ON THE EXPONENTIAL DECAY FOR COMPRESSIBLE NAVIER-STOKES-KORTEWEG EQUATIONS WITH A DRAG TERM

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Abstract. In this paper, we consider global weak solutions to compressible Navier-Stokes-Korteweg equations with density dependent viscosities, in a periodic domain $\Omega = \mathbb{T}^3$, with a linear drag term with respect to the velocity. The main result concerns the exponential decay to equilibrium of such solutions using log-sobolev type inequalities. In order to show such a result, the starting point is a global weak-entropy solutions definition introduced in D. Bresch, A. Vasseur and C. Yu [arXiv:1905.02701 (2019)]. Assuming extra assumptions on the shear viscosity when the density is close to vacuum and when the density tends to infinity, we conclude the exponential decay to equilibrium. Note that our result covers the quantum Navier-Stokes system with a drag term.

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1. Introduction

The goal of this paper is to study the long-time behaviour of solutions of Navier-Stokes-Korteweg models with a drag term of type $r_3\rho u$ in a periodic domain $\Omega = \mathbb{T}^3$ using a κ relative entropy generalizing the one by [11]. More precisely the system under consideration reads

$$\begin{aligned}
&\partial_t \rho + \mathrm{div}(\rho u) = 0,\\
&\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p(\rho)\\
&\qquad -2\mathrm{div}(\mu(\rho)\mathbb{D}(u)) - \nabla(\lambda(\rho)\mathrm{div} u)\\
&\qquad -2\varepsilon\rho\nabla\Big(\sqrt{K(\rho)}\Delta\int_0^\rho \sqrt{K(s)}\,ds\Big) + r_3\rho u = 0
\end{aligned}$$

where $r_3 \geq 0$ is constant, the pressure state law reads $p(\rho) = a\rho^\gamma$ where $a > 0$ and $\gamma > 1$ are constants, the capillarity coefficient given by $K(\rho) = (\mu'(\rho))^2/\rho$ and the bulk viscosity given by the BD relation $\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho))$.

Note that such system includes the quantum Navier-Stokes system choosing $K(\rho) = 1/\rho$. In this situation, note that A. Vasseur and C. Yu (see in [29]) have proven the global-in-time existence of weak solutions of the quantum Navier-Stokes equations with two drag terms of type $r_0 u$ and $r_1\rho|u|^2u$. Their result is still valuable in the case $r_1 = 0$ and their proof is based on a Faedo-Galerkin approximation (following the ideas of [20]) and the BD entropy (see [8, 9]). Note that the result is also still valuable with the add of a third drag force term of type $r_3\rho u$ because the terme does not perturb the uniform estimates and stable in limit process. In [30], the authors use the result obtained in [29] and pass to the limits ε, r_0, r_1 tend to zero to prove the existence of global-in-time weak solutions to degenerate compressible Navier-Stokes Equations with $\mu(\rho) = \mu\rho$ and $\lambda(\rho) = 0$. Such existence of global existence of weak solutions has also been obtained at the same time and independently in [24]. Note that to prove the result in [30], they need uniform (with respect to r_0, r_1) estimates to pass to the limit r_0, r_1 tend to 0. To this end they have to firstly pass to the limit ε tends to 0. Recently in [23], global existence of weak solutions for the quantum Navier-Stokes Equations has been proved without drag terms. The method is based on the construction of weak solutions that are renormalized in the velocity variable for the system with drag terms $r_0 u$ and $r_1\rho|u|^2 u$ and uses the stability part of the result to pass to the limit r_0, r_1 tend to zero to obtain weak renormalized solutions of the system without drag terms. All weak renormalized solution being a weak solution, the existence of global weak solutions for the quantum Navier-Stokes system (namely with $\mu(\rho) = \mu\rho$ and $\lambda(\rho) = 0$) without drag terms is shown. Adding an extra drag term $r_3\rho u$ to the system does not change the result: it does not perturb the estimates and is stable in limit passage. Note that the construction being uniform with respect to the Planck constant ε, the authors also perform the semi-classical limit $\varepsilon \to 0$ to the associated compressible Navier-Stokes equations. Note also the paper [4] concerning the global existence for the quantum Navier-Stokes system using the approximate procedure introduced by J. Li and Z.P. Xin (see[24]). We also mention the recent paper [5] where they consider the global existence of weak solutions to Navier-Stokes-Korteweg model extending the method of truncation, regularization and renormalization introduced by A. Vasseur and C. Yu in [30] and generalized by I. Lacroix-Violet and A. Vasseur in [23].

It is important to remark that a global weak solution of the quantum Navier-Stokes equations in the sense of [23] is also weak solution of the corresponding augmented system as introduced in [10]. Quantum fluid models have attracted a lot of attention in the last decades due to the variety of applications. Indeed, such models can be used to describe superfluids [25], quantum semiconductors [16], weakly interacting Bose gases [18] and quantum trajectories of Bohmian mechanics [31]. Recently some dissipative quantum fluid models have been derived. In particular, under some assumptions and using a Chapman-Enskog expansion in Wigner equation, the authors have obtained in [13] the so-called quantum Navier-Stokes model. Roughly speaking, it corresponds to the classical Navier-Stokes equations with a quantum correction term. The main difficulties of such models lie in the highly nonlinear structure of the third order quantum term and the proof of positivity (or non-negativity) of the particle density. Note that formally, the quantum Euler system corresponds to the limit of the quantum Navier-Stokes model when the viscosity coefficient tends to zero. This type of models belong to more general classes of models: the Navier-Stokes-Korteweg and the Euler-Korteweg systems. Readers interested by Korteweg type systems are referred to the following articles and books: [22, 28, 14, 15, 27, 26, 19, 2, 3, 4, 5] and references cited therein.

