# A trace formula and classical solutions to the KdV equation

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*Dedicated to the memory of Sergey Naboko, my teacher and friend*

**Abstract** We show that if the initial profile $q(x)$ for the Korteweg-de Vries (KdV) equation is supported on $(a, \infty), a > -\infty$, and $\int_a^\infty x^{7/4} |q(x)| \, dx < \infty$, then the time evolved $q(x,t)$ is a classical solution of the KdV equation.

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## 1 Introduction

We are concerned with the Cauchy problem for the Korteweg-de Vries (KdV) equation

$$\begin{cases} \partial_t u - 6u\partial_x u + \partial_x^3 u = 0, & x \in \mathbb{R}, t \geq 0 \\ u(x,0) = q(x). \end{cases} \tag{1.1}$$

As is well-known, (1.1) is the first nonlinear evolution PDE solved in the seminal 1967 Gardner-Greene-Kruskal-Miura paper [7] by the method which is now referred to as the inverse scattering transform (IST). Much of the original work was done under generous assumptions on initial data $q$ (typically from the Schwarz class) for

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which the well-posedness of (1.1) was not an issue even in the classical sense[1]. But well-posedness in less nice function classes becomes a problem. From the harmonic analysis point of view, the natural function class for well-posedness results is the $L^2$ based Sobolev space $H^s(\mathbb{R})$. Before 1993 well-posedness was proven for $s > 0$. The $s = 0$ bar was reached in 1993 in the seminal papers by Bourgain [3], where, among others, he proved that (1.1) is well-posed in $H^0(\mathbb{R}) = L^2(\mathbb{R})$. Moreover his trademark harmonic analysis techniques could be pushed below $s = 0$. We refer the interested reader to the influential [2] for the extensive literature prior to 2003. Until very recently, the best well-posedness Sobolev space for (1.1) remained $H^{-3/4}(\mathbb{R})$. Note that harmonic analysis methods break down while crossing $s = -3/4$ in an irreparable way. Further improvements required utilizing complete integrability of the KdV equation. The breakthrough occurred in 2019 in Killip-Visan [11] where $s = -1$ was reached. That is, (1.1) is well-posed for initial data of the form $q = v + w'$ where $v, w \in L^2(\mathbb{R})$. For $s < -1$ the KdV equation is ill-posed in $H^s(\mathbb{R})$ scale (see [11] for relevant discussions and the literature cited therein). While [11] relies on the complete integrability of the KdV equation it does not utilize the IST transform as all Sobolev spaces allow for the rate of decay that is slower than what the IST requires. As it was shown by Naboko [13] slower than $q(x) = O\left(|x|^{-1}\right)$ may produce dense singular spectrum filling $(0, \infty)$ leaving any hope that a suitable IST can include such a situation.

Well-posedness in $H^s(\mathbb{R})$ with $s \leq 0$ does not of course provide any regularity of KdV solutions and hence solutions can not possibly be understood in the classical sense which requires continuity of at least three spatial and one temporal derivatives of $q(x,t)$. On the other hand, from the physical point of view, we want to solve (1.1) by the IST and want our solution to satisfy (1.1) in the classical sense. Well-posedness results in the $H^s$ scale do not help. This issue drew much of attention once (1.1) became in the spot light. For the earlier literature account we refer the reader to the substantial 1987 paper [1] by Cohen-Kappeler. The main result of [1] says that if[2]

$$\int_{-\infty}^{\infty} (1 + |x|)\, |q(x)|\, dx < \infty, \tag{1.2}$$

$$\int^{\infty} (1 + |x|)^N\, |q(x)|\, dx < \infty, \quad N \geq 11/4, \tag{1.3}$$

