WELL-POSEDNESS OF THE RIEMANN PROBLEM WITH TWO SHOCKS FOR THE ISENTROPIC EULER SYSTEM IN A CLASS OF VANISHING PHYSICAL VISCOSITY LIMITS

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ABSTRACT. We consider the Riemann problem composed of two shocks for the 1D Euler system. We show that the Riemann solution with two shocks is stable and unique in the class of weak inviscid limits of solutions to the Navier-Stokes equations with initial data with bounded energy. This work extends to the case of two shocks a previous result of the authors in the case of a single shock. It is based on the method of weighted relative entropy with shifts known as a-contraction theory. A major difficulty due to the method is that very little control is available on the shifts. A modification of the construction of the shifts is needed to ensure that the two shock waves are well separated, at the level of the Navier-Stokes system, even when subjected to large perturbations. This work put the foundations needed to consider a large family of interacting waves. It is a key result in the program to solve the Bianchini-Bressan conjecture, that is the inviscid limit of solutions to the Navier-Stokes equation to the unique BV solution of the Euler equation, in the case of small BV initial values.

CONTENTS

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1. Introduction

Consider the one-dimensional barotropic Navier-Stokes system in the Lagrangian coordinates:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = \nu\Big(\dfrac{\mu(v)}{v}u_x\Big)_x, \qquad t>0,\ x\in\mathbb{R}, \end{cases} \tag{1.1}$$

where v denotes the specific volume, u is the fluid velocity, and $p(v)$ is the pressure law. We consider the case of a polytropic perfect gas where the pressure is given by

$$p(v) = v^{-\gamma}, \quad \gamma > 1, \tag{1.2}$$

with γ the adiabatic constant. Here, μ denotes the viscosity coefficient given by

$$\mu(v) = bv^{-\alpha}, \qquad b > 0. \tag{1.3}$$

We assume the following relating between α and γ:

$$0 < \alpha \le \gamma \le \alpha + 1. \tag{1.4}$$

This includes, for instance, the case of the viscous shallow water equation $\gamma = 2$, $\alpha = 1$ (see [16]).

We are interested in studying on the well-posedness of the inviscid limits $\nu \to 0$. At least formally, the limit system of (1.1) is given by the isentropic Euler system:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = 0. \end{cases} \tag{1.5}$$

For small BV initial values, global weak solutions for conservation laws as (1.5) have been constructed in 1965 by Glimm [17]. The uniqueness of these weak entropy solutions in the class of small BV functions has been proved later by Bressan, Crasta, and Piccoli in 2000 [6] (see also Liu and Yang [32], Bressan Lui and Yang [7], and Bressan [5]). In 2005, Bianchini and Bressan considered the inviscid limit of a fully parabolic model (with viscosity, and artificial diffusion in the v equation), and showed that the solutions converge to the unique BV solution in the case of small BV initial values [1]. In this work, they mention the problem of the inviscid limit of Navier-Stokes as an open problem. As today, the problem is still unsolved.

This paper is a key step in our program to solve this problem. One key difficulty for the problem is that obtaining a uniform BV estimate on the solutions of Navier-Stokes seems unattainable. Our general philosophy is to avoid completely this step, by working only with the natural energy estimates via our method of a-contraction with shifts. In [24], we used the method to show the stability, uniformly in ν, of a single shock wave. In [26], we apply the result to show that shock waves of (1.5) are unique in the class of weak limits of (1.1) whose initial values converges to the shock wave. The basic idea is now to combine several such waves. Because the limit equation (1.5) has a finite speed of propagation, it is expected to be possible to estimate such evolution, and pass into the limits in the number of waves. However, many difficulties stem from features of the method itself. First, at the level of the Navier-Stokes equation, the waves do interact at long distance, and are not independent anymore. More importantly, we can obtain only rough a priori control on the artificial shifts induced by the method. Since we do not have a priori control on the solutions, uniform in L^∞ with respect to ν, these shifts can be very oscillatory at the limit. For the limit problems (1.5), due to the separation of shock speeds, a 1-shock can never collide from the left with a 2-shock. This allows, in particular, to solve the Riemann problem. Due to the artificial shifts, and the lack of uniform bounds in ν on the solutions of (1.1), the separation of waves is far more complicated at the level of Navier-Stokes. A key point is to show that shifts associated to two different families of shocks cannot produce artificial collisions. We need to ensure that a 1-shock cannot be pushed through the artificial shift, so much that it would collide with a 2-shock from the left. This is crucial to recover the Riemann problem at the limit, and this is the problem solved in this paper.

Consider the Riemann problem for (1.5) with the Riemann initial data:

$$(\bar{v}, \bar{u})(x) = \begin{cases} (v_-, u_-) & \text{if } x < 0, \\ (v_+, u_+) & \text{if } x > 0. \end{cases} \tag{1.6}$$

Here, the two end states (v_-, u_-) and (v_+, u_+) are given constants such that the following holds: there exists an intermediate (constant) state (v_m, u_m) such that (v_-, u_-) connects with (v_m, u_m) by the 1-Hugoniot curve, and (v_m, u_m) connects with (v_+, u_+) by the 2-Hugoniot curve, that is, those satisfy the two Rankine-Hugoniot conditions and Lax entropy conditions, respectively:

$$\begin{aligned} &\begin{cases} -\sigma_1(v_m - v_-) - (u_m - u_-) = 0, \\ -\sigma_1(u_m - u_-) + p(v_m) - p(v_-) = 0, \end{cases} \\ &\text{where} \quad \sigma_1 = -\sqrt{-\frac{p(v_m) - p(v_-)}{v_m - v_-}}, \quad v_- > v_m, \; u_- > u_m; \\ &\begin{cases} -\sigma_2(v_+ - v_m) - (u_+ - u_m) = 0, \\ -\sigma_2(u_+ - u_m) + p(v_+) - p(v_m) = 0, \end{cases} \\ &\text{where} \quad \sigma_2 = \sqrt{-\frac{p(v_+) - p(v_m)}{v_+ - v_m}}, \quad v_m < v_+, \; u_m > u_+. \end{aligned} \tag{1.7}$$

The Riemann solution $(\bar v,\bar u)(t,x)$, corresponding to the above data, is the self-similar entropy solution for (1.5) consisting of the 1-shock wave and the 2-shock wave as follows:

$$
(\bar v,\bar u)(t,x)=\begin{cases}(v_-,u_-) & \text{if } x/t<\sigma_1,\\ (v_m,u_m) & \text{if } \sigma_1<x/t<\sigma_2,\\ (v_+,u_+) & \text{if } x/t>\sigma_2.\end{cases} \tag{1.8}
$$

In this paper, we show the stability and uniqueness of the Riemann solution (1.8) in the class of inviscid limits of Navier-Stokes. To achieve our goal, the main step is to get the uniform (in ν) stability for large perturbations of a composite viscous wave related to the Riemann solution. More precisely, we first recall the fact (see Matsumura-Wang [36]) that the system (1.1) admits the 1-viscous shock wave $(\tilde v_1^\nu,\tilde u_1^\nu)(x-\sigma_1 t)$ and the 2-viscous shock wave $(\tilde v_2^\nu,\tilde u_2^\nu)(x-\sigma_2 t)$ as traveling wave solutions:

$$
\begin{aligned}
&\begin{cases}-\sigma_1(\tilde v_1^\nu)_x-(\tilde u_1^\nu)_x=0,\\ -\sigma_1(\tilde u_1^\nu)_x+p(\tilde v_1^\nu)_x=\nu\Big(\frac{\mu(\tilde v_1^\nu)}{\tilde v_1^\nu}(\tilde u_1^\nu)_x\Big)_x\\ \lim_{x\to-\infty}(\tilde v_1^\nu,\tilde u_1^\nu)(x-\sigma_1 t)=(v_-,u_-),\quad \lim_{x\to+\infty}(\tilde v_1^\nu,\tilde u_1^\nu)(x-\sigma_1 t)=(v_m,u_m),\end{cases}\\
&\begin{cases}-\sigma_2(\tilde v_2^\nu)_x-(\tilde u_2^\nu)_x=0,\\ -\sigma_2(\tilde u_2^\nu)_x+p(\tilde v_2^\nu)_x=\nu\Big(\frac{\mu(\tilde v_2^\nu)}{\tilde v_2^\nu}(\tilde u_2^\nu)_x\Big)_x\\ \lim_{x\to-\infty}(\tilde v_2^\nu,\tilde u_2^\nu)(x-\sigma_2 t)=(v_m,u_m),\quad \lim_{x\to+\infty}(\tilde v_2^\nu,\tilde u_2^\nu)(x-\sigma_2 t)=(v_+,u_+).\end{cases}
\end{aligned} \tag{1.9}
$$

Then, as the viscous version of (1.8), we consider the composite wave consisting of the two viscous shock waves:

$$
(\tilde v^\nu,\tilde u^\nu)(t,x):=\Big(\tilde v_1^\nu(x-\sigma_1 t)+\tilde v_2^\nu(x-\sigma_2 t)-v_m,\tilde u_1^\nu(x-\sigma_1 t)+\tilde u_2^\nu(x-\sigma_2 t)-u_m\Big). \tag{1.10}
$$

For the global-in-time existence of solutions to (1.1), we introduce the function space:

$$
\begin{aligned}
\mathcal{X}_T:=\{(v,u)\ |\ & v-\underline{v},\ u-\underline{u}\in C(0,T;H^1(\mathbb{R})),\\
& u-\underline{u}\in L^2(0,T;H^2(\mathbb{R})),\ 0<v^{-1}\in L^\infty((0,T)\times\mathbb{R})\},
\end{aligned}
$$

where $\underline{v}$ and $\underline{u}$ are smooth monotone functions such that

$$
\underline{v}(x)=v_\pm \quad\text{and}\quad \underline{u}(x)=u_\pm \quad \text{for } \pm x\ge 1. \tag{1.11}
$$

Define the relative potential energy as

$$
Q(v|\underline{v})=Q(v)-Q(\underline{v})-Q'(\underline{v})(v-\underline{v}),
$$

where $Q'=p$. Thanks to [25], for any initial value (v_0,u_0) with finite relative energy

$$
\int_{\mathbb{R}}\left(\frac{|u_0-\underline{u}|^2}{2}+Q(v_0|\underline{v})\right)dx<\infty, \tag{1.12}
$$

and such that

$$
\begin{aligned}
& v_0-\underline{v},u_0-\underline{u}\in H^k(\mathbb{R}), \qquad \text{for some } k\ge 4,\\
& 0<\underline{\kappa}_0\nu\le v_0(x)\le \overline{\kappa}_0/\nu, \qquad \forall x\in\mathbb{R}, \qquad \text{for some constants } \underline{\kappa}_0,\overline{\kappa}_0,\\
& \partial_x u_0(x)\le \frac{v_0^{\alpha-\gamma}}{\nu}, \qquad \forall x\in\mathbb{R},
\end{aligned} \tag{1.13}
$$

there exists a unique global solution to (1.1). Moreover, for any time $T>0$, this solution lies in $\mathcal{X}_T$.

1.1. **Main results.** To estimate the stability and uniqueness of the Riemann solution, we use the relative entropy associated to the entropy of (1.5) as follows: For any functions v_1, u_1, v_2, u_2,

$$\eta((v_1,u_1)|(v_2,u_2)) := \frac{|u_1-u_2|^2}{2} + Q(v_1|v_2), \tag{1.14}$$

where $Q(v_1|v_2)$ is the relative functional associated with the strictly convex function

$$Q(v) := \frac{v^{-\gamma+1}}{\gamma-1}, \quad v>0.$$

However, in the inviscid limit, the first components v_1 are limit of Navier-Stokes equations, which can be a measure in t, x. So, we should extend the definition of the relative entropy to the case of measures defined on $\mathbb{R}^+\times\mathbb{R}$ as in [26]. We will restrict the definition in the case where we compare a measure dv with a simple function $\bar{\mathbf{v}}$ only taking three values v_-, v_+ and v_m satisfying $v_\pm \geq v_m$ (as the values of (1.7)). Let v_a denote the Radon-Nikodym derivative of dv with respect to the Lebesgue measure and dv_s its singular part, i.e., $dv = v_a\,dt\,dx + dv_s$. The relative potential energy is then itself a measure defined as

$$dQ(v|\bar{\mathbf{v}})(t,x) := Q\left(v_a|\bar{\mathbf{v}}\right)dtdx + |Q'(\overline{V}(t,x))|dv_s(t,x), \tag{1.15}$$

where $\overline{V}$ is defined everywhere on $\mathbb{R}^+\times\mathbb{R}$ by

$$\overline{V}(t,x) = \begin{cases} v_- & \text{for } (t,x)\in\overline{\{\bar{\mathbf{v}}=v_-\}} \quad (=\text{the closure of } \{\bar{\mathbf{v}}=v_-\}),\\ v_+ & \text{for } (t,x)\in\overline{\{\bar{\mathbf{v}}=v_+\}},\\ v_m & \text{for } (t,x)\in\left(\overline{\{\bar{\mathbf{v}}=v_-\}}\cup\overline{\{\bar{\mathbf{v}}=v_+\}}\right)^c, \end{cases}$$

Note that $|Q'(v_\pm)| \leq |Q'(v_m)|$. Also, note that if $v \in L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}) + \mathcal{M}(\mathbb{R}))$, then $dQ(v|\bar{v})$ is defined in $L^\infty(\mathbb{R}^+; L^\infty(\mathbb{R}) + \mathcal{M}(\mathbb{R}))$, where $\mathcal{M}$ denotes the space of bounded Radon measures.

The main result of this paper is the following.

Theorem 1.1. *For each $\nu>0$, consider the system* (1.1)-(1.3) *with the assumption* (1.4)*. For a given constant $U_* := (v_*, u_*) \in \mathbb{R}^+\times\mathbb{R}$, there exists a constant $\varepsilon_0>0$ such that the following holds.*
Let $U_- := (v_-,u_-), U_m := (v_m,u_m), U_+ := (v_+,u_+) \in \mathbb{R}^+\times\mathbb{R}$ be any constant states such that (1.7) *holds true, and $U_-, U_m, U_+ \in B_{\varepsilon_0}(U_*)$.*
Then for a given initial datum (v^0,u^0) of (1.5) *satisfying*

$$\mathcal{E}_0 := \int_{-\infty}^{\infty} \eta\big((v^0,u^0)|(\bar{v},\bar{u})\big)dx < \infty, \tag{1.16}$$

as a perturbation of the Riemann initial datum (1.6)*, the following is true.*
(i) (Well-prepared initial data) There exists a sequence of smooth functions $\{(v_0^\nu, u_0^\nu)\}_{\nu>0}$ such that

$$\lim_{\nu\to 0} v_0^\nu = v^0, \quad \lim_{\nu\to 0} u_0^\nu = u^0 \quad a.e., \quad v_0^\nu>0,$$
$$\lim_{\nu\to 0}\int_{\mathbb{R}}\left[\frac{1}{2}\left(u_0^\nu + \nu\left(p(v_0^\nu)^{\frac{\alpha}{\gamma}}\right)_x - \tilde{u}^\nu(0,x) - \nu\left(p(\tilde{v}^\nu(0,x))^{\frac{\alpha}{\gamma}}\right)_x\right)^2 + Q(v_0^\nu|\tilde{v}^\nu(0,x))\right]dx = \mathcal{E}_0, \tag{1.17}$$

where $(\tilde{v}^\nu, \tilde{u}^\nu)$ is the composite wave (1.10) *of the two viscous shocks* (1.9)*.*
(ii) For a given $T>0$, let $\{(v^\nu,u^\nu)\}_{\nu>0}$ be a sequence of solutions in $\mathcal{X}_T$ to (1.1) *with the*

initial datum (v_0^ν, u_0^ν) *as above. Then there exist limits* v_∞ *and* u_∞ *such that as* $\nu \to 0$ (*up to a subsequence*),

(1.18)
$$v^\nu \rightharpoonup v_\infty, \quad u^\nu \rightharpoonup u_\infty \quad \text{in } \mathcal{M}_{\mathrm{loc}}((0,T)\times\mathbb{R}) \text{ (space of locally bounded Radon measures)},$$

where v_∞ *lies in* $L^\infty(0,T,L^\infty(\mathbb{R})+\mathcal{M}(\mathbb{R}))$, *and* u_∞ *lies in* $L^\infty(0,T,L^\infty(\mathbb{R})+L^2(\mathbb{R}))$.
In addition, there exist shifts $X_1^\infty, X_2^\infty \in BV((0,T))$ *and constant* $C>0$ *such that for a.e.* $t\in(0,T]$,

$$\sigma_1 t + X_1^\infty(t) \le \frac{\sigma_1}{2}t < 0 < \frac{\sigma_2}{2}t \le \sigma_2 t + X_2^\infty(t), \tag{1.19}$$

and

$$\int_{\mathbb{R}} \frac{|u_\infty(t,x) - \bar u^{X_1^\infty,X_2^\infty}(t,x)|^2}{2}dx + \left(\int_{x\in\mathbb{R}} dQ(v_\infty|\bar v^{X_1^\infty,X_2^\infty})(t,x)\right)(t) \ \le C\mathcal{E}_0, \tag{1.20}$$

where $(\bar v^{X_1^\infty,X_2^\infty}, \bar u^{X_1^\infty,X_2^\infty})$ *denotes the shifted Riemann solution, that is,*

$$(\bar v^{X_1^\infty,X_2^\infty}, \bar u^{X_1^\infty,X_2^\infty})(t,x) = \begin{cases} (v_-,u_-) & \text{if } x<\sigma_1 t + X_1^\infty(t),\\ (v_m,u_m) & \text{if } \sigma_1 t + X_1^\infty(t) < x < \sigma_2 t + X_2^\infty(t),\\ (v_+,u_+) & \text{if } x > \sigma_2 t + X_2^\infty(t). \end{cases}$$

Moreover, for a.e. $t_0>0$, *there exists a positive constant* $C(t_0)$ *such that*

$$|X_1^\infty(t)| + |X_2^\infty(t)| \le Ct_0 + C(t_0)\Big(\mathcal{E}_0 + (1+t)\sqrt{\mathcal{E}_0}\Big), \quad \text{for a.e. } t\in(0,T). \tag{1.21}$$

Therefore, the Riemann solution (1.8) *is stable (up to shifts) and unique in the class of weak inviscid limits of solutions to the Navier-Stokes systems* (1.1)-(1.4).

Remark 1.1. *1. Since the shifts* X_1^∞, X_2^∞ *are of* BV *on* $(0,T)$, *we have*

$$\overline{\{\bar v^{X_1^\infty,X_2^\infty} = v_-\}} = \{x \le \sigma_1 t + X_1^\infty(t)\},$$
$$\overline{\{\bar v^{X_1^\infty,X_2^\infty} = v_+\}} = \{x \ge \sigma_2 t + X_2^\infty(t)\},$$
$$\left(\overline{\{\bar v^{X_1^\infty,X_2^\infty} = v_-\}} \cup \overline{\{\bar v^{X_1^\infty,X_2^\infty} = v_+\}}\right)^c = \{\sigma_1 t + X_1^\infty(t) < x < \sigma_2 t + X_2^\infty(t)\}.$$

Thus it follows from (1.15) *that the measure* $dQ(v_\infty|\bar v^{X_1^\infty,X_2^\infty})$ *in the stability estimate* (1.20) *is written as follows: for the decomposition* $dv_\infty = v_a\,dt\,dx + dv_s$,

$$dQ(v_\infty|\bar v^{X_1^\infty,X_2^\infty})(t,x) := Q\left(v_a|\bar v^{X_1^\infty,X_2^\infty}\right)dtdx + |Q'(\overline{V}(t,x))|dv_s(t,x),$$

where

$$\overline{V}(t,x) = \begin{cases} v_- & \text{for } \ x\le \sigma_1 t + X_1^\infty(t),\\ v_+ & \text{for } \ x \ge \sigma_2 t + X_2^\infty(t),\\ v_m & \text{for } \ \sigma_1 t + X_1^\infty(t) < x < \sigma_2 t + X_2^\infty(t). \end{cases} \tag{1.22}$$

2. Theorem 1.1 provides the stability and uniqueness of the Riemann solution (1.8) *in the wide class of weak inviscid limits of solutions to the Navier-Stokes system.*
Indeed, for the uniqueness, if $\mathcal{E}_0 = 0$, *then* (1.21) *implies that for a.e.* $t_0>0$,

$$|X_1^\infty(t)| + |X_2^\infty(t)| \le Ct_0, \qquad \text{for a.e. } t\in(0,T),$$

and so,

$$X_1^\infty(t) = 0, \quad X_2^\infty(t)=0, \qquad \text{for a.e. } t\in(0,T).$$

This together with (1.20) *implies that for a.e.* $t \in (0,T]$,

$$\int_{\mathbb{R}} \frac{|u_\infty(t,x) - \bar{u}(t,x)|^2}{2} dx + \int_{\mathbb{R}} Q(v_a(t,x)|\bar{v}(t,x))dx = 0,$$

where the singular part v_s *of* v_∞ *vanishes. Therefore, we have*

$$u_\infty(t,x) = \bar{u}(t,x), \qquad v_\infty(t,x) = \bar{v}(t,x), \qquad a.e.\ (t,x) \in (0,T] \times \mathbb{R}.$$

3. By (1.20)*, the limits* v_∞, u_∞ *satisfy* $v_\infty \in \bar{v} + L^\infty(0,T;L^\infty(\mathbb{R}) + \mathcal{M}(\mathbb{R}))$ *and* $u_\infty \in \bar{u} + L^\infty(0,T;L^2(\mathbb{R}))$*. The control in measure of* v_∞ *is due to the fact that* $Q(v|\overline{v}) \geq c_2|v - \overline{v}|$ *for* $v \geq 3v_-$ *(see (A.1) in Lemma A.1). Especially,* v_∞ *may have some measure concentrated at infinity. This corresponds physically to cavitation and appearance of vacuum.*
Note that we do not need to know whether the weak inviscid limits (u,v) *are solutions to the system* (1.5)*, nor any a priory regualrity. The stability of the Riemann's problem needs only that the perturbations are generated through inviscid limits of the Navier-Stokes equation* (1.1)*. This is very different in spirit from results obtained via compensated compactness, see for instance Chen and Perepelitsa* [10]*. The compensated compactness method shows that a certain limit verifies the equation. But it does not provide any information on the stability of these functions.*
4. In fact, the smallness of amplitude of shocks is not needed for the proof of Theorem 1.1. The constraint is due to Theorem 1.2.

The starting point for the proof of Theorem 1.1 is to derive the uniform (in ν) stability of any large perturbations of the composite wave for (1.1). It is equivalent to obtaining the contraction property of any large perturbations of the composite wave to (1.1) with a fixed $\nu = 1$:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = \left(\frac{\mu(v)}{v} u_x\right)_x. \end{cases} \tag{1.23}$$

As in [26], we consider the following relative functional $E(\cdot|\cdot)$ to measure the stability:

for any functions v_1, u_1, v_2, u_2,

$$E((v_1,u_1)|(v_2,u_2)) := \frac{1}{2}\left(u_1 + \left(p(v_1)^{\frac{\alpha}{\gamma}}\right)_x - u_2 - \left(p(v_2)^{\frac{\alpha}{\gamma}}\right)_x\right)^2 + Q(v_1|v_2), \tag{1.24}$$

where the constants γ, α are in (1.2) and (1.3). Since $Q(v_1|v_2)$ is positive definite, so is the functional $E(\cdot|\cdot)$, that is, for any functions (v_1,u_1) and (v_2,u_2) we have $E((v_1,u_1)|(v_2,u_2)) \geq 0$, and

$$E((v_1,u_1)|(v_2,u_2)) = 0 \text{ a.e.} \quad \Leftrightarrow \quad (v_1,u_1) = (v_2,u_2) \text{ a.e.}$$

The functional E is associated to the BD entropy (see (4.1)). The following main result provides the uniform stability of any large perturbations of the composite wave (1.10) with $\nu = 1$.

Theorem 1.2. *Assume* $\gamma > 1$ *and* $\alpha > 0$ *satisfying* $\alpha \leq \gamma \leq \alpha + 1$*. For a given constant* $U_* := (v_*, u_*) \in \mathbb{R}^+ \times \mathbb{R}$*, there exists constant* $\delta_0 \in (0, 1/2)$ *such that the following holds. Let* $U_- := (v_-, u_-), U_m := (v_m, u_m), U_+ := (v_+, u_+) \in \mathbb{R}^+ \times \mathbb{R}$ *be any constant states such that* (1.7) *and* $U_-, U_m, U_+ \in B_{\delta_0}(U_*)$*. Let* $\varepsilon_1 := |p(v_-) - p(v_m)|$ *and* $\varepsilon_2 := |p(v_m) - p(v_+)|$*. For any* $\lambda > 0$ *with* $\varepsilon_1/\lambda, \varepsilon_2/\lambda < \delta_0$ *and* $\lambda < \delta_0$*, there exist a constant* $C > 0$ *and smooth monotone functions* a_1, a_2 *with* $a_1(x), a_2(x) \in [1 - \lambda, 1]$ *for all* $x \in \mathbb{R}$ *such that the following holds.*

Let $\tilde{U}(t,x) := (\tilde{v},\tilde{u})(t,x)$ *be the composite wave* (1.10) *with* $\nu = 1$.
Let

$$a(t,x) := a_1(x-\sigma_1 t) + a_2(x-\sigma_2 t) - 1. \tag{1.25}$$

For a given $T > 0$, *let* $U := (v,h)$ *be a solution in* $\mathcal{X}_T$ *to* (1.23) *with a initial datum* $U_0 := \begin{pmatrix} v_0 \\ u_0 \end{pmatrix}$ *satisfying* $\int_{-\infty}^{\infty} E(U_0(x)|\tilde{U}(0,x))dx < \infty$. *Then, there exist shift functions* $X_1, X_2 \in W^{1,1}((0,T))$ *with* $X_1(0) = X_2(0) = 0$ *such that for the shifted composite wave*

$$\tilde{U}^{X_1,X_2}(t,x) := \begin{pmatrix} \tilde{v}^{X_1,X_2}(t,x) \\ \tilde{u}^{X_1,X_2}(t,x) \end{pmatrix} := \begin{pmatrix} \tilde{v}_1(x-\sigma_1 t - X_1(t)) + \tilde{v}_2(x-\sigma_2 t - X_2(t)) - v_m \\ \tilde{u}_1(x-\sigma_1 t - X_1(t)) + \tilde{u}_2(x-\sigma_2 t - X_2(t)) - u_m \end{pmatrix},$$

and the shifted weight

$$a^{X_1,X_2}(t,x) := a_1(x-\sigma_1 t - X_1(t)) + a_2(x-\sigma_2 t - X_2(t)) - 1,$$

we have the uniform stability:

$$\begin{aligned}
&\int_{\mathbb{R}} a^{X_1,X_2}(t,x) E\big(U(t,x)|\tilde{U}^{X_1,X_2}(t,x)\big)dx \\
&\qquad + \int_0^T \int_{-\infty}^{\infty} |\partial_x a^{X_1,X_2}(t,x)| Q\left(v(t,x)|\tilde{v}^{X_1,X_2}(t,x)\right) dxdt \\
&\qquad + \int_0^T \int_{-\infty}^{\infty} v^{\gamma-\alpha}(t,x) \Big|\partial_x\big(p(v(t,x))\big) - p(\tilde{v}^{X_1,X_2}(t,x))\big)\Big|^2 dxdt \\
&\quad \le C \int_{\mathbb{R}} a(0,x) E\big(U_0(x)|\tilde{U}(0,x)\big)dx + C,
\end{aligned} \tag{1.26}$$

and

$$X_1(t) \le -\frac{\sigma_1}{2} t, \quad X_2(t) \ge -\frac{\sigma_2}{2} t, \quad \forall t > 0, \tag{1.27}$$

in addition, for each $i = 1,2$,

$$\begin{aligned}
&|\dot{X}_i(t)| \le C\left[f(t) + \int_{\mathbb{R}} \eta(U_0(x)|\tilde{U}(0,x))dx + 1\right] \qquad \text{for a.e. } t \in [0,T], \\
&\text{for some positive function } f \text{ satisfying } \quad \|f\|_{L^1(0,T)} \le C \int_{-\infty}^{\infty} \eta\big(U_0(x)|\tilde{U}(0,x)\big)dx.
\end{aligned} \tag{1.28}$$

Remark 1.2. *1. The stability of the viscous shock waves for the Navier-Stokes system is a very important issue in both mathematical and physical viewpoints. Theorem 1.2 provides the first result on stability, independent of the size of the perturbation, for composite wave of two viscous shocks of the compressible Navier-Stokes system. To the best of our knowledge, all the previous results on stability of composite wave of viscous shocks (even a single shock) for the Navier-Stokes require smallness conditon of initial perturbations (see for instance* [19, 20, 33, 34, 35, 42]*).*
2. Notice that the uniform stability (1.26) *is not a contraction estimate, contrary to the case of a single shock in* [26]. *This is natural because the composite wave* (1.10) *is not a solution to* (1.23) *(due to the nonlinearity of* (1.23)*).*

The rest of the paper is as follows. We first explain the background of our problem in Section 2. Section 3 provides scenario of proofs of the mains results. In Section 4, we present a transformation of the system (1.23), and the statement of Theorem 4.1 as an equivalent version of Theorem 1.2. In Section 5, we show that proving Theorem 4.1 boils down to showing Proposition 5.1. The proof of the main Proposition 5.1 is presented in Section 7. In Section 6, as a special section, we provide useful propositions written in an abstract manner. Finally, Section 8 is dedicated to the proof of Theorem 1.1.

2. Background and previous results

The method of proof is based on the relative entropy. Consider U the state variable for conservation laws, and η an associated strictly convex entropy functional. In the case of the Euler system (1.5) (or Navier-Stokes (1.1)), it corresponds to $U = (v, u)$, $\eta(U) = u^2/2 + Q(v)$. The relative entropy of a state U compared to another state $\tilde{U}$ is defined as

$$\eta(U|\tilde{U}) = \eta(U) - \eta(\tilde{U}) - d\eta(\tilde{U}) \cdot (U - \tilde{U}).$$

Note that the formula is not symmetric in U and $\tilde{U}$. Because of convexity of η, this quantity provides a pseudo-distance between the state U and $\tilde{U}$ (if U and $\tilde{U}$ stay bounded in a compact set, then it is actually equivalent to $|U - \tilde{U}|^2$). This quantity was used by Dafermos [13] to show the weak/strong uniqueness principle, a stability result for smooth solutions among the large class of merely weak solutions. Especially, it shows that smooth solutions $\tilde{U}$ are unique among any weak solutions U. Our main idea is to extend this principle to the situation where the solution $\tilde{U}$ has some discontinuities, and U is an inviscid limit of (1.1) (possibly not solution to (1.5)).

This idea is similar to a program developed in parallel, and initiated in [40], to show the stability (and so uniqueness) of small BV solutions of (1.5) in the large class of bounded weak solutions of (1.5) verifying a strong trace property. This program is inspired by the early work of DiPerna [14] who showed the uniqueness of shocks among this family of weak solutions (see also Chen-Frid-Li [9] for the extension to the Riemann problem).

Let us first describe a bit the history of this second program which deals directly with the hyperbolic equation as (1.5) (without considering any approximation as (1.1)). When $\tilde{U}$ is a a fixed constant, the relative entropy is an affine perturbation of the entropy η, and so is an entropy on its own right. Therefore, the relative entropy $\eta(\cdot|\tilde{U})$ is a large family of entropies in the spirit of the Kruzkov entropies for the scalar case $|\cdot - \tilde{u}|$ (see [30]). The Kruzkov theory generates contraction properties of the weak solutions in L^1. Such contraction property cannot be obtained in general from the relative entropy. This is because the L^2 norm is incompatible with the Rankine-Huguoniot condition which dictates the displacement of singularities, and is based on the conservation of linear combinations of the conserved quantities. To understand this obstruction, we first considered the case where $\tilde{U}$ is a single shock. In [31], Leger showed that, in the scalar case, the relative entropy generates a contraction in L^2 up to a shift. This artificial shift is very sensitive to the weak perturbation U, and is not uniquely defined. Typically, it can be obtained by solving a generalized ODE (Fillippov [15]) depending on the left and right values of the weak solution at a single point. This is why, in the purely hyperbolic case, the theory needs the notion of strong traces. Unfortunately, this property is known to holds only for the scalar case [39], or the isentropic system with $\gamma = 3$ [38]. In [37], it has been showed that L^2-type contraction for shocks, even up to a shift, is not true for most of the systems, including (1.5). However, we showed in [22, 41] that this contraction property can be recovered by weighting the relative entropy, leading to the theory of a-contraction with shifts. More precisely, it has been shown that for any shocks $\tilde{U} = (U_l, U_r, \sigma)$, there exist weights $a_1, a_2 > 0$ such that for any weak solution U we can build a shift $t \to X(t)$ such that

$$a_1 \int_{-\infty}^{X(t)} \eta(U(t,x)|U_l)\, dx + a_2 \int_{X(t)}^{+\infty} \eta(U(t,x)|U_r)\, dx \qquad \text{is non increasing in time.}$$

In [29], it was showed how to generalize this statement to obtain a stability result for any solution U in the scalar case. To extend the result to the system case where U is of BV, two subtle properties are needed. When considering several waves, we need to associate an artificial shift for each of the shock curves. Note that we have very little control on the artificial shifts $X_i(t)$. A crucial needed property is that, even with these artificial shifts, waves which are not supposed to collide (like a 1-shock on the left of a 2-shock) will never do collide. This has been achieved by Krupa in [28]. Note that the large family of waves will generate a large family of weights a_i. It is then important to control the strength of variations of $a_i - a_{i-1}$. It can be shown that these variations can be chosen proportionally to the strength of the associated shock wave. Surprisingly, this problem was first solved in the context of Navier-Stokes in [24]. The proof for the hyperbolic case is very different, and will be available in [18]. The final result of stability of BV solutions in the class of weak bounded solutions with strong trace can then been proved [8].

The program to solve the Bianchini-Bressan conjecture consists of following the same strategy, but aiming for a function U which is an inviscid limit of (1.1) instead of being a solution of (1.5). Note that the results in this context, as Theorem 1.1, are stronger than the one obtain directly on the hyperbolic system. Indeed no a priori assumption is needed on the inviscid limit (not even L^∞ bounds or strong trace property). This is because the relative entropy calculus is done at the level of the Navier-Stokes with $\nu > 0$ where enough regularity on the solutions ensures that all the computations hold true. But the price is a far higher level of sophistication in the proofs.

Even in the case of a single shock, the contraction involves a subtle balance between the hyperbolic structure (forcing toward the singularity), and the parabolic one (fighting against it). An important point is to ensure that estimates are uniform with respect to ν. However, we will show that it is enough to consider the case $\nu = 1$ (in this paper it corresponds to Theorems 1.2, 4.1), at the price of considering general large perturbations. For this reason, Theorem 1.2 (and Theorem 4.1) is a far stronger result than a standard stability result, since it does not assume any smallness on the initial perturbation. This shows the strength of replacing the notion of stability, by the notion of contraction (without smallness on the initial value). The scaling argument is as follows. Consider $\tilde{U}^\nu$ a traveling wave (viscous shock) of (1.1). Assume that we want to show that for any solution U^ν of (1.1), we have a contraction up to a weight function a_ν and a shift X_ν:

$$\int_{\mathbb{R}} a_\nu(x - X_\nu(t))\eta(U^\nu(t,x)|\tilde{U}^\nu(x - X_\nu(t)))\, dx \qquad \text{is non increasing in time.}$$

Then, the function $U(t,x) = U^\nu(\nu t, \nu x)$ is a solution to (1.1) with $\nu = 1$, and $\tilde{U}(x) = \tilde{U}^\nu(\nu x)$ is a corresponding traveling wave. Therefore, using the change of variable in x, it is equivalent to showing that up to a weight function ($a(x) = a_\nu(\nu x)$) and a shift ($X(t) = X_\nu(\nu t)/\nu$), we have

$$\int_{\mathbb{R}} a(x - X(t))\eta(U(t,x)|\tilde{U}(x - X(t)))\, dx \qquad \text{is non increasing in time.}$$

However, even if the initial perturbation for the ν problem is small, let say

$$\int_{\mathbb{R}} \eta(U^\nu(0,x)|\tilde{U}^\nu(x))\,dx = \delta,$$

The associated initial perturbation for the rescaled problem is very big as

$$\int_{\mathbb{R}} \eta(U(0,x)|\tilde{U}(x))\,dx = \frac{\delta}{\nu}.$$

Especially, every method based on linearization will fail.

In the context of viscous models, the method was first introduced for the viscous scalar case (without weight) in [23] (improved in [21]), and for the multi-dimensional scalar case in [27]. In the case of Navier-Stokes, the a-contraction with shift for large initial perturbation was proved in [24]. The method was also applied in [11] to the Keller-Segel-type model. It provided the key tool to show the global-in-time existence of solutions for non-homogenous boundary data [12]. To obtain the stability of shocks of (1.5) in the family of inviscid limits of (1.1), we need to pass into the limit ν goes to 0. This step, performed in [26] is also delicate due to the lack of uniform bound both on the solutions U^ν, and on the shifts X_ν. Especially, nothing prevents cavitations, which corresponds to concentration in measure of v. It is remarkable that the stability result can handle even this effect. This paper is dedicated to the last crucial property needed before considering a large family of waves. We show the a-contraction property with shifts for the Riemann problem consisting of two shocks. The important point is to show that we can construct two shifts (one for each shock) which will never collide. This result is the counter part of [28] for Navier-Stokes. As always in this program, the philosophy is similar to the hyperbolic case, but the techniques and the results are very different and far more technical.

3. An overview of the proof

We here describe the main steps of the proof.

The Stability for $\nu = 1$: Theorem 1.2. As explained in the previous section, the main results of this paper boil down to the proof of stability of a composite wave, consisting of the superposition of two viscous shocks waves, to the Navier-Stokes equation UNIFORMLY with respect to the strength of the viscosity. A mentioned, this is equivalent to the stability for the case $\nu = 1$, if we consider LARGE perturbations (Theorem 1.2). One difficulty due to considering a composite wave, is that at the level of Navier-Stokes, superpositions of exact shock waves are not exact solutions to Navier-Stokes (because of the viscosity term, the waves should interact). This explains the extra constant term on the right-hand side of (1.26). However, after rescaling to obtain the result of small viscosity ν, this term becomes $C\nu$ and so converges to 0 in the inviscid limit. This is consistent with the fact that there are no interactions of far away waves vanish for the inviscid equation. The proof of the theorem can be split into several steps.

Step one: Introducing a new velocity variable: Section 4. The growth of the perturbation is partly due to hyperbolic terms (flux functionals). Thanks to the relative entropy method, the linear fluxes are easier to handle (the relative functional of linear quantity vanishes). Therefore, the main hyperbolic quantities to control are the pressure terms depending only on the specific volume v. At the core of the method, we are using a generalized Poincaré inequality Proposition B.1, first proved in [24]. The Navier-Stokes system can be seen as a degenerate parabolic system. But the diffusion is in the other variable, the velocity variable u. Bresch and Desjardins (see [4, 3]) showed that compressible Navier-Stokes systems have a natural perturbed velocity quantity associated to the viscosity:

$$h^\nu = u^\nu + \nu \left(p(v^\nu)^{\frac{\alpha}{\gamma}} \right)_x.$$

Remarkably, the system in the variables (v^ν, h^ν) exhibits a diffusion in the v variable (the Smoluchowski equation), rather than in the velocity variable. For this reason, we are working with the natural relative entropy of this system, which corresponds to the usual relative entropy of the associated p-system in the $U_h^\nu = (v^\nu, h^\nu)$ variable:

$$\eta(U_h^\nu | \tilde{U}_h^\nu) = E_\nu(U^\nu | \tilde{U}^\nu).$$

To simplify the notation, we denote now $U = (v, h)$, since we work on the new system, and only with $\nu = 1$. The associated shifted composite wave is then denoted by $\tilde{U}^{X_1, X_2}$. It is the superposition, in the (v, h) variables, of the two shocks $\tilde{U}_i$, each subjected to an artificial shift $X_i(t)$ to be determined.