Finally, let us cite [12] where the existence of global weak solutions for some Navier-Stokes-Korteweg system in $\Omega = \mathbb{T}^3$ is obtained. More precisely they consider shear and bulk viscosities such that:

$$\mu \in \mathcal{C}^0(\mathbb{R}_+;\mathbb{R}_+) \cap \mathcal{C}^2(\mathbb{R}_+^*,\mathbb{R}), \tag{1}$$

where $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{R}_+^* = (0,\infty)$. The authors also assume that there exists two positive numbers α_1, α_2 such that

$$\begin{aligned} &\frac{2}{3} < \alpha_1 < \alpha_2 < 4 \\ &\text{and for any } \rho > 0, \qquad 0 < \frac{1}{\alpha_2}\rho\mu'(\rho) \le \mu(\rho) \le \frac{1}{\alpha_1}\rho\mu'(\rho), \end{aligned} \tag{2}$$

and there exists a constant $C > 0$ such that

$$\Big|\frac{\rho\mu''(\rho)}{\mu'(\rho)}\Big| \le C < +\infty. \tag{3}$$

The shear and bulk viscosities are assumed to satisfy the BD relation

$$\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho)). \tag{4}$$

and therefore thanks to (2) and (3), there exists $\widetilde{\nu} > 0$ such that

$$\lambda(\rho) + 2\mu(\rho)/3 \ge \widetilde{\nu}\mu(\rho) \tag{5}$$

and

$$\mu(0) = \lambda(0) = 0.$$

The method to prove global existence of weak solution to the Navier-Stokes-Korteweg is linked to the two-hydrodynamic system introduced in [10], the extension of the Böhm identity proved in [6], the generalization of the dissipation inequality used in [20] for the quantum Navier-Stokes system and established in [21] and the renormalized solutions introduced in [23]. The addition of a drag term $r_3\rho u$ does not change the result because the κ-entropy remains uniform and the term ρu is already in the time derivative for compactness.

Remark 1. *Note the recent paper* [1] *where the authors have proved a sobolev inequality which could help to enlarge the range of viscosities namely*

$$\mu(\rho) = \rho^\alpha, \qquad \lambda(\rho) = 2(\alpha - 1)\rho^\alpha$$

with $2/3 < \alpha < +\infty$.

Let us recall the definition of the global weak solutions for the Navier-Stokes-Korteweg system similar to what has been developped in [12]

Definition 2. *We say that* (ρ, u) *is a global weak solution to the compressible Navier-Stokes-Korteweg equations as constructed in* [12] *if*

- *The density satisfies* $\rho \in \mathcal{C}^0([0,+\infty); L^\gamma_{\rm weak}(\Omega))$ *with* $\rho \ge 0$ *in* $(0,+\infty)\times\Omega$, $\rho|_{t=0} = \rho_0$ *a.e. in* Ω *with the viscosity* $\mu(\rho) \in \mathcal{C}^0([0,+\infty); L^{3/2}_{\rm weak}(\Omega))$.
- *The momentum satisfies* $\rho\, u \in \mathcal{C}([0,+\infty); L^{2\gamma/(\gamma+1)}_{\rm weak}(\Omega))$ *with* $\rho u|_{t=0} = m_0$.

- *The following energy estimate holds a.e* $t \in [0, +\infty)$

$$
\begin{aligned}
&\int_\Omega \Big[\rho\Big(\frac{|w|^2}{2} + [(1-\kappa)\kappa + \varepsilon]\frac{|v|^2}{2}\Big) + H(\rho) + 2r_3\kappa\, h(\rho)\Big](t)\, dx \\
&+\tilde{\nu}\int_0^t\int_\Omega |\mathbb{T}_\mu|^2\, dx\, dt + r_3\int_0^t\int_\Omega \rho|u|^2 dx\, dt \\
&+\varepsilon\, C_{\alpha_1,\tilde{\nu}}\int_0^t\int_\Omega \Big[|\nabla^2 Z(\rho)|^2 + |\nabla Z_1(\rho)|^4 + |\mathbb{T}_\varepsilon|^2\Big]\, dx\, dt \\
&+2\kappa\int_0^t\int_\Omega \frac{\mu'(\rho)p'(\rho)}{\rho}|\nabla\rho|^2\, dx\, dt \\
&\le \int_\Omega \Big[\rho_0\Big(\frac{|w_0|^2}{2} + [(1-\kappa)\kappa + \varepsilon]\frac{|v_0|^2}{2}\Big) + H(\rho_0) + 2r_3\kappa\, h(\rho_0)\Big]\, dx
\end{aligned} \tag{6}
$$

for some $C_{\alpha_1,\widetilde{\nu}} > 0$ *depending on* α_1 *and* $\widetilde{\nu}$ *and some* $\kappa \in (0,1)$ *with*

$$w = u + 2\kappa\nabla s(\rho) \quad \text{and} \quad v = 2\nabla s(\rho) \quad \text{where} \quad s'(\rho) = \mu'(\rho)/\rho \tag{7}$$

$$H(\rho) = \rho\int_0^\rho p(s)/s^2\, ds \quad \text{and} \quad h(\rho) = \rho\int_0^\rho \mu(s)/s^2\, ds \tag{8}$$

$$Z(\rho) = \int_0^\rho \frac{\sqrt{\mu(s)}\mu'(s)}{s}\, ds \quad \text{and} \quad Z_1(\rho) = \int_0^\rho \frac{\mu'(s)}{(\mu(s))^{1/4}s^{1/2}}\, ds \tag{9}$$

and where $\mathbb{T}_\mu$ *is defined through*

$$\sqrt{\mu(\rho)}\mathbb{T}_\mu = \nabla(\sqrt{\rho}u\frac{\mu(\rho)}{\sqrt{\rho}}) - \sqrt{\rho}u \otimes \sqrt{\rho}\nabla s(\rho) \tag{10}$$

$$
\begin{aligned}
&\frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}\mathbb{T}_\mu)\mathrm{Id} \\
&\qquad = \Big[\mathrm{div}(\frac{\lambda(\rho)}{\mu(\rho)}\sqrt{\rho}u\frac{\mu(\rho)}{\sqrt{\rho}}) - \sqrt{\rho}u\cdot\sqrt{\rho}\nabla s(\rho)\frac{\rho\mu''(\rho)}{\mu'(\rho)}\Big]\mathrm{Id}.
\end{aligned} \tag{11}
$$

The same definitions and compatibility condition are satisfied for $\mathbb{T}_\varepsilon$*, replacing* u *by* $v = 2\nabla s(\rho)$ *respectively in* (10) *and* (11).

