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An Initial Framework for Analyzing Students' Reasoning with Equivalence Across Mathematical Domains

The concept of equivalence is foundational in mathematics and is pervasive in the K-16 curriculum. Though much research has focused on equivalence, nearly all of it is domain-specific, and it is therefore unclear how students' reasoning about equivalence in one domain might influence their reasoning about it in another, if at all. This highlights a need for increased theoretical unity and coherence. In this theoretical paper, we propose an initial framework for analyzing students' reasoning about equivalence across domains. We use the framework to highlight commonalities amongst the ways in which equivalence is interpreted with respect to fractions, K-12 algebra, modular arithmetic, and linear algebra. We demonstrate the framework's strength as an analytical tool by using it to conduct detailed analyses of student data from already-published studies in combinatorics and abstract algebra. We conclude by suggesting ways in which this framework lays a rich foundation for future research.

1: Introduction

“Equivalence,” wrote the eminent mathematician Paul Halmos (1982), “is one of the basic building blocks out of which all mathematical thought is constructed” (p. 246). Indeed, equivalence is one of the most fundamental, far-reaching concepts in mathematics, and it emerges in a variety of content areas across the K-16 spectrum. There is ample evidence in the literature and in policy documents that highlights the centrality and ubiquity of equivalence in K-12 mathematics (e.g., Carpenter, Franke, & Levi, 2003; Herscovics & Kieran, 1980; Kara, Simon, & Placa, 2018; Kieran & Sfard, 1999; Knuth et al., 2006; McNeil & Alibali, 2005; National Governors' Association Center for Best Practices & Council of Chief State School Officers, 2010; Ni, 2001; J.P. Smith, 1995; Steffe, 2004; Wong & Evans, 2007). As one example, Charles (2005) argued that equivalence is one of the ‘big ideas’ at the K-8 level that “links numerous mathematical understandings into a coherent whole” (p. 10). In postsecondary

mathematics, equivalence plays a key role in learning about such concepts as angle measure (e.g., Moore, 2013), logic (e.g., Stylianides et al., 2004), number theory (e.g., Smith, 2006), linear algebra (e.g., Berman et al., 2013), combinatorics (e.g., Lockwood & Reed, 2020), and abstract algebra (e.g., Cook, 2018; Larsen, 2013). Equivalence and its many manifestations also have important implications for teacher preparation (e.g., AMTE, 2017; NCTM, 2020). For example, it is generally agreed upon that future K-12 teachers should come to “understand the critical importance of equivalence and approach the teaching of algebraic concepts by *explicitly* attending to equivalence,” (AMTE, 2017, p. 97, emphasis added).

However, in spite of the importance of equivalence across the K-16 curriculum, research suggests that students at all levels experience considerable difficulty with equivalence (e.g.,; Carpenter, Franke, & Levi, 2003; Chick, 2003; Chesney et al., 2013; Godfrey & Thomas, 2008; Kieran, 1981; McNeil et al., 2006; Pomerantsev & Korostoleva, 2003; Stephens, 2006; Weinberg, 2009). Moreover, while the literature emphasizes the importance of equivalence with respect to a variety of topics, there is a need for more overall coherence: characterizations of equivalence are often either implicit, underspecified, or domain-specific. This is notable given that, for students, “equivalence has had many different faces and [...] many different names” (Asghari, 2019, p. 4675). As a result, it is currently unclear how the understandings of equivalence students develop in one domain might influence their reasoning about equivalence in another. Each time students encounter equivalence in a new context, we as a field have no theoretical recourse but to assume that they are encountering the concept for the first time. We therefore interpret that the equivalence literature as a whole is currently less than the sum of its parts. That is, potential ways in which equivalence may be framed in a unifying, overarching manner have not been leveraged in the literature so far. This highlights the need for a coherent theoretical approach to equivalence that researchers could leverage across multiple domains.

In this theoretical paper, we propose an initial attempt at such a framework. The framework takes the form of a conceptual analysis (Thompson, 2008), in which we explicitly articulate three key interpretations of equivalence that are common across mathematical domains. In addition to describing these three interpretations in detail, our primary objective is to illustrate that these interpretations (1) capture aspects of productive reasoning with equivalence, and (2) account for students' reasoning with equivalence in a variety of domains.

2: Background Literature

A 1948 article by Burington provided the motivating, central idea that underpins our objectives here: “the literature abounds in such phrases as A is equivalent to B, which, unless properly defined, are often meaningless or misleading” (p. 1). This sentiment harmonizes with the calls of researchers throughout the equivalence literature who have underscored the need to be explicit about what equivalence entails (Kieran, 1981; Knuth et al., 2006; McNeil & Alibali, 2005; McNeil et. al., 2006; Ni, 2001; J.P. Smith; 1995; Solares & Kieran, 2013; Stephens, 2006). Indeed, what *does* it mean to say that “A is equivalent to B”? In considering this question, we situate our work within a particular domain that has received substantial attention in the equivalence literature: the equivalence of expressions in algebra. Our primary intention here is not to examine the equivalent expressions literature per se, but to use this substantial body of literature to generate hypotheses about the general concept of equivalence. We focus here on three such insights that informed our development and use of the conceptual analysis that follows in Section 3.2.

First, we note that there are several forms of equivalence that are prominent in algebra, each of which depends on the types of objects that are being related. To illustrate this point, we consider two fundamental algebraic objects: expressions and equations (with at most one

variable). We find it useful to distinguish between expressions that are *numerical* (numbers only, no variables) or *algebraic* (with variables). Numerical expressions are equivalent¹ when they correspond to the same value (e.g., Liebenberg, Linchevski, Sasman, & Oliver, 1999; Saldana & Kieran, 2005), and algebraic expressions are equivalent if “for any admissible number that replaces x , each of the expressions gives the same value” (Saldanha & Kieran, 2005, p. 3). We note that these characterizations are undeniably similar – they both, for example, focus on the value of the expressions in question – yet are ultimately different in subtle ways. Thus, reasoning productively with equivalence requires explicit attention to the objects involved and the warrants and interpretations by which one establishes that two objects are equivalent.

Second, these forms of equivalence necessarily build upon one another. For example, the characterization of the equivalence of algebraic expressions implicitly depends on the equivalence of numerical expressions, as illustrated in the following reformulation: algebraic expressions $f(x)$ and $g(x)$ are equivalent if they are equivalent as numerical expressions for each admissible value of x (Solares & Kieran, 2013). Additionally, understanding the concept of equations first requires understanding the equivalence of expressions. For example, equations can be *conditional* (i.e., for a proper subset of the set of all possible admissible values for x , the two expressions are equivalent as numerical expressions) or *identical* (i.e., the two expressions are equivalent as algebraic expressions) (Godfrey & Thomas, 2008). This key distinction is important for such fundamental tasks as solving linear equations. For instance, verifying that $x = 8$ is a solution to the conditional equation $2(x + 1) = 18$ requires attention to the equivalence of numerical expressions (e.g., the solution set $\{8\}$ is precisely the set of values for which both sides of $2(x + 1)$ and 18 are equivalent as numerical expressions). Similarly, understanding why it is

¹ Some researchers refer to this form of equivalence as ‘equality’ or ‘numerical equality’ (e.g., Molina & Castro, 2009; Saldanha & Kieran, 2005). We follow these researchers in viewing equality as a specific form of equivalence.

acceptable to replace $2(x + 1)$ with $2x + 2$ to obtain $2x + 2 = 18$ requires recognizing that the two are equivalent algebraic expressions – put another way, $2(x + 1) = 2x + 2$ is an identical equation (in the sense of Godfrey & Thomas, 2008). This kind of reasoning requires careful attention to both the objects and the form of equivalence in question. Reasoning in this way is not a trivial endeavor; rather, it is complex, multi-layered, and interrelated.

