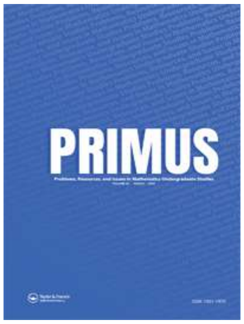

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Modeling Memorization: A Data Collection and Mathematical Modeling Experience

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ABSTRACT
This article presents a mathematical modeling activity for students related to the process of memorization in which students collect their own data to drive their model development, parameterization, and validation. Engaging in the data collection gives them insight to critique and evaluate various models. This task is a low-floor high ceiling problem that offers both a relatable context and a window to quantitative approaches in cognitive science. Experimental results of students participation in this activity are discussed. This article also includes pedagogical recommendations with a focus on fostering equitable teaching practices and a detailed analysis of the situation comprised of several mathematical approaches to model the memorization process that highlight the richness of the problem. Instructors can adapt and implement this modeling exploration for use in various undergraduate courses, from introductory to advanced, depending on the emphasis of the lesson.

KEYWORDS
Memorization; mathematical modeling; mathematics education; data collection; project-based learning

1. INTRODUCTION

Mathematical modeling in the undergraduate curriculum has been shown to increase student interest through authentic problem-solving by promoting critical thinking, communication skills, and creativity. However, resources for instructors remain sparse. Mathematical modeling tasks included in typical undergraduate courses are often met with superficial treatment. Even courses titled "Mathematical Modeling" are often about mathematical models rather than the process of mathematical modeling.

The lesson presented in this article is part of the mathematical modeling course designed for the NSF-funded Mathematics of Doing, Understanding, Learning and Educating for Secondary Schools (MODULE(S2)) [4]. MODULE(S2) aims to improve future teachers' mathematical modeling knowledge and pedagogical knowledge by creating course materials that address the process of modeling along with the mathematical knowledge, skills, and competencies necessary to build and analyze mathematical models of real-world phenomena. Our general pedagogical approach is to present students with a series of modeling activities and address

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mathematical content required for modeling as it arises from the students' work on the activity. Although the project was designed with preservice teachers in mind, the topic is of broad interest to students in many disciplines.

Mathematical modeling has long been a component of post-secondary mathematics curricula [24]. It helps students learn complex problem-solving skills used in daily life and industry. Research suggests that a way to develop mathematical modeling competencies is through the practice of doing modeling activities [2, 18]. Recent studies of mathematical curriculum have found that secondary and post-secondary mathematics curriculum may not be appropriately addressing mathematical modeling [17], and that current activities are likely to provide students with a "cynical view" of mathematical applications [29, p. 368]. For example, typical textbook real-world problems fail to offer genuine mathematical modeling experiences. As such, it is up to educators to find modeling problems that are relevant to students and offer opportunities to engage in multiple facets of mathematical modeling. Thus, the intent of this paper is to describe a novel approach to a task about memorization from Blanchard, Devaney, and Hall [7] that provide a context relevant to many STEM programs and to teacher education. This task can be implemented in various courses, including but not limited to college algebra, differential equations, mathematical modeling, and a capstone course for mathematics majors with an emphasis of education. The lesson implementations described in this article were conducted with undergraduate and graduate students with a teaching emphasis.

This experience focuses on leading students to reflect on the process of learning and build a predictive model to describe the memorization of information. In everyday life, memorization arises from learning basic facts, definitions, and small pieces of information like telephone numbers or identification numbers (e.g., student identification codes). If placed in the context of teacher education, psychology, neuroscience, or computer science, the process of memorization provides a starting point for understanding how learning takes place and how to describe this process mathematically. As noted by Blanchard et al. who offer this context as a lab experience in their differential equations textbook [7, pp. 142–143], learning and cognition are complex processes that are not fully understood and continue to be subjects of active investigation by researchers in multiple disciplines, including mathematics. Some examples exist of mathematical models of learning as a whole, but these tend to be theoretical and do not typically involve analyzing data due to the difficulty of gaining and measuring the necessary data (see [30]). Conversely, memorization (which is directly connected to learning [5]) lends itself to data collection and analysis.

Although memorization is not as central in schools today as it was at the turn of the 20th century, it still has an important role. Effective teachers in Mainland China, Hong Kong, Australia, and the United States believe that memorization is an important part of the learning process [10]. Memorization can be seen in the practice of teaching both cognitively and pedagogically. A basic tenet of behaviorism [27] which informs practice in education is that subjects remember actions and consequences. For teachers who aim to design lessons and assess students' understanding, Cangelosi [11] categorizes recall and memorization as the simple knowledge learning level of mathematical cognition.

For these reasons, the task described here focuses on the process of memorization. Our primary goals for students are the following.

(1) Reflect on the process of memorizing information and use mathematical modeling to gain understanding;
(2) Interpret and explain different mathematical formalisms (e.g., ordinary differential equations (ODEs) or random variables) in the context of developing memory;
(3) Apply parameter fitting procedures;
(4) Understand and describe the limitations of various models.

The process of memorization has been modeled with various functions [25], differential equations [14, 20], and Markov chains [6, 21] that are accessible to students at a variety of levels. These various entry-points allow for memorization to be a low floor, high ceiling problem situation to be modeled, and the project described here is designed with equitable teaching practices in mind that engage students deeply in mathematical content, leverage students' mathematical knowledge and competencies in parallel to affirming and valuing their contributions and identities [1]. In what follows, we present the lesson plan with our commentary regarding the implementation of equitable teaching practices, mathematical approaches with student-generated data, and assessment strategies with an analysis of student work.

2. THE LESSON AND PEDAGOGICAL APPROACH

This section includes a lesson progression and addresses our pedagogical approach and recommendations, highlighting our use of equitable teaching practices that specifically focus on distributed student participation and collaborative mathematical thinking. Aguirre, Mayfield-Ingram, and Martin's [1] equity-based mathematics teaching practices encompass multiple, interconnected dimensions that are foundational to learners' achievement and agency. The five equity-based practices are listed below with abbreviations.

- Going deep with mathematics (GDM)
- Leveraging multiple mathematical competencies (LMC)
- Affirming mathematics learners' identities (ALI)
- Challenging spaces of marginality (CSM)
- Drawing on multiple resources of knowledge (DMR).

We posit that these equity-based practices integrate organically into the process of mathematical modeling and position the learner as an active participant and contributor to the team solution. We designed the lesson to be delivered in two 75-min class sessions with the following agenda for those class meetings.

2.1. Introduce the Task

On day 1, the lesson begins with a class discussion on learning. Some questions to guide this discussion may include: What does learning mean? How do we know when learning has occurred? How is memory associated with learning?