- *The following extra estimates hold*

$$
\begin{aligned}
&\|\mu(\rho)\|_{L^\infty(0,+\infty;W^{1,1}(\Omega))} + \|\mu(\rho)u\|_{L^\infty(0,\infty;L^{3/2}(\Omega))\cap L^2(0,+\infty;W^{1,1}(\Omega))} < +\infty \\
&\|\partial_t\mu(\rho)\|_{L^\infty(0,+\infty;W^{-1,1}(\Omega))} + \|Z(\rho)\|_{L^{1+}(0,+\infty;L^1(\Omega))} < +\infty.
\end{aligned}
$$

- *The density* ρ *satisfies the mass equation in* $\mathcal{D}'((0,+\infty)\times\Omega)$*:*

$$\partial_t\rho + \mathrm{div}(\rho u) = 0. \tag{12}$$

- *The velocity satisfies the momentum in* $\mathcal{D}'((0,+\infty)\times\Omega)$ *:*

$$
\begin{aligned}
&\partial_t(\rho u) + \mathrm{div}(\rho u\otimes u) + \nabla p(\rho) + r_3\rho u \\
&\qquad -2\mathrm{div}(\sqrt{\mu(\rho)}\mathbb{S}_\mu + \frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}\mathbb{S}_\mu)\mathrm{Id}) \\
&\qquad -2\varepsilon\mathrm{div}(\sqrt{\mu(\rho)}\mathbb{S}_\varepsilon + \frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}\mathbb{S}_\varepsilon)\mathrm{Id}) \\
&\qquad = 0
\end{aligned} \tag{13}
$$

with

$$\mathbb{S}_\mu = \frac{(\mathbb{T}_\mu + \mathbb{T}_\mu^t)}{2}, \qquad \mathbb{S}_\varepsilon = \frac{(\mathbb{T}_\varepsilon + \mathbb{T}_\varepsilon^t)}{2}.$$

- *The viscosity satisfies in* $\mathcal{D}'((0,+\infty)\times\Omega)$*:*

$$\partial_t \mu(\rho) + \operatorname{div}(\mu(\rho)u) + \frac{\lambda(\rho)}{2\mu(\rho)}\operatorname{Tr}(\sqrt{\mu(\rho)}\mathbb{T}_\mu) = 0. \tag{14}$$

- *The global weak solution is obtained through regularized solutions* $(\rho_\zeta, w_\zeta, v_\zeta)$ *(letting* ζ *tend to zero) satisfying*

$$\frac{d\mathcal{E}_\zeta}{dt}(t) + \varepsilon C_{\widetilde{\nu},\alpha_1}\int_\Omega |\nabla^2 Z(\rho_\zeta)|^2 + \kappa\int_\Omega \mu'(\rho_\zeta)H''(\rho_\zeta)|\nabla\rho_\zeta|^2 + \int_\Omega r_3\rho_\zeta|u_\zeta|^2 \le 0 \tag{15}$$

where

$$\mathcal{E}_\zeta(t) = \int_\Omega \Big[\rho_\zeta(|w_\zeta|^2 + (\kappa(1-\kappa)+\varepsilon)|v_\zeta|^2) + H(\rho_\zeta|r) + 2r_3\kappa h(\rho_\zeta|r)\Big] \tag{16}$$

where $H(\rho_\zeta|r)$ *and* $h(\rho_\zeta|r)$ *is defined by:*

$$H(\rho_\zeta|r) = H(\rho_\zeta) - H(r) - H'(r)(\rho_\zeta - r),$$
$$h(\rho_\zeta|r) = h(\rho_\zeta) - h(r) - h'(r)(\rho_\zeta - r).$$

Remark 3. *In the present paper we assume the presence of the drag term* $r_3\rho u$ *with* $r_3 > 0$. *Such quantity implies (using the* κ*-entropy estimate) the control of* $\sup_{t\in(0,+\infty)}\int_\Omega h(\rho)(t,x)\,dx$ *if* $\int_\Omega h(\rho_0)\,dx$ *initially. This reads*

$$\sup_{t\in(0,+\infty)}\int_\Omega \rho\int_1^\rho \mu(s)/s^2\,ds \le C < +\infty.$$

Using Hypothesis (2)*, this implies that*

$$\mu(\rho) \in L^\infty(0,+\infty; L^1(\Omega)).$$

This is an important information compared to previous papers such as [24] *or* [12]. *Indeed, to control* $\mu(\rho)$ *in* $L^\infty(0,+\infty;L^1(\Omega))$ *without using drag terms information, we would need as in* [24] *to consider a pressure law as*

$$p(\rho) = a\rho^\gamma \text{ with } \gamma \ge \max(2\alpha_2 - 1, 1)$$

but this condition is restrictive. Note that $\rho^\gamma \in L^\infty(0,+\infty;L^1(\Omega))$ *implies* $\mu(\rho) \in L^\infty(0,+\infty;L^1(\Omega))$ *because* $\sqrt{\rho} \in L^\infty(0,+\infty;L^2(\Omega))$ *deduced from the mass conservation. Note that the hypothesis* $\gamma \ge 2\alpha_2 - 1$ *is not present in* [12]. *It suffices to assume* $\mu(\rho_0) \in L^1(\Omega)$ *and use the equation satisfied by* $\mu(\rho)$ *to get the result* $\mu(\rho) \in L^\infty(0,T;L^1(\Omega))$ *for all* $T < +\infty$ *but this result is not valid for* $T = +\infty$.

Remark 4. *Note that the quantum case corresponds to the case where* $K(\rho) = 1/\rho$ *which gives* $\mu(\rho) = \rho$ *and, with the BD relation* (4) *this implies* $\lambda(\rho) = 0$.

Without loss of generality we fix $|\Omega| = 1$. In order to be able to get exponential decay, we add the following assumptions on the shear viscosity:

- *Behaviour of $\mu(\rho)$ for ρ small enough:*

$$\lim_{\rho\to 0} \frac{\rho\mu'(\rho)}{\mu(\rho)} = \alpha, \qquad \text{with } \frac{2}{3} < \alpha \leq 4, \tag{17}$$

assuming
  - *If $\alpha < 1$ or if there exists $\rho_0 > 0$ uch that $\mu(\rho) \geq C\rho$ for $\rho < \rho_0$ that the pressure exponent satisfies $\gamma \geq 1$,*
  - *If $\alpha \geq 1$ that the pressure exponent satisfies $\gamma < \alpha$.*

- *Behaviour of $\mu(\rho)$ for ρ big enough:*

$$\lim_{\rho\to\infty} \frac{\rho\mu'(\rho)}{\mu(\rho)} = \beta, \qquad \text{with } 1 < \beta < 4 \tag{18}$$

or letting $M > 0$ a given constant

$$\mu(\rho) = \rho \text{ for any } \rho \geq M. \tag{19}$$

Remark 5. *The quantum case satisfies* (17) *and* (18) *without any restriction on the pressure law namely $p(\rho) = a\rho^\gamma$ with $\gamma \geq 1$ is elligible.*