A final point that highlights the complexity of equivalence is that it can (and should) be understood in various ways (e.g., Harel, 2008; Kieran & Sfard, 1999; Solares & Kieran, 2013; Zwetschler & Prediger, 2013). For simplicity, in the preceding paragraphs we intentionally focused on characterizations of equivalence in terms of a shared numerical value (or set of numerical values); these characterizations all stem from a *numeric* interpretation of equivalence (e.g., Saldanha & Kieran, 2005). Previous efforts to identify coherence in the many manifestations of equivalence in the K-8 curriculum have emphasized the potential productivity of this view – for example, Charles (2005) highlighted the centrality of recognizing that “any number, measure, numerical expression, algebraic expression, or equation can be represented in an infinite number of ways that *have the same value*” (p. 10, emphasis added). In addition to this *numeric* interpretation, researchers have called attention to two other key interpretations: a *transformational* (e.g., Solares & Kieran, 2013) and a *descriptive* (e.g., Zwetschler & Prediger, 2013) interpretation of equivalence. We elaborate all three of these interpretations in Table 1 and Section 3.2.² The fact that equivalence can and should be interpreted in various ways provided the basis on which we developed the cross-domain conceptual analysis. Specifically, the numeric, descriptive, and transformational interpretations of equivalence – which previously had

² All three of these constructs appear under different names. For example, some researchers refer to a *numeric* interpretation as *insertion* equivalence (e.g., Zwetschler & Prediger, 2010). Additionally, a *transformational* interpretation is referred to by some as a *syntactic* interpretation (e.g., Solares & Kieran, 2013). Though the names are different, the meanings are compatible.

only been applied within the domain of K-12 algebra – form the foundations of our cross-domain theory.

3: A framework for characterizing productive interpretations of equivalence across domains

3.1: Theoretical underpinnings

Our framework takes the form of a *conceptual analysis* (Thompson, 2008). At its core, a conceptual analysis is a form of theory that explicitly articulates “what students might understand when they know a particular idea in various ways” (Thompson, 2008, p. 43). A conceptual analysis can be used to analyze and address the fit and coherence of mathematical meanings expressed in an instructional sequence, curriculum, or, in this case, body of literature. As noted by O’Bryan (2018), “supporting students in seeing the *coherence within and across* mathematics courses demands that curriculum designers and instructors first clearly articulate for themselves the general ways of reasoning mathematics instruction should support” (p. 123, emphasis added). The conceptual analysis we propose below seeks to address the fragmented nature of the equivalence literature by explicitly articulating detailed descriptions of three interpretations of equivalence that, we claim, capture key aspects of equivalence that are common across mathematical domains (this is the focus of the mathematical examples we present in Section 3).

A conceptual analysis also provides researchers with a means by which to explicitly account for, articulate, and understand students’ reasoning about a particular mathematical idea. That is, it can be used to create *models of students’ thinking* (e.g., Clement, 2000; Steffe & Thompson, 2000). By ‘model,’ we refer to the inferences that researchers make about the cognitive structures – in this case, the interpretations of equivalence – that might underlie and inform students’ language and actions. Thus, in addition to addressing issues of coherence, we will

illustrate the utility of our framework by using it to build models of students' thinking that capture meaningful aspects of their reasoning about equivalence (this is the purpose of the detailed analyses of students' activity we present in Section 4). We situate each interpretation within relevant literature, and for each interpretation we describe criteria – in the form of characterizations of students' language and actions – to use to determine if a student is enacting a particular understanding.

3.2: A cross-domain framework for analyzing students' reasoning with equivalence

In this section we present our framework for students' interpretations of equivalence. From our perspective as researchers, we specify the scope of mathematical situations to which this framework applies by operationalizing the formal, abstract notion of an equivalence relation³. Such a relation partitions a given set S into subsets (equivalence classes); two elements of a set S are equivalent if they belong to the same equivalence class. This is consistent with what Hamdan (2006) called a *global* view of equivalence, which (implicitly) underlies all of its manifestations in the primary, secondary, and postsecondary mathematics curriculum. These manifestations include (but are certainly not limited to) the equivalence of fractions, numerical expressions, algebraic expressions, algebraic equations, and integers (modulo k).

Thompson (2002) noted that a conceptual analysis should be grounded in *students'* conceptual experiences. Thus, while the notion of equivalence relation specifies the scope of mathematical situations that we (as *researchers*) associate with this framework, we intend for the framework to explicate the various ways in which *students* might interpret equivalence in these situations. The interpretations that we set forth therefore all follow from a *local*, element-to-element view of equivalence (Hamdan, 2006), primarily because this is the predominant view of

³ Formally, an *equivalence relation* on a set S is a relation $\sim$ that is reflexive (i.e., for all $x \in S$, $x \sim x$), symmetric (i.e., for all $x, y \in S$, if $x \sim y$ then $y \sim x$), and transitive (i.e., for all $x, y, z \in S$, if $x \sim y$ and $y \sim z$, then $x \sim z$).

equivalence in students' experiences and is therefore, we argue, better suited to capture meaningful aspects of students' reasoning about equivalence across domains. These interpretations are based upon those in the literature on algebraic expressions that we briefly previewed in Section 2: *common characteristic* (a generalization of *numeric*), *descriptive*, and *transformational* interpretations of equivalence. While these components were initially articulated in terms of numerical and quantitative contexts in K-12 algebra, we have adapted them here to address students' reasoning about equivalence in a wide variety of mathematical domains. We note that we do not presume that these interpretations comprehensively account for all aspects of students' reasoning about equivalence. Instead, we propose that these interpretations can capture key aspects of students' activity with equivalence across domains.

Table 1. A framework for analyzing students' reasoning about equivalence.

Interpretation of equivalence	Description	Associated observable behaviors (criteria)	Example
Common characteristic	Interpreting or determining equivalence based upon a perceived attribute that the objects in question have in common	Criterion C1: use of descriptors like <i>same</i>, <i>common</i>, <i>similar</i>, <i>identical</i>, <i>invariant</i>, or <i>shared</i> Criterion C2: sameness is explained by identifying an attribute that the objects themselves share	Viewing two systems of linear equations as equivalent because they share the same solution set (Harel, 2008).
Descriptive	Interpreting or determining that objects are equivalent because they describe the same quantity or serve the same purpose with respect to a given situation	Criterion D1: use of descriptors like <i>same</i>, <i>common</i>, <i>similar</i>, <i>identical</i>, <i>invariant</i>, or <i>shared</i> Criterion D2: sameness is explained by referencing an aspect of the given mathematical situation or task setting	Proving combinatorial identities by using two different counting processes to count the same set of outcomes (Lockwood, Caughman, & Weber, 2020).

Transformational	Interpreting or determining equivalence based upon the relationship between objects in terms of the actions by which one object has been or might be transformed into another	Criterion T: a sequence of actions (either already performed or imagined) by which one object might or can be changed into another is enacted or described.	Solving the equation $5x + 1 = 3x + 5$ by “undoing”: subtracting 1 from both sides (to obtain $5x = 3x + 4$), then subtracting $3x$ from both sides (to obtain $2x = 4$), then dividing both sides by 2 (to obtain $x = 2$) (de Lima & Tall, 2008).
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A *common characteristic* interpretation of equivalence involves interpreting or determining equivalence on the basis of a perceived attribute that the objects in question share. The term *common characteristic* was used by Piaget (1997) – and subsequently by Hamdan (2006) – to describe how a person classifies objects based upon that person’s perceptions of similarity⁴, however informal or intuitive, amongst the objects. In this context, it serves as an adaptation of the previously mentioned *numeric* view of the equivalence of algebraic expressions, in which two expressions are seen as equivalent if they have the same value for all admissible values of the relevant variable(s) (Saldanha & Kieran, 2005; Solares & Kieran, 2013; Zwetschler & Prediger, 2013). We consider a student to be employing a *common characteristic* interpretation of equivalence if both of the following criteria are satisfied.

