

Symplectic fillings of asymptotically dynamically convex manifolds I

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Dedicated to the memory of Chenxue.

ABSTRACT

We consider exact fillings with vanishing first Chern class of asymptotically dynamically convex (ADC) manifolds. We construct two structure maps on the positive symplectic cohomology and prove that they are independent of the filling for ADC manifolds. The invariance of the structure maps implies that the vanishing of symplectic cohomology and the existence of symplectic dilations are properties independent of the filling for ADC manifolds. Using them, various topological applications on symplectic fillings are obtained, including the uniqueness of diffeomorphism types of fillings for many contact manifolds. We use the structure maps to define the first symplectic obstructions to Weinstein fillability. In particular, we show that for all dimension $4k + 3, k \geq 1$, there exist infinitely many contact manifolds that are exactly fillable, almost Weinstein fillable but not Weinstein fillable. The invariance of the structure maps generalizes to strong fillings with vanishing first Chern class. We show that any strong filling with vanishing first Chern class of a class of manifolds, including $(S^{2n-1}, \xi_{\text{std}}), \partial(T^*L \times \mathbb{C}^n)$ with L simply connected, must be exact and have unique diffeomorphism type. As an application of the proof, we show that the existence of symplectic dilation implies uniruledness. In particular, any affine exotic $\mathbb{C}^n$ with nonnegative log Kodaira dimension is a symplectic exotic $\mathbb{C}^n$.

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1. Introduction

One natural question in symplectic topology is understanding symplectic fillings of a contact manifold. One aspect of the question is understanding the existence of symplectic fillings. Contact obstructions to the existence of symplectic fillings were first discovered by Eliashberg [20]. There are various obstructions to fillings of different flavors, cf. [40] and references therein. There are also topological obstructions to the existence of almost Weinstein fillings [13].

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Another aspect of the question is understanding the uniqueness of symplectic fillings. The first result along this line is by Gromov [27] that exact fillings of the standard contact 3-sphere are unique. In [20], Eliashberg proved fillings of (S^3, ξ_{std}) are diffeomorphic to blow-ups of the ball. Shortly after, McDuff [41] generalized the result to lens space $L(p, 1)$. In dimension 3, several uniqueness or finiteness results of symplectic fillings were obtained on T^3 [62], lens space $L(p, q)$ [39] and $S^*\Sigma_g$ [59]. Only the results on $S^3, L(p, 1), T^3$ obtained uniqueness of symplectic fillings, while other results are about topological types of fillings. The dimension 3 case is special since we have more tools like intersection theory of holomorphic curves and Seiberg–Witten theory. In higher dimensions, Eliashberg–Floer–McDuff [21, 42] proved that any exact filling of $(S^{2n-1}, \xi_{\text{std}})$ is diffeomorphic to the ball B^{2n} . The Eliashberg–Floer–McDuff method was generalized by Oancea–Viterbo [49] to obtain homological information for symplectic aspherical fillings of simply connected subcritically fillable contact manifolds. Barth–Geiges–Zehmisch [7] extracted refined homological information and showed that symplectic aspherical fillings of simply connected subcritically fillable contact manifolds have unique diffeomorphism types via h-cobordism.

The symplectic aspect of the uniqueness in higher dimensions remains largely unknown. However, there are some evidences. Seidel–Smith [55] showed that any exact filling of $(S^{2n-1}, \xi_{\text{std}})$ has vanishing symplectic cohomology. In [68], we showed that any exact filling with vanishing first Chern class of a simply connected flexibly fillable contact manifold has vanishing symplectic cohomology. It turns out that those contact manifolds are asymptotically dynamically convex (ADC) in the sense of Lazarev [37].[†] The ADC condition is a generalization of the index-positive condition introduced in [18, §9.5]. A contact manifold Y^{2n-1} is called index-positive, if there is a nondegenerate contact form so that every Reeb orbit γ has positive degree, that is, $\mu_{CZ}(\gamma) + n - 3 > 0$. There are many natural ADC contact manifolds, for example, boundaries of cotangent bundles of manifolds of dimension at least 4, boundaries of flexible Weinstein domains and links of terminal singularities (Remark 3.11). The ADC property is a condition on the Conley–Zehnder index, which is suitable for Floer theoretic study. The importance of index-positive/ADC is that positive symplectic cohomology is independent of the filling, hence a contact invariant [18, 37] via neck-stretching. Combining invariance of positive symplectic cohomology and vanishing of symplectic cohomology, the tautological long exact sequence of symplectic cohomology yields that any exact filling with vanishing first Chern class of a simply connected flexibly fillable contact manifold has the same cohomology group as the flexible filling.

The above result serves as a basic prototype of studying symplectic fillings of ADC manifolds: We first prove some invariance results on the Floer theory of fillings of ADC manifolds, then we infer invariant symplectic or topological properties from there. The key point in this paper is that the invariance is not limited to some Floer cohomology like positive symplectic cohomology, but also structure maps on those Floer cohomology. The substance of this paper is constructing two structure maps and proving their invariance with respect to fillings for ADC manifolds. Then we will derive various symplectic and topological applications from them on both uniqueness and existence aspects of symplectic fillings.

1.1. Invariance of restriction and persistence of vanishing

Let W be an exact filling of Y , the first structure map δ_{∂} is the composition of $\delta : SH_+^*(W) \rightarrow H^{*+1}(W)$ and the restriction $H^{*+1}(W) \rightarrow H^{*+1}(Y)$, where δ is the connecting map in the tautological long exact sequence $\dots \rightarrow SH^*(W) \rightarrow SH_+^*(W) \rightarrow H^{*+1}(W) \rightarrow \dots$. In the following, we will restrict to topologically simple fillings of ADC manifolds, that is, those

[†]In fact, all contact manifolds in the first paragraph are ADC, when $c_1 = 0$.

fillings W such that $c_1(W) = 0$ and $\pi_1(Y) \rightarrow \pi_1(W)$ is injective (when Y is strongly ADC (Definition 3.5), we only require $c_1(W) = 0^\dagger$). Our first theorem is the following.

THEOREM A. *Let Y be a (strongly) ADC contact manifold. Then $\delta_\partial : SH_+^*(W) \rightarrow H^{*+1}(Y)$ is independent of topologically simple exact fillings.*

REMARK 1.1. Unless specified, our coefficient can be any ring with a default setting of $\mathbb{Z}$.

Since whether $1 \in \text{im } \delta$ is equivalent to whether 1 is mapped to 0 in the unital map $H^*(W) \rightarrow SH^*(W)$, $1 \in \text{im } \delta_\partial$ is equivalent to $SH^*(W) = 0$. This reproves the vanishing result in [68]. Moreover, unlike the proof based on the formal properties of symplectic cohomology in [68], the proof here explains the background geometry to some extent by finding a persistent holomorphic curve. Moreover, we have the following finer invariance result on the topology of the filling.