Main objectives of the paper. Denoting by r the mean value of ρ on Ω *i.e.* $r = \int_\Omega \rho\, dx$, the goal is to show that a global weak solution (ρ, u) in the sense of Definition 2 with $p(\rho) = a\rho^\gamma$ with $\gamma \geq 1$ and the shear and bulk viscosities satisfying (1)–(4) and (17)–(18) is exponentially decaying in time to $(r, 0)$. To obtain such a result we use the augmented formulation of the system introduced by [10] and we use a relative entropy entropy version of the κ entropy. More precisely, the goal is to prove the following result:

Theorem 6. *Let (ρ, u) be a global weak solution of the compressible Navier-Stokes-Korteweg System in the sense of Definition* 2 *with $p(\rho) = a\rho^\gamma$ with $\gamma \geq 1$ and the shear and bulk viscosities satisfying* (1)–(4) *and* (17)–(18) *then defining*

$$\mathcal{E}(\rho, u, v|r, 0, 0) = \mathcal{E}(t) = \int_\Omega [\rho(|w|^2 + (\kappa(1-\kappa)+\varepsilon)|v|^2) + H(\rho|r) + 2\kappa r_3\, h(\rho|r)](t)\, dx$$

where v, w are given by (7) *and $H(\rho|r)$ and $h(\rho|r)$ defined by*

$$H(\rho|r) = H(\rho) - H(r) - H'(r)(\rho - r), \qquad h(\rho|r) = h(\rho) - h(r) - h'(r)(\rho - r)$$

with h, H given by (8) *satisfies*

$$\mathcal{E}(t) \leq \mathcal{E}(0)\exp(-Ct)$$

for some constant $C > 0$.

The paper is organized as follows: In section 2 we prove Theorem 6. Finally we give in Appendix the definition of all the operators and the proof of a technical lemma (lemma 11) given in section 2. Note that in all the paper the notation C is used for constants independent of time t but which can be different from line to line.

2. The exponential decay of the solutions

The goal of this section is to prove Theorem 6. To do so, our goal is to prove that there exists a constant $C > 0$ such that for every $t > 0$

$$\frac{d\mathcal{E}}{dt}(t) \leq -C\mathcal{E}(t),$$

in order to be able to apply the Gronwall's Lemma. As global weak solutions are constructed using regular enough solutions, the formal calculations we will do may be justified using these approximate solutions, prove the result on $\mathcal{E}_\zeta$, write the exponential decay and then pass to the limit with respect to ζ.

Let us start with the inequality satisfied by the regular approximation where we skip the indice ζ in all the sequel for the sake of simplicity

$$\frac{d\mathcal{E}}{dt}(t) \quad \leq \quad -\varepsilon C_{\widetilde{\nu},\alpha_1} \int_\Omega |\nabla^2 Z(\rho)|^2 - \kappa \int_\Omega \mu'(\rho) H''(\rho)|\nabla\rho|^2 - \int_\Omega r_3 \rho |u|^2 \tag{20}$$

It remains then to prove that there exists a constant $C > 0$ such that, for all $t > 0$ the right hand side is smaller than $-C\mathcal{E}(t)$.

Let us first prove the following Lemma

Lemma 7. *The exists a constant $C > 0$ such that for every $t > 0$*

(21)

$$-\varepsilon C_{\widetilde{\nu},\alpha_1} \int_\Omega |\nabla^2 Z(\rho)|^2 - \kappa \int_\Omega \mu'(\rho)H''(\rho)|\nabla\rho|^2 - \int_\Omega r_3\rho|u|^2 \leq -C \int_\Omega \frac{\rho}{2}\left(|w|^2 + (\kappa(1-\kappa)+\varepsilon)|v|^2\right)$$

Proof. From Poincaré's inequality, we find

$$\int_\Omega |\nabla^2 Z|^2 \geq C \int_\Omega |\nabla Z|^2 = C\int_\Omega \frac{\mu(\rho)}{\rho}\rho|v|^2.$$

Moreover,

$$\mu'(\rho)H''(\rho)|\nabla\rho|^2 = \frac{\rho}{\mu'(\rho)} H''(\rho)\rho|v|^2,$$

hence there exists $C > 0$ such that

$$-\varepsilon C_{\widetilde{\nu},\alpha_1} \int_\Omega |\nabla^2 Z(\rho)|^2 - \kappa \int_\Omega \mu'(\rho)H''(\rho)|\nabla\rho|^2 \leq -C \int_\Omega \left(\frac{\mu(\rho)}{\rho} + \frac{\rho H''(\rho)}{\mu'(\rho)}\right)\rho|v|^2.$$

Let us prove that there exists a constant $C > 0$ such that for any $\rho \geq 0$

$$\frac{\mu(\rho)}{\rho} + \frac{\rho H''(\rho)}{\mu'(\rho)} \geq C.$$

and so

$$-\varepsilon C_{\widetilde{\nu},\alpha_1} \int_\Omega |\nabla^2 Z(\rho)|^2 - \kappa \int_\Omega \mu'(\rho)H''(\rho)|\nabla\rho|^2 \leq -C \int_\Omega \rho|v|^2. \tag{22}$$

To do so let us consider large and small density values. The inequality being true for intermediate values.

Large density value. From (18) or (19), for $M > 0$ big enough and $\rho > M$, we have $\rho\mu'(\rho) \geq \mu(\rho)$. Integrating the ODE gives $\mu(\rho) \geq C\rho$ for $\rho > M$. Since μ is positive on $\mathbb{R}_*^+$, for every $\rho_0 > 0$, there exists $C > 0$ such that for every $\rho \geq \rho_0$,

$$\frac{\mu(\rho)}{\rho} \geq C.$$

Small density value. We have to consider two cases to control the small values of ρ. If $\alpha \le 1$ in (17), then, for $\rho_0 > 0$ small enough and $\rho < \rho_0$, we have $\rho\mu'(\rho) \ge \mu(\rho)$. Integrating the ODE gives $\mu(\rho) \ge C\rho$ for all values of $\rho \ge 0$. Now, we consider the case $\alpha > 1$ in (17). For any $\eta > 0$ there exists $\rho_\eta > 0$ such that for any $\rho < \rho_\eta$ we have

$$\rho\mu'(\rho) \le (\alpha - \eta)\mu(\rho).$$

Integrating backward in ρ the ODE gives that for $\rho < \rho_\eta$:

$$\mu(\rho) \ge C\rho^{\alpha-\eta}.$$

Using (17) again gives that for $\rho < \rho_\eta$:

$$\frac{\rho H''(\rho)}{\mu'(\rho)} \ge C\rho^{\gamma-\alpha+\eta}.$$

If $\gamma < \alpha$, taking $\eta > 0$ small enough

$$\frac{\rho H''(\rho)}{\mu'(\rho)} \ge 1, \qquad \text{for } \rho > \rho_\eta.$$

This gives Estimate (22).