Step 2: Evolution of the relative entropy: Lemma 5.2. The evolution of the relative entropy, modulated by a weight a which has also to be determined, can be roughly represented as

$$\begin{aligned} \frac{d}{dt} \int_{-\infty}^{\infty} a(x) \eta\big(U(t,x) | \tilde{U}^{X_1,X_2}(x)\big) dx \\ = \sum_{i=1}^{2} \dot{X}_i(t) Y_i(U(t)) + \mathcal{J}^{bad}(U(t)) - \mathcal{J}^{good}(U(t)). \end{aligned}$$

The functional $\mathcal{J}^{good}(U)$ is non-negative (good term) and can be split into three terms:

$$\mathcal{J}^{good}(U) = \mathcal{J}_1^{good}(U) + \mathcal{G}_2(U) + \mathcal{D}(U),$$

where only $\mathcal{J}_1^{good}(U)$ depends on h (and actually does not depend on v). The term $\mathcal{D}(U)$ corresponds to the diffusive term (which depends on v only, thanks to the transformation of the system).

Step 3: Construction of the shifts and the weight function a. The shifts $X_i(t)$ together with choice of weight function a produce the terms $\dot{X}_i(t) Y_i(U)$. The key idea of the technique is to take advantage of these terms, when the $Y_i(U(t))$ are not too small, by compensating all the other terms via the choice of the velocity of the shift (see (5.29)). Specifically, we algebraically ensure that the contraction holds as long as one of the conditions $(-1)^{i-1} Y_i(U(t)) \geq \varepsilon_i^2$ holds, while ensuring that the two shifts keep the two shocks waves apart. This last property is crucial to avoid unnatural collisions between the two waves, and is due to Proposition 6.2. The rest of the analysis is to ensure that when both $(-1)^{i-1} Y_i(U(t)) \leq \varepsilon_i^2$ hold, the uniform stability still holds.

The conditions $(-1)^{i-1} Y_i(U(t)) \leq \varepsilon_i^2$ ensure smallness conditions that we want to fully exploit. This is where the non-homogeneity of the semi-norm is crucial. In the case where the function a is constant, $Y_i(U)$ are linear functional in U. The smallness of linear part of $Y_i(U)$ gives only that a certain weighted mean value of U is almost null. However, when a has the right monotonicity, $Y_i(U)$ becomes convex. The condition $(-1)^{i-1} Y_i(U(t)) \leq \varepsilon_i^2$ implies, for this fixed time t, a control in L^2 for moderate values of v, and in L^1 for big values of v, in the two layer regions ($|\xi - X_i(t)| \lesssim 1/\varepsilon_i$).

The problem now looks, at first glance, as a typical problem of stability with a smallness condition. There are, however, three major difficulties: The bad term $\mathcal{J}^{bad}(U)$ has some terms depending on the variable h for which we do not have diffusion, we have some smallness in v, only for a very weak norm, and only localized in the layer regions. More importantly, the smallness is measured with respect to the smallness of the shocks. It basically says that, considering only the moderate values of v: the perturbation is not bigger than ε/λ (which is still very big with respect to the size of the biggest shock ε). Actually, as we will see later, it is not possible to consider only the linearized problem: Third order terms appear in the expansion using the smallness condition (the energy method involving the linearization would have only second order term in ε).

In the argument, for the values of t satisfying both $(-1)^{i-1}Y_i(U(t)) \le \varepsilon_i^2$, we construct the shifts as solutions to the ODEs:

$$\dot{X}_i(t) = \begin{cases} Y_i(U(t,\cdot + X_i(t)))/\varepsilon^4, & \text{if } 0 \le (-1)^{i-1}Y_i(U) \le \varepsilon_i^2, \\ \frac{(-1)^{i-1}\sigma_i}{2\varepsilon_i^2} Y_i(U(t,\cdot + X_i(t))), & \text{if } -\varepsilon_i^2 \le (-1)^{i-1}Y_i(U) \le 0, \\ -\frac{1}{2}\sigma_i, & \text{if } (-1)^{i-1}Y_i(U) \le -\varepsilon_i^2, \end{cases}$$

From this point, we forget that $U = U(t,\xi)$ is a solution to the equation and that $X_i(t)$ are the shifts. That is, we leave out the $X_i(t)$ and the t-variable of U. Then we show that for any function U satisfying both $(-1)^{i-1}Y_i(U(t)) \le \varepsilon_i^2$ for $i = 1, 2$, we have

$$\begin{aligned} \sum_{i=1}^{2} \Bigg(&- \frac{1}{\varepsilon_i^4} |Y_i(U)|^2 \mathbf{1}_{\{0 \le (-1)^{i-1}Y_i(U) \le \varepsilon_1^2\}} + \frac{(-1)^{i-1}\sigma_i}{2\varepsilon_i^2} |Y_i(U)|^2 \mathbf{1}_{\{-\varepsilon_i^2 \le (-1)^{i-1}Y_i(U) \le 0\}} \\ &- \frac{\sigma_i}{2} Y_i(U) \mathbf{1}_{\{(-1)^{i-1}Y_i(U) \le -\varepsilon_i^2\}} \Bigg) + \mathcal{J}^{bad}(U) - \mathcal{J}^{good}(U) \\ &\le C \left[g_1(t) \int_{\mathbb{R}} \eta(U|\tilde{U}) dx + g_2(t) \right] \mathbf{1}_{t \ge t_0} + C \mathbf{1}_{t \le t_0}, \end{aligned}$$

where g_1, g_2 are some integrable functions. This is the main Proposition 5.1 (actually, the proposition is slightly stronger to ensure the control of the shift). This implies clearly the uniform estimate as desired. From now on, we are focusing on the proof of this statement.

Step 4: Maximization in h for v fixed. We recall that the new system is parabolic only in the variable v. Therefore, we need to get rid of the dependence on the h variable from the bad parts $\mathcal{J}^{bad}(U)$. It is done in two different ways depending of the value of $p(v) - p(\tilde{v})$ with respect to a threshold δ_1 to be determined (and depending on the Poincaré inequality). When $p(v) - p(\tilde{v}) \ge \delta_1$, the bad terms involving h can be controlled using additional information from the unconditional estimates $(-1)^{i-1}Y_i(U(t)) \le \varepsilon_i^2$. However, when $p(v) - p(\tilde{v}) \le \delta_1$, the idea is to maximize the bad term with respect to h for v fixed:

$$\mathcal{B}(v) = \sup_h \left(\mathcal{J}^{bad}(v,h) - \mathcal{J}_1^{good}(h) \right).$$

We then have an inequality depending only on v and $\partial_x v$ (through $\mathcal{D}(U)$) for which we can apply the generalized Poincaré inequality.

Step 5: Expansion in ε_i. Although we have no control on the supremum of $|p(v) - p(\tilde{v})|$, we can control independently the contribution of the values $|p(v) - p(\tilde{v})| \ge \delta_1$ in Proposition 7.1 (for the same δ_1 related to the maximization process above. The coefficient δ_1 can be chosen very small, but independent of ε_i and of ε_i/λ). The last step is to perform an expansion in the size of the shocks ε_i, uniformly in v (but for a fixed small value of δ).

The expansion is done for each shock wave separately. The generalized nonlinear Poincaré inequality, Proposition B.1 concludes the proof.

The inviscid limit: Theorem 1.1.

We have now a stability result uniform with respect to the viscosity. It is natural to expect a stability result on the corresponding inviscid limit. The result, however, is not immediate. Several difficulties have to be overcome. First, due to the BD representation as above, the stability result for ν fixed is on the quantities:

$$U_h^\nu = (v^\nu, h^\nu), \quad h^\nu = u^\nu + \nu \left(p(v^\nu)^{\frac{\alpha}{\gamma}}\right)_x.$$

This is the reason we need a compatibility condition on the family of initial values U_0^ν. This also leads to a very weak convergence (in measure in (t,x) only). The next difficulty is that for small values of v, the relative entropy controls only the L^1 norm of $Q(v) = 1/v^{\gamma-1}$. Therefore the pressure $p(v) = 1/v^\gamma$ cannot be controlled at all. Therefore, we do not control the time derivative of u in any distributional sense in x. We need to show that the shifts vanish when the perturbation converges to 0. This can be obtained, thanks to the convergence of v in $C^0(\mathbb{R}^+, W^{-s,1}_{\text{loc}}(\mathbb{R}))$. It is interesting to note that the continuity (in time) of v is enough. We do not obtain any such control on u (nor h).

4. Reformulation for Theorem 1.2

Following [24], the first step consists of making a change of variables to work on an equivalent system where the diffusion is in the v equation (instead of u). This is important, since the nonlinearity of the hyperbolic term of (1.23) are in v only (through the pressure). First of all, since the strength of the coefficient b in $\mu(v)$ does not affect our analysis, as in [24], we set $b = \gamma$ (for simplicity). Then, any solution (v,u) to (1.1), we consider

$$h := u + \left(p(v)^{\frac{\alpha}{\gamma}}\right)_x. \tag{4.1}$$

This quantity is the modulated velocity associated to the BD entropy (see Bresch and Desjardins [2]). Let $\beta := \gamma - \alpha$. The new unknown (v,h) is then a solution to the system:

$$\begin{cases} v_t - h_x = -\left(v^\beta p(v)_x\right)_x \\ h_t + p(v)_x = 0. \end{cases} \tag{4.2}$$

Then, the two viscous traveling waves in the variables (v,h) associated to (1.9) are as follows:

$$\begin{cases} -\sigma_i \partial_x \tilde{v}_i(x-\sigma_i t) - \partial_x \tilde{h}_i(x-\sigma_i t) = -\partial_x\left(\tilde{v}_i(x-\sigma_i t)^\beta \partial_x p(\tilde{v}_i(x-\sigma_i t))\right) \\ -\sigma_i \partial_x \tilde{h}_i(x-\sigma_i t) + \partial_x p(\tilde{v}_i(x-\sigma_i t)) = 0, \end{cases} \tag{4.3}$$

together with

$$\lim_{x\to-\infty}(\tilde{v}_1,\tilde{h}_1)(x-\sigma_1 t) = (v_-,u_-), \qquad \lim_{x\to+\infty}(\tilde{v}_1,\tilde{h}_1)(x-\sigma_1 t) = (v_m,u_m),$$

$$\lim_{x\to-\infty}(\tilde{v}_2,\tilde{h}_2)(x-\sigma_2 t) = (v_m,u_m), \qquad \lim_{x\to+\infty}(\tilde{v}_2,\tilde{h}_2)(x-\sigma_2 t) = (v_+,u_+).$$

Note from the space $\mathcal{X}_T$ that the global solutions to (4.2) are in the following function space: (see Remark 4.1)

$$\begin{aligned}\mathcal{H}_T := \{(v,h) \mid v-\underline{v} \in C(0,T;H^1(\mathbb{R})),\\ h-\underline{u} \in C(0,T;L^2(\mathbb{R})),\ 0<v^{-1}\in L^\infty((0,T)\times\mathbb{R})\},\end{aligned} \tag{4.4}$$

where $\underline{v}$ and $\underline{u}$ are as in (1.11).

Theorem 1.2 is an immediate consequence of the following theorem.

Theorem 4.1. *Assume $\gamma>1$ and $\alpha>0$ satisfying $\alpha\le\gamma\le\alpha+1$. For a given constant $U_*:=(v_*,u_*)\in\mathbb{R}^+\times\mathbb{R}$, there exists constant $\delta_0\in(0,1/2)$ such that the following holds. Let $U_-:=(v_-,u_-), U_m:=(v_m,u_m), U_+:=(v_+,u_+)\in\mathbb{R}^+\times\mathbb{R}$ be any constant states such that (1.7) and $U_-,U_m,U_+\in B_{\delta_0}(U_*)$. Let $\varepsilon_1:=|p(v_-)-p(v_m)|$ and $\varepsilon_1:=|p(v_m)-p(v_+)|$. For any $\lambda>0$ with $\varepsilon_1/\lambda,\varepsilon_2/\lambda<\delta_0$ and $\lambda<\delta_0$, there exist a constant $C>0$ and smooth monotone functions a_1,a_2 with $a_1(x),a_2(x)\in[1-\lambda,1]$ for all $x\in\mathbb{R}$ such that the following holds.*
Let $\tilde{U}$ be the composite wave connecting U_- and U_+ as:

$$\tilde{U}(t,x) := \begin{pmatrix}\tilde{v}(t,x)\\ \tilde{h}(t,x)\end{pmatrix} := \begin{pmatrix}\tilde{v}_1(x-\sigma_1t)+\tilde{v}_2(x-\sigma_2t)-v_m\\ \tilde{h}_1(x-\sigma_1t)+\tilde{h}_2(x-\sigma_2t)-u_m\end{pmatrix}. \tag{4.5}$$

Let

$$a(t,x) := a_1(x-\sigma_1t)+a_2(x-\sigma_2t)-1. \tag{4.6}$$

For a given $T>0$, let $U:=(v,h)$ be a solution in $\mathcal{H}_T$ to (4.2) with a initial datum $U_0:=\begin{pmatrix}v_0\\u_0\end{pmatrix}$ satisfying $\int_{-\infty}^{\infty}\eta(U_0(x)|\tilde{U}(0,x))dx<\infty$. Then, there exist shift functions $X_1,X_2\in W^{1,1}((0,T))$ with $X_1(0)=X_2(0)=0$ such that for the shifted composite wave

$$\tilde{U}^{X_1,X_2}(t,x) := \begin{pmatrix}\tilde{v}^{X_1,X_2}(t,x)\\ \tilde{h}^{X_1,X_2}(t,x)\end{pmatrix} := \begin{pmatrix}\tilde{v}_1(x-\sigma_1t-X_1(t))+\tilde{v}_2(x-\sigma_2t-X_2(t))-v_m\\ \tilde{h}_1(x-\sigma_1t-X_1(t))+\tilde{h}_2(x-\sigma_2t-X_2(t))-u_m\end{pmatrix},$$

and the shifted weight

$$a^{X_1,X_2}(t,x) := a_1(x-\sigma_1t-X_1(t))+a_2(x-\sigma_2t-X_2(t))-1,$$

we have the uniform stability:

$$\begin{aligned}&\int_{\mathbb{R}} a^{X_1,X_2}(t,x)\eta\big(U(t,x)|\tilde{U}^{X_1,X_2}(t,x)\big)dx\\ &\qquad+\int_0^T\int_{-\infty}^{\infty}|\partial_x a^{X_1,X_2}(t,x)|Q\left(v(t,x)|\tilde{v}^{X_1,X_2}(t,x)\right)dxdt\\ &\qquad+\int_0^T\int_{-\infty}^{\infty}v^{\gamma-\alpha}(t,x)\Big|\partial_x\big(p(v(t,x)))-p(\tilde{v}^{X_1,X_2}(t,x))\big)\Big|^2dxdt\\ &\quad\le C\int_{\mathbb{R}}a(0,x)\eta\big(U_0(x)|\tilde{U}(0,x)\big)dx+C,\end{aligned} \tag{4.7}$$

and

$$X_1(t)\le-\frac{\sigma_1}{2}t,\quad X_2(t)\ge-\frac{\sigma_2}{2}t,\quad\forall t>0, \tag{4.8}$$

in addition, for each $i = 1, 2$,

$$
\begin{aligned}
&|\dot{X}_i(t)| \le C\left[f(t) + \int_{\mathbb{R}} \eta(U_0(x)|\tilde{U}(0,x))dx + 1\right] \qquad \textit{for a.e. } t\in[0,T],\\
&\textit{for some positive function } f \textit{ satisfying } \quad \|f\|_{L^1(0,T)} \le C\int_{-\infty}^{\infty}\eta\big(U_0(x)|\tilde{U}(0,x)\big)dx.
\end{aligned}
\tag{4.9}
$$

Remark 4.1. *1. Theorem 4.1 provides the uniform stability for composite waves with suitably small amplitude parametrized by $|p(v_-) - p(v_m)| = \varepsilon_1$ and $|p(v_m) - p(v_+)| = \varepsilon_2$. This smallness together with (1.7) implies that*

$$
|v_- - v_m| = \mathcal{O}(\varepsilon_1), \quad |u_- - u_m| = \mathcal{O}(\varepsilon_1), \quad |v_+ - v_m| = \mathcal{O}(\varepsilon_2), \quad |u_+ - u_m| = \mathcal{O}(\varepsilon_2). \tag{4.10}
$$

2. If we consider the solution $(v,u)\in\mathcal{X}_T$ to (1.23). Then, (4.2) admits the solution (v,h) in $\mathcal{H}_T$. Indeed, since $v_t = u_x \in L^2(0,T;H^1(\mathbb{R}))$ by $(1.23)_1$, we have $v - \underline{v} \in C(0,T;H^1(\mathbb{R}))$. To show $h - \underline{u} \in C(0,T;L^2(\mathbb{R}))$, we first find that for $(v,u)\in\mathcal{X}_T$,

$$
h - \underline{u} = u - \underline{u} + \frac{\alpha}{\gamma}p(v)^{\frac{\alpha}{\gamma}-1}v_x \in L^\infty(0,T;L^2(\mathbb{R})).
$$

Moreover, together with the fact that $v\in L^\infty((0,T)\times\mathbb{R})$ by Sobolev embedding, we find that

$$
\begin{aligned}
&u_t = -p'(v)v_x + \frac{d}{dv}\Big(\frac{\mu(v)}{v}\Big)v_x u_x + \frac{\mu(v)}{v}u_{xx} \in L^2(0,T;L^2(\mathbb{R})),\\
&\Big(p(v)^{\frac{\alpha}{\gamma}-1}v_x\Big)_t = (\frac{\alpha}{\gamma}-1)p(v)^{\frac{\alpha}{\gamma}-2}v_t v_x + p(v)^{\frac{\alpha}{\gamma}-1}v_{xt} \in L^2(0,T;L^2(\mathbb{R})),
\end{aligned}
$$

which implies $h_t \in L^2(0,T;L^2(\mathbb{R}))$, and therefore $h - \underline{u} \in C(0,T;L^2(\mathbb{R}))$.

• **Notation:** In what follows, C denotes a positive constant which may change from line to line, but which is independent of $\varepsilon_1, \varepsilon_2$ (the sizes of shocks) and λ (the total variation of the weights a_i).

5. Proof of Theorem 4.1

5.1. **Properties of small shock waves.** In this subsection, we present useful properties of the i-shock waves $(\tilde{v}_i, \tilde{h}_i)$ with small amplitude ε_i. In the sequel, we assume that the 1-shock (resp. 2-shock) satisfy $\tilde{v}_1(0) = \frac{v_-+v_m}{2}$ (resp. $\tilde{v}_2(0) = \frac{v_m+v_+}{2}$) without loss of generality (by the translation invarience).

Lemma 5.1. *For a given constant $U_* := (v_*, u_*) \in \mathbb{R}^+\times\mathbb{R}$, there exist positive constants $\varepsilon_0, C, C_1, C_2$ such that the following holds.*
Let $U_- := (v_-, u_-), U_m := (v_m, u_m), U_+ := (v_+, u_+) \in \mathbb{R}^+\times\mathbb{R}$ be any constant such that (1.7) and $U_-, U_m, U_+ \in B_{\varepsilon_0}(U_*)$, and $|p(v_-) - p(v_m)| =: \varepsilon_1 < \varepsilon_0$ and $|p(v_m) - p(v_+)| =: \varepsilon_2 < \varepsilon_0$. Let $(\tilde{v}_1, \tilde{h}_1)$ and $(\tilde{v}_2, \tilde{h}_2)$ be the 1- and 2-shocks respectively connecting from U_- to U_m, and from U_m to U_+ such that $\tilde{v}_1(0) = \frac{v_-+v_m}{2}$ and $\tilde{v}_2(0) = \frac{v_m+v_+}{2}$. Then, the following estimates hold.

$$
C^{-1}\varepsilon_1 e^{-C_1\varepsilon_1|x-\sigma_1 t|} \le \tilde{v}_1(x-\sigma_1 t) - v_m \le C\varepsilon_1 e^{-C_2\varepsilon_1|x-\sigma_1 t|}, \quad x \ge \sigma_1 t, \tag{5.1}
$$

$$
-C^{-1}\varepsilon_1^2 e^{-C_1\varepsilon_1|x-\sigma_1 t|} \le \partial_x\tilde{v}_1(x-\sigma_1 t) \le -C\varepsilon_1^2 e^{-C_2\varepsilon_1|x-\sigma_1 t|}, \quad x\in\mathbb{R},\ t>0, \tag{5.2}
$$

and

$$C^{-1}\varepsilon_2 e^{-C_1\varepsilon_2|x-\sigma_2 t|} \le \tilde{v}_2(x-\sigma_2 t) - v_m \le C\varepsilon_2 e^{-C_2\varepsilon_2|x-\sigma_2 t|}, \quad x \le \sigma_2 t, \tag{5.3}$$

$$C^{-1}\varepsilon_2^2 e^{-C_1\varepsilon_2|x-\sigma_2 t|} \le \partial_x\tilde{v}_2(x-\sigma_2 t) \le C\varepsilon_2^2 e^{-C_2\varepsilon_2|x-\sigma_2 t|}, \quad x \in \mathbb{R},\ t>0. \tag{5.4}$$

As a consequence, for each $i = 1, 2$,

$$\inf_{[-\varepsilon_i^{-1},\varepsilon_i^{-1}]} |\partial_x \tilde{v}_i| \ge C\varepsilon_i^2. \tag{5.5}$$

In addition, for each $i = 1, 2$,

$$|(\tilde{v}_i)_{xx}(x-\sigma_i t)| \le C\varepsilon_i |(\tilde{v}_i)_x(x-\sigma_i t)|, \quad x \in \mathbb{R},\ t>0. \tag{5.6}$$

Proof. Since $v_*/2 < \tilde{v}_i < 2v_*$ with choosing ε_0 small enough, the proofs of (5.1)-(5.5) use the same computations as in the proof of [24, Lemma 2.1]. Therefore, we omit the details. To show (5.6), we first observe from (4.3) that

$$\begin{aligned}\left(\frac{\sigma_i^2}{p'(\tilde{v}_i)}+1\right)\frac{p'(\tilde{v}_i)\partial_x\tilde{v}_i}{\sigma_i} &= \sigma_i\partial_x\tilde{v}_i + \partial_x\tilde{h}_i = \partial_x\left(\tilde{v}_i^\beta\partial_x p(\tilde{v}_i)\right)\\ &= \beta\tilde{v}_i^{\beta-1}p'(\tilde{v}_i)|(\tilde{v}_i)_x|^2 + \tilde{v}_i^\beta p''(\tilde{v}_i)|(\tilde{v}_i)_x|^2 + \tilde{v}_i^\beta p'(\tilde{v}_i)(\tilde{v}_i)_{xx},\end{aligned}$$

where note that the above waves are evaluated at $x - \sigma_i t$.
Then, using (5.2), (5.4) and $v_*/2 \le \tilde{v}_i \le 2v_*$, we find

$$|(\tilde{v}_i)_{xx}| \le C\varepsilon_i^2|(\tilde{v}_i)_x| + \left|\frac{\sigma_i^2}{p'(\tilde{v}_i)}+1\right||(\tilde{v}_i)_x|.$$

Using Taylor theorem together with (4.10) and (1.7), we have

$$\sigma_1 = -\sqrt{-p'(v_-)} + \mathcal{O}(\varepsilon_1), \qquad \sigma_2 = -\sqrt{-p'(v_m)} + \mathcal{O}(\varepsilon_2),$$

and

$$p'(\tilde{v}_1)^{-1} = p'(v_-)^{-1} + \mathcal{O}(\varepsilon_1), \qquad p'(\tilde{v}_2)^{-1} = p'(v_m)^{-1} + \mathcal{O}(\varepsilon_2).$$

In addition, since $|v_- - v_*| \le \varepsilon_0$ and $|v_m - v_*| \le \varepsilon_0$, we have

$$\left|\frac{\sigma_i^2}{p'(\tilde{v}_i)}+1\right| \le C\varepsilon_i \quad \text{for each } i = 1, 2.$$

Therefore we have (5.6). □

Remark 5.1. *Notice that Lemma 5.1 also holds for* $\tilde{h}_i$*, since*

$$C^{-1}|\partial_x\tilde{v}_i| \le |\partial_x\tilde{h}_i| \le C|\partial_x\tilde{v}_i|, \tag{5.7}$$

which comes from the fact that $\partial_x\tilde{h}_i = \frac{p'(\tilde{v}_i)}{\sigma_i}\partial_x\tilde{v}_i$ *and* $\frac{1}{2}\frac{p'(v_*)}{\sigma_*} \le \frac{p'(\tilde{v}_i)}{\sigma_i} \le 2\frac{p'(v_*)}{\sigma_*}$ *by* (1.7) *and* (4.10) *with* $\varepsilon_1, \varepsilon_2 < \varepsilon_0 \ll 1$. *Especially, notice that* $\tilde{v}_i$ *and* $\tilde{h}_i$ *are monotone by* (5.2), (5.4) *and* $\partial_x\tilde{h}_i = \frac{p'(\tilde{v}_i)}{\sigma_i}\partial_x\tilde{v}_i$.

5.2. **Relative entropy method.** A starting point of our analysis is to use the relative entropy method. The method is purely nonlinear, and allows to handle rough and large perturbations.

To use the relative entropy method, we rewrite (4.2) into the viscous hyperbolic system of conservation laws:

$$\partial_t U + \partial_x A(U) = \begin{pmatrix} -\partial_\xi\big(v^\beta \partial_\xi p(v)\big) \\ 0 \end{pmatrix}, \tag{5.8}$$

where

$$U := \begin{pmatrix} v \\ h \end{pmatrix}, \quad A(U) := \begin{pmatrix} -h \\ p(v) \end{pmatrix}.$$

The system (5.8) has a convex entropy $\eta(U) := \frac{h^2}{2} + Q(v)$, where $Q(v) = \frac{v^{-\gamma+1}}{\gamma-1}$, i.e., $Q'(v) = -p(v)$.
Using the derivative of the entropy as

$$\nabla\eta(U) = \begin{pmatrix} -p(v) \\ h \end{pmatrix}, \tag{5.9}$$

the above system (5.8) can be rewritten as

$$\partial_t U + \partial_x A(U) = \partial_x\Big(M(U)\partial_x \nabla\eta(U)\Big), \tag{5.10}$$

where $M(U) = \begin{pmatrix} v^\beta & 0 \\ 0 & 0 \end{pmatrix}$.
Also, for each wave

$$\tilde{U}_i(x-\sigma_i t) := \begin{pmatrix} \tilde{v}_i(x-\sigma_i t) \\ \tilde{h}_i(x-\sigma_i t) \end{pmatrix},$$

the system (4.3) can be rewritten as:

$$-\sigma_i \partial_x \tilde{U}_i + \partial_x A(\tilde{U}_i) = \partial_x\Big(M(\tilde{U}_i)\partial_x \nabla\eta(\tilde{U}_i)\Big). \tag{5.11}$$

Thus, the composite wave

$$\tilde{U}(t,x) = \tilde{U}_1(x-\sigma_1 t) + \tilde{U}_2(x-\sigma_2 t) - \begin{pmatrix} v_m \\ u_m \end{pmatrix}$$

satisfies

$$\partial_t \tilde{U} = \sum_{i=1}^{2}\Big(-\partial_x A(\tilde{U}_i) + \partial_x\Big(M(\tilde{U}_i)\partial_x\nabla\eta(\tilde{U}_i)\Big)\Big). \tag{5.12}$$

We define the relative entropy function by

$$\eta(U|V) = \eta(U) - \eta(V) - \nabla\eta(V)(U-V),$$

and the relative flux by

$$A(U|V) = A(U) - A(V) - \nabla A(V)(U-V).$$

Let $G(\cdot;\cdot)$ be the flux of the relative entropy defined by

$$G(U;V) = G(U) - G(V) - \nabla\eta(V)(A(U) - A(V)),$$

where G is the entropy flux of η, i.e., $\partial_i G(U) = \sum_{k=1}^{2} \partial_k \eta(U) \partial_i A_k(U), \quad 1 \le i \le 2$. Then, for our system (5.8), we have

$$\begin{aligned} \eta(U|\tilde{U}) &= \frac{|h-\tilde{h}|^2}{2} + Q(v|\tilde{v}), \\ A(U|\tilde{U}) &= \begin{pmatrix} 0 \\ p(v|\tilde{v}) \end{pmatrix}, \\ G(U;\tilde{U}) &= (p(v)-p(\tilde{v}))(h-\tilde{h}), \end{aligned} \tag{5.13}$$

where the relative pressure is defined as

$$p(v|w) = p(v) - p(w) - p'(w)(v-w). \tag{5.14}$$

We will consider a weighted relative entropy between the solution U of (5.10) and the shifted composite wave $\tilde{U}^{X_1,X_2}$ as follows: for shifts X_1, X_2 (to be determined),

$$a^{X_1,X_2}(t,x)\eta\big(U(t,x)|\tilde{U}^{X_1,X_2}(t,x)\big),$$

where the wave $\tilde{U}^{X_1,X_2}$ and the weight a^{X_1,X_2} have the form of

$$\tilde{U}^{X_1,X_2}(t,x) := \begin{pmatrix} \tilde{v}^{X_1,X_2}(t,x) \\ \tilde{h}^{X_1,X_2}(t,x) \end{pmatrix} := \begin{pmatrix} \tilde{v}_1(x-\sigma_1 t - X_1(t)) + \tilde{v}_2(x-\sigma_2 t - X_2(t)) - v_m \\ \tilde{h}_1(x-\sigma_1 t - X_1(t)) + \tilde{h}_2(x-\sigma_2 t - X_2(t)) - u_m \end{pmatrix}, \tag{5.15}$$

and

$$a^{X_1,X_2}(t,x) := a_1(x-\sigma_1 t - X_1(t)) + a_2(x-\sigma_2 t - X_2(t)) - 1. \tag{5.16}$$

In Lemma 5.2, we will derive a quadratic structure on

$$\frac{d}{dt}\int_{\mathbb{R}} a^{X_1,X_2}(t,x)\eta\big(U(t,x)|\tilde{U}^{X_1,X_2}(t,x)\big)dx.$$

For notational simplicity, we will use the following notations: for the waves $\tilde{U}_1, \tilde{U}_2$ and the functions a_1, a_2, and any shifts X_1, X_2,

$$\tilde{U}_i^{X_i} := \tilde{U}_i(x-\sigma_i t - X_i(t)), \quad a_i^{X_i} := a_i(x-\sigma_i t - X_i(t))$$

Lemma 5.2. *Let a_1 and a_2 be any positive smooth bounded functions whose derivative is bounded and integrable. Let $U \in \mathcal{H}_T$ be a solution to* (5.10)*, and X_1, X_2 be any absolutely continuous functions on $[0,T]$. Let $\tilde{U}^{X_1,X_2}$ and a^{X_1,X_2} as in* (5.15) *and* (5.16)*. Then,*

$$\frac{d}{dt}\int_{\mathbb{R}} a^{X_1,X_2}(t,x)\eta(U(t,x)|\tilde{U}^{X_1,X_2}(t,x))dx = \sum_{i=1}^{2}\big(\dot{X}_i(t)Y_i(U)\big) + \mathcal{J}^{bad}(U) - \mathcal{J}^{good}(U), \tag{5.17}$$

where

(5.18)

$$
\begin{aligned}
Y_i(U) &:= -\int_{\mathbb{R}} (a_i)_x^{X_i} \eta(U|\tilde{U}^{X_1,X_2}) dx + \int_{\mathbb{R}} a^{X_1,X_2} (\tilde{U}_i)_x^{X_i} \nabla^2 \eta(\tilde{U}^{X_1,X_2})(U - \tilde{U}^{X_1,X_2}) dx, \\
\mathcal{J}^{bad}(U) &:= \sum_{i=1}^{2} \bigg[\int_{\mathbb{R}} (a_i)_x^{X_i} \big(p(v) - p(\tilde{v}^{X_1,X_2})\big)\big(h - \tilde{h}^{X_1,X_2}\big) dx + \sigma_i \int_{\mathbb{R}} a^{X_1,X_2} (\tilde{v}_i)_x^{X_i} p(v|\tilde{v}^{X_1,X_2}) dx \\
&\quad - \int_{\mathbb{R}} (a_i)_x^{X_i} v^\beta \big(p(v) - p(\tilde{v}^{X_1,X_2})\big) \partial_x \big(p(v) - p(\tilde{v}^{X_1,X_2})\big) dx \\
&\quad - \int_{\mathbb{R}} (a_i)_x^{X_i} \big(p(v) - p(\tilde{v}^{X_1,X_2})\big)(v^\beta - (\tilde{v}^\beta)^{X_1,X_2}) \partial_x p(\tilde{v}^{X_1,X_2}) dx \bigg] \\
&\quad - \int_{\mathbb{R}} a^{X_1,X_2} \partial_x \big(p(v) - p(\tilde{v}^{X_1,X_2})\big)(v^\beta - (\tilde{v}^\beta)^{X_1,X_2}) \partial_x p(\tilde{v}^{X_1,X_2}) dx \\
&\quad + \int_{\mathbb{R}} a^{X_1,X_2} \big(p(v) - p(\tilde{v}^{X_1,X_2})\big) \tilde{E}_1 dx - \int_{\mathbb{R}} a^{X_1,X_2} (h - \tilde{h}^{X_1,X_2}) \tilde{E}_2 dx, \\
\mathcal{J}^{good}(U) &:= \sum_{i=1}^{2} \left(\frac{\sigma_i}{2} \int_{\mathbb{R}} (a_i)_x^{X_i} \left| h - \tilde{h}^{X_1,X_2} \right|^2 dx + \sigma_i \int_{\mathbb{R}} (a_i)_x^{X_i} Q(v|\tilde{v}^{X_1,X_2}) dx \right) \\
&\quad + \int_{\mathbb{R}} a^{X_1,X_2} v^\beta |\partial_x \big(p(v) - p(\tilde{v}^{X_1,X_2})\big)|^2 dx,
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{E}_1 &:= \partial_x \big((\tilde{v}^\beta)^{X_1,X_2} \partial_x p(\tilde{v}^{X_1,X_2})\big) - \partial_x \big((\tilde{v}_1^\beta)^{X_1} \partial_x p(\tilde{v}_1^{X_1})\big) - \partial_x \big((\tilde{v}_2^\beta)^{X_2} \partial_x p(\tilde{v}_2^{X_2})\big), \\
\tilde{E}_2 &:= \partial_x p(\tilde{v}^{X_1,X_2}) - \partial_x p(\tilde{v}_1^{X_1}) - \partial_x p(\tilde{v}_2^{X_2}).
\end{aligned}
$$

Remark 5.2. *In what follows, we will define the weight function a such that $\sigma_\varepsilon a' > 0$. Therefore, $-\mathcal{J}^{good}$ consists of three good terms, while $\mathcal{J}^{bad}$ consists of bad terms.*

Proof. By the definition of the relative entropy, we first have

$$
\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}} a^{X_1,X_2}(t,x) \eta(U(t,x)|\tilde{U}^{X_1,X_2}(t,x)) dx &= \int_{\mathbb{R}} \partial_t a^{X_1,X_2} \eta(U|\tilde{U}^{X_1,X_2}) dx \\
&\quad + \int_{\mathbb{R}} a^{X_1,X_2} \bigg[\Big(\nabla \eta(U) - \nabla \eta(\tilde{U}^{X_1,X_2}) \Big) \partial_t U - \nabla^2 \eta(\tilde{U}^{X_1,X_2})(U - \tilde{U}^{X_1,X_2}) \partial_t \tilde{U}^{X_1,X_2} \bigg] dx.
\end{aligned}
$$

Since

$$
\partial_t a^{X_1,X_2}(t,x) = -\sum_{i=1}^{2} (\sigma_i + \dot{X}_i)(a_i)_x^{X_i},
$$

and it follows from (5.12) that

$$
\partial_t \tilde{U}^{X_1,X_2}(t,x) = \sum_{i=1}^{2} \left(-\dot{X}_i (\tilde{U}_i)_x^{X_i} - \partial_x A(\tilde{U}_i^{X_i}) + \partial_x \Big(M(\tilde{U}_i^{X_i}) \partial_x \nabla \eta(\tilde{U}_i^{X_i}) \Big) \right),
$$

we have

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} a^{X_1,X_2}(t,x)\eta(U(t,x)|\tilde{U}^{X_1,X_2}(t,x))dx \\
&= \sum_{i=1}^{2}(\dot{X}_i Y_i) - \sum_{i=1}^{2}\sigma_i\int_{\mathbb{R}}(a_i)_x^{X_i}\eta(U|\tilde{U}^{X_1,X_2})dx \\
&\qquad + \int_{\mathbb{R}} a^{X_1,X_2}\bigg[\Big(\nabla\eta(U)-\nabla\eta(\tilde{U}^{X_1,X_2})\Big)\Big(-\partial_x A(U)+\partial_x\Big(M(U)\partial_x\nabla\eta(U)\Big)\Big) \\
&\qquad - \nabla^2\eta(\tilde{U}^{X_1,X_2})(U-\tilde{U}^{X_1,X_2})\underbrace{\sum_{i=1}^{2}\Big(-\partial_x A(\tilde{U}_i^{X_i})+\partial_x\Big(M(\tilde{U}_i^{X_i})\partial_x\nabla\eta(\tilde{U}_i^{X_i})\Big)\Big)}_{=:J}\bigg]dx,
\end{aligned}
$$

where

$$
Y_i := -\int_{\mathbb{R}}(a_i)_x^{X_i}\eta(U|\tilde{U}^{X_1,X_2})dx + \int_{\mathbb{R}} a^{X_1,X_2}(\tilde{U}_i)_x^{X_i}\nabla^2\eta(\tilde{U}^{X_1,X_2})(U-\tilde{U}^{X_1,X_2})dx.
$$

Since

$$
J = -\partial_x A(\tilde{U}^{X_1,X_2}) + \partial_x\Big(M(\tilde{U}^{X_1,X_2})\partial_x\nabla\eta(\tilde{U}^{X_1,X_2})\Big) + \begin{pmatrix}\tilde{E}_1\\ \tilde{E}_2\end{pmatrix},
$$

where

$$
\begin{pmatrix}\tilde{E}_1\\ \tilde{E}_2\end{pmatrix} := \begin{pmatrix}\partial_x\big((\tilde{v}^{\beta})^{X_1,X_2}\partial_x p(\tilde{v}^{X_1,X_2})\big) - \partial_x\big((\tilde{v}_1^{\beta})^{X_1}\partial_x p(\tilde{v}_1^{X_1})\big) - \partial_x\big((\tilde{v}_2^{\beta})^{X_2}\partial_x p(\tilde{v}_2^{X_2})\big)\\ \partial_x p(\tilde{v}^{X_1,X_2}) - \partial_x p(\tilde{v}_1^{X_1}) - \partial_x p(\tilde{v}_2^{X_2})\end{pmatrix}.
$$