- Criterion C1 (recognition of sameness): in referring to the objects in question, descriptors like *same*, *common*, *similar*, *identical*, *invariant*, or *shared* (or a reasonable synonym) are used.
- Criterion C2 (explanation of sameness): the sameness of the objects in question is explained by identifying an attribute that the objects themselves share.

To exemplify this interpretation, we note that Hamdan (2006) reports on a classroom episode in which the instructor asks students to consider a situation in which lines are equivalent if they

⁴ This characterization intentionally emphasizes the importance of attending to how an *individual* might intuitively or informally perceive similarity. While we believe this is a generally productive approach to analyzing students’ reasoning and supporting students’ learning, certainly not all perceptions of common characteristics are productive. See the Appendix (Section 6) to see how we, as researchers, formally define the scope of the *common characteristic* interpretation using equivalence relations.

are parallel⁵. A student reformulates the notion of parallel by noting that equivalent lines a common “direction, of course, or slope. The common property will be the slope” (p. 143). As another example, in linear algebra, two linear systems are equivalent if they “have the same solution sets” (Poole, 2014, p. 60), or, analogously, if their corresponding coefficient matrices “have the same reduced row echelon form” (Berman et al., 2013, p. 4). Here we note the use of the word “same” (criterion C1) as well as an indication of the shared attribute⁶ (“solution sets,” “reduced row echelon form,” criterion C2).

A *descriptive* interpretation of equivalence involves viewing objects as equivalent on the basis that they describe the same quantity or serve the same purpose in a given external situation. Descriptive interpretations of equivalence differ from common characteristic interpretations on the grounds that, whereas a common characteristic interpretation focuses on an attribute of the objects themselves, a descriptive interpretation focuses on an attribute of a given situation. This is reflected in our criteria; the first focuses on the recognition of some kind of sameness, whereas the second clarifies that the recognition of sameness must be based upon aspects of the given situation:

- Criterion D1 (recognition of sameness): in referring to the objects in question, descriptors like *same*, *common*, *similar*, *identical*, *invariant*, or *shared* (or a reasonable synonym) are used.
- Criterion D2 (explanation of sameness): the sameness of the objects in question is explained by referencing an aspect of the given mathematical situation or task setting.

Descriptive interpretations of equivalence are particularly useful when reasoning about measurement. For example, consider the equation “1 week = 7 days.” A productive way to

⁵ For example, lines of slope 2 can be viewed as equivalent classes with respect to the following equivalence relation on $\mathbb{R}^2$: $(x_1, y_1) \sim (x_2, y_2)$ whenever $-2x_1 + y_1 = -2x_2 + y_2$.

⁶ We note, however, that a focus on the sequence of row operations by which the matrices in question are manipulated into reduced row echelon form would be classified as *transformational* activity related to equivalence. The constructs in the framework are not disjoint and can supplement each other. We elaborate further on the nature of transformational activity related to equivalence below.

interpret this equality is to view both expressions as measurements of the same magnitude of time: “1” is the measurement when measuring in weeks, and “7” is the measurement when measuring in days. From this perspective, the equivalence of 1 week and 7 days “makes explicit [use of] the fact that the magnitude of a quantity is invariant with respect to a change of unit” (Thompson et al., 2014, p. 5). This is descriptive in nature because the two expressions describe an invariant (recognition of sameness, criterion D1) magnitude as measured in different units (an aspect of the situation or task setting, criterion D2). As another example, consider combinatorial proofs of binomial identities. A common strategy is to reason that “if two different combinatorial expressions can be represented as different ways of counting the *same set of objects*, then the two combinatorial expressions are equal” (Lockwood, Caughman, & Weber, 2020, p. 179, emphasis added). We consider this a use of a descriptive interpretation because two different yet legitimate counting processes (attributes of an external situation, criterion D2) are used to count the same set of outcomes and are therefore equal (explicit references to sameness, criterion D1).

A *transformational* interpretation⁷ of equivalence involves viewing the relationship between objects in terms of the actions by which one object has been or might be transformed into another, according to a set of rules. Rather than focusing exclusively on the fact that two objects are in some way the *same*, transformational activity focuses on how one object can be *changed* into another. This typically involves “a sense of the actions needed in order to reach a desired [...] form” (Harel, 2008, p. 15). Transformational activity has been identified across the equivalence literature as one that supports productive reasoning, largely because it enables one to generate equivalent representations of objects that are easier to work with or serve an explicit

⁷ See the Appendix for a discussion of how informal notions of transformations on objects align with the formal equivalence relation concept.

purpose. And yet, despite its benefits, researchers have also cautioned that over-relying on transformational activity can preclude the necessary emphasis on equivalence (e.g. Alibali et al., 2007; Carpenter, Franke, & Levi, 2003; Kieran, 1981). Said another way, an overemphasis on change overshadows an emphasis on sameness. For example, de Lima and Tall (2008) reported that

many students respond to the general principle of ‘doing the same thing to both sides’ in specific terms in which the +1 in $5x+1=3x+5$ is shifted over the other side with a ‘change in sign’ and the 3 in $3x = 6$ is ‘moved over the other side and ‘put underneath’. [This] may be seen as moving symbols with a kind of additional ‘magic’ to get the correct solution (p. 4).

The key distinction between whether or not one’s transformational activity is productive ultimately hinges on the extent to which one recognizes that the transformations that are applied preserve some key feature of the object or related context (e.g., Alibali et al., 2007; Carpenter et al., 2003). Harel (2008) encapsulated this idea well by arguing that a hallmark of productive transformational activity is “changing the form of an entity without changing a certain property of the entity” (p. 16) – this is at the heart of what he calls an *algebraic invariance* way of thinking. Through the lens of our framework, we frame “a certain property” in terms of the other two interpretations: common characteristic and descriptive. For instance, Harel pointed out that solving $ax^2 + bx + c = 0$ by using familiar actions (e.g., adding the same element to both sides, completing the square) to transform it into the more tractable $(x + T)^2 = L$ requires an awareness that one is changing the appearance of the equation but “maintaining the solution set unchanged” (p. 14). In the language of our framework, we observe that Harel is suggesting that such transformational activity be paired with a common characteristic interpretation: the focus is on the solution set, a common attribute of the equations themselves (criterion C2) that remains unchanged (criterion C1). Additionally, regarding the pairing of transformational activity with a descriptive interpretation, consider again the aforementioned ‘duration of time’ scenario. If

asked how many days are in 12 weeks, a transformational approach might focus on changing 12 weeks into the appropriate number of days by multiplying by a conversion factor: $12 \text{ weeks} \cdot 7 \text{ days/week} = 12 \cdot 7 \text{ days} = 84 \text{ days}$. Supplementing with a descriptive approach, “12” is the measurement when the given duration of time is measured in weeks. The duration of time as measured in days can be productively visualized as a partitioning of each of the 12 weeks into 7 days, so that “84” is the measurement of the given duration of time in days. 12 weeks and 84 days are equivalent, then, because they are measurements of the same, invariant magnitude (recognition of sameness, criterion D1) in different units (reference to an aspect of the task setting, criterion D2).

We therefore propose that transformational activity can indeed be productive when reasoning about equivalence because generating equivalent forms is a fundamental mathematical strategy. But it is more productive and less constrained when it is accompanied by a notion of what these transformations preserve. This could include supplementing one’s transformational activity with a common characteristic interpretation (as in the quadratic equation example, in which it is the solution set that is preserved) and/or a descriptive interpretation (as in the measurement example, in which it is the given duration of time that is preserved). For the purposes of our framework, then, we distinguish between transformational activity that is accompanied by an explicit recognition of sameness (such as those described in the common characteristic and descriptive interpretations) and transformational activity that is enacted without an explicit recognition of sameness. We reflect this distinction as follows:

- We reserve the term *transformational interpretation of equivalence* to refer to instances in which the student recognizes that the new forms being generated by the transformations are equivalent in some way – that is, the student is aware that the

transformations preserve some aspect of sameness. To specify the aspect of sameness, we supplement our descriptions with one (or both) of the other interpretations as appropriate in order to specify the relevant form of sameness in question.