COROLLARY B. *Let Y be a (strongly) ADC contact manifold, then $SH^*(W) = 0$ is a property independent of topologically simple exact fillings. In that case, $H^*(W) \rightarrow H^*(Y)$ is independent of topologically simple exact fillings, that is, for two topologically simple exact fillings W, W' , we have an isomorphism $H^*(W) \simeq H^*(W')$ such that the following commutes:*

$$\begin{array}{ccc} H^*(W) & \longrightarrow & H^*(Y) \\ \downarrow \simeq & & \downarrow = \\ H^*(W') & \longrightarrow & H^*(Y). \end{array}$$

The invariance of $H^*(W) \rightarrow H^*(Y)$ is a very strong topological constraint, especially when it is injective. In particular, by the universal coefficient theorem, Corollary B implies [7, Theorem 1.2(a)] if the exact filling is topologically simple. Combining with Theorem E below yields a Floer theoretic proof of exact and $c_1 = 0$ fillings of simply connected subcritically fillable contact manifolds have unique diffeomorphism type by the h-cobordism argument in [7, §5]. Moreover, we can extract the topology condition required for the h-cobordism argument to reach the following uniqueness result for some only Liouville fillable contact manifolds (Corollary 3.14) and flexible fillable contact manifolds.

THEOREM C. *Exact fillings W with vanishing first Chern class of the following contact manifolds Y with $\dim Y \geq 5$ have unique diffeomorphism types.*

- (i) $Y = \partial(V \times \mathbb{C})$ simply connected, where V is a simply connected Liouville domain such that $c_1(V) = 0$ and $\dim V > 0$.
- (ii) Y is the boundary of the flexible cotangent bundle $\text{Flex}(T^*Q)$ for simply connected Q with $\chi(Q) = 0$.

REMARK 1.2. The contact boundary $Y := \partial(V \times \mathbb{C})$ was also considered in [7, §2.6], where Barth-Geiges-Zehmisch proved that for every symplectic aspherical filling W of Y , we must have that $H_*(Y) \rightarrow H_*(W)$ is surjective. Using Corollary B, if we assume $c_1(V) = 0$ and W is topologically simple, then $H^*(W) \rightarrow H^*(Y)$ is invariant and injective, see Corollary 6.10.

REMARK 1.3. $SH^*(W) = 0$ is a restrictive condition, the major classes of examples are flexible Weinstein domains [9, 47], $V \times \mathbb{C}$ for Liouville domains V [48]. These two classes

[†]Roughly speaking, strongly ADC requires that Reeb orbits are contractible in addition to the asymptotically dynamically convex condition.

provide many examples with ADC boundaries as long as the first Chern class vanishes. Recall that if $V \subset W$ is an exact subdomain, the Viterbo transfer map $SH^*(W) \rightarrow SH^*(V)$ is a unital ring map. Since the 0 ring is the terminal object in the category of unital rings, a symplectic manifold W with $SH^*(W) = 0$ should be understood as the simplest symplectic manifold and the contact boundary has the best chance of having unique exact fillings.

Theorem A also holds for symplectic cohomology with local systems, and Corollary B holds if the ring structure still exists. In particular, those results can be applied to cotangent bundles considered in [5], whose symplectic cohomology without local system does not vanish. In particular, we have the following invariance result for cotangent bundles.

THEOREM D. *Let Q be a manifold such that the Hurewicz map $\pi_2(Q) \rightarrow H_2(Q)$ is nonzero and ST^*Q is ADC (for example, $\dim Q \geq 4$), then we have the following.*

- (i) $H^*(W; \mathbb{C}) \rightarrow H^*(ST^*Q; \mathbb{C})$ is independent of the topologically simple exact filling W as long as $H^2(W; \mathbb{Z}/p) \rightarrow H^2(ST^*Q; \mathbb{Z}/p)$ is surjective for every prime p .
- (ii) If $\pi_1(Q) = 0$, and Q is spin, then the independence of $H^*(W; \mathbb{C}) \rightarrow H^*(ST^*Q; \mathbb{C})$ for topologically simple W holds as long as $H^2(W) \rightarrow H^2(ST^*Q)$ is nonzero. If Q is not spin, we need to assume $\text{im}(H^2(W; \mathbb{Z}/2) \rightarrow H^2(ST^*Q; \mathbb{Z}/2))$ contains $\pi^*w_2(Q)|_{ST^*Q}$, where $\pi : T^*Q \rightarrow Q$ is the projection and $w_2(Q) \in H^2(Q; \mathbb{Z}/2)$ is the second Stiefel–Whitney class of Q .
- (iii) If, in addition, we have $\chi(Q) = 0$, then the rational homotopy type of W with same conditions above is T^*Q .

We also study the symplectic cohomology of covering spaces and prove an analogous statement to Theorem A, which implies the following theorem. It can be viewed as a generalization of [7, Theorem 1.2(b)].

THEOREM E. *Assume Y is an ADC contact manifold, with a topologically simple exact filling W such that $SH^*(W) = 0$ (integer coefficient without local systems) and $\pi_1(Y) \rightarrow \pi_1(W)$ is an isomorphism. Then $\pi_1(Y) \rightarrow \pi_1(W')$ is an isomorphism for any other topologically simple exact filling W' . If Y is strongly ADC with the same property and $\pi_1(Y)$ is abelian, then $\pi_1(Y) \rightarrow \pi_1(W')$ is an isomorphism for any other topologically simple exact filling W' .[†]*

The results above can be put under one theme: understand whether the symplectic filling is unique. It is conjectured that exact fillings of flexibly fillable contact manifolds are unique. Since Theorems A–E can be applied to a larger class of contact manifolds in addition to flexibly fillable contact manifolds (see §6), they suggest that contact manifolds with unique exact fillings may go beyond flexibly fillable contact manifolds.

For (S^3, ξ_{std}) , Eliashberg [20] and McDuff [41] showed that symplectic fillings of the standard contact 3-sphere are symplectic blow-ups of the standard ball. Hence it has a unique exact filling. However, the procedure of blow-up destroys both exactness and $c_1 = 0$. Hence in this special case, one can trade the exactness condition to $c_1 = 0$ condition and still has the uniqueness of fillings, in particular exactness is equivalent to $c_1 = 0$. In higher dimension, the exactness/symplectic asphericity plays an important role in [7, 49] while $c_1 = 0$ is not required or used. But if we study it using Floer theory, nonexactness only adds technical difficulties and can be overcome by a dimension argument [30] when $c_1 = 0$. However, $c_1 = 0$ plays a fundamental rule, since the ADC property is an index property. Therefore, we can generalize Eliashberg–McDuff’s result in the $c_1 = 0$ direction to higher dimensions as follows.

[†]Note that in the strongly ADC context, topologically simple filling only requires $c_1 = 0$, hence it is a stronger conclusion compared to the ADC case above.