Using the same computation in [24, Lemma 2.3] (see also [40, Lemma 4]), we have

$$
\frac{d}{dt}\int_{\mathbb{R}} a^{X_1,X_2}(t,x)\eta(U(t,x)|\tilde{U}^{X_1,X_2}(t,x))dx = \sum_{i=1}^{2}(\dot{X}_i Y_i) + \sum_{i=1}^{6} I_i,
$$

where

$$
\begin{aligned}
I_1 &:= -\sum_{i=1}^{2}\sigma_i\int_{\mathbb{R}}(a_i)_x^{X_i}\eta(U|\tilde{U}^{X_1,X_2})dx - \int_{\mathbb{R}} a^{X_1,X_2}\partial_x G(U;\tilde{U}^{X_1,X_2})dx, \\
I_2 &:= -\int_{\mathbb{R}} a^{X_1,X_2}\partial_x\nabla\eta(\tilde{U}^{X_1,X_2})A(U|\tilde{U}^{X_1,X_2})dx, \\
I_3 &:= \int_{\mathbb{R}} a^{X_1,X_2}\Big(\nabla\eta(U)-\nabla\eta(\tilde{U}^{X_1,X_2})\Big)\partial_x\Big(M(U)\partial_x\big(\nabla\eta(U)-\nabla\eta(\tilde{U}^{X_1,X_2})\big)\Big)dx, \\
I_4 &:= \int_{\mathbb{R}} a^{X_1,X_2}\Big(\nabla\eta(U)-\nabla\eta(\tilde{U}^{X_1,X_2})\Big)\partial_x\Big(\big(M(U)-M(\tilde{U}^{X_1,X_2})\big)\partial_x\nabla\eta(\tilde{U}^{X_1,X_2})\Big)dx, \\
I_5 &:= \int_{\mathbb{R}} a^{X_1,X_2}(\nabla\eta)(U|\tilde{U}^{X_1,X_2})\partial_x\Big(M(\tilde{U}^{X_1,X_2})\partial_x\nabla\eta(\tilde{U}^{X_1,X_2})\Big)dx, \\
I_6 &:= -\int_{\mathbb{R}} a^{X_1,X_2}\nabla^2\eta(\tilde{U}^{X_1,X_2})(U-\tilde{U}^{X_1,X_2})\begin{pmatrix}\tilde{E}_1\\ \tilde{E}_2\end{pmatrix}dx.
\end{aligned}
$$

Using $\partial_x a^{X_1,X_2} = (a_1)_x^{X_1} + (a_2)_x^{X_2}$ and (5.13) with (5.9), we have

$$
I_1 = \sum_{i=1}^{2}\left(-\sigma_i\int_{\mathbb{R}}(a_i)_x^{X_i}\eta(U|\tilde{U}^{X_1,X_2})dx + \int_{\mathbb{R}}(a_i)_x^{X_i}\big(p(v)-p(\tilde{v}^{X_1,X_2})\big)\big(h-\tilde{h}^{X_1,X_2}\big)dx\right),
$$

and

$$I_2 = -\sum_{i=1}^{2}\int_{\mathbb{R}} a^{X_1,X_2}(\tilde{h}_i)_x^{X_i} p(v|\tilde{v}^{X_1,X_2})dx.$$

By integration by parts, we have

$$\begin{aligned}
I_3 &= \int_{\mathbb{R}} a^{X_1,X_2}\big(p(v)-p(\tilde{v}^{X_1,X_2})\big)\partial_x\Big(v^\beta\partial_x\big(p(v)-p(\tilde{v}^{X_1,X_2})\big)\Big)dx\\
&= -\int_{\mathbb{R}} a^{X_1,X_2}v^\beta|\partial_x\big(p(v)-p(\tilde{v}^{X_1,X_2})\big)|^2dx\\
&\quad -\sum_{i=1}^{2}\int_{\mathbb{R}}(a_i)_x^{X_i}v^\beta\big(p(v)-p(\tilde{v}^{X_1,X_2})\big)\partial_x\big(p(v)-p(\tilde{v}^{X_1,X_2})\big)dx,
\end{aligned}$$

and

$$\begin{aligned}
I_4 &= \int_{\mathbb{R}} a^{X_1,X_2}\big(p(v)-p(\tilde{v}^{X_1,X_2})\big)\partial_x\Big((v^\beta-(\tilde{v}^\beta)^{X_1,X_2})\partial_x p(\tilde{v}^{X_1,X_2})\Big)dx\\
&= -\sum_{i=1}^{2}\int_{\mathbb{R}}(a_i)_x^{X_i}\big(p(v)-p(\tilde{v}^{X_1,X_2})\big)(v^\beta-(\tilde{v}^\beta)^{X_1,X_2})\partial_x p(\tilde{v}^{X_1,X_2})dx\\
&\quad -\int_{\mathbb{R}} a^{X_1,X_2}\partial_x\big(p(v)-p(\tilde{v}^{X_1,X_2})\big)(v^\beta-(\tilde{v}^\beta)^{X_1,X_2})\partial_x p(\tilde{v}^{X_1,X_2})dx.
\end{aligned}$$

It follows from (5.11) and (5.9) that

$$\begin{aligned}
I_5 &= \int_{\mathbb{R}} a^{X_1,X_2}(\nabla\eta)(U|\tilde{U}^{X_1,X_2})\Bigg[\sum_{i=1}^{2}\Big(-\sigma_i(\tilde{U}_i)_x^{X_i}+\partial_x A(\tilde{U}_i^{X_i})\\
&\quad -\partial_x\Big(M(\tilde{U}_i^{X_i})\partial_x\nabla\eta(\tilde{U}_i^{X_i})\Big)\Big)+\partial_x\Big(M(\tilde{U}^{X_1,X_2})\partial_x\nabla\eta(\tilde{U}^{X_1,X_2})\Big)\Bigg]dx\\
&= \sum_{i=1}^{2}\left[\sigma_i\int_{\mathbb{R}} a^{X_1,X_2}(\tilde{v}_i)_x^{X_i}p(v|\tilde{v}^{X_1,X_2})dx+\int_{\mathbb{R}} a^{X_1,X_2}(\tilde{h}_i)_x^{X_i}p(v|\tilde{v}^{X_1,X_2})dx\right]\\
&\quad +\int_{\mathbb{R}} a^{X_1,X_2}p(v|\tilde{v}^{X_1,X_2})\tilde{E}_1dx.
\end{aligned}$$

Since

$$\nabla^2\eta(\tilde{U}^{X_1,X_2}) = \begin{pmatrix} -p'(\tilde{v}^{X_1,X_2}) & 0\\ 0 & 1\end{pmatrix}, \tag{5.19}$$

we have

$$I_6 = \int_{\mathbb{R}} a^{X_1,X_2}p'(\tilde{v}^{X_1,X_2})(v-\tilde{v}^{X_1,X_2})\tilde{E}_1dx-\int_{\mathbb{R}} a^{X_1,X_2}(h-\tilde{h}^{X_1,X_2})\tilde{E}_2dx.$$

Thus we have some cancellation

$$\begin{aligned}
I_2+I_5+I_6 &= \sum_{i=1}^{2}\left[\sigma_i\int_{\mathbb{R}} a^{X_1,X_2}(\tilde{v}_i)_x^{X_i}p(v|\tilde{v}^{X_1,X_2})dx\right]\\
&\quad +\int_{\mathbb{R}} a^{X_1,X_2}\big(p(v)-p(\tilde{v}^{X_1,X_2})\big)\tilde{E}_1dx-\int_{\mathbb{R}} a^{X_1,X_2}(h-\tilde{h}^{X_1,X_2})\tilde{E}_2dx.
\end{aligned}$$

Therefore, we have the desired representation. □

5.3. **Construction of the weight function.** We define the weight function a by

$$
\begin{aligned}
&a(t,x) = a_1(x-\sigma_1 t) + a_2(x-\sigma_2 t) - 1,\\
&a_1(x-\sigma_1 t) := 1 - \lambda\frac{p(\tilde v_1(x-\sigma_1 t)) - p(v_-)}{\varepsilon_1},\\
&a_2(x-\sigma_2 t) := 1 - \lambda\frac{p(\tilde v_2(x-\sigma_2 t)) - p(v_+)}{\varepsilon_2}.
\end{aligned}
\tag{5.20}
$$

That is,

$$
a(t,x) = 1 - \lambda\frac{p(\tilde v_1(x-\sigma_1 t)) - p(v_-)}{\varepsilon_1} - \lambda\frac{p(\tilde v_2(x-\sigma_2 t)) - p(v_+)}{\varepsilon_2}.
$$

We briefly present some useful properties on the weight a.
First of all, a_1 decreases from 1 to $1-\lambda$ on $\mathbb{R}$, but a_2 increases from $1-\lambda$ to 1 on $\mathbb{R}$. Thus, the weight function a satisfies $1-\lambda \le a \le 1$, and

$$
\int_{\mathbb{R}} |(a_i)_x| dx = \lambda, \quad \text{for } i = 1, 2. \tag{5.21}
$$

Note that for $\varepsilon_1, \varepsilon_2 < \delta_0 \ll 1$, $p'(v_*/2) \le p'(\tilde v_i) \le p'(2v_*) < 0$ for all $i = 1, 2$. Thus, for each $i = 1, 2$, $\sigma_i (a_i)_x > 0$, and

$$
(a_i)_x = -\frac{\lambda}{\varepsilon_i} p'(\tilde v_i)(\tilde v_i)_x, \tag{5.22}
$$

we have

$$
C^{-1}\frac{\lambda}{\varepsilon_i}|(\tilde v_i)_x| \le |(a_i)_x| \le C\frac{\lambda}{\varepsilon_i}|(\tilde v_i)_x|. \tag{5.23}
$$

5.4. **Maximization in terms of $h - \tilde h$.** In order to estimate the right-hand side of (5.17), we will use Proposition 6.1, i.e., a sharp estimate with respect to $p(v) - p(\tilde v)$ near $p(\tilde v)$. For that, we need to rewrite $\mathcal{J}^{bad}$ on the right-hand side of (5.17) only in terms of $p(v)$ near $p(\tilde v)$, by separating $h - \tilde h$ from the first term of $\mathcal{J}^{bad}$. Therefore, we will rewrite $\mathcal{J}^{bad}$ into the maximized representation in terms of $h - \tilde h$ in the following lemma. However, we will keep all terms of $\mathcal{J}^{bad}$ in a region $\{p(v) - p(\tilde v) > \delta\}$ for small values of v.

Lemma 5.3. *Let $\tilde U^{X_1,X_2}$ and a^{X_1,X_2} as in (5.15) and (5.16). Let δ be any positive constant. Then, for any $U \in \mathcal{H}_T$,*

$$
\mathcal{J}^{bad}(U) - \mathcal{J}^{good}(U) = B_\delta(U) - G_\delta(U), \tag{5.24}
$$

where

$$
\begin{aligned}
B_\delta(U) :=& \sum_{i=1}^2 \Big[\sigma_i \int_{\mathbb{R}} a^{X_1,X_2} (\tilde v_i)_x^{X_i} p(v|\tilde v^{X_1,X_2}) dx \\
&+ \frac{1}{2\sigma_i}\int_{\mathbb{R}} (a_i)_x^{X_i} |p(v)-p(\tilde v^{X_1,X_2})|^2 \mathbf{1}_{\{p(v)-p(\tilde v^{X_1,X_2})\le\delta\}} dx \\
&+ \int_{\mathbb{R}} (a_i)_x^{X_i} \big(p(v)-p(\tilde v^{X_1,X_2})\big)\big(h-\tilde h^{X_1,X_2}\big) \mathbf{1}_{\{p(v)-p(\tilde v^{X_1,X_2})>\delta\}} dx \\
&- \int_{\mathbb{R}} (a_i)_x^{X_i} v^\beta \big(p(v)-p(\tilde v^{X_1,X_2})\big)\partial_x\big(p(v)-p(\tilde v^{X_1,X_2})\big) dx \\
&- \int_{\mathbb{R}} (a_i)_x^{X_i} \big(p(v)-p(\tilde v^{X_1,X_2})\big)(v^\beta-(\tilde v^\beta)^{X_1,X_2})\partial_x p(\tilde v^{X_1,X_2}) dx\Big] \\
&- \int_{\mathbb{R}} a^{X_1,X_2} \partial_x\big(p(v)-p(\tilde v^{X_1,X_2})\big)(v^\beta-(\tilde v^\beta)^{X_1,X_2})\partial_x p(\tilde v^{X_1,X_2}) dx \\
&+ \int_{\mathbb{R}} a^{X_1,X_2}\big(p(v)-p(\tilde v^{X_1,X_2})\big)\tilde E_1 dx - \int_{\mathbb{R}} a^{X_1,X_2}(h-\tilde h^{X_1,X_2})\tilde E_2 dx,
\end{aligned} \tag{5.25}
$$

and

$$
\begin{aligned}
G_\delta(U) :=& \sum_{i=1}^2 \bigg(\frac{\sigma_i}{2}\int_{\mathbb{R}} (a_i)_x^{X_i} \left| h-\tilde h^{X_1,X_2} - \frac{p(v)-p(\tilde v^{X_1,X_2})}{\sigma_i}\right|^2 \mathbf{1}_{\{p(v)-p(\tilde v^{X_1,X_2})\le\delta\}} dx \\
&+ \frac{\sigma_i}{2}\int_{\mathbb{R}} (a_i)_x^{X_i} \left| h-\tilde h^{X_1,X_2}\right|^2 \mathbf{1}_{\{p(v)-p(\tilde v^{X_1,X_2})>\delta\}} dx + \sigma_i \int_{\mathbb{R}} (a_i)_x^{X_i} Q(v|\tilde v^{X_1,X_2}) dx\bigg) \\
&+ \int_{\mathbb{R}} a^{X_1,X_2} v^\beta |\partial_x\big(p(v)-p(\tilde v^{X_1,X_2})\big)|^2 dx,
\end{aligned} \tag{5.26}
$$

Remark 5.3. *Since $\sigma_i(a_i)_x > 0$ and $a > 0$, $-G_\delta$ consists of good terms.*

Proof. For a given $\delta > 0$, we split the first terms of $\mathcal{J}^{bad}$ and $-\mathcal{J}^{good}$ as follows:

$$
\begin{aligned}
&\sum_{i=1}^2 \int_{\mathbb{R}} (a_i)_x^{X_i}\big(p(v)-p(\tilde v^{X_1,X_2})\big)\big(h-\tilde h^{X_1,X_2}\big) dx \\
&= \sum_{i=1}^2 \Big[\underbrace{\int_{\mathbb{R}} (a_i)_x^{X_i}\big(p(v)-p(\tilde v^{X_1,X_2})\big)\big(h-\tilde h^{X_1,X_2}\big)\mathbf{1}_{\{p(v)-p(\tilde v^{X_1,X_2})\le\delta\}} dx}_{=:J_i} \\
&\quad + \int_{\mathbb{R}} (a_i)_x^{X_i}\big(p(v)-p(\tilde v^{X_1,X_2})\big)\big(h-\tilde h^{X_1,X_2}\big)\mathbf{1}_{\{p(v)-p(\tilde v^{X_1,X_2})>\delta\}} dx\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
&-\sum_{i=1}^{2}\frac{\sigma_i}{2}\int_{\mathbb{R}}(a_i)_x^{X_i}\left|h-\tilde{h}^{X_1,X_2}\right|^2dx\\
&=-\sum_{i=1}^{2}\Bigg[\underbrace{\sum_{i=1}^{2}\frac{\sigma_i}{2}\int_{\mathbb{R}}(a_i)_x^{X_i}\left|h-\tilde{h}^{X_1,X_2}\right|^2\mathbf{1}_{\{p(v)-p(\tilde{v}^{X_1,X_2})\le\delta\}}dx}_{=:K_i}\\
&\quad-\sum_{i=1}^{2}\frac{\sigma_i}{2}\int_{\mathbb{R}}(a_i)_x^{X_i}\left|h-\tilde{h}^{X_1,X_2}\right|^2\mathbf{1}_{\{p(v)-p(\tilde{v}^{X_1,X_2})>\delta\}}dx\Bigg]
\end{aligned}
$$

Applying the quadratic identity $\alpha x^2+\beta x=\alpha(x+\frac{\beta}{2\alpha})^2-\frac{\beta^2}{4\alpha}$ with $x:=h-\tilde{h}^{X_1,X_2}$ to the integrands of J_i+K_i, we find

$$
\begin{aligned}
&-\frac{\sigma_i}{2}\left|h-\tilde{h}^{X_1,X_2}\right|^2+\big(p(v)-p(\tilde{v}^{X_1,X_2})\big)\big(h-\tilde{h}^{X_1,X_2}\big)\\
&=-\frac{\sigma_i}{2}\left|h-\tilde{h}^{X_1,X_2}-\frac{p(v)-p(\tilde{v}^{X_1,X_2})}{\sigma_i}\right|^2+\frac{1}{2\sigma_i}|p(v)-p(\tilde{v}^{X_1,X_2})|^2.
\end{aligned}
$$

Therefore, we have the desired representation. □

5.5. **Construction of shifts.** For a given $\varepsilon>0$, we consider a continuous function Φ_ε defined by

$$
\Phi_\varepsilon(y)=\begin{cases}0, & \text{if } y\le 0,\\ -\frac{1}{\varepsilon^4}y, & \text{if } 0\le y\le\varepsilon^2,\\ -\frac{1}{\varepsilon^2}, & \text{if } y\ge\varepsilon^2.\end{cases}\tag{5.27}
$$

$$
\Psi_\varepsilon(y)=\begin{cases}1, & \text{if } y\le -\varepsilon^2,\\ -\frac{1}{\varepsilon^2}y, & \text{if } -\varepsilon^2\le y\le 0,\\ 0, & \text{if } y\ge 0.\end{cases}\tag{5.28}
$$

For any fixed $\varepsilon_1,\varepsilon_2>0$, and $U\in\mathcal{H}_T$, we define a pair of shift functions $\binom{X_1}{X_2}$ as a solution to the system of nonlinear ODEs:

$$
\begin{cases}\dot{X}_1(t)=\Phi_{\varepsilon_1}(Y_1(U))\Big(2|\mathcal{J}^{bad}(U)|+1\Big)-\frac{\sigma_1}{2}\Psi_{\varepsilon_1}(Y_1(U)),\\ \dot{X}_2(t)=-\Phi_{\varepsilon_2}(-Y_2(U))\Big(2|\mathcal{J}^{bad}(U)|+1\Big)-\frac{\sigma_2}{2}\Psi_{\varepsilon_2}(-Y_2(U)),\\ X_1(0)=X_2(0)=0,\end{cases}\tag{5.29}
$$

where Y_1,Y_2 and $\mathcal{J}^{bad}$ are as in (5.18).
Then, it is shown in Appendix C that the system (5.29) has a unique absolutely continuous solution $\binom{X_1}{X_2}$ on $[0,T]$.

Since it follows from (5.29) that for each $i=1,2$,

$$
\dot{X}_i(t)=\begin{cases}(-1)^i\varepsilon_i^{-2}\big(2|\mathcal{J}^{bad}(U)|+1\big), & \text{if } (-1)^{i-1}Y_i(U)\ge\varepsilon_i^2,\\ -\varepsilon_i^{-4}Y_i(U)\big(2|\mathcal{J}^{bad}(U)|+1\big), & \text{if } 0\le(-1)^{i-1}Y_i(U)\le\varepsilon_i^2,\\ (-1)^{i-1}\frac{1}{2}\sigma_i\varepsilon_i^{-2}Y_i(U), & \text{if } -\varepsilon_i^2\le(-1)^{i-1}Y_i(U)\le 0,\\ -\frac{1}{2}\sigma_i, & \text{if } (-1)^{i-1}Y_i(U)\le-\varepsilon_i^2,\end{cases}\tag{5.30}
$$

the shifts satisfy the bounds:

$$\dot{X}_1(t) \leq -\frac{\sigma_1}{2}, \quad \dot{X}_2(t) \geq -\frac{\sigma_2}{2}, \quad \forall t > 0.$$

Thus,

$$X_1(t) \leq -\frac{\sigma_1}{2}t, \quad X_2(t) \geq -\frac{\sigma_2}{2}t, \quad \forall t > 0,$$

which gives (4.8).
Especially, we have

$$X_1(t) + \sigma_1 t \leq \frac{\sigma_1}{2}t < 0, \quad X_2(t) + \sigma_2 t \geq \frac{\sigma_2}{2}t > 0, \quad \forall t > 0. \tag{5.31}$$

5.6. **Main proposition.** The main proposition for the proof of Theorem 4.1 is the following.

Proposition 5.1. *For the given constant $U_* := (v_*, u_*) \in \mathbb{R}^+ \times \mathbb{R}$, there exist positive constants $\delta_0, \delta_1 \in (0, 1/2)$ such that the following holds. Let $\tilde{U}^{X_1,X_2}$ be the composite wave for the given constant states $U_-, U_m, U_+ \in B_{\delta_0}(U_*)$, where $\varepsilon_1 = |p(v_-) - p(v_m)|$ and $\varepsilon_2 = |p(v_m) - p(v_+)|$. Then, for any $\lambda > 0$ with $\varepsilon_1/\lambda, \varepsilon_2/\lambda < \delta_0$ and $\lambda < \delta_0$, there exist positive constants C_{ε,δ_0} and t_0 such that the following holds.*
For any $U \in \mathcal{H}_T$, let $\binom{X_1}{X_2}$ be the solution to (5.29). Then, for all $U \in \mathcal{H}_T$ satisfying $Y_1(U) \leq \varepsilon_1^2$ and $Y_2(U) \geq -\varepsilon_2^2$,

(5.32)

$$\begin{aligned}
\mathcal{R}(U) :=& -\frac{1}{\varepsilon_1^4}|Y_1(U)|^2 \mathbf{1}_{\{0 \leq Y_1(U) \leq \varepsilon_1^2\}} + \frac{\sigma_1}{2\varepsilon_1^2}|Y_1(U)|^2 \mathbf{1}_{\{-\varepsilon_1^2 \leq Y_1(U) \leq 0\}} - \frac{\sigma_1}{2} Y_1(U) \mathbf{1}_{\{Y_1(U) \leq -\varepsilon_1^2\}} \\
& - \frac{1}{\varepsilon_2^4}|Y_2(U)|^2 \mathbf{1}_{\{-\varepsilon_2^2 \leq Y_2(U) \leq 0\}} - \frac{\sigma_2}{2\varepsilon_2^2}|Y_2(U)|^2 \mathbf{1}_{\{0 \leq Y_2(U) \leq \varepsilon_2^2\}} - \frac{\sigma_2}{2} Y_2(U) \mathbf{1}_{\{Y_2(U) \geq \varepsilon_2^2\}} \\
& + B_{\delta_1}(U) + \delta_0 \frac{\min(\varepsilon_1, \varepsilon_2)}{\lambda} |B_{\delta_1}(U)| - G_{11}^-(U) - G_{11}^+(U) - G_{12}^-(U) - G_{12}^+(U) \\
& - \left(1 - \delta_0 \frac{\varepsilon_1}{\lambda}\right) G_{21}(U) - \left(1 - \delta_0 \frac{\varepsilon_2}{\lambda}\right) G_{22}(U) - (1 - \delta_0) D(U) \\
\leq& \, C_{\varepsilon,\delta_0} \left[\frac{1}{t^2} \int_{\mathbb{R}} \eta(U|\tilde{U}) dx + \left(\exp\big(-C_\varepsilon t\big) + \frac{1}{t^4}\right)\right] \mathbf{1}_{t \geq t_0} + C \mathbf{1}_{t \leq t_0},
\end{aligned}$$

where Y_i and B_{δ_1} are as in (5.18) and (5.25), and $G_{11}^-, G_{11}^+, G_{12}^-, G_{12}^+, G_{21}, G_{22}, D$ denote the good terms of G_{δ_1} in (5.26) as follows: for each $i = 1, 2$,

$$\begin{aligned}
G_{1i}^-(U) &:= \frac{\sigma_i}{2} \int_{\mathbb{R}} (a_i)_x^{X_i} \left| h - \tilde{h}^{X_1,X_2} \right|^2 \mathbf{1}_{\{p(v) - p(\tilde{v}^{X_1,X_2}) > \delta_1\}} dx, \\
G_{1i}^+(U) &:= \frac{\sigma_i}{2} \int_{\mathbb{R}} (a_i)_x^{X_i} \left| h - \tilde{h}^{X_1,X_2} - \frac{p(v) - p(\tilde{v}^{X_1,X_2})}{\sigma_i} \right|^2 \mathbf{1}_{\{p(v) - p(\tilde{v}^{X_1,X_2}) \leq \delta_1\}} dx, \\
G_{2i}(U) &:= \sigma_i \int_{\mathbb{R}} (a_i)_x^{X_i} Q(v|\tilde{v}^{X_1,X_2}) dx, \\
D(U) &:= \int_{\mathbb{R}} a^{X_1,X_2} v^\beta |\partial_x \big(p(v) - p(\tilde{v}^{X_1,X_2})\big)|^2 dx.
\end{aligned} \tag{5.33}$$

5.7. **Proof of Theorem 4.1 from the main Proposition.** We here show how Proposition 5.1 implies Theorem 4.1.

To prove the contraction estimate (4.7), following (5.17), we may estimate

$$\mathcal{F}(U) := \sum_{i=1}^{2} \big(\dot{X}_i(t) Y_i(U)\big) + \mathcal{J}^{bad}(U) - \mathcal{J}^{good}(U). \tag{5.34}$$

First, it follows from (5.30) that for each $i = 1, 2$,

$$\dot{X}_i(t) Y_i(U) \le \begin{cases} -2|\mathcal{J}^{bad}(U)|, & \text{if } (-1)^{i-1} Y_i(U) \ge \varepsilon_i^2, \\ -\varepsilon_i^{-4} |Y_i(U)|^2, & \text{if } 0 \le (-1)^{i-1} Y_i(U) \le \varepsilon_i^2, \\ (-1)^{i-1} \frac{1}{2} \sigma_i \varepsilon_i^{-2} |Y_i(U)|^2, & \text{if } -\varepsilon_i^2 \le (-1)^{i-1} Y_i(U) \le 0, \\ -\frac{1}{2} \sigma_i Y_i(U), & \text{if } (-1)^{i-1} Y_i(U) \le -\varepsilon_i^2. \end{cases}$$

Then we first find that for all $U \in \mathcal{H}_T$ satisfying $Y_1(U) \ge \varepsilon_1^2$ or $Y_2(U) \le -\varepsilon_2^2$,

$$\mathcal{F}(U) \le -|\mathcal{J}^{bad}(U)| - \mathcal{J}^{good}(U) \le 0.$$

Since (5.24) with $\delta = \delta_1$ yields that for all $U \in \mathcal{H}_T$ satisfying $Y_1(U) \le \varepsilon_1^2$ and $Y_2(U) \ge -\varepsilon_2^2$,

$$\begin{aligned} \mathcal{F}(U) \le & -\frac{1}{\varepsilon_1^4} |Y_1(U)|^2 \mathbf{1}_{\{0 \le Y_1(U) \le \varepsilon_1^2\}} + \frac{\sigma_1}{2\varepsilon_1^2} |Y_1(U)|^2 \mathbf{1}_{\{-\varepsilon_1^2 \le Y_1(U) \le 0\}} - \frac{\sigma_1}{2} Y_1(U) \mathbf{1}_{\{Y_1(U) \le -\varepsilon_1^2\}} \\ & - \frac{1}{\varepsilon_2^4} |Y_2(U)|^2 \mathbf{1}_{\{-\varepsilon_2^2 \le Y_2(U) \le 0\}} - \frac{\sigma_2}{2\varepsilon_2^2} |Y_2(U)|^2 \mathbf{1}_{\{0 \le Y_2(U) \le \varepsilon_2^2\}} - \frac{\sigma_2}{2} Y_2(U) \mathbf{1}_{\{Y_2(U) \ge \varepsilon_2^2\}} \\ & + B_{\delta_1}(U) - G_{\delta_1}(U), \end{aligned}$$

Proposition 5.1 implies that for all $U \in \mathcal{H}_T$ satisfying $Y_1(U) \le \varepsilon_1^2$ and $Y_2(U) \ge -\varepsilon_2^2$,

$$\begin{aligned} & \mathcal{F}(U) + \delta_0 \frac{\min(\varepsilon_1, \varepsilon_2)}{\lambda} |B_{\delta_1}(U)| + \delta_0 \frac{\varepsilon_1}{\lambda} G_{21}(U) + \delta_0 \frac{\varepsilon_2}{\lambda} G_{22}(U) - \delta_0 D(U) \\ & \le C_{\varepsilon,\delta_0} \left[\frac{1}{t^2} \int_{\mathbb{R}} \eta(U|\tilde{U}) dx + \left(\exp\big(-C_\varepsilon t\big) + \frac{1}{t^4} \right) \right] \mathbf{1}_{t \ge t_0} + C \mathbf{1}_{t \le t_0}. \end{aligned}$$

Thus, using the above estimates together with $\varepsilon_i/\lambda < \delta_0 < 1$ and the definition of $\mathcal{J}^{good}$, we find that for all $U \in \mathcal{H}_T$,

$$\begin{aligned} & \mathcal{F}(U) + \delta_0 \frac{\varepsilon_1}{\lambda} G_{21}(U) + \delta_0 \frac{\varepsilon_2}{\lambda} G_{22}(U) + \delta_0 D(U) \\ & \quad + |\mathcal{J}^{bad}(U)| \mathbf{1}_{\{Y_1(U) \ge \varepsilon_1^2\} \cup \{Y_2(U) \le -\varepsilon_1^2\}} + \delta_0 \frac{\min(\varepsilon_1, \varepsilon_2)}{\lambda} |B_{\delta_1}(U)| \mathbf{1}_{\{Y_1(U) \le \varepsilon_1^2\} \cap \{Y_2(U) \ge -\varepsilon_1^2\}} \\ & \le C_{\varepsilon,\delta_0} \left[\frac{1}{t^2} \int_{\mathbb{R}} \eta(U|\tilde{U}) dx + \left(\exp\big(-C_\varepsilon t\big) + \frac{1}{t^4} \right) \right] \mathbf{1}_{t \ge t_0} + C \mathbf{1}_{t \le t_0}. \end{aligned}$$

This together with (5.17), (5.34) and $1/2 \le a^{X_1,X_2} \le 1$ implies that for a.e. $t > 0$,

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} a^{X_1,X_2}\eta(U|\tilde{U}^{X_1,X_2})dx + \delta_0\frac{\varepsilon_1}{\lambda}G_{21}(U) + \delta_0\frac{\varepsilon_2}{\lambda}G_{22}(U) + \delta_0 D(U)\\
&\quad + |\mathcal{J}^{bad}(U)|\mathbf{1}_{\{Y_1(U)\ge\varepsilon_1^2\}\cup\{Y_2(U)\le-\varepsilon_1^2\}} + \delta_0\frac{\min(\varepsilon_1,\varepsilon_2)}{\lambda}|B_{\delta_1}(U)|\mathbf{1}_{\{Y_1(U)\le\varepsilon_1^2\}\cap\{Y_2(U)\ge-\varepsilon_1^2\}}\\
&= \mathcal{F}(U) + \delta_0\frac{\varepsilon_1}{\lambda}G_{21}(U) + \delta_0\frac{\varepsilon_2}{\lambda}G_{22}(U) + \delta_0 D(U)\\
&\quad + |\mathcal{J}^{bad}(U)|\mathbf{1}_{\{Y_1(U)\ge\varepsilon_1^2\}\cup\{Y_2(U)\le-\varepsilon_1^2\}} + \delta_0\frac{\min(\varepsilon_1,\varepsilon_2)}{\lambda}|B_{\delta_1}(U)|\mathbf{1}_{\{Y_1(U)\le\varepsilon_1^2\}\cap\{Y_2(U)\ge-\varepsilon_1^2\}}\\
&\le C_{\varepsilon,\delta_0}\left[\frac{1}{t^2}\int_{\mathbb{R}}\eta(U|\tilde{U})dx + \left(\exp\big(-C_\varepsilon t\big) + \frac{1}{t^4}\right)\right]\mathbf{1}_{t\ge t_0} + C\mathbf{1}_{t\le t_0}.
\end{aligned}
$$

Since t^{-2} and $\exp\big(-C_\varepsilon t\big) + t^{-4}$ are integrable on $[t_0,\infty)$, Grönwall's inequality implies that there exists a positive constant $C(\varepsilon_1,\varepsilon_2,\lambda,\delta_0)$ such that

(5.35)
$$
\begin{aligned}
&\int_{\mathbb{R}} a^{X_1,X_2}\eta(U^X|\tilde{U}^{X_1,X_2})dx + \int_0^t G_{21}(U)ds + \int_0^t G_{22}(U)ds + \delta_0\int_0^t D(U)ds\\
&\quad + \int_0^t |\mathcal{J}^{bad}(U)|\mathbf{1}_{\{Y_1(U)\ge\varepsilon_1^2\}\cup\{Y_2(U)\le-\varepsilon_1^2\}}ds + \int_0^t |B_{\delta_1}(U)|\mathbf{1}_{\{Y_1(U)\le\varepsilon_1^2\}\cap\{Y_2(U)\ge-\varepsilon_1^2\}}ds\\
&\le C(\varepsilon_1,\varepsilon_2,\lambda,\delta_0)\left[\int_{\mathbb{R}} a(0,x)\eta(U_0(x)|\tilde{U}(0,x))dx + 1\right],
\end{aligned}
$$

which completes (4.7).

To estimate $|\dot{X}_i|$, we first observe that (5.30) implies that for each $i = 1, 2$,

$$
|\dot{X}_i(t)| \le \max\left(\frac{1}{\varepsilon_i^2}(2|\mathcal{J}^{bad}(U)|+1), \frac{|\sigma_i|}{2}\right), \qquad \text{for a.e. } t\in(0,T). \tag{5.36}
$$

Notice that it follows from (5.35) that

$$
\int_0^T f(t)dt \le C(\varepsilon_1,\varepsilon_2,\lambda,\delta_0)\int_{\mathbb{R}}\eta(U_0|\tilde{U}(0,x))dx, \tag{5.37}
$$

where

$$
f(t) := |\mathcal{J}^{bad}(U)|\mathbf{1}_{\{Y_1(U)\ge\varepsilon_1^2\}\cup\{Y_2(U)\le-\varepsilon_1^2\}} + |\mathcal{B}_{\delta_1}(U)|\mathbf{1}_{\{Y_1(U)\le\varepsilon_1^2\}\cap\{Y_2(U)\ge-\varepsilon_1^2\}}.
$$

To estimate $|\mathcal{J}^{bad}(U)|$ globally in time, using (5.24) with the definitions of $\mathcal{J}^{good}$ and G_{δ_1}, we find that

$$
\begin{aligned}
&|\mathcal{J}^{bad}(U)|\\
&\le |\mathcal{J}^{bad}(U)|\mathbf{1}_{\{Y_1(U)\ge\varepsilon_1^2\}\cup\{Y_2(U)\le-\varepsilon_1^2\}} + |\mathcal{J}^{bad}(U)|\mathbf{1}_{\{Y_1(U)\le\varepsilon_1^2\}\cap\{Y_2(U)\ge-\varepsilon_1^2\}}\\
&= |\mathcal{J}^{bad}(U)|\mathbf{1}_{\{Y_1(U)\ge\varepsilon_1^2\}\cup\{Y_2(U)\le-\varepsilon_1^2\}} + |\mathcal{J}^{good}(U)+B_{\delta_1}(U)-G_{\delta_1}(U)|\mathbf{1}_{\{Y_1(U)\le\varepsilon_1^2\}\cap\{Y_2(U)\ge-\varepsilon_1^2\}}\\
&\le |\mathcal{J}^{bad}(U)|\mathbf{1}_{\{Y_1(U)\ge\varepsilon_1^2\}\cup\{Y_2(U)\le-\varepsilon_1^2\}} + |\mathcal{B}_{\delta_1}(U)|\mathbf{1}_{\{Y_1(U)\le\varepsilon_1^2\}\cap\{Y_2(U)\ge-\varepsilon_1^2\}}\\
&+\frac{|\sigma_i|}{2}\int_{\mathbb{R}}|(a_i)_x^{X_i}|\left|\left(h-\tilde{h}^{X_1,X_2}\right)^2-\left(h-\tilde{h}^{X_1,X_2}-\frac{p(v)-p(\tilde{v}^{X_1,X_2})}{\sigma_i}\right)^2\right|\mathbf{1}_{\{p(v)-p(\tilde{v}^{X_1,X_2})\le\delta_1\}}dx\\
&\le f(t)+C\int_{\mathbb{R}}|(a_i)_x^{X_i}|\Big(\left(h-\tilde{h}^{X_1,X_2}\right)^2+\left(p(v)-p(\tilde{v}^{X_1,X_2})\right)^2\Big)\mathbf{1}_{\{p(v)-p(\tilde{v}^{X_1,X_2})\le\delta_1\}}dx.
\end{aligned}
$$

Since for any v satisfying $p(v)-p(\tilde{v})\le\delta_1$, there exists a positive constant c_* such that $v>c_*^{-1}$ and $|p(v)-p(\tilde{v})|\le c_*$, we use (A.4) and (A.1) to have

$$
\begin{aligned}
&\int_{\mathbb{R}}|(a_i)_x^{X_i}|\left(p(v)-p(\tilde{v}^{X_1,X_2})\right)^2\mathbf{1}_{\{p(v)-p(\tilde{v}^{X_1,X_2})\le\delta_1\}}dx\\
&\quad\le c_*\int_{v>c_*^{-1}}\left|p(v)-p(\tilde{v}^{X_1,X_2})\right|\mathbf{1}_{\{v\ge 3v_-\}}dx+\int_{v>c_*^{-1}}|(a_i)_x^{X_i}|\left|p(v)-p(\tilde{v}^{X_1,X_2})\right|^2\mathbf{1}_{\{v\le 3v_-\}}dx\\
&\quad\le C\int_{v>c_*^{-1}}|(a_i)_x^{X_i}|\Big(|v-\tilde{v}^{X_1,X_2}|\mathbf{1}_{\{v\ge 3v_-\}}+|v-\tilde{v}^{X_1,X_2}|^2\mathbf{1}_{\{v\le 3v_-\}}\Big)dx\\
&\quad\le C\int_{\mathbb{R}}|(a_i)_x^{X_i}|Q(v|\tilde{v}^{X_1,X_2})dx.
\end{aligned}
$$

Therefore, using $|(a_i)_x^{X_i}|\le C\delta_0$ and $\delta_0\le\frac{1}{2}\le a^{X_1,X_2}$, we have

$$|\mathcal{J}^{bad}(U)|\le f(t)+C\int_{\mathbb{R}}a^{X_1,X_2}\eta(U|\tilde{U}^{X_1,X_2})dx,$$

which together with (5.35) and (8.4) implies that

$$|\dot{X}_i(t)|\le C(\varepsilon_1,\varepsilon_2,\lambda,\delta_0)\left[f(t)+\int_{\mathbb{R}}\eta(U_0(x)|\tilde{U}(0,x))dx+1\right].$$

This and (5.37) give (4.9).

6. Abstract propositions for a general setting

This section provides some useful propositions that will be all used in Section 7 for the proof of Proposition 5.1. The propositions are generalizations of [26, Propositions 4.2 and 4.3 and Lemmas 4.4-4.8]. Those are stated below in an abstract setting for future application in a general context, for example, for studies on various composite waves.

6.1. **Sharp estimate near a shock wave.** The following proposition is a generalization of [26, Proposition 4.2].

