf{v}}_n$ for simplicity), such that $\tilde{\mathbf{v}}_n \to \tilde{\mathbf{v}}_0$ weakly in $H^1$ and strongly in $L^2$. Moreover, in view of (2.7), we also see that $\tilde{\mathbf{v}}_n \to \tilde{\mathbf{v}}_0$ strongly in $L^2(\partial\Omega)$. Combining the convergence in $L^2$, $H^1$ and $L^2(\partial\Omega)$ with the lower semi-continuity, we have

$$\begin{aligned} 0 < \sup_{\tilde{\mathbf{v}} \in \mathcal{A}} E(\tilde{\mathbf{v}}) &= \limsup_{n\to\infty} E(\tilde{\mathbf{v}}_n) \\ &= -\frac{\mu}{2} \liminf_{n\to\infty} \int |S(\tilde{\mathbf{v}}_n)|^2 \mathrm{d}x - \zeta \lim_{n\to\infty} \int_{\partial\Omega} |\tilde{\mathbf{v}}_n|^2 \mathrm{d}\sigma \leq E(\tilde{\mathbf{v}}_0), \end{aligned} \tag{2.11}$$

which also implies $\int_{\partial\Omega} |\tilde{\mathbf{v}}_0|^2 d\sigma > 0$. In addition, we can also show that $\tilde{\mathbf{v}}_0 \cdot n = 0$, $\operatorname{div} \tilde{\mathbf{v}}_0 = 0$ and $\int |\tilde{\mathbf{v}}_0|^2 dx = 1$. Thus, $\tilde{\mathbf{v}}_0 \in \mathcal{A}$ and

$$E(\tilde{\mathbf{v}}_0) \leq \sup_{\tilde{\mathbf{v}} \in \mathcal{A}} E(\tilde{\mathbf{v}}). \tag{2.12}$$

Consequently, $E(\tilde{\mathbf{v}})$ achieves its maximum on $\mathcal{A}$. □

Next, we will prove the maximizer constructed above satisfying the problem (2.2)–(2.3).

**Proposition 2.2** *Let $\tilde{\mathbf{v}} \in \mathcal{A}$ be the maximizer of $E(\cdot)$ constructed in Proposition 2.1 and $\Lambda = E(\tilde{\mathbf{v}})$. Then there exists a corresponding pressure field $\tilde{q}$ associated to $\tilde{\mathbf{v}}$, such that $(\tilde{\mathbf{v}}, \tilde{q}) \in H^2 \times H^1$ solves the boundary value problem* (2.2)–(2.3).

***Proof*** For any $\tilde{\mathbf{w}}_0 \in \mathcal{A}$ and $t, r \in \mathbb{R}$, we define

$$j(t, r) := \int |\tilde{\mathbf{v}} + t\tilde{\mathbf{w}}_0 + r\tilde{\mathbf{v}}|^2 \mathrm{d}x.$$

Then $j(t, r)$ is smooth and $j(0, 0) = 1$. Also, notice that

$$\partial_t j(t, r)\Big|_{(t,r)=(0,0)} = 2 \int \tilde{\mathbf{v}} \cdot \tilde{\mathbf{w}}_0 \mathrm{d}x, \quad \partial_r j(t, r)\Big|_{(t,r)=(0,0)} = 2. \tag{2.13}$$

Hence, in view of the implicit function theorem, there exists a smooth function $r = r(t)$ defined near 0, such that $r(0) = 0$ and $j(t, r(t)) = 1$. Since $\tilde{\mathbf{v}}$ is a maximizer of $E(\mathbf{v})$, a direct calculation leads to

$$\begin{aligned} 0 =& \frac{\mathrm{d}}{\mathrm{d}t} E\big(\tilde{\mathbf{v}} + t\tilde{\mathbf{w}}_0 + r(t)\tilde{\mathbf{v}}\big)\Big|_{t=0} \\ =& -2\zeta \int_{\partial\Omega} \tilde{\mathbf{v}} \cdot \tilde{\mathbf{w}}_0 \mathrm{d}\sigma - 2\zeta r'(0) \int_{\partial\Omega} |\tilde{\mathbf{v}}|^2 \mathrm{d}\sigma - \mu r'(0) \int |S(\tilde{\mathbf{v}})|^2 \mathrm{d}x \\ & - \mu \int S(\tilde{\mathbf{v}}) : S(\tilde{\mathbf{w}}_0) \mathrm{d}x \\ =& -2\zeta \int_{\partial\Omega} \tilde{\mathbf{v}} \cdot \tilde{\mathbf{w}}_0 \mathrm{d}\sigma + 2\Lambda r'(0) - \mu \int S(\tilde{\mathbf{v}}) : S(\tilde{\mathbf{w}}_0) \mathrm{d}x. \end{aligned} \tag{2.14}$$

Here we have used the definition of $\Lambda$ and the fact that

$$\int |\tilde{\mathbf{v}}|^2 \mathrm{d}x = 1.$$

Now an implicit differentiation from $j(t, r(t)) = 1$ gives

$$r'(0) = -\int \tilde{\mathbf{v}} \cdot \tilde{\mathbf{w}}_0 \mathrm{d}x. \tag{2.15}$$

Inserting (2.15) into (2.14) yields

$$\Lambda \int \tilde{\mathbf{v}} \cdot \tilde{\mathbf{w}}_0 \mathrm{d}x = -\zeta \int_{\partial\Omega} \tilde{\mathbf{v}} \cdot \tilde{\mathbf{w}}_0 \mathrm{d}\sigma - \frac{\mu}{2} \int S(\tilde{\mathbf{v}}) : S(\tilde{\mathbf{w}}_0) \mathrm{d}x, \tag{2.16}$$

which implies that there exists a pair of functions $(\tilde{\mathbf{v}}, \tilde{q}) \in H^1 \times L^2$ solving (2.2) in weak sense and $\tilde{\mathbf{v}} \cdot n = 0$ on $\Omega$. It remains to show that $\tilde{\mathbf{v}}$ satisfies the boundary condition $(2.3)_2$ and $(\tilde{\mathbf{v}}, \tilde{q}) \in H^2 \times H^1$. However, since the remaining arguments are similar to the following Proposition 3.1, we omit the details here. □

The proof of Theorem 1.1 follows from Propositions 2.1 and 2.2.

## 3 The nonlinear instability

In this section, we investigate the nonlinear instability of the perturbed problem (1.4)–(1.5). As a starting point, one needs the local existence and regularity theory for (1.4)–(1.5), which was known for the case $\zeta \geq 0$, and for some cases when $\zeta < 0$ under certain condition such as $\mu\zeta^2 \leq 1$, see Theorem 1 in [26]. Here, we would need a theory for all $\zeta$, in particular for $\zeta < -\mathcal{C}_r\mu$. In the study of problems of Navier-Stokes equations, the theory of corresponding Stokes problem plays an important role. In this case, the Stokes problem reads as

$$\begin{cases} -\mu\,\Delta\mathbf{v} + \nabla q = f, \\ \operatorname{div}\mathbf{v} = 0, \quad x \in \Omega, \end{cases} \tag{3.1}$$

with Navier boundary conditions

$$\begin{cases} \mathbf{v} \cdot n = 0, \quad x \in \partial\Omega, \\ \mu\big(S(\mathbf{v}) \cdot n\big)_\tau = -\zeta \mathbf{v}_\tau. \end{cases} \tag{3.2}$$

When $\zeta > -\mathcal{C}_r\mu$, the existence of weak solutions to the Stokes problem (3.1)–(3.2) can be directly derived from a Lax-Milgram argument, which fails for the case $\zeta < -\mathcal{C}_r\mu$. To establish a local existence and regularity theory for (1.4)–(1.5), we will proceed with a classical Faedo-Galerkin approximation method based on the existence of a nice basis satisfying the Navier boundary conditions (1.5). Traditionally, such a basis is constructed from the Stokes problem (3.1)–(3.2). In absence of the existence of weak solutions to the Stokes problem (3.1)–(3.2), we will study the auxiliary problem (3.4)–(3.5), a stationary problem of Navier-Stokes equation with a large damping term, in Proposition 3.1, which would help us to construct the basis used in Faedo-Galerkin approximation toward a local existence and regularity theory for (1.4)–(1.5) in this section. The basis also would help to establish the existence of weak solutions to (3.1)–(3.2) under some additional assumptions. The latter is presented in the Appendix.