THEOREM F. *Let (Y, ξ) be a tamed asymptotically dynamically convex (TADC) manifold (Definition 6.2) with one topologically simple exact filling W , such that $SH^*(W; \mathbb{Q}) = 0$. Assume $H^2(W; \mathbb{Q}) \rightarrow H^2(Y; \mathbb{Q})$ is injective and $H^1(W; \mathbb{Q}) \rightarrow H^1(Y; \mathbb{Q})$ is surjective. Then any topologically simple strong filling of Y is exact.*

The strategy of proving Theorem F is showing the invariance of $H^*(W; \mathbb{Q}) \rightarrow H^*(Y; \mathbb{Q})$ like Corollary B for strong fillings. Then exactness is a consequence of such invariance. TADC manifolds are more general than index-positive manifolds, but more restricted than ADC manifolds. Examples of TADC manifolds that Theorem F can be applied are boundaries of $\mathbb{C}^n (n \geq 2)$, $T^*M \times \mathbb{C}$, $W \times \mathbb{C}$, where W is the Milnor fiber of $\sum x_i^{a_i} = 0$, $a_i \in \mathbb{N}$ with $\sum \frac{1}{a_i} \geq 1$, and products among them. In particular, Theorem F implies that any strong filling of $(S^{2n-1}, \xi_{\text{std}})$ with vanishing first Chern class must be exact[†], hence is diffeomorphic to B^{2n} . Theorem F can also be applied to non-Weinstein example, for example, $\partial(V \times \mathbb{C})$, where V is the exact but not Weinstein domain in [40], since $\partial(V \times \mathbb{C})$ is TADC by Theorem 6.3. Another source of examples are cotangent bundles with local systems, we can get exactness from $c_1 = 0$ and similar conditions in Theorem D, see Theorem 8.14.

REMARK 1.4. Theorem F is expected to hold for ADC manifolds. In the general ADC case, we need to stretch along expanding contact hypersurfaces since nonexact filling may not contain the whole negative end of the symplectization of Y . Then we need more functoriality than we are able to get. However, the expanding issue indicates using an SFT description may be a better way to prove the generalization, see Remarks 1.7 and 8.15.

REMARK 1.5. It is worthwhile to compare our method with the method in [7, 42, 49], where the method can be summarized as finding a ‘homological foliation’ by rational curves in a partial compactification of the filling. Such method has the benefit of only assuming exactness/symplectic asphericity. Our method is from a very different perspective, which essentially only assumes $c_1 = 0$ as in Theorem F, while the exactness in assumptions of previous theorems are only for the simplicity of the setup. Therefore we cover different aspects of the uniqueness of filling, that is, $c_1 = 0$ versus symplectic asphericity. Moreover, we are able to extract some symplectic invariance through the study of symplectic cohomology. By [7], exact filling W of subcritically fillable contact manifold Y must have $c_1(W) = 0$ if $c_1(Y) = 0$ when $\dim W \geq 6$. Combining Theorem F and Remark 1.4, it suggests that for fillings of subcritically fillable contact manifolds, exactness is equivalent to $c_1(W) = 0$. In particular, if there is any procedure of modifying a filling, exactness and $c_1(W) = 0$ must be destroyed at the same time. This is the case for blow-up.

REMARK 1.6. The condition of $c_1 = 0$ is necessary for the results above to hold. For example, the once blow-up of the standard ball B^{2n} , that is, the total space $\mathcal{O}(-1)$ of the degree -1 bundle over $\mathbb{CP}^{n-1}$, has nonvanishing symplectic cohomology [53] and $H^*(\mathcal{O}(-1)) \rightarrow H^*(S^{2n-1})$ is different from $H^*(B^{2n}) \rightarrow H^*(S^{2n-1})$. By the Viterbo transfer map, such phenomena persist for all exact domains considered above after once blow-up.

1.2. Persistence of dilation

The second structure map is related to the symplectic dilation introduced by Seidel–Solomon [58]. To explain the structure map, first recall that symplectic cohomology is equipped with a degree -1 BV operator Δ . Then a symplectic dilation is an element $x \in SH^1(W)$ such

[†]Although in this special case, proving $H^*(W; \mathbb{Q}) = H^*(B^{2n}; \mathbb{Q})$ is enough. The general case requires the invariance of $H^*(W; \mathbb{Q}) \rightarrow H^*(Y; \mathbb{Q})$.

that $\Delta(x) = 1$. The existence of symplectic dilation puts strong restrictions on Lagrangians that can be embedded exactly [56, 58]. On the cochain level, Δ also respects the splitting into positive and zero symplectic cohomology. Therefore we have a well-defined degree -1 map $\Delta_+ : SH_+^*(W) \rightarrow SH_+^{*-1}(W)$. Then there is a well-defined degree -1 map $\Delta_\partial : \ker \Delta_+ \rightarrow \operatorname{coker} \delta_\partial$. When Y is ADC and W is topologically simple exact, we have that Δ_+ is independent of the filling. Moreover, by Theorem A, δ_∂ is independent of the filling. Then our second main result is the following.

THEOREM G. *Let Y be an ADC manifold, then for any two topologically simple exact fillings W_1, W_2 , we have an isomorphism $\Gamma : SH_+^*(W_1) \rightarrow SH_+^*(W_2)$, such that:*

- (i) $\delta_\partial \circ \Gamma = \delta_\partial$;
- (ii) $\Delta_+ \circ \Gamma = \Gamma \circ \Delta_+$;
- (iii) $\Delta_\partial \circ \Gamma = \Delta_\partial$.

The property of whether $1 \in \operatorname{im} \Delta_\partial$ is closely related to the existence of symplectic dilation. We have the following corollary. A stronger version concerning symplectic dilation on exact fillings can be found in Corollary 4.17.

COROLLARY H. *Let Y be an ADC contact manifold of dimension ≥ 5 . Then the existence of symplectic dilation is independent of Weinstein fillings.*

The vanishing of symplectic cohomology and the existence of symplectic dilations can be understood as the first two levels of indications of the complexity of symplectic manifolds. In fact, there exists a whole hierarchy of structures after them called higher dilations, all of them have associated structure maps similar to $\delta_\partial, \Delta_\partial$, which are also independent of the topologically simple exact filling for ADC manifolds. Details of the construction will appear in the sequel paper [69].

REMARK 1.7. Following [10], positive symplectic cohomology should be understood as the nonequivariant linearized contact homology. When Y is ADC, positive symplectic cohomology can be viewed as nonequivariant cylindrical contact homology, since the augmentation from filling is trivial by degree reason. $\delta_\partial, \Delta_\partial$ should have an equivalent description using SFT on Y . In particular, when the analytic foundation for the full SFT is completed (in a forthcoming paper by Fish and Hofer), one should be able to strengthen results in this paper to contact manifolds admitting a Reeb flow without a degree zero orbit and its asymptotic version based on the same argument. From SFT point of view, for any linearized nonequivariant contact homology $HC_*(Y)$, one should be able to define a map $HC_*(Y) \rightarrow H^{n+1-*}(Y)$ by counting holomorphic curves (with one positive puncture and multiple negative punctures) with one marked point mapped to $Y \times \{0\}$ along with the augmentation. When the augmentation is from a filling, then the map is the composition $SH_+^*(W) \rightarrow H^{*+1}(W) \rightarrow H^{*+1}(Y)$. It will imply Theorem A, since ADC contact manifolds should have no nontrivial augmentation by degree reason. Roughly speaking, the set of fillings is closely related to the set of augmentations. The theme of this paper can be summarized as: if a contact manifold admits a unique augmentation, then many Floer theoretic properties of the filling are independent of the filling. It is possible to prove such claim by reformulizing constructions in this paper using SFT.