- We use the term *transformational activity* to refer to instances in which we (as researchers) recognize that the transformations the student applies do indeed produce new, equivalent forms, but the student does not explicitly attend to such notions. In this way, the term *transformational activity* is more general than *transformational interpretation*.

This approach has two immediate benefits: it coheres with and respects the findings regarding transformational activity in previous research (i.e., students who are engaging in transformational activity are not necessarily attentive to equivalence), and it also provides more details about how students are interpreting the accompanying notion of sameness, if at all (see Table 2). As such, we require only a single criterion for the identification of transformational activity (and leave open the possibility that it could be paired with notions of sameness as specified in one of the other interpretations):

- Criterion T: a sequence of actions (either already performed or imagined) by which one object might or can be changed into another is enacted or described.

Table 2. Possible ways in which a transformational interpretation of equivalence might appear in conjunction with a common characteristic or descriptive interpretation

Terms that describe how transformations might manifest in students' reasoning about equivalence	Corresponding description of the student's reasoning	Example
Transformational activity only	A student applies transformations that generate new, equivalent forms of an object but does not necessarily recognize or attend to the fact that these forms are equivalent.	Transforming $5x + 1 = 3x + 5$ into $3x = 6$ via “moving symbols with a kind of additional ‘magic’ to get the correct solution” (de Lima & Tall, 2008, p. 4).
Transformational interpretation (supplemented with a common characteristic interpretation)	A student is aware that the forms produced by the transformations she is applying share a common attribute – that is, the transformations preserve a key aspect of these objects.	Adding L to both sides of $(x + T)^2 - L = 0$ to obtain $(x + T)^2 = L$ and recognizing that the solution set does not change (e.g., Harel, 2008).
Transformational interpretation (supplemented with a descriptive interpretation)	A student is aware that the forms produced by the transformations she is applying both describe the same quantity or serve the same purpose in a given situation.	Supplementing the transformation $12 \text{ weeks} \cdot 7 \text{ days/week} = 84 \text{ days}$ by reasoning that “12 weeks” and “84 days” are equivalent because they are measurements of the same magnitude of time in different units (e.g., Thompson et al., 2014).

3.3: Examples of the three interpretations from the literature on fraction equivalence and modular congruence

To illustrate how the framework captures meaningful aspects of reasoning with equivalence *across* domains as well as how these three interpretations might work in tandem *within* domains, we now use our framework to consider examples from the literature – including interpretations from both researchers and students – on fraction equivalence and modular congruence. We summarize our discussion of the three interpretations of equivalence in these two domains in Table 3.

In the case of fractions, consider $\frac{1}{2}$ and $\frac{3}{6}$. As noted by Ni (2001), “in many classrooms the acquisition of the concept of equivalent fraction is [...] reduced to the mastery of the rule ‘multiply or divide the numerator and denominator of a fraction by the same number’” (p. 413). Using this interpretation, $\frac{1}{2}$ and $\frac{3}{6}$ are equivalent because multiplying the numerator and denominator of $\frac{1}{2}$ by 3 yields $\frac{3}{6}$. In this way, this interpretation focuses on “[altering] the numerical representation of fractions” (Smith, 1995, p. 20). This is an example of a *transformational activity* because the central focus is on changing (i.e., altering) one fraction into another (criterion T). Another way to interpret the equivalence of these two fractions is to reason that “they have the same value” (Martinez & Yeong, 2018, p. 48) or, analogously, that their respective quotients are invariant (e.g., Behr, Harel, Post, & Lesh, 1992). Smith (1995) also identified this as a common student interpretation, associating it with a procedure in which students “divide the numerator of each fraction by its denominator. If the resulting decimals are identical, the original fractions are equal” (p. 47). From this perspective, $\frac{1}{2}$ and $\frac{3}{6}$ are equivalent because they both have the same quotient of .5. This is an example of a *common characteristic* interpretation because explicit reference is made to sameness (e.g., same, invariant, identical) (criterion C1), and the attribute on which sameness is focused (the quotient) is framed as an attribute of the fractions $\frac{1}{2}$ and $\frac{3}{6}$ themselves (criterion C2).

Lastly, the equivalence of $\frac{1}{2}$ and $\frac{3}{6}$ can be warranted via reference to the preservation of a quantity (e.g., Steffe, 2004). For example, Simon & Tzur (2004) presented a learning trajectory based upon such as tasks as: “Draw a rectangle with $\frac{1}{2}$ shaded. Draw lines on the rectangle so that it is divided into sixths. Determine how many sixths are in $\frac{1}{2}$ ” (p. 97). From this perspective, students might interpret that $\frac{1}{2}$ and $\frac{3}{6}$ are equivalent because “the areas of a whole and shaded part never change, but the number of equal parts into which the whole is divided can alter dramatically” (Wong & Evans, 2007, p. 827) – see Figure 1. Interpreting

equivalence in this way leads to the conclusion that $\frac{1}{2}$ and $\frac{3}{6}$ are equivalent because they both describe the same amount of shaded area. This interpretation is inherently descriptive because it attends to the fact that the two fractions represent the *same* amount of shaded area – that is, the shaded areas across each diagram “never change” (criterion D1). Shaded area is an attribute of the external task setting and not, as in the case of the invariant quotient interpretation above, of the fractions themselves (criterion D2).



Figure 1. Illustrating a descriptive interpretation of equivalence: $\frac{1}{2}$ and $\frac{3}{6}$ are equivalent because they represent the same amount of shaded area in relation to the area of the whole rectangle.

In the domain of modular arithmetic, consider a comment made by one of Smith’s (2006) participants, who conceptualized equivalence modulo 11 as follows: “we’re looking for the set of numbers such that when you divide them by 11, you’re left with a remainder of 1” (p. 271). This interpretation relies upon the idea that two integers are equivalent because they “leave the same remainder upon division by n ” (Smith, 2006, p. 260). This is an example of a *common characteristic* interpretation because it attends to the sameness (criterion C1) of two integers based upon a common attribute of those integers – specifically, that they have the same remainder (criterion C2). The researcher also noted that many students immediately associated equivalence with “a computation or a transformation that had occurred” (p. 260). For example, one participant she interviewed interpreted the equivalence of a and b modulo n as follows: “if I take n and I multiply it with a whole number and then I add b to it, I should get a ” (p. 262). This student is exhibiting *transformational activity* because she is interpreting the

relationship between a and b in terms of the sequence of actions by which one might be changed into the other (criterion T). Lastly, we consider an example from Cook's (2012) study of how students might develop formal notions of algebraic structure. The study initiated by having students – who had no formal exposure to modular congruence – solve equations using their intuitive notions of clock arithmetic. When solving the linear equation $x + 3 = 9$, the students began discussing the additive inverse of 3 (so that they could add it to both sides in order to isolate x). Their discussion led them to conclude that “ $-3=9$ ” (p. 103) on the grounds that both -3 and 9 function as the additive inverse of 3. They eventually extrapolated this idea and identified other such pairs (such as $-1 = 11$ and $-2 = 10$) en route to more formal notions of modular congruence. We claim that this is a descriptive interpretation. The students claim that two elements are equivalent (“ $-3=9$,” satisfying Criterion D1) because they both function as an additive inverse for 3. In this context, additive inverse was a key component of the external task setting in which they were engaged (criterion D2). Put another way, both -3 and 9 served the same purpose in the situation of solving $x + 3 = 9$.

Table 3. Illustrating the components of the framework in the domains of fraction equivalence and modular congruence.