then (1.1) can be solved by the IST and the solution $q(x,t)$ satisfies the KdV equation in the classical sense, the initial condition being satisfied in $H^{-1}(\mathbb{R})$. Note that uniqueness was not proven in [1] and in fact it was stated as an open problem. The best known uniqueness result back then was available for $H^{3/2}(\mathbb{R})$ which of course assumes some smoothness whereas the conditions (1.2)-(1.3) do not. Since any function subject to (1.2)-(1.3) can be properly included in $H^s(\mathbb{R})$ with $s < -1/2$, a well-posedness statement in $H^s(\mathbb{R})\,, s < -1/2$, would turn the Cohen-Kappeler. existence result into a well-posedness but not in the class (1.2)-(1.3) where initial data belong to. A peculiar consequence of [1] is that a rough initial profile subject to (1.2)-(1.3) turns instantaneously into a three times differentiable function showing a strong smoothing effect of the KdV flow. However, the payoff for the gain of regularity is a slower decay at $+\infty$ (a smaller $N$ in (1.3) for $t > 0$). In [14] we show that essentially any (non-decaying) initial profile supported on $(-\infty, a)$ turns into a spatially meromorphic function suggesting that it is the decay at $+\infty$ that affects smoothness

[1] Here and below by a classical solution to the KdV equation we mean a solution at least three times continuously differentiable in $x$ and once in $t$ for any $t > 0$ (but not necessarily $t = 0$).

[2] $\int^{\infty} |f(x)|\, dx < \infty$ means that $\int_a^{\infty} |f(x)|\, dx < \infty$ for all finite $a$.

of the solution. In the recent [8] we improve (1.3) to $N \geq 5/2$. The current note is devoted to improving the power even further. More specifically we prove

**Theorem 1.1 (Main Theorem)**

*Suppose that a real locally integrable initial profile $q$ in (1.1) satisfies for some $a > -\infty$*

$$q(x) = 0,\ x \leq a \quad \textit{(restricted support)}; \tag{1.4}$$

$$\int_a^{\infty} (1 + |x|)^N |q(x)|\, dx < \infty, \quad N \geq 7/4 \quad \textit{(decay at} + \infty\textit{)}. \tag{1.5}$$

*If the unique solution $q(x,t)$ of the initial-value problem (1.1) admits representation (4.1) then it is the classical one (i.e. tree times continuously differentiable in $x$ and once in $t$), the initial condition being satisfied in the sense*

$$\lim_{t \to +0} q(x,t) = q(x) \ \textit{ in } H^{-1}(\mathbb{R}). \tag{1.6}$$

The condition (1.4) seems to be very restrictive but it can actually be replaced with

$$\operatorname*{Sup}_{|I|=1} \int_I \max(-q(x), 0)\ dx < \infty,$$

which does not require any decay at $-\infty$. The main reason why we assume (1.4) is that we rely on a recent result from [6] which is proven under the condition (1.4). Note, that the validity of the trace formula (4.1) is assumed to avoid some technical complications and can be removed. We also believe that the initial condition (1.6) can be understood in a much more specific sense. We plan to come back to these and other interesting questions elsewhere.

Let us briefly discuss our arguments. Recall that in [8] we improve $N$ in (1.3) by $1/4$. While we rely in [8] on the Faddeev-Marchenko inverse scattering theory, we do it within the Hankel operator approach, which we develop in [10]. Note that the Cohen-Kappeler approach [1] is also based upon the Marchenko integral equation in its classical form, which does not offer a transparent way of relaxation of the condition (1.3), while ours does. Besides, the treatment is quite involved. Our approach in [8] rests on the Dyson (second log determinant) formula for $q(x,t)$ that is written in terms of Hankel operators. Its analysis requires the membership in the trace class of our Hankel operator and its $x$-derivatives of order up to five. Here we rely on the trace formula introduced in [5] which allows us to get by with three $x$-derivatives saving extra $3/4$. The Hankel operator however still works behind the scene as it is crucially used in [6] to prove Proposition.3.2.

The paper is organized as follows. The short Section 2 is devoted to our agreement on notation. In Section 3.1 we present some background information on scattering theory and give an important asymptotic formula. Section 4 is devoted to the proof of Theorem 1.1.