1.3. Obstructions to Weinstein fillings and cobordisms

One natural question in the study of symplectic fillings is understanding the difference between exact fillability and Weinstein fillability. In dimension 3, exact fillable but not Weinstein

fillable manifold was found by Bowden [12]. In higher dimension, such examples were found by Bowden–Crowley–Stipsicz [13]. Their obstruction is topological in nature and their examples are exactly fillable, but not almost Weinstein fillable. Hence the next question we could ask is whether the topological obstruction is sufficient, or whether there is a contact manifold with exact filling and almost Weinstein filling, but no Weinstein filling. In § 6, we answer the question by proving the following.

THEOREM I. *Let $k \geq 1$, there exist infinitely many pairwise noncontactomorphic $4k + 3$ dimensional contact manifolds, such that they are exactly fillable, almost Weinstein fillable, but not Weinstein fillable.*

To prove Theorem I, we need new obstructions to Weinstein fillability beyond topological obstructions. Using Theorems A and G, for ADC contact manifolds, $\text{im } \delta_\partial, \text{im } \Delta_\partial$ contain a nontrivial element of grading higher than $\frac{1}{2} \dim W$ that are symplectic obstructions to the existence of Weinstein fillings (Corollaries 3.14 and 4.19). Such obstructions, to our best knowledge, are the first symplectic obstructions to Weinstein fillability. This answers a Wendl’s question [63, Question 14] on the existence of obstructions to Weinstein fillability of contact structures in higher dimensions. With such obstructions, in addition to proving Theorem I, we give simple constructions of many exactly fillable, but not Weinstein fillable manifolds in dimension ≥ 7 . Hence they exist in abundance.

We can also use similar ideas to study symplectic cobordism. Corollary B and Theorem G show that whether $1 \in \text{im } \delta_\partial, \text{im } \Delta_\partial$ are actually contact invariants for ADC manifolds. Hence they can be used to develop obstructions to symplectic cobordisms. In particular, we have the following.

COROLLARY J. *Let Y^{2n-1} be an ADC contact manifold with a Weinstein filling W such that $c_1(W) = 0$ for $n \geq 3$. Let V be a Weinstein domain. If one of the following conditions holds, then there is no Weinstein cobordism from ∂V to Y .*

- (i) *If $1 \in \text{im } \delta_\partial$ for W , and $1 \notin \text{im } \delta_\partial$ for V .*
- (ii) *If $1 \in \text{im } \Delta_\partial$ for W , and $1 \notin \text{im } \Delta_\partial$ for V .*
- (iii) *If W admits a symplectic dilation, and V does not admit a symplectic dilation.*

A stronger version of Corollary J concerning obstructions to exact cobordisms can be found in Theorem 7.1. Note that usually, an algebraic obstruction to cobordism will come from SFT type invariants [36], which is difficult to define and compute. However, our obstruction is based on symplectic cohomology, hence is relatively easy to define and compute compared to the SFT. As explained in Remark 1.7, it can be understood as an easy case of some SFT obstructions. In § 7, we use Corollary J to give many pairs of contact manifolds that admit almost Weinstein cobordisms but no Weinstein cobordism in every dimension ≥ 5 .

1.4. Constructions of ADC manifolds

ADC contact manifolds exist in abundance. Moreover, subcritical and flexible surgeries preserve the ADC property by the work of Lazarev [37]. In order to provide examples to Theorem I, we prove two more constructions of ADC manifolds, which bear independent interests.

THEOREM K. *Let V be an exact domain such that $c_1(V) = 0, \dim V > 0$, then $\partial(V \times \mathbb{C})$ is ADC. If V, W are two exact ADC domains (Definition 3.8) of dimension at least 4 with vanishing first Chern classes, then $\partial(V \times W)$ is ADC.*

When V is a Weinstein domain and $V \times \mathbb{C}$ is a subcritical Weinstein domain, then the theorem above follows from [37, 64]. Repeatedly using subcritical and flexible surgeries and Theorem K, we have a lot of examples of ADC contact manifolds, and many of them have either vanishing symplectic cohomology or symplectic dilation, hence Theorems A and G can be applied.

1.5. Uniruledness

At last, we discuss a byproduct of proofs of Theorems A and G. Uniruledness in the symplectic setting was studied by McLean [45], an exact domain W is called (k, Λ) -uniruled if and only if for every point p in the interior W^0 , there is a proper holomorphic curve in W^0 passing through p with area at most Λ and the domain Riemann surface S has the property that $\text{rank } H_1(S; \mathbb{Q}) \leq k - 1$. McLean showed that the algebraic uniruledness for affine varieties is rather a symplectic property. Hence it is reasonable to look for symplectic characterization of uniruledness.

THEOREM L. *Let W be an exact domain and there exists a symplectic dilation, then W is $(1, \Lambda)$ -uniruled, for some $\Lambda \in \mathbb{R}_+$.*

The specific value of Λ is not relevant, as it is not an invariant with respect to the homotopy of Liouville forms. The key property needed in order to use results from [45] is that we have such a Λ that will bound areas of curves from above. The existence of symplectic dilation is very far from being equivalent to $(1, \Lambda)$ -uniruledness. In fact, the existence of k -dilation in [69] will also imply uniruledness. On the other hand, if W is not 1-uniruled, then $SH^*(W) \neq 0$. By [45], the algebraic uniruledness is equivalent to symplectic uniruledness. Since the log Kodaira dimension provides obstruction to algebraic uniruledness for affine varieties, we have the following corollary, which provides a simple proof of the existence of exotic Stein $\mathbb{C}^n$ for $n \geq 3$ by taking complex exotic $\mathbb{C}^n$ with nonnegative log Kodaira dimension, which exists in abundance [66].

COROLLARY M. *Let V be an affine variety of nonnegative log Kodaira dimension, then $SH^*(V) \neq 0$ and there is no symplectic dilation. In particular, any complex exotic $\mathbb{C}^n$ with nonnegative log Kodaira dimension is a symplectic exotic $\mathbb{C}^n$.*

In particular, any affine variety admitting a Calabi–Yau compactification has nonzero symplectic cohomology and hence is not displaceable by [32]. Similar results are investigated in [60].

Organization of the paper

In §2, we review symplectic cohomology and give a different treatment of the cochain complex on the zero action part. In §3, we construct an alternative description of δ_∂ and prove Theorems A–E via neck-stretching. Since we will compare cochain complexes on two different fillings, naturality is very important. We carry out a detailed discussion on naturality in §§2 and 3. In §4, we review the BV operator and symplectic dilation and construct Δ_∂ . Then we prove Theorem G and show that $\Delta_+, \Delta_\partial$ are invariants of exact domains up to exact symplectomorphisms. In §5, we discuss uniruledness and prove Theorem L and its corollaries. In §6, we prove Theorem K and use the symplectic obstructions introduced in §§3 and 4 to prove Theorem I. In §7, we discuss obstructions to symplectic cobordisms. In §8, we generalize the construction to strong fillings with vanishing first Chern class and finish the proof of Theorem F. We explain our orientation conventions in the Appendix.