Interpretation of equivalence	Fraction equivalence	Modular congruence
Common characteristic	$\frac{1}{2}$ and $\frac{3}{6}$ are equivalent because their respective quotients are the same (e.g. Behr, Harel, Post, & Lesh, 1992; Martinez & Yeong, 2018).	Integers -3 and 9 are congruent modulo 12 because they share the same remainder (i.e. 9) when divided by 12 (e.g. J.C. Smith, 2006).
Descriptive	$\frac{1}{2}$ and $\frac{3}{6}$ are equivalent because they both represent the same amount of shaded area in relation to the area of the whole rectangle (e.g. Simon & Tzur, 2004; Wong & Evans, 2007).	Integers -3 and 9 are congruent modulo 12 because they both function as the additive inverse of 3 when solving equations (e.g. Cook, 2012).

Transformational	$\frac{1}{2}$ and $\frac{3}{6}$ are equivalent because multiplying both the numerator and denominator of $\frac{1}{2}$ by 3 yields $\frac{3}{6}$ (e.g. Ni, 2001; J.P. Smith, 1995).	Integers -3 and 9 are congruent modulo 12 because adding the modulus, 12 , to -3 yields 9 (e.g. J.C. Smith, 2006).
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4: Detailed Analyses of Students' Reasoning about Equivalence in Abstract Algebra and Combinatorics

We now show the utility of the framework by conducting detailed analyses of students' activity from two previous studies in undergraduate mathematics: one in abstract algebra (Cook, 2018), and one in combinatorics (Lockwood & Reed, 2020). We use the framework to classify students' reasoning with equivalence and demonstrate the usefulness of the three interpretations.

4.1: Equivalence in abstract algebra

The examples of student activity in 4.1.1 and 4.1.2 comes from a study (Cook, 2018) designed to investigate abstract algebra students' reasoning with the zero-product property (i.e., for all elements a and b in a ring R , if $a \cdot b = 0_R$, then $a = 0_R$ or $b = 0_R$). The instructional tasks centered on leveraging the properties needed to solve basic polynomial equations to gain insight into abstract algebraic structures. For example, the zero-product property functions both as a key tool for solving polynomial equations and as the defining characteristic of integral domains, one of the foundational structures of abstract algebra.

These episodes focus on the activity of one student, Matt (pseudonym), a junior mathematics education major who had completed linear algebra but had no formal exposure to abstract algebra, as he solved equations. Matt leveraged equivalence (in the form of the equivalence of equations) as a means to explain his general inattention to the structural implications of his algebraic manipulations. In this context, a *common characteristic* interpretation involves viewing that equations are equivalent when they share the same solution set. For example, the equations

$(x + 2)(x + 3) = 0$ and $(x + 2)^5(x + 3)^5 = 0$ are equivalent because they share the solution set $\{-2, -3\}$ – the solution set is a common characteristic. *Transformational activity* in this context would focus on the algebraic manipulation of one equation into another, such as the fact that $(x + 2)^5(x + 3)^5 = 0$ can be generated by raising both sides of $(x + 2)(x + 3) = 0$ to the fifth power. A *transformational interpretation* would involve accompanying this kind of activity with an awareness of the feature that these transformations preserve (in this case, the transformations preserve the solution set, which is central to the aforementioned *common characteristic* interpretation).

4.1.1: Episode 1: Employing transformational activity

One of the first equation-solving tasks Matt was given prompted him to solve the pre-factored quadratic equation $(x + 2)(x + 3) = 0$. In response, he distributed the left-hand side of the equation to obtain $x^2 + 5x + 6 = 0$ and (correctly) applied the quadratic formula to obtain $x = -2$ and $x = -3$. Hoping to necessitate use of the zero-product property, the researcher designed a follow-up task prompting Matt to solve the polynomial equation $(x + 2)^5(x + 3)^5 = 0$. Instead, however, Matt began a series of unsuccessful attempts to solve the equation, one of which, for example, involved expanding the $(x + 2)^5$ term into $(x + 2)^2(x + 2)^2(x + 2)$ and then dividing both sides by $(x + 3)^5$ (see Figure 2). Upon deciding that this approach was not productive, he considered taking “log base 5 of either side.” Ultimately, Matt decided to take the fifth root of both sides of the equation to obtain $(x + 2)(x + 3) = 0$, and then applied the quadratic formula (again) to obtain the (correct) solutions $x = -2$ and $x = -3$. Subsequently, when prompted to solve the equation $(x + 2)^5(x + 3)^7 = 0$ (another task designed to incentivize use of the zero-product property), Matt tried a host of similar approaches (e.g., expanding the

polynomial, taking roots of both sides), before coming to an impasse where he did not know how to proceed (see Figure 3).

$$\begin{aligned} (x+2)^2 (x+2)^2 (x+2) (x+3)^5 &= 0 \\ (x^2 + 4x + 4)(x^2 + 4x + 4)(x+2)(x+3)^5 &= 0 \\ (x^2 + 4x + 4)(x^2 + 4x + 4)(x+2) &= 0 \end{aligned}$$

Figure 2. Matt's initial attempt to solve $(x + 2)^5(x + 3)^5 = 0$ in $\mathbb{R}$.

$$\begin{aligned} [(x+2)(x+3)^{7/5}]^5 &= 0 \\ (x+2)(x+3)^{7/5} &= 0 \\ \rightarrow x(x+3)^{7/5} + 2(x+3)^{7/5} &= 0 \end{aligned}$$

Figure 3. Matt's initial attempt to solve $(x + 2)^5(x + 3)^7 = 0$ in $\mathbb{R}$.

In Cook's (2018) original analysis, the primary explanation for Matt's activity focused on Matt's preference for expansion and algebraic manipulation instead of a more efficient route via the zero-product property. Here, we focus on the interpretations of equivalence that Matt was exhibiting. We claim that Matt was relying on *transformational* activity because he focused exclusively on the sequence of actions by which he was generating new equations (criterion T). We consider his commentary while working on these tasks – such as his earlier mention of “taking log base 5” – as evidence in support of this claim. As additional evidence, we consider the following excerpt in which Matt explained what he saw as the path to eventually finding a solution to $(x + 2)^5(x + 3)^7 = 0$.

Matt: I mean I've tried foiling it, and I've tried taking the roots of that beast [...]. If I had kept with this ... I think it would have just fallen out. I would have had to have worked with it a little bit, but [...] if I had, just a little further and I could have gotten it.

Researcher: So, if you stuck, if you had stuck with the algebraic manipulation, then ...

Matt: We would have been here all night, but, yeah, I think I would have gotten something.

Notice that he focused exclusively on the actions he performed (e.g., “foiling it”, “taking the roots”) and how he viewed continued effort with these manipulations as the key to eventually solving the equation (e.g., when “it would have just fallen out”, “worked with it a little bit”).

4.1.2: Episode 2: Absence of a common characteristic interpretation

Matt exhibited some initial signs that he was unaware of the invariance of the solution set when he was asked to verify that $x = -2$ and $x = -3$ were both solutions to $(x + 2)(x + 3) = 0$ (the researcher asked this in hopes of focusing Matt's attention on the original equation and, thus, the potential use of the zero-product property as an efficient way to identify solutions). Curiously, Matt responded by using $x^2 + 5x + 6 = 0$ to verify the solutions. Matt explained that “that's where I got my solutions. From the quadratic formula. So that's where I wanted to put them back. [...] It seemed like the right thing to do. Since I got [the solutions] from the expanded version.” The following excerpt provides additional insight into Matt's thinking:

Researcher: Are -2 and -3 solutions to the original equation as well?

Matt: I don't ... I don't think so. It's, um, well it's different now, it's just not the same at this point.

Researcher: Okay, um, why is it not the same? Or in what way is it, is it different?

Matt: Well, here I foiled it, so now we have this x squared term and this 5x plus 6 and so they, they don't, they don't seem to be the same equation anymore.
[...]

Researcher: Okay, so, in your view, you're saying that these solutions, -2 and -3, are solutions to, um, the most recent one, or the expanded one, but not the original?

Matt: Yes. They're solutions to the expanded, but not the original.