Proposition 6.1. *For any constant $C_2>0$ and any constant $U_*:=(v_*,u_*)\in\mathbb{R}^+\times\mathbb{R}$, there exists $\delta_1>0$ such that for any $\varepsilon>0$ and any $\lambda,\delta\in(0,\delta_1]$ satisfying $\varepsilon/\lambda\le\delta_1$, the following holds.*
Let $U_l:=(v_l,u_l)$ and $U_r:=(v_r,u_r)$ be any constants satisfying $U_l,U_r\in B_{\delta_1}(U_*)$ and

$\varepsilon = |p(v_l)-p(v_r)|$, and one of the two conditions (1.7) with the velocity σ_0. Let $\tilde{U}_0 := (\tilde{v}_0, \tilde{h}_0)$ be the viscous shock as a solution to the equation (5.11) connecting the left end state U_l and the right end state U_r.
Let a_0 be a function such that

$$\partial_x a_0 = -\lambda \frac{\partial_x p(\tilde{v}_0)}{\varepsilon}.$$

Let a be any positive function such that $\|a-1\|_{L^\infty(\mathbb{R})} \le 2\lambda$.
Let ϕ be any Lipschitz function, and let

$$\begin{aligned}
\mathcal{Y}^g(v) &:= -\frac{1}{2\sigma_0^2}\int_{\mathbb{R}} (a_0)_x \phi^2(x)|p(v)-p(\tilde{v}_0)|^2 dx - \int_{\mathbb{R}} (a_0)_x \phi^2(x) Q(v|\tilde{v}_0) dx\\
&\qquad - \int_{\mathbb{R}} a\partial_x p(\tilde{v}_0)\phi(x)(v-\tilde{v}_0)dx + \frac{1}{\sigma_0}\int_{\mathbb{R}} a(\tilde{h}_0)_x \phi(x)\big(p(v)-p(\tilde{v}_0)\big)dx,\\
\mathcal{I}_1(v) &:= \sigma_0 \int_{\mathbb{R}} a(\tilde{v}_0)_x \phi^2(x) p(v|\tilde{v}_0) dx,\\
\mathcal{I}_2(v) &:= \frac{1}{2\sigma_0}\int_{\mathbb{R}} (a_0)_x \phi^2(x) |p(v)-p(\tilde{v}_0)|^2 dx,\\
\mathcal{G}_2(v) &:= \sigma_0 \int_{\mathbb{R}} (a_0)_x \bigg(\frac{1}{2\gamma} p(\tilde{v}_0)^{-\frac{1}{\gamma}-1}\phi^2(x)\big(p(v)-p(\tilde{v}_0)\big)^2\\
&\qquad - \frac{1+\gamma}{3\gamma^2} p(\tilde{v}_0)^{-\frac{1}{\gamma}-2}\phi^3(x)\big(p(v)-p(\tilde{v}_0)\big)^3\bigg) dx,\\
\mathcal{D}(v) &:= \int_{\mathbb{R}} a\, v^\beta |\partial_x\big(\phi(x)(p(v)-p(\tilde{v}_0))\big)|^2 dx,
\end{aligned} \tag{6.1}$$

For any function $v : \mathbb{R} \to \mathbb{R}^+$ such that $\mathcal{D}(v) + \mathcal{G}_2(v)$ is finite, if

$$|\mathcal{Y}^g(v)| \le C_2 \frac{\varepsilon^2}{\lambda}, \qquad \|p(v)-p(\tilde{v}_0)\|_{L^\infty(\mathbb{R})} \le 2\delta_1, \tag{6.2}$$

then

$$\begin{aligned}
\mathcal{R}_\delta(v) &:= -\frac{1}{\varepsilon\delta}|\mathcal{Y}^g(v)|^2 + \mathcal{I}_1(v) + \delta|\mathcal{I}_1(v)|\\
&\qquad + \mathcal{I}_2(v) + \delta\left(\frac{\varepsilon}{\lambda}\right)|\mathcal{I}_2(v)| - \left(1-\delta\left(\frac{\varepsilon}{\lambda}\right)\right)\mathcal{G}_2(v) - (1-\delta)\mathcal{D}(v)\\
&\le 0.
\end{aligned} \tag{6.3}$$

Proof. The proof is essentially the same as that of [26, Proposition 4.2 and Appendix A] through a generalization. Notice that the functionals $\mathcal{Y}^g, \mathcal{I}_1, \mathcal{I}_2, \mathcal{D}$, by putting $\phi \equiv 1$ in their integrands, are respectively the same as $Y_g, \mathcal{I}_1, \mathcal{I}_2, \mathcal{D}$ in [26, Proposition 4.2]. On the other hand, the integrand of the functional $\mathcal{G}_2$ is the same as the approximation for $Q(v|\tilde{v})$ of $\mathcal{G}_2$ in [26, Proposition 4.2] by (A.8) of Lemma A.3 together with $\phi \equiv 1$.
For completeness, the main parts of the proof are given in Appendix B. □

6.2. **Smallness of the weighted relative entropy.** The following proposition is a generalization of [26, Lemma 4.4].

Proposition 6.2. *For a given constant $U_* := (v_*, u_*) \in \mathbb{R}^+ \times \mathbb{R}$, there exist positive constants δ_0, C, C_0 such that for any $\varepsilon, \lambda > 0$ satisfying $\varepsilon/\lambda < \delta_0$ and $\lambda < \delta_0$, the following holds.*

Let $\tilde{U}_0 := (\tilde{v}_0, \tilde{h}_0) : \mathbb{R} \to \mathbb{R}^+ \times \mathbb{R}$*,* $\mathbf{w} : \mathbb{R} \to \mathbb{R}$*,* $\mathbf{v} := (\mathbf{v}_1, \mathbf{v}_2) : \mathbb{R} \to \mathbb{R}^2$ *and* $\mathbf{a} : \mathbb{R} \to \mathbb{R}$ *be any functions such that*

$$|\tilde{U}_0(x) - U_*| \le C\delta_0, \quad \tilde{v}_0(x) \ge C^{-1}, \quad |\mathbf{v}(x)| \le C\frac{\varepsilon}{\lambda}|\mathbf{w}(x)|, \quad \forall x \in \mathbb{R}, \tag{6.4}$$

$$\mathbf{w} \text{ is either positive or negative globally on } \mathbb{R}, \tag{6.5}$$

$$\mathbf{v}_1, \mathbf{v}_2 \in L^1(\mathbb{R}), \quad \int_{\mathbb{R}} |\mathbf{w}| dx = \lambda, \quad \|\mathbf{a}\|_{L^\infty(\mathbb{R})} \le 1. \tag{6.6}$$

Let $U := (v, h) : \mathbb{R} \to \mathbb{R}^+ \times \mathbb{R}$ *be any function such that* $\mathrm{sgn}(\mathbf{w})\mathbf{Y}(U) \le \varepsilon^2$*, where*

$$\mathbf{Y}(U) := \int_{\mathbb{R}} \mathbf{w}\eta(U|\tilde{U}_0)dx + \int_{\mathbb{R}} \mathbf{a}\mathbf{v}^T \nabla^2\eta(\tilde{U}_0)(U - \tilde{U}_0)dx.$$

Then,

$$\int_{\mathbb{R}} |\mathbf{w}||h - \tilde{h}_0|^2\, dx + \int_{\mathbb{R}} |\mathbf{w}| Q(v|\tilde{v}_0)\, dx \le C\frac{\varepsilon^2}{\lambda}, \tag{6.7}$$

and

$$|\mathbf{Y}(U)| \le C_0\frac{\varepsilon^2}{\lambda}. \tag{6.8}$$

Proof. • *Proof of* (6.7) : We first use (A.1) to have

$$\int_{\mathbb{R}} |\mathbf{w}|\eta(U|\tilde{U}_0) \ge \int_{\mathbb{R}} |\mathbf{w}|\frac{|h - \tilde{h}_0|^2}{2} + c_1\int_{v \le 3v_*} |\mathbf{w}||v - \tilde{v}_0|^2 + c_2\int_{v > 3v_*} |\mathbf{w}||v - \tilde{v}_0|. \tag{6.9}$$

Using (6.5) and

$$\int_{\mathbb{R}} \mathbf{w}\eta(U|\tilde{U}_0)dx = \mathbf{Y}(U) - \int_{\mathbb{R}} \mathbf{a}\mathbf{v}^T \nabla^2\eta(\tilde{U}_0)(U - \tilde{U}_0)dx,$$

we have

$$\begin{aligned}\int_{\mathbb{R}} |\mathbf{w}|\eta(U|\tilde{U}_0)dx &= \mathrm{sgn}(\mathbf{w})\int_{\mathbb{R}} \mathbf{w}\eta(U|\tilde{U}_0)dx \\ &= \mathrm{sgn}(\mathbf{w})\mathbf{Y}(U) - \mathrm{sgn}(\mathbf{w})\int_{\mathbb{R}} \mathbf{a}\mathbf{v}^T \nabla^2\eta(\tilde{U}_0)(U - \tilde{U}_0)dx.\end{aligned}$$

Thus, for all U satisfying $\mathrm{sgn}(\mathbf{w})\mathbf{Y}(U) \le \varepsilon^2$,

$$\int_{\mathbb{R}} |\mathbf{w}|\eta(U|\tilde{U}_0)dx \le \varepsilon^2 + \int_{\mathbb{R}} |\mathbf{a}||\mathbf{v}^T \nabla^2\eta(\tilde{U}_0)(U - \tilde{U}_0)|dx.$$

Note that the assumptions (6.4) and (6.6) with (5.19) implies

$$\int_{\mathbb{R}} |\mathbf{w}|\eta(U|\tilde{U}_0)dx \le \varepsilon^2 + C\frac{\varepsilon}{\lambda}\int_{\mathbb{R}} |\mathbf{w}|(|v - \tilde{v}_0| + |h - \tilde{h}_0|)dx.$$

Then using (6.6) and Young's inequality, we have

(6.10)

$$\begin{aligned}\int_{\mathbb{R}} |\mathbf{w}|\eta(U|\tilde{U}_0)dx &\le \varepsilon^2 + C\frac{\varepsilon}{\lambda}\int_{v>3v_*} |\mathbf{w}||v-\tilde{v}_0|dx \\ &\quad + C\frac{\varepsilon}{\lambda}\Big(\int_{v\le 3v_*} |\mathbf{w}||v-\tilde{v}_0|^2dx + \int_{\mathbb{R}} |\mathbf{w}||h-\tilde{h}_0|^2dx\Big)^{1/2}\Big(\int_{\mathbb{R}} |\mathbf{w}|dx\Big)^{1/2} \\ &\le \varepsilon^2 + C\frac{\varepsilon}{\lambda}\int_{v>3v_*} |\mathbf{w}||v-\tilde{v}_0|dx \\ &\quad + \frac{c_1}{2}\int_{v\le 3v_*} |\mathbf{w}||v-\tilde{v}_0|^2dx + \frac{1}{4}\int_{\mathbb{R}} |\mathbf{w}||h-\tilde{h}_0|^2dx + C\frac{\varepsilon^2}{\lambda}.\end{aligned}$$

Now, choosing δ_0 small enough such that $C\varepsilon/\lambda < c_2/2$, and then combining the two estimates (6.9) and (6.10), we have

$$\int_{\mathbb{R}} |\mathbf{w}||h-\tilde{h}_0|^2 + \int_{v\le 3v_*} |\mathbf{w}||v-\tilde{v}_0|^2 + \int_{v>3v_*} |\mathbf{w}||v-\tilde{v}_0| \le C\frac{\varepsilon^2}{\lambda}. \tag{6.11}$$

Applying (6.11) to (6.10), we have (6.7).

- *Proof of* (6.8) : We first use (5.19) and the definition of $\eta(\cdot|\cdot)$ to rewrite $\mathbf{Y}(U)$ as

$$\mathbf{Y}(U) = \underbrace{\int_{\mathbb{R}} \mathbf{w}\Big(\frac{|h-\tilde{h}_0|^2}{2} + Q(v|\tilde{v}_0)\Big)dx}_{=:J_1} + \underbrace{\int_{\mathbb{R}} \mathbf{a}\Big(-\mathbf{v}_1 p'(\tilde{v}_0)(v-\tilde{v}_0) + \mathbf{v}_2(h-\tilde{h}_0)\Big)dx}_{=:J_2}.$$

It follows from (6.7) that

$$|J_1| \le C\frac{\varepsilon^2}{\lambda}.$$

As done before, we have

$$\begin{aligned}|J_2| &\le C\frac{\varepsilon}{\lambda}\int_{\mathbb{R}} |\mathbf{w}|(|v-\tilde{v}_0| + |h-\tilde{h}_0|)dx \\ &\le C\frac{\varepsilon}{\lambda}\int_{v>3v_*} |\mathbf{w}||v-\tilde{v}_0|dx \\ &\quad + C\frac{\varepsilon}{\lambda}\Big(\int_{v\le 3v_*} |\mathbf{w}||v-\tilde{v}_0|^2dx + \int_{\mathbb{R}} |\mathbf{w}||h-\tilde{h}_0|^2dx\Big)^{1/2}\Big(\int_{\mathbb{R}} |\mathbf{w}|dx\Big)^{1/2} \\ &\le C\frac{\varepsilon}{\lambda}\int_{\mathbb{R}} |\mathbf{w}|Q(v|\tilde{v}_0)dx + C\frac{\varepsilon}{\sqrt{\lambda}}\Big(\int_{\mathbb{R}} |\mathbf{w}|\big(Q(v|\tilde{v}_0) + |h-\tilde{h}_0|^2\big)dx\Big)^{1/2} \le C\frac{\varepsilon^2}{\lambda}.\end{aligned}$$

□

6.3. **Estimates outside truncation.** The following proposition is a generalization of [26, Lemmas 4.5-4.8].

Proposition 6.3. *For a given constant $U_* := (v_*, u_*) \in \mathbb{R}^+ \times \mathbb{R}$, there exist positive constants δ_0, δ_1 and C (depending on δ_1) such that for any $\varepsilon, \lambda > 0$ satisfying $\varepsilon/\lambda < \delta_0$ and $\lambda < \delta_0$, the following holds.*

Let $\tilde{U}_0 := (\tilde{v}_0, \tilde{h}_0) : \mathbb{R} \to \mathbb{R}^+ \times \mathbb{R}$ *and* $\mathbf{w} : \mathbb{R} \to \mathbb{R}$ *be any functions such that*

$$|\tilde{U}_0(x) - U_*| \le C\delta_0, \quad \tilde{v}_0(x) \ge C^{-1}, \quad \forall x \in \mathbb{R}, \tag{6.12}$$

$$|(\tilde{v}_0)_x| \le \delta_0^2, \quad |\mathbf{w}(x)| \le C\varepsilon\lambda \exp\big(-C^{-1}\varepsilon|x|\big), \quad \forall x \in \mathbb{R}, \tag{6.13}$$

$$\inf_{-\varepsilon^{-1} \le x \le \varepsilon^{-1}} |\mathbf{w}(x)| \ge C\varepsilon\lambda, \quad \int_{\mathbb{R}} |\mathbf{w}| dx = \lambda. \tag{6.14}$$

Let $U := (v, h) : \mathbb{R} \to \mathbb{R}^+ \times \mathbb{R}$ *be any function such that*

$$\int_{\mathbb{R}} |\mathbf{w}||h - \tilde{h}_0|^2 \, dx + \int_{\mathbb{R}} |\mathbf{w}| Q(v|\tilde{v}_0) \, dx \le C\frac{\varepsilon^2}{\lambda}. \tag{6.15}$$

Let $\bar{\mathbf{v}}$ *be a* δ_1*-truncation of* v *defined by (well-defined since the function* p *is one to one)*

$$p(\bar{\mathbf{v}}) - p(\tilde{v}_0) = \bar{\psi}\big(p(v) - p(\tilde{v}_0)\big), \qquad \text{where} \quad \bar{\psi}(y) = \inf\left(\delta_1, \sup(-\delta_1, y)\right). \tag{6.16}$$

Let $\mathbf{\Omega} := \{x \mid p(v) - p(\tilde{v}_0) \le \delta_1\}$*, and*

$$\begin{aligned}
\mathbf{G}_1^-(U) &:= \int_{\mathbf{\Omega}^c} |\mathbf{w}| \left| h - \tilde{h}_0 \right|^2 dx, \\
\mathbf{G}_2(U) &:= \int_{\mathbb{R}} |\mathbf{w}| Q(v|\tilde{v}_0) dx, \\
\tilde{\mathbf{G}}_2(U) &:= \int_{\mathbb{R}} |(\tilde{v}_0)_x| Q(v|\tilde{v}_0) dx, \\
\mathbf{D}(U) &:= \int_{\mathbb{R}} v^\beta |\partial_x\big(p(v) - p(\tilde{v}_0)\big)|^2 dx.
\end{aligned}$$

Then, the following estimates hold, where $\bar{U} := (\bar{\mathbf{v}}, h)$*.*

$$\int_{\mathbf{\Omega}} |\mathbf{w}|\big|p(v) - p(\bar{\mathbf{v}})\big|^2 dx + \int_{\mathbb{R}} |\mathbf{w}|\big|p(v) - p(\bar{\mathbf{v}})\big| dx \le \sqrt{\frac{\varepsilon}{\lambda}} \mathbf{D}(U), \tag{6.17}$$

$$\int_{\mathbf{\Omega}} |\mathbf{w}| \Big| |p(v) - p(\tilde{v}_0)|^2 - |p(\bar{\mathbf{v}}) - p(\tilde{v}_0)|^2 \Big| \, dx \le \sqrt{\frac{\varepsilon}{\lambda}} \mathbf{D}(U), \tag{6.18}$$

$$\begin{aligned}
&\int_{\mathbb{R}} |\mathbf{w}|^2 v^\beta |p(v) - p(\bar{\mathbf{v}})|^2 dx + \int_{\mathbb{R}} |\mathbf{w}|^2 v^\beta |p(v) - p(\bar{\mathbf{v}})| \, dx \\
&\qquad \le C\lambda^2 \Big(\mathbf{D}(U) + \delta_0 \tilde{\mathbf{G}}_2(U)\Big) + C\varepsilon\lambda \mathbf{G}_2(U),
\end{aligned} \tag{6.19}$$

$$\begin{aligned}
&\int_{\mathbb{R}} |\mathbf{w}|^2 \Big| v^\beta |p(v) - p(\tilde{v}_0)|^2 - \bar{\mathbf{v}}^\beta |p(\bar{\mathbf{v}}) - p(\tilde{v}_0)|^2 \Big| \, dx \\
&\qquad \le C\lambda^2 \Big(\mathbf{D}(U) + \delta_0 \tilde{\mathbf{G}}_2(U)\Big) + C\varepsilon\lambda \mathbf{G}_2(U),
\end{aligned} \tag{6.20}$$

$$\begin{aligned}
&\int_{\mathbb{R}} |\mathbf{w}| \left| p(v|\tilde{v}_0) - p(\bar{\mathbf{v}}|\tilde{v}_0) \right| \, dx \\
&\qquad \le C\sqrt{\frac{\varepsilon}{\lambda}} \Big(\mathbf{D}(U) + \delta_0 \tilde{\mathbf{G}}_2(U)\Big) + C\left(\mathbf{G}_2(U) - \mathbf{G}_2(\bar{U})\right),
\end{aligned} \tag{6.21}$$

$$\int_{\mathbb{R}} |\mathbf{w}| \left| Q(v|\tilde{v}_0) - Q(\bar{\mathbf{v}}|\tilde{v}_0) \right| \, dx + \int_{\mathbb{R}} |\mathbf{w}||v - \bar{\mathbf{v}}| \, dx \le C\left(\mathbf{G}_2(U) - \mathbf{G}_2(\bar{U})\right), \tag{6.22}$$

$$\int_{\mathbf{\Omega}^c} |\mathbf{w}|\big|p(v)-p(\bar{\mathbf{v}})\big|^2 dx \le C\frac{\varepsilon^{2-q}}{\lambda}\Big(\mathbf{D}(U)+\delta_0\tilde{\mathbf{G}}_2(U)\Big)^q, \quad q:=\frac{2\gamma}{\gamma+\alpha}, \tag{6.23}$$

$$\int_{\mathbf{\Omega}^c} |\mathbf{w}|\big|p(v)-p(\tilde{v}_0)\big||h-\tilde{h}_0|dx \le \delta_0\Big(\mathbf{D}(U)+\delta_0\tilde{\mathbf{G}}_2(U)\Big)+(\delta_1+C\delta_0)\mathbf{G}_1^-(U), \tag{6.24}$$

$$\int_{\mathbf{\Omega}^c} |\mathbf{w}|\left(Q(\bar{\mathbf{v}}|\tilde{v}_0)+|\bar{\mathbf{v}}-\tilde{v}_0|\right)dx \le C\sqrt{\frac{\varepsilon}{\lambda}}\Big(\mathbf{D}(U)+\delta_0\tilde{\mathbf{G}}_2(U)\Big), \tag{6.25}$$

$$\int_{\mathbb{R}} |\mathbf{w}|^2\frac{|v^\beta-\bar{\mathbf{v}}^\beta|^2}{v^\beta}\,dx \le C\lambda\bigg(\mathbf{D}(U)+\delta_0\tilde{\mathbf{G}}_2(U)\bigg)+C\lambda\left(\mathbf{G}_2(U)-\mathbf{G}_2(\bar{U})\right), \tag{6.26}$$

$$\begin{aligned}&\int_{\mathbb{R}} |\mathbf{w}|^2\Big|\frac{|v^\beta-\tilde{v}_0^\beta|^2}{v^\beta}-\frac{|\bar{\mathbf{v}}^\beta-\tilde{v}_0^\beta|^2}{\bar{\mathbf{v}}^\beta}\Big|\,dx\\ &\qquad\le C\lambda\bigg(\mathbf{D}(U)+\delta_0\tilde{\mathbf{G}}_2(U)\bigg)+C\lambda\left(\mathbf{G}_2(U)-\mathbf{G}_2(\bar{U})\right).\end{aligned} \tag{6.27}$$

Proof. First, let $\bar{\mathbf{v}}_s$ and $\bar{\mathbf{v}}_b$ be one-sided truncations of v defined by

$$p(\bar{\mathbf{v}}_s)-p(\tilde{v}_0):=\bar{\psi}_s\big(p(v)-p(\tilde{v}_0)\big), \qquad \text{where} \quad \bar{\psi}_s(y)=\inf(\delta_1,y), \tag{6.28}$$

and

$$p(\bar{\mathbf{v}}_b)-p(\tilde{v}_0):=\bar{\psi}_b\big(p(v)-p(\tilde{v}_0)\big), \qquad \text{where} \quad \bar{\psi}_b(y)=\sup(-\delta_1,y). \tag{6.29}$$

Notice that the function $\bar{\mathbf{v}}_s$ (resp. $\bar{\mathbf{v}}_b$) represents the truncation of small (resp. big) values of v corresponding to $|p(v)-p(\tilde{v}_0)|\ge\delta_1$.
By comparing the definitions of (6.16), (6.28) and (6.29), we see

$$\bar{\mathbf{v}}=\begin{cases}\bar{\mathbf{v}}_b, & \text{on } \mathbf{\Omega},\\ \bar{\mathbf{v}}_s, & \text{on } \mathbf{\Omega}^c,\end{cases} \tag{6.30}$$

and

$$\begin{aligned}&\left(p(\bar{\mathbf{v}}_s)-p(\tilde{v}_0)\right)\mathbf{1}_{\{p(v)-p(\tilde{v}_0)\ge-\delta_1\}}=\left(p(\bar{\mathbf{v}})-p(\tilde{v}_0)\right)\mathbf{1}_{\{p(v)-p(\tilde{v}_0)\ge-\delta_1\}},\\ &\left(p(\bar{\mathbf{v}}_b)-p(\tilde{v}_0)\right)\mathbf{1}_{\{p(v)-p(\tilde{v}_0)\le\delta_1\}}=\left(p(\bar{\mathbf{v}})-p(\tilde{v}_0)\right)\mathbf{1}_{\{p(v)-p(\tilde{v}_0)\le\delta_1\}}.\end{aligned}$$

We also note that

$$\begin{aligned} p(v)-p(\bar{\mathbf{v}}_s) = {} & \left(p(v)-p(\tilde{v}_0)\right)+\left(p(\tilde{v}_0)-p(\bar{\mathbf{v}}_s)\right)\\ = {} & \left(I-\bar{\psi}_s\right)\left(p(v)-p(\tilde{v}_0)\right)\\ = {} & \left(\left(p(v)-p(\tilde{v}_0)\right)-\delta_1\right)_+,\\ p(\bar{\mathbf{v}}_b)-p(v) = {} & \left(p(\bar{\mathbf{v}}_b)-p(\tilde{v}_0)\right)+\left(p(\tilde{v}_0)-p(v)\right)\\ = {} & \left(\bar{\psi}_b-I\right)\left(p(v)-p(\tilde{v}_0)\right)\\ = {} & \left(-\left(p(v)-p(\tilde{v}_0)\right)-\delta_1\right)_+,\\ |p(v)-p(\bar{\mathbf{v}})| = {} & |(p(v)-p(\tilde{v}_0))+(p(\tilde{v}_0)-p(\bar{\mathbf{v}}))|\\ = {} & |(I-\bar{\psi})(p(v)-p(\tilde{v}_0))|\\ = {} & (|p(v)-p(\tilde{v}_0)|-\delta_1)_+.\end{aligned} \tag{6.31}$$

Therefore, using (6.16), (6.29), (6.28) and (6.31), we have

(6.32)
$$
\begin{aligned}
\mathbf{D}(U) &= \int_{\mathbb{R}} v^\beta |\partial_x (p(v) - p(\tilde{v}_0))|^2 dx \\
&= \int_{\mathbb{R}} v^\beta |\partial_x (p(v) - p(\tilde{v}_0))|^2 (\mathbf{1}_{\{|p(v)-p(\tilde{v}_0)| \le \delta_1\}} + \mathbf{1}_{\{p(v)-p(\tilde{v}_0) > \delta_1\}} + \mathbf{1}_{\{p(v)-p(\tilde{v}_0) < -\delta_1\}}) dx \\
&= \mathbf{D}(\bar{U}) + \int_{\mathbb{R}} v^\beta |\partial_x (p(v) - p(\bar{\mathbf{v}}_s))|^2 dx + \int_{\mathbb{R}} v^\beta |\partial_x (p(v) - p(\bar{\mathbf{v}}_b))|^2 dx \\
&\ge \int_{\mathbb{R}} v^\beta |\partial_x (p(v) - p(\bar{\mathbf{v}}_s))|^2 dx + \int_{\mathbb{R}} v^\beta |\partial_x (p(v) - p(\bar{\mathbf{v}}_b))|^2 dx,
\end{aligned}
$$

which also yields

$$
\mathbf{D}(U) \ge \mathbf{D}(\bar{U}). \tag{6.33}
$$

On the other hand, since $Q(v|\tilde{v}_0) \ge Q(\bar{v}|\tilde{v}_0)$, we have

$$
\mathbf{G}_2(U) - \mathbf{G}_2(\bar{U}) = \int_{\mathbb{R}} |\mathbf{w}| \left(Q(v|\tilde{v}_0) - Q(\bar{v}|\tilde{v}_0) \right) \, dx \ge 0,
$$

which together with (6.7) yields

$$
0 \le \mathbf{G}_2(U) - \mathbf{G}_2(\bar{U}) \le \mathbf{G}_2(U) \le C \int_{\mathbb{R}} |\mathbf{w}| Q(v|\tilde{v}_0) \, dx \le C \frac{\varepsilon^2}{\lambda}. \tag{6.34}
$$

To get the desired estimates, we use the same computations as in the proofs of [26, Lemmas 4.5, 4.6, 4.7, 4.8] by considering $\mathbf{w}$ the spatial derivative of each weight. Indeed, the main ideas for the proofs of [26, Lemmas 4.5, 4.6, 4.7, 4.8] are based on the smallness of the weighted relative entropy (6.15), and the following point-wise estimates (6.35) and (6.36):
Since (6.15) and (6.14) imply

$$
\begin{aligned}
2\varepsilon \int_{-1/\varepsilon}^{1/\varepsilon} Q(v|\tilde{v}_0) \, dx &\le \frac{2\varepsilon}{\inf_{[-\varepsilon^{-1}, \varepsilon^{-1}]} |\mathbf{w}|} \int_{\mathbb{R}} |\mathbf{w}| Q(v|\tilde{v}_0) \, dx \\
&\le C \frac{\varepsilon}{\lambda \varepsilon} \frac{\varepsilon^2}{\lambda} = C \left(\frac{\varepsilon}{\lambda} \right)^2,
\end{aligned}
$$

there exists $x_0 \in [-\varepsilon^{-1}, \varepsilon^{-1}]$ such that $Q(v|\tilde{v}_0)(x_0) \le C(\varepsilon/\lambda)^2$. For δ_0 small enough, and using (A.10), we have

$$
|(p(v) - p(\tilde{v}_0))(x_0)| \le C \frac{\varepsilon}{\lambda}.
$$

Thus, if δ_0 is small enough such that $C\varepsilon/\lambda \le \delta_1/2$, then we find from (6.16) that

$$
|(p(v) - p(\bar{\mathbf{v}}))(x_0)| = 0,
$$

which together with (6.31) implies

$$
|(p(v) - p(\bar{\mathbf{v}}_b))(x_0)| = 0, \qquad |(p(v) - p(\bar{\mathbf{v}}_s))(x_0)| = 0.
$$

Therefore, using (6.32), we find

$$
\begin{aligned}
\forall x \in \mathbb{R}, \quad |(p(v)-p(\bar{\mathbf{v}}_b))(x)| &\le \int_{x_0}^{x} \left|\partial_y\big(p(v)-p(\bar{\mathbf{v}}_b)\big)\right| \mathbf{1}_{\{p(v)-p(\tilde{v}_0)<-\delta_1\}}\, dy \\
&\le C\int_{x_0}^{x} v^{\beta/2} \left|\partial_y\big(p(v)-p(\bar{\mathbf{v}}_b)\big)\right| \mathbf{1}_{\{p(v)-p(\tilde{v}_0)<-\delta_1\}}\, dy \\
&\le C\sqrt{|x|+\frac{1}{\varepsilon}}\sqrt{\mathbf{D}(U)}.
\end{aligned} \tag{6.35}
$$

To get a point-wise estimate for $|v^{\beta/2}(p(v)-p(\bar{\mathbf{v}}_s))(x)|$, we use

$$
|v^{\beta/2}(p(v)-p(\bar{\mathbf{v}}_s))(x)| = \left|\int_{x_0}^{x} \partial_y\big(v^{\beta/2}(p(v)-p(\bar{\mathbf{v}}_s))\big)\, dy\right|.
$$

To control the right-hand side by the good terms, we observe that since $v^{\beta/2} = p(v)^{-(\gamma-\alpha)/2\gamma}$, we have

$$
\begin{aligned}
&\partial_y\big(v^{\beta/2}(p(v)-p(\bar{\mathbf{v}}_s))\big) = \partial_y\big(p(v)^{-(\gamma-\alpha)/2\gamma}(p(v)-p(\bar{\mathbf{v}}_s))\big) \\
&\quad = p(v)^{-(\gamma-\alpha)/2\gamma}\partial_y(p(v)-p(\bar{\mathbf{v}}_s)) \\
&\qquad -\frac{\gamma-\alpha}{2\gamma}p(v)^{-(\gamma-\alpha)/2\gamma}\frac{p(v)-p(\bar{\mathbf{v}}_s)}{p(v)}\partial_y\big[(p(v)-p(\tilde{v}_0))+p(\tilde{v}_0)\big] \\
&\quad = v^{\beta/2}\partial_y(p(v)-p(\bar{\mathbf{v}}_s)) - \frac{\gamma-\alpha}{2\gamma}v^{\beta/2}\underbrace{\frac{p(v)-p(\bar{\mathbf{v}}_s)}{p(v)}\partial_y(p(v)-p(\tilde{v}_0))}_{=:K} \\
&\qquad -\frac{\gamma-\alpha}{2\gamma}v^{\beta/2}\frac{p(v)-p(\bar{\mathbf{v}}_s)}{p(v)}\partial_y p(\tilde{v}_0).
\end{aligned}
$$

Using the fact that

$$
p(\bar{\mathbf{v}}_s) = p(\tilde{v}_0)+\delta_1 \quad \text{and} \quad \frac{p(v)-p(\bar{\mathbf{v}}_s)}{p(v)} \le C \quad \text{on } \{p(v)-p(\tilde{v}_0)>\delta_1\},
$$

we have

$$
K = \frac{p(v)-p(\bar{\mathbf{v}}_s)}{p(v)}\mathbf{1}_{\{p(v)-p(\tilde{v}_0)>\delta_1\}}\partial_y(p(v)-p(\tilde{v}_0)) = \frac{p(v)-p(\bar{\mathbf{v}}_s)}{p(v)}\partial_y(p(v)-p(\bar{\mathbf{v}}_s)),
$$

and so,

$$
|K| \le C|\partial_y(p(v)-p(\bar{\mathbf{v}}_s))|.
$$

In addition, using $|\partial_y p(\tilde{v}_0)| = |p'(\tilde{v}_0)||(\tilde{v}_0)_y| \le C|(\tilde{v}_0)_y|$, we have

$$
|\partial_y\big(v^{\beta/2}(p(v)-p(\bar{\mathbf{v}}_s))\big)| \le Cv^{\beta/2}(|\partial_y(p(v)-p(\bar{\mathbf{v}}_s))|+|(\tilde{v}_0)_y|).
$$

Therefore, using (6.32), we have that for any $x \in \mathbb{R}$,

$$\begin{aligned}
|v^{\beta/2}(p(v)-p(\bar{\mathbf{v}}_s))(x)| &= \left|\int_{x_0}^x \partial_y\big(v^{\beta/2}(p(v)-p(\bar{\mathbf{v}}_s))\big)\,dy\right| \\
&\le \int_{x_0}^x \big|\partial_y\big(v^{\beta/2}(p(v)-p(\bar{\mathbf{v}}_s))\big)\big|\mathbf{1}_{\{p(v)-p(\tilde{v}_0)>\delta_1\}}dy \\
&\le C\int_{x_0}^x v^{\beta/2}(|\partial_y((p(v)-p(\bar{\mathbf{v}}_s))|+|(\tilde{v}_0)_y|)\mathbf{1}_{\{p(v)-p(\tilde{v}_0)>\delta_1\}}dy \\
&\le C\sqrt{|x|+\frac{1}{\varepsilon}}\left(\sqrt{\mathbf{D}(U)}+\sqrt{\int_{\mathbb{R}}|(\tilde{v}_0)_y|^2 v^\beta \mathbf{1}_{\{p(v)-p(\tilde{v}_0)>\delta_1\}}dy}\right).
\end{aligned}$$

Using the condition $\beta=\gamma-\alpha>0$, we have

$$\begin{aligned}
\int_{\mathbb{R}}|(\tilde{v}_0)_x|^2 v^\beta \mathbf{1}_{\{p(v)-p(\tilde{v}_0)>\delta_1\}}dx &= \int_{\mathbb{R}}|(\tilde{v}_0)_x|^2\frac{v^\beta}{|v-\tilde{v}_0|^2}|v-\tilde{v}_0|^2\mathbf{1}_{\{p(v)-p(\tilde{v}_0)>\delta_1\}}dx \\
&\le C\int_{\mathbb{R}}|(\tilde{v}_0)_x|^2|v-\tilde{v}_0|^2\mathbf{1}_{\{p(v)-p(\tilde{v}_0)>\delta_1\}}dx.
\end{aligned}$$

In addition, since $\{p(v)-p(\tilde{v}_0)>\delta_1\}=\{v\le C\}$ for some constant C, (6.13) and (A.1) yield

$$\int_{\mathbb{R}}|(\tilde{v}_0)_x|^2|v-\tilde{v}_0|^2\mathbf{1}_{\{p(v)-p(\tilde{v}_0)>\delta_1\}}d\xi \le C\delta_0^2\int_{\mathbb{R}}|(\tilde{v}_0)_x|Q(v|\tilde{v}_0)dx = C\delta_0^2\tilde{\mathbf{G}}_2(U).$$

Therefore we obtain that

$$\forall i=1,2,\ \forall x\in\mathbb{R},\quad |v^{\beta/2}(p(v)-p(\bar{\mathbf{v}}_s))(x)| \le C\sqrt{|x|+\frac{1}{\varepsilon}}\left(\sqrt{\mathbf{D}(U)}+\delta_0\sqrt{\tilde{\mathbf{G}}_2(U)}\right). \tag{6.36}$$

The remaining parts use the same computations as in the proofs of [26, Lemmas 4.5, 4.6, 4.7, 4.8]. Especially, for the estimate (6.23), note from (6.30) that

$$\int_{\mathbf{\Omega}^c}|\mathbf{w}|\big|p(v)-p(\bar{\mathbf{v}})\big|^2dx = \int_{\mathbf{\Omega}^c}|\mathbf{w}|\big|p(v)-p(\bar{\mathbf{v}}_s)\big|^2dx,$$

and so, its proof follows from the proof of [26, Lemmas 4.7]. We omit those details. ☐

The following proposition is a generalization of [26, Proposition 4.3].