### 3.1 Local well-posedness

Recalling that from Korn's inequality (1.8) and the trace theorem (2.7), we can define $C_\Omega$ as the best constant of the following inequality

$$\int_{\partial\Omega} |\mathbf{v}|^2 \mathrm{d}\sigma \le C_\Omega \|S(\mathbf{v})\|_{L^2} \|\mathbf{v}\|_{L^2} \quad \text{for all} \quad \mathbf{v} \in H^1_\sigma. \tag{3.3}$$

As first step, we consider the following auxiliary equations:

$$\begin{cases} -\mu\,\Delta\mathbf{v} + \nabla q + \gamma\mathbf{v} = f, \\ \operatorname{div}\mathbf{v} = 0, \quad x \in \Omega, \end{cases} \tag{3.4}$$

where $\gamma = 1 + \frac{C_\Omega^2\zeta^2}{\mu}$, supplemented with Navier boundary conditions

$$\begin{cases} \mathbf{v}\cdot n = 0, \quad x \in \partial\Omega, \\ \mu\big(S(\mathbf{v})\cdot n\big)_\tau = -\zeta\mathbf{v}_\tau. \end{cases} \tag{3.5}$$

**Proposition 3.1** *For each $f \in L^2$, there exists a unique strong solution $(\mathbf{v}, q) \in H^2 \times H^1$ to the problem (3.4)–(3.5) satisfying*

$$\|\mathbf{v}\|_{H^2} + \|q\|_{H^1} \le C\|f\|_{L^2}, \tag{3.6}$$

*where $C > 0$ depends only on $\Omega$, $\mu$ and $\zeta$.*

***Proof*** We are seeking $\mathbf{v} \in H^1_\sigma$ such that, for any $\mathbf{w} \in H^1_\sigma$, it holds that

$$\frac{\mu}{2}\int S(\mathbf{v}) : S(\mathbf{w})\mathrm{d}x + \zeta\int_{\partial\Omega} \mathbf{v}\cdot\mathbf{w}\mathrm{d}\sigma + \gamma\int \mathbf{v}\cdot\mathbf{w}\mathrm{d}x = \int f\cdot\mathbf{w}\mathrm{d}x. \tag{3.7}$$

To this end, we define the bilinear form

$$\mathcal{B}(\mathbf{v}, \mathbf{w}) := \frac{\mu}{2}\int S(\mathbf{v}) : S(\mathbf{w})\mathrm{d}x + \zeta\int_{\partial\Omega} \mathbf{v}\cdot\mathbf{w}\mathrm{d}\sigma + \gamma\int \mathbf{v}\cdot\mathbf{w}\mathrm{d}x.$$

Obviously, $\mathcal{B}(\cdot,\cdot)$ is continuous and symmetric on $H^1_\sigma \times H^1_\sigma$. Using Cauchy's inequality and (3.3), one has

$$\zeta\int_{\partial\Omega} |\mathbf{v}_\tau|^2 d\sigma \ge -\frac{\mu}{4}\|S(\mathbf{v})\|_{L^2}^2 - \frac{C_\Omega^2\zeta^2}{\mu}\|\mathbf{v}\|_{L^2}^2.$$

Then, since $\gamma = 1 + \frac{C_\Omega^2 \zeta^2}{\mu}$, the bilinear form $\mathcal{B}(\cdot,\cdot)$ satisfies

$$\mathcal{B}(\mathbf{v},\mathbf{v}) \geq \frac{\mu}{4}\|S(\mathbf{v})\|_{L^2}^2 + (\gamma - \frac{C_\Omega^2\zeta^2}{\mu})\|\mathbf{v}\|_{L^2}^2 \geq C\|\mathbf{v}\|_{H^1}^2,$$

which implies that the bilinear form $\mathcal{B}(\cdot,\cdot)$ is coercive on $H_\sigma^1 \times H_\sigma^1$. Hence, Lax-Milgram theorem ensures that there exists a unique function $\mathbf{v} \in H_\sigma^1$ such that $\mathcal{B}(\mathbf{v},\mathbf{w}) = (f,\mathbf{w})$ for any $\mathbf{w} \in H_\sigma^1$. Choosing $\varphi \in \mathcal{D}_\sigma$ as a test function in (3.7), we have

$$\big\langle -\mu\Delta\mathbf{v} + \gamma\mathbf{v}, \varphi\big\rangle_{(\mathcal{C}_0^\infty)',\mathcal{C}_0^\infty} = \frac{\mu}{2}\int S(\mathbf{v}) : S(\varphi)\mathrm{d}x + \gamma\int \mathbf{v}\cdot\varphi\mathrm{d}x = \int f\cdot\varphi\mathrm{d}x.$$

Then, by virtue of Theorem IV.2.4 in [4], there exists a distribution $q \in (\mathcal{C}_0^\infty)'$ such that

$$-\mu\Delta\mathbf{v} + \nabla q + \gamma\mathbf{v} = f, \quad \text{in } \Omega, \tag{3.8}$$

and $-\mu\Delta\mathbf{v} + \gamma\mathbf{v} - f \in H^{-1}$ yields that $q \in L^2$ which is defined uniquely up to an additive constant. Also, $\mathbf{v} \in H_\sigma^1$ implies $\operatorname{div}\mathbf{v} = 0$ in $\Omega$ and $\mathbf{v}\cdot n = 0$ on $\partial\Omega$. Thus, it remains to prove that $\mathbf{v}$ satisfies the boundary condition $(3.5)_2$. Since $-\mu\Delta\mathbf{v} + \nabla q + \gamma\mathbf{v} \in L^2$, taking dual product of the equation (3.8) with $\varphi \in H_\sigma^1$ and using the Green's formula in [31], we have

$$\big\langle \mu\big(S(\mathbf{v})\cdot n\big)_\tau, \varphi\big\rangle_{\partial\Omega} + \zeta\int_{\partial\Omega}\mathbf{v}_\tau\cdot\varphi d\sigma = 0,$$

for any $\varphi \in H_\sigma^1$. Now, for $\phi \in H^{\frac{1}{2}}(\partial\Omega)$, there exists $\tilde{\varphi} \in H^1(\Omega)$ such that $\operatorname{div}\tilde{\varphi} = 0$ in $\Omega$ and $\tilde{\varphi} = \phi_\tau$ on $\partial\Omega$. Then, $\tilde{\varphi} \in H_\sigma^1$ and

$$\begin{aligned}\big\langle \mu\big(S(\mathbf{v})\cdot n\big)_\tau + \zeta\mathbf{v}_\tau, \phi\big\rangle_{\partial\Omega} &= \big\langle \mu\big(S(\mathbf{v})\cdot n\big)_\tau + \zeta\mathbf{v}_\tau, \phi_\tau\big\rangle_{\partial\Omega} \\ &= \big\langle \mu\big(S(\mathbf{v})\cdot n\big)_\tau + \zeta\mathbf{v}_\tau, \tilde{\varphi}\big\rangle_{\partial\Omega} = 0,\end{aligned}$$

which yields that

$$\mu(S(\mathbf{v})\cdot n)_\tau = -\zeta\mathbf{v}_\tau, \quad \text{in } H^{-\frac{1}{2}}(\partial\Omega).$$

Thus, there exists a unique weak solution $(\mathbf{v}, q) \in H^1 \times L^2$ to the problem (3.4)–(3.5), which satisfies

$$\|\mathbf{v}\|_{H^1} \leq C\|f\|_{L^2}. \tag{3.9}$$

Next, we show that $(\mathbf{v}, q) \in H^2 \times H^1$. For this purpose, we consider the following Stokes problem:

$$\begin{cases} -\mu\,\Delta\mathbf{w} + \nabla\pi = f - \gamma\mathbf{v}, \\ \operatorname{div}\mathbf{w} = 0, \quad x \in \Omega, \end{cases} \tag{3.10}$$

with Navier boundary conditions

$$\begin{cases} \mathbf{w}\cdot n = 0, \quad x \in \partial\Omega, \\ \mu\big(S(\mathbf{w})\cdot n\big)_\tau = -\zeta\mathbf{v}_\tau. \end{cases} \tag{3.11}$$

Since $\mathbf{v} \in H^1$, we have $\mathbf{v}_\tau \in H^{\frac{1}{2}}(\partial\Omega)$. Then, by virtue of Theorem 4.5 in [31], the problem (3.10)–(3.11) has a unique strong solution $(\tilde{\mathbf{w}}, \tilde{\pi}) \in H^2 \times H^1$, and we have

$$\|\tilde{\mathbf{w}}\|_{H^2} + \|\tilde{\pi}\|_{H^1} \leq C\big(\|f - \gamma\mathbf{v}\|_{L^2} + \|\mathbf{v}\|_{H^{\frac{1}{2}}(\partial\Omega)}\big). \tag{3.12}$$

It follows from trace theorem, (3.9) and (3.12) that

$$\|\tilde{\mathbf{w}}\|_{H^2} + \|\tilde{\pi}\|_{H^1} \leq C\|f\|_{L^2}. \tag{3.13}$$

On the other hand, $(\mathbf{v}, q)$ is also a weak solution of the problem (3.10)–(3.11). Thus, it remains to prove $(\mathbf{v}, q) = (\tilde{\mathbf{w}}, \tilde{\pi})$. In fact, we can see that $(\mathbf{v} - \tilde{\mathbf{w}}, q - \tilde{\pi})$ satisfies

$$\begin{cases} -\mu\Delta(\mathbf{v} - \tilde{\mathbf{w}}) + \nabla(q - \tilde{\pi}) = 0, \\ \operatorname{div}(\mathbf{v} - \tilde{\mathbf{w}}) = 0, \quad x \in \Omega, \end{cases} \tag{3.14}$$

with the slip boundary conditions

$$\begin{cases} (\mathbf{v} - \tilde{\mathbf{w}}) \cdot n = 0, \quad x \in \partial\Omega, \\ \mu\big(S(\mathbf{v} - \tilde{\mathbf{w}}) \cdot n\big)_\tau = 0. \end{cases} \tag{3.15}$$

Multiplying (3.14) by $(\mathbf{v} - \tilde{\mathbf{w}})$ and integrating by parts, we have

$$\frac{\mu}{2}\int |S(\mathbf{v} - \tilde{\mathbf{w}})|^2 \mathrm{d}x = 0,$$

which implies $\mathbf{v} = \tilde{\mathbf{w}} \in H^2$. Furthermore, by (3.13), we have

$$\|\mathbf{v}\|_{H^2} \leq C\|f\|_{L^2}. \tag{3.16}$$

As for $q$, it follows from $(3.4)_1$ and (3.16) that $q \in H^1$ and

$$\|q\|_{H^1} \leq C\|f\|_{L^2}.$$

The proof of this proposition is completed. □

Next, in order to use the classical Faedo-Galerkin approximation method to prove the local well-posedeness of the perturbed problem (1.4)–(1.5), we need a basis $\{\mathbf{w}_i\}_{i=1}^\infty \subset H^2$ for $H^1_\sigma$ satisfying the Navier boundary conditions (1.5) including the case when $\zeta < -\mathcal{C}_r\mu$. This is achieved with the help of Proposition 3.1.