Matt's belief that solutions solve only the 'most recent' equation that has emerged was consistent across tasks. When, for example, he was asked by the researcher whether or not $x = -2$ is a solution to $(x + 3)^5(x + 2)^7 = 0$, he substituted $x = -2$ into the most recent equation he had generated, $x(x + 3)^7 + 2(x + 3)^5 = 0$ (see Figure 3).

The image shows a series of handwritten equations in green ink on lined paper. The first line is $x = -2$. The second line is $-2(-2+3)^{7/5} + 2(-2+3)^{7/5} = 0$. The third line is $-2(1)^{7/5} + 2(1)^{7/5} = 0$. The fourth line is $-2(1) + 2 = 0$. The fifth line is $0 = 0 \checkmark$.

Figure 3. Matt justifies that $x = -2$ is a solution by substituting it back into the equation he most recently produced via manipulations (instead of the original equation).

As a result, we claim that Matt had not yet developed a *common characteristic* interpretation of equivalence. Consider Matt's comment in the above excerpt that -2 and -3 are "solutions to the expanded, but not the original." Here, "the original" referred to $(x + 2)(x + 3) = 0$ and "the expanded" referred to $x^2 + 5x + 6 = 0$. As we argued previously, Matt seemed to see these equations as related via the transformations he had applied, which was evidenced in this case by his comment that "I foiled it." Consider, however, his comment that "they're solutions to the expanded, but not the original," which we took to mean that Matt was viewing -2 and -3 as solutions to $x^2 + 5x + 6 = 0$ but not $(x + 2)(x + 3) = 0$. An immediate implication of this view is that the solution set of $x^2 + 5x + 6 = 0$ is not the same as the solution set of $(x + 2)(x + 3) = 0$ and, thus, Matt was not viewing the solution set as a *common characteristic* of both equations (because his activity does not satisfy criterion C2). Additionally, Matt's

comment that “they don’t seem to be the same equation anymore” makes it clear that criterion C1 (explicit recognition of sameness) is not satisfied.

Matt’s claims were challenged shortly thereafter when he was asked to prove that the solutions to the expanded version were, as he previously asserted, not solutions to the original equation. When attempting to prove that this was indeed the case, he paused before exclaiming “ x is a solution to the original equation!” He repeated this procedure – substituting the solutions back into the original equation – for all of the equations he had previously solved. Though we do not assert that Matt had fully developed a *common characteristic* interpretation of equivalence at this point, what is clear is that Matt showed signs of a shift in thinking, focusing more and more on the original equation and how it shared the same solutions as the most recent equation he had generated (that is, he showed initial signs related to criterion C2). The benefits of this shift were immediately noticeable, as Matt was able to solve equations similar to those which he had previously been unable to solve. For example, when given $(x - 1)^{19}(x - 4)^{23} = 0$, an equation similar to $(x + 3)^5(x + 2)^7 = 0$ that would have been quite cumbersome (if not impossible) to solve by expansion, he immediately identified $x = 1$ and $x = 4$ as solutions by reasoning only with the original equation itself.

4.1.3: Episode 3: Extending notions of equivalence to $\mathbb{Z}_{12}$

The shift in Matt’s thinking that resulted in an increased focus on the original equation and its solutions remained stable across different algebraic contexts as well. Later in the teaching experiment he was asked to solve equations in $\mathbb{Z}_{12}$. One of these tasks involved the equation $3(x - 2) = 0$, an equation that, because 3 is a zero-divisor, has multiple solutions. After inspecting the original equation and immediately identifying $x = 2$ as a solution, however, he expressed uncertainty about how to proceed. Matt, aware by this point of the existence of zero-

divisors in $\mathbb{Z}_{12}$, stated that he knew “there are other solutions” to $3(x - 2) = 0$ but that he did not “know how you would get those other numbers.” We claim that the uncertainty he was expressing indicates a desire for a process to employ to find these elusive solutions. As evidence, consider language he used to express his uncertainty: he stated “I don’t know *how to go* from here to there,” “I just don’t know *how to use* that information,” and “I don’t know *how you would get* those other numbers.” All of these statements involve Matt’s admission that he did not know *how* to perform the actions necessary to find the other solutions. These statements also highlight his focus on determining a sequence of actions to apply to the equation (criterion T). Indeed, this was exactly the function that Matt’s transformational activity served earlier, enabling him to find solutions to a variety of polynomial equations in $\mathbb{R}$. Now operating in a different algebraic structure, however, Matt appeared to be unsure which transformations are admissible and which were not. Thus, even though he was showing signs of developing a *common characteristic* interpretation of equivalence, in this situation he did not solve the equation completely because he was not sure of how to operationalize his *transformational* activity to find the remaining solutions.

4.1.4: On the importance of a *common characteristic* and a *transformational* interpretation of equivalence

We have thus far used the framework to illuminate facets of Matt’s reasoning about equivalence as he solved equations in an abstract algebra setting. We now argue that these facets are an integral part of reasoning productively about equivalence. Matt’s initial activity was primarily transformational – his overemphasis on *change* precluded his attention to *sameness*. Although far from the most efficient approach he could have used, his transformational activity was in some ways beneficial. For example, it contributed to his correct identification of the solutions to each equation in $\mathbb{R}$ he was given except for $(x + 2)^5(x + 3)^7 = 0$.

Matt’s transformational activity, though undeniably useful, was not accompanied by a *common characteristic* interpretation and was ultimately insufficient for solving equations and attending to the underlying algebraic structure. Interestingly, Matt asserted that the original equation was ‘not the same’ as the equations he was generating with his transformations. We would have expected a student who had developed a *common characteristic* interpretation to perhaps recognize that the equations differ in appearance but have the same solution set. The fact that Matt had not yet developed a common characteristic interpretation led to a situation in which he had indeed found the correct solutions to the original equation but did not realize it (and, in fact, explicitly asserted otherwise). Once Matt had started to show initial signs of developing a common characteristic interpretation (e.g., his focus on the original equation towards the end of the Episode 2), he was able to make quick work of the equations he was given, including some equations (e.g. $(x + 1)^{19}(x + 4)^{23} = 0$) similar to those he had struggled to solve earlier (e.g. $(x + 2)^5(x + 3)^7 = 0$). In addition to illustrating these interpretations in a student’s activity, our analysis here therefore provides some empirical support for our claim in Section 3.2 that transformational activity can be more productive and less constrained when accompanied by a common characteristic interpretation or a descriptive interpretation.

4.2: Equivalence in Combinatorics

The examples in 4.2.1 and 4.2.2 come from a study (Lockwood & Reed, 2020) of the mathematical activity of novice counters⁸ as they engaged in combinatorial tasks. The particular episodes presented were from sessions wherein two undergraduate students (pseudonyms Rose and Sanjeev) developed understandings of four basic counting formulas as well as what

⁸ By “novice counter” we mean undergraduate students who have not received formal combinatorial instruction beyond what can be expected from high school algebra.

Lockwood & Reed (2020) called an *equivalence way of thinking* in combinatorics. Our focus here, however, is on using the examples of students' activity from previous studies to illustrate aspects of our framework and its utility for illuminating key aspects of students' reasoning about equivalence.

We focus on instances of activity in which the students began to determine certain outcomes in a counting process as being “duplicates,” “redundant,” or “the same,” depending on the constraints they imputed to the set of outcomes from their interpretations of the problem being solved. We focus on illustrating and showcasing the utility of a descriptive interpretation of equivalence; this both helps us clarify our characterization of descriptive equivalence and demonstrates the utility of the framework.

Lockwood and Reed (2020) inferred that Rose and Sanjeev employed an equivalence way of thinking from observing the students' regular propensity to identify initially perceived disparate outcomes as “duplicates” and to then adjust their enumeration of an outcome set accordingly. Here, we draw from episodes highlighted in that paper to suggest that the students' development of equivalence ways of thinking is supported by descriptive interpretations of equivalence.