Proposition 6.4. *For a given constant $U_*:=(v_*,u_*)\in\mathbb{R}^+\times\mathbb{R}$, there exist positive constants $\delta_0,\delta_1,\sigma,C^*$ and C (in particular, C depends on the constant δ_1, but C and C^* are independent of δ_0) such that for any $\varepsilon,\lambda>0$ satisfying $\varepsilon/\lambda<\delta_0$ and $\lambda<\delta_0$, the same hypotheses as in Proposition 6.3 hold. In addition, let $\mathbf{v}_1,\mathbf{v}_2:\mathbb{R}\to\mathbb{R}$ be any functions such that*

$$|\mathbf{v}_i|\le C\frac{\varepsilon}{\lambda}|\mathbf{w}(x)|,\quad \forall x\in\mathbb{R},\quad i=1,2, \tag{6.37}$$

and assume

$$\tilde{\mathbf{G}}_2(U)\le\frac{\varepsilon^2}{\lambda}. \tag{6.38}$$

Consider the following functionals:

$$\mathbf{B}_1(U) := \sigma \int_{\mathbb{R}} \mathbf{v}_1 p(v|\tilde{v}_0) dx,$$
$$\mathbf{B}_2^-(U) := \int_{\mathbf{\Omega}^c} \mathbf{w}\big(p(v) - p(\tilde{v}_0)\big)\big(h - \tilde{h}_0\big) dx, \qquad \mathbf{B}_2^+(U) := \frac{1}{2\sigma}\int_{\mathbf{\Omega}} \mathbf{w}|p(v) - p(\tilde{v}_0)|^2 dx,$$
$$\mathbf{B}_3(U) := -\int_{\mathbb{R}} \mathbf{w} v^\beta \big(p(v) - p(\tilde{v}_0)\big)\partial_x\big(p(v) - p(\tilde{v}_0)\big) dx,$$
$$\mathbf{B}_4(U) := \int_{\mathbb{R}} |\mathbf{w}||\mathbf{v}_1|\big|p(v) - p(\tilde{v}_0)\big|\big|v^\beta - \tilde{v}_0^\beta\big| dx,$$
$$\mathbf{B}_5(U) := \int_{\mathbb{R}} |\mathbf{v}_1|\big|\partial_x\big(p(v) - p(\tilde{v}_0)\big)\big|\big|v^\beta - \tilde{v}_0^\beta\big| dx,$$

$$\mathbf{Y}^g(U) := -\frac{1}{2\sigma^2}\int_{\mathbf{\Omega}} \mathbf{w}|p(v) - p(\tilde{v}_0)|^2 dx - \int_{\mathbf{\Omega}} \mathbf{w} Q(v|\tilde{v}_0) dx$$
$$\qquad - \int_{\mathbf{\Omega}} \mathbf{v}_1 p'(\tilde{v})(v - \tilde{v}_0) dx + \frac{1}{\sigma}\int_{\mathbf{\Omega}} \mathbf{v}_2\big(p(v) - p(\tilde{v}_0)\big) dx,$$
$$\mathbf{Y}^s(U) := -\int_{\mathbf{\Omega}^c} \mathbf{w} Q(v|\tilde{v}_0) dx - \int_{\mathbf{\Omega}^c} \mathbf{v}_1 p'(\tilde{v}_0)(v - \tilde{v}_0) dx - \int_{\mathbf{\Omega}^c} \mathbf{w}\frac{|h - \tilde{h}_0|^2}{2} dx + \int_{\mathbf{\Omega}^c} \mathbf{v}_2 (h - \tilde{h}_0) dx,$$
$$\mathbf{Y}^b(U) := -\frac{1}{2}\int_{\mathbf{\Omega}} \mathbf{w}\Big(h - \tilde{h}_0 - \frac{p(v) - p(\tilde{v}_0)}{\sigma}\Big)^2 dx - \frac{1}{\sigma}\int_{\mathbf{\Omega}} \mathbf{w}\big(p(v) - p(\tilde{v}_0)\big)\Big(h - \tilde{h}_0 - \frac{p(v) - p(\tilde{v}_0)}{\sigma}\Big) dx,$$
$$\mathbf{Y}^l(U) := \int_{\mathbf{\Omega}} \mathbf{v}_2\Big(h - \tilde{h}_0 - \frac{p(v) - p(\tilde{v}_0)}{\sigma}\Big) dx,$$

and

$$\mathbf{G}_1^+(U) := \int_{\mathbf{\Omega}} |\mathbf{w}| \left| h - \tilde{h}_0 - \frac{p(v) - p(\tilde{v}_0)}{\sigma}\right|^2 dx.$$

Then, the following estimates hold:

$$\begin{aligned} &|\mathbf{B}_1(U) - \mathbf{B}_1(\bar{U})| \le C\delta_0 \left(\mathbf{D}(U) + \tilde{\mathbf{G}}_2(U) + \big(\mathbf{G}_2(U) - \mathbf{G}_2(\bar{U})\big)\right), \\ &|\mathbf{B}_2^-(U)| \le \delta_0 \left(\mathbf{D}(U) + \tilde{\mathbf{G}}_2(U)\right) + (\delta_1 + C\delta_0)\mathbf{G}_1^-(U), \\ &|\mathbf{B}_2^+(U) - \mathbf{B}_2^+(\bar{U})| \le \sqrt{\delta_0}\mathbf{D}(U), \end{aligned} \tag{6.39}$$

$$|\mathbf{B}_1(\bar{U})| + |\mathbf{B}_2^+(\bar{U})| \le C\int_{\mathbb{R}} |\mathbf{w}| Q(\bar{\mathbf{v}}|\tilde{v}_0)\, dx \le C^*\frac{\varepsilon^2}{\lambda}, \tag{6.40}$$

(6.41)
$$|\mathbf{B}_3(U)| + |\mathbf{B}_4(U)| + |\mathbf{B}_5(U)| \le C\delta_0 \left(\mathbf{D}(U) + \tilde{\mathbf{G}}_2(U) + \big(\mathbf{G}_2(U) - \mathbf{G}_2(\bar{U})\big) + \frac{\varepsilon}{\lambda}\mathbf{G}_2(\bar{U})\right).$$

If, in addition, $\mathbf{D}(U) \le \frac{4C^*}{\sqrt{\delta_0}}\frac{\varepsilon^2}{\lambda}$*, then*

$$
\begin{aligned}
&|\mathbf{Y}^g(U) - \mathbf{Y}^g(\bar{U})|^2 + |\mathbf{Y}^b(U)|^2 + |\mathbf{Y}^l(U)|^2 + |\mathbf{Y}^s(U)|^2 \\
&\le C\frac{\varepsilon^2}{\lambda}\bigg(\sqrt{\delta_0}\mathbf{D}(U) + \big(\mathbf{G}_2(U) - \mathbf{G}_2(\bar{U})\big) + \delta_0\tilde{\mathbf{G}}_2(U) + \Big(\frac{\varepsilon}{\lambda}\Big)^{1/4}\mathbf{G}_2(\bar{U}) \\
&\qquad + \mathbf{G}_1^-(U) + \Big(\frac{\lambda}{\varepsilon}\Big)^{1/4}\mathbf{G}_1^+(U)\bigg).
\end{aligned}
\tag{6.42}
$$

Proof. • *Proof of* (6.39) : First, using (6.21) with (6.37), and (6.24), we have

$$
\begin{aligned}
&|\mathbf{B}_1(U) - \mathbf{B}_1(\bar{U})| \le C\frac{\varepsilon}{\lambda}\Big(\mathbf{D}(U) + \tilde{\mathbf{G}}_2(U) + \big(\mathbf{G}_2(U) - \mathbf{G}_2(\bar{U})\big)\Big), \\
&|\mathbf{B}_2^-(U)| \le \delta_0\Big(\mathbf{D}(U) + \tilde{\mathbf{G}}_2(U)\Big) + (\delta_1 + C\delta_0)\mathbf{G}_1^-(U).
\end{aligned}
$$

Using (6.18), we have

$$
|\mathbf{B}_2^+(U) - \mathbf{B}_2^+(\bar{U})| = \int_{\mathbf{\Omega}} |\mathbf{w}|\,\big||p(v) - p(\tilde{v}_0)|^2 - |p(\bar{\mathbf{v}}) - p(\tilde{v}_0)|^2\big|\,dx \le \sqrt{\frac{\varepsilon}{\lambda}}\mathbf{D}(U).
$$

Thus, for any ε, λ satisfying $\varepsilon/\lambda < \delta_0$, we have the desired estimates.

• *Proof of* (6.40): Using (A.7), (A.10) and (6.15), we have

$$
|\mathbf{B}_1(\bar{U})| + |\mathbf{B}_2^+(\bar{U})| \le C\int_{\mathbb{R}} |\mathbf{w}|Q(\bar{\mathbf{v}}|\tilde{v}_0)\,dx \le C\int_{\mathbb{R}} |\mathbf{w}|Q(v|\tilde{v}_0)\,dx \le C\frac{\varepsilon^2}{\lambda}.
\tag{6.43}
$$

• *Proof of* (6.41): For $\mathbf{B}_3$, Young's inequality implies

$$
|\mathbf{B}_3(U)| \le \delta_0\mathbf{D}(U) + \frac{C}{\delta_0}\underbrace{\int_{\mathbb{R}} |\mathbf{w}|^2 v^\beta |p(v) - p(\tilde{v}_0)|^2\,dx}_{=:J_1(U)}.
$$

Note from (6.20) that

$$
|J_1(U) - J_1(\bar{U})| \le C\lambda^2\Big(\mathbf{D}(U) + \delta_0\tilde{\mathbf{G}}_2(U)\Big) + C\varepsilon\lambda\mathbf{G}_2(U).
$$

For $\mathbf{B}_4$ and $\mathbf{B}_5$, we first have

$$
\begin{aligned}
&|\mathbf{B}_4(U)| \le CJ_1(U) + C\underbrace{\int_{\mathbb{R}} |\mathbf{v}_1|^2\frac{|v^\beta - \tilde{v}_0^\beta|^2}{v^\beta}\,d\xi}_{=:J_2(U)}, \\
&|\mathbf{B}_5(U)| \le \delta_0\mathbf{D}(U) + \frac{C}{\delta_0}J_2(U).
\end{aligned}
$$

Note that (6.37) and (6.27) yield

$$
\begin{aligned}
|J_2(U) - J_2(\bar{U})| &\le \Big(\frac{\varepsilon}{\lambda}\Big)^2\int_{\mathbb{R}} |\mathbf{w}|^2\Big|\frac{|v^\beta - \tilde{v}_0^\beta|^2}{v^\beta} - \frac{|\bar{\mathbf{v}}^\beta - \tilde{v}_0^\beta|^2}{\bar{\mathbf{v}}^\beta}\Big|\,dx \\
&\le C\frac{\varepsilon^2}{\lambda}\bigg(\mathbf{D}(U) + \delta_0\tilde{\mathbf{G}}_2(U) + \big(\mathbf{G}_2(U) - \mathbf{G}_2(\bar{U})\big)\bigg).
\end{aligned}
$$

Therefore, using $\lambda < \delta_0$ and $\varepsilon/\lambda < \delta_0$, we have

$$\begin{aligned}&|\mathbf{B}_3(U)| + |\mathbf{B}_4(U)| + |\mathbf{B}_5(U)| \\ &\quad \le \frac{C}{\delta_0}\left(|J_1(\bar{U})| + |J_2(\bar{U})|\right) + C\delta_0\Big(\mathbf{D}(U) + \tilde{\mathbf{G}}_2(U) + \big(\mathbf{G}_2(U) - \mathbf{G}_2(\bar{U})\big)\Big).\end{aligned}$$

For the remaining terms, we first use the assumptions (6.13) and (6.37) to have

$$\begin{aligned}\frac{C}{\delta_0}\left(|J_1(\bar{U})| + |J_2(\bar{U})|\right) &\le \frac{C}{\delta_0}\left[\varepsilon\lambda\int_{\mathbb{R}}|\mathbf{w}|\bar{\mathbf{v}}^\beta|p(\bar{\mathbf{v}}) - p(\tilde{v}_0)|^2\,dx + \frac{\varepsilon^3}{\lambda}\int_{\mathbb{R}}|\mathbf{w}|\frac{|\bar{\mathbf{v}}^\beta - \tilde{v}_0^\beta|^2}{\bar{\mathbf{v}}^\beta}\,dx\right] \\ &\le C\delta_0\frac{\varepsilon}{\lambda}\bigg[\int_{\mathbb{R}}|\mathbf{w}|\bar{\mathbf{v}}^\beta|p(\bar{\mathbf{v}}) - p(\tilde{v}_0)|^2\,dx + \int_{\mathbb{R}}|\mathbf{w}|\frac{|\bar{\mathbf{v}}^\beta - \tilde{v}_0^\beta|^2}{\bar{\mathbf{v}}^\beta}\,dx\bigg].\end{aligned}$$

Using $C^{-1} \le \bar{\mathbf{v}} \le C$ and $Q(\bar{\mathbf{v}}|\tilde{v}_0) \ge C|\bar{\mathbf{v}} - \tilde{v}_0|^2 \ge C|\bar{\mathbf{v}}^\beta - \tilde{v}_0^\beta|^2$, we have

$$\int_{\mathbb{R}}|\mathbf{w}|\frac{|\bar{\mathbf{v}}^\beta - \tilde{v}_0^\beta|^2}{\bar{\mathbf{v}}^\beta}\,dx \le C\int_{\mathbb{R}}|\mathbf{w}|Q(\bar{\mathbf{v}}|\tilde{v}_0)\,dx \le C\mathbf{G}_2(\bar{U}),$$

Using (A.10) with $|p(\bar{\mathbf{v}}) - p(\tilde{v}_0)| \le \delta_1$, we have

$$\int_{\mathbb{R}}|\mathbf{w}|\bar{\mathbf{v}}^\beta|p(\bar{\mathbf{v}}) - p(\tilde{v}_0)|^2\,dx \le C\int_{\mathbb{R}}|\mathbf{w}|Q(\bar{\mathbf{v}}|\tilde{v}_0)\,dx \le C\mathbf{G}_2(\bar{U}). \tag{6.44}$$

Hence we have

$$\begin{aligned}&|\mathbf{B}_3(U)| + |\mathbf{B}_4(U)| + |\mathbf{B}_5(U)| \\ &\quad \le C\delta_0\Big(\mathbf{D}(U) + \tilde{\mathbf{G}}_2(U) + \big(\mathbf{G}_2(U) - \mathbf{G}_2(\bar{U})\big) + \frac{\varepsilon}{\lambda}\mathbf{G}_2(\bar{U})\Big).\end{aligned}$$

• *Proof of* (6.42): We split the proof into three steps.

Step 1: First of all, we will use notations Y_1^s, Y_2^s, Y_3^s and Y_4^s for the terms of $\mathbf{Y}^s$ as follows : set $\mathbf{Y}^s = Y_1^s + Y_2^s + Y_3^s + Y_4^s$ where

$$\begin{aligned}Y_1^s &:= -\int_{\mathbf{\Omega}^c}\mathbf{w}Q(v|\tilde{v}_0)dx, \qquad Y_2^s := -\int_{\mathbf{\Omega}^c}\mathbf{v}_1 p'(\tilde{v}_0)(v - \tilde{v}_0)dx, \\ Y_3^s &:= -\int_{\mathbf{\Omega}^c}\mathbf{w}\frac{|h - \tilde{h}_0|^2}{2}dx, \qquad Y_4^s := \int_{\mathbf{\Omega}^c}\mathbf{v}_2(h - \tilde{h}_0)dx.\end{aligned}$$

Using (6.17), (6.22) with (6.12) and (6.37), and (6.18), we find that

$$\begin{aligned}&|\mathbf{Y}^g(U) - \mathbf{Y}^g(\bar{U})| + |Y_1^s(U) - Y_1^s(\bar{U})| + |Y_2^s(U) - Y_2^s(\bar{U})| \\ &\quad \le C\int_{\mathbf{\Omega}}|\mathbf{w}|\Big||p(v) - p(\tilde{v}_0)|^2 - |p(\bar{\mathbf{v}}) - p(\tilde{v}_0)|^2\Big|dx + C\int_{\mathbf{\Omega}}|\mathbf{w}||p(v) - p(\bar{\mathbf{v}})|dx \\ &\qquad + C\int_{\mathbb{R}}|\mathbf{w}|\Big(\big|Q(v|\tilde{v}_0) - Q(\bar{\mathbf{v}}|\tilde{v}_0)\big| + |v - \bar{\mathbf{v}}|\Big)\,dx \\ &\quad \le C\sqrt{\frac{\varepsilon}{\lambda}}\mathbf{D}(U) + C\left(\mathbf{G}_2(U) - \mathbf{G}_2(\bar{U})\right).\end{aligned} \tag{6.45}$$

Note that (6.25) with (6.12) and (6.37) yields

$$|Y_1^s(\bar{U})| + |Y_2^s(\bar{U})| \le C\int_{\mathbf{\Omega}^c}|\mathbf{w}|\left(Q(\bar{\mathbf{v}}|\tilde{v}_0) + |\bar{\mathbf{v}} - \tilde{v}_0|\right)dx \le C\sqrt{\frac{\varepsilon}{\lambda}}\Big(\mathbf{D}(U) + \delta_0\tilde{\mathbf{G}}_2(U)\Big). \tag{6.46}$$

Next, since

$$
\begin{aligned}
|Y_3^s(U)| + |\mathbf{Y}^b(U)| &\le C\int_{\mathbf{\Omega}^c} |\mathbf{w}||h-\tilde{h}_0|^2\,dx + C\int_{\mathbf{\Omega}} |\mathbf{w}|\left(|h-\tilde{h}_0|^2 + |p(v)-p(\tilde{v}_0)|^2\right)\,dx \\
&\le C\int_{\mathbb{R}} |\mathbf{w}||h-\tilde{h}_0|^2\,dx + C|\mathbf{B}_2^+(U)|,
\end{aligned}
$$

it follows from (6.39) and (6.40) that

$$
|Y_3^s(U)| + |\mathbf{Y}^b(U)| \le C\int_{\mathbb{R}} |\mathbf{w}||h-\tilde{h}_0|^2\,dx + C\sqrt{\delta_0}\mathbf{D}(U) + C\frac{\varepsilon^2}{\lambda}.
$$

Therefore, using (6.15), (6.34) and the assumptions (6.38) and $\mathbf{D}(U) \le 2\frac{C^*}{\sqrt{\delta_0}}\frac{\varepsilon^2}{\lambda}$, together with combining (6.45), (6.46) and the above estimates, we have

$$
|\mathbf{Y}^g(U) - \mathbf{Y}^g(\bar{U})| + |Y_1^s(U)| + |Y_2^s(U)| + |Y_3^s(U)| + |\mathbf{Y}^b(U)| \le C\frac{\varepsilon^2}{\lambda}.
$$

Step 2: First of all, using Young's inequality, (6.39) and (6.43), we have

$$
\begin{aligned}
&\left|\int_{\mathbf{\Omega}} \mathbf{w}\big(p(v)-p(\tilde{v}_0)\big)\Big(h-\tilde{h}_0 - \frac{p(v)-p(\tilde{v}_0)}{\sigma}\Big)dx\right| \\
&\le \left(\frac{\lambda}{\varepsilon}\right)^{1/4}\mathbf{G}_1^+(U) + C\left(\frac{\varepsilon}{\lambda}\right)^{1/4}\int_{\mathbf{\Omega}} |\mathbf{w}||p(v)-p(\tilde{v}_0)|^2\,dx \\
&\le \left(\frac{\lambda}{\varepsilon}\right)^{1/4}\mathbf{G}_1^+(U) + C\left(\frac{\varepsilon}{\lambda}\right)^{1/4}\left(\mathbf{B}_2^+(\bar{U}) + \big(\mathbf{B}_2^+(U) - \mathbf{B}_2^+(\bar{U})\big)\right) \\
&\le \left(\frac{\lambda}{\varepsilon}\right)^{1/4}\mathbf{G}_1^+(U) + C\left(\frac{\varepsilon}{\lambda}\right)^{1/4}\left(\mathbf{G}_2(\bar{U}) + \sqrt{\delta_0}\mathbf{D}(U)\right),
\end{aligned}
$$

which yields

$$
|\mathbf{Y}^b(U)| \le C\left(\frac{\lambda}{\varepsilon}\right)^{1/4}\mathbf{G}_1^+(U) + C\left(\frac{\varepsilon}{\lambda}\right)^{1/4}\left(\mathbf{G}_2(\bar{U}) + \sqrt{\delta_0}\mathbf{D}(U)\right).
$$

Thus, this and (6.45)-(6.46) together with $|Y_3^s(U)| \le C\mathbf{G}_1^-(U)$ imply

$$
\begin{aligned}
&|\mathbf{Y}^g(U) - \mathbf{Y}^g(\bar{U})| + |Y_1^s(U)| + |Y_2^s(U)| + |Y_3^s(U)| + |\mathbf{Y}^b(U)| \\
&\le C\bigg(\sqrt{\delta_0}\mathbf{D}(U) + \big(\mathbf{G}_2(U) - \mathbf{G}_2(\bar{U})\big) + \delta_0\tilde{\mathbf{G}}_2(U) + \left(\frac{\varepsilon}{\lambda}\right)^{1/4}\mathbf{G}_2(\bar{U}) \\
&\qquad + \mathbf{G}_1^-(U) + \left(\frac{\lambda}{\varepsilon}\right)^{1/4}\mathbf{G}_1^+(U)\bigg).
\end{aligned}
$$

Step 3: For the remaining terms, using Hölder's inequality together with (6.37) and (6.14), we have

$$
\begin{aligned}
|Y_4^s(U)|^2 &\le C\left(\frac{\varepsilon}{\lambda}\right)^2\left(\int_{\mathbb{R}} |\mathbf{w}|\,dx\right)\int_{\mathbf{\Omega}^c} |\mathbf{w}||h-\tilde{h}_0|^2\,dx \le C\frac{\varepsilon^2}{\lambda}\mathbf{G}_1^-(U), \\
|\mathbf{Y}^l(U)|^2 &\le C\left(\frac{\varepsilon}{\lambda}\right)^2\left(\int_{\mathbb{R}} |\mathbf{w}|\,dx\right)\int_{\mathbf{\Omega}} |\mathbf{w}|\left(h-\tilde{h}_0 - \frac{p(\bar{v})-p(\tilde{v}_0)}{\sigma}\right)^2\,dx \le C\frac{\varepsilon^2}{\lambda}\mathbf{G}_1^+(U).
\end{aligned}
$$

Therefore, this together with Step1 and Step2 yields

$$
\begin{aligned}
&|\mathbf{Y}^g(U)-\mathbf{Y}^g(\bar U)|^2+|\mathbf{Y}^b(U)|^2+|\mathbf{Y}^l(U)|^2+|\mathbf{Y}^s(U)|^2\\
&\le 2\Big(|\mathbf{Y}^g(U)-\mathbf{Y}^g(\bar U)|+|Y_1^s(U)|+|Y_2^s(U)|+|Y_3^s(U)|+|\mathbf{Y}^b(U)|\Big)^2+2|Y_4^s(U)|^2+|\mathbf{Y}^l(U)|^2\\
&\le C\frac{\varepsilon^2}{\lambda}\bigg(\sqrt{\delta_0}\mathbf{D}(U)+\big(\mathbf{G}_2(U)-\mathbf{G}_2(\bar U)\big)+\delta_0\tilde{\mathbf{G}}_2(U)+\Big(\frac{\varepsilon}{\lambda}\Big)^{1/4}\mathbf{G}_2(\bar U)\\
&\qquad\qquad+\mathbf{G}_1^-(U)+\Big(\frac{\lambda}{\varepsilon}\Big)^{1/4}\mathbf{G}_1^+(U)\bigg).
\end{aligned}
$$

□

7. Proof of Proposition 5.1

This section is dedicated to the proof of Proposition 5.1.

7.1. **Smallness of the localized relative entropy.** In order to use Proposition 6.1 in the proof of Proposition 5.1, we need to show that all bad terms on the region $\{|p(v)-p(\tilde v^{X_1,X_2})|\ge\delta_1\}$ are absorbed by a very small portion of the good terms. For that, we will crucially use the following lemma on smallness of the relative entropy localized by the space-derivative of each weight a_i, under an assumption that the functionals Y_i of (5.18) are bounded below or above. The following lemma is analogous to [24, Lemma 3.2]. Notice that the below assumption $(-1)^{i-1}Y_i(U)\le\varepsilon_i^2$ is weaker than the condition $|Y_i(U)|\le\varepsilon_i^2$ of [24, Lemma 3.2].

Lemma 7.1. *For the given constant $U_*:=(v_*,u_*)\in\mathbb{R}^+\times\mathbb{R}$, there exist positive constants δ_0, C, C_0 such that the following holds. Let $\tilde U^{X_1,X_2}$ be the composite wave for the given constant states $U_-,U_m,U_+\in B_{\delta_0}(U_*)$, where $\varepsilon_1=|p(v_-)-p(v_m)|$ and $\varepsilon_1=|p(v_m)-p(v_+)|$. Then, for any $\lambda>0$ with $\varepsilon_1/\lambda,\varepsilon_2/\lambda<\delta_0$ and $\lambda<\delta_0$, the following estimates hold.*
For each $i=1,2$, and for all $U\in\mathcal{H}_T$ satisfying $(-1)^{i-1}Y_i(U)\le\varepsilon_i^2$,

$$
\int_{\mathbb{R}}|(a_i)_x^{X_i}||h-\tilde h^{X_1,X_2}|^2\,dx+\int_{\mathbb{R}}|(a_i)_x^{X_i}|Q(v|\tilde v^{X_1,X_2})\,dx\le C\frac{\varepsilon_i^2}{\lambda},\quad\forall t>0,\tag{7.1}
$$

and

$$
|Y_i(U)|\le C_0\frac{\varepsilon_i^2}{\lambda},\quad\forall t>0.\tag{7.2}
$$

Proof. We will apply Proposition 6.2 to the two cases where for each $i=1,2$, $\mathbf{w}=-(a_i)_x^{X_i}, \mathbf{v}_1=(\tilde v_i)_x^{X_i}, \mathbf{v}_2=(\tilde h_i)_x^{X_i}$, $\mathbf{a}=a^{X_1,X_2}$, $\varepsilon=\varepsilon_i$, and $\tilde U_0=\tilde U^{X_1,X_2}$ for the composite wave $\tilde U^{X_1,X_2}$.
First, since $U_-,U_m,U_+\in B_{\delta_0}(U_*)$, and $\tilde v_i,\tilde h_i$ are monotone (see Remark 5.1), we find that for each $i=1,2$,

$$
\|\tilde v_i^{X_i}-v_*\|_{L^\infty(\mathbb{R})}<\delta_0,\quad \|\tilde h_i^{X_i}-u_*\|_{L^\infty(\mathbb{R})}<\delta_0,
$$

which implies

$$
\|\tilde v^{X_1,X_2}-v_*\|_{L^\infty(\mathbb{R})}\le\|\tilde v_1^{X_1}-v_m\|_{L^\infty(\mathbb{R})}+\|\tilde v_2^{X_2}-v_*\|_{L^\infty(\mathbb{R})}\le C\varepsilon_1+\delta_0\le C\delta_0,
$$

and similarly,

$$
\|\tilde h^{X_1,X_2}-u_*\|_{L^\infty(\mathbb{R})}\le C\delta_0.
$$

Moreover, note that

$$\tilde{v}^{X_1,X_2}(x) \geq \frac{v_*}{2}, \quad \forall x \in \mathbb{R}.$$

Thus, the assumptions of (6.4) hold by the above facts and the first inequality of (5.23). By the properties of the weights (see Section **??**) and of the waves (see Lemma 5.1 and Remark 5.1), the remaining assumptions (6.5)-(6.6) also hold.
For any fixed $i = 1, 2$, notice that

$$\mathbf{Y}(U) = -\int_{\mathbb{R}} (a_i)_x^{X_i} \eta(U|\tilde{U}^{X_1,X_2})dx + \int_{\mathbb{R}} a^{X_1,X_2} (\tilde{U}_i)_x^{X_i} \nabla^2 \eta(\tilde{U}^{X_1,X_2})(U - \tilde{U}^{X_1,X_2})dx = Y_i(U),$$

and $\mathrm{sgn}(\mathbf{w}) = -\mathrm{sgn}\big((a_i)_x^{X_i}\big) = (-1)^{i-1}$, and thus,

$$\mathrm{sgn}(\mathbf{w})\mathbf{Y}(U) = (-1)^{i-1} Y_i(U).$$

Hence, Proposition 6.2 implies the desired estimates. □

7.2. **Estimates for fixing the size of truncation.** We will apply Proposition 6.1 to each of the waves $\tilde{v}_1$ and $\tilde{v}_2$ later on. More precisely, for the weights a, a_1, a_2 as in (5.20), and for the Lipschitz functions $\phi_{1,t}$ and $\phi_{2,t}$ as in (7.26) and (7.27), we consider the following functionals: for each $i = 1, 2$,

$$\begin{aligned}
\mathcal{Y}_i^g(v) &:= -\frac{1}{2\sigma_i^2} \int_{\mathbb{R}} (a_i)_x \phi_{i,t}^2(x) |p(v) - p(\tilde{v}_i)|^2 dx - \int_{\mathbb{R}} (a_i)_x \phi_{i,t}^2(x) Q(v|\tilde{v}_i) dx \\
&\quad - \int_{\mathbb{R}} a \partial_x p(\tilde{v}_i) \phi_{i,t}(x)(v - \tilde{v}_i) dx + \frac{1}{\sigma_i} \int_{\mathbb{R}} a (\tilde{h}_i)_x \phi_{i,t}(x) \big(p(v) - p(\tilde{v}_i)\big) dx, \\
\mathcal{I}_{1i}(v) &:= \sigma_i \int_{\mathbb{R}} a (\tilde{v}_i)_x \phi_{i,t}^2(x) p(v|\tilde{v}_i) dx, \\
\mathcal{I}_{2i}(v) &:= \frac{1}{2\sigma_i} \int_{\mathbb{R}} (a_i)_x \phi_{i,t}^2(x) |p(v) - p(\tilde{v}_i)|^2 dx, \qquad (7.3)\\
\mathcal{G}_{2i}(v) &:= \sigma_i \int_{\mathbb{R}} (a_i)_x \bigg(\frac{1}{2\gamma} p(\tilde{v}_i)^{-\frac{1}{\gamma}-1} \phi_{i,t}^2(x) \big(p(v) - p(\tilde{v}_i)\big)^2 \\
&\quad - \frac{1+\gamma}{3\gamma^2} p(\tilde{v}_i)^{-\frac{1}{\gamma}-2} \phi_{i,t}^3(x) \big(p(v) - p(\tilde{v}_i)\big)^3 \bigg) dx, \\
\mathcal{D}_i(v) &:= \int_{\mathbb{R}} a\, v^\beta |\partial_x \big(\phi_{i,t}(x)(p(v) - p(\tilde{v}_i))\big)|^2 dx,
\end{aligned}$$

where the functions $a, (a_i)_x, \tilde{v}_i, (\tilde{v}_i)_x, (\tilde{h}_i)_x$ are evaluated at $x - \sigma_i t - X_i(t)$.

However, since the value of δ_1 is itself conditioned to the constant C_2 of Proposition 6.1, we should find the bound of $\mathcal{Y}_i^g$ on the unconditional level (for the assumption (6.2)).

For that, we will first define a truncation on $|p(v) - p(\tilde{v}^{X_1,X_2})|$ with any $k > 0$, and then the special case $k = \delta_1$ as in Proposition 6.1. But for now, we consider a general case k to estimate the constant C_2. For that, let ψ_k be a continuous function on $\mathbb{R}$ defined by

$$\psi_k(y) = \inf\left(k, \sup(-k, y)\right), \quad k > 0. \qquad (7.4)$$

We then define the function $\bar{v}_k$ uniquely (since the function p is one to one) by

$$p(\bar{v}_k) - p(\tilde{v}^{X_1,X_2}) = \psi_k\big(p(v) - p(\tilde{v}^{X_1,X_2})\big). \qquad (7.5)$$

Notice that $\|p(\bar{v}_k) - p(\tilde{v}^{X_1,X_2})\|_\infty \le k$.

We have the following lemma from Lemma 7.1.

Lemma 7.2. *For the given constant $U_* := (v_*, u_*) \in \mathbb{R}^+ \times \mathbb{R}$, there exists positive constants δ_0, C_2, k_0 such that the following holds. For any $\varepsilon_1, \varepsilon_2, \lambda > 0$ satisfying $\varepsilon_1/\lambda, \varepsilon_2/\lambda < \delta_0$ and $\lambda < \delta_0$, there exists $t_0 > 0$ such that the following estimates hold.*

$$|\mathcal{Y}_i^g(\bar{v}_k)| \le C_2 \frac{\varepsilon_i^2}{\lambda}, \qquad \forall k \le k_0, \quad \forall t \ge t_0. \tag{7.6}$$

Proof. For simplicity, we omit the dependence of the wave and weight on shifts, that is, $\tilde{v} := \tilde{v}^{X_1,X_2}, \tilde{v}_i := \tilde{v}_i^{X_i}$ and $(a_i)_x := (a_i)_x^{X_i}$, etc.

To get the desired estimate, we will use (7.1). For that, we need to replace each wave $\tilde{v}_i$ by the composite wave $\tilde{v}$ as follows.
First, using (5.23), (5.7) and $|\phi_{i,t}| \le 1$, we have

$$\begin{aligned} |\mathcal{Y}_i^g(\bar{v}_k)| &= \bigg| -\frac{1}{2\sigma_i^2}\int_{\mathbb{R}} (a_i)_x \phi_{i,t}^2 |p(\bar{v}_k) - p(\tilde{v}_i)|^2 dx - \int_{\mathbb{R}} (a_i)_x \phi_{i,t}^2 Q(\bar{v}_k|\tilde{v}_i) dx \\ &\qquad - \int_{\mathbb{R}} a \partial_x p(\tilde{v}_i) \phi_{i,t} (\bar{v}_k - \tilde{v}_i) dx + \frac{1}{\sigma_i}\int_{\mathbb{R}} a (\tilde{h}_i)_x \phi_{i,t} \big(p(\bar{v}_k) - p(\tilde{v}_i)\big) dx \bigg| \\ &\le C(I_1 + I_2 + \cdots + I_5), \end{aligned}$$

where

$$\begin{aligned} I_1 &:= \int_{\mathbb{R}} |(a_i)_x| |p(\bar{v}_k) - p(\tilde{v})|^2 \, dx, \\ I_2 &:= \int_{\mathbb{R}} \frac{\lambda}{\varepsilon_i} |(\tilde{v}_i)_x| |p(\tilde{v}) - p(\tilde{v}_i)|^2 \, dx, \\ I_3 &:= \int_{\mathbb{R}} |(a_i)_x| Q(\bar{v}_k|\tilde{v}_i) \, dx, \\ I_4 &:= \int_{\mathbb{R}} \frac{\varepsilon_i}{\lambda} |(a_i)_x| \Big(|\bar{v}_k - \tilde{v}| + |p(\bar{v}_k) - p(\tilde{v})| \Big) dx, \\ I_5 &:= \int_{\mathbb{R}} |(\tilde{v}_i)_x| \Big(|\tilde{v} - \tilde{v}_i| + |p(\tilde{v}) - p(\tilde{v}_i)| \Big) dx. \end{aligned}$$

Choose $k_0 \le \delta_*/2$ for δ_* of Lemma A.3.
Then, for any $k \le k_0$, we have $|p(\bar{v}_k) - p(\tilde{v})| \le k \le \frac{\delta_*}{2}$.
Thus, using (A.10) with δ_0 small enough, we have

$$I_1 \le C \int_{\mathbb{R}} |(a_i)_x| Q(\bar{v}_k|\tilde{v}) \, dx.$$

For I_3, we use Lemma A.5 to have

$$I_3 = \int_{\mathbb{R}} |(a_i)_x| \Big(Q(\bar{v}_k|\tilde{v}) - Q(\tilde{v}_i|\tilde{v}) + \big(Q'(\tilde{v}_i) - Q'(\tilde{v})\big)(\tilde{v}_i - \bar{v}_k) \Big) \, dx.$$

Using (A.9) and $|\bar{v}_k| \le C$, we have

$$I_3 \le \int_{\mathbb{R}} |(a_i)_x| Q(\bar{v}_k|\tilde{v}) dx + C \int_{\mathbb{R}} |(a_i)_x| \big(|p(\tilde{v}) - p(\tilde{v}_i)|^2 + |\tilde{v}_i - \tilde{v}| \big) dx =: I_{31} + I_{32}.$$

Using (A.1) and (A.10), we have

$$I_4 \le \sqrt{\int_{\mathbb{R}} \Big(\frac{\varepsilon_i}{\lambda}\Big)^2 |(a_i)_x|\,dx}\sqrt{\int_{\mathbb{R}} |(a_i)_x|\Big(|\bar v_k - \tilde v|^2 + |p(\bar v_k) - p(\tilde v)|^2\Big)\,dx}$$
$$\le C\sqrt{\frac{\varepsilon_i^2}{\lambda}}\sqrt{\int_{\mathbb{R}} |(a_i)_x| Q(\bar v_k|\tilde v)\,dx}.$$

Notice that since the definition of $\bar v_k$ implies either $\tilde v \le \bar v_k \le v$ or $v \le \bar v_k \le \tilde v$, it follows from (A.2) that

$$Q(v|\tilde v) \ge Q(\bar v_k|\tilde v).$$

This and (7.1) imply

$$I_1 + I_{31} + I_4 \le C\int_{\mathbb{R}} |(a_i)_x| Q(v|\tilde v)\,dx + C\sqrt{\frac{\varepsilon_i^2}{\lambda}}\sqrt{\int_{\mathbb{R}} |(a_i)_x| Q(v|\tilde v)\,dx} \le C\frac{\varepsilon_i^2}{\lambda}.$$

For the remaining terms, we use $\tilde v, \tilde v_i \in (v_-/2, 2v_-)$ to have

$$I_2 + I_{32} + I_5 \le C\frac{\lambda}{\varepsilon_i}\int_{\mathbb{R}} |(\tilde v_i)_x||\tilde v - \tilde v_i|dx.$$

Then, using (7.7) of the following lemma together with taking $\delta_0 < \varepsilon_0$, we have

$$I_2 + I_{32} + I_5 \le C\lambda \exp\Big(-C\min(\varepsilon_1, \varepsilon_2)t\Big).$$

We now choose t_0 big enough such that

$$\lambda \exp\Big(-C\min(\varepsilon_1, \varepsilon_2)t_0\Big) \le C\frac{(\min(\varepsilon_1, \varepsilon_2))^2}{\lambda}.$$

Then, for all $t \ge t_0$,

$$I_2 + I_{32} + I_5 \le C\frac{\varepsilon_i^2}{\lambda}.$$

Hence, for some $C_2 > 0$,

$$|\mathcal{Y}_i^g(\bar v_k)| \le C_2\frac{\varepsilon_i^2}{\lambda}, \qquad \forall k \le k_0, \quad \forall t \ge t_0.$$

□

The following lemma provides inequalities on the interaction of waves, which are useful in the proofs of Lemma 7.2, the estimate (7.21) and Lemma 7.5.