## 2 Notations

We follow standard notation accepted in Analysis. $\mathbb{R}$ is the real line, $\mathbb{R}_{\pm} = (0, \pm\infty)$, $\mathbb{C}$ is the complex plane, $\mathbb{C}^{\pm} = \{z \in \mathbb{C} : \pm \operatorname{Im} z > 0\}$. $\overline{z}$ is the complex conjugate of $z$. As always, $\partial_x^n := \partial^n / \partial x^n$.

As usual, $L^p(S)$, $0 < p \leq \infty$, is the Lebesgue space on a set $S$. We will also deal with the weighted $L^1$ spaces $L^1_N(S), N > 0$, of functions summable with the $N$th order moments

$$\int_S \left(1+|x|^N\right)|f(x)|\,dx<\infty.$$

This function class is basic for direct/inverse scattering theory for Schrödinger operators on the line. Besides, we use Hardy spaces. We recall that a function $f$ analytic in $\mathbb{C}^{\pm}$ is in the Hardy space $H^2_{\pm}$ if

$$\sup_{y>0}\int_{-\infty}^{\infty}|f(x\pm iy)|^2\ dx<\infty.$$

We will also need $H^{\infty}_{\pm}$, the algebra of analytic functions uniformly bounded in $\mathbb{C}^{\pm}$. We set $H^2=H^2_+$, $H^{\infty}=H^{\infty}_+$.

We occasionally write $f(x)\sim g(x), x\to x_0$ (finite or infinite) if $f(x)-g(x)\to 0, x\to x_0$.

## 3 Our framework and main ingredients

In this section we briefly review the necessary material and introduce our main ingredients.

### 3.1 Scattering data

Through this section we assume that $q$ is short-range, i.e. $q\in L^1_1(\mathbb{R})$. Associate with $q$ the full line Schrödinger operator $\mathbb{L}_q=-\partial_x^2+q(x)$. As is well-known, $\mathbb{L}_q$ is self-adjoint on $L^2(\mathbb{R})$ and its spectrum consists of a finite number of simple negative eigenvalues $\{-\kappa_n^2\}$, called bound states, and two fold absolutely continuous component filling $\mathbb{R}_+$. There is no singular continuous spectrum.

Two linearly independent (generalized) eigenfunctions of the a.c. spectrum $\psi_{\pm}(x,k),\ k\in\mathbb{R}$, can be chosen to satisfy

$$\psi_{\pm}(x,k)=e^{\pm ikx}+o(1),\ \partial_x\psi_{\pm}(x,k)\mp ik\psi_{\pm}(x,k)=o(1),\quad x\to\pm\infty. \tag{3.1}$$

The functions $\psi_{\pm}$ are referred to as Jost solutions of the Schrödinger equation

$$\mathbb{L}_q\psi=k^2\psi. \tag{3.2}$$

Since $q$ is real, $\overline{\psi_{\pm}}$ also solves (3.2) and one can easily see that the pairs $\{\psi_+,\overline{\psi_+}\}$ and $\{\psi_-,\overline{\psi_-}\}$ form fundamental sets for (3.2). Hence $\psi_{\mp}$ is a linear combination of $\{\psi_{\pm},\overline{\psi_{\pm}}\}$. We write this fact as follows ($k\in\mathbb{R}$)

$$T(k)\psi_-(x,k)=\overline{\psi_+(x,k)}+R(k)\psi_+(x,k), \tag{3.3}$$

where $T$ and $R$ are called transmission and (right) reflection coefficients respectively. The identity (3.3) is totally elementary but serves as a basis for inverse scattering theory and for this reason it is commonly referred to as the basic scattering relation. As is well-known (see, e.g. [12]), the triple $\{R,(\kappa_n,c_n)\}$, where $c_n=\|\psi_+(\cdot,i\kappa_n)\|^{-1}$, determines $q$ uniquely and is called the scattering data for $\mathbb{L}_q$. We will need the following statement from [8].