Note now that we have

$$|w|^2 = |u + \kappa v|^2 \le 2(|u|^2 + \kappa^2|v|^2).$$

So

$$-|u|^2 \le \kappa^2|v|^2 - \frac{|w|^2}{2}.$$

Hence

$$\begin{aligned} -\varepsilon C_{\widetilde{\nu},\alpha_1} \int_\Omega |\nabla^2 Z(\rho)|^2 - \kappa \int_\Omega \mu'(\rho)H''(\rho)|\nabla\rho|^2 - \int_\Omega \rho|u|^2 \\ \le -C\Big(\int_\Omega \rho|v|^2 + \int_\Omega \rho|u|^2\Big) \\ \le -C\int_\Omega \frac{\rho}{2}\left(|w|^2 + (\kappa(1-\kappa)+\varepsilon)|v|^2\right) \end{aligned}$$

This last inequality together with (22) gives the result of Lemma 7. □

The goal is now to prove that there exists a constant $C > 0$ such that

$$\int_\Omega 2\kappa r_3\, h(\rho|r) + \int_\Omega H(\rho|r) \le C\int_\Omega |\nabla^2 Z(\rho)|^2 + C\int_\Omega H''(\rho)\mu'(\rho)|\nabla\rho|^2. \tag{23}$$

This will be done with the proof of two lemmas. Firstly, we prove the following one.

Lemma 8. *For any constant $\eta > 0$ small enough, there exist $0 < \rho_0 < r < M$ and $C > 0$ such that*

$$\begin{aligned} \int_\Omega |\nabla^2 Z|^2 \quad \ge \quad & C\int_\Omega |\rho - r|^2 \mathbf{1}_{\{\rho_0 \le \rho \le M\}} \\ & + C\int_\Omega \mathbf{1}_{\{\rho < \rho_0\}} + C\int_\Omega \rho^{3(\beta-\eta)-2}\mathbf{1}_{\{\rho \ge M\}}. \end{aligned} \tag{24}$$

Moreover, if $\beta \ge 1$ then

$$\int_\Omega H''(\rho)\mu'(\rho)|\nabla\rho|^2 \ge C\int_\Omega H(\rho)\mathbf{1}_{\{\rho \ge M\}}. \tag{25}$$

Proof. The proof is split into three parts: I) The first one deals with the large values of ρ, II) the second one will be dedicated to the small values of ρ, and finally in part III) the intermediate values of ρ are considered.

First part. In this part we consider the large values of ρ. We first focus on the proof of (24) and then, using a generalization of the classical log-Sobolev inequality we prove (25).

Proof of (24): Since the function $\rho \to Z(\rho)$ is an increasing function, we have $|\{Z(\rho) \geq Z(2r)\}| = |\{\rho \geq 2r\}|$. By the Tchebychev's inequality, we get

$$|\{Z(\rho) \geq Z(2r)\}| = |\{\rho \geq 2r\}| \leq \frac{1}{2r}\int_\Omega \rho = \frac{1}{2}. \tag{26}$$

Since $|\Omega| = 1$, using (26) we have

$$|\{Z(\rho) < Z(2r)\}| \geq \frac{1}{2}.$$

Using the classical technical Lemma (Lemma 11) given in appendix 3.1 on $(Z(\rho) - Z(2r))_+$ we get

$$\int_\Omega (Z(\rho) - Z(2r))_+^2 \leq C \int_\Omega |\nabla Z|^2.$$

Thanks to assumptions (18) or (19), the function $\rho \to Z(\rho)$ converges to infinity when ρ goes to infinity. So there exists $M > 0$ such that for any $\rho \geq M$

$$Z(\rho)^2 \leq 2(Z(\rho) - Z(2r))_+^2.$$

Let us assume (18) or (19). Then for any $\eta > 0$, there exists a possibly bigger $M > 0$ such that we have for any $\rho > M$:

$$Z(\rho)^2 \geq C\rho^{3(\beta-\eta)-2}.$$

Therefore:

$$\int_\Omega \rho^{3(\beta-\eta)-2}\mathbf{1}_{\{\rho\geq M\}} \leq C\int_\Omega |Z(\rho)|^2\mathbf{1}_{\{\rho\geq M\}} \leq C\int_\Omega |\nabla Z|^2.$$

Proof of (25): If $\gamma > 1$, working with the function $J(\rho)$ instead of $Z(\rho)$ where $J(\rho)$ is defined such that

$$J'(\rho) = \sqrt{H''(\rho)\mu'(\rho)},$$

we obtain (25). If $\gamma = 1$, then $H(\rho) = \rho \ln \rho$, and we need a generalization of the classical log-Sobolev inequality. Let us denote $f = \sqrt{\rho}$. As in (26), we have

$$|\{f \leq \sqrt{2r}\}| \geq |\{\rho \leq 2r\}| \geq \frac{1}{2}. \tag{27}$$

So, from Sobolev injections and Lemma 11, we obtain

$$\begin{aligned}
\|1 + (f - \sqrt{2r})_+^2\|_{L^3} &\leq 1 + \|(f - \sqrt{2r})_+^2\|_{L^3} \leq 1 + \|(f - \sqrt{2r})_+\|_{L^6}^2 \\
&\leq 1 + C\left(\|\nabla(f - \sqrt{2r})_+\|_{L^2}^2 + \|(f - \sqrt{2r})_+\|_{L^2}^2\right) \\
&\leq 1 + C\|\nabla(f - \sqrt{2r})_+\|_{L^2}^2.
\end{aligned}$$

Denoting by

$$m = \int_\Omega 1 + (f - \sqrt{2r})_+^2 \, dx, \qquad d\mu = \frac{1 + (f - \sqrt{2r})_+^2}{m}\, dx,$$