**Proposition 3.2** *There exists a basis $\{\mathbf{w}_i\}_{i=1}^\infty \subset H^2$ to $H^1_\sigma$, which is also an orthonormal basis to $L^2_\sigma$ satisfying*

$$\mu\big(S(\mathbf{w}_i) \cdot n\big)_\tau = -\zeta(\mathbf{w}_i)_\tau, \quad x \in \partial\Omega. \tag{3.17}$$

***Proof*** From the proof of Proposition 3.1, we have that for each $f \in L^2_\sigma$, there exists a unique function $\tilde{\mathbf{w}} \in H^1_\sigma$ such that $\mathcal{B}(\tilde{\mathbf{w}}, \mathbf{w}) = (f, \mathbf{w})$ for any $\mathbf{w} \in H^1_\sigma$. Thus, we can define a linear invertible operator $\mathcal{L}$ by $\mathcal{L}\tilde{\mathbf{w}} = f$ or $\mathcal{L}^{-1}f = \tilde{\mathbf{w}}$. Since the embedding map $H^1_\sigma \hookrightarrow L^2_\sigma$ is compact, the operator $\mathcal{L}^{-1}$ is a bounded linear compact operator from $L^2_\sigma$ to $L^2_\sigma$. The symmetry of $\mathcal{L}$ can also be deduced by the symmetry of the bilinear form $\mathcal{B}(\cdot, \cdot)$. Thus, by virtue of the spectral theory of operators, we infer that $\mathcal{L}^{-1}$ possesses countable real positive eigenvalues $\{\eta_i\}_{i=1}^\infty$ with $\eta_i \to +\infty$ as $i \to +\infty$. The corresponding eigenfunctions $\{\mathbf{w}_i\}_{i=1}^\infty$ form an orthonormal basis to $L^2_\sigma$. Furthermore, we have $\mathcal{L}\mathbf{w}_i = \frac{1}{\eta_i}\mathbf{w}_i$. Therefore, we also deduce that $\mathcal{L}$ has a countable set of eigenvalues $\{\frac{1}{\eta_i}\}_{i=1}^\infty$ with the corresponding eigenfunctions $\{\mathbf{w}_i\}_{i=1}^\infty$, which also constitute a basis of $H^1_\sigma$. Then the following eigenvalue problem:

$$\begin{cases} -\Delta\mathbf{v} + \nabla p = \lambda\mathbf{v}, \\ \operatorname{div}\mathbf{v} = 0, \quad x \in \Omega, \end{cases}$$

with the Navier boundary conditions (1.5) possesses countable eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$, where $\lambda_i = -\gamma + \frac{1}{\eta_i}$, with the corresponding eigenfunctions $\{\mathbf{w}_i\}_{i=1}^{\infty}$. Finally, by Proposition 3.1, we can prove that $\{\mathbf{w}_i\}_{i=1}^{\infty}$ also lie in $H^2$. The proof of this proposition is completed. □

Finally, with the help of the above preparations, we are ready to apply the Faedo-Galerkin approximation method to prove the local well-posedeness of the perturbed problem (1.4)–(1.5).

**Proposition 3.3** *For any given initial datum $\mathbf{v}_0 \in \mathcal{W}$, there exist a constant $\widetilde{T} > 0$ and a unique strong solution $(\mathbf{v}, q) \in \mathcal{C}([0, \widetilde{T}), H^2 \times H^1)$ to the perturbed problem (1.4)–(1.5).*

***Remark 3.1*** In [26], when $\mu\zeta^2 \leq 1$, a local well-posedness theorem has been established in conormal Sobolev spaces, aiming at a uniform theory with respect to the viscosity $\mu$. Here, we establish local well-posedness theory for all $\zeta \in \mathbb{R}$ with a fixed positive viscosity $\mu$ in standard Sobolev spaces. The difficulty of boundary values for $\nabla\mathbf{v}$ was avoided by utilizing estimates on time derivatives of $\mathbf{v}$, and then apply the Stokes type estimates given by Proposition 3.1 to achieve higher order derivative estimates on $\mathbf{v}$.

***Proof*** We take an orthonormal basis $\{\mathbf{w}_i\}_{i=1}^{\infty} \subset H^2$ of $L^2_\sigma$ which, from Proposition 3.2, is also a basis of $H^1_\sigma$. For each $m \in \mathbb{N}_+$, we search an approximate function $\mathbf{v}_m$ in the form

$$\mathbf{v}_m = \sum_{i=1}^{m} \mathbf{v}_{i,m}(t)\mathbf{w}_i, \tag{3.18}$$

where $\mathbf{v}_{i,m}(t)$ are the functions to be determined. $\mathbf{v}_m$ is obtained by solving the following differential equation:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int \mathbf{v}_m \cdot \mathbf{w}_i \mathrm{d}x + \frac{\mu}{2}\int S(\mathbf{v}_m) : S(\mathbf{w}_i)\mathrm{d}x \\ \qquad\qquad + \displaystyle\int \mathbf{v}_m \cdot \nabla\mathbf{v}_m \cdot \mathbf{w}_i \mathrm{d}x = -\zeta \int_{\partial\Omega} \mathbf{v}_m \cdot \mathbf{w}_i \mathrm{d}\sigma, \\ \mathbf{v}_m(0) = \displaystyle\sum_{i}^{m} (\mathbf{v}_0, \mathbf{w}_i)_{H^2}\, \mathbf{w}_i, \end{cases} \tag{3.19}$$

i.e., $\mathbf{v}_{i,m}(t)$ with $i = 1, 2, ..., m$ are determined by solving the following nonlinear ODEs:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_{i,m}(t) + \displaystyle\sum_{j=1}^{m} A_{j,i}\mathbf{v}_{j,m}(t) + \sum_{j,k=1}^{m} B_{j,k,i}\mathbf{v}_{j,m}(t)\mathbf{v}_{k,m}(t) = 0, \\ \mathbf{v}_{i,m}(0) = (\mathbf{v}_0, \mathbf{w}_i)_{H^2}, \end{cases} \tag{3.20}$$

where

$$\begin{aligned} A_{j,i} &= \frac{\mu}{2}\int S(\mathbf{w}_j) : S(\mathbf{w}_i)\mathrm{d}x + \zeta \int_{\partial\Omega} \mathbf{w}_j \cdot \mathbf{w}_i \mathrm{d}\sigma, \\ B_{j,k,i} &= \int \mathbf{w}_j \cdot \nabla\mathbf{w}_k \cdot \mathbf{w}_i \mathrm{d}x. \end{aligned}$$

From the structure of (3.20), in view of the standard nonlinear ODEs theory, there exists a positive time $\widetilde{T}_m$, such that the problem (3.20) possesses a unique solution $(\mathbf{v}_{1,m}(t), \mathbf{v}_{2,m}(t), \cdots, \mathbf{v}_{m,m}(t)) \in \mathcal{C}^1([0, \widetilde{T}_m))$, i.e., there exists a unique solution $\mathbf{v}_m \in \mathcal{C}^1([0, \widetilde{T}_m), H^2)$ to the problem (3.19). Now, we need to show that $\widetilde{T}_m$ can be extended to any positive constant $T$ $(> \widetilde{T}_m)$. To this end, we give the following uniform energy estimate. Multiplying (3.19) by $\mathbf{v}_{i,m}(t)$, summing them up from $i = 1$ to $m$ and integrating by parts, we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\mathbf{v}_m|^2\mathrm{d}x + \frac{\mu}{2}\int |S(\mathbf{v}_m)|^2\mathrm{d}x = -\zeta\int_{\partial\Omega}|\mathbf{v}_m|^2\mathrm{d}\sigma. \tag{3.21}$$

Using (3.3), we get

$$-\zeta\int_{\partial\Omega}|\mathbf{v}_m|^2\mathrm{d}\sigma \le \frac{\zeta^2 C_\Omega^2}{\mu}\|\mathbf{v}_m\|_{L^2}^2 + \frac{\mu}{4}\|S(\mathbf{v}_m)\|_{L^2}^2,$$

which together with (3.21) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\mathbf{v}_m|^2\mathrm{d}x + \frac{\mu}{4}\int |S(\mathbf{v}_m)|^2\mathrm{d}x \le \frac{\zeta^2 C_\Omega^2}{\mu}\|\mathbf{v}_m\|_{L^2}^2. \tag{3.22}$$

An application of Gronwall's inequality to (3.22) then implies that, for $0 < t < \widetilde{T}_m$,

$$\sup_{0\le t\le \widetilde{T}_m}\|\mathbf{v}_m\|_{L^2}^2 + \int_0^{\widetilde{T}_m}\|\nabla\mathbf{v}_m\|_{L^2}^2\mathrm{d}t \le e^{C\widetilde{T}_m}\|\mathbf{v}_0\|_{H^2}^2 \le e^{CT}\|\mathbf{v}_0\|_{H^2}^2, \tag{3.23}$$

which means that the maximum life span $\widetilde{T}_m$ can be extended to $T$ and $\{\mathbf{v}_m\}_{m=1}^\infty$ is bounded in $L^\infty(0,T;L^2_\sigma)\cap L^2(0,T;H^1_\sigma)$. Furthermore, there exists a function $\mathbf{v}\in L^\infty(0,T;L^2_\sigma)\cap L^2(0,T;H^1_\sigma)$ and a subsequence (still denoted by $\{\mathbf{v}_m\}_{m=1}^\infty$) such that as $m\to+\infty$,

$$\mathbf{v}_m \to \mathbf{v} \text{ weakly} - \star \text{ in } L^\infty(0,T;L^2_\sigma) \text{ and } \mathbf{v}_m \to \mathbf{v} \text{ weakly in } L^2(0,T;H^1_\sigma).$$