We begin with their use of equivalence in two qualitatively different ways while solving the “Subsets” problem, which states *How many 4 element subsets are contained in the set 0, 1, 2, 3, 4, 5, 6, 7, 8, 9?*. We highlight their attention to the “sameness” of particular outcomes as onset by their interpretations of the mathematical context, particularly the constraints the students put on to the outcome set from the counting problem itself. In working on the Subsets problem, Rose and Sanjeev quickly and correctly determined $\frac{10!}{6!4!}$ to be an enumeration of the outcome set. In their description of their solution (in the following exchange), we see that their two-stage

division was intentional, and indicated two different ways they engaged in interpretations from descriptive equivalence. Below is Sanjeev's description of their first step $\left(\frac{10!}{6!}\right)$.

- Sanjeev: ... So let's say you have 10 numbers, and you have 10 spots. And you want to look for all the possible combinations of numbers, well, then you'd continue this all the way down to your last spot where you have one number left [Sanjeev drew 10 dashes on the board, writing the numbers 10 to 1 under the dashes in descending order from left to right]. And that would give you all the possible arrangements you could make with 10 numbers.
- Int.: Uh-huh.
- Sanjeev: But now, say, **you're just looking for 4 of those numbers**, well, then you'd isolate these 4 numbers and ignore anything here [referring to the last 6 positions], **because any numbers you place here could mean whatever arrangement – you want them to be. And so these numbers [refers to the last 6 positions], it doesn't matter how they're arranged, because you're all looking for a group of 4 [refers to the first 4 positions].**
- Int.: Uh-huh.
- Sanjeev: And so then you ended up with 10 factorial, and then you divide out 6 factorial, because this portion right here, 6 factorial, and so that – **that's basically saying you have 10 factorial total combinations possible with 10 numbers, but you're dividing by 6 factorial combinations, because those are the numbers that are irrelevant, because it doesn't matter how they're arranged.**
- Int.: Okay.
- Sanjeev: It's only the first **4 that you care about that are arranged.**

We note that Sanjeev considered arrangements of the final 6 numbers in a 10-number set to be “irrelevant” following arrangement of 4 numbers. We interpret from this statement that “some arrangements are ‘irrelevant’” is the complementary version of Sanjeev attributing “sameness” to arrangements of 10 numbers where the first four numbers are fixed. Sanjeev said as much by describing that, “...these numbers [refers to the last 6 positions], it doesn't matter how they're arranged, because you're all looking for a group of 4.” Sanjeev envisioned a process of first arranging 10 numbers and then using division by 6! to account for “duplicate” outcomes (language they used throughout the teaching experiment), where the duplication came from rearrangement of the final 6 numbers in the set of 10 numbers. Lockwood & Reed (2020) discussed the sameness here in terms of Sanjeev's equivalence way of thinking, but they did not

specify a particular interpretation in terms of our framework. We elaborate this point to demonstrate the framework's potential to highlight additional aspects of students' reasoning about equivalence.

Sanjeev's words and actions satisfy criterion D1 for a descriptive interpretation of equivalence. Specifically, Sanjeev contended that once 4 numbers from a 10-number set were arranged, further arrangements of the final 6 numbers would all be equivalent to each other. Moreover, we also suggest that Sanjeev viewed the equivalence of the outcomes as determined by the problem constraints, specifically "It's only the first 4 that you care about that are arranged," and "you're just looking for 4 of those numbers." That is, he seemed to perceive that whether or not two outcomes were equivalent was determined by particular problem he was solving – in the Subsets problem, he only cared about arrangements of 4 and not 10 numbers. The distinguishing feature of descriptive equivalence (criterion D2) draws from interpretations of the mathematical context or task to determine objects as similar or dissimilar. In this case, Sanjeev described that he "cared" only about ordering of the first four, thus conceiving of other possible counting situations where further arrangement of up to 10 numbers might be desired. It is unlikely that Sanjeev saw "sameness" as intrinsic to the particular objects being counted, but rather that he viewed "sameness" as a flexible property to be imputed to outcomes depending on the problem – here, arrangements of 10 numbers.

We continue with the rest of the students' explanation of the Subsets problem, focusing on Rose's contribution, again providing an instance of a descriptive interpretation of equivalence, but from a new attribution of "sameness". In describing the final division by $4!$, Rose said the following as a direct continuation of what Sanjeev had done:

Rose: Because when you're arranging the first 4 digits you could end up with 0, 1, 2, 3 or 3, 2, 1, 0. And it – **without the division it – those would be treated as separate results even though they're the same thing. And if they're the same**

4 element subset, so we divided by 24, because there would be 24 subsets that are essentially the same, [...] **so divided by 24 we're left with just unique subsets.**

Rose provides a clear statement of “sameness,” thus satisfying criterion D1. This is an example of a tendency that Rose and Sanjeev developed when solving counting problems to routinely check whether outcome construction produced “duplicate” outcomes (See Lockwood & Reed, 2020; Reed & Lockwood, 2021). Moreover, in this excerpt Rose compared two possible outcomes resulting from their $\frac{10!}{6!}$ process, and considered whether the constraints of the problem dictated that they should instead be counted as the same (this was typical of their counting activity in the teaching experiment). Rose then completed the problem by explaining that the division by 24 ($4!$) ensured that they counted only a single 4-element subset rather than the 24 that result from arrangement. We interpret her activity and explanation as also satisfying criterion D2, as Rose interpreted that she was counting 4-element subsets, and that different orderings of elements within a subset do not make for distinct outcomes. Her interpretation of a feature of the desired final outcome stems from her interpretation of the counting problem, which situated her mathematical setting.

We conclude this section by noting that the students operationalized “sameness” in two distinct ways. First, the students determined how many elements they sought to arrange after imagining an arrangement of the entire set, thus creating duplicates via arrangement of the remaining “undesirable” set elements, once the “desirable” elements had been arranged. Then, the students determined whether arrangement of their chosen set elements created duplicate outcomes as described by the counting problem. In each case, they employed division as a means of creating a single desired outcome from among the duplicate outcomes. This is akin to reasoning with a representative element of an equivalence class (see Lockwood & Reed, 2020).

5: Discussion

In this paper, we have proposed a theoretical framework – specifically, a conceptual analysis – for analyzing students’ activity with equivalence across mathematical domains. Although we specify the scope of mathematical situations for which our framework is relevant using the formal equivalence relation concept, our framework is intended to capture the more informal ways in which students interpret equivalence. In our framework, we have tried to demonstrate both that these interpretations each capture relevant aspects of students’ activity with equivalence across mathematical domains, and that these interpretations are an essential part of reasoning productively with equivalence. To do this, we illustrated how these interpretations highlighted commonalities amongst the various characterizations of equivalence across multiple domains spanning primary to postsecondary mathematics. For example, in Sections 3.2 and 3.3 we pointed out that the three interpretations – which emerged in and had previously only been applied within the domain of K-12 algebra – each feature prominently in the literature on fractions, modular arithmetic, combinatorics, and linear algebra. In section 4, we used the framework to conduct a detailed analysis of students’ activity in abstract algebra and combinatorics, specifically linking these interpretations to students’ successful reasoning with tasks in these two mathematical domains. Importantly, we note that the framework extends yet maintains coherence with the equivalence-related findings in these domains.

The primary contribution of the framework is that it provides researchers with a theoretical means with which to account for students’ activity with equivalence across mathematical domains. This contribution is particularly notable because, as we have previously stated, the equivalence literature, while substantial, is largely context-specific. In other words, the literature provides little insight into how interpretations of equivalence in one domain might relate to (and

perhaps influence) those in another. This framework has also incorporated, maintained consistency with, and extended previous efforts to identify coherence amongst the many manifestations of equivalence. For example, Charles (2005) pointed out that the equivalence of fractions, numerical expressions, and algebraic expressions could all be framed in terms of a common numerical value – in our framework, this is classified as a *common characteristic* interpretation. We therefore see this framework as an important initial step toward specifying ways in which to consider equivalence more broadly across domains.