(note that μ is a positive measure of mass 1), the ln function being increasing and concave, using Jensen's inequality, we have

$$\begin{aligned} C\|\nabla(f-\sqrt{2r})_+\|^2_{L^2} &\geq \ln(1+C\|\nabla(f-\sqrt{2r})_+\|^2_{L^2}) \\ &\geq \frac{1}{3}\ln\left[\int_\Omega \left(1+(f-\sqrt{2r})^2_+\right)^3 dx\right] = \frac{1}{3}\ln\left[\int_\Omega m\left(1+(f-\sqrt{2r})^2_+\right)^2 d\mu\right] \\ &\geq \int_\Omega \frac{1}{3}\ln\left(m\left(1+(f-\sqrt{2r})^2_+\right)^2\right) d\mu = \frac{2}{3}\int_\Omega \ln\left(\sqrt{m}(1+(f-\sqrt{2r})^2_+)\right)\frac{1+(f-\sqrt{2r})^2_+}{m}\,dx \\ &\geq \frac{2}{3(1+r)}\int_\Omega \ln\left((1+(f-\sqrt{2r})^2_+)\right)(1+(f-\sqrt{2r})^2_+)\,dx, \end{aligned} \tag{28}$$

where we have used that $1 \leq m \leq 1+r$. Fix $M = \sup(8r, 16))$. Then for $f = \sqrt{\rho} \geq \sqrt{M}$, we have both

$$1+(f-\sqrt{2r})^2_+ \geq \frac{\rho}{4} \qquad \text{and} \qquad \ln(1+(f-\sqrt{2r})^2_+) \geq \frac{1}{2}\ln\rho.$$

Therefore, (28) gives the following control

$$C\int_\Omega H(\rho)\mathbf{1}_{\{\rho>M\}} = C\int_\Omega \rho\ln\rho\,\mathbf{1}_{\{\rho>M\}} \leq C\|\nabla(f-\sqrt{2r})_+\|^2_{L^2}.$$

From properties (18) and (19)

$$\sup_{\rho\geq 2r}\frac{1/\rho}{H''(\rho)\mu'(\rho)} \leq C,$$

and so

$$C\int_\Omega H(\rho)\mathbf{1}_{\{\rho>M\}} \leq C\|\nabla(f-\sqrt{2r})_+\|^2_{L^2} \leq C\|\nabla\sqrt{\rho}\mathbf{1}_{\{\rho\geq 2r\}}\|^2_{L^2} \leq C\int_\Omega H''(\rho)\mu'(\rho)|\nabla\rho|^2$$

which gives (25).

Second part (small density values). In this part we consider the small values of ρ. From (17), there exists $\sigma > 0$ such that

$$Z'(\rho) \geq C\rho^{\sigma-1},$$

and so

$$\int_\Omega |\nabla\rho^\sigma|^2 \leq C\int_\Omega |\nabla Z|^2. \tag{29}$$

Let us fix $\rho_0 = r/4$ and $\delta = \inf(r/16, 1/2)$. Note that, if $C\int_\Omega |\nabla^2 Z|^2 \geq \delta$,

$$\int_\Omega \mathbf{1}_{\{\rho\leq\rho_0\}} \leq 1 \leq \frac{C}{\delta}\int_\Omega |\nabla^2 Z|^2,$$

which immediately gives (24) for $\rho < \rho_0$. Then, without loss of generality, we can assume that

$$C\int_\Omega |\nabla^2 Z|^2 \leq \delta,$$

such that the result of part 1 implies

$$\int_\Omega \mathbf{1}_{\{\rho\geq M\}}(\rho-r)_+ \leq M^{-3(\beta-\eta)+3}\int_\Omega \rho^{3(\beta-\eta)-2}\mathbf{1}_{\{\rho\geq M\}} \leq C\int_\Omega |\nabla^2 Z|^2 \leq \delta.$$

We claim that then

$$|\{\rho > r/2\}| \geq \frac{\delta}{M+1}. \tag{30}$$

To prove this claim let us show that if it is not the case we obtain a contradiction. Assuming

$$|\{\rho > r/2\}| < \frac{\delta}{M+1},$$

we have

$$M|\{\rho \geq r\}| \leq M|\{\rho > r/2\}| \leq \delta,$$

and so

$$\int_\Omega (\rho - r)_+ \leq \int_\Omega \mathbf{1}_{\{\rho \geq M\}}(\rho - r)_+ + M|\{\rho \geq r\}| \leq 2\delta.$$

Since $r = \displaystyle\int_\Omega \rho$, we have $\int_\Omega (\rho - r)_+ = \int_\Omega (r-\rho)_+$. Using Tchebychev inequality, we get

$$|\{\rho \leq r/2\}| = |\{(r-\rho)_+ \geq r/2\}| \leq \frac{2}{r}\int_\Omega (r-\rho)_+ \leq \frac{4\delta}{r} \leq \frac{1}{4}.$$

Therefore

$$|\{\rho > r/2\}| \geq 1 - 1/4 = 3/4$$

which is a contradiction, since $\delta/(M+1) \leq \delta \leq 1/2$. Therefore the claim (30) is proved. We rewrite the claim (since $\sigma > 0$)

$$|\{(r/2)^\sigma - \rho^\sigma < 0\}| \geq \frac{\delta}{M+1}.$$

Using Lemma 11 with the function $((r/2)^\sigma - \rho^\sigma)_+$ and (29), we get

$$\int_\Omega ((r/2)^\sigma - \rho^\sigma)_+^2 \leq C\int_\Omega |\nabla((r/2)^\sigma - \rho^\sigma)_+|^2 \leq C\int_\Omega |\nabla \rho^\sigma|^2 \leq C\int_\Omega |\nabla^2 Z|^2.$$

Using the Tchebychev's inequality and the previous one we get

$$|\{\rho \leq r/4\}| \leq \frac{C}{(2^\sigma - 1)^2}\left(\frac{4}{r}\right)^{2\sigma} \int_\Omega |\nabla^2 Z|^2.$$

This gives (24) for $\rho < \rho_0$.