To pass to the limit in the nonlinear term, it is necessary to obtain a strong convergence result. Here, the compactness theorem of Aubin-Lions-Simon will be used. For this purpose, we need to provide an estimate on the derivative with respect to time of $\{\mathbf{v}_m\}_{m=1}^\infty$. Note that for any $\varphi\in H^1_\sigma$, we have

$$\begin{aligned}\Big| -\mu\int\Delta\mathbf{v}_m\cdot\varphi\mathrm{d}x\Big| &= \Big|\frac{\mu}{2}\int S(\mathbf{v}_m):S(\varphi)\mathrm{d}x + \zeta\int_{\partial\Omega}\mathbf{v}_m\cdot\varphi\mathrm{d}\sigma\Big|\\ &\le C\|\mathbf{v}_m\|_{H^1}\|\varphi\|_{H^1},\end{aligned}$$

which implies that

$$\{-\Delta\mathbf{v}_m\}_{m=1}^\infty \text{ is bounded in } L^2(0,T;(H^1_\sigma)'). \tag{3.24}$$

For the nonlinear term, we have, for any $\varphi\in H^1_\sigma$,

$$\Big|\int\mathbf{v}_m\cdot\nabla\mathbf{v}_m\cdot\varphi\mathrm{d}x\Big| \le C\|\mathbf{v}_m\|_{L^3}\|\nabla\mathbf{v}_m\|_{L^2}\|\varphi\|_{L^6} \le C\|\mathbf{v}_m\|_{L^2}^{\frac{1}{2}}\|\mathbf{v}_m\|_{H^1}^{\frac{3}{2}}\|\varphi\|_{H^1}.$$

Thus, thanks to the bound for $\{\mathbf{v}_m\}_{m=1}^\infty$ in (3.23), we conclude that

$$\{\mathbf{v}_m\cdot\nabla\mathbf{v}_m\}_{m=1}^\infty \text{ is bounded in } L^{\frac{4}{3}}(0,T;(H^1_\sigma)'). \tag{3.25}$$

Therefore, it follows from (3.24) and (3.25) that

$$\Big\{\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_m\Big\}_{m=1}^\infty \text{ is bounded in } L^{\frac{4}{3}}(0,T;(H^1_\sigma)'). \tag{3.26}$$

The rest of arguments is completely analogous to the case for the Navier-Stokes equations with non-slip boundary condition, see for instance, Sect. 1.3 of Chapter V in [4]. We can now pass to the limit in the equations satisfied by approximate solutions, and prove that $\mathbf{v} \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$ is a global weak solution to the problem (1.4)–(1.5) with the initial datum $\mathbf{v}_0$.

Next, we show that this weak solution $\mathbf{v}$ is actually a local strong solution. To this end, we should obtain higher order energy estimates. We differentiate $(3.19)_1$ with respect to $t$, multiply the resulting equations by $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_{i,m}(t)$ and then obtain the following estimate

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\partial_t\mathbf{v}_m|^2\mathrm{d}x+\frac{\mu}{2}\int|S(\partial_t\mathbf{v}_m)|^2\mathrm{d}x\\&\quad\le|\zeta|\int_{\partial\Omega}|\partial_t\mathbf{v}_m|^2\mathrm{d}\sigma+\int|\partial_t\mathbf{v}_m|^2|\nabla\mathbf{v}_m|\mathrm{d}x.\end{aligned}\tag{3.27}$$

The terms on the right hand side of (3.27) can be bounded as follows:

$$|\zeta|\int_{\partial\Omega}|\partial_t\mathbf{v}_m|^2\mathrm{d}\sigma\le C|\zeta|\|S(\partial_t\mathbf{v}_m)\|_{L^2}\|\partial_t\mathbf{v}_m\|_{L^2},\tag{3.28}$$

$$\int|\partial_t\mathbf{v}_m|^2|\nabla\mathbf{v}_m|\mathrm{d}x\le C\|\partial_t\mathbf{v}_m\|_{L^2}^{\frac{1}{2}}\|S(\partial_t\mathbf{v}_m)\|_{L^2}^{\frac{3}{2}}\|S(\mathbf{v}_m)\|_{L^2}.\tag{3.29}$$

Substituting the above two inequalities back into (3.27) and using Young's inequality, we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\partial_t\mathbf{v}_m|^2\mathrm{d}x+\frac{\mu}{4}\int|S(\partial_t\mathbf{v}_m)|^2\mathrm{d}x\le C(1+\|S(\mathbf{v}_m)\|_{L^2}^4)\|\partial_t\mathbf{v}_m\|_{L^2}^2.\tag{3.30}$$

Note that $\|S(\mathbf{v}_m)\|_{L^2}$ appears on the right hand side of (3.30), thus we also need to give its estimate. Multiplying both sides of $(3.19)_1$ by $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_{i,m}(t)$, adding them up from $i=1$ to $m$, and integrating by parts, we obtain that

$$\begin{aligned}&\frac{\mu}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int|S(\mathbf{v}_m)|^2\mathrm{d}x+\int|\partial_t\mathbf{v}_m|^2\mathrm{d}x\\&\quad\le\int|\mathbf{v}_m|^2|\nabla\partial_t\mathbf{v}_m|\mathrm{d}x+|\zeta|\int_{\partial\Omega}|\mathbf{v}_m||\partial_t\mathbf{v}_m|\mathrm{d}\sigma.\end{aligned}\tag{3.31}$$

Arguing analogously to (3.28) and (3.29), we have

$$\begin{aligned}&|\zeta|\int_{\partial\Omega}|\mathbf{v}_m||\partial_t\mathbf{v}_m|\mathrm{d}\sigma\\&\le C|\zeta|\|S(\mathbf{v}_m)\|_{L^2}^{\frac{1}{2}}\|\mathbf{v}_m\|_{L^2}^{\frac{1}{2}}\|S(\partial_t\mathbf{v}_m)\|_{L^2}^{\frac{1}{2}}\|\partial_t\mathbf{v}_m\|_{L^2}^{\frac{1}{2}}\\&\le C(\varepsilon)(\|S(\mathbf{v}_m)\|_{L^2}^2+\|\mathbf{v}_m\|_{L^2}^2)+\frac{1}{2}\|\partial_t\mathbf{v}_m\|_{L^2}^2+\frac{\varepsilon}{2}\|S(\partial_t\mathbf{v}_m)\|_{L^2}^2,\end{aligned}\tag{3.32}$$

and

$$\begin{aligned}\int|\mathbf{v}_m|^2|\nabla\partial_t\mathbf{v}_m|\mathrm{d}x&\le C\|\mathbf{v}_m\|_{L^2}^{\frac{1}{2}}\|\mathbf{v}_m\|_{L^6}^{\frac{3}{2}}\|\nabla\partial_t\mathbf{v}_m\|_{L^2}\\&\le C(\varepsilon)\|\mathbf{v}_m\|_{L^2}\|S(\mathbf{v}_m)\|_{L^2}^3+\frac{\varepsilon}{2}\|S(\partial_t\mathbf{v}_m)\|_{L^2}^2.\end{aligned}\tag{3.33}$$

It follows from (3.32) and (3.33) that

$$\begin{aligned}&\frac{\mu}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int|S(\mathbf{v}_m)|^2\mathrm{d}x+\frac{1}{2}\int|\partial_t\mathbf{v}_m|^2\mathrm{d}x\\&\quad\le C(\varepsilon)\big(\|S(\mathbf{v}_m)\|_{L^2}^2+\|\mathbf{v}_m\|_{L^2}^2+\|\mathbf{v}_m\|_{L^2}\|S(\mathbf{v}_m)\|_{L^2}^3\big)+\varepsilon\|S(\partial_t\mathbf{v}_m)\|_{L^2}^2.\end{aligned}\tag{3.34}$$

Adding (3.22), (3.30) and (3.34) together and choosing $\varepsilon = \frac{\mu}{8}$, we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (|\mathbf{v}_m|^2 + |\partial_t \mathbf{v}_m|^2)\mathrm{d}x + \frac{\mu}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int |S(\mathbf{v}_m)|^2\mathrm{d}x \\
&\quad + \frac{1}{2}\int |\partial_t \mathbf{v}_m|^2\mathrm{d}x + \frac{\mu}{4}\int |S(\mathbf{v}_m)|^2\mathrm{d}x + \frac{\mu}{8}\int |S(\partial_t \mathbf{v}_m)|^2\mathrm{d}x \\
&\leq C\big(\|S(\mathbf{v}_m)\|_{L^2}^2 + \|\mathbf{v}_m\|_{L^2}^2 + \|\mathbf{v}_m\|_{L^2}\|S(\mathbf{v}_m)\|_{L^2}^3\big) \\
&\quad + C\big(1 + \|S(\mathbf{v}_m)\|_{L^2}^4\big)\|\partial_t \mathbf{v}_m\|_{L^2}^2. 
\end{aligned} \tag{3.35}$$

In order to integrate the above inequality in time, we need to estimate $\|\partial_t \mathbf{v}_m(0)\|_{L^2}$. Multiplying $(3.19)_1$ by $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_{i,m}$, adding them up from $i = 1$ to $m$, and integrating by parts, we arrive at

$$\begin{aligned}
\int \big|\partial_t \mathbf{v}_m\big|^2\mathrm{d}x &= -\mu\int \Delta\mathbf{v}_m\cdot\partial_t\mathbf{v}_m\mathrm{d}x - \int \mathbf{v}_m\cdot\nabla\mathbf{v}_m\cdot\partial_t\mathbf{v}_m\mathrm{d}x \\
&\leq \frac{1}{2}\|\partial_t\mathbf{v}_m\|_{L^2}^2 + C\big(\|\Delta\mathbf{v}_m\|_{L^2}^2 + \|\mathbf{v}_m\cdot\nabla\mathbf{v}_m\|_{L^2}^2\big).
\end{aligned} \tag{3.36}$$