2. Symplectic cohomology

In this section, we review the basics of symplectic cohomology. However, on the zero action part, we will use a Morse–Bott construction following [19]. Such modification is important to the proof of our main theorem and applications to uniruledness. We will first focus on the theory of exact fillings, the situation for nonexact fillings will be discussed in § 8. We assume our contact manifolds and fillings are connected throughout the paper unless specified otherwise.

2.1. Hamiltonian–Floer cohomology using cascades

Let W be a Liouville domain with a Liouville form λ . Then we have the completion $\widehat{W} := W \cup \partial W \times [1, \infty)$ with the completed Liouville form $\widehat{\lambda}$. Unless specified otherwise, the Reeb flow on the contact boundary $(\partial W, \lambda|_{\partial W})$ is nondegenerate throughout this paper. Let $\mathcal{S}$ denote the length spectrum of the Reeb orbits, that is, the set of periods of periodic orbits. Given a time-dependent Hamiltonian H on $\widehat{W}$, the symplectic action of a *contractible* loop $x : S^1 \rightarrow \widehat{W}$ is defined by

$$\mathcal{A}_H(x) := - \int_{S^1} x^* \widehat{\lambda} + \int_{S^1} H_t \circ x(t) dt. \quad (2.1)$$

Symplectic cohomology is defined as the Floer cohomology of (2.1) for a Hamiltonian $H = r^2, r \gg 0$ [55]. Alternatively, symplectic cohomology can be defined as the direct limit of Floer cohomology of $H = Dr, r \gg 0$ for $D \rightarrow \infty$ [18, 52]. In this paper, we will fix one special Hamiltonian, and our convention for Hamiltonian vector field is $\omega(\cdot, X_H) = dH$. First we consider a Hamiltonian $H = h(r)$ such that $H = 0$ on W and $H = h(r)$ when $r > 1$ such that $h''(r) > 0$ and $\lim_{r \rightarrow \infty} h'(r) = \infty$. Then the periodic orbits of X_H are all points in W , and S^1 families of nonconstant periodic orbits corresponding to Reeb orbits γ on ∂W shifted to the level r , where r is given by $h'(r) = \int_{\gamma} \lambda|_{\partial W}$. The action of such orbit is then given by $-rh'(r) + h(r)$, which is a strictly decreasing function since $h''(r) > 0$. Then following [11], we can put a small time-dependent perturbation supported near each S^1 family of nonconstant periodic orbits, such that the periodic orbits of the perturbed Hamiltonian are points in W and pairs of nondegenerate nonconstant orbits near each S^1 family of nonconstant periodic orbits of H . We will fix one such perturbed Hamiltonian and call it $\mathbf{H}$. Let $\mathcal{P}^*(\mathbf{H})$ denote the set of nonconstant *contractible* periodic orbits of $\mathbf{H}$, then $\mathbf{H}$ has the following properties.

- (1) $\mathbf{H} = 0$ on W .
- (2) $\mathbf{H} = h(r)$, such that $h''(r) > 0$ on $\partial W \times (1, \rho]$ for some ρ and $h'(\rho) < \min \mathcal{S}$.
- (3) There are nonempty disjoint intervals (a_i, b_i) moving toward infinity with $(a_0, b_0) = (1, \rho)$, such that $\mathbf{H}|_{\partial W \times (a_i, b_i)}$ is a function $f(r)$ with $f''(r) > 0$ and $f'(r) \notin \mathcal{S}$.
- (4) There exists $0 = D_0 < D_1 \dots$ converging to ∞ , such that all periodic orbits of action $\geq -D_i$ are contained in $W^i := \{r \leq a_i\}$.

DEFINITION 2.1. A Morse function f on W is admissible if $\partial_r f > 0$ on ∂W and f has a unique local minimum. The class of admissible Morse functions is denoted by $\mathcal{M}(W)$.

REMARK 2.2. $\mathbf{H}$ is not strictly Morse–Bott, since the critical points of $\mathcal{A}_H$ are not Morse–Bott nondegenerate along $\partial W \subset W$. We will see in Proposition 2.6 that such points are invisible to our moduli spaces, also see [19].

On the symplectization $\partial W \times (0, \infty)$, a compatible almost complex structure is called cylindrical convex if J preserves $\xi = \ker \lambda|_{\partial W}$ and $J(r\partial_r) = R_\lambda$, where R_λ is the Reeb vector field on $(\partial W, \lambda|_{\partial W})$.

DEFINITION 2.3. A time-dependent almost complex structure $J : S^1 \rightarrow \text{End}(T\widehat{W})$ is admissible if and only if the following holds.

- (i) J is compatible with $d\widehat{\lambda}$ on $\widehat{W}$.
- (ii) J is cylindrical convex on $\partial W \times (a_i, b_i)$.
- (iii) J is S^1 -independent on W .

The class of admissible almost complex structure is denoted by $\mathcal{J}(W)$.

For the Hamiltonian $\mathbf{H}$, we are almost in a Morse–Bott case. We adopt the cascades treatment [19] of the Morse–Bott situation, hence we pick any admissible Morse function f on W . Let $\mathcal{C}(f)$ denote the set of critical points of f . Let g be a metric on W . Then we define the following three moduli spaces.

$$M_{x,y} := \{u : \mathbb{R}_s \times S^1_t \rightarrow \widehat{W} \mid \partial_s u + J(\partial_t u - X_{\mathbf{H}}) = 0, \lim_{s \rightarrow -\infty} u = x, \lim_{s \rightarrow \infty} u = y\} / \mathbb{R}; \quad (2.2)$$

$$M_{p,q} := \{\gamma : \mathbb{R}_s \rightarrow W \mid \frac{d}{ds} \gamma + \nabla_g f = 0, \lim_{s \rightarrow -\infty} \gamma = p, \lim_{s \rightarrow \infty} \gamma = q\} / \mathbb{R}; \quad (2.3)$$

$$M_{p,y} := \left\{ u : \mathbb{C} \rightarrow \widehat{W}, \quad \begin{cases} \partial_s u + J(\partial_t u - X_{\mathbf{H}}) = 0, \partial_s \gamma + \nabla_g f = 0, \\ \gamma : (-\infty, 0] \rightarrow W \mid \lim_{s \rightarrow -\infty} \gamma = p, u(0) = \gamma(0), \lim_{s \rightarrow \infty} u = y, \end{cases} \right\} / \mathbb{R}; \quad (2.4)$$

where $p, q \in \mathcal{C}(f)$, $x, y \in \mathcal{P}^*(\mathbf{H})$. In (2.4), the $\mathbb{R}$ action is the dilation preserving 0, which under the polar coordinate $z = e^{2\pi(s+it)}$ is the translation in s . Since $\partial_r f > 0$ on ∂W , the Floer equation is well defined on $\mathbb{C}$, because the condition $u(0) = \gamma(0)$ implies $u(0) \in W^0$ (the interior of W), where $X_{\mathbf{H}} = 0$ near it. In particular, the Floer equation is the Cauchy–Riemann equation near $0 \in \mathbb{C}$. There will be no $M_{x,q}$, for otherwise the Floer part u will have negative energy.