**Proposition 3.1 (On reflection coefficient)**

*Suppose $q$ is real and in $L^1_1(\mathbb{R})$, $q|_{\mathbb{R}_-}=0$, and $T(0)=0$*[3]*. Then*

$$R(k)=T(k)\left\{\frac{1}{2ik}\int_0^\infty e^{-2ikx}q(x)\,dx+\frac{1}{(2ik)^2}\int_0^\infty e^{-2ikx}Q'(x)\,dx\right\}, \quad (3.4)$$

*where $Q$ is an absolutely continuous function subject to*

$$|Q'(x)|\le C_1|q(x)|+C_2\int_x^\infty |q|, \quad x\ge 0, \quad (3.5)$$

*with some (finite) constants $C_1,C_2$ dependent on $\|q\|_{L^1}$ and $\|q\|_{L^1_1}$ only.*

Note that (3.5) implies

$$q\in L^1_N \Longrightarrow Q'\in L^1_{N-1}. \quad (3.6)$$

Note that for $q$ supported on the full line, it was proven in [5] that

$$R(k)=\frac{T(k)}{2ik}\int_{-\infty}^\infty e^{-2ikx}g(x)\,dx,$$

where $g$ satisfies

$$|g(x)|\le|q(x)|+const\begin{cases}\int_x^\infty|q| & ,\ x\ge 0\\ \int_{-\infty}^x|q| & ,\ x<0\end{cases}, \quad (3.7)$$

and nothing better can be said about $g$ in general. In the case of $q$ supported on $(0,\infty)$ this statement can be improved. Indeed, (3.4) implies that

$$g(x)=q(x)+Q(x)$$

with some absolutely continuous on $(0,\infty)$ function which derivative $Q'$ satisfies (3.5).

We will rely on the following statement from [6, Theorem 8.2].

**Proposition 3.2 (On Jost solution)**

*Let*

$$y(x,t,k)=e^{-ikx}\psi_+(x,t,k)-1,$$

*where $\psi_+(x,t,k)$ is the right Jost solution corresponding to the solution of (1.1)*[4]*. Suppose that $q(x)$ is subject to the conditions of Theorem 1.1. Then*[5]

$$\partial_x^n y(x,t,k),\partial_t y(x,t,k)\in H^2\cap H^\infty, n=0,1,2,3.$$

We emphasize that this proposition holds only for $t>0$ in general. For $t=0$ it fails unless we assume extra regularity conditions on $q$. Note that the existence of $\psi_+(x,t,k)$ is guaranteed by a much more general result from [4, Theorem 1.4]. In particular, it holds even for $q(x)\in L^1$. We however believe that it was known already in the 1970s for $q(x)\in L^1_1$. At least it follows from [1] for $q(x)\in L^1_{11/4}$.

Finally, recall that uniformly in the upper half-plane

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[3] In fact, $T(0)=0$ happens generically and is not a real restriction. Recall that the transmission coefficient doesn't vanish at $k=0$ only for the so-called exceptional potentials but an arbitrarily small perturbation turns such a potential into generic. In our case it can be achieved by merely shifting the data $q$.

[4] Such a solution is also commonly referred to as the time evolved Jost solution under the KdV flow.

[5] The Hardy spaces are with respect to $k$

$$T(k) = 1 + \frac{1}{2ik}\int_{-\infty}^{\infty} q(x)\,dx + O\left(\frac{1}{k^2}\right), k \to \infty. \tag{3.8}$$

## 3.2 An oscillatory integral

The following lemma is an immediate consequence of the method of steepest descent [15].

**Lemma 3.3** *Let $\Gamma$ be a deformation of $\mathbb{R}$ providing the absolute convergence of the integral*

$$F_n(x,t) := \int_\Gamma \lambda^n \exp\left(i\lambda^3 t - i\lambda x\right) d\lambda. \tag{3.9}$$

*Then for a fixed $t > 0$ as $x \to \infty$ :*

$$F_n(x,t) \sim \sqrt{\frac{12}{\pi}} t^{-(2n+1)/4}\ x^{(2n-1)/4} \begin{cases} -i\sin\phi(x,t), & n \text{ is odd} \\ \cos\phi(x,t), & n \text{ is even} \end{cases}, \tag{3.10}$$