Lemma 7.3. *For given $v_- > 0$ and $u_- \in \mathbb{R}$, there exist positive constants ε_0, C such that for any $\varepsilon_1, \varepsilon_2 \in (0, \varepsilon_0)$, the following estimates hold.*
For each $i = 1, 2$,

$$\int_{\mathbb{R}} |(\tilde v_i)_x^{X_i}||\tilde v^{X_1,X_2} - \tilde v_i^{X_i}|dx \le C\varepsilon_1\varepsilon_2 \exp\Big(-C\min(\varepsilon_1, \varepsilon_2)t\Big), \qquad t > 0, \tag{7.7}$$

and

$$\int_{\mathbb{R}} |(\tilde v_1)_x^{X_1}||(\tilde v_2)_x^{X_2}|dx \le C\varepsilon_1\varepsilon_2 \exp\Big(-C\min(\varepsilon_1, \varepsilon_2)t\Big), \qquad t > 0. \tag{7.8}$$

Proof. Proof of (7.7) We first consider $i=1$. By (5.2), there exists $C>0$ such that

$$(7.9)\quad |(\tilde{v}_1)_x^{X_1}| = |\partial_x \tilde{v}_1(x-\sigma_1 t - X_1(t))| \le C\varepsilon_1^2 \exp\big(-C\varepsilon_1|x-\sigma_1 t - X_1(t)|\big), \quad \forall x\in\mathbb{R},\ t>0.$$

Since $\tilde{v}^{X_1,X_2} = \tilde{v}_1^{X_1} + \tilde{v}_2^{X_2} - v_m$ (by (5.15)), it follows from Lemma 5.1 that

$$(7.10)\quad |\tilde{v}^{X_1,X_2} - \tilde{v}_1^{X_1}| = |\tilde{v}_2^{X_2} - v_m| \le \begin{cases} C\varepsilon_2 \exp\big(-C\varepsilon_2|x-\sigma_2 t - X_2(t)|\big), & \text{if } x\le \sigma_2 t + X_2(t),\\ C\varepsilon_2, & \text{if } x\ge \sigma_2 t + X_2(t).\end{cases}$$

Thus, using the above estimates together with the fact that $\sigma_2 t + X_2(t) > 0$ by (5.31), we find

$$|(\tilde{v}_1)_x^{X_1}|^{1/2} |\tilde{v}^{X_1,X_2} - \tilde{v}_1^{X_1}| \le \begin{cases} C\varepsilon_1\varepsilon_2 \exp\big(-C\varepsilon_2|x-\sigma_2 t - X_2(t)|\big), & \text{if } x\le 0,\\ C\varepsilon_1\varepsilon_2 \exp\big(-C\varepsilon_1|x-\sigma_1 t - X_1(t)|\big), & \text{if } x\ge 0.\end{cases}$$

Since (5.31) implies that

$$(7.11)\quad \begin{aligned} x\le 0 \quad &\Rightarrow \quad x-(\sigma_2 t + X_2(t)) \le x - \frac{\sigma_2}{2}t \le -\frac{\sigma_2}{2}t < 0,\\ x\ge 0 \quad &\Rightarrow \quad x-(\sigma_1 t + X_1(t)) \ge x - \frac{\sigma_1}{2}t \ge -\frac{\sigma_1}{2}t > 0,\end{aligned}$$

we have

$$|(\tilde{v}_1)_x^{X_1}|^{1/2} |\tilde{v}^{X_1,X_2} - \tilde{v}_1^{X_1}| \le C\varepsilon_1\varepsilon_2 \exp\big(-C\min(\varepsilon_1,\varepsilon_2)t\big), \quad \forall x\in\mathbb{R},\ t>0.$$

Therefore, using

$$(7.12)\quad \int_{\mathbb{R}} |(\tilde{v}_1)_x^{X_1}|^{1/2} dx \le C\int_{\mathbb{R}} \varepsilon_1 \exp\big(-C\varepsilon_1|x-\sigma_1 t - X_1(t)|\big) dx \le C,$$

we have (7.7) for $i=1$.
Likewise, for $i=2$, we use Lemma 5.1 to have

$$(7.13)\quad |(\tilde{v}_2)_x^{X_2}| \le C\varepsilon_2^2 \exp\big(-C\varepsilon_2|x-\sigma_2 t - X_2(t)|\big), \quad \forall x\in\mathbb{R},\ t>0,$$

and

$$(7.14)\quad |\tilde{v}^{X_1,X_2} - \tilde{v}_2^{X_2}| \le \begin{cases} C\varepsilon_1, & \text{if } x\le \sigma_1 t + X_1(t),\\ C\varepsilon_1 \exp\big(-C\varepsilon_1|x-\sigma_1 t - X_1(t)|\big), & \text{if } x\ge \sigma_1 t + X_1(t),\end{cases}$$

which imply

$$|(\tilde{v}_2)_x^{X_2}|^{1/2} |\tilde{v}^{X_1,X_2} - \tilde{v}_2^{X_2}| \le \begin{cases} C\varepsilon_1\varepsilon_2 \exp\big(-C\varepsilon_2|x-\sigma_2 t - X_2(t)|\big), & \text{if } x\le 0,\\ C\varepsilon_1\varepsilon_2 \exp\big(-C\varepsilon_1|x-\sigma_1 t - X_1(t)|\big), & \text{if } x\ge 0.\end{cases}$$

Therefore we have the desired estimate for $i=2$.
Proof of (7.8) First, we use (5.2) and (5.4) to have

$$|(\tilde{v}_1)_x^{X_1}|^{1/2} |(\tilde{v}_2)_x^{X_2}| \le C\varepsilon_1\varepsilon_2^2 \exp\big(-C\varepsilon_1|x-\sigma_1 t - X_1(t)| - C\varepsilon_2|x-\sigma_2 t - X_2(t)|\big), \quad \forall x\in\mathbb{R},\ t>0.$$

Then using (7.11), we have

$$|(\tilde{v}_1)_x^{X_1}|^{1/2} |(\tilde{v}_2)_x^{X_2}| \le C\varepsilon_1\varepsilon_2^2 \exp\big(-C\min(\varepsilon_1,\varepsilon_2)t\big), \quad \forall x\in\mathbb{R},\ t>0.$$

Therefore, using (7.12), we have (7.8). □

7.3. **Estimates for big values of $|p(v) - p(\tilde{v}^{X_1,X_2})|$.** We now fix the constant δ_1 for Proposition 6.1 associated to the constant C_2 of Lemma 7.2. If needed, we retake δ_1 such that $\delta_1 < k_0$ for the constant k_0 of (7.6). (since Proposition 6.1 is valid for any smaller δ_1). From now on, we set (without confusion)

$$\bar{v} := \bar{v}_{\delta_1}, \qquad \bar{U} := (\bar{v}, h).$$

In what follows, for simplicity we use the notation:

$$\Omega := \{x \mid (p(v) - p(\tilde{v}^{X_1,X_2}))(x) \le \delta_1\},$$

and omit the dependence of the waves and weights on shifts without confusion, for example, $\tilde{U} := \tilde{U}^{X_1,X_2}, \tilde{U}_i := \tilde{U}_i^{X_i}$ and $(a_i)_x := (a_i)_x^{X_i}$, etc.

In order to present all terms to be controlled on the region $\{|p(v) - p(\tilde{v}^{X_1,X_2})| \ge \delta_1\}$, we split Y_i into four parts Y_i^g, Y_i^b, Y_i^l and Y_i^s as follows: for each $i = 1, 2$,

(7.15)

$$\begin{aligned} Y_i &= -\int_{\mathbb{R}} (a_i)_x \Big(\frac{|h - \tilde{h}|^2}{2} + Q(v|\tilde{v}) \Big) dx + \int_{\mathbb{R}} a \Big(-(\tilde{v}_i)_x p'(\tilde{v})(v - \tilde{v}) + (\tilde{h}_i)_x (h - \tilde{h}) \Big) dx \\ &= Y_i^g + Y_i^b + Y_i^l + Y_i^s, \end{aligned}$$

where

$$\begin{aligned} Y_i^g &:= -\frac{1}{2\sigma_i^2} \int_\Omega (a_i)_x |p(v) - p(\tilde{v})|^2 dx - \int_\Omega (a_i)_x Q(v|\tilde{v}) dx - \int_\Omega a(\tilde{v}_i)_x p'(\tilde{v})(v - \tilde{v}) dx \\ &\quad + \frac{1}{\sigma_i} \int_\Omega a(\tilde{h}_i)_x \big(p(v) - p(\tilde{v})\big) dx, \\ Y_i^b &:= -\frac{1}{2} \int_\Omega (a_i)_x \Big(h - \tilde{h} - \frac{p(v) - p(\tilde{v})}{\sigma_i} \Big)^2 dx \\ &\quad - \frac{1}{\sigma_i} \int_\Omega (a_i)_x \big(p(v) - p(\tilde{v})\big) \Big(h - \tilde{h} - \frac{p(v) - p(\tilde{v})}{\sigma_i} \Big) dx, \\ Y_i^l &:= \int_\Omega a(\tilde{h}_i)_x \Big(h - \tilde{h} - \frac{p(v) - p(\tilde{v})}{\sigma_i} \Big) dx, \end{aligned}$$

and

$$Y_i^s := -\int_{\Omega^c} (a_i)_x Q(v|\tilde{v}) dx - \int_{\Omega^c} a(\tilde{v}_i)_x p'(\tilde{v})(v - \tilde{v}) dx - \int_{\Omega^c} (a_i)_x \frac{|h - \tilde{h}|^2}{2} dx + \int_{\Omega^c} a(\tilde{h}_i)_x (h - \tilde{h}) dx.$$

Notice that Y_i^g consists of the terms related to $v - \tilde{v}$, while Y_i^b and Y_i^l consist of terms related to $h - \tilde{h}$, where Y_i^b is quadratic, and Y_i^l is linear in $h - \tilde{h}$. In Proposition 7.1, we will show that $Y_i^g(U) - Y_i^g(\bar{U})$, $Y_i^b(U)$, $Y_i^l(U)$ and $Y_i^s(U)$ are negligible by the good terms.

For the bad terms B_{δ_1} of (5.25) with $\delta = \delta_1$, we will use the following notations :

$$B_{\delta_1} = \sum_{i=1}^{2} \Big(B_{1i} + B_{2i}^- + B_{2i}^+ + B_{3i} + B_{4i} \Big) + B_5 + B_6, \tag{7.16}$$

where

$$
\begin{aligned}
&B_{1i} := \sigma_i \int_{\mathbb{R}} a(\tilde{v}_i)_x p(v|\tilde{v}) dx,\\
&B_{2i}^- := \int_{\Omega^c} (a_i)_x \big(p(v) - p(\tilde{v})\big)\big(h - \tilde{h}\big) dx, \qquad B_{2i}^+ := \frac{1}{2\sigma_i} \int_{\Omega} (a_i)_x |p(v) - p(\tilde{v})|^2 dx,\\
&B_{3i} := - \int_{\mathbb{R}} (a_i)_x v^\beta \big(p(v) - p(\tilde{v})\big) \partial_x \big(p(v) - p(\tilde{v})\big) dx,\\
&B_{4i} := - \int_{\mathbb{R}} (a_i)_x \big(p(v) - p(\tilde{v})\big)(v^\beta - \tilde{v}^\beta) \partial_x p(\tilde{v}) dx,\\
&B_5 := - \int_{\mathbb{R}} a \partial_x \big(p(v) - p(\tilde{v})\big)(v^\beta - \tilde{v}^\beta) \partial_x p(\tilde{v}) dx,
\end{aligned}
$$

and

$$
B_6 := \int_{\mathbb{R}} a\big(p(v) - p(\tilde{v})\big) \tilde{E}_1 dx - \int_{\mathbb{R}} a(h - \tilde{h}) \tilde{E}_2 dx
$$

with

$$
\begin{aligned}
&\tilde{E}_1 := \partial_x\big(\tilde{v}^\beta \partial_x p(\tilde{v})\big) - \partial_x\big(\tilde{v}_1^\beta \partial_x p(\tilde{v}_1)\big) - \partial_x\big(\tilde{v}_2^\beta \partial_x p(\tilde{v}_2)\big),\\
&\tilde{E}_2 := \partial_x p(\tilde{v}) - \partial_x p(\tilde{v}_1) - \partial_x p(\tilde{v}_2).
\end{aligned}
$$

We also recall the functionals $G_{1i}^-, G_{1i}^+, G_{2i}, D$ of (5.33) for the good terms.
Note that

$$
\begin{aligned}
D(U) &= \int_{\mathbb{R}} a v^\beta |\partial_\xi (p(v) - p(\tilde{v}))|^2 dx\\
&= \int_{\mathbb{R}} a v^\beta |\partial_\xi (p(v) - p(\tilde{v}))|^2 (\mathbf{1}_{\{|p(v)-p(\tilde{v})|\le\delta_1\}} + \mathbf{1}_{\{p(v)-p(\tilde{v})>\delta_1\}} + \mathbf{1}_{\{p(v)-p(\tilde{v})<-\delta_1\}}) dx\\
&= D(\bar{U}) + \int_{\mathbb{R}} a v^\beta |\partial_\xi (p(v) - p(\tilde{v}))|^2 (\mathbf{1}_{\{p(v)-p(\tilde{v})>\delta_1\}} + \mathbf{1}_{\{p(v)-p(\tilde{v})<-\delta_1\}}) dx\\
&\ge D(\bar{U}),
\end{aligned}
$$

and it follow from $Q(v|\tilde{v}) \ge Q(\bar{v}|\tilde{v})$ that

$$
G_{2i}(U) - G_{2i}(\bar{U}) = |\sigma_i| \int_{\mathbb{R}} |(a_i)_x| \left(Q(v|\tilde{v}) - Q(\bar{v}|\tilde{v})\right) dx \ge 0, \tag{7.17}
$$

We now state the following proposition.

Proposition 7.1. *There exist constants $\delta_0, C, C^* > 0$ (in particular, C depends on the constant δ_1) such that for any $\varepsilon_1, \varepsilon_2, \lambda > 0$ satisfying $\varepsilon_1/\lambda, \varepsilon_2/\lambda < \delta_0$ and $\lambda < \delta_0$, the following statements hold.*

1. For each $i=1,2$, for all U satisfying $(-1)^{i-1}Y_i(U)\le\varepsilon_i^2$, the following estimates hold:

$$
\begin{aligned}
&|B_{1i}(U)-B_{1i}(\bar U)|\le C\delta_0\left(D(U)+\big(G_{2i}(U)-G_{2i}(\bar U)\big)+\sum_{j=1}^{2}\frac{\varepsilon_j}{\lambda}G_{2j}(U)\right),\\
&|B_{2i}^-(U)|\le\delta_0\left(D(U)+\sum_{j=1}^{2}\frac{\varepsilon_j}{\lambda}G_{2j}(U)\right)+(\delta_1+C\delta_0)G_{1i}^-(U),\\
&|B_{2i}^+(U)-B_{2i}^+(\bar U)|\le\sqrt{\delta_0}D(U),
\end{aligned}
\tag{7.18}
$$

and

$$
|B_{1i}(\bar U)|+|B_{2i}^+(\bar U)|\le C\int_{\mathbb{R}}|(a_i)_x|Q(\bar v|\tilde v)\,dx\le C^*\frac{\varepsilon_i^2}{\lambda}.
\tag{7.19}
$$

2. For all U such that $(-1)^{i-1}Y_i(U)\le\varepsilon_i^2$ for all $i=1,2$, the following estimates hold:

$$
\begin{aligned}
&\sum_{i=1}^{2}\Big(|B_{3i}(U)|+|B_{4i}(U)|\Big)+|B_5(U)|\\
&\qquad\le C\delta_0D(U)+C\delta_0\sum_{i=1}^{2}\Big(\big(G_{2i}(U)-G_{2i}(\bar U)\big)+\frac{\varepsilon_i}{\lambda}G_{2i}(\bar U)\Big),
\end{aligned}
\tag{7.20}
$$

$$
\begin{aligned}
|B_6(U)|&\le C\delta_0D(U)+C\delta_0\sum_{i=1}^{2}\Big(G_{1i}^-(U)+G_{1i}^+(U)+\big(G_{2i}(U)-G_{2i}(\bar U)\big)+\frac{\varepsilon_i}{\lambda}G_{2i}(\bar U)\Big)\\
&\quad+C\exp\Big(-C\min(\varepsilon_1,\varepsilon_2)t\Big),\qquad t>0,
\end{aligned}
\tag{7.21}
$$

and

$$
|B_{\delta_1}(U)|\le C\sqrt{\delta_0}D(U)+C.
\tag{7.22}
$$

3. For each $i=1,2$, for all U satisfying $(-1)^{i-1}Y_i(U)\le\varepsilon_i^2$ and $D(U)\le 2\frac{C^*}{\sqrt{\delta_0}}\frac{\varepsilon_i^2}{\lambda}$,

$$
\begin{aligned}
&|Y_i^g(U)-Y_i^g(\bar U)|^2+|Y_i^b(U)|^2+|Y_i^l(U)|^2+|Y_i^s(U)|^2\\
&\le C\frac{\varepsilon_i^2}{\lambda}\Bigg(\sqrt{\delta_0}D(U)+\big(G_{2i}(U)-G_{2i}(\bar U)\big)+\delta_0\sum_{j=1}^{2}\frac{\varepsilon_j}{\lambda}G_{2j}(U)+\Big(\frac{\varepsilon_i}{\lambda}\Big)^{1/4}G_{2i}(\bar U)\\
&\qquad+G_{1i}^-(U)+\Big(\frac{\lambda}{\varepsilon_i}\Big)^{1/4}G_{1i}^+(U)\Bigg).
\end{aligned}
\tag{7.23}
$$

Proof. We will apply Proposition 6.3 and Proposition 6.4 to the two cases where for each $i=1,2$, $\mathbf{w}=(a_i)_x^{X_i}$, $\mathbf{v}_1=a^{X_1,X_2}(\tilde v_i)_x^{X_i}$, $\mathbf{v}_2=a^{X_1,X_2}(\tilde h_i)_x^{X_i}$, $\varepsilon=\varepsilon_i$, $\bar{\mathbf{v}}=\bar v$, and $\tilde U_0=\tilde U^{X_1,X_2}$ for the composite wave $\tilde U^{X_1,X_2}$, in addition, $\mathbf{\Omega}=\Omega$.
First of all, by Lemma 5.1, (5.7), (5.23) and (5.21), the assumptions (6.12)-(6.14) and (6.37) of the Propositions hold. In addition, for each $i=1,2$, and any U satisfying $(-1)^{i-1}Y_i(U)\le\varepsilon_i^2$, it follows from (7.1) of Lemma 7.1 that the assumption (6.15) holds.

Then, by (6.39) and (6.40) together with $1/2 \le a \le 1$ and $C^{-1} \le |\sigma_j| \le C$, we have the desired estimates (7.18)-(7.19), where note that since $(\tilde v_0)_x = (\tilde v)_x = (\tilde v_1)_x + (\tilde v_2)_x$,

$$\tilde{\mathbf{G}}_2(U) \le \sum_{j=1}^{2} \frac{\varepsilon_j}{\lambda} \int_{\mathbb{R}} |(a_j)_x| Q(v|\tilde v) dx \le \sum_{j=1}^{2} \frac{\varepsilon_j}{\lambda} G_{2j}(U). \tag{7.24}$$

To show (7.20), we first observe that

$$\begin{aligned} B_{4i}(U) &= -\int_{\mathbb{R}} (a_i)_x \big(p(v) - p(\tilde v)\big)(v^\beta - \tilde v^\beta) p'(\tilde v)\big((\tilde v_1)_x + (\tilde v_2)_x\big) dx, \\ B_5(U) &= -\int_{\mathbb{R}} a\partial_x \big(p(v) - p(\tilde v)\big)(v^\beta - \tilde v^\beta) p'(\tilde v)\big((\tilde v_1)_x + (\tilde v_2)_x\big) dx, \end{aligned}$$

which together with $|p'(\tilde v)| \le C$ yield

$$\begin{aligned} |B_{4i}(U)| &\le C \sum_{j=1}^{2} \int_{\mathbb{R}} |(a_i)_x| |(\tilde v_j)_x| \big|p(v) - p(\tilde v)\big| \big|v^\beta - \tilde v^\beta\big| dx, \\ |B_5(U)| &\le C \sum_{j=1}^{2} \int_{\mathbb{R}} |(\tilde v_j)_x| \big|\partial_x \big(p(v) - p(\tilde v)\big)\big| \big|v^\beta - \tilde v^\beta\big| dx. \end{aligned}$$

Then, applying (6.41) together with $\mathbf{B}_3 = B_{3j}$, $|B_{4i}(U)| \le C \sum_{j=1}^2 \mathbf{B}_4(U)$ and $|B_5(U)| \le C \sum_{j=1}^2 \mathbf{B}_5(U)$ for each i, j, and then summing them up together with (7.24), we obtain the desired estimate (7.20).

For (7.23), since $D(U) \le 2\frac{C^*}{\sqrt{\delta_0}} \frac{\varepsilon_i^2}{\lambda}$, applying (6.42) together with $\mathbf{D}(U) \le 2D(U)$ (by $1/2 \le a$), we have (7.23).

To show (7.21), we set

$$B_6 = \underbrace{\int_{\mathbb{R}} a\big(p(v) - p(\tilde v)\big)\tilde E_1 dx}_{=:B_{61}} - \underbrace{\int_{\mathbb{R}} a(h - \tilde h)\tilde E_2 dx}_{=:B_{62}},$$

with

$$\begin{aligned} \tilde E_1 &:= \partial_x\big(\tilde v^\beta \partial_x p(\tilde v)\big) - \partial_x\big(\tilde v_1^\beta \partial_x p(\tilde v_1)\big) - \partial_x\big(\tilde v_2^\beta \partial_x p(\tilde v_2)\big), \\ \tilde E_2 &:= \partial_x p(\tilde v) - \partial_x p(\tilde v_1) - \partial_x p(\tilde v_2). \end{aligned}$$

Estimate of B_{61} : First, using $\partial_x \tilde v = \partial_x \tilde v_1 + \partial_x \tilde v_2$, we observe that by setting $f(y) := y^\beta p'(y)$,

$$\begin{aligned} \tilde E_1 &= \partial_x\Big(\big(f(\tilde v) - f(\tilde v_1)\big)\partial_x \tilde v_1 + \big(f(\tilde v) - f(\tilde v_2)\big)\partial_x \tilde v_2\Big) \\ &= \big(f(\tilde v) - f(\tilde v_1)\big)(\tilde v_1)_{xx} + \big(f(\tilde v) - f(\tilde v_2)\big)(\tilde v_2)_{xx} \\ &\quad + \big(f'(\tilde v) - f'(\tilde v_1)\big)|(\tilde v_1)_x|^2 + \big(f'(\tilde v) - f'(\tilde v_2)\big)|(\tilde v_2)_x|^2 + 2f'(\tilde v)(\tilde v_1)_x(\tilde v_2)_x, \end{aligned}$$

which together with (5.2), (5.4), (5.6) and $C^{-1} < \tilde v, \tilde v_i < C$ implies

$$|\tilde E_1| \le C\varepsilon_1 |(\tilde v_1)_x| |\tilde v - \tilde v_1| + C\varepsilon_2 |(\tilde v_2)_x| |\tilde v - \tilde v_2| + C|(\tilde v_1)_x| |(\tilde v_2)_x|.$$

Then, using (5.23) and $\varepsilon_1, \varepsilon_2 < \delta_0$, we have

$$
\begin{aligned}
|B_{61}| &= \left| \int_\Omega a\big(p(v) - p(\tilde v)\big)\tilde E_1 dx + \int_{\Omega^c} a\big(p(v) - p(\tilde v)\big)\tilde E_1 dx \right| \\
&\le C\sum_{i=1}^2 \Bigg(\underbrace{\delta_0 \frac{\varepsilon_i}{\lambda}\int_\Omega |(a_i)_x|\big|p(v) - p(\tilde v)\big|^2 dx}_{=:K_1} + \underbrace{\int_\Omega |(\tilde v_i)_x||\tilde v - \tilde v_i|^2 dx + \int_\Omega |(\tilde v_1)_x||(\tilde v_2)_x| dx}_{=:K_2} \Bigg) \\
&\quad + C\sum_{i=1}^2 \underbrace{\left(\int_{\Omega^c} |(a_i)_x|\big|p(v) - p(\tilde v)\big|^2 dx\right)^{1/2} \left(\int_{\Omega^c} |(\tilde v_i)_x||\tilde v - \tilde v_i|^2 dx + \int_\Omega |(\tilde v_1)_x||(\tilde v_2)_x| dx\right)^{1/2}}_{=:K_3}.
\end{aligned}
$$

Using (7.18), (7.19) and the definition of $B_{2i}^+(U)$ with $\sigma_i (a_i)_x > 0$, we have

(7.25)

$$
\begin{aligned}
K_1 &\le \delta_0 \frac{\varepsilon_i}{\lambda}\left(\int_\Omega |(a_i)_x| \Big| |p(v) - p(\tilde v)|^2 - |p(\bar v) - p(\tilde v)|^2 \Big| dx + \int_\Omega |(a_i)_x||p(\bar v) - p(\tilde v)|^2 dx \right) \\
&\le \delta_0 \frac{\varepsilon_i}{\lambda}\left(|B_{2i}^+(U) - B_{2i}^+(\bar U)| + |B_{2i}^+(\bar U)| \right) \\
&\le \delta_0 \frac{\varepsilon_i}{\lambda}\left(\sqrt{\delta_0} D(U) + \int_\Omega |(a_i)_x| Q(\bar v|\tilde v)\, d\xi \right) \le \delta_0 \frac{\varepsilon_i}{\lambda}\left(\sqrt{\delta_0} D(U) + G_{2i}(\bar U) \right).
\end{aligned}
$$

By Lemma 7.3 with $|\tilde v - \tilde v_i|^2 \le C|\tilde v - \tilde v_i|$, we have

$$
K_2 \le C\varepsilon_1\varepsilon_2 \exp\Big(- C\min(\varepsilon_1, \varepsilon_2)t \Big), \qquad t > 0.
$$

For K_3, we first see that (as in K_2)

$$
K_3 \le C\sqrt{\varepsilon_1\varepsilon_2}\exp\Big(- C\min(\varepsilon_1, \varepsilon_2)t \Big) \left(\int_{\Omega^c} |(a_i)_x|\big|p(v) - p(\tilde v)\big|^2 dx \right)^{1/2}, \qquad t > 0.
$$

Using $\big|p(\bar v) - p(\tilde v)\big| \le \delta_1$ and (5.21), we have

$$
\begin{aligned}
\int_{\Omega^c} |(a_i)_x|\big|p(v) - p(\tilde v)\big|^2 dx &\le 2\int_{\Omega^c} |(a_i)_x|\big|p(v) - p(\bar v)\big|^2 dx + 2\int_{\Omega^c} |(a_i)_x|\big|p(\bar v) - p(\tilde v)\big|^2 dx \\
&\le 2\int_{\Omega^c} |(a_i)_x|\big|p(v) - p(\bar v)\big|^2 dx + 2\delta_1^2\lambda.
\end{aligned}
$$

Applying (6.23) with (7.24), we have

$$
\int_{\Omega^c} |(a_i)_x|\big|p(v) - p(\bar v)\big|^2 dx \le C\frac{\varepsilon_i^{2-q}}{\lambda}\left(D(U) + \sum_{j=1}^2 \frac{\varepsilon_j}{\lambda} G_{2j}(U) \right)^q,
$$

where $q := \frac{2\gamma}{\gamma+\alpha}$, and note that $1 < q < 2$ by $0 < \alpha < \gamma$.
Therefore, using Young's inequality, we have

$$\begin{aligned}
K_3 &\le C\sqrt{\varepsilon_1\varepsilon_2}\exp\Big(-C\min(\varepsilon_1,\varepsilon_2)t\Big)\left[\sqrt{\frac{\varepsilon_i^{2-q}}{\lambda}}\left(D(U)+\sum_{j=1}^2\frac{\varepsilon_j}{\lambda}G_{2j}(U)\right)^{q/2}+C\sqrt{\lambda}\right]\\
&\le C\sqrt{\frac{\varepsilon_1\varepsilon_2}{\lambda}}\left(D(U)+\sum_{j=1}^2\frac{\varepsilon_j}{\lambda}G_{2j}(U)\right)^{q/2}\sqrt{\varepsilon_1\varepsilon_2}\exp\Big(-C\min(\varepsilon_1,\varepsilon_2)t\Big)\\
&\qquad +C\exp\Big(-C\min(\varepsilon_1,\varepsilon_2)t\Big)\\
&\le \delta_0\bigg(D(U)+\sum_{j=1}^2\frac{\varepsilon_j}{\lambda}G_{2j}(U)\bigg)+C\exp\Big(-C\min(\varepsilon_1,\varepsilon_2)t\Big).
\end{aligned}$$

Hence,

$$|B_{61}| \le C\delta_0 D(U) + C\delta_0\sum_{i=1}^2\Big(\big(G_{2i}(U)-G_{2i}(\bar U)\big)+\frac{\varepsilon_i}{\lambda}G_{2i}(\bar U)\Big)+C\exp\Big(-C\min(\varepsilon_1,\varepsilon_2)t\Big).$$

Estimate of B_{62} : Likewise, since

$$|\tilde E_2| \le \big|p'(\tilde v)-p'(\tilde v_1)\big||(\tilde v_1)_x| + \big|p'(\tilde v)-p'(\tilde v_2)\big||(\tilde v_2)_x| \le C|(\tilde v_1)_x||\tilde v-\tilde v_1| + C|(\tilde v_2)_x||\tilde v-\tilde v_2|,$$

we use the Young's inequality and Lemma 7.3 to have

$$\begin{aligned}
|B_{62}| &\le \sum_{i=1}^2\left(\delta_0\frac{\varepsilon_i}{\lambda}\int_{\mathbb{R}}|(a_i)_x||h-\tilde h|^2dx+\frac{C}{\delta_0}\int_\Omega|(\tilde v_i)_x||\tilde v-\tilde v_i|^2dx\right)\\
&\le C\sum_{i=1}^2\delta_0\frac{\varepsilon_i}{\lambda}\int_{\mathbb{R}}|(a_i)_x||h-\tilde h|^2dx+C\exp\Big(-C\min(\varepsilon_1,\varepsilon_2)t\Big).
\end{aligned}$$

Note that using the good terms G_{1i}^- and G_{1i}^+,

$$\begin{aligned}
\int_{\mathbb{R}}|(a_i)_x||h-\tilde h|^2dx &= \int_{\Omega^c}|(a_i)_x||h-\tilde h|^2dx+\int_\Omega|(a_i)_x||h-\tilde h|^2dx\\
&\le CG_{1i}^-(U)+CG_{1i}^+(U)+C\int_\Omega|(a_i)_x||p(v)-p(\tilde v)|^2dx.
\end{aligned}$$

Therefore, using (7.25), we have

$$|B_{62}| \le C\delta_0\frac{\varepsilon_i}{\lambda}\Big(G_{1i}^-(U)+G_{1i}^+(U)+\sqrt{\delta_0}D(U)+G_{2i}(\bar U)\Big)+C\exp\Big(-C\min(\varepsilon_1,\varepsilon_2)t\Big).$$

Proof of (7.22)**:** First, it follows from (7.19) and (7.18) with (7.1) that

$$\begin{aligned}
G_{1i}^+(U) &\le C\int_\Omega|(a_i)_x||h-\tilde h|^2dx+\int_\Omega|(a_i)_x||p(v)-p(\tilde v)|^2dx\\
&\le C\frac{\varepsilon_i^2}{\lambda}+C|B_{2i}^+(U)| \le C\frac{\varepsilon_i^2}{\lambda}+C\sqrt{\delta_0}D(U).
\end{aligned}$$

This together with using (7.18)-(7.21), (7.1), (7.17) and recalling (7.16), implies

$$|B_{\delta_1}(U)| \le C\sqrt{\delta_0}D(U) + C\sum_{i=1}^{2}\frac{\varepsilon_i^2}{\lambda} + C\exp\Big(-C\min(\varepsilon_1,\varepsilon_2)t\Big), \quad t>0.$$

Therefore, we have the rough bound (7.22).

□

7.4. **Estimates for separation of waves.** We define a non-negative Lipschitz monotone function $\phi_{1,t}$ on $\mathbb{R}$ as follows: for any fixed $t>0$,

$$\phi_{1,t}(x) = \begin{cases} 1 & \text{if } x < \frac{1}{2}(X_1(t)+\sigma_1 t), \\ \text{linear} & \text{if } \frac{1}{2}(X_1(t)+\sigma_1 t) \le x \le \frac{1}{2}(X_2(t)+\sigma_2 t), \\ 0 & \text{if } x > \frac{1}{2}(X_2(t)+\sigma_2 t). \end{cases} \tag{7.26}$$

Likewise, we define a non-negative Lipschitz monotone function $\phi_{2,t}$ on $\mathbb{R}$ such that $\phi_{1,t}(x)+\phi_{2,t}(x)=1$ for all $x\in\mathbb{R}$ and $t>0$. Thus, $\phi_{2,t}$ satisfies

$$\phi_{2,t}(x) = \begin{cases} 0 & \text{if } x < \frac{1}{2}(X_1(t)+\sigma_1 t), \\ \text{linear} & \text{if } \frac{1}{2}(X_1(t)+\sigma_1 t) \le x \le \frac{1}{2}(X_2(t)+\sigma_2 t), \\ 1 & \text{if } x > \frac{1}{2}(X_2(t)+\sigma_2 t). \end{cases} \tag{7.27}$$

Lemma 7.4. *Let $\phi_{1,t}$ and $\phi_{2,t}$ be the non-negative Lipschitz monotone functions such that (7.26)-(7.27). For given $v_->0$ and $u_-\in\mathbb{R}$, there exist positive constants ε_0, C such that for any $\varepsilon_1,\varepsilon_2\in(0,\varepsilon_0)$ and for all $t>0$,*

$$\int_{\mathbb{R}} |(\tilde{v}_1)_x^{X_1}|\phi_{2,t}dx \le C\varepsilon_1\exp\big(-C\varepsilon_1 t\big),$$
$$\int_{\mathbb{R}} |(\tilde{v}_2)_x^{X_2}|\phi_{1,t}dx \le C\varepsilon_2\exp\big(-C\varepsilon_2 t\big).$$

Proof. The proof is similar to the one of Lemma 7.3.
First, by (7.9),

$$\phi_{2,t}|(\tilde{v}_1)_x^{X_1}|^{1/2} \le C\phi_{2,t}\varepsilon_1\exp\big(-C\varepsilon_1|x-\sigma_1 t - X_1(t)|\big).$$

Note from (7.27) that $0\le\phi_{2,t}\le 1$ and

$$\phi_{2,t} = \phi_{2,t}\mathbf{1}_{\{x\ge(X_1(t)+\sigma_1 t)/2\}}.$$

Since (5.31) implies that

$$x \ge \frac{X_1(t)+\sigma_1 t}{2} \quad \Rightarrow \quad x-(\sigma_1 t + X_1(t)) \ge -\frac{X_1(t)+\sigma_1 t}{2} \ge -\frac{\sigma_1}{4}t > 0,$$

we have

$$\phi_{2,t}|(\tilde{v}_1)_x^{X_1}|^{1/2} \le C\varepsilon_1\exp\big(-C\varepsilon_1 t\big), \quad \forall x\in\mathbb{R}, \quad t>0.$$

Thus, using (7.12) again, we have the desired estimate.
Likewise, using (7.13) and

$$\phi_{1,t} = \phi_{1,t}\mathbf{1}_{\{x\le(X_2(t)+\sigma_2 t)/2\}},$$

and

$$x \le \frac{X_2(t)+\sigma_2 t}{2} \quad \Rightarrow \quad x-(\sigma_2 t + X_2(t)) \le -\frac{X_2(t)+\sigma_2 t}{2} \le -\frac{\sigma_2}{4}t < 0,$$

we have the desired estimate. □

Lemma 7.5. *Let $\phi_{1,t}$ and $\phi_{2,t}$ be the non-negative Lipschitz monotone functions such that (7.26)-(7.27) and $\phi_{1,t}+\phi_{2,t}=1$. For given $v_->0$ and $u_-\in\mathbb{R}$, there exist positive constants C and δ_0 such that for any $\varepsilon_1,\varepsilon_2,\lambda>0$ satisfying $\varepsilon_1/\lambda,\varepsilon_2/\lambda<\delta_0$ and $\lambda<\delta_0$, there exists a constant C_ε depending on $\varepsilon_1,\varepsilon_2$ such that the following estimates hold.*
For each $i=1,2$ and for all $t>0$,

$$\begin{aligned}&\big|B_{1i}(\bar U)-\mathcal{I}_{1i}(\bar v)\big|+\big|B_{2i}^+(\bar U)-\mathcal{I}_{2i}(\bar v)\big|+\big|Y_i^g(\bar U)-\mathcal{Y}_i^g(\bar v)\big|\le C\delta_0\exp\Big(-C_\varepsilon t\Big),\\&-\big(G_{2i}(\bar U)-\mathcal{G}_{2i}(\bar v)\big)\le C\delta_0\exp\big(-C_\varepsilon t\big),\end{aligned}\tag{7.28}$$

and

$$-D(\bar U)\le-\frac{1}{(1+\delta_0)^2}\sum_{i=1}^2\mathcal{D}_i(\bar v)+C\delta_0\left(\exp\big(-C_\varepsilon t\big)+\frac{1}{t^4}\right)+\frac{C}{\delta_0}\frac{1}{t^2}\int_{\mathbb{R}}\eta(U|\tilde U)dx.\tag{7.29}$$

Proof. **Proof of** (7.28)**:** For simplicity, we here use the following notations: for each $i=1,2$,

$$\begin{aligned}\overline{B_{1i}}&:=\sigma_i\int_{\mathbb{R}}a(\tilde v_i)_x\phi_{i,t}^2p(\bar v|\tilde v)dx,\\\overline{B_{2i}^+}&:=\frac{1}{2\sigma_i}\int_{\mathbb{R}}(a_i)_x\phi_{i,t}^2\big|p(\bar v)-p(\tilde v)\big|^2dx,\\\overline{G_{2i}}&:=\sigma_i\int_{\mathbb{R}}(a_i)_x\left(\frac{1}{2\gamma}p(\tilde v)^{-\frac{1}{\gamma}-1}\phi_{i,t}^2\big(p(\bar v)-p(\tilde v)\big)^2-\frac{1+\gamma}{3\gamma^2}p(\tilde v)^{-\frac{1}{\gamma}-2}\phi_{i,t}^3\big(p(\bar v)-p(\tilde v)\big)^3\right)dx,\\\overline{Y_i^g}&:=-\frac{1}{2\sigma_i^2}\int_{\mathbb{R}}(a_i)_x\phi_{i,t}^2|p(\bar v)-p(\tilde v)|^2dx-\int_{\mathbb{R}}(a_i)_x\phi_{i,t}^2Q(\bar v|\tilde v)dx\\&\quad-\int_{\mathbb{R}}a\partial_xp(\tilde v_i)\phi_{i,t}(\bar v-\tilde v)dx+\frac{1}{\sigma_i}\int_{\mathbb{R}}a(\tilde h_i)_x\phi_{i,t}\big(p(\bar v)-p(\tilde v)\big)dx.\end{aligned}$$

We will only prove the case when $i=1$, since the other case can be shown in the same way. To estimate $\big|B_{11}(\bar U)-\mathcal{I}_{11}(\bar v)\big|$, we first separate it into two parts:

$$\big|B_{11}(\bar U)-\mathcal{I}_{11}(\bar v)\big|\le\big|B_{11}(\bar U)-\overline{B_{11}}\big|+\big|\overline{B_{11}}-\mathcal{I}_{11}(\bar v)\big|.$$

Take δ_0 such that $\delta_0<\sqrt{\varepsilon_0}$ for the constant ε_0 of Lemma 7.4. Since $\phi_{1,t}+\phi_{2,t}=1$ with $|\phi_{i,t}|\le1$ for any $i=1,2$, and $p(\bar v|\tilde v)\le C$, we use Lemma 7.4 to have

$$\big|B_{11}(\bar U)-\overline{B_{11}}\big|\le C\int_{\mathbb{R}}|(\tilde v_1)_x|\phi_{2,t}p(\bar v|\tilde v)dx\le C\int_{\mathbb{R}}|(\tilde v_1)_x|\phi_{2,t}dx\le C\varepsilon_1\exp\big(-C\varepsilon_1t\big).$$

To control $\big|\overline{B_{11}}-\mathcal{I}_{11}(\bar v)\big|$, we first use Lemma A.5 to find

$$p(\bar v|\tilde v)-p(\bar v|\tilde v_1)=-p(\tilde v|\tilde v_1)+(p'(\tilde v)-p'(\tilde v_1))(\tilde v-\bar v).$$

Then, using $\bar v,\tilde v,\tilde v_i\in(v_-/2,2v_-)$ and (A.5), we have

$$\big|\overline{B_{11}}-\mathcal{I}_{11}(\bar v)\big|\le C\int_{\mathbb{R}}|(\tilde v_1)_x|\Big(p(\tilde v|\tilde v_1)+|p'(\tilde v)-p'(\tilde v_1)||\tilde v-\bar v|\Big)dx\le C\int_{\mathbb{R}}|(\tilde v_1)_x||\tilde v-\tilde v_1|dx,$$

Therefore, using (7.7), there exists a constant $C_\varepsilon>0$ such that

$$\begin{aligned}\big|B_{11}(\bar U)-\mathcal{I}_{11}(\bar v)\big|&\le C\varepsilon_1\exp\big(-C\varepsilon_1t\big)+C\varepsilon_1\varepsilon_2\exp\Big(-C\min(\varepsilon_1,\varepsilon_2)t\Big)\\&\le C\delta_0\exp\Big(-C_\varepsilon t\Big).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\left|B_{21}^+(\bar U)-\mathcal{I}_{21}(\bar v)\right| &\le \left|B_{21}^+(\bar U)-\overline{B_{21}^+}\right|+\left|\overline{B_{2i}^+}-\mathcal{I}_{21}(\bar v)\right|\\
&\le C\frac{\lambda}{\varepsilon_1}\int_{\mathbb{R}}|(\tilde v_1)_x|\Big(\phi_{2,t}\big|p(\bar v)-p(\tilde v)\big|^2+\Big|\big|p(\bar v)-p(\tilde v)\big|^2-\big|p(\bar v)-p(\tilde v_1)\big|^2\Big|\Big)dx\\
&\le C\frac{\lambda}{\varepsilon_1}\int_{\mathbb{R}}|(\tilde v_1)_x|\Big(\phi_{2,t}+|p(\tilde v)-p(\tilde v_1)|\Big)dx\\
&\le C\frac{\lambda}{\varepsilon_1}\int_{\mathbb{R}}|(\tilde v_1)_x|\Big(\phi_{2,t}+|\tilde v-\tilde v_1|\Big)dx\le C\delta_0\exp\Big(-C_\varepsilon t\Big).\end{aligned}$$