REMARK 2.4. The Morse–Bott situation here is much simpler than [19], since we only have one nonisolated critical manifold W and its action is the maximum among all critical points. Therefore only three types of moduli spaces above need to be considered, that is, there is no cascades with multiple levels. Moreover, when one end is asymptotic to a point in W^0 , that is, in (2.4), the equation degenerates to the Cauchy–Riemann equation, hence there is no new analysis.

The Gromov–Floer compactification of them is standard and was considered in [19], the only place that needs attention is (2.4). In particular, the curve u in (2.4) could break at a point in W . Since we choose J to be convex near ∂W and $u(0) \in W^0$, then the integrated maximum principle below implies that the component breaking at a point in W is contained in W . But since W is exact, such configuration cannot exist in the compactification. Moreover, since J is cylindrical on $\partial W \times (a_i, b_i)$, the integrated maximum principle prevent curves from escaping to infinity. The following is a special form of the integrated maximum principle of Abouzaid–Seidel [3], we recall it from [18] and state it for strong fillings, since we will use it in §8.

LEMMA 2.5 [18, Lemma 2.2]. *Let (W, ω) be a strong filling of (Y, α) with completion $(\widehat{W}, \widehat{\omega})$. Let $H : \widehat{W} \rightarrow \mathbb{R}$ be a Hamiltonian such that $H = h(r)$ near $r = r_0$. Let J be a $\widehat{\omega}$ -compatible almost complex structure that is cylindrical convex on $Y \times (r_0, r_0 + \epsilon)$ for some $\epsilon > 0$. If both ends of a Floer cylinder u (that is, the u component in (2.2) and (2.4)) are contained inside $Y \times \{r_0\}$, then u is contained inside $Y \times \{r_0\}$. This also holds for H_s depending on $s \in \mathbb{R}$ if $\partial_s H_s \leq 0$ on $r > r_0$ and $H_s = h_s(r)$ on $(r_0, r_0 + \epsilon)$ such that $\partial_s(rh'_s(r) - h_s(r)) \leq 0$ and J_s is cylindrical convex on $Y \times (r_0, r_0 + \epsilon)$.*

The original statement of [18, Lemma 2.2] requires that W is exact and the conditions on H and J hold on a neighborhood of the hypersurface $r = r_0$. However, their proof is based on integration of the curve outside r_0 . Moreover precisely, we can choose a sequence of regular values $r_i > r_0$ of $\pi_{\mathbb{R}} \circ u$ such that $r_i \rightarrow r_0$. Then the integration argument in [18, Lemma 2.2] implies that the curve is inside every $r = r_i$ hypersurface as long the symplectic manifold is exact outside $r = r_0$ and the conditions on H and J hold on the $(r_0, r_0 + \epsilon)$ for some $\epsilon > 0$.

PROPOSITION 2.6. *The Gromov–Floer compactification of the moduli spaces (2.2)–(2.4) above are the following.*

- (i) $\mathcal{M}_{x,y} := \bigcup_{z_1, \dots, z_k \in \mathcal{P}^*(\mathbf{H})} M_{x,z_1} \times \dots \times M_{z_k,y}$, for $x, y \in \mathcal{P}^*(\mathbf{H})$.
- (ii) $\mathcal{M}_{p,q} := \bigcup_{r_1, \dots, r_k \in \mathcal{C}(f)} M_{p,r_1} \times \dots \times M_{r_k,q}$, for $p, q \in \mathcal{C}(f)$.
- (iii) $\mathcal{M}_{p,y} := \bigcup_{\substack{r_1, \dots, r_j \in \mathcal{C}(f) \\ z_1, \dots, z_k \in \mathcal{P}^*(\mathbf{H})}} M_{p,r_1} \times \dots \times M_{r_j,z_1} \times \dots \times M_{z_k,y}$, for $y \in \mathcal{P}^*(\mathbf{H})$ and $p \in \mathcal{C}(f)$.

Proof. The claim for (ii) is the standard result in Morse theory. In (i), there are no other breakings due to action reasons. Curves in $\mathcal{M}_{x,y}$ will not escape to infinity, since Lemma 2.5 can be applied to $r \in (a_i, b_i)$ for some i big enough. For (iii), there exists $\delta > 0$, such that on $\partial W \times [1 - \delta, 1]$ we have $\partial_r f > 0$. Then the curve u in (2.4) has the property that $u(0) \in W_\delta := W \setminus (\partial W \times [1 - \delta, 1])$. Then for $x \in \mathcal{P}^*(\mathbf{H})$, we can consider the following moduli space which will also be used later:

$$B_x := \left\{ u : \mathbb{C} \rightarrow \widehat{W} \mid \partial_s u + J(\partial_t u - X_H) = 0, \lim_{s \rightarrow \infty} u = x, u(0) \in W_\delta \right\} / \mathbb{R}. \quad (2.5)$$

Then we claim the Gromov–Floer compactification of B_x is given by

$$\mathcal{B}_x := \bigcup_{x_1, \dots, x_k \in \mathcal{P}^*(\mathbf{H})} B_{x_1} \times M_{x_1, x_2} \times \dots \times M_{x_k, x}.$$

This is because if there is a limit curve with a nontrivial component, u breaks in W . Since $\mathcal{A}_H$ is not Morse–Bott along ∂W , hence it may not be true that $\lim_{s \rightarrow \infty} u$ exists. However, we still have the limit set of u for $s \rightarrow \infty$ is contained in W . Since we can apply Lemma 2.5 to $r = 1$, hence u is contained in W and then the equation degenerates to the Cauchy–Riemann equation and u extends to $\mathbb{CP}^1$ by removal of singularity for J -curves. Due to the exactness assumption, there is no such nontrivial curve. It is clear that the compactness of $\mathcal{B}_x$ implies the claim on (iii). $\square$

If $c_1(W) = 0$, then there is a $\mathbb{Z}$ -grading assigned to elements of $\mathcal{C}(f)$, $\mathcal{P}^*(\mathbf{H})$ by

$$|p| = \text{ind } p, \quad p \in \mathcal{C}(f); \quad |x| = n - \mu_{CZ}(x), \quad x \in \mathcal{P}^*(\mathbf{H}).$$

The convention here is consistent with the grading rule in [52], that is, the unit will have grading 0. For each component of $\mathcal{M}_{a,b}$, we can assign a well-defined virtual dimension $\text{virdim } \mathcal{M}_{a,b} := |a| - |b| - 1$. In general, the above assignment defines a $\mathbb{Z}/2$ -grading.

REMARK 2.7. Most of the results in this paper except those in § 5 require a $\mathbb{Z}$ -grading, hence we will impose the $c_1(W) = 0$ condition in most situations. To get a $\mathbb{Z}$ -grading on symplectic cohomology generated by contractible orbits, one only needs $c_1(W)|_{\pi_2(W)} = 0$. One special case of such condition is $c_1(W)$ is torsion. All results in the paper holds for this generalized condition. However, it seems that such generalization does not add many new examples. Hence we state all results in the simplified condition $c_1(W) = 0$.