*where*

$$\phi(x,t) := \frac{2x^{3/2}}{3\sqrt{3}t^{1/2}} - \frac{\pi}{4}.$$

***Proof*** Rescaling in (3.9) $\lambda \to (x/3t)^{1/2}\,\lambda$ and setting $\omega = x\,(x/3t)^{1/2}$ yield

$$F_n(x,t) = \left(\frac{\omega}{3t}\right)^{(n+1)/3} \int_\Gamma \lambda^n e^{i\omega S(\lambda)} d\lambda,$$

where

$$S(\lambda) := \lambda^3/3 - \lambda.$$

The phase $S(\lambda)$ has two stationary points $\lambda = \pm 1$ and hence by the steepest descent

$$\int_\Gamma \lambda^n e^{i\omega S(\lambda)} d\lambda = \sqrt{\frac{\pi}{\omega}} \left[e^{-2i\omega/3 + i\pi/4} + (-1)^n e^{2i\omega/3 - i\pi/4}\right]$$
$$+O\left(\omega^{-3/2}\right), \omega \to \infty.$$

Therefore

$$F_n(x,t)$$
$$= \sqrt{\frac{3}{\pi t}} \left(\frac{\omega}{3t}\right)^{(2n-1)/6} \left[e^{-2i\omega/3 + i\pi/4} + (-1)^n e^{2i\omega/3 - i\pi/4} + o\left(\omega^{-1/2}\right)\right],$$
$$\omega \to \infty$$

Switching back from $\omega$ to $x$ we arrive at (3.10). □

## 4 Prove of the Main Theorem

Since the KdV equation is translation invariant without loss of generality we may assume that $a = 0$. Existence, uniqueness, and (1.6) follow from the general results (see e.g. [11]). Our proof is based on the trace formula [5]

$$q(x,t) = q_c(x,t) + q_d(x,t), \tag{4.1}$$

where

$$q_c(x,t) = \frac{2i}{\pi}\int_{-\infty}^{\infty} kR(k)\,\xi_{x,t}(k)\,(1+y(x,t;k))^2\,dk,$$
$$q_d(x,t) = -4\sum_j \kappa_j c_j^2 \xi_{x,t}(i\kappa_j)\,(1+y(x,t;i\kappa_j))^2.$$

Here $y(x,t;k)$ (refereed sometimes to as the Faddeev function) is as in Proposition 3.2 and

$$\xi_{x,t}(k) := \exp\left(8ik^3t + 2ikx\right).$$

The integral in (4.1) is understood in the sense of Cesàro means. We however use a more convenient way to regularize this integral. Consider $q_c$ first. By the Cauchy-Green formula applied to the strip $0 \le \operatorname{Im}\lambda \le 1$ we have ($\lambda = u + iv$)

$$\begin{aligned} q_c(x,t) &= \frac{2i}{\pi}\int_{0\le \operatorname{Im}\lambda\le 1} 2i\lambda\xi_{x,t}(\lambda)\,(1+y(x,t;\lambda))^2\,\overline{\partial}R\left(\lambda,\overline{\lambda}\right)dudv \\ &+ \frac{1}{\pi}\int_{\mathbb{R}+i} 2i\lambda\xi_{x,t}(\lambda)\,R\left(\lambda,\overline{\lambda}\right)(1+y(x,t;\lambda))^2\,d\lambda \\ &=: q_c^{(0)}(x,t) + q_c^{(1)}(x,t), \end{aligned}$$

where $R\left(\lambda,\overline{\lambda}\right)$ is the (not analytic) continuation of the reflection coefficient $R(k)$ given by

$$R\left(\lambda,\overline{\lambda}\right) = \frac{T(\lambda)}{2i\lambda}\left\{\int_0^\infty e^{-2i\overline{\lambda}s}g(s)\,ds + \frac{1}{2i\lambda}\int_0^\infty e^{-2i\overline{\lambda}s}Q'(x)\,dx\right\}. \tag{4.2}$$

Note that such continuation guaranties $R\left(\lambda,\overline{\lambda}\right) = o(1/|\lambda|)$ in the closed upper half-plane.