Thus, taking $t \to 0^+$ in (3.36), we have

$$\|\partial_t\mathbf{v}_m(0)\|_{L^2} \leq C(\|\mathbf{v}_0\|_{H^2} + \|\mathbf{v}_0\|_{H^2}^2). \tag{3.37}$$

Now, defining

$$Q(t) = \frac{1}{2}\int (|\mathbf{v}_m|^2 + |\partial_t\mathbf{v}_m|^2)\mathrm{d}x + \frac{\mu}{4}\int |S(\mathbf{v}_m)|^2\mathrm{d}x, \tag{3.38}$$

one reads, with the help of Young's inequality, from (3.35) that

$$Q'(t) \leq \widetilde{C}_1 Q(t) + \widetilde{C}_2 Q^3(t), \tag{3.39}$$

for some positive constants $\widetilde{C}_1$ and $\widetilde{C}_2$. Therefore, it is clear from (3.39) that there exists a constant $\widetilde{T} > 0$, which is finite due to the nonlinearity on the right hand side of (3.39), such that

$$\begin{aligned}
&\sup_{0\leq t\leq\widetilde{T}} \|(\mathbf{v}_m, \partial_t\mathbf{v}_\mathbf{m}, S(\mathbf{v}_m))(t)\|_{L^2}^2 \\
&\quad + \int_0^{\widetilde{T}} \|(\partial_t\mathbf{v}_m, S(\mathbf{v}_\mathbf{m}), S(\partial_t\mathbf{v}_m))\|_{L^2}^2\mathrm{d}t \leq C.
\end{aligned} \tag{3.40}$$

In view of the lower semi-continuity of norms to the bounds (3.40), we obtain that

$$\sup_{0\leq t\leq\widetilde{T}} \|(\mathbf{v}, \mathbf{v}_t, S(\mathbf{v}))(t)\|_{L^2}^2 + \int_0^{\widetilde{T}} \|(\mathbf{v}_t, S(\mathbf{v}), S(\mathbf{v}_t))\|_{L^2}^2\mathrm{d}t \leq C. \tag{3.41}$$

Finally, we establish the $H^2$-regularity for $\mathbf{v}$. To this end, we rewrite the equations (1.4) as

$$\begin{cases} -\mu\Delta \mathbf{v} + \nabla q + \gamma \mathbf{v} = \gamma \mathbf{v} - \partial_t \mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{v}, \\ \operatorname{div}\mathbf{v} = 0, \quad x \in \Omega, \end{cases} \tag{3.42}$$

with Navier boundary conditions

$$\begin{cases} \mathbf{v}\cdot n = 0, \quad x \in \partial\Omega, \\ \mu\big(S(\mathbf{v})\cdot n\big)_\tau = -\zeta \mathbf{v}_\tau. \end{cases} \tag{3.43}$$

Based on the above arguments, $\mathbf{v} \in H^1_\sigma$ is the weak solution to the Stokes problem (3.42)–(3.43). Then, it follows from Proposition 3.1 that $\mathbf{v}(t) \in H^2$ for $t \in (0, \widetilde{T})$ and

$$\begin{aligned} \|\mathbf{v}\|^2_{H^2} + \|q\|^2_{H^1} &\le C\|\mathbf{v}_t\|^2_{L^2} + C\|\mathbf{v}\cdot\nabla\mathbf{v}\|^2_{L^2} + C\|\mathbf{v}\|^2_{L^2} \\ &\le C\|\mathbf{v}_t\|^2_{L^2} + C\|\mathbf{v}\|^2_{L^6}\|\nabla\mathbf{v}\|^2_{L^3} + C\|\mathbf{v}\|^2_{L^2} \\ &\le C\big(\|\mathbf{v}_t\|^2_{L^2} + \|\mathbf{v}\|^2_{L^2}\big) + C\|\nabla\mathbf{v}\|^6_{L^2} + \frac{1}{2}\|\nabla\mathbf{v}\|^2_{H^1}, \end{aligned} \tag{3.44}$$

which combined with (3.41) yields

$$\sup_{0\le t\le \widetilde{T}} (\|\mathbf{v}(t)\|^2_{H^2} + \|q(t)\|^2_{H^1}) \le C.$$

Thus, the whole proof is completed. □

### 3.2 Nonlinear energy estimates

Next, we establish some nonlinear energy estimates for the perturbed problem (1.4)–(1.5), which will be used in the proof of Theorem 1.2. For this purpose, we assume that $(\mathbf{v}, q)$ is the strong solution of the perturbed problem (1.4)–(1.5) with the initial datum $\mathbf{v}_0$, which is obtained in Proposition 3.3. Arguing analogously to (3.22), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}\|^2_{L^2} + \frac{\mu}{2}\|S(\mathbf{v})\|^2_{L^2} \le 2\frac{\zeta^2 C^2_\Omega}{\mu}\|\mathbf{v}\|^2_{L^2}. \tag{3.45}$$

Differentiating the equation $(1.4)_1$ in time, multiplying the resulting equations by $\mathbf{v}_t$ and then integrating over $\Omega$ imply that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\mathbf{v}_t|^2\mathrm{d}x + \frac{\mu}{2}\int |S(\mathbf{v}_t)|^2\mathrm{d}x = -\zeta\int_{\partial\Omega} |(\mathbf{v}_t)_\tau|^2\mathrm{d}\sigma - \int \mathbf{v}_t\cdot\nabla\mathbf{v}\cdot\mathbf{v}_t\mathrm{d}x. \tag{3.46}$$

In view of Korn's inequality, Hölder's inequality and Young's inequality, we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}_t\|^2_{L^2} + \frac{\mu}{2}\|S(\mathbf{v}_t)\|^2_{L^2} \\ &\le C|\zeta|\|S(\mathbf{v}_t)\|_{L^2}\|\mathbf{v}_t\|_{L^2} + \|\mathbf{v}\|_{L^\infty}\|\nabla\mathbf{v}_t\|_{L^2}\|\mathbf{v}_t\|_{L^2} \\ &\le \frac{\mu}{8}\|S(\mathbf{v}_t)\|^2_{L^2} + \frac{4C|\zeta|^2}{\mu}\|\mathbf{v}_t\|^2_{L^2} + \frac{4C}{\mu}\|\mathbf{v}\|^2_{L^\infty}\|\mathbf{v}_t\|^2_{L^2}. \end{aligned} \tag{3.47}$$

Furthermore, multiplying $(1.4)_1$ by $\mathbf{v}_t$, integrating by parts and recalling that $\operatorname{div}\mathbf{v}_t = 0$, we have, for any positive $\varepsilon$, that

$$\begin{aligned} &\frac{\mu}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int |S(\mathbf{v})|^2\mathrm{d}x + \int |\mathbf{v}_t|^2\mathrm{d}x \\ &= -\int \mathbf{v}\cdot\nabla\mathbf{v}\cdot\mathbf{v}_t\mathrm{d}x - \zeta\int_{\partial\Omega} \mathbf{v}_\tau\cdot(\mathbf{v}_t)_\tau\mathrm{d}\sigma \end{aligned}$$

$$
\begin{aligned}
&\le C|\zeta|\|S(\mathbf{v})\|_{L^2}^{\frac{1}{2}}\|\mathbf{v}\|_{L^2}^{\frac{1}{2}}\|S(\mathbf{v}_t)\|_{L^2}^{\frac{1}{2}}\|\mathbf{v}_t\|_{L^2}^{\frac{1}{2}} + C\|\mathbf{v}\|_{L^\infty}\|\mathbf{v}\|_{L^2}\|\nabla\mathbf{v}_t\|_{L^2}\\
&\le \frac{8C}{\varepsilon\mu}\|\mathbf{v}\|_{L^\infty}^2\|\mathbf{v}\|_{L^2}^2 + \frac{\varepsilon\mu}{16}\|S(\mathbf{v}_t)\|_{L^2}^2 + \frac{8C|\zeta|^4}{\varepsilon^2\mu}\|\mathbf{v}\|_{L^2}^2\\
&\quad + \frac{\mu}{8}\|S(\mathbf{v})\|_{L^2}^2 + \frac{\varepsilon}{8\mu}\|\mathbf{v}_t\|_{L^2}^2.
\end{aligned}
\tag{3.48}
$$

If we assume that

$$
\|\mathbf{v}(t)\|_{H^2} \le \bar{\delta}, \quad \forall\, t \in [0, T],
$$

for some positive $\bar{\delta}$ and $T > 0$, then there is a positive constant $C$ such that

$$
\|\mathbf{v}(t)\|_{L^\infty} \le C\bar{\delta}.
$$

Under the condition $\|\mathbf{v}(t)\|_{H^2} \le \bar{\delta}$, we sum (3.45), $\varepsilon\times$ (3.47) and (3.48) up and then take $\varepsilon > 0$ small enough to obtain that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\mathbf{v}\|_{L^2}^2 + \frac{\varepsilon}{2}\|\mathbf{v}_t\|_{L^2}^2 + \frac{\mu}{4}\|S(\mathbf{v})\|_{L^2}^2\Big)\\
&\quad + \Big(\frac{3\mu}{8}\|S(\mathbf{v})\|_{L^2}^2 + \frac{\varepsilon\mu}{8}\|S(\mathbf{v}_t)\|_{L^2}^2 + \frac{1}{2}\|\mathbf{v}_t\|^2\Big) \le C\Big(\frac{\zeta^2}{\mu} + \frac{\bar{\delta}^2}{\varepsilon\mu} + \frac{|\zeta|^4}{\varepsilon^2\mu}\Big)\|\mathbf{v}\|_{L^2}^2.
\end{aligned}
\tag{3.49}
$$