We suggest that the framework can be used to gain additional, refined insight into students' thinking about equivalence. Recall Burington's (1948) question (in Section 2), which spurred us to ask: what does it really mean to say that 'A is equivalent to B'? Answering this question afforded insights into the nature of equivalence itself and the ways in which students reason about it. This can be seen in our analyses of the examples of students' activity provided in this paper. For example, Cook (2018) originally argued that viewing the equations one is generating via algebraic manipulation as equivalent is a necessary precursor to attending to the underlying algebraic structure that is so often the focus of abstract algebra. The framework provides a mechanism with which further refine this analysis: Matt initially related the equations he was generating purely in terms of the *transformational activity* that produced them; he then gradually shifted to interpreting the relationship between equations in terms of the *common characteristic* (i.e., the solution set) that they all share. Lockwood and Reed's (2020) example of students' activity in combinatorics similarly illustrates the utility in accounting for students' descriptive interpretations of equivalence.

In the introduction to this paper, we also highlighted how important it is for pre- and in-service teachers to explicitly attend to the meanings of equivalence that are manifest in their instruction. We see the cross-domain framework presented here as a valuable tool in this regard

because it provides a mechanism that can support “the grounding of a teacher’s mathematics content knowledge and their teaching practices” (Charles, 2005, p. 9) upon a coherent, unifying notion of equivalence. We also see it as a useful tool for connecting the advanced mathematics coursework required for secondary teacher preparation with secondary mathematics itself.

Noting that it is far from certain that teachers will make these connections (e.g., Zazkis & Leikin, 2010), Wasserman (2018) argued that “the study of advanced mathematics does not appear to be *inherently* beneficial to secondary teachers. [...] But that is not the same as claiming that it *cannot* be beneficial” (p. 6, emphasis in original). He suggested that, rather than relying on broad assumptions that advanced mathematics is beneficial, we as a field should be much more explicit about connections we want future teachers to make between advanced and secondary mathematics. One way to do this is to conduct cognitive analyses of content-related connections, which involve identifying understandings of advanced mathematics content that mirror the understandings needed to reason productively in school mathematics. Indeed, the equivalence framework proposed in this paper highlights that the interpretations of equivalence needed to reason productively in abstract algebra and combinatorics mirror those needed to reason productively about, for example, fractions and equations in K-12 mathematics. In other words, the framework provides a vehicle for connecting two areas that researchers have in general struggled to connect: advanced mathematics and school mathematics. In this way, the cross-domain framework of equivalence interpretations that we presented here can “help depict the kinds of rich mathematical understandings that need to be developed by secondary mathematics students, and help inform how the study of [advanced mathematics] might enrich secondary teachers’ mathematical knowledge” (Wasserman, 2018, p. 11). We expect that investigating these claims directly could be a fruitful topic for further research.

This cross-domain framework opens up many productive avenues for future research. For example, it can inform the design of instructional sequences – by, for example, informing the identification of desirable targets of instruction – involving equivalence in various mathematical domains. It could also inform analyses of how interpretations of equivalence that students develop in one context might influence their reasoning about equivalence in another or how students might develop and enact a context-independent understanding of equivalence. In a related vein, the framework could be used to identify the specific interpretations of equivalence that can support the learning of other concepts. That is, it could be used to gain insight into the question: which interpretations are advantageous for learning about and reasoning with certain concepts or in certain mathematical domains? The examples of students’ activity in this paper certainly illustrate the essential supporting role that equivalence can play when learning about other concepts. In the case of zero-divisors in Cook (2018), for example, the common characteristic and transformational interpretations were essential. In the combinatorial example from Lockwood and Reed (2020), a descriptive interpretation featured prominently in the students’ success. There is much room for future research in this regard.

We acknowledge that this framework is not exhaustive; we do not expect it to account comprehensively for all aspects of students’ activity with equivalence in all contexts. We also note that our analyses of students’ activity in this paper were entirely retrospective. These limitations, we believe, offer several additional opportunities for expansion of the ideas presented here. The interpretations in the framework could, for example, be elaborated and refined within specific domains, or applied to other domains to which we have not explicitly attended (such as calculus). There is also room for extension to additional topics within abstract algebra and combinatorics (such as isomorphism and quotient groups).

Finally, future research could identify other cross-domain interpretations of equivalence. As we pointed out in Section 3.2, the framework is exclusively concerned with students' *local*, element-wise reasoning (Hamdan, 2006) about equivalence. Though we used the notion of an equivalence relation to specify the mathematical scope of the framework, the framework is not intended to (immediately) apply to students' reasoning about this more *global*, equivalence class-based perspective (Hamdan, 2006). Future research could therefore examine how the interpretations in this local framework might be adapted to account for students' global reasoning about equivalence, as well as, perhaps, the reasoning involved in moving flexibly between a local and a global perspective on equivalence. One promising possibility in this regard comes from research on *substitution-based* meanings for the equal sign (e.g. Bishop et al., 2016; Jones & Pratt, 2012; Musgrave, Hatfield, & Thompson, 2015), which are not directly accounted for in our framework. Substitution-based meanings involve interpreting "statements such as $i^2 = -1$ [...] as a rule for substitution" (Jones & Pratt, 2012, p. 4). We consider substitution to be particularly related to the transitive property of equivalence relations (e.g., if $a \sim b$ and $b \sim c$, then $a \sim c$), and thus it could play an important role in bridging the gap and examining the relationship between local and global thinking about equivalence.

6: References

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7: Appendix: more formal treatments of the *common characteristic* and *transformational* interpretations via the equivalence relation concept

In our framework, we note that the notion of a ‘common characteristic’ can be based upon a students’ intuitive, informal perception of similarity amongst objects. However, there is a direct analog with the formal equivalence relation concept. It is based upon a key theorem: a relation $\sim$ on a set A is an equivalence relation if and only if there exists a function f with domain A such that $a \sim b$ if and only if $f(a) = f(b)$. Now, as explained by Hamdan (2006):

R classifies the elements of A into equivalence classes according to certain characteristics that they have in common (e.g. integers are classified into even and odd numbers). The function f is then constructed based on these characteristics, in the sense that it will map an element to that identified characteristic (continuing the example: f maps numbers divisible by 2 onto “even” and numbers not divisible by 2 onto “odd”). Consequently, it will naturally follow that elements in A that are in the same class are mapped to the same image. In a way, this construction relies on the range since we are considering that $f^{-1}(\text{characteristic } x) =$ the set of those elements of A which possess the characteristic x . (p. 134)

We can also define a ‘transformation’ in a more formal way that is broadly compatible with our transformational interpretation as described in the framework and the equivalence relation concept: a transformation on a set A is an invertible function $T: A \rightarrow A$. This enables us to define a relation as follows: $a \sim b$ if and only if there exists a transformation $T: A \rightarrow A$ such that $T(a) = b$. We note that a relation $\sim$ is indeed an equivalence relation because it is:

- *Reflexive*: $a \sim a$ because the identity function $i: A \rightarrow A$ given by $i(a) = a$ is a transformation on A
- *Symmetric*: if $a \sim b$ then there exists a transformation T on A such that $T(a) = b$. Since T is invertible, T^{-1} is also a transformation on A and, by definition, $T^{-1}(b) = a$. Therefore $b \sim a$.
- *Transitive*: if $a \sim b$ and $b \sim c$ then there exist transformations T_1 and T_2 on A such that $T_1(a) = b$ and $T_2(b) = c$. Since T_1 and T_2 are transformations on A , $T_2 \circ T_1: A \rightarrow A$ is also a transformation on A . Now, observe that $c = T_2(b) = T_2(T_1(a)) = (T_2 \circ T_1)(a)$. Therefore $a \sim c$.

Though not a centerpiece of our framework, we believe these treatments are useful for two reasons. First, they specify in greater detail what is meant by ‘common characteristic’ and ‘transformational.’ Second, they highlight how intuitive, informal interpretations of equivalence can anticipate and mirror formal aspects of the equivalence relation concept.