Beginning the lesson with a discussion on learning provides an opportunity for students to bring prior knowledge about learning and memorization to the class (DMR). Our recommendation is for the instructor to focus on qualitative rather than quantitative aspects of learning and memorization. Recording students' ideas on a board can serve as a reference for when they are formulating their models. The recording of this information will provide greater access to others' experiences and backgrounds in learning and memorization and actively challenges spaces of marginality by acknowledging and assigning status to all students' experiences (DMR, CSM).

2.2. Pose the Problem

Blumenfeld et al. [8] state that driving questions motivate student participation in a project-based experience. After the initial discussion, we raise these driving questions to introduce our exploration: How can we better understand memorization using mathematics? What model can we create that explains the process of memorization? How can we create a predictive model for memorization? What experiment can we conduct to validate our model?

Modeling the memorization process is of high cognitive demand, placing mathematics central to the lesson yet multiple solution approaches and strategies help students stay engaged for longer periods to deeply explore the mathematics (GDM). By supporting student thinking through collaborative work in low-risk environments, students become empowered to continue and sustain their mathematical thinking.

A brief literature search on memorization reveals three main memory processes: acquisition, consolidation, and storage which arise as a learner acquires the information, encodes it, and stores it for later retrieval [5]. Figure 1 displays a simplified schematic of the process. Information enters the brain (acquisition) through sensory processes where a memory trace is formed (consolidation). This sensory memory lasts between 0.5 and 3 s, only long enough to transmit sensations to short-term memory. As a memory trace persists it is transferred from short-term to long-term memory (storage). Short-term memory refers to a span of 15–30 s whereas long-term memory could last for hours, days, or decades [16]. It is important to note that at any stage, information can be lost or forgotten.

Suppose that a person was to learn a list of numbers. Studying the list can be considered a "rehearsal" in which the memory is strengthened. A memory may be retrieved if it was moved to storage, depending on the robustness of the representation. The retrieval of stored information can be considered a "performance" (e.g., recording as many numbers as one can remember).

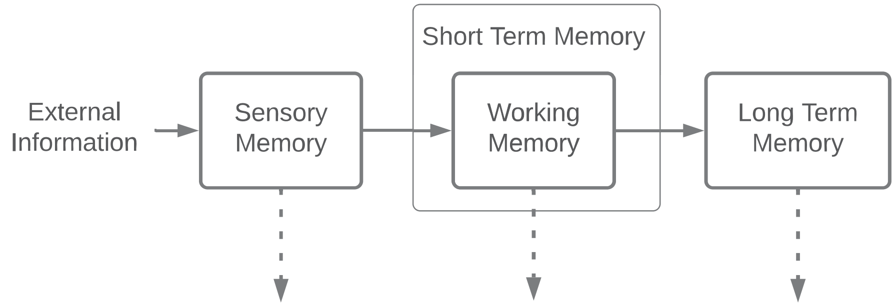

Figure 1. The memorization process. External information is acquired through the senses and briefly exists as sensory memory. Some information enters the working memory, a component of short-term memory, where further consolidation and encoding can result in transfer to long-term memory. Dashed arrows represent information lost or forgotten. Figure adapted from Murre et al. [21] and Shrestha [26].

Students can learn more about the topic by reading [32], exploring an interactive web cartoon [12], or referencing any of the mathematical model articles cited in this report.

2.3. Data Collection

At this point, it is useful to have a discussion about the experimental design, and how the experiment might influence the model development. Even with what seems like a clearly outlined procedure, subtleties in the experiment can make a difference. Next, the students begin implementing their data collection procedure. We recommend doing one trial in class and additional trials outside of class for homework because this part of the process can be time-consuming. The materials needed for data collection include a timer, pencil, paper, and a list of 20 "words" or items to be memorized. We use 3-digit numbers as "words." A sample data collection procedure handout with multiple lists of 3-digit numbers is included as Appendix 1. (Students can also generate more lists of their own or perhaps try to memorize digits of π, Euler's number e, or the golden ratio ϕ.)

When discussing data collection and experimental design, interesting details and assumptions about the environment naturally arise. Some examples include distractions that may cause difficulty in memorization, the amount of time after rehearsal before testing (recording immediately following the studying or inserting time before testing), the amount of time after testing and going back to studying the list or even checking results after each test before the next rehearsal which may affirm beliefs about certain items on the list. Other issues that may come up in discussion could be the number of times needed to perform this exercise (e.g., Is one trial good enough? If not, how many is good enough?) and the order in which the list is memorized (e.g., Will the list be necessarily repeated verbatim or does the order of recalled items not matter?). These are some common ideas brought up in the implementations of the lesson. Another interesting thought is that the basis of this experiment was a psychological test [23] that was performed with trigrams (3-letter combinations) instead of 3-digit numbers and interference was provided in between studying and performance (participants had to count backwards). Does this make a difference? Acknowledging the process collectively provides students with ownership of the problem and their solutions.

This lesson and final report can be done in groups or individually. The data is easy to collect individually so it makes the construction of the model feasible to do individually. However, there are many benefits of allowing and encouraging small group discussions before individual work time. Furthermore, doing this as a group assignment encourages more student-to-student interaction (CSM) with a collaborative goal, and thus promotes persistence in problem-solving (ALI).

2.4. Model Development, Student Presentations, and Discussion

Once students have collected their data, on day 2 of the lesson, we recommend employing the five teaching practices described by Smith and Stein [28] for conducting rich mathematical tasks: anticipating, monitoring, selecting, sequencing, and connecting. The models of the memorization process described in the next section provide a basis for anticipating students' approaches. As the students engage in the modeling experience, the instructor monitors their work and helps them think through their ideas. Equity-based instruction calls for instructors to value students' approaches and entry points to this problem (LMC), so we recommend building the mathematics from student models. Leveraging multiple mathematical competencies is a strength in the cognitive space created by the mathematical modeling process. Different approaches taken by students can be selected to highlight for the whole class as this allows for students to see the variation in the solution.

The sequencing of presenting student work can take different forms, but we recommend grouping the models by underlying mathematical structures so that discussion flows in a hierarchical fashion. Selected students present their preliminary models, and the instructor connects these approaches by offering an opportunity for students to compare and contrast models and offer comments about their peers' work. When done in a productive environment, this allows students to sustain a deep mathematical level (GDM). It is important for students to reflect on the assumptions of their models and discuss possible methods of eliminating limitations. The students then revise and improve their models, and in doing so, they actively participate in collaborative mathematical modeling and see themselves as valued contributors (ALI). A written modeling report of their final models and their modeling process is assigned for homework. Report guidelines are provided under Section 3.1.