Then, for any bounded $\bar{\delta}$, we get under the condition $\|\mathbf{v}(t)\|_{H^2} \le \bar{\delta}$ that

$$
\begin{aligned}
&\|\big(\mathbf{v}, \mathbf{v}_t, S(\mathbf{v})\big)(t)\|_{L^2}^2 + \int_0^t \|\big(\mathbf{v}_t, S(\mathbf{v}), S(\mathbf{v}_t)\big)(s)\|_{L^2}^2\mathrm{d}s\\
&\qquad \le C\|\big(\mathbf{v}, \mathbf{v}_t, S(\mathbf{v})\big)(0)\|_{L^2}^2 + C\int_0^t \|\mathbf{v}\|_{L^2}^2\mathrm{d}s.
\end{aligned}
\tag{3.50}
$$

On the other hand, multiplying $(1.4)_1$ by $\mathbf{v}_t$, integrating over $\Omega$ and recalling that $\operatorname{div}\mathbf{v}_t = 0$, we arrive at

$$
\int |\mathbf{v}_t|^2\mathrm{d}x = \int \big(\mu\Delta\mathbf{v}\cdot\mathbf{v}_t - \mathbf{v}\cdot\nabla\mathbf{v}\cdot\mathbf{v}_t\big)\mathrm{d}x \le \int \big(|\mathbf{v}||\nabla\mathbf{v}| + \mu|\Delta\mathbf{v}|\big)|\mathbf{v}_t|\mathrm{d}x.
\tag{3.51}
$$

By virtue of Sobolev's inequality, Hölder's inequality and Cauchy's inequality, we have

$$
\|\mathbf{v}_t(t)\|_{L^2}^2 \le C(\bar{\delta} + \mu^2)\|\mathbf{v}(t)\|_{H^2}^2.
$$

Taking $t \to 0^+$ in the above inequality yields

$$
\limsup_{t\to 0^+}\|\mathbf{v}_t(t)\|_{L^2}^2 \le C(\bar{\delta} + \mu^2)\|\mathbf{v}_0\|_{H^2}^2.
\tag{3.52}
$$

Therefore, we conclude from (3.50) and (3.52) that, if $\|\mathbf{v}(t)\|_{H^2} \le \bar{\delta}$, then

$$
\begin{aligned}
&\|\big(\mathbf{v}, \mathbf{v}_t, S(\mathbf{v})\big)(t)\|_{L^2}^2 + \int_0^t \|\big(\mathbf{v}_t, S(\mathbf{v}), S(\mathbf{v}_t)\big)(s)\|_{L^2}^2\mathrm{d}s\\
&\quad \le C\|\mathbf{v}_0\|_{H^2}^2 + C\int_0^t \|\mathbf{v}\|_{L^2}^2\mathrm{d}s.
\end{aligned}
\tag{3.53}
$$

Finally, it follows from (3.44) and the assumption $\|\mathbf{v}(t)\|_{H^2} \le \bar{\delta}$ that

$$
\frac{1}{2}\|\nabla^2\mathbf{v}\|_{L^2}^2 + \|q\|_{H^1}^2 \le C\big(\|\mathbf{v}_t\|_{L^2}^2 + \|\mathbf{v}\|_{L^2}^2 + (\bar{\delta})^2\|\nabla\mathbf{v}\|_{L^2}^2\big),
$$

which, together with (3.53), implies that for $t \in [0, T]$,

$$\|\mathbf{v}(t)\|_{H^2}^2 + \|\mathbf{v}_t(t)\|_{L^2}^2 + \|q(t)\|_{H^1}^2 + \int_0^t \|\big(\mathbf{v}_t, S(\mathbf{v}), S(\mathbf{v}_t)\big)(s)\|_{L^2}^2 \mathrm{d}s$$
$$\leq C\Big(\|\mathbf{v}_0\|_{H^2}^2 + \int_0^t \|\mathbf{v}(s)\|_{L^2}^2 \mathrm{d}s\Big). \tag{3.54}$$

Therefore, summarizing the above conclusions, we arrive at the following theorem.

**Theorem 3.1** *For any given initial datum* $\mathbf{v}_0 \in H^2$ *satisfying* $\operatorname{div} \mathbf{v}_0 = 0$ *and the boundary compatibility conditions, there exist a constant* $T > 0$ *and a unique strong solution* $(\mathbf{v}, q) \in \mathcal{C}^0([0, T], H^2 \times H^1)$ *to the perturbed problem* (1.4)–(1.5)*. Moreover, there exists a constant* $\bar{\delta} > 0$*, such that if* $\|\mathbf{v}(t)\|_{H^2} \leq \bar{\delta}$ *for* $t \in [0, T]$*, then the strong solution satisfies the energy inequality* (3.54) *for some constant* $C \geq 1$.

### 3.3 Proof of the nonlinear instability

Now, we adopt some ideas in [14] to prove the nonlinear instability. To this end, in view of Theorem 1.1, we first construct a linear solution

$$\mathbf{v}^l = e^{\Lambda t}\bar{\mathbf{v}}_0 \in H^2, \tag{3.55}$$

to the problem (1.5)–(1.6) with the initial datum $\bar{\mathbf{v}}_0 \in H^2$ satisfying $\operatorname{div} \bar{\mathbf{v}}_0 = 0$ and $\|\bar{\mathbf{v}}_0\|_{H^2} = 1$.

Denote $\mathbf{v}_0^\delta := \delta\bar{\mathbf{v}}_0$ and $C_1 := \|\bar{\mathbf{v}}_0\|_{L^2}$. By Theorem 3.1, there is a $\tilde{\delta} > 0$ such that, for any $\delta < \tilde{\delta}$, there exists a unique local solution $(\mathbf{v}^\delta, q^\delta) \in \mathcal{C}^0([0, T], H^2 \times H^1)$ to the problem (1.4)–(1.5) with the initial datum $\mathbf{v}_0^\delta$ satisfying $\|\mathbf{v}_0^\delta\|_{H^2} = \delta$. Let $C > 0$ and $\bar{\delta}$ be the same constants as in Theorem 3.1 and $\delta_0 = \min\{\bar{\delta}, \tilde{\delta}, 1\}$. Then, for any $\delta \in (0, \delta_0)$, we define

$$T^\delta = \frac{1}{\Lambda}\ln\frac{2\epsilon_0}{\delta}, \quad \text{i.e.} \quad \delta e^{\Lambda T^\delta} = 2\epsilon_0, \tag{3.56}$$

where $\epsilon_0$, independent of $\delta$, is a sufficiently small positive number to be determined later. Furthermore, we define

$$T^* = \sup\left\{t \in (0, T^{\max}]\big|\|\mathbf{v}^\delta\|_{H^2} \leq \delta_0\right\},$$

and

$$T^{**} = \sup\left\{t \in (0, T^{\max}]\big|\|\mathbf{v}^\delta\|_{L^2} \leq 2\delta C_1 e^{\Lambda t}\right\},$$

where $T^{\max}$ stands for the maximal time of existence. Apparently, $T^*, T^{**} > 0$ and

$$\|\mathbf{v}^\delta(T^*)\|_{H^2} = \delta_0 \quad \text{if} \quad T^* < \infty, \tag{3.57}$$
$$\|\mathbf{v}^\delta(T^{**})\|_{L^2} = 2\delta C_1 e^{\Lambda T^{**}} \quad \text{if} \quad T^{**} < \infty. \tag{3.58}$$

Thus, in view of (3.54) and the definitions of $T^*$ and $T^{**}$, we obtain that for any $t \leq \min\{T^\delta, T^*, T^{**}\}$,

$$\|\mathbf{v}^\delta(t)\|_{H^2}^2 + \|\mathbf{v}_t^\delta(t)\|_{L^2}^2 \leq C\delta^2\|\bar{\mathbf{v}}_0\|_{H^2}^2 + C\int_0^t \|\mathbf{v}^\delta\|_{L^2}^2 \mathrm{d}s$$
$$\leq C\delta^2 + \frac{4C\delta^2 C_1^2 e^{2\Lambda t}}{2\Lambda} \leq C_2\delta^2 e^{2\Lambda t}, \tag{3.59}$$

where $C_2$, independent of $\delta$, is a positive constant.

Denote $\mathbf{v}^d = \mathbf{v}^\delta - \delta \mathbf{v}^l$ and $q^d = q^\delta - \delta q^l$. Notice that $(\delta \mathbf{v}^l, \delta q^l)$ is also a solution to the problem (1.5)–(1.6) with the initial datum $\mathbf{v}_0^\delta \in H^2$. Hence, $\mathbf{v}^d$ satisfies the following equations

$$\begin{cases} \partial_t \mathbf{v}^d - \mu \Delta \mathbf{v}^d + \nabla q^d + \mathbf{v}^\delta \cdot \nabla \mathbf{v}^\delta = 0, \\ \operatorname{div} \mathbf{v}^d = 0, \quad x \in \Omega, \end{cases} \tag{3.60}$$

with Navier boundary conditions

$$\begin{cases} \mathbf{v}^d \cdot n = 0, \quad x \in \partial\Omega, \\ \mu \big(S(\mathbf{v}^d) \cdot n\big)_\tau = -\zeta \mathbf{v}_\tau^d, \end{cases} \tag{3.61}$$

and the initial condition $\mathbf{v}^d|_{t=0} = 0$. Multiplying $(3.60)_1$ by $\mathbf{v}^d$ and integrating by parts over $\Omega$, we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int |\mathbf{v}^d|^2 \mathrm{d}x + \frac{\mu}{2} \int |S(\mathbf{v}^d)|^2 \mathrm{d}x &= - \int \mathbf{v}^\delta \cdot \nabla \mathbf{v}^\delta \cdot \mathbf{v}^d \mathrm{d}x - \zeta \int_{\partial\Omega} |\mathbf{v}_\tau^d|^2 \mathrm{d}\sigma \\ &\le -\zeta \int_{\partial\Omega} |\mathbf{v}_\tau^d|^2 \mathrm{d}\sigma + \|\mathbf{v}^\delta\|_{L^\infty} \|\nabla \mathbf{v}^\delta\|_{L^2} \|\mathbf{v}^d\|_{L^2} \\ &\le -\zeta \int_{\partial\Omega} |\mathbf{v}_\tau^d|^2 \mathrm{d}\sigma + C_3 \|\mathbf{v}^\delta\|_{H^2}^2 \|\mathbf{v}^d\|_{L^2}. \end{aligned} \tag{3.62}$$