Likewise, using (5.7) and Lemma A.5, we have

$$\begin{aligned}&\left|Y_1^g(\bar U)-\mathcal{Y}_1^g(\bar v)\right|\le\left|Y_1^g(\bar U)-\overline{Y_1^g}\right|+\left|\overline{Y_1^g}-\mathcal{Y}_1^g(\bar v)\right|\\
&\le C\frac{\lambda}{\varepsilon_1}\int_{\mathbb{R}}|(\tilde v_1)_x|\phi_{2,t}dx+C\frac{\lambda}{\varepsilon_1}\int_{\mathbb{R}}|(\tilde v_1)_x|\Big(|p(\tilde v)-p(\tilde v_1)|+Q(\tilde v|\tilde v_1)+|Q'(\tilde v)-Q'(\tilde v_1)|\Big)dx\\
&\quad+C\int_{\mathbb{R}}|(\tilde v_1)_x|\Big(|\tilde v-\tilde v_1|+|p(\tilde v)-p(\tilde v_1)|\Big)dx\\
&\le C\frac{\lambda}{\varepsilon_1}\int_{\mathbb{R}}|(\tilde v_1)_x|\Big(\phi_{2,t}+|\tilde v-\tilde v_1|\Big)dx\le C\delta_0\exp\Big(-C_\varepsilon t\Big).\end{aligned}$$

Next, to estimate

$$-\big(G_{21}(\bar U)-\mathcal{G}_{21}(\bar v)\big)=-\big(G_{21}(\bar U)-\overline{G_{21}}\big)-\big(\overline{G_{21}}-\mathcal{G}_{2i}(\bar v)\big)\,,$$

we first observe that (A.8) with $\sigma_i(a_i)_x>0$ for any i implies

$$\begin{aligned}&G_{21}(\bar U)-\overline{G_{21}}\\
&=\sigma_1\int_{\mathbb{R}}(a_1)_x\left[Q(\bar v|\tilde v)-\left(\frac{1}{2\gamma}p(\tilde v)^{-\frac{1}{\gamma}-1}\phi_{1,t}^2\big(p(\bar v)-p(\tilde v)\big)^2-\frac{1+\gamma}{3\gamma^2}p(\tilde v)^{-\frac{1}{\gamma}-2}\phi_{1,t}^3\big(p(\bar v)-p(\tilde v)\big)^3\right)\right]dx\\
&\ge\sigma_1\int_{\mathbb{R}}(a_1)_x\left[\frac{1}{2\gamma}p(\tilde v)^{-\frac{1}{\gamma}-1}(1-\phi_{1,t}^2)\big(p(\bar v)-p(\tilde v)\big)^2-\frac{1+\gamma}{3\gamma^2}p(\tilde v)^{-\frac{1}{\gamma}-2}(1-\phi_{1,t}^3)\big(p(\bar v)-p(\tilde v)\big)^3\right]dx,\end{aligned}$$

and thus,

$$-\big(G_{21}(\bar U)-\overline{G_{21}}\big)\le C\frac{\lambda}{\varepsilon_1}\int_{\mathbb{R}}|(\tilde v_1)_x|\phi_{2,t}dx\le C\delta_0\exp\big(-C\varepsilon_1 t\big).$$

Moreover, since

$$\begin{aligned}\left|\overline{G_{21}}-\mathcal{G}_{21}(\bar v)\right|\le C\frac{\lambda}{\varepsilon_1}\int_{\mathbb{R}}|(\tilde v_1)_x|\Big(&\left|p(\tilde v)^{-\frac{1}{\gamma}-1}-p(\tilde v_1)^{-\frac{1}{\gamma}-1}\right|+|p(\tilde v)-p(\tilde v_1)|\\
&+\left|p(\tilde v)^{-\frac{1}{\gamma}-2}-p(\tilde v_1)^{-\frac{1}{\gamma}-2}\right|\Big)dx\le C\frac{\lambda}{\varepsilon_1}\int_{\mathbb{R}}|(\tilde v_1)_x||\tilde v-\tilde v_1|dx,\end{aligned}$$

we have

$$-\big(G_{21}(\bar U)-\mathcal{G}_{21}(\bar v)\big)\le C\delta_0\exp\Big(-C_\varepsilon t\Big).$$

Proof of (7.29)**:** First, using the fact that $\phi_{1,t}+\phi_{2,t}=1$ and $1\ge\phi_{i,t}\ge\phi_{i,t}^2\ge0$ for any i, we separate $D(\bar U)$ into

$$D(\bar U)=\int_{\mathbb{R}}a\,\bar v^\beta(\phi_{1,t}+\phi_{2,t})|\partial_x\big(p(\bar v)-p(\tilde v)\big)|^2dx\ge\sum_{i=1}^2\int_{\mathbb{R}}a\,\bar v^\beta\phi_{i,t}^2|\partial_x\big(p(\bar v)-p(\tilde v)\big)|^2dx.$$

Since Young's inequality yields

$$\int_{\mathbb{R}} a\,\bar{v}^\beta|\partial_x\big(\phi_{i,t}(p(\bar{v})-p(\tilde{v}))\big)|^2dx \le (1+\delta_0)\int_{\mathbb{R}} a\,\bar{v}^\beta\phi_{i,t}^2|\partial_x\big(p(\bar{v})-p(\tilde{v})\big)|^2dx + \frac{C}{\delta_0}\int_{\mathbb{R}} a\,\bar{v}^\beta|\partial_x\phi_{i,t}|^2|p(\bar{v})-p(\tilde{v})|^2dx,$$

we have

$$-D(\bar{U}) \le -\frac{1}{1+\delta_0}\sum_{i=1}^2 \underbrace{\int_{\mathbb{R}} a\,\bar{v}^\beta|\partial_x\big(\phi_{i,t}(p(\bar{v})-p(\tilde{v}))\big)|^2dx}_{=:\overline{D_i}} + \frac{C}{\delta_0}\sum_{i=1}^2\int_{\mathbb{R}} a\,\bar{v}^\beta|\partial_x\phi_{i,t}|^2|p(\bar{v})-p(\tilde{v})|^2dx.$$

Note that since (5.31) yields

$$\frac{1}{2}\Big((X_2(t)+\sigma_2 t)-(X_1(t)+\sigma_1 t)\Big) \ge \frac{\sigma_2-\sigma_1}{4}t>0,$$

it follows from (7.26)-(7.27) that for each $i=1,2$,

$$|\partial_x\phi_{i,t}(x)| \le \frac{4}{\sigma_2-\sigma_1}\frac{1}{t},\quad \forall x\in\mathbb{R},\quad t>0. \tag{7.30}$$

This together with $C^{-1}\le a\bar{v}^\beta\le C$ implies

$$-D(\bar{U}) \le -\frac{1}{1+\delta_0}\sum_{i=1}^2\overline{D_i} + \frac{C}{\delta_0}\frac{1}{t^2}\int_{\mathbb{R}}\eta(U|\tilde{U})dx. \tag{7.31}$$

To estimate $\overline{D_i}$, since Young's inequality yields

$$\int_{\mathbb{R}} a\,\bar{v}^\beta|\partial_x\big(\phi_{i,t}(p(\bar{v})-p(\tilde{v}_i))\big)|^2dx \le (1+\delta_0)\int_{\mathbb{R}} a\,\bar{v}^\beta|\partial_x\big(\phi_{i,t}(p(\bar{v})-p(\tilde{v}))\big)|^2dx + \frac{C}{\delta_0}\int_{\mathbb{R}} a\,\bar{v}^\beta|\partial_x\big(\phi_{i,t}(p(\tilde{v})-p(\tilde{v}_i))\big)|^2dx,$$

we have

$$-\overline{D_i} \le -\frac{1}{1+\delta_0}\int_{\mathbb{R}} a\,\bar{v}^\beta|\partial_x\big(\phi_{i,t}(p(\bar{v})-p(\tilde{v}_i))\big)|^2dx + \underbrace{\frac{C}{\delta_0}\int_{\mathbb{R}} a\,\bar{v}^\beta|\partial_x\big(\phi_{i,t}(p(\tilde{v})-p(\tilde{v}_i))\big)|^2dx}_{=:J_i}.$$

First, when $i=1$, we observe that

$$\begin{aligned} J_1 &\le \frac{C}{\delta_0}\int_{\mathbb{R}}|\partial_x\phi_{1,t}|^2|p(\tilde{v})-p(\tilde{v}_1)|^2dx + \frac{C}{\delta_0}\int_{\mathbb{R}}\phi_{1,t}^2|\partial_x\big(p(\tilde{v})-p(\tilde{v}_1)\big)|^2dx \\ &\le \frac{C}{\delta_0}\underbrace{\int_{\mathbb{R}}|\partial_x\phi_{1,t}|^2|\tilde{v}-\tilde{v}_1|^2dx}_{=:J_{11}} + \frac{C}{\delta_0}\underbrace{\int_{\mathbb{R}}\Big(|p'(\tilde{v})-p'(\tilde{v}_1)|^2|(\tilde{v}_1)_x|^2+\phi_{1,t}^2|p'(\tilde{v})||(\tilde{v}_2)_x|^2\Big)dx}_{=:J_{12}}. \end{aligned}$$

Using (7.7) and Lemma 7.4, we have

$$\begin{aligned} J_{12} &\le C\int_{\mathbb{R}}\Big(|\tilde{v}-\tilde{v}_1|^2|(\tilde{v}_1)_x|^2+|\phi_{1,t}(\tilde{v}_2)_x|^2\Big)dx \\ &\le C\varepsilon_1\varepsilon_2\exp\Big(-C\min(\varepsilon_1,\varepsilon_2)t\Big)+C\varepsilon_2\exp\big(-C\varepsilon_2 t\big). \end{aligned}$$

Since (7.26)-(7.27) implies that for each $i=1,2$,

$$\partial_x\phi_{i,t}(x)=\partial_x\phi_{i,t}(x)\mathbf{1}_{\{\frac12(X_1(t)+\sigma_1 t)\le x\le\frac12(X_2(t)+\sigma_2 t)\}},$$

it follows from (7.10) and (7.30) with (7.26) that

$$|\partial_x\phi_{1,t}||\tilde v-\tilde v_1|\le\frac{C}{t}\varepsilon_2\exp\big(-C\varepsilon_2|x-\sigma_2 t-X_2(t)|\big)\mathbf{1}_{\{\frac12(X_1(t)+\sigma_1 t)\le x\le\frac12(X_2(t)+\sigma_2 t)\}}. \tag{7.32}$$

Then, using

$$x\le\frac{X_2(t)+\sigma_2 t}{2}\quad\Rightarrow\quad x-(\sigma_2 t+X_2(t))\le-\frac{X_2(t)+\sigma_2 t}{2}\le-\frac{\sigma_2}{4}t<0,$$

we have

$$|\partial_x\phi_{1,t}||\tilde v-\tilde v_1|\le\frac{C}{t}\varepsilon_2\exp\big(-C\varepsilon_2 t\big),$$

which together with (7.32) implies

$$|\partial_x\phi_{1,t}|^2|\tilde v-\tilde v_1|^2\le\frac{C}{t^2}\varepsilon_2^2\exp\big(-C\varepsilon_2 t\big)\exp\big(-C\varepsilon_2|x-\sigma_2 t-X_2(t)|\big).$$

Thus, we have

$$J_{11}\le\frac{C}{t^2}\varepsilon_2\exp\big(-C\varepsilon_2 t\big).$$

Therefore,

$$J_1\le C\delta_0\left(\exp\big(-C_\varepsilon t\big)+\frac{1}{t^4}\right).$$

Likewise, using (7.7), Lemma 7.4 and (7.14), we have

$$J_2\le C\delta_0\left(\exp\big(-C_\varepsilon t\big)+\frac{1}{t^4}\right).$$

Hence we have

$$-\overline{D_i}\le-\frac{1}{1+\delta_0}\mathcal{D}_i(\bar v)+C\delta_0\left(\exp\big(-C_\varepsilon t\big)+\frac{1}{t^4}\right).$$

which together with (7.31) gives the desired result.

□

7.5. **Proof of Proposition 5.1.** We first apply Proposition 6.1 to the functionals of (7.3) for each waves, and to the weights of (5.20). Let us take δ_0 small enough as $\delta_0<\delta_1$. Then, $U_-,U_m,U_+\in B_{\delta_0}(U_*)\subset B_{\delta_1}(U_*)$. Note that it follows from (5.20) that for each $i=1,2$,

$$\partial_x a_i=-\lambda\frac{\partial_x p(\tilde v_i)}{\varepsilon_i},\quad\text{where } \varepsilon_1=|p(v_-)-p(v_m)|,\quad \varepsilon_2=|p(v_m)-p(v_+)|,$$

and the weight a satisfies $\|a-1\|_{L^\infty(\mathbb{R})}\le\|a_1-1\|_{L^\infty(\mathbb{R})}+\|a_2-1\|_{L^\infty(\mathbb{R})}\le 2\lambda$.
Moreover, it follows from Lemma 7.2 that for each $i=1,2$,

$$|\mathcal{Y}_i^g(\bar v)|\le C_2\frac{\varepsilon_i^2}{\lambda},\quad\forall t\ge t_0. \tag{7.33}$$

In addition, using (7.10), (7.14) with $\varepsilon_i\le\delta_0\ll\delta_1$ (by taking δ_0 small enough as $\delta_0\ll\delta_1$), we have

$$|p(\bar v)-p(\tilde v_i)|\le|p(\bar v)-p(\tilde v)|+|p(\tilde v)-p(\tilde v_i)|\le\delta_1+C|\tilde v-\tilde v_i|\le\delta_1+C\delta_0\le 2\delta_1.$$

Therefore, since the two waves $\tilde{U}_1, \tilde{U}_2$ and the $\bar{v}$ satisfy the hypotheses of Proposition 6.1 when $t \geq t_0$, we find that for each $i = 1, 2$,

$$\begin{aligned} \mathcal{R}^i_{\delta_1}(\bar{v}) &:= -\frac{1}{\varepsilon_i \delta_1} |\mathcal{Y}^g_i(\bar{v})|^2 + \mathcal{I}_{1i}(\bar{v}) + \delta_1 |\mathcal{I}_{1i}(\bar{v})| \\ &\qquad + \mathcal{I}_{2i}(\bar{v}) + \delta_1 \left(\frac{\varepsilon_i}{\lambda}\right) |\mathcal{I}_{2i}(\bar{v})| - \left(1 - \delta_1 \left(\frac{\varepsilon_i}{\lambda}\right)\right) \mathcal{G}_{2i}(\bar{v}) - (1-\delta_1)\mathcal{D}_i(\bar{v}) \\ &\leq 0, \qquad \forall t \geq t_0. \end{aligned} \tag{7.34}$$

For simplicity, we here use the following notations G_{Y_i} to denote the parts of Y_i in $\mathcal{R}$:

$$\begin{aligned} G_{Y_1}(U) &:= -\frac{1}{\varepsilon_1^4} |Y_1(U)|^2 \mathbf{1}_{\{0 \leq Y_1(U) \leq \varepsilon_1^2\}} + \frac{\sigma_1}{2\varepsilon_1^2} |Y_1(U)|^2 \mathbf{1}_{\{-\varepsilon_1^2 \leq Y_1(U) \leq 0\}} - \frac{\sigma_1}{2} Y_1(U) \mathbf{1}_{\{Y_1(U) \leq -\varepsilon_1^2\}}, \\ G_{Y_2}(U) &:= -\frac{1}{\varepsilon_2^4} |Y_2(U)|^2 \mathbf{1}_{\{-\varepsilon_2^2 \leq Y_2(U) \leq 0\}} - \frac{\sigma_2}{2\varepsilon_2^2} |Y_2(U)|^2 \mathbf{1}_{\{0 \leq Y_2(U) \leq \varepsilon_2^2\}} - \frac{\sigma_2}{2} Y_2(U) \mathbf{1}_{\{Y_2(U) \geq \varepsilon_2^2\}}. \end{aligned}$$

That is,

$$\begin{aligned} \mathcal{R}(U) &= G_{Y_1}(U) + G_{Y_2}(U) + B_{\delta_1}(U) + \delta_0 \frac{\min(\varepsilon_1, \varepsilon_2)}{\lambda} |B_{\delta_1}(U)| - G^-_{11}(U) - G^+_{11}(U) \\ &\quad - G^-_{12}(U) - G^+_{12}(U) - \left(1 - \delta_0 \frac{\varepsilon_1}{\lambda}\right) G_{21}(U) - \left(1 - \delta_0 \frac{\varepsilon_2}{\lambda}\right) G_{22}(U) - (1-\delta_0) D(U). \end{aligned}$$

Step 1) We first get a rough bound for short time $t \leq t_0$ as follows.
Using $G_{Y_1}(U) \leq 0$ and $G_{Y_2}(U) \leq 0$ (by $\sigma_1 < 0 < \sigma_2$), and $\delta_0 < 1/2$, we find that for all $t > 0$,

$$\mathcal{R}(U) \leq 2|B_{\delta_1}(U)| - (1-\delta_0) D(U).$$

Thus, using (7.22) of Proposition 7.1 and taking $\delta_0 \ll 1$, we find that for all U satisfying $Y_1(U) \leq \varepsilon_1^2$ and $Y_2(U) \geq -\varepsilon_2^2$,

$$\mathcal{R}(U) \leq C.$$

Step 2) This step is for long time $t \geq t_0$. Without loss of generality, we assume $\varepsilon_1 < \varepsilon_2$. We split this step into the following three cases, depending on the strength of the dissipation term $D(U)$: (i) $D(U) > \frac{2C^*}{\sqrt{\delta_0}} \frac{\varepsilon_2^2}{\lambda}$; (ii) $D(U) \leq \frac{2C^*}{\sqrt{\delta_0}} \frac{\varepsilon_1^2}{\lambda}$; (iii) $\frac{2C^*}{\sqrt{\delta_0}} \frac{\varepsilon_2^2}{\lambda} \geq D(U) > \frac{2C^*}{\sqrt{\delta_0}} \frac{\varepsilon_1^2}{\lambda}$, where C^* is the constant of Proposition 7.1.

Case i) Assume $D(U) > \frac{2C^*}{\sqrt{\delta_0}} \frac{\varepsilon_2^2}{\lambda}$. Then, $D(U) > \frac{2C^*}{\sqrt{\delta_0}} \frac{\varepsilon_i^2}{\lambda}$ for all $i = 1, 2$ (by $\varepsilon_1 < \varepsilon_2$). We first have

$$\mathcal{R}(U) \leq 2|B_{\delta_1}(U)| - \sum_{i=1}^{2} \Big(G^-_{1i}(U) + G^+_{1i}(U) + \frac{1}{2}\big(G_{2i}(U) - G_{2i}(\bar{U})\big) + \frac{1}{2} G_{2i}(\bar{U}) \Big) - \frac{1}{2} D(U).$$

Since it follows from (7.18), (7.19), (7.20) and (7.21) with $\varepsilon_i/\lambda < \delta_0 \le \delta_1 \ll 1$ that for some constant C_ε (depending on $\varepsilon_1, \varepsilon_2$),

$$
\begin{aligned}
|B_{\delta_1}(U)| &\le 2\sum_{i=1}^{2}\Big(|B_{1i}(U)-B_{1i}(\bar U)| + |B_{2i}^{+}(U)-B_{2i}^{+}(\bar U)| + |B_{2i}^{-}(U)| + |B_{3i}(U)| + |B_{4i}(U)|\Big)\\
&\quad + 2\sum_{i=1}^{2}\big(|B_{1i}(\bar U)| + |B_{2i}^{+}(\bar U)|\big) + 2|B_5(U)| + 2|B_6(U)|\\
&\le C\sqrt{\delta_0}D(U) + C\delta_1\sum_{i=1}^{2}\Big(G_{1i}^{-}(U) + G_{1i}^{+}(U) + \big(G_{2i}(U)-G_{2i}(\bar U)\big)\Big) + C\delta_0\sum_{i=1}^{2}\frac{\varepsilon_i}{\lambda}G_{2i}(\bar U)\\
&\quad + 2C^*\sum_{i=1}^{2}\frac{\varepsilon_i^2}{\lambda} + C\exp\Big(-C_\varepsilon t\Big),
\end{aligned}
$$

we find that for any $t>0$,

$$
\begin{aligned}
\mathcal{R}(U) &\le 2C^*\sum_{i=1}^{2}\frac{\varepsilon_i^2}{\lambda} - \frac14 D(U) + C\exp\Big(-C_\varepsilon t\Big)\\
&\le C\exp\Big(-C_\varepsilon t\Big).
\end{aligned}
$$

Case ii) We first notice that (7.2) of Lemma 7.1 implies that

(7.35)
$$
\begin{aligned}
-\frac{\sigma_1}{2}Y_1(U)\mathbf{1}_{\{Y_1(U)\le -\varepsilon_1^2\}} &\le -\frac{\sigma_1}{2}Y_1(U)\mathbf{1}_{\{Y_1(U)\in[-C_0\varepsilon_1^2/\lambda,-\varepsilon_1^2]\}}\\
&\le \frac{\sigma_1\lambda}{2C_0\varepsilon_1^2}|Y_1(U)|^2\mathbf{1}_{\{Y_1(U)\in[-C_0\varepsilon_1^2/\lambda,-\varepsilon_1^2]\}},\\
-\frac{\sigma_2}{2}Y_2(U)\mathbf{1}_{\{Y_2(U)\ge \varepsilon_2^2\}} \le -\frac{\sigma_2}{2}Y_2(U)&\mathbf{1}_{\{Y_2(U)\in[\varepsilon_2^2,C_0\varepsilon_2^2/\lambda]\}} \le -\frac{\sigma_2\lambda}{2C_0\varepsilon_2^2}|Y_2(U)|^2\mathbf{1}_{\{Y_2(U)\in[\varepsilon_2^2,C_0\varepsilon_2^2/\lambda]\}}.
\end{aligned}
$$

Since it follows from (7.15) that for each $i=1,2$,

$$
Y_i(U) = \mathcal{Y}_i^g(\bar v) + \big(Y_i^g(U)-Y_i^g(\bar U)\big) + \big(Y_i^g(\bar U)-\mathcal{Y}_i^g(\bar v)\big) + Y_i^b(U) + Y_i^l(U) + Y_i^s(U),
$$

we have

$$
\begin{aligned}
|\mathcal{Y}_i^g(\bar v)|^2 &\le 4\Big(|Y_i(U)|^2 + \big|Y_i^g(U)-Y_i^g(\bar U)\big|^2 + \big|Y_i^g(\bar U)-\mathcal{Y}_i^g(\bar v)\big|^2\\
&\qquad + |Y_i^b(U)|^2 + |Y_i^l(U)|^2 + |Y_i^s(U)|^2\Big),
\end{aligned}
$$

and so,

$$
\begin{aligned}
-4|Y_i(U)|^2 &\le -|\mathcal{Y}_i^g(\bar v)|^2 + 4\Big(\big|Y_i^g(U)-Y_i^g(\bar U)\big|^2 + \big|Y_i^g(\bar U)-\mathcal{Y}_i^g(\bar v)\big|^2\\
&\qquad + |Y_i^b(U)|^2 + |Y_i^l(U)|^2 + |Y_i^s(U)|^2\Big).
\end{aligned}
\tag{7.36}
$$

Then, using (7.35), we find that for any $\varepsilon_i/\lambda < \delta_0$,

$$
\begin{aligned}
G_{Y_1}(U) &\le -\frac{4}{\varepsilon_1\delta_1}|Y_1(U)|^2\mathbf{1}_{\{Y_1(U)\in[-C_0\varepsilon_1^2/\lambda,\varepsilon_1^2]\}},\\
G_{Y_2}(U) &\le -\frac{4}{\varepsilon_2\delta_1}|Y_2(U)|^2\mathbf{1}_{\{Y_2(U)\in[-\varepsilon_2^2,C_0\varepsilon_2^2/\lambda]\}}.
\end{aligned}
$$

Thus, it follows from (7.36) and (7.2) of Lemma 7.1 that for any U satisfying $Y_1(U) \le \varepsilon_1^2$ and $Y_2(U) \ge -\varepsilon_2^2$,

$$
\begin{aligned}
G_{Y_i}(U) \le& -\frac{|\mathcal{Y}_i^g(\bar v)|^2}{\varepsilon_i\delta_1} + \frac{4}{\varepsilon_i\delta_1}\Big(\big|Y_i^g(U)-Y_i^g(\bar U)\big|^2 + \big|Y_i^g(\bar U) - \overline{Y_i^g}\big|^2 \\
&+ \big|\overline{Y_i^g} - \mathcal{Y}_i^g(\bar v)\big|^2 + |Y_i^b(U)|^2 + |Y_i^l(U)|^2 + |Y_i^s(U)|^2\Big).
\end{aligned}
\tag{7.37}
$$

Next, for the diffusion term, we use (7.29) to have

$$
\begin{aligned}
&-(1-\delta_0)D(U) \le -((\delta_1/2)-\delta_0)D(U) - (1-\delta_1/2)D(\bar U)\\
&\quad\le -\Big(\frac{\delta_1}{2}-\delta_0\Big)D(U) - \frac{1-\delta_1/2}{(1+\delta_0)^2}\sum_{i=1}^2 \mathcal{D}_i(\bar v) + C\left(\exp\big(-C_\varepsilon t\big)+\frac{1}{t^4}\right) + \frac{C}{\delta_0}\frac{1}{t^2}\int_{\mathbb{R}}\eta(U|\tilde U)dx\\
&\quad\le -\Big(\frac{\delta_1}{2}-\delta_0\Big)D(U) - (1-\delta_1)\sum_{i=1}^2 \mathcal{D}_i(\bar v) + C\left(\exp\big(-C_\varepsilon t\big)+\frac{1}{t^4}\right) + \frac{C}{\delta_0}\frac{1}{t^2}\int_{\mathbb{R}}\eta(U|\tilde U)dx.
\end{aligned}
$$

Therefore, this and (7.37) imply that for any $\varepsilon_i/\lambda < \delta_0$,

$$
\mathcal{R}(U) \le \sum_{i=1}^2\big(\mathcal{R}^i_{\delta_1}(\bar v) + \mathbb{Y}_i + 2\mathbb{B}_i + \mathbb{G}_i\big) + \mathbb{D} + C\left(\exp\big(-C_\varepsilon t\big)+\frac{1}{t^4}\right) + \frac{C}{\delta_0}\frac{1}{t^2}\int_{\mathbb{R}}\eta(U|\tilde U)dx,
$$

where $\mathcal{R}^i_{\delta_1}$ denotes the functional in (7.34) as

$$
\begin{aligned}
\mathcal{R}^i_{\delta_1}(\bar v) =& -\frac{1}{\varepsilon_i\delta_1}|\mathcal{Y}_i^g(\bar v)|^2 + \mathcal{I}_{1i}(\bar v) + \delta_1|\mathcal{I}_{1i}(\bar v)|\\
&+ \mathcal{I}_{2i}(\bar v) + \delta_1\left(\frac{\varepsilon_i}{\lambda}\right)|\mathcal{I}_{2i}(\bar v)| - \left(1-\delta_1\left(\frac{\varepsilon_i}{\lambda}\right)\right)\mathcal{G}_{2i}(\bar v) - (1-\delta_1)\mathcal{D}_i(\bar v),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{Y}_i :=& \frac{4}{\varepsilon_i\delta_1}\Big(\big|Y_i^g(U)-Y_i^g(\bar U)\big|^2 + \big|Y_i^g(\bar U)-\mathcal{Y}_i^g(\bar v)\big|^2 + |Y_i^b(U)|^2 + |Y_i^l(U)|^2 + |Y_i^s(U)|^2\Big),\\
\mathbb{B}_i :=& |B_{1i}(U)-B_{1i}(\bar U)| + \big|B_{1i}(\bar U)-\mathcal{I}_{1i}(\bar v)\big| + |B_{2i}^+(U)-B_{2i}^+(\bar U)|\\
&+ \big|B_{2i}^+(\bar U)-\mathcal{I}_{2i}(\bar v)\big| + |B_{2i}^-(U)| + |B_{3i}(U)| + |B_{4i}(U)| + |B_5(U)| + |B_6(U)|\\
&- \left(1-\delta_1\frac{\varepsilon_i}{\lambda}\right)\Big(G_{2i}(\bar U)-\mathcal{G}_{2i}(\bar v)\Big),
\end{aligned}
$$

and $\mathbb{G}_i$ and $\mathbb{D}$ contain the good terms as

$$
\begin{aligned}
\mathbb{G}_i :=& -G_{1i}^-(U) - G_{1i}^+(U) - \left(1-\delta_0\frac{\varepsilon_i}{\lambda}\right)\big(G_{2i}(U)-G_{2i}(\bar U)\big) - (\delta_1-\delta_0)\frac{\varepsilon_i}{\lambda}G_{2i}(\bar U),\\
\mathbb{D} :=& -\Big(\frac{\delta_1}{2}-\delta_0\Big)D(U).
\end{aligned}
$$

Since we consider the case when $D(U) \le \frac{2C^*}{\sqrt{\delta_0}}\frac{\varepsilon_1^2}{\lambda}$, we have $D(U) \le \frac{2C^*}{\sqrt{\delta_0}}\frac{\varepsilon_i^2}{\lambda}$ for all $i = 1, 2$ (by $\varepsilon_1 < \varepsilon_2$). Thus, using Proposition 7.1 and Lemma 7.5, we find that for any $\varepsilon_i/\lambda < \delta_0$ and

$\lambda < \delta_0$,

$$
\begin{aligned}
&\sum_{i=1}^{2} \mathbb{Y}_i \\
&\le \frac{C}{\delta_1}\sum_{i=1}^{2}\frac{\varepsilon_i}{\lambda}\bigg(\sqrt{\frac{\varepsilon_i}{\lambda}}D(U) + \big(G_{2i}(U) - G_{2i}(\bar{U})\big) + \Big(\frac{\varepsilon_i}{\lambda}\Big)^{1/4} G_{2i}(\bar{U}) + G_{1i}^{-}(U) + \bigg(\frac{\lambda}{\varepsilon_i}\bigg)^{1/4} G_{1i}^{+}(U)\bigg) \\
&\quad + C_\varepsilon \exp\Big(- C_\varepsilon t\Big) \\
&\le \frac{C}{\delta_1}\sum_{i=1}^{2}\Big(\frac{\varepsilon_i}{\lambda}\Big)^{1/4}\Big(D(U) + \big(G_{2i}(U) - G_{2i}(\bar{U})\big) + G_{1i}^{-}(U) + G_{1i}^{+}(U) + \frac{\varepsilon_i}{\lambda}G_{2i}(\bar{U})\Big) + C_\varepsilon e^{-C_\varepsilon t},
\end{aligned}
$$

which together with taking δ_0 small enough as $\delta_0 \ll \delta_1^8$ implies

$$
\begin{aligned}
\sum_{i=1}^{2} \mathbb{Y}_i &\le C\frac{\delta_0^{1/4}}{\delta_1}\sum_{i=1}^{2}\Big(D(U) + \big(G_{2i}(U) - G_{2i}(\bar{U})\big) + G_{1i}^{-}(U) + G_{1i}^{+}(U) + \frac{\varepsilon_i}{\lambda}G_{2i}(\bar{U})\Big) + C_\varepsilon e^{-C_\varepsilon t} \\
&\le -\frac{1}{4}\Big(\mathbb{D} + \sum_{i=1}^{2}\mathbb{G}_i\Big) + C_\varepsilon e^{-C_\varepsilon t}.
\end{aligned}
$$

Likewise, we have

$$
\begin{aligned}
&2\sum_{i=1}^{2}\mathbb{B}_i \\
&\le C\sqrt{\delta_0}D(U) + C\delta_1\sum_{i=1}^{2}\Big(G_{1i}^{-}(U) + G_{1i}^{+}(U) + \big(G_{2i}(U) - G_{2i}(\bar{U})\big)\Big) + C\delta_0\sum_{i=1}^{2}\frac{\varepsilon_i}{\lambda}G_{2i}(\bar{U}) \\
&\quad + Ce^{-C_\varepsilon t} \\
&\le -\frac{1}{4}\Big(\mathbb{D} + \sum_{i=1}^{2}\mathbb{G}_i\Big) + Ce^{-C_\varepsilon t}.
\end{aligned}
$$

Therefore,

$$
\mathcal{R}(U) \le \sum_{i=1}^{2}\mathcal{R}_{\delta_1}^{i}(\bar{v}) + C_\varepsilon\left(\exp\big(- C_\varepsilon t\big) + \frac{1}{t^4}\right) + \frac{C}{\delta_0}\frac{1}{t^2}\int_{\mathbb{R}}\eta(U|\tilde{U})dx.
$$

Hence, it follows from (7.34) that

$$
\mathcal{R}(U) \le C_\varepsilon\left(\exp\big(- C_\varepsilon t\big) + \frac{1}{t^4}\right) + \frac{C}{\delta_0}\frac{1}{t^2}\int_{\mathbb{R}}\eta(U|\tilde{U})dx, \quad \forall t \ge t_0.
$$

Case iii) Since $\frac{2C^*}{\sqrt{\delta_0}}\frac{\varepsilon_2^2}{\lambda} \ge D(U) > \frac{2C^*}{\sqrt{\delta_0}}\frac{\varepsilon_1^2}{\lambda}$, we may combine the strategies of the previous cases. We first have

$$
\begin{aligned}
\mathcal{R}(U) \le& \big(\mathcal{R}_{\delta_1}^{2}(\bar{v}) + \mathbb{Y}_2 + 2\mathbb{B}_2 + \mathbb{G}_2\big) + \mathbb{D} + C\left(\exp\big(- C_\varepsilon t\big) + \frac{1}{t^4}\right) + \frac{C}{\delta_0}\frac{1}{t^2}\int_{\mathbb{R}}\eta(U|\tilde{U})dx \\
&+ 2\Big(|B_{11}(U) - B_{11}(\bar{U})| + |B_{21}^{+}(U) - B_{21}^{+}(\bar{U})| + |B_{21}^{-}(U)| + |B_{31}(U)| + |B_{41}(U)|\Big) \\
&+ 2\big(|B_{11}(\bar{U})| + |B_{21}^{+}(\bar{U})|\big) - \Big(G_{11}^{-}(U) + G_{11}^{+}(U) + \frac{1}{2}\big(G_{21}(U) - G_{21}(\bar{U})\big) + \frac{1}{2}G_{21}(\bar{U})\Big)
\end{aligned}
$$

Since $D(U) \leq \frac{2C^*}{\sqrt{\delta_0}} \frac{\varepsilon_2^2}{\lambda}$, using the same estimates as in Case i) and ii) together with (7.34), we have

$$\mathbb{Y}_2 \leq -\frac{1}{4}\Big(\mathbb{D} + \mathbb{G}_2\Big) + \delta_0 G_{21}(U) + C_\varepsilon e^{-C_\varepsilon t},$$

and

$$\begin{aligned} &2\Big(\mathbb{B}_2 + |B_{11}(U) - B_{11}(\bar{U})| + |B_{21}^+(U) - B_{21}^+(\bar{U})| + |B_{21}^-(U)| + |B_{31}(U)| + |B_{41}(U)|\Big) \\ &\quad \leq -\frac{1}{4}\Big(\mathbb{D} + \mathbb{G}_2\Big) + C_\varepsilon e^{-C_\varepsilon t} + \frac{1}{2}\Big(G_{11}^-(U) + G_{11}^+(U) + \frac{1}{2}\big(G_{21}(U) - G_{21}(\bar{U})\big) + \frac{1}{2}G_{21}(\bar{U})\Big). \end{aligned}$$

In addition, using (7.19) and (7.34), we have

$$\begin{aligned} \mathcal{R}(U) \leq \frac{1}{2}\mathbb{D} + C_\varepsilon\left(\exp\big(-C_\varepsilon t\big) + \frac{1}{t^4}\right) + \frac{C}{\delta_0}\frac{1}{t^2}\int_{\mathbb{R}} \eta(U|\tilde{U})dx \\ + 2C^*\frac{\varepsilon_1^2}{\lambda} + \sqrt{\delta_0}D(U). \end{aligned}$$

Since δ_0 is small enough (as $\delta_0 \ll \delta_1$) such that

$$2\sqrt{\delta_0}D(U) \leq -\frac{1}{2}\mathbb{D},$$

we have

$$\mathcal{R}(U) \leq C_\varepsilon\left(\exp\big(-C_\varepsilon t\big) + \frac{1}{t^4}\right) + \frac{C}{\delta_0}\frac{1}{t^2}\int_{\mathbb{R}} \eta(U|\tilde{U})dx + 2C^*\frac{\varepsilon_1^2}{\lambda} - \sqrt{\delta_0}D(U).$$

Therefore, by $D(U) > \frac{2C^*}{\sqrt{\delta_0}} \frac{\varepsilon_1^2}{\lambda}$, we have

$$\mathcal{R}(U) \leq C_\varepsilon\left(\exp\big(-C_\varepsilon t\big) + \frac{1}{t^4}\right) + \frac{C}{\delta_0}\frac{1}{t^2}\int_{\mathbb{R}} \eta(U|\tilde{U})dx.$$

Hence we complete the proof of Proposition 5.1.

8. Proof of Theorem 1.1

This section basically follows the same proof as in [26, Section 5]. Therefore, we omit the details of the proof, but briefly present non-trivial parts of the proof for completeness. Contrary to [26, Theorem 1.1], we need to show the estimates (1.19) and (1.21) by using the separation property (1.27).