3. STUDENT WORK AND ANALYSIS OF REPORTS

An important component of the task is the data collection by the students. After several implementations of this lesson, we have found that while students have similar experiences, collected data show significant variation from one individual to the

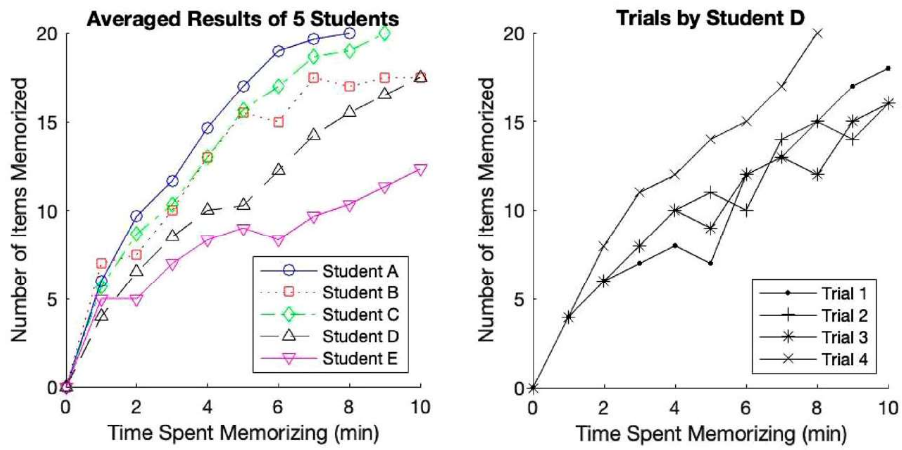

Figure 2. Data collected by students showing the number of correctly recalled items as a function of time spent learning the list. Each student memorized multiple lists. The graph on the left shows the average results for each student. The graph on the right shows the variation of one student across multiple lists.

next. Figure 2 shows a sample collected by five students. Each trial consists of studying a list of numbers one minute at a time for up to 10 min and reporting the number of correctly memorized items after each minute of studying. The students repeated the experiment over several trials using different lists. The panel on the right of the figure shows the results of four trials by one of the student participants. The left panel of Figure 2 shows the trial-average data for five participants.

We provide report writing guidelines below and a rubric for assessing student mathematical modeling reports in Appendix 2. We conclude this section with our analysis of some sample student reports in light of the lesson goals.

3.1. Assessment of Student Learning

Students are expected to write a mathematical modeling report describing their process, solutions, and interpretations. We suggest using a report outline similar to that of Bruder and Kohler [9].

- Background Information on the Problem (State the necessary information about the problem)
- Experimental Setup (Describe the experiment in your own words)
- Data (Include a table or a plot of the data collected)
- Model(s) (Include assumptions)
- Results (Answer the driving questions and provide insights from the model as a solution)
- Discussion (Discuss strengths and limitations of the model and potential future revisions)

For assessing the work of students, we took a holistic approach focusing on the mathematical modeling process. We identified evidence of competencies that students were developing within the elements of mathematical modeling. We considered Niss, Blum, and Galbraith's view of modeling competency as the ability to identify and translate relevant questions, variables, and assumptions about a situation into mathematics and interpret and validate the solution, which offers a holistic perspective on assessment rather than exclusively assessing the model itself [22]. We recognized Maaß's framework for defining individual competencies within the modeling elements as "skills and abilities to perform modelling processes appropriately and are goal oriented as well as the willingness to put these into action" [19, p. 117].

We created a rubric for assessing student work on the modeling process by considering elements of the modeling process and competency development (see Appendix 2). The rubric outlines seven elements of modeling, which include: (1) Understanding of the problem situation, including the goals of the solution and factors that affect the solution, for example; (2) Consideration of a simplified version of the situation, including making use of appropriate assumptions and choices; (3) Creation of the model, which includes decontextualizing, that is, translating information into mathematical notation; (4) Computation using the model and checking for precision; (5) Interpretation of the solution and drawing conclusions, which includes contextualizing the mathematical solution back into the problem situation; (6) Validation of the conclusions, which includes determining if the solution makes sense and if it is within a valid range of values; and (7) Reporting out the solution, which entails communicating the model and solution with justifications for the assumptions and choices made.

Research underscores that prospective teachers, in addition to placing value on the model as a product, placed value on the modeling process [3]. Placing value on the process allows for assessment of students' creativity, decision-making, initiative, and communication, thus giving a broader perspective of students' competences in multiple areas of mathematics within modeling tasks. This finding aligns with the recommendations from the *Guidelines for Assessment and Instruction in Mathematical Modeling Education*, "Assessment should be in service of helping students improve their ability to model, which will, in time, translate to a better product" [15, p. 21].

In our work, we used the rubric to provide formative feedback while students were engaged in the modeling process and to assess the final report that students created on the memorization modeling task. Although this paper does not focus on the assessment of mathematical modeling, we include this section to provide a tool for formative assessment and a framework for assessing students' work in mathematical modeling as a final product.

3.2. Student Created Models

We have implemented this lesson multiple times in different institutions in the Southwest and Southeast of the United States over the past two years. In this section, we highlight the work of the five students (labeled A-E, whose data is displayed in

Table 1. Five students' modeling approaches to the memorization task in a mathematical modeling course for prospective teachers.

Student	Model
A	$\frac{dL}{dt} = a(1 - L)$ $t = \left(\frac{L(t)}{a}\right)^{\frac{1}{b}}$
B	$\frac{dL}{dt} = a(1 - L)$ with the initial condition $L(0) = 0$
C	$S(t) = at^2 + bt + c,$ $S(t) = at^3 + bt^2 + ct + d$ $S(t) = \frac{a}{1 + be^{-ct}}$ $S(t) = \frac{a}{b - t} + c$ *
D	$S(t) = a(b)^t + c$ $S(t) = \frac{a}{1 - e^b t} + c$ $S(t) = -\frac{a}{b + t} + c$ *
E	$\frac{dS}{dt} = aS + b$ with the initial condition $S(0)=0$ $S(t) = ae^{bt} + M$ where $b < 0$ $S(t) = at^b e^{ct}$ *

Note: Variables are t representing time, L, the fraction of the list memorized, S, the number of items memorized, and M, the number of items on the list. Parameters are a, b, c, and d. An asterisk * indicates which model was deemed best by the student.

Figure 2) in light of the four goals of the lesson. The students wrote about a variety of mathematical models in their reports – differential equations, polynomials, exponential, rational, and power functions – which are listed in Table 1. The following section describes evidence of progress on lesson goals found in the students' reports.

Mathematical Modeling to Gain Contextual Understanding

Goal 1: Reflect on the process of memorizing information and use mathematical modeling to gain understanding about memorization

Students C and E researched the memorization process and wrote about what they learned in their reports. Student C cited literature to inform their stance that through practice and technique, the rate at which one memorizes can increase. Student E decided to discuss the notion of forgetting in their model exposition. This student also referred to researched information suggesting that a curve shaped like a Gamma distribution would be an appropriate model of memorization.