Additionally, from the definition of $\Lambda$, we obtain that

$$-\frac{\mu}{2} \int \big|S(\mathbf{v}^d)|\mathrm{d}x - \zeta \int_{\partial\Omega} |\mathbf{v}_\tau^d|^2 \mathrm{d}\sigma \le \Lambda \int |\mathbf{v}^d|^2 \mathrm{d}x,$$

which, together with (3.62), implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\mathbf{v}^d\|_{L^2} \le \Lambda \|\mathbf{v}^d\|_{L^2} + C_3 \|\mathbf{v}^\delta\|_{H^2}^2.$$

By Gronwall's inequality, we arrive at

$$\|\mathbf{v}^d\|_{L^2} \le C_3 e^{\Lambda t} \int_0^t e^{-\Lambda s} \|\mathbf{v}^\delta\|_{H^2}^2 \mathrm{d}s \le C_3 e^{\Lambda t} C_2 \delta^2 \int_0^t e^{\Lambda s} \mathrm{d}s \le C_4 \delta^2 e^{2\Lambda t}. \tag{3.63}$$

Now we claim that

$$T^\delta = \min\left\{T^\delta, T^*, T^{**}\right\}, \tag{3.64}$$

provided that $\epsilon_0$ is taken as

$$\epsilon_0 = \min\left\{\frac{\delta_0}{4\sqrt{C_2}}, \frac{C_1}{4C_4}\right\}. \tag{3.65}$$

Indeed, if $T^* = \min\left\{T^\delta, T^*, T^{**}\right\}$, then $T^* < \infty$. In fact, in view of (3.59), we have

$$\|\mathbf{v}^\delta(T^*)\|_{H^2} \le \sqrt{C_2} \delta e^{\Lambda T^*} \le \sqrt{C_2} \delta e^{\Lambda T^\delta} = 2\epsilon_0 \sqrt{C_2} < \delta_0,$$

which contradicts with (3.57). On the other hand, if $T^{**} = \min\left\{T^\delta, T^*, T^{**}\right\}$, then $T^{**} < \infty$. It follows from (3.56) and (3.58) that

$$\begin{aligned} \|\mathbf{v}^\delta(T^{**})\|_{L^2} &\le \delta \|\mathbf{v}^l(T^{**})\|_{L^2} + \|\mathbf{v}^d(T^{**})\|_{L^2} \\ &\le \delta \|\mathbf{v}^l(T^{**})\|_{L^2} + C_4 \delta^2 e^{2\Lambda T^{**}} \end{aligned}$$

$$\leq \delta C_1 e^{\Lambda T^{**}} + C_4 \delta^2 e^{2\Lambda T^{**}}$$
$$\leq \delta e^{\Lambda T^{**}} \left( C_1 + C_4 \delta e^{\Lambda T^{\delta}} \right) < 2\delta C_1 e^{\Lambda T^{**}},$$

which contradicts with (3.58). Therefore, $T^\delta = \min\left\{T^\delta, T^*, T^{**}\right\}$.

Finally, we get from (3.63) and (3.65) that

$$\begin{aligned}\|\mathbf{v}^\delta(T^\delta)\|_{L^2} \geq \|\delta \mathbf{v}^l(T^\delta)\|_{L^2} - \|\mathbf{v}^d(T^\delta)\|_{L^2} &\geq C_1 \delta e^{\Lambda T^\delta} - C_4 \delta^2 e^{2\Lambda T^\delta} \\ &\geq 2C_1\epsilon_0 - 2C_4\epsilon_0^2 \geq C_1\epsilon_0,\end{aligned} \tag{3.66}$$

which completes the proof of Theorem 1.2 by choosing $\epsilon = \epsilon_0$.

## 4 The linear and nonlinear stability

In this section, we give the proof of Theorem 1.3, namely, the stability of the linear and nonlinear problems under the assumption (1.16): $\zeta > -\mathcal{C}_r\mu$.

It follows from the definition of $\mathcal{C}_r$ that for any $\mathbf{u} \in H^1_\sigma$, and for $\zeta < 0$, it holds

$$\zeta \mathcal{C}_r \int_{\partial\Omega} |\mathbf{u}|^2 \mathrm{d}\sigma \geq \frac{\zeta}{2} \int |S(\mathbf{u})|^2 \mathrm{d}x. \tag{4.1}$$

This is crucial for the proof of the stability.

The local well-posedness theory for both the linearized problem (1.5)–(1.6) and the nonlinear problem (1.4)–(1.5) can be obtained by the Faedo-Galerkin approximation method as Proposition 3.3. In order to prove Theorem 1.3, we only need to derive the a priori estimates similar to those of Theorem 3.1.

***Proof of Theorem 1.3*** Since $\mathcal{C}_r > 0$, we divide the condition (1.16) into the following two cases: $-\mathcal{C}_r\mu < \zeta < 0$ and $\zeta \geq 0$.

We start with the linear stability. Let $\mathbf{v}$ be a solution of the linearized problem (1.5)–(1.6) with an associated pressure $q$. Then the solution $(\mathbf{v}, q)$ possesses proper regularity such that the procedure of formal calculations makes sense. Taking inner product between the equation $(1.6)_1$ and $\mathbf{v}$, using the boundary condition (1.5) and the constraint $\operatorname{div}\mathbf{v} = 0$, and integrating by parts, we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\mathbf{v}|^2 \mathrm{d}x + \frac{\mu}{2}\int |S(\mathbf{v})|^2 \mathrm{d}x = -\zeta \int_{\partial\Omega} |\mathbf{v}_\tau|^2 \mathrm{d}\sigma. \tag{4.2}$$

In the case of $\zeta \geq 0$, (4.2) implies that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\mathbf{v}|^2 \mathrm{d}x + \frac{\mu}{2}\int |S(\mathbf{v})|^2 \mathrm{d}x \leq 0.$$

In the case of $-\mathcal{C}_r\mu < \zeta < 0$, (4.2) gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\mathbf{v}|^2 \mathrm{d}x + \frac{1}{2}\left(\mu + \frac{\zeta}{\mathcal{C}_r}\right)\int |S(\mathbf{v})|^2 \mathrm{d}x \leq 0. \tag{4.3}$$

Therefore, in either case, as long as (1.16) holds, there is a positive constant $\sigma_1 > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |\mathbf{v}|^2 \mathrm{d}x + \sigma_1 \int |S(\mathbf{v})|^2 \mathrm{d}x \leq 0, \tag{4.4}$$

which, with the help of Korn's inequality (1.8), implies that there is a positive constant $\alpha > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}\|_{L^2}^2 + \alpha\|\mathbf{v}(t)\|_{H^1}^2 \leq 0. \tag{4.5}$$

We thus conclude from (4.5) that, for any $t > 0$, it holds

$$\int_0^t \|\mathbf{v}(\tau)\|_{H^1}^2 \mathrm{d}\tau \leq C\|\mathbf{v}_0\|_{L^2}^2, \quad \|\mathbf{v}(t)\|_{L^2} \leq Ce^{-\frac{\alpha}{2}t}\|\mathbf{v}_0\|_{L^2}. \tag{4.6}$$

Next, taking time derivative on $(1.6)_1$, multiplying the resulting equations by $\mathbf{v}_t$ in $L^2$, and then using the calculations similar to that in the derivation of (4.4), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |\mathbf{v}_t|^2 \mathrm{d}x + \sigma_1 \int |S(\mathbf{v}_t)|^2 \mathrm{d}x \leq 0, \tag{4.7}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}_t\|_{L^2}^2 + \alpha\|\mathbf{v}_t(t)\|_{H^1}^2 \leq 0. \tag{4.8}$$

Analogously to (3.52), we also have

$$\|\mathbf{v}_t(0)\|_{L^2}^2 \leq C\|\Delta\mathbf{v}_0\|_{L^2}^2. \tag{4.9}$$

Hence, (4.8) implies that, for any $t > 0$,

$$\int_0^t \|\mathbf{v}_t(\tau)\|_{H^1}^2 \mathrm{d}\tau \leq C\|\mathbf{v}_0\|_{H^2}^2, \quad \|\mathbf{v}_t(t)\|_{L^2} \leq Ce^{-\frac{\alpha}{2}t}\|\mathbf{v}_0\|_{H^2}. \tag{4.10}$$

Now, using the estimates in Proposition 3.1, we have

$$\|\mathbf{v}\|_{H^2}^2 + \|q\|_{H^1}^2 \leq C(\|\mathbf{v}\|_{L^2}^2 + \|\mathbf{v}_t\|_{L^2}^2) \leq Ce^{-\alpha t}\|\mathbf{v}_0\|_{H^2}^2. \tag{4.11}$$

The first assertion (linear stability) in Theorem 1.3 follows.

Now, we turn to prove the second assertion (nonlinear stability) in Theorem 1.3. From a standard local well-posedness theory on Navier-Stokes equations with Navier boundary conditions, we assume that $\mathbf{v} \in H^2$ is a solution of the perturbed problem (1.4)–(1.5) with the associated pressure $q \in H^1$, up to some time $T > 0$. We now proceed under the following a priori hypothesis: *There is a sufficiently small positive number* $\delta_1$ *such that*

$$\|\mathbf{v}(t)\|_{H^2} \leq \delta_1, \ \forall\ t \in [0, T]. \tag{4.12}$$

The choice of $\delta_1$ will be given later.