Consider $q_c^{(1)}(x,t)$ first. Since $\xi_{x,t}(\lambda)$ rapidly decays along any line in the upper half-plane parallel to the real line, $q_c^{(1)}(x,t)$ is differentiable in $x$ as many times as $y(x,t;\lambda)$. By Proposition 3.2 then $q_c^{(1)}(x,t)$ is at least three times continuously differentiable in $x$. Turn now to $q_c^{(0)}(x,t)$. Since by Proposition 3.2

$$\partial_x^n y(x,t;\lambda) \in H^2 \cap H^\infty, \text{ for } n = 0,1,2,3,$$

it follows from

$$\begin{aligned} &\partial_x^3\left[\xi_{x,t}(\lambda)\,(1+y(x,t;\lambda))\right]^2 \\ &= (2i\lambda)^3\,\xi_{x,t}(\lambda) \\ &+ (2i\lambda)^3\,\xi_{x,t}(\lambda)\left[2y(x,t;\lambda) + y(x,t;\lambda)^2\right] \\ &+ 6\,(2i\lambda)^2\,\xi_{x,t}(\lambda)\,[1+y(x,t;\lambda)]\,\partial_x y(x,t;\lambda) \\ &+ 3\,(2i\lambda)\,\xi_{x,t}(\lambda)\left\{[1+y(x,t;\lambda)]\,\partial_x^2 y(x,t;\lambda) + [\partial_x y(x,t;\lambda)]^2\right\} \\ &+ \xi_{x,t}(\lambda)\left\{3\partial_x y(x,t;\lambda)\,\partial_x^2 y(x,t;\lambda) + [1+y(x,t;\lambda)]\,\partial_x^2 y(x,t;\lambda)\right\} \end{aligned}$$

that for a fixed real $x$ and a positive $t$, uniformly in the strip $0 \leq \operatorname{Im}\lambda \leq 1$

$$\partial_x^3 \left[\xi_{x,t}(\lambda)\left(1 + y(x,t;\lambda)\right)\right]^2 = (2i\lambda)^3 \xi_{x,t}(\lambda)\left[1 + o(1/\lambda)\right], \lambda \to \infty.$$

Thus, if the integral

$$\frac{2i}{\pi} \int_{0 \leq \operatorname{Im}\lambda \leq 1} (2i\lambda)^n \xi_{x,t}(\lambda) \overline{\partial} R\left(\lambda, \overline{\lambda}\right) du dv,$$

converges for $n = 1, 2, 3, 4$, so do the integrals representing $\partial_x^n q_c^{(0)}(x,t)$ for $n = 0, 1, 2, 3$. Apparently it is enough to consider the largest derivative $\partial_x^3$ only. Due to (4.2), we have

$$\begin{aligned}
&\frac{2i}{\pi} \int_{0 \leq \operatorname{Im}\lambda \leq 1} (2i\lambda)^4 \xi_{x,t}(\lambda) \overline{\partial} R\left(\lambda, \overline{\lambda}\right) du dv \\
&= \frac{1}{\pi} \int_{0 \leq \operatorname{Im}\lambda \leq 1} (2i\lambda)^3 \xi_{x,t}(\lambda) T(\lambda) \left[\int_0^\infty (-2is) e^{-2i\overline{\lambda}s} q(s) ds\right] du dv \\
&+ \frac{1}{\pi} \int_{0 \leq \operatorname{Im}\lambda \leq 1} (2i\lambda)^2 \xi_{x,t}(\lambda) T(\lambda) \left[\int_0^\infty (-2is) e^{-2i\overline{\lambda}s} Q'(x) dx\right] du dv \\
&= I_0(x,t) + I_1(x,t) + I_2(x,t),
\end{aligned}$$