Student D discussed that everyone will have a different saturation limit and looked at the average of data from several participants to mitigate the "memor[ies] and emotional connection[s] to specific numbers" that were contained with certain lists that made those easier to remember. They also discussed that mental state would affect the rate of memorization.

Student B observed that the rate of memorization would not necessarily improve or decrease with practice. They, like Student D, conjectured that the rate of memorization is highly dependent on the values in the list. However, they point out that the numbers may have relationships with others on the list. For example, "there were 4 numbers at the end which were all in the 200's. This allowed me to know they all started the same and then just learn the two-digit number at the end. The more easily seen connections that exist, the easier it is to learn."

In all these excerpts, students made sense of memorization. This occurred prior to developing their models by researching information or reflecting on their experience from the data-gathering portion of this project, and they gained more insight due to their attempts to go back and forth between the mathematical model and the real situation.

Interpreting Mathematical Formalisms

Goal 2: Interpret and explain different mathematical formalisms in a real-world context

In the majority of student samples, differential equations were used to represent and explain the rate at which individuals memorize information. Throughout their exposition, Student E was concerned with making sure that their models were consistent with their understanding of memorization. They justified and then made adjustments to models to make sure the resulting function had features they were expecting. For example, the exponential function "appeared to fit the data and had the property of a horizontal asymptote." This property of a horizontal asymptote was important to Student E because it is impossible to learn more than the number of items on the list.

Similarly, student D claimed that memorization will have a saturation limit or a carrying capacity saying, "[I]n this activity memorizing 20 random numbers was a stretch …[h]owever, I suspect that …it is possible that I would not be able to memorize past a certain number of words." They evaluated the quality of models by looking at what saturation limit was more realistic after parameter fitting.

The students used various families of functions like quadratic, cubic, rational, exponential, and logistic. Student C dismissed the quadratic and cubic functions because "once [the curves representing the amount memorized] reach their maximum or relative maximum they start to decrease." They also chose to pick a model that went through the origin, the only known data point for this model.

The students seemed to write about models similar to those discussed in the various mathematical approaches section (Section 4). They attempted to connect their models with their conception and assumptions about memorization.

Applying Parameter Fitting Techniques

Goal 3: Apply parameter fitting procedures

Most of the reports we analyzed used a blackbox fitting tool like Desmos to find the parameters that best fit the data. The advantage is that this is easy to use and the students can focus energy on interpreting the output. Student B, on the other hand, wrote their own MATLAB code to fit their model to data using three methods and then compared the best fit parameter value that resulted from each method. The fitting techniques included (1) transforming the data to be linear (with a logarithm) and using linear regression, (2) minimizing the sum of squared errors with the data, and (3) using the maximum likelihood estimator. Method 2 and 3 produced nearly identical results for the parameter, but Method 1 produced a different result. In later revisions of their work, Student B used method 2 as it was an easier technique to apply and provided satisfactory results when compared to the other methods. This is an example of a student developing a deep understanding of the details of these fitting methods because of the added challenge of coding them from scratch.

Understanding Limitations of the Models

Goal 4: Describe limitations of models

Most of these students' reports described limitations of their models. Student B felt that their model was limited because the memorization rate (which was modeled as a constant) depended on the length of the list. In each attempt to understand how long it takes to memorize a list of a new length, they needed to re-scale the data and re-fit their parameter to the new data. They didn't understand that the parameter could simply scale by the list length.

Student A discussed that their model fails to account for memorizing the entire list (in finite time). They needed to develop a strategy for dealing with their collected data when $L = 1$. They wrote about options and decided to use a model variation depending on the goal. To understand the overall trend of the data and to investigate if the rate of acquisition improves after practice, student A used the model described by the function:

$$L(t) = 1 - e^{-k\cdot t+c}.$$

When investigating the amount of time it would take for memorizing a list of some arbitrary length, student A encountered the same limitation as student B. They decided to go with a different model to overcome this obstacle :

$$t = \left(\frac{L}{a}\right)^{\frac{1}{b}}.$$

Student A discussed extending the model where the rate of acquisition would be a function of the number of complete trials.

Student D felt that there was a limitation in the experimental design that hindered the validation of the model and fitting of the model. They felt that the first trial of memorization was an outlier and was lowering the average. They concluded that if they exclude it, they could treat it as a primer for the data collection. However, once this data is excluded, student D believed the averages were skewed and seemed unrealistic. They suggested that the lists be longer and that during a trial more data points be recorded.

These students analyzed their models and found limitations. In some cases, they were able to devise and execute methods for overcoming these limitations.

4. POSSIBLE MATHEMATICAL APPROACHES, MODELS, AND ANALYSES

In this section, we provide a description of various mathematical models so that instructors might be aware of possible approaches. In our experience, undergraduates are often unprepared to use the mathematics they know in their models. Understanding various approaches allow instructors to see the richness of this modeling challenge, it helps them guide students through their own modeling approach without steering them to a predetermined model, it provides a basis for anticipating student questions, and also increases the instructors' overall preparation for the lesson. Analysis of a variety of models may help instructors promote connections between approaches taken by different groups of students, and mathematical and statistical content that is often studied separately, ultimately to improve undergraduate teaching.

4.1. Empirical Modeling Approach

When looking at data on a graph, students often look for functions that seem to fit the data. Typical functions considered are linear, rational, exponential, logarithmic, and power functions. This provides an opportunity to discuss the mathematics of what it means to fit a model to these data and the features required of the function. In introductory courses, the software Desmos is a good choice due to its relative ease and accessibility. The data can be uploaded from Microsoft Excel, Google Sheets, or can be entered manually. The students can type in the function and parameters. Another useful option is the curve fitting app in MATLAB shown in Figure 3. Engaging in the empirical modeling approach allows the instructor to challenge students to propose functions that have certain properties (asymptotes, local maximum, etc.), which is the reverse of the more common problem of listing properties of a given function. The former approach also provides opportunities to discuss least-squares optimization techniques.