First, we choose $\{(v_0^\nu, u_0^\nu)\}_{\nu>0}$ a sequence of smooth functions satisfying (1.17). Then, consider $\{(v^\nu, u^\nu)\}_{\nu>0}$ a sequence of solutions on $(0,T)$ to (1.1) corresponding to the initial datum (v_0^ν, u_0^ν). We use the uniform estimate (1.26) and (1.17) together with the scaling argument to find that

$$
\begin{aligned}
&\text{for any } \delta \in (0,1), \text{ there exists } \nu_* \text{ such that for all } \nu < \nu_*,\\
&\int_{-\infty}^{\infty} \eta\big((v^\nu, h^\nu)(t,x) | (\tilde{v}^\nu, \tilde{h}^\nu)^{X_1^\nu, X_2^\nu}(t,x)\big) dx\\
&\qquad + \int_0^T \int_{-\infty}^{\infty} |\partial_x (\tilde{v}^\nu)^{X_1^\nu, X_2^\nu}(t,x)|\, Q\left(v^\nu(t,x) | (\tilde{v}^\nu)^{X_1^\nu, X_2^\nu}(t,x)\right) dxdt\\
&\qquad + \nu \int_0^T \int_{-\infty}^{\infty} (v^\nu)^{\gamma-\alpha} \big|\partial_x \big(p(v^\nu(t,x)) - p((\tilde{v}^\nu)^{X_1^\nu, X_2^\nu}(t,x))\big)\big|^2 dxdt\\
&\quad \le C\mathcal{E}_0 + \delta,
\end{aligned}
\tag{8.1}
$$

where

$$
h^\nu := u^\nu + \nu\Big(p(v^\nu)^{\frac{\alpha}{\gamma}}\Big)_x, \quad \tilde{h}^\nu := \tilde{u}^\nu + \nu\Big(p(\tilde{v}^\nu)^{\frac{\alpha}{\gamma}}\Big)_x, \quad X_i^\nu(t) = \nu X_i(t/\nu).
$$

As in the proof of [26, Lemma 5.1], we use (1.28) to have

$$
\left|\frac{d}{dt} X_i^\nu(t)\right| = |X_i'(t/\nu)| \le C\left(f_\nu(t) + \int_{-\infty}^{\infty} E_1\big((v_0^1, u_0^1) | (\tilde{v}, \tilde{u})\big) dx + 1\right),
$$

where $f_\nu(t) := f\left(\frac{t}{\nu}\right)$. Since (1.28) implies that

$$
\|f_\nu\|_{L^1(0,T)} = \nu \|f\|_{L^1(0,T/\nu)} \le C\nu,
$$

f_ν is uniformly bounded in $L^1(0,T)$. Therefore, $\frac{d}{dt} X_i^\nu$ is uniformly bounded in $L^1(0,T)$. Moreover, since $X_i^\nu(0) = 0$ and so,

$$
|X_i^\nu(t)| \le Ct + C\int_0^t f_\nu(s) ds \le C(t+\nu), \tag{8.2}
$$

X_i^ν is also uniformly bounded in $L^1(0,T)$.
Therefore, by the compactness of BV, there exist $X_1^\infty, X_2^\infty \in BV(0,T)$ such that for each $i = 1, 2$,

$$
X_i^\nu \to X_i^\infty \quad \text{in } L^1(0,T), \quad \text{up to subsequence as } \nu \to 0. \tag{8.3}
$$

Especially, (8.3) and the uniform (in ν) bound (8.2) imply that

$$
|X_i^\infty(t)| \le Ct \quad \text{for a.e. } t \in [0,T]. \tag{8.4}
$$

Since it follows from (1.27) that

$$
X_1^\nu(t) \le -\frac{\sigma_1}{2} t, \quad X_2^\nu(t) \ge -\frac{\sigma_2}{2} t, \quad \forall t \in (0,T],
$$

we use (8.3) to have

$$
X_1^\infty(t) \le -\frac{\sigma_1}{2} t, \quad X_2^\infty(t) \ge -\frac{\sigma_2}{2} t \quad \text{for a.e. } t \in (0,T].
$$

Thus we have

$$
\sigma_1 t + X_1^\infty(t) \le \frac{\sigma_1}{2} t < 0 < \frac{\sigma_2}{2} t \le \sigma_2 t + X_2^\infty(t) \quad \text{for a.e. } t \in (0,T], \tag{8.5}
$$

which proves (1.19).

As in [26, Section 5], we use the uniform estimate (8.1) to show that there exist limits (v_∞, u_∞) of the sequence $\{(v^\nu, u^\nu)\}_{\nu>0}$ as $\nu \to 0$ in the sense of (1.18) such that the stability estimate (1.20) holds. Therefore, it remains to prove (1.21). This will be shown by using the fact that it follows from (1.18) and the first (linear) equation of $(1.1)_1$ that the limits v_∞, u_∞ satisfy $\partial_t v_\infty - \partial_x u_\infty = 0$ in the sense of distributions, and by using the stability estimate (1.20) together with (8.5) on the separation estimate and (8.4) on the continuity at $t = 0$ of the shifts.

• *proof of* (1.21) : First, thanks to the fact that $X_1^\infty, X_2^\infty \in BV((0,T))$, we choose positive constants $r_1 = r_1(T)$ and $r_2 = r_2(T)$ such that $\|X_i^\infty\|_{L^\infty((0,T))} \le r_i$ for each $i = 1, 2$.
Let $t_0 \in (0,T)$ be any constant at which (8.4)-(8.5) and (1.20) hold (Note that (8.4)-(8.5) and (1.20) hold for almost everywhere on $(0,T)$). We consider $\psi_1, \psi_2 : \mathbb{R} \to \mathbb{R}$ nonnegative smooth functions defined by

$$\psi_1(x) = \begin{cases} 1 & \text{if } x \in [-r_1 + \sigma_1 T, \frac{\sigma_1 t_0}{2}], \\ 0 & \text{if } x \le -2r_1 + \sigma_1 T \text{ or } x \ge 0, \end{cases} \qquad \text{and} \quad \|\psi_1'\|_{L^\infty(\mathbb{R})} \le \max\left(\frac{2}{|r_1|}, \frac{4}{|\sigma_1| t_0}\right),$$

$$\psi_2(x) = \begin{cases} 1 & \text{if } x \in [\frac{\sigma_2 t_0}{2}, r_2 + \sigma_2 T], \\ 0 & \text{if } x \le 0 \text{ or } x \ge 2r_2 + \sigma_2 T, \end{cases} \qquad \text{and} \quad \|\psi_2'\|_{L^\infty(\mathbb{R})} \le \max\left(\frac{2}{r_2}, \frac{4}{\sigma_2 t_0}\right)$$

Let $\theta : \mathbb{R} \to \mathbb{R}$ be a nonnegative smooth function such that $\theta(s) = \theta(-s)$, $\int_{\mathbb{R}} \theta = 1$ and $\operatorname{supp} \theta = [-1, 1]$, and let

$$\theta_\delta(s) := \frac{1}{\delta}\theta\Big(\frac{s-\delta}{\delta}\Big) \quad \text{for any } \delta > 0.$$

For any $t \in (t_0, T)$ at which (8.5) and (1.20) hold, and for any $\delta < \min(t - t_0, T - t)$, we define a nonnegative smooth function

$$\varphi_{t,\delta}(s) := \int_0^s \Big(\theta_\delta(\tau - t_0) - \theta_\delta(\tau - t)\Big) d\tau.$$

Since $\partial_t v_\infty - \partial_x u_\infty = 0$ in the sense of distributions, we have

$$\int_{(0,T)\times\mathbb{R}} \big(\varphi_{t,\delta}'(s)\psi_i(x) dv_\infty(s,x) - \varphi_{t,\delta}(s)\psi_i'(x) u_\infty(s,x) ds dx\big) = 0, \quad \text{for each } i = 1, 2. \tag{8.6}$$

We rewrite the left-hand side above as

$$I_1^\delta + I_2^\delta + I_3^\delta = 0,$$

where

$$\begin{aligned} I_1^\delta &:= \int_{(0,T)\times\mathbb{R}} \theta_\delta(s - t_0)\psi_i(x) dv_\infty(s,x), \\ I_2^\delta &:= -\int_{(0,T)\times\mathbb{R}} \theta_\delta(s - t)\psi_i(x) dv_\infty(s,x), \\ I_3^\delta &:= -\int_{(0,T)\times\mathbb{R}} \varphi_{t,\delta}(s)\psi_i'(x) u_\infty(s,x) dx ds. \end{aligned}$$

Since v_∞ is weakly continuous in time (as in [26, Section 5]), we find that as $\delta \to 0$:

$$I_1^\delta \to \int_{\mathbb{R}} \psi_i(x) v_\infty(t_0, dx), \quad I_2^\delta \to -\int_{\mathbb{R}} \psi_i(x) v_\infty(t, dx),$$

and

$$I_3^\delta \to -\int_{t_0}^t \int_{\mathbb{R}} \psi_i'(x) u_\infty(s,x) dx ds.$$

Therefore, it follows from (8.6) that

$$\underbrace{\int_{\mathbb{R}} \psi_i(x)\big(v_\infty(t,dx) - v_\infty(t_0,dx)\big)}_{=:J_1} + \underbrace{\int_{t_0}^{t}\int_{\mathbb{R}} \psi_i'(x) u_\infty(s,x)dxds}_{=:J_2} = 0.$$

Our strategy is to use the stability estimate (1.20) and the Rankine-Hugoniot condition. For that, we use the shifted Riemann solution to decompose J_1 into three parts:

$$J_1 = J_{11}^i + J_{12}^i + J_{13}^i,$$

where

$$\begin{aligned} J_{11}^i &= \int_{\mathbb{R}} \psi_i(x)\big(v_\infty(t,dx) - \bar{v}^{X_1^\infty,X_2^\infty}(t,x)dx\big),\\ J_{12}^i &= \int_{\mathbb{R}} \psi_i(x)\big(\bar{v}^{X_1^\infty,X_2^\infty}(t,x) - \bar{v}^{X_1^\infty,X_2^\infty}(t_0,x)\big)dx,\\ J_{13}^i &= \int_{\mathbb{R}} \psi_i(x)\big(\bar{v}^{X_1^\infty,X_2^\infty}(t_0,x)dx - v_\infty(t_0,dx)\big). \end{aligned}$$

Likewise, we decompose J_2 into two parts:

$$J_2 = J_{21}^i + J_{22}^i,$$

where

$$\begin{aligned} J_{21}^i &= \int_{t_0}^{t}\int_{\mathbb{R}} \psi_i'(x)\big(u_\infty(s,x) - \bar{u}^{X_1^\infty,X_2^\infty}(s,x)\big)dxds,\\ J_{22}^i &= \int_{t_0}^{t}\int_{\mathbb{R}} \psi_i'(x)\bar{u}^{X_1^\infty,X_2^\infty}(s,x)dxds. \end{aligned}$$

Since it follows from (8.5) that

$$\begin{aligned} &-r_1 + \sigma_1 T \le \sigma_1 t + X_1^\infty(t) \le \frac{\sigma_1}{2}t_0 < 0\\ &\qquad < \frac{\sigma_2}{2}t_0 \le \sigma_2 t + X_2^\infty(t) \le r_2 + \sigma_2 T \quad \text{for a.e. } t \in [t_0,T), \end{aligned}$$

we have

$$J_{12}^i = \begin{cases} (v_- - v_m)(X_1^\infty(t) - X_1^\infty(t_0) + \sigma_1(t-t_0)) & \text{when } i=1,\\ (v_m - v_+)(X_2^\infty(t) - X_2^\infty(t_0) + \sigma_2(t-t_0)) & \text{when } i=2, \end{cases}$$

$$J_{22}^i = \begin{cases} (t-t_0)(u_- - u_m) & \text{when } i=1,\\ (t-t_0)(u_m - u_+) & \text{when } i=2, \end{cases}$$

Then using $\sigma_1 = -\frac{u_m-u_-}{v_m-v_-}, \sigma_2 = -\frac{u_+-u_m}{v_+-v_m}$ by the condition (1.7), we have

$$J_{12}^i + J_{22}^i = \begin{cases} (v_- - v_m)(X_1^\infty(t) - X_1^\infty(t_0)) & \text{when } i=1,\\ (v_m - v_+)(X_2^\infty(t) - X_2^\infty(t_0)) & \text{when } i=2, \end{cases}$$

Now, it remains to control the remaining terms by the initial perturbation $\mathcal{E}_0 = \int_{-\infty}^{\infty} \eta\big((v^0,u^0)|(\bar{v},\bar{u})\big)dx$ as follows. First, recall the (unique) decomposition of the measure v_∞ by

$$dv_\infty(t,dx) = v_a(t,x)dx + v_s(t,dx).$$

Using (A.1), we have

$$
\begin{aligned}
|J_{11}^1| &\le \int_{-2r_1+\sigma_1 T}^0 \big|v_a(t,x) - \bar v^{X_1^\infty, X_2^\infty}(t,x)\big| \mathbf{1}_{\{v\le 3v_-\}} dx \\
&\quad + \int_{-2r_1+\sigma_1 T}^0 \big|v_a(t,x) - \bar v(x - X_\infty(t))\big| \mathbf{1}_{\{v\ge 3v_-\}} dx + \int_{\mathbb{R}} \psi(x) v_s(t,dx) \\
&\le \frac{1}{\sqrt{c_1}} \int_{-2r_1+\sigma_1 T}^0 \sqrt{Q\big(v_a(t,x)|\bar v(x - X_\infty(t))\big)} dx \\
&\quad + \frac{1}{c_2} \int_{-2r_1+\sigma_1 T}^0 Q\big(v_a(t,x)|\bar v(x - X_\infty(t))\big) dx + \frac{1}{|Q'(\max(v_-,v_+))|} \int_{\mathbb{R}} \psi(x) |Q'(\overline{V})| v_s(t,dx),
\end{aligned}
$$

where note that $|Q'(\overline{V})| \ge |Q'(\max(v_-,v_+))| > 0$ by (1.22).
Thus, we use the stability estimate (1.20) to have

$$|J_{11}^1| \le C\sqrt{\mathcal{E}_0} + C\mathcal{E}_0.$$

Likewise, we have

$$|J_{11}^2| \le C\sqrt{\mathcal{E}_0} + C\mathcal{E}_0,$$

and

$$|J_{13}^i| \le C\sqrt{\mathcal{E}_0} + C\mathcal{E}_0 \quad \text{for } i = 1,2.$$

Similarly, there exists a constant $C(t_0)$ depending on t_0 such that

$$|J_{21}^1| \le \|\psi_1'\|_{L^\infty(\mathbb{R})} \int_{t_0}^t \int_{[-2r_1+\sigma_1 T, -r_1+\sigma_1 T]\cup[\sigma_1 t_0/2, 0]} \big|u_\infty(s,x) - \bar u^{X_1^\infty, X_2^\infty}(s,x)\big| dx ds \le C(t_0) t \sqrt{\mathcal{E}_0},$$

and

$$|J_{21}^2| \le C(t_0) t \sqrt{\mathcal{E}_0}.$$

Therefore, we have shown that for each $i = 1,2$,

$$|X_i^\infty(t) - X_i^\infty(t_0)| \le C(t_0)\Big(\mathcal{E}_0 + (1+t)\sqrt{\mathcal{E}_0}\Big), \quad \text{for a.e. } t \in (t_0, T).$$

Since it follows from (8.4) that

$$|X_i^\infty(t_0)| \le C t_0,$$

we have

$$|X_i^\infty(t)| \le C(t_0)\Big(\mathcal{E}_0 + (1+t)\sqrt{\mathcal{E}_0}\Big), \quad \text{for a.e. } t \in (t_0, T).$$

Hence, this and (8.4) imply the desired estimate (1.21).

Appendix A. Useful inequalities

We here present the useful inequalities developed in [24, Section 2.4 and Lemma 2.2]. First, the following lemma provides some global inequalities on the relative function $Q(\cdot|\cdot)$ corresponding to the convex function $Q(v) = \frac{v^{-\gamma+1}}{\gamma-1}$, $v > 0$, $\gamma > 1$.

Lemma A.1. [24, Lemma 2.4] *For given constants $\gamma > 1$, and $v_* > 0$, there exists constants $c_1, c_2 > 0$ such that the following inequalities hold.*
1) For any $w \in (0, 2v_*)$,

$$
\begin{aligned}
&Q(v|w) \ge c_1 |v-w|^2, \quad \textit{for all } 0 < v \le 3v_*, \\
&Q(v|w) \ge c_2 |v-w|, \quad \textit{for all } v \ge 3v_*.
\end{aligned}
\tag{A.1}
$$

2) Moreover if $0 < w \le u \le v$ *or* $0 < v \le u \le w$ *then*

$$Q(v|w) \ge Q(u|w), \tag{A.2}$$

and for any $\delta_* > 0$ *there exists a constant* $C > 0$ *such that if, in addition,* $|w - v_*| \le \delta_*/2$ *and* $|w - u| > \delta_*$*, we have*

$$Q(v|w) - Q(u|w) \ge C|u - v|. \tag{A.3}$$

The following lemma provides some global inequalities on the pressure $p(v) = v^{-\gamma}$, $v > 0$, $\gamma > 1$, and on the associated relative function $p(\cdot|\cdot)$.

Lemma A.2. [24, Lemma 2.5] *For given constants* $\gamma > 1$*, and* $v_* > 0$*, there exist constants* $c_3, C > 0$ *such that the following inequalities hold.*
For any $w > v_*/2$*,*

$$|p(v) - p(w)| \le c_3|v - w|, \qquad \forall v \ge v_*/2, \tag{A.4}$$

$$p(v|w) \le C|v - w|^2, \qquad \forall v \ge v_*/2, \tag{A.5}$$

$$p(v|w) \le C(|v - w| + |p(v) - p(w)|), \qquad \forall v > 0. \tag{A.6}$$

The following lemma presents some local estimates on $p(v|w)$ and $Q(v|w)$ for $|v-w| \ll 1$, based on Taylor expansions.

Lemma A.3. *For given constants* $\gamma > 1$ *and* $v_* > 0$ *there exist positive constants* C *and* δ_* *such that for any* $0 < \delta < \delta_*$*, the following is true.*
1) For any $(v, w) \in \mathbb{R}_+^2$ *satisfying* $|p(v) - p(w)| < \delta$*, and* $|p(w) - p(v_*)| < \delta$ *the following estimates* (A.7)-(A.9) *hold:*

$$p(v|w) \le \left(\frac{\gamma+1}{2\gamma}\frac{1}{p(w)} + C\delta\right)|p(v) - p(w)|^2, \tag{A.7}$$

$$Q(v|w) \ge \frac{p(w)^{-\frac{1}{\gamma}-1}}{2\gamma}|p(v) - p(w)|^2 - \frac{1+\gamma}{3\gamma^2}p(w)^{-\frac{1}{\gamma}-2}(p(v) - p(w))^3, \tag{A.8}$$

$$Q(v|w) \le \left(\frac{p(w)^{-\frac{1}{\gamma}-1}}{2\gamma} + C\delta\right)|p(v) - p(w)|^2. \tag{A.9}$$

2) For any $(v, w) \in \mathbb{R}_+^2$ *such that* $|p(w) - p(v_*)| \le \delta$*, and satisfying either* $Q(v|w) < \delta$ *or* $|p(v) - p(w)| < \delta$*,*

$$|p(v) - p(w)|^2 \le CQ(v|w). \tag{A.10}$$

The following lemma presents an estimate based on the inverse of the pressure function, which will be used in Appendix B.

Lemma A.4. *For any* $r > 0$*, there exist* $\varepsilon_0 > 0$ *and* $C > 0$ *such that the following holds. For any* $p_-, p_+, p > 0$ *such that* $p_- \in (r/2, 2r)$*,* $p_+ - p_- =: \varepsilon \in (0, \varepsilon_0)$*,* $p_- \le p \le p_+$*, and* v, v_-, v_+ *such that* $p(v) = p, p(v_\pm) = p_\pm$*, we have*

$$\left|\frac{v - v_-}{p - p_-} + \frac{v - v_+}{p_+ - p} + \frac{1}{2}\frac{p''(v_-)}{p'(v_-)^2}(v_- - v_+)\right| \le C\varepsilon^2.$$

The following identity of [22, Lemma 4.1] will be used when splitting the composite wave.

Lemma A.5. [22, Lemma 4.1] *For any function* F*, its relative function* $F(\cdot|\cdot)$ *satisfies*

$$F(u|w) + F(w|v) = F(u|v) + (F'(w) - F'(v))(w - u), \quad \forall u, v, w.$$

Appendix B. Proof of Proposition 6.1

First of all, given (v_*, u_*), fix the value of ε_0 corresponding to the constant $r := p(v_*)$ in Lemma A.4. Then we consider any constant $\delta_1 \in (0, 1/2)$ such that

$$\delta_1 < \min\left(\frac{\delta_*}{|p'(v_*/2)|}, \frac{\delta_*}{2}, \frac{p(v_*)}{2}, \sqrt{\varepsilon_0}\right),$$

where δ_* is the constant as in Lemma A.3.
Note that using $|\tilde{v}_0 - v_*| \le \delta_1$, we have

$$|p(\tilde{v}_0) - p(v_*)| \le |p'(v_*/2)||\tilde{v}_0 - v_*| \le |p'(v_*/2)|\delta_1 < \delta_*.$$

Moreover, since $|p(v) - p(\tilde{v}_0)| \le 2\delta_1 < \delta_*$, we can apply the results of Lemma A.3 to the case of $w = \tilde{v}_0$. Since $\varepsilon \le \lambda\delta_1 < \delta_1^2 < \varepsilon_0$, we can also apply Lemma A.4 to the case of $\{p_-, p_+\} = \{p(v_l), p(v_r)\}$.

Without loss of generality, we assume $v_l < v_r$, and so, $\sigma_0 = \sqrt{-\frac{p(v_r)-p(v_l)}{v_r - v_l}} > 0$ (that is, assume the case of 2-shock).
We will rewrite the functionals $\mathcal{Y}^g, \mathcal{I}_1, \mathcal{I}_2, \mathcal{G}_2, \mathcal{D}$ with respect to the following variables:

$$w := \big(p(v) - p(\tilde{v}_0)\big)\phi, \qquad W := \frac{\lambda}{\varepsilon}w, \qquad y := \frac{p(\tilde{v}_0) - p(v_r)}{\varepsilon}, \tag{B.1}$$

where note that $\varepsilon = p(v_l) - p(v_r)$.
Since $p(\tilde{v}_0)$ is decreasing in x, we can use the change of variables $x \mapsto y \in [0, 1]$.
Notice from the assumptions that

$$\partial_x a_0 = -\lambda\frac{\partial_x p(\tilde{v}_0)}{\varepsilon}, \qquad \partial_x y = \frac{\partial_x p(\tilde{v}_0)}{\varepsilon}, \tag{B.2}$$

and

$$\|a - 1\|_{L^\infty(\mathbb{R})} \le 2\lambda \le 2\delta_1. \tag{B.3}$$

First of all, note that the functionals $\mathcal{Y}^g, \mathcal{I}_1, \mathcal{I}_2$ with $\phi \equiv 1$ are the same as $Y_g, \mathcal{I}_1, \mathcal{I}_2$ in [26, Proposition 4.2] (and as $Y_g, \mathcal{B}_1, \mathcal{B}_{21}$ in [24, Proposition 4.2]). As in [24, Proposition 3.4] and [26, Appendix A], we use the notations:

$$\sigma_* = \sqrt{-p'(v_*)}, \qquad \alpha_* := \frac{\gamma\sqrt{-p'(v_*)}p(v_*)}{\gamma + 1}.$$

Note that

$$|\sigma_0 - \sigma_*| \le C\delta_1, \tag{B.4}$$

and

$$\|\sigma_*^2 + p'(\tilde{v}_0)\|_\infty \le C\delta_1, \qquad \left\|\frac{1}{\sigma_*^2} - \frac{p(\tilde{v}_0)^{-\frac{1}{\gamma}-1}}{\gamma}\right\|_\infty \le C\delta_1. \tag{B.5}$$

Thus, following the same estimates as in [24, (3.30), (3.33), (3.34)] together with using Lemma A.3 and (B.1)-(B.5) above, we find that

(B.6) $$\left|\sigma_*^2\frac{\lambda}{\varepsilon^2}\mathcal{Y}^g - \int_0^1 W^2\,dy - 2\int_0^1 W\,dy\right| \le C\delta_1\left(\int_0^1 W^2\,dy + \int_0^1 |W|\,dy\right),$$

(B.7) $$2\alpha_*\frac{\lambda^2}{\varepsilon^3}|\mathcal{I}_1| \le (1+C\delta_1)\int_0^1 W^2\,dy,$$

(B.8) $$2\alpha_*\frac{\lambda^2}{\varepsilon^3}|\mathcal{I}_2| \le \left(\frac{\alpha_\gamma}{\sigma_*}\left(\frac{\lambda}{\varepsilon}\right) + C\delta_1\right)\int_0^1 W^2\,dy.$$

Using (B.6) and the assumption (6.2), we have

$$\int_0^1 W^2\,dy - 2\left|\int_0^1 W\,dy\right| \le C_2\sigma_*^2 + C\delta_1\left(\int_0^1 W^2\,dy + \int_0^1 |W|\,dy\right).$$

Using

$$\left|\int_0^1 W\,dy\right| \le \int_0^1 |W|\,dy \le \frac{1}{8}\int_0^1 W^2\,dy + 8,$$

we have

$$\int_0^1 W^2\,dy \le 2\left|\int_0^1 W\,dy\right| + C_2\sigma^2 + C\delta_1\left(\int_0^1 W^2\,dy + \int_0^1 |W|\,dy\right) \le C + 24 + \frac{1}{2}\int_0^1 W^2\,dy,$$

for δ_1 small enough. Thus there exists a constant $C_1 > 0$ depending on C_2 (but not on ε nor λ) such that

(B.9) $$\int_0^1 W^2\,dy \le C_1.$$

For an estimate on the $|\mathcal{Y}^g|^2$ terms, we use (B.6) to have

$$\begin{aligned}
-2\alpha_*\left(\frac{\lambda^2}{\varepsilon^3}\right)\frac{|\mathcal{Y}^g|^2}{\varepsilon\delta_1} &= -\frac{2\alpha_*}{\delta_1\sigma_*^4}\left|\frac{\sigma_*^2\lambda}{\varepsilon^2}\mathcal{Y}^g\right|^2 \\
&\le -\frac{\alpha_*}{\delta_1\sigma_*^4}\left|\int_0^1 W^2\,dy + 2\int_0^1 W\,dy\right|^2 + C\delta_1\left(\int_0^1 W^2\,dy + \int_0^1 |W|\,dy\right)^2,
\end{aligned}$$

which together with (B.9) yields

(B.10) $$-2\alpha_*\left(\frac{\lambda^2}{\varepsilon^3}\right)\frac{|\mathcal{Y}^g|^2}{\varepsilon\delta_1} \le -\frac{\alpha_*}{\delta_1\sigma_*^4}\left|\int_0^1 W^2\,dy + 2\int_0^1 W\,dy\right|^2 + C\delta_1\int_0^1 W^2\,dy.$$

For the $\mathcal{G}_2$, we use (B.1)-(B.5) to have

$$\begin{aligned}
\mathcal{G}_2 &= -\frac{\sigma_0\lambda}{2\gamma}\int_0^1 p(\tilde{v}_0)^{-\frac{1}{\gamma}-1}w^2dy + \sigma_0\lambda\frac{1+\gamma}{3\gamma^2}\int_0^1 p(\tilde{v}_0)^{-\frac{1}{\gamma}-2}w^3dy \\
&\ge \left(\frac{\lambda}{2\sigma_*} - C\varepsilon\delta_1\right)\int_0^1 w^2dy - \frac{\lambda}{3\alpha_*}\int_0^1 w^3dy - C\frac{\varepsilon\lambda}{\alpha_*}\int_0^1 |w|^3dy,
\end{aligned}$$

which yields

(B.11) $$-2\alpha_*\frac{\lambda^2}{\varepsilon^3}\mathcal{G}_2 \le \left(-\frac{\alpha_*}{\sigma_*}\left(\frac{\lambda}{\varepsilon}\right) + C\delta_1\right)\int_0^1 W^2\,dy + \frac{2}{3}\int_0^1 W^3\,dy + C\varepsilon\int_0^1 |W|^3\,dy.$$

Therefore, it remains to estimate the diffusion $\mathcal{D}$ as follows.
First, by the change of variable, the diffusion $\mathcal{D}$ is written as

$$\mathcal{D}=\int_0^1 a|\partial_y w|^2 v^\beta\Big(-\frac{\partial y}{\partial x}\Big)dy.$$

Since integrating (5.11) over $(-\infty,x]$ yields that

$$\tilde{v}_0^\beta\partial_x p(\tilde{v}_0)=\sigma_0(\tilde{v}_0-v_l)+\frac{p(\tilde{v}_0)-p(v_l)}{\sigma_0},$$

we use (B.2) to have

$$\varepsilon\,\tilde{v}_0^\beta\frac{\partial y}{\partial x}=\frac{1}{\sigma_0}\Big(\sigma_0^2(\tilde{v}_0-v_l)+p(\tilde{v}_0)-p(v_l)\Big).$$

Following the proof of [24, Lemma 3.1], with $1-y=\frac{p(v_l)-p(\tilde{v}_0)}{\varepsilon}$, we have

$$\frac{\tilde{v}_0^\beta}{y(1-y)}\Big(-\frac{\partial y}{\partial x}\Big)=\frac{\varepsilon}{\sigma_0(v_l-v_r)}\left(\frac{\tilde{v}_0-v_r}{p(\tilde{v}_0)-p(v_r)}+\frac{\tilde{v}_0-v_l}{p(v_l)-p(\tilde{v}_0)}\right).$$

Then

$$\begin{aligned}&\left|\frac{\tilde{v}_0^\beta}{y(1-y)}\Big(-\frac{\partial y}{\partial x}\Big)-\varepsilon\frac{p''(v_*)}{2p'(v_*)^2\sigma_*}\right|\\&\qquad\le\underbrace{\left|\frac{\tilde{v}_0^\beta}{y(1-y)}\Big(-\frac{\partial y}{\partial x}\Big)-\varepsilon\frac{p''(v_r)}{2p'(v_r)^2\sigma_0}\right|}_{=:I_1}+\underbrace{\frac{\varepsilon}{2}\left|\frac{p''(v_r)}{p'(v_r)^2\sigma_0}-\frac{p''(v_*)}{p'(v_*)^2\sigma_*}\right|}_{=:I_2}.\end{aligned}$$

Applying Lemma A.4 to the case where $p_-=p(v_r)$, $p_+=p(v_l)$ and $p=p(\tilde{v}_0)$, we have

$$I_1=\frac{\varepsilon}{|\sigma_0|(v_r-v_l)}\left|\frac{\tilde{v}_0-v_r}{p(\tilde{v}_0)-p(v_r)}+\frac{\tilde{v}_0-v_l}{p(v_l)-p(\tilde{v}_0)}+\frac{p''(v_r)}{2p'(v_r)^2}(v_r-v_l)\right|\le C\varepsilon^2.$$

Using (B.4) and $|v_*-v_r|\le\delta_1$, we have $I_2\le C\varepsilon\delta_1$. Thus, we get

$$\left|\frac{\tilde{v}_0^\beta}{y(1-y)}\Big(-\frac{\partial y}{\partial x}\Big)-\varepsilon\frac{p''(v_*)}{2p'(v_*)^2\sigma_*}\right|\le C\varepsilon\delta_1.$$

Since $p(v)=v^{-\gamma}$, we have

$$\frac{p''(v_*)}{p'(v_*)^2\sigma_*}=\frac{\gamma+1}{\gamma\sigma_* p(v_*)}=\frac{1}{\alpha_*},$$

which yields

$$\left|\frac{\tilde{v}_0^\beta}{y(1-y)}\Big(-\frac{\partial y}{\partial x}\Big)-\frac{\varepsilon}{2\alpha_*}\right|\le C\varepsilon\delta_1.$$

Thus, using $|(v^\beta/\tilde{v}_0^\beta)-1|\le C\delta_1$ and (B.3), we have

$$\begin{aligned}
\mathcal{D} &\ge (1-2\delta_1)\int_0^1 |\partial_y w|^2 v^\beta\Big(-\frac{\partial y}{\partial x}\Big)dy\\
&= (1-2\delta_1)\int_0^1 |\partial_y w|^2 \frac{v^\beta}{\tilde{v}_0^\beta}\tilde{v}_0^\beta\Big(-\frac{\partial y}{\partial x}\Big)dy\\
&\ge (1-2\delta_1)\left(\frac{\varepsilon}{2\alpha_*}-C\varepsilon\delta_1\right)\int_0^1 y(1-y)|\partial_y w|^2\,dy\\
&\ge \frac{\varepsilon}{2\alpha_*}(1-C\delta_1)\int_0^1 y(1-y)|\partial_y w|^2\,dy.
\end{aligned}$$

After the normalization, we obtain

$$-2\alpha_*\frac{\lambda^2}{\varepsilon^3}\mathcal{D}\le -(1-C\delta_1)\int_0^1 y(1-y)|\partial_y W|^2\,dy. \tag{B.12}$$

To finish the proof, we first observe that for any $\delta\le\delta_1$,

$$\begin{aligned}
\mathcal{R}_\delta(v) &\le -\frac{1}{\varepsilon\delta_1}|\mathcal{Y}^g(v)|^2+\mathcal{I}_1(v)+\delta_1|\mathcal{I}_1(v)|\\
&\quad+\mathcal{I}_2(v)+\delta_1\left(\frac{\varepsilon}{\lambda}\right)|\mathcal{I}_2(v)|-\left(1-\delta_1\left(\frac{\varepsilon}{\lambda}\right)\right)\mathcal{G}_2(v)-(1-\delta_1)\mathcal{D}(v).
\end{aligned}$$

Then, (B.7), (B.8), (B.10), (B.11), (B.12) imply that for some constants $C_\gamma, C_*>0$,

$$\begin{aligned}
\mathcal{R}_\delta(v)\le \frac{\varepsilon^3}{2\alpha_*\lambda^2}\bigg[&-\frac{1}{C_\gamma\delta_1}\left(\int_0^1 W^2\,dy+2\int_0^1 W\,dy\right)^2+(1+C_*\delta_1)\int_0^1 W^2\,dy\\
&+\frac{2}{3}\int_0^1 W^3\,dy+C_*\delta_1\int_0^1|W|^3\,dy-(1-C_*\delta_1)\int_0^1 y(1-y)|\partial_y W|^2\,dy\bigg].
\end{aligned}$$

To finish the proof, we will use the nonlinear Poincaré type inequality [24, Proposition 3.3] as follow:

Proposition B.1. [24, Proposition 3.3] *For a given $C_1>0$, there exists $\delta_2>0$, such that for any $\delta<\delta_2$ the following is true.*
For any $W\in L^2(0,1)$ such that $\sqrt{y(1-y)}\partial_y W\in L^2(0,1)$, if $\int_0^1|W(y)|^2\,dy\le C_1$, then

$$\begin{aligned}
&-\frac{1}{\delta}\left(\int_0^1 W^2\,dy+2\int_0^1 W\,dy\right)^2+(1+\delta)\int_0^1 W^2\,dy\\
&\qquad+\frac{2}{3}\int_0^1 W^3\,dy+\delta\int_0^1|W|^3\,dy-(1-\delta)\int_0^1 y(1-y)|\partial_y W|^2\,dy\le 0.
\end{aligned} \tag{B.13}$$

To apply the Proposition, let us fix the value of the δ_2 of Proposition B.1 corresponding to the constant C_1 of (B.9).
Then we retake δ_1 small enough such that $\max(C_\gamma, C_*)\delta_1\le\delta_2$. Thus we find that

$$\begin{aligned}
\mathcal{R}_\delta(v)\le \frac{\varepsilon^3}{2\alpha_*\lambda^2}\bigg[&-\frac{1}{\delta_2}\left(\int_0^1 W^2\,dy+2\int_0^1 W\,dy\right)^2+(1+\delta_2)\int_0^1 W^2\,dy\\
&+\frac{2}{3}\int_0^1 W^3\,dy+\delta_2\int_0^1|W|^3\,dy-(1-\delta_2)\int_0^1 y(1-y)|\partial_y W|^2\,dy\bigg].
\end{aligned}$$

Therefore, using Proposition B.1, we have

$$\mathcal{R}_\delta(v) \le 0,$$

which completes the proof.

Appendix C. Existence and uniqueness of shifts

We here prove the existence and uniqueness of the shift functions defined by the non-autonomous system (5.29). For any fixed $\varepsilon_1, \varepsilon_2$ and $U \in \mathcal{X}_T$, let $F : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ denote the right-hand side of (5.29) by

$$F\big(t, X_1, X_2\big) = \begin{pmatrix} \Phi_{\varepsilon_1}(Y_1(U))\Big(2|\mathcal{J}^{bad}(U)| + 1\Big) - \frac{\sigma_1}{2}\Psi_{\varepsilon_1}(Y_1(U)) \\ -\Phi_{\varepsilon_2}(-Y_2(U))\Big(2|\mathcal{J}^{bad}(U)| + 1\Big) - \frac{\sigma_2}{2}\Psi_{\varepsilon_2}(-Y_2(U)) \end{pmatrix}.$$

We may show that there exists $a, b \in L^q(0,T)$ (for some $q \ge 1$) such that

$$\sup_{(X_1,X_2)\in\mathbb{R}^2} |F(t, X_1, X_2)| \le a(t), \qquad \sup_{(X_1,X_2)\in\mathbb{R}^2} |\nabla_{X_1,X_2} F(t, X_1, X_2)| \le b(t).$$

To this end, we use the facts that

Φ_{ε_i} and Ψ_{ε_i} are Lipschitz and bounded;

for the solution $(v,h) \in \mathcal{X}_T$, $\quad v, v^{-1} \in L^\infty((0,T)\times\mathbb{R})$ $\quad$ and $\quad h - \tilde{h}, v_x \in L^\infty(0,T; L^2(\mathbb{R}))$;

$|a^{X_1,X_2}| + |\tilde{h}^{X_1,X_2}| \le C, C^{-1} \le \tilde{v}^{X_1,X_2} \le C,$ where the constant C is uniform w.r.t. X_1, X_2;

for any $r \in [1,\infty]$, $\quad L^r(\mathbb{R})$-norms of $(a_i)_x^{X_i}, (a_i)_{xx}^{X_i}, (\tilde{v}_i)_x^{X_i}, (\tilde{v}_i)_{xx}^{X_i}, (\tilde{v}_i)_{xxx}^{X_i}, (\tilde{h}_i)_x^{X_i}, (\tilde{h}_i)_{xx}^{X_i}$ are uniform w.r.t. X_1, X_2.

Thus, there exists a constant C_* (uniform w.r.t. X_1, X_2) such that

$$\begin{aligned} |F(t, X_1, X_2)| \le C_* \sum_{i=1}^2 \Bigg[&\|(a_i)_x^{X_i}\|_{L^2(\mathbb{R})} \|h - \tilde{h}\|_{L^\infty(0,T;L^2(\mathbb{R}))} + \|(a_i)_x^{X_i}\|_{L^1(\mathbb{R})} \|\tilde{h} - \tilde{h}^{X_1,X_2}\|_{L^\infty(\mathbb{R})} \\ &+ \Big(\|(a_i)_x^{X_i}\|_{L^2(\mathbb{R})} + \|(\tilde{v}_i)_x^{X_i}\|_{L^2(\mathbb{R})}\Big)\Big(\|v_x\|_{L^\infty(0,T;L^2(\mathbb{R}))} + \|(\tilde{v}_i)_x^{X_i}\|_{L^2(\mathbb{R})}\Big) + 1 \Bigg] \\ &\le C_*, \end{aligned}$$

which especially implies

$$\sup_{(X_1,X_2)\in\mathbb{R}^2} |F(t, X_1, X_2)| \le a(t) \in L^1(0,T).$$

Likewise, we have

$$|\nabla_{X_1,X_2} F(t, X_1, X_2)| \le C_*,$$

which implies

$$\sup_{(X_1,X_2)\in\mathbb{R}^2} |\nabla_{X_1,X_2} F(t, X_1, X_2)| \le b(t) \in L^2(0,T).$$

Therefore, the system (5.29) has a unique absolutely continuous solution thanks to the following lemma. This lemma is a simple extension of [11, Lemma A.1] (which is for scalar ODE). So, we omit the proof.

Lemma C.1. *Let $p > 1$, $T > 0$ and $n \in \mathbb{N}$. Suppose that a function $F : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies that*

$$\sup_{X\in\mathbb{R}^n} |F(t, X)| \le a(t), \qquad \sup_{X\in\mathbb{R}^n} |\nabla_X F(t, X)| \le b(t),$$

for some functions $a \in L^1(0, T)$ and $b \in L^p(0, T)$. Then for any $x_0 \in \mathbb{R}$, there exists a unique absolutely continuous solution $X : [0, T] \to \mathbb{R}^n$ to the system of ODEs:

$$\begin{cases} \dot{X}(t) = F(t, X(t)) & \text{for a.e. } t \in [0, T], \\ X(0) = x_0. \end{cases}$$

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