Taking inner product between $(1.4)_1$ and $\mathbf{v}$, the standard energy estimate and a similar argument to that for (4.4) yield that, for any $t \in [0, T]$, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |\mathbf{v}|^2 \mathrm{d}x + \sigma_1 \int |S(\mathbf{v})|^2 \mathrm{d}x \leq 0, \tag{4.13}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}\|_{L^2}^2 + \alpha\|\mathbf{v}(t)\|_{H^1}^2 \leq 0. \tag{4.14}$$

Therefore, for any $t \in [0, T]$, it holds

$$\int_0^t \|\mathbf{v}(\tau)\|_{H^1}^2 \mathrm{d}\tau \leq C\|\mathbf{v}_0\|_{L^2}^2, \quad \|\mathbf{v}(t)\|_{L^2} \leq Ce^{-\frac{\alpha}{2}t}\|\mathbf{v}_0\|_{L^2}. \tag{4.15}$$

Applying $\partial_t$ to (1.4)$_1$, taking the inner product of the result with $\mathbf{v}_t$ and a similar argument to that in the derivation of (4.7), we obtain that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int|\mathbf{v}_t|^2\mathrm{d}x+\sigma_1\int|S(\mathbf{v}_t)|^2\mathrm{d}x\\&\leq C\Big|\int\mathbf{v}_t\cdot\nabla\mathbf{v}\cdot\mathbf{v}_t\mathrm{d}x\Big|\leq C\|\nabla\mathbf{v}\|_{L^2}\|\mathbf{v}_t\|_{L^4}^2\leq C\delta_1\|S(\mathbf{v}_t)\|_{L^2}^2,\end{aligned}$$

which, with the help of the smallness of $\delta_1$, implies that there is a positive constant $\beta$ such that for any $t\in[0,T]$, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\int|\mathbf{v}_t|^2\mathrm{d}x+\beta\int|S(\mathbf{v}_t)|^2\mathrm{d}x\leq 0. \tag{4.16}$$

From (3.52), we have

$$\|\mathbf{v}_t(0)\|_{L^2}^2\leq C\|\mathbf{v}_0\|_{H^2}^2. \tag{4.17}$$

Therefore, for any $t\in[0,T]$, it holds

$$\int_0^t\|\mathbf{v}_\tau(\tau)\|_{H^1}^2\mathrm{d}\tau\leq C\|\mathbf{v}_0\|_{H^2}^2,\quad\|\mathbf{v}_t(t)\|_{L^2}\leq Ce^{-\frac{\beta}{2}t}\|\mathbf{v}_0\|_{H^2}. \tag{4.18}$$

Finally, from the estimates in Proposition 3.1, we have

$$\begin{aligned}\|\mathbf{v}\|_{H^2}^2+\|q\|_{H^1}^2&\leq C\big(\|\mathbf{v}_t+\mathbf{v}\cdot\nabla\mathbf{v}\|_{L^2}^2+\|\mathbf{v}\|_{L^2}^2\big)\\&\leq C\big(\|\mathbf{v}_t\|_{L^2}^2+\|\mathbf{v}\|_{L^2}^2+\|\mathbf{v}\|_{L^\infty}^2\|\nabla\mathbf{v}\|_{L^2}^2\big)\\&\leq C\big(\|\mathbf{v}_t\|_{L^2}^2+\|\mathbf{v}\|_{L^2}^2\big)+C\delta_1^2\|\nabla\mathbf{v}\|_{L^2}^2,\end{aligned} \tag{4.19}$$

which, with the help of the smallness of $\delta_1$, (4.15) and (4.18), gives, for any $t\in[0,T]$, that

$$\|\mathbf{v}(t)\|_{H^2}+\|q(t)\|_{H^1}\leq C(\|\mathbf{v}(t)\|_{L^2}+\|\mathbf{v}_t(t)\|_{L^2})\leq Ce^{-\frac{\gamma}{2}t}\|\mathbf{v}_0\|_{H^2}, \tag{4.20}$$

for $\gamma=\min\{\alpha,\beta\}$.

Now, as assumed in Theorem 1.3, $\|\mathbf{v}_0\|_{H^2}\leq\varepsilon_1$, we find from (4.20) that, if $\|\mathbf{v}(t)\|_{H^2}\leq\delta_1$ for $t\in[0,T]$, then

$$\|\mathbf{v}(t)\|_{H^2}\leq C\varepsilon_1,\ \forall\, t\in[0,T],$$

for a positive constant $C$ independent of $t$. Therefore, the a priori hypothesis (4.12) is fulfilled if we choose $\delta_1=\sqrt{\varepsilon_1}$ for $\varepsilon_1$ small enough. A standard continuity argument, with the help of the local well-posedness theory, the choice of $\delta_1=\sqrt{\varepsilon_1}$, the smallness of $\varepsilon_1$, and the uniform estimate (4.20), gives the global existence of the unique strong solution $(\mathbf{v},q)\in H^2\times H^1$ to the nonlinear problem (1.4)–(1.5) satisfying the estimates (1.18). Thus, the proof of Theorem 1.3 is completed. □

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## 5 Appendix

In this section, we study the existence of solutions to the Stokes problem (3.1)–(3.2) with the help of Proposition 3.2 and some additional assumptions.

We first recall the Stokes problem (3.1)–(3.2) as

$$\begin{cases} -\mu\Delta\mathbf{v}+\nabla q = f, \\ \operatorname{div}\mathbf{v}=0, \quad x\in\Omega, \end{cases} \tag{5.1}$$

with Navier boundary conditions

$$\begin{cases} \mathbf{v}\cdot n = 0, \quad x\in\partial\Omega, \\ \mu\big(S(\mathbf{v})\cdot n\big)_\tau = -\zeta\mathbf{v}_\tau. \end{cases} \tag{5.2}$$

From Proposition 3.2, the following eigenvalue problem:

$$\begin{cases} -\Delta\mathbf{v}+\nabla p = \lambda\mathbf{v}, \\ \operatorname{div}\mathbf{v}=0, \quad x\in\Omega, \end{cases}$$

with the Navier boundary conditions (5.2) possesses countable eigenvalues $\{\lambda_i\}_{i=1}^\infty$, with the corresponding eigenfunctions $\{\mathbf{w}_i\}_{i=1}^\infty$. In particular, $\{\mathbf{w}_i\}_{i=1}^\infty \subset H^2$ is an orthonormal basis of $L^2_\sigma$, and also a basis of $H^1_\sigma$.

**Proposition 5.1** *Assume that $f\in L^2$ and $\zeta < -\mathcal{C}_r\mu$.*

*(1) If $0\notin\{\lambda_i\}_{i=1}^\infty$, then the Stokes problem* (5.1)–(5.2) *possesses a unique strong solution $(\mathbf{v},q)\in H^2\times H^1$ and*

$$\|\mathbf{v}\|_{H^2}+\|q\|_{H^1}\le C\big(\|\mathbf{v}\|_{L^2}+\|f\|_{L^2}\big).$$

*(2) If $0\in\{\lambda_i\}_{i=1}^\infty$ with the corresponding eigenfunction $\mathbf{w}_0$ and*

$$\int f\cdot\mathbf{w}_0\mathrm{d}x = 0,$$

*then the conclusions in* (1) *also hold.*

***Proof*** We first give the proof of the first assertion. We assume that the weak solution of the Stokes problem (5.1)–(5.2) has the following form:

$$\mathbf{v}=\sum_{i=1}^\infty v_i\mathbf{w}_i, \tag{5.3}$$

where the constants $v_i$ will be determined by solving the following equations:

$$v_i\Big(\frac{\mu}{2}\int|S(\mathbf{w}_i)|^2\mathrm{d}x+\zeta\int_{\partial\Omega}|\mathbf{w}_i|^2\mathrm{d}\sigma\Big)=\int f\cdot\mathbf{w}_i\mathrm{d}x,\quad \text{for any } i\in\mathbb{N}_+. \tag{5.4}$$

On the other hand, we have that for each $\mathbf{w}_i$,

$$\frac{\mu}{2}\int|S(\mathbf{w}_i)|^2\mathrm{d}x+\zeta\int_{\partial\Omega}|\mathbf{w}_i|^2\mathrm{d}\sigma=\lambda_i. \tag{5.5}$$

Thus, it follows from (5.4) and (5.5) that for any $i\in\mathbb{N}_+$,

$$v_i=\frac{\int f\cdot\mathbf{w}_i\mathrm{d}x}{\lambda_i}.$$

As a result, we deduce that

$$\mathbf{v} = \sum_{i=1}^{\infty} \frac{\int f \cdot \mathbf{w}_i \mathrm{d}x}{\lambda_i} \mathbf{w}_i \in L^2.$$

Multiplying (5.4) by $v_i$, summing them up from $i = 1$ to $\infty$ and using Korn's inequality, we infer that $\mathbf{v}$ also belongs to $H_\sigma^1$ and

$$\|\mathbf{v}\|_{H^1} \leq C\big(\|\mathbf{v}\|_{L^2} + \|f\|_{L^2}\big).$$

It is also easy to check that for any $\mathbf{w} \in H_\sigma^1$,

$$\frac{\mu}{2} \int S(\mathbf{v}) : S(\mathbf{w}) \mathrm{d}x + \zeta \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{w} \mathrm{d}\sigma = \int f \cdot \mathbf{w} \mathrm{d}x.$$

The rest of arguments are similar to Proposition 3.1. The proof of the first assertion is completed. We can prove the second assertion in the same way. □

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