4.2. Differential Equations Modeling Approach

Deterministic models based on differential equations can be derived from considering the rate at which one memorizes items (e.g., words) in a list. We start by defining M to be the total number of items on the list and $S(t)$ to be the number of items we have memorized after spending time t acquiring and consolidating or actively memorizing. Then, the model is developed by making an assumption on how the rate of memorization, $\mathrm{d}S/\mathrm{d}t$, relates to M and $S(t)$. For instance, we may assume that the rate of memorization is constant, leading to the model

$$\frac{\mathrm{d}S}{\mathrm{d}t} = c.$$

In order to compare results from experiments that may have been conducted with lists of different lengths, it is useful to express the model in terms of $L(t) = S(t)/M$,

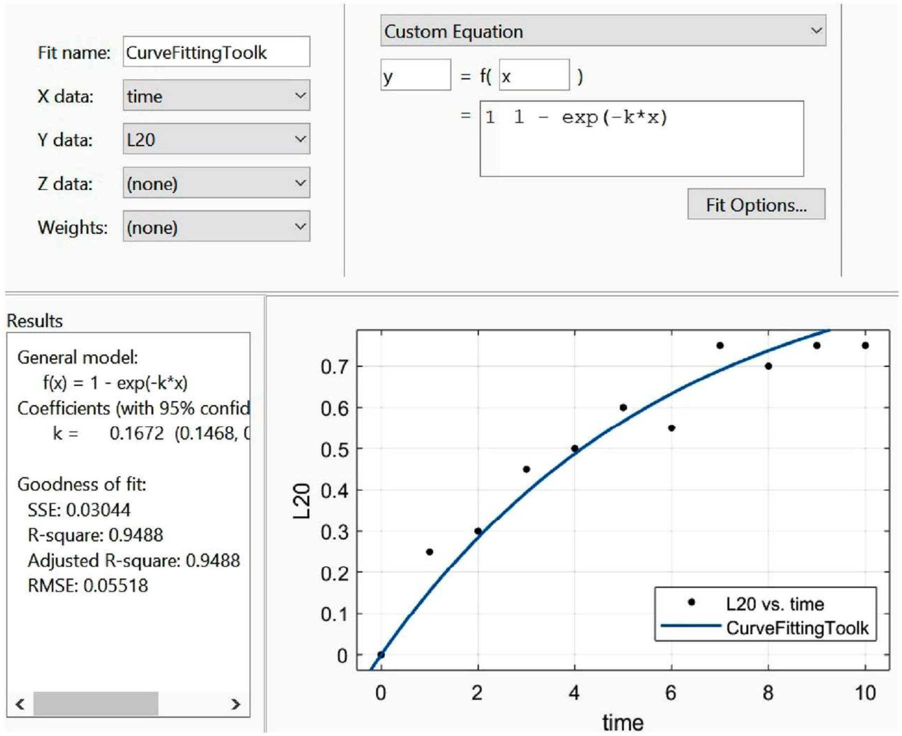

Figure 3. Screenshot of the Curve Fitting App in MATLAB. Students can enter their data, choose a model type from a list or create their own custom model and try different fit options.

which is the fraction of M that has been memorized. This gives

$$\frac{\mathrm{d}L(t)}{\mathrm{d}t} = c/M, \quad \text{with } L(0) = 0.$$

The constant c/M is the memorization rate which is unique for each person. The initial condition is based on the assertion that no information has been memorized before looking at the list. The solution is $L(t) = (c/M)t$ and is only valid until $L(t) = 1$ which occurs at $t = M/c$. At that time the list is fully memorized, making this simple model applicable for sufficiently short lists.

A model that is valid for longer lists of words requires different assumptions. One possibility is to assume that the memorization rate is proportional to the number of items on the list that remain to be memorized. This gives

$$\frac{\mathrm{d}L}{\mathrm{d}t} = k\Big(1 - L(t)\Big) \quad \text{with } L(0) = 0.$$

The parameter, k, can be interpreted as an individual's memorization rate. The solution

$$L(t) = 1 - e^{-k \cdot t}$$

can be used in several ways. First, the students can use it to fit the model to the data, as is done in the empirical modeling approach. Second, the properties of the function can be analyzed in the context of the task. The asymptote at $L = 1$ indicates that, according to this model, the list is never 100% memorized, and hence it may be more applicable for long lists. A Taylor series at $t = 0$ gives $L(t) = kt + O(t^2)$, which indicates that the memorization rate is approximately constant in early times. Finally, the function $L(t)$ is increasing, indicating that the rate of forgetting is either unaccounted for or always smaller than the rate of memorization.

A more comprehensive model can be derived from the following set of assumptions:

(1) M is the total number of items on the list and $S(t)$ is the number of items we have memorized after spending time t.
(2) Each person has a maximum number of words that can be memorized during this experiment, denoted by W.
(3) The memorization rate is proportional to the number of items on the list that have not been memorized, $k(W - S(t))$. This process is active only when the person is memorizing words (acquiring information).
(4) The forgetting rate is proportional to the number of items on the list that have been memorized, $rS(t)$. This process is always active, regardless of whether the person is memorizing words or not.
(5) The first few words on the list are memorized very quickly compared to the long-term memorization rate. We assume that a small number of words, denoted by β, is memorized instantly at $t = 0$.

The model is $S(0) = \beta$ and

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} = k(W - S(t)) - rS(t)$$

with $k \neq 0$ only while memorizing words, and $k = 0$ otherwise. Since the data is presented as proportions of M, we will scale the equation by defining $L(t) = S(t)/M$, $w = W/M$, $b = \beta/M$, to get

$$\begin{aligned}\frac{\mathrm{d}L(t)}{\mathrm{d}t} &= k(w - L(t)) - rL(t), \quad L(0) = b\\ &= kw - (k + r)L(t)\\ &= K(B - L(t)) \quad K = k + r, \quad B = \frac{kw}{k + r}\end{aligned}$$

The solution is

$$L(t) = B + (b - B)\, e^{-Kt} \tag{1}$$

with $K = r$ and $B = 0$ when the person is not memorizing words. The parameter B corresponds to the horizontal asymptote. When the value of w is greater than one more word can be memorized than what is included on the list. When w is less than 1 not all items can possibly be memorized. The value of b indicates how much of the list is memorized instantaneously. And K is an individual's net rate of memorization. Solutions of equation (1) with different parameters are depicted

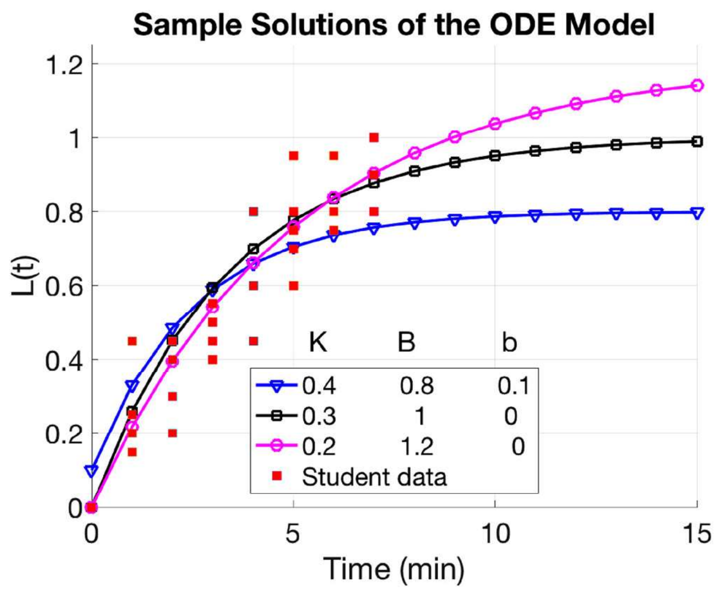

Figure 4. Three solutions to the ordinary differential equation (ODE) model. The graphs of $L(t)$ (see Equation (1)) with various parameter values are shown along with some data from students (indicated with squares).

in Figure 4. The values of the parameter K were selected so that the value of $L(1)$ was near 0.25, which is consistent with the student data. The parameter B is the asymptotic value of $L(t)$ so its values were chosen near $B = 1$. Finally, the parameter b is chosen to be zero for the solution to go through the origin but a small value captures the quick memorization of a few words at the beginning of the process.