The following transversality is standard in symplectic cohomology. If we allow the Morse function to be generic, then transversality is easy to see. Using generic Morse function is

enough for this paper, but for simplicity, we show transversality for any admissible Morse function f , this is made possible by that the evaluation map on the universal moduli spaces is submersive [43, §3.4].

PROPOSITION 2.8. *For every admissible Morse function f , let g be a metric on W such that (f, g) is a Morse–Smale pair. Then there exists a subset $\mathcal{J}_{reg}^{\leq 1}(\mathbf{H}, f, g) \subset \mathcal{J}(W)$ of second category (in particular it is dense), such that for every $J \in \mathcal{J}_{reg}^{\leq 1}(\mathbf{H}, f, g)$, $\mathcal{M}_{x,y}$ is a compact manifold of dimension $|x| - |y| - 1$ with boundary when $|x| - |y| \leq 2$ for $x, y \in \mathcal{C}(f) \cup \mathcal{P}^*(\mathbf{H})$ and $\partial \mathcal{M}_{x,y} = \cup_{z \in \mathcal{C}(f) \cup \mathcal{P}^*(\mathbf{H})} \mathcal{M}_{x,z} \times \mathcal{M}_{z,y}$. Here ≤ 1 indicates that transversality holds for moduli spaces up to dimension 1.*

Proof. The situation for $\mathcal{M}_{p,q}, p, q \in \mathcal{C}(f)$ follows from the Morse–Smale assumption, and the situation for $\mathcal{M}_{x,y}, x, y \in \mathcal{P}^*(\mathbf{H})$ is standard on symplectic cohomology. Hence we need to prove transversality for $\mathcal{M}_{p,y}, y \in \mathcal{P}^*(\mathbf{H}), p \in \mathcal{C}(f)$. We consider the uncompactified moduli space B_y in (2.5). Using the standard argument by universal moduli spaces and u is somewhere injective [25], the universal moduli space

$$B_{y, \mathcal{J}^l} := \{u : \mathbb{C} \rightarrow \widehat{W}, J \in \mathcal{J}^l(W) | \partial_s u + J(\partial_t u - X_H) = 0, \lim_{s \rightarrow \infty} u = y, u(0) \in W^0\} / \mathbb{R}. \quad (2.6)$$

is cut out transversely, where $\mathcal{J}^l(W)$ is the Banach manifold of C^l admissible almost complex structures. Moreover, by [43, Proposition 3.4.2, Lemma 3.4.3], $ev_0 : B_{x, \mathcal{J}^l} \rightarrow W^0, u \mapsto u(0)$ is transverse to all stable manifolds of $\nabla_g f$. Then by the Sard–Smale theorem, there is a second category subset of $\mathcal{J}^l(W)$, such that the uncompactified $M_{p,y}$ is cut out transversely. Then using the argument of Taubes [25, Theorem 5.1], we get a second category subset $\mathcal{J}_{reg}^{\leq 1}(f, g)$ of smooth admissible almost complex structures where transversality holds[†]. Since the breaking in $\mathcal{M}_{p,y}$ is either a Morse breaking at a critical point of f or a Floer breaking at a nonconstant periodic orbit of H , in particular no new gluing analysis is required. This finishes the proof. $\square$

Moreover, $\mathcal{M}_{*,*}$ can be equipped with orientations following [24]. We use $\pm$ instead of $\cup$ to indicate the relation on orientations in a union. For example, $\mathcal{M}_{*,*}$ is oriented such that $\partial \mathcal{M}_{x,y} = \sum \mathcal{M}_{x,z} \times \mathcal{M}_{z,y}$, that is, the boundary orientation is the product orientation. We explain our orientation convention in the Appendix. In the following, we will use the coherent orientations without proof.

Using the almost complex structure J in Proposition 2.8, we can define the following cochain complexes by counting $\mathcal{M}_{*,*}$.

- (1) $C(\mathbf{H}, J, f)$ is a free $\mathbb{Z}$ -module generated by $\mathcal{P}^*(\mathbf{H}) \cup \mathcal{C}(f)$.
- (2) $C_0(\mathbf{H}, J, f)$ is a free $\mathbb{Z}$ -module generated by $\mathcal{C}(f)$.
- (3) $C_+(\mathbf{H}, J, f)$ is a free $\mathbb{Z}$ -module generated by $\mathcal{P}^*(\mathbf{H})$.

The differentials are defined by

$$dy = \sum_x \# \mathcal{M}_{x,y} x.$$

Since each Floer cylinder has nonnegative energy, we have that $\mathcal{M}_{x,y} \neq \emptyset$ implies $\mathcal{A}_H(x) \geq \mathcal{A}_H(y)$. Hence there are finitely many x such that $\mathcal{M}_{x,y} \neq \emptyset$ for a fixed y . In particular, the differential is well defined. $C_0(\mathbf{H}, J, f)$ is the Morse complex of f on W and a subcomplex of $C(\mathbf{H}, J, f)$, hence we abbreviate it to $C_0(f)$. We abbreviate the quotient complex $C_+(\mathbf{H}, J, f)$

[†]An alternative approach to get smooth almost complex structures is by considering smooth complex structure with derivative controlled by a sequence $\epsilon_0, \epsilon_1, \dots$, see [23, §5]. For results in this paper, using C^l almost complex structures is also sufficient.

to $C_+(\mathbf{H}, J)$ since it does not depend on f (even the regularity requirement for J to define $C_+(\mathbf{H}, J)$ does not depend on f).

REMARK 2.9. In this paper, the default coefficient is $\mathbb{Z}$, since usually the $\mathbb{Z}$ -coefficient theory carries more information. However, all results in this paper hold for any coefficient except for the results in § 8, where we need to use the Novikov coefficient over $\mathbb{Q}$.

By construction, there is a tautological short exact sequence,

$$0 \rightarrow C_0(f) \rightarrow C(\mathbf{H}, J, f) \rightarrow C_+(\mathbf{H}, J) \rightarrow 0, \quad (2.7)$$

We use d_0, d_+ to denote the differential on $C_0(f)$ and $C_+(\mathbf{H}, J)$, respectively. Let $d_{+,0}$ denote the map from $C_+(\mathbf{H}, J)$ to $C_0(f)$ in the definition of d for $C(\mathbf{H}, J, f)$. Then $d_{+,0}$ is a cochain map $C_+(\mathbf{H}, J) \rightarrow C_0(f)[1]$, and it induces the connecting map $\delta : H^*(C_+(\mathbf{H}, J)) \rightarrow H^{*+1}(C_0(f))$ in the induced long exact sequence. $d_{+,0}$ is defined by counting $\mathcal{M}_{p,y}$ for $y \in \mathcal{P}^*(\mathbf{H})$ and $p \in \mathcal{C}(f)$. Moreover, since the differential always increases action, for every $i \in \mathbb{N}$, we can define $C^{D_i}(\mathbf{H}, J, f)$ and $C_+^{D_i}(\mathbf{H}, J)$ to be the subcomplexes of $C(\mathbf{H}, J, f), C_+(\mathbf{H}, J)$ generated by orbits with action $\geq -D_i$, or equivalently those contained in W^i . Moreover, by Lemma 2.5, the curve appears in the differential of $C_+^{D_i}$ is contained in W^i . We will call it the length filtration, and in the exact case, it coincides the action filtration. Then we have

$$\varinjlim H^*(C^{D_i}(\mathbf{H}, J, f)) = H^*(C(\mathbf{H}, J, f)), \quad \varinjlim H^*(C_+^{D_i}(\mathbf{H}, J)) = H^*(C_+(\mathbf{H})),$$

similarly for the tautological long exact sequence.