Third part (Intermediate density values). In this part we deal withs the intermediate values of ρ. We introduce

$$\rho_T(t,x) = \sup(\rho, \inf(M, \rho(t,x))).$$

Note that $|Z'(\rho)|$ is bounded by below on $[\rho_0, M]$, hence

$$|\nabla \rho_T| = |\nabla \rho| 1_{\{\rho_0 \leq \rho \leq M\}} = \left|\frac{\nabla Z(\rho)}{Z'(\rho)}\right| 1_{\{\rho_0 \leq \rho \leq M\}} \leq C|\nabla Z|.$$

Therefore, using Poincaré's inequality,

$$\begin{aligned}\int_\Omega |\rho - r|^2 \mathbf{1}_{\{\rho_0\le\rho\le M\}} &\le \int_\Omega |\rho_T - r|^2 \\ &\le 2\int_\Omega \left|\rho_T - \int_\Omega \rho_T\right|^2 + 2\int_\Omega \left|\int_\Omega \rho_T - r\right|^2 \\ &\le C\int_\Omega |\nabla Z|^2 + 2\left(\int_\Omega \rho_T - \int_\Omega \rho\right)^2 \\ &\le C\int_\Omega |\nabla Z|^2 + 2\left(\int_{\{\rho>M\}} \rho + \int_{\{\rho<\rho_0\}} \rho\right)^2 \\ &\le C\int_\Omega |\nabla Z|^2 + 2r\left(\int_{\{\rho>M\}} \rho + \int_{\{\rho<\rho_0\}} \rho\right) \\ &\le C\int_\Omega |\nabla Z|^2 + 2r\left(M^{-3(\beta-\eta)+3}\int_\Omega \mathbf{1}_{\{\rho>M\}}\rho^{3(\beta-\eta)-2} + \rho_0\int_\Omega \mathbf{1}_{\{\rho<\rho_0\}}\right).\end{aligned}$$

We get the desired result using the part and second part of the proof. □

The second lemma that we want to show in order to obtain (23) is the following.

Lemma 9. *Assume that G is a function of ρ verifying G is C^2 on $[\rho_0, M]$, and that there exists a constant $C > 0$ such that*

$$G(\rho) \le C\sup(\rho^{3(\beta-\eta)-2}, H(\rho)), \qquad \text{for} \quad \rho \ge M, \tag{31}$$

$$G(\rho) \le C, \qquad \text{for} \quad \rho \le \rho_0. \tag{32}$$

Then, there exists an other constant $C > 0$ such that

$$\int_\Omega G(\rho|r) \le C\int_\Omega |\nabla^2 Z|^2 + C\int_\Omega H''(\rho)\mu'(\rho)|\nabla\rho|^2,$$

where $G(\rho|r)$ is given by

$$G(\rho|r) = G(\rho) - G(r) - G'(r)(\rho - r).$$

Proof. We have

$$\begin{aligned}\int_\Omega G(\rho|r) &= \int_\Omega G(\rho|r)\mathbf{1}_{\{\rho_0\le\rho\le M\}} + \int_\Omega G(\rho|r)\mathbf{1}_{\{\rho<\rho_0\}} + \int_\Omega G(\rho|r)\mathbf{1}_{\{\rho>M\}} \\ &\le C\left(\sup_{[\rho_0,M]} |G''|\right)\int_\Omega |\rho - r|^2 \mathbf{1}_{\{\rho_0\le\rho\le M\}} \\ &\qquad + C\int_\Omega \mathbf{1}_{\{\rho<\rho_0\}} + C\int_\Omega \sup(\rho^{3(\beta-\eta)-2}, H(\rho))\mathbf{1}_{\{\rho>M\}}.\end{aligned}$$

Lemma 8 finishes the proof. □

We can now state and prove the proposition which is the goal of this section since it allows us to obtain the exponential time decay given in Theorem 6 letting ζ tend to zero if we obtain the exponential decay for $\mathcal{E}_\zeta$ with constant independent on ζ. Here we adopt the indice ζ for reader's convenience.

Proposition 10. *Let the hypothesis on the viscosities and on the pressure law be satisfied namely $p(s) = as^\gamma$ with $\gamma \ge 1$ and the shear and bulk viscosities satisfying*

(1)–(4) *and* (17)–(18) *Then there exists a constant* $C > 0$ *such that for every* $t > 0$, *the quantity* $\mathcal{E}(t)$ *defined in* (16) *verifies*

$$\frac{d\mathcal{E}_\zeta(t)}{dt} \leq -C\mathcal{E}_\zeta(t).$$

Proof. With Lemma 7 in (20), we get that

$$\begin{aligned}&\frac{d\mathcal{E}_\zeta(t)}{dt}\\ &\leq -C\int_\Omega \Big(\frac{\rho_\zeta}{2}((\kappa(1-\kappa)+\varepsilon)|v_\zeta|^2 + \frac{\rho_\zeta}{2}|w_\eta|^2\Big)\\ &\qquad\qquad -C\int_\Omega |\nabla^2 Z(\rho_\zeta)|^2 - C\int_\Omega \mu'(\rho_\zeta)H''(\rho_\zeta)|\nabla\rho_\zeta|^2.\end{aligned}$$

The idea is now to apply Lemma 9 with G equals h and H. Note that (32) is valid for both quantities in all configurations. We want to show that (31) is also true for both quantities in all configurations. To this end, we consider three cases.

Case 1. Let us first assume that we have (17)–(18), with $\gamma \geq 1$. Since $\beta > 1$, for η small enough,

$$1 < \beta + \eta < 3(\beta - \eta) - 2. \tag{33}$$

Solving the ODE (18) for $\rho_\zeta > r$, we find that there exists a constant $C > 0$ such that for any $\rho_\zeta > r$:

$$\mu(\rho_\zeta) \leq C\rho_\zeta^{\beta+\eta},$$

and using again (18) to get a bound by above on μ' from the bound on μ, we get

$$\mu'(\rho_\zeta) \leq C\rho_\zeta^{\beta+\eta-1}.$$

Since $h''(\rho_\zeta) = \mu'(\rho_\zeta)/\rho_\zeta$, using the bound above on μ' and integrating in ρ_ζ twice, we find that there exists a constant $C > 0$ such that for every $\rho_\zeta > M$, we have

$$h(\rho_\zeta) \leq C\rho_\zeta^{\beta+\eta},$$

which, thanks to (33), leads to

$$h(\rho_\zeta) \leq C\sup(\rho_\zeta^{3(\beta-\eta)-2}, \rho_\zeta^\gamma).$$

This proves (31) for h. If $\gamma > 1$, then obviously,

$$H(\rho_\zeta) = \frac{\rho_\zeta^\gamma}{\gamma - 1} \leq C\rho_\zeta^\gamma.$$

But if $\gamma = 1$, then, since $3(\beta - \eta) - 2 > 1$, for ρ big enough

$$H(\rho_\zeta) = \rho_\zeta \ln\rho \leq \rho_\zeta^{3(\beta-\eta)-2},$$

and H verifies (31).