This can be viewed as the alternation between two states, actively memorizing and pausing the memorization process. Specifically, let $t_0 = 0$ and assume the person memorizes words until t_1. No new information is acquired until t_2 when the person continues to memorize words until t_3. Following this pattern, we look for a solution $L(t)$ that is continuous and of the form $L(t) = B + (b - B)e^{-Kt}$ for $t_{2n} \leq t \leq t_{2n+1}$ and of the form $L(t) = be^{-rt}$ for $t_{2n+1} \leq t \leq t_{2n+2}$. This gives $L(t_0) = b$ and for $n = 0, 1, 2, \ldots$

$$L(t) = \begin{cases} B + (L(t_{2n}) - B)\, e^{-K(t-t_{2n})}, & t_{2n} \leq t < t_{2n+1} \\ L(t_{2n+1})e^{-r(t-t_{2n+1})}, & t_{2n+1} \leq t \leq t_{2n+2} \end{cases} \tag{2}$$

A sample solution is shown in Figure 5.

4.3. A Probabilistic Modeling Approach

This approach considers a probability associated with each word being remembered at any moment in time in the context of this task.

Each word on the list has a probability p of being remembered after a 60 s time period, but the probability changes in time. Time may be divided into three different intervals: (1) before looking at the word, when the probability p is zero; (2) during a period of time when actively trying to remember the word, the probability of being

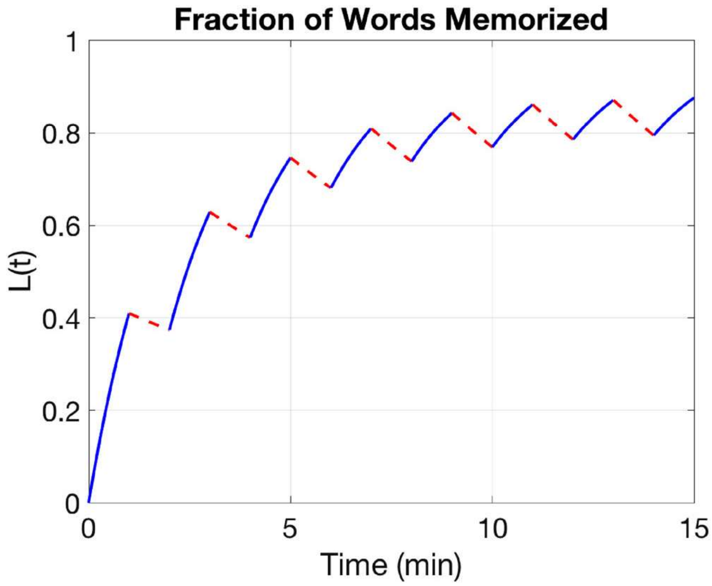

Figure 5. Graph of $L(t)$ or the fraction of items memorized, M, (Equation (2)) of the model where memorization periods are followed by breaks where only forgetting takes place. The parameters $(k, r, w, b) = (0.44, 0.092, 1.2, 0)$ were chosen to illustrate the process. Solid parts of the curve correspond to memorization periods, while the dashed parts correspond to breaks in the memorization process where only forgetting takes place.

remembered increases; and (3) after one moves on to memorize other words, the probability starts to decrease because forgetting continues.

The probability of remembering a word after 60 s might look like the curves shown in Figure 6. After 60 s each word (indexed with i) has a probability $p_i(60)$ of being remembered (the dots in the figure).

The functions that represent the probabilities in the different time intervals can be chosen in different ways. Here we borrow the functions from the previous model with $i = 1, \ldots, W$

$$p_i(t) = \begin{cases} 1 - \left(1 - p_i(t_0)\right) e^{-k_i(t-t_0)}, & t_0 \le t < t_1 \\ p_i(t_1) e^{-r_i(t-t_1)}, & t_1 \le t \le 60 \end{cases} \tag{3}$$

where k_i and r_i are parameters associated with word i, which depend on the person doing the memorizing. The values of t_0 and t_1 are part of the data collection. Typically, $p_i(t) = 0$ for $0 \le t < t_0$ if the word has not been seen before, but $p_i(t_0) > 0$ if the word has been visited before. In this model, each word can have its own parameters (k_i, r_i) because some words are easy for individuals to memorize (like ones that match our area code, birthdate, etc.) while other words are difficult to memorize.

At the end of the 60 s each word will have a probability $p_i(60)$ of being remembered. We translate this probability into a binary variable by choosing a random number between 0 and 1 from a uniform distribution and declaring that word i is remembered if the random number is less than the probability $p_i(60)$. This model also allows for the remembering and forgetting rates to change after the first 60 s depending on the words that have been memorized. For example, Figure 7 shows a

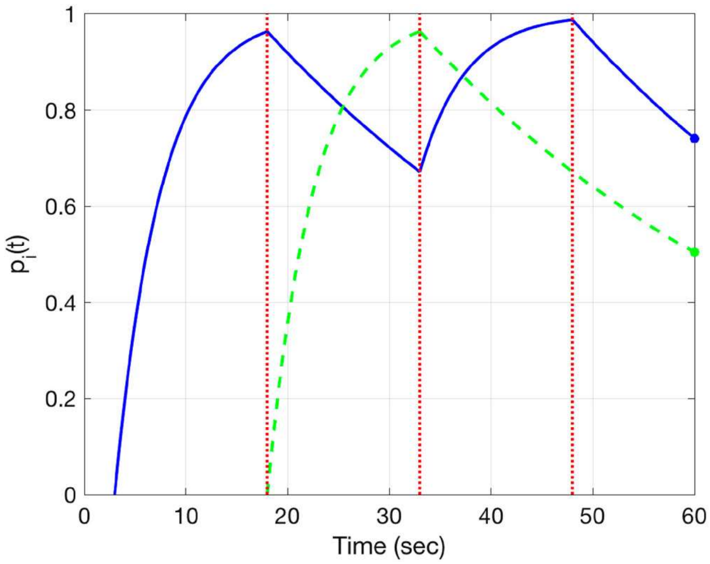

Figure 6. Sample solution of the probabilistic model (see Equation (3)) where the dashed curve is the probability of remembering a word that is actively being memorized in the interval $18 < t < 33$. The solid curve shows the probability of remembering a different word that is actively being memorized during $3 < t < 18$ and again during $33 < t < 49$. The dots at $t = 60$ are the probabilities of remembering these words after 60 s.