where

$$I_0(x,t) = \frac{-2i}{\pi} \int_{0 \leq \operatorname{Im}\lambda \leq 1} (2i\lambda)^3 \xi_{x,t}(\lambda) \left[\int_0^\infty e^{-2i\overline{\lambda}s} s q(s) ds\right] du dv,$$

$$I_1(x,t) = \frac{-2i}{\pi} \int_{0 \leq \operatorname{Im}\lambda \leq 1} (2i\lambda)^2 \xi_{x,t}(\lambda) \left[\int_0^\infty e^{-2i\overline{\lambda}s} s Q'(x) dx\right] du dv,$$

$$\begin{aligned}
I_2(x,t) = &\frac{-2i}{\pi} \int_{0 \leq \operatorname{Im}\lambda \leq 1} (2i\lambda)^3 \xi_{x,t}(\lambda) (T(\lambda) - 1) \left[\int_0^\infty e^{-2i\overline{\lambda}s} s q(s) ds\right] du dv \\
&- \frac{2i}{\pi} \int_{0 \leq \operatorname{Im}\lambda \leq 1} (2i\lambda)^2 \xi_{x,t}(\lambda) (T(\lambda) - 1) \left[\int_0^\infty e^{-2i\overline{\lambda}s} s Q'(x) dx\right] du dv.
\end{aligned}$$

Due to (3.8), it is enough to study $I_0(x,t)$ and $I_1(x,t)$. For $I_0(x,t)$ one has

$$\begin{aligned}
I_0(x,t) &= \frac{-2i}{\pi} \int_0^\infty ds\ q(s) \int_{0 \leq \operatorname{Im}\lambda \leq 1} (2i\lambda)^3 \xi_{x-s,t}(\lambda) e^{-4vs} du dv \\
&= \frac{-2i}{\pi} \int_0^\infty ds\ q(s) \int_0^1 dv\ e^{-4vs} \int_{\mathbb{R}+iv} (2i\lambda)^3 \xi_{x-s,t}(\lambda) d\lambda
\end{aligned}$$

Here we have used that $\overline{\lambda} = u - i = \lambda - 2i$ and $du = d\lambda$ for each fixed $v$. Since $(2i\lambda)^3 \xi_{x-s,t}(\lambda)$ is analytic and rapidly decaying on $\mathbb{R} + iv$ we can deform $\mathbb{R} + iv$ to some contour $\Gamma$ independent of $v$, to be chosen later. Thus we have arrived at

$$\begin{aligned}
I_0(x,t) &= \frac{-i}{2\pi} \int_0^\infty ds\ q(s) \left(1 - e^{-4s}\right) \int_\Gamma (2i\lambda)^3 \xi_{x-s,t}(\lambda) d\lambda \\
&= -\frac{1}{4\pi} \int_0^\infty ds\ q(s) \left(1 - e^{-4s}\right) \int_\Gamma \lambda^3 \xi_{x-s,t}(\lambda/2) d\lambda
\end{aligned}$$

Applying Lemma 3.3 to the inner integral we conclude that the integral $I_0(x,t)$ converges if

$$\int^{\infty} s^{5/4} |q(s)| \, ds < \infty.$$

Turn now to $I_1(x,t)$. Similarly, one gets

$$I_1(x,t) = -\frac{1}{2\pi} \int_0^{\infty} ds \; Q'(s) \left(1 - e^{-4s}\right) \int_{\Gamma} \lambda^2 \xi_{x-s,t}(\lambda/2) \, d\lambda$$

and hence, as above, $I_1(x,t)$ converges if

$$\int^{\infty} s^{3/4} |Q'(s)| \, ds < \infty. \tag{4.3}$$

Due to (3.6) we see that if $q \in L^1_{7/4}$ then $Q' \in L^1_{3/4}$, and hence (4.3) holds.

The term $q_d(x,t)$ is easy. It is differentiable in $x$ as many many as $y(x,t;i\kappa_j)$ and thus we have at least three continuos derivatives.

Continuous differentiability of $q(x,t)$ in $t > 0$ follows from $\partial_t \xi_{x,t} = -\partial_x^3 \xi_{x,t}$.

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