Case 2: If we have (17)–(18), with $\gamma > 1$, then, for ρ_ζ big enough, we have both

$$H(\rho_\zeta) = \frac{\rho_\zeta^\gamma}{\gamma - 1} \leq C\rho_\zeta^\gamma,$$

and

$$h(\rho_\zeta) = \rho_\zeta \ln\rho_\zeta \leq C\rho_\zeta^\gamma.$$

Case 3: It remains to treat the case (17)–(18), with $\gamma = 1$. Then we have $h(\rho_\zeta) = H(\rho_\zeta) = \rho_\zeta \ln \rho_\zeta$, which ends the proof.

Then in all cases, we can apply Lemma 9 to find

$$\int_\Omega 2\kappa r_3 \, h(\rho_\zeta | r) + \int_\Omega H(\rho_\zeta | r) \leq C \left(\int_\Omega |\nabla^2 Z(\rho_\zeta)|^2 + \int_\Omega H''(\rho_\zeta)\mu'(\rho_\zeta)|\nabla \rho_\zeta|^2 \right),$$

and so, together with (33), this ends the proof. □

3. Appendix

3.1. A classical technical Lemma.

Lemma 11. *For any function $f \geq 0$ such that $|\{f = 0\}| \geq \delta > 0$, there exists a constant C depending on δ and Ω, such that*

$$\int_\Omega f^2 \, dx \leq C \int_\Omega |\nabla f|^2 \, dx.$$

Proof. From the fundamental theorem of calculus, for any $x, y \in \Omega$

$$f(x) - f(y) = \int_0^1 \nabla f(tx + (1-t)y) \cdot (x - y) \, dt.$$

Therefore

$$\begin{aligned}
\int_\Omega |f(x)|^2 \, dx &\leq \frac{1}{\delta} \int_{\{f=0\}} \left(\int_\Omega |f(x)|^2 \, dx \right) dy = \frac{1}{\delta} \int_{\{f=0\}} \int_\Omega |f(x) - f(y)|^2 \, dx \, dy \\
&\leq \frac{1}{\delta} \int_\Omega \int_\Omega |f(x) - f(y)|^2 \, dx \, dy \leq \frac{1}{\delta} \int_\Omega \int_\Omega \int_0^1 |\nabla f(tx + (1-t)y)|^2 |x - y|^2 \, dx \, dy \, dt \\
&\leq C \int_\Omega \int_\Omega \left(\int_0^{1/2} |\nabla f(tx + (1-t)y)|^2 \, dt + \int_{1/2}^1 |\nabla f(tx + (1-t)y)|^2 \, dt \right) dx \, dy \\
&\leq 2C \int_\Omega \int_\Omega \int_0^{1/2} |\nabla f(tx + (1-t)y)|^2 \, dt \, dx \, dy \\
&\leq 2C \int_\Omega \int_0^{1/2} \left(\int_\Omega |\nabla f(tx + (1-t)y)|^2 \, dy \right) dt \, dx \\
&\leq 2C \int_\Omega \int_0^{1/2} \left(\int_{(1-t)\Omega} |\nabla f(tx + z)|^2 \, \frac{dz}{(1-t)^3} \right) dt \, dx \\
&\leq 2C \int_\Omega \int_0^{1/2} \left(\int_\Omega |\nabla f(tx + z)|^2 \, \frac{dz}{(1-t)^3} \right) dt \, dx \\
&\leq 16C \int_\Omega \int_0^{1/2} \left(\int_\Omega |\nabla f(z)|^2 \, dz \right) dt \, dx \\
&\leq 8C \int_\Omega |\nabla f(z)|^2 \, dz.
\end{aligned}$$

□

3.2. **Definition of the operators.** For the convenience of the reader we recall in this Section all the definitions of the operators used in this article.

Let f be a scalar, u, v two vectors and $\sigma = (\sigma_{ij})_{1\le i,j\le d}$ a tensor field defined on $\Omega \subset \mathbb{R}^d$ smooth enough.

- Denoting by $v_1, \cdots, v_d$ the coordinates of v, we call *divergence* of v the scalar given by:
$$\mathrm{div}(v) = \sum_{i=1}^{d} \frac{\partial v_i}{\partial x_i}.$$
- We call *laplacian* of f the scalar given by:
$$\Delta f = \mathrm{div}(\nabla f) = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}.$$
- We call *gradient* of v the tensor given by:
$$\nabla v = \left(\frac{\partial v_i}{\partial x_j}\right)_{1\le i,j\le d}.$$
- We call *divergence* of σ the vector given by:
$$\mathrm{div}(\sigma) = \left(\sum_{j=1}^{d} \frac{\partial \sigma_{ij}}{\partial x_j}\right)_{1\le i\le d}.$$
- We call *laplacian* of v the vector given by:
$$\Delta v = \mathrm{div}(\nabla v).$$
- We call *tensor product* of u and v the tensor given by:
$$u \otimes v = (u_i v_j)_{1\le i,j\le d}.$$

Proposition 12. *Let u, v, w three smooth enough vectors on Ω and r a scalar smooth enough on Ω. We have the following properties.*

- $(u \otimes v)w = (v \cdot w)u$,
- $\mathrm{div}(u \otimes v) = (\mathrm{div}\, v)u + (v \cdot \nabla)u$,
- $\mathrm{div}(r\, u) = \nabla r \cdot u + r \,\mathrm{div}\, u$,
- $\mathrm{div}(r\, u \otimes v) = (\nabla r \cdot v)u + r(v \cdot \nabla)u + r\, \mathrm{div}(v)u$.

Definition 13. *Let τ and σ be two tensors of order 2. We call scalar product of the two tensors the real defined by:*
$$\sigma : \tau = \sum_{1\le i,j\le d} \sigma_{ij}\tau_{ij}.$$
The norm associated to this scalar product is simply denoted by $|\cdot|$ in such a way that
$$|\sigma|^2 = \sigma : \sigma.$$

Remark 14. *By definition we have*
$$\sigma : \tau = {}^t\sigma : {}^t\tau$$

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