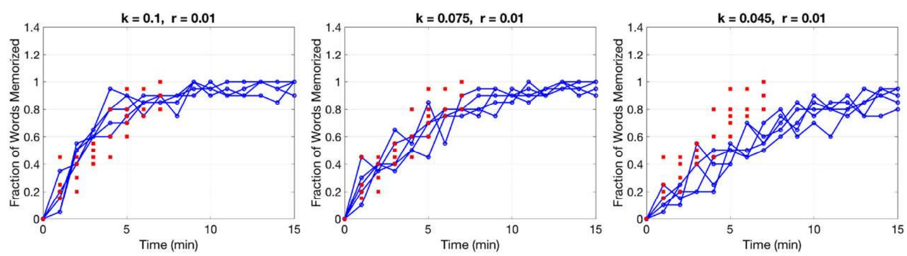

Figure 7. Simulations using the probabilistic model. Five simulations for each value of the learning rate k, are shown, with k high in the left panel and low in the right panel. The student data (squares) are also shown in each plot.

simulation using this model when all learning rates k_i are equal to k (see figure caption) and all forgetting rates r_i are equal to $r = 0.01$. The time intervals for studying a word before moving on to the next word have been set to 3 s.

4.4. Markov Chain Approach

"[A]ccording to cognitive informatics, the logical architecture of memories in the brain can be classified into …(a) the sensory buffer memory, (b) short-term memory, (c) the long term memory, and (d) the action buffer memory" [31, p. 82]. We can use this structure of discrete states to model the memorization process. The short-term, long-term and action buffer memory will be considered states of memory. The sensory buffer memory "refers to the short-lived memory for sensory details of events" [13, p. 23] which include things like how some event looked, sounded,

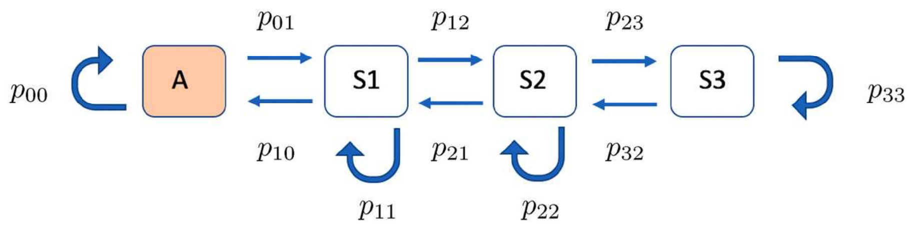

Figure 8. Visualization of the transition probabilities for a Markov model with four states. The acquisition state A includes items not memorized, and $S1$–$S3$ are storage states. The transition probabilities are indicated. For example, the words in $S2$ move to $S1$ with probability p_{21} and from $S2$ to $S3$ with probability p_{23}.

smelled or tasted. Information in this state is not considered as a memory state in our situation.

In the model, A^n represents the fraction of words left to be memorized at the nth minute of the experiment. The variables S_1^n through S_3^n represent the fraction of words in the three memory categories (sensory, short term, long term). From one minute to the next, words in one category have a probability of moving to another category, as shown in Figure 8. Since everything in $S2$ either stays or moves to other locations, the condition $p_{21} + p_{22} + p_{23} = 1$ must be satisfied. More generally, the probabilities of arrows pointing out of any location must add up to 1.

At time interval n, each location contains some words, which get re-distributed at the next time interval according to the transition matrix from state n to state $n+1$

$$\begin{pmatrix} A \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}^{(n+1)} = \begin{pmatrix} p_{00} & p_{10} & 0 & 0 \\ p_{01} & p_{11} & p_{21} & 0 \\ 0 & p_{12} & p_{22} & p_{32} \\ 0 & 0 & p_{23} & p_{33} \end{pmatrix} \begin{pmatrix} A \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}^{(n)}$$

However, the columns have to add up to 1, so the number of parameters is reduced. Moreover, according to our assumptions, the contents of $S1$, $S2$, and $S3$ are words memorized, so we are interested in the quantity $Q = S1 + S2 + S3$. Adding the last three equations give

$$Q^{n+1} = Q^n + (1 - p_{00})A^n - p_{10}S_1$$

which shows that p_{11}, p_{21}, p_{23} and p_{22} are irrelevant since we do not distinguish between words in S_1 or S_2 or S_3. Therefore, the process reduces to the simpler model

$$\begin{pmatrix} A \\ S_1 \end{pmatrix}^{(n+1)} = \begin{pmatrix} p_{00} & p_{10} \\ 1 - p_{00} & 1 - p_{10} \end{pmatrix} \begin{pmatrix} A \\ S_1 \end{pmatrix}^{(n)}$$

where the value of S_1 represents the words memorized. The starting point is (A^0, S_1^0) (Figure 9).

Finding a solution: Since the columns of the matrix add to 1, every iteration redistributes the amounts in A^n and S_1^n into A^{n+1} and S_1^{n+1} so that the total remains the same, $A^n + S_1^n = A^0 + S_1^0$. This can be verified by adding the two equations.

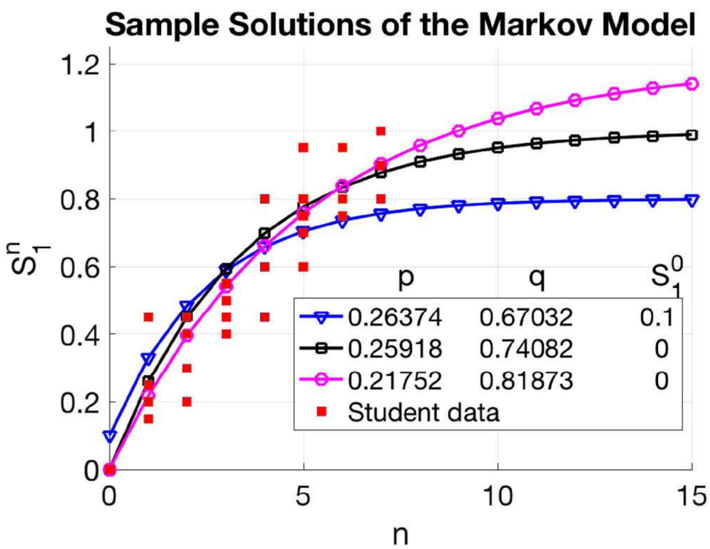

Figure 9. Simulations of the Markov chain model (see Equation (4)). The parameters have been defined in terms of the parameters of the ODE model as $S_1^0 = b$, $q = e^{-K}$, and $p = B(1 - e^{-K})$ for comparison.