The next proposition shows that what we defined is a model for the symplectic cohomology.

PROPOSITION 2.10. *There are isomorphisms $SH^*(W) \rightarrow H^*(C(\mathbf{H}, J, f))$ and $SH_+^*(W) \rightarrow H^*(C_+(\mathbf{H}, J))$, such that the following long exact sequences commutes*

$$\begin{array}{ccccccccc} \dots & \longrightarrow & H^*(W) & \longrightarrow & SH^*(W) & \longrightarrow & SH_+^*(W) & \longrightarrow & H^{*+1}(W) & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & H^*(C_0(f)) & \longrightarrow & H^*(C(\mathbf{H}, J, f)) & \longrightarrow & H^*(C_+(\mathbf{H}, J)) & \longrightarrow & H^{*+1}(C_0(f)) & \longrightarrow & \dots \end{array}$$

Proof. We prove the isomorphism using a continuation argument from a nondegenerate Hamiltonian. We use another Hamiltonian H , such that $H = \mathbf{H}$ on $\partial W \times [\rho, \infty)$, $H = h(r)$ with $h''(r) > 0$ on $\partial W \times [1, \rho]$ and $h'(\rho) < \min \mathcal{S}$. Moreover, $H \leq \mathbf{H}$ everywhere and H is a C^2 small admissible Morse function on W . Then for a generic choice of J_1 , the Floer cohomology of $\mathcal{A}_H$ defines $SH^*(W)$ and $SH_+^*(W)$. Then we choose a homotopy H_s such that $H_s = \mathbf{H}$ when $s < 0$, $H_s = H$ when $s > 1$ and $H_s = \mathbf{H} = H$ when $r \geq \rho$ and $\partial_s H_s \leq 0$. Let J_s be a homotopy of admissible almost complex structures such that $J_s = J, s < 0$ and $J_s = J_1, s > 1$. Let $\mathcal{P}(H)$ denote the set of contractible periodic orbits of H , then we have $\mathcal{P}^*(H) = \mathcal{P}^*(\mathbf{H})$ and $\mathcal{P}(H) = \mathcal{P}^*(H) \cup \mathcal{C}(H|_W)$. Then we define the following moduli spaces,

$$N_{x,y} := \{u \mid \partial_s u + J_s(\partial_t u - X_{H_s}) = 0, \lim_{s \rightarrow -\infty} u = x, \lim_{s \rightarrow \infty} u = y\}, \quad (2.8)$$

$$x \in \mathcal{P}^*(\mathbf{H}), y \in \mathcal{P}^*(H).$$

$$N_{p,y} := \left\{ u : \mathbb{C} \rightarrow \widehat{W}, \begin{array}{l} \partial_s u + J_s(\partial_t u - X_{H_s}) = 0, \partial_s \gamma + \nabla_g f = 0, \\ \gamma : (-\infty, 0] \rightarrow W \left| \begin{array}{l} \lim_{s \rightarrow -\infty} \gamma = p, u(0) = \gamma(0), \lim_{s \rightarrow \infty} u = y \end{array} \right. \end{array} \right\}, \quad (2.9)$$

$$p \in \mathcal{C}(f), y \in \mathcal{P}(H).$$

Since we assume $\partial_s H \leq 0$, $N_{x,y} \neq \emptyset$ implies that $\mathcal{A}_H(x) \geq \mathcal{A}_H(y)$ by energy reasons. Therefore if $x \in \mathcal{P}^*(H)$, $y \in \mathcal{C}(H|_W)$, then $N_{x,y} = \emptyset$. By the same reason in Proposition 2.6, we have the following compactification,

$$\mathcal{N}_{a,b} = \bigcup_{\substack{a_1 \in \mathcal{C}(f) \cup \mathcal{P}^*(H), \\ b_1 \in \mathcal{P}(H)}} \mathcal{M}_{a,a_1} \times N_{a_1,b_1} \times \mathcal{M}_{b_1,b}.$$

Similarly, we can find a generic J_s such that transversality holds for all moduli spaces of dimension ≤ 1 . This would yield a cochain map Φ from $C(H, J)$ to $C(H, J, f)$. Note that Φ preserves the length filtration and induces isomorphism on the first page of the spectral sequence. This is because on the action between $-D_{i+1}$ and $-D_i$ part, Φ induces identity on the first page, since $H_s = H = H$ there. This shows Φ is a quasi-isomorphism on the positive symplectic cohomology. On the part with action close to 0, the contribution to Φ is from $\mathcal{N}_{p,y}$ for $y \in \mathcal{C}(H|_W)$, then we can apply Lemma 2.5 to $r = \rho$. Hence everything is contained in $W \cup \partial W \times [1, \rho]$. Then following [25], for $H_s|_{W \cup \partial W \times [1, \rho]}$ sufficiently C^2 -small, there exists regular J_s independent of $t \in S^1$. Then the moduli space can be identified with the usual continuation map in Morse theory. This shows Φ is a quasi-isomorphism and induces the commutative diagram of exact sequences. $\square$

REMARK 2.11. If we consider the continuation map from $C(H, J, f)$ to $C(H, J_1)$, then we need to flow in the direction of $\nabla_g f$ from $p \in \mathcal{C}(f)$. Thus the degenerate ∂W may not be invisible anymore. This could cause problem in compactness and Fredholm setups. As pointed out in [19, §5], the difficulty can be overcome by choosing a particular homotopy.

2.2. Naturality of the construction

Since we will need to change almost complex structure in the neck-stretching process and compare things in two different domains, to keep track of the naturality of maps we write down, it is important to specify the choice of almost complex structure and the regularity requirement. To emphasize this aspect, we will always spell out the almost complex structure when needed.

In order to prove $d^2 = 0$, we need transversality for moduli spaces up to dimension 1. However, to define d_+ or d , we only need transversality for moduli spaces up to dimension 0. Therefore we have another two larger second category subsets of $\mathcal{J}(W)$: $\mathcal{J}_{\text{reg},+}(H) \supset \mathcal{J}_{\text{reg}}(H, f, g) \supset \mathcal{J}^{\leq 1}(H, f, g)$ so that d_+ and d are defined, respectively. Similarly, we define $\mathcal{J}_{\text{reg},+}^{D_i}(H)$, $\mathcal{J}_{\text{reg}}^{D_i}(H, f, g)$, that is