We can use this in two ways. First, a transition matrix always has 1 as an eigenvalue and the corresponding eigenvector is a fixed point of the iteration. Therefore, it is related to an asymptote. In this case, the eigenvector is $(p_{10}, 1 - p_{00})$. The iterations converge to a multiple of this vector. Specifically, the final state is

$$(A^\infty, S_1^\infty) = \frac{A^0 + S_1^0}{p_{10} + 1 - p_{00}}(p_{10}, 1 - p_{00}).$$

Second, the conservation implies that $A^n = A^0 + S_1^0 - S_1^n$ and we can re-write the second of the matrix equations as

$$S_1^{n+1} = (1 - p_{00})(A^0 + S_1^0) + (p_{00} - p_{10})S_1^n$$

This suggests that by defining $p = (1 - p_{00})(A^0 + S_1^0)$ and $q = (p_{00} - p_{10})$ the model reduces further to three parameters:

$$S_1^n = p + qS_1^{n-1}, \quad S_1^0 \text{ given}$$

Using this expression recursively, we find that

$$S_1^n = \frac{p}{1 - q} + q^n \left(S_1^0 - \frac{p}{1 - q} \right) \tag{4}$$

Comparing this solution with the ODE model (associating n with t)

$$L(t) = B + (b - B)e^{-Kt}$$

we can deduce that both models give the same solution when $S_1^0 = b$, $q = e^{-K}$, and $p = B(1 - e^{-K})$.

We have demonstrated that many mathematical topics relevant to building and analyzing mathematical models come up naturally in this task. Instructors of a variety of courses can select this task to fit the various content and level of sophistication in the course. We suggest that multiple model approaches be highlighted, especially those that come from students.

5. CONCLUSION

This article details a data-driven mathematical modeling experience based on the context of memorization. The application of memorization as a mathematical modeling challenge provides a valuable opportunity for students in teacher education, neuroscience, computer science, mathematics, and various interdisciplinary fields. We included a sample agenda, data collection procedures, a rubric for assessment of students' modeling reports, and we share various pedagogical supports for instructors that promote an equitable and productive learning environment. We discussed the student progress on the learning goals based on our analysis of several modeling reports. Lastly, we include an assortment of mathematical approaches and models that could serve as a resource for readers to consider. Our summary of the lesson progression infused with equitable teaching practices, discussion on student-created models, and presentation of resources are intended to entice instructors to consider implementing this data collection and mathematical modeling activity with their students.

This lesson is one example of the variety of mathematical modeling activities created for secondary teacher teacher preparation through the MODULE(S2) project. We posit that while these mathematical modeling activities are intended for future teachers that they have value in contexts outside of teacher education and can broadly enhance mathematics education at the collegiate level.

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APPENDICES

APPENDIX 1. DATA COLLECTION WORKSHEET

A.1. Materials

Timer, pencil, strips of paper

A.2. Instructions

Read all of the instructions before beginning.

(1) Decide on one of the lists, do not switch lists until completion.
(2) Set a timer for 1 minute, and begin to memorize the list of 3 digit numbers. When the 1 minute is up, put the list away.
(3) On a separate sheet, record what you remember without looking.
(4) Repeat the previous 2 steps until one of two things is accomplished:
  (a) You memorize and successfully record the entire list.
  (b) You record data 10 times.

A.3. Additional Instructions

For the class to clarify after a trial run.

A.4. Lists

Random 3-digit numbers are tabulated here.

	List 1	List 2	List 3	List 4
1	527	091	725	790
2	986	887	161	254
3	115	230	763	594
4	292	948	790	425
5	928	121	444	366
6	213	581	553	093
7	346	815	678	798
8	558	536	583	320
9	729	204	893	716
10	537	211	565	461
11	500	904	432	977
12	197	449	182	638
13	045	393	212	701
14	713	371	934	945
15	153	042	914	866
16	626	991	149	158
17	476	205	542	082
18	920	083	762	914
19	581	610	543	457
20	409	467	820	060

APPENDIX 2: MATHEMATICAL MODELING RUBRIC

Rubric for Assessment of Mathematical Modeling

Modeling Element	4 Proficient Evidence	3 Emergent Evidence	2 Limited Evidence	1/0 Little/no evidence
Understand the problem situation	Fully identifies the context, objective of the solution, and factors that affect the solution. Uses background knowledge of the context or researches the context.	Partially identifies the context, objective of the solution, or factors that affect the solution. Uses partial background knowledge of context or partially researches the context.	Limited in identifying the context, objective of the solution, and factors that affect the solution. Does not use available background knowledge and does little to no research of the context.	Demonstrates little to no evidence.
Pose a simplified version of the situation	Fully determines useful information given. Makes useful, appropriate assumptions and choices. Uses background knowledge and experience or research as additional information.	Partially determines useful given information Partially makes useful, appropriate assumptions and choices Partially uses background knowledge and experience or research as additional information	Limited in determining given information, making necessary assumptions, and making appropriate choices,	Demonstrates little to no evidence.
Develop a model	Translates the information into mathematical notation (decontextualize). The model uses all relevant assumptions made.	Partially translates the information into mathematical notation (decontextualize). The model uses some assumptions made.	Limited in translating the information into mathematical notation (decontextualize). The model uses assumptions different from those made (modified or implicit).	Does not translate information into mathematics.
Compute a solution of the model	Performs calculations correctly in the model (possibly one minor error). Checks for precision.	Performs calculations correctly in the model (with few errors). Is aware of checking for precision.	Limited with calculations in the model (with multiple errors) and is not aware of needing to check for precision.	Demonstrates little to no evidence.
Interpret the solution and draw conclusions	Interprets the mathematical solution in terms of the original situation (contextualizing). Draws conclusions that the solution implies about the original situation.	Partially interprets the mathematical solution in terms of the original situation (contextualizing). Is aware of drawing conclusions that the solution implies about the original situation.	Limited with interpreting the mathematical solution in terms of the original situation (contextualizing) and with drawing conclusions that the solution implies about the original situation.	Demonstrates little to no evidence.
Validate the conclusions	Determines if the mathematical answer makes sense in terms of the original situation and verifies the answer is within a valid range of values. Determines if conclusions are satisfactory in all respects; if not, shows evidence of iteration of the process to improve the model.	Partially determines if the mathematical answer makes sense in terms of the original situation. Demonstrates awareness that the answer is within a valid range of values. Partially determines if conclusions are satisfactory.	Limited in determining if the mathematical answer makes sense in terms of the original situation. Shows little awareness that verification of the solution should be made.	Demonstrates little to no evidence.
Report the solution	Communicates the model with full explanations and justifications of assumptions and choices made.	Communicates the model with partial explanations and justifications of assumptions and choices made.	Limited in communicating the model, explanations, and justifications demonstrating little understanding of the problem and solution.	Demonstrates little to no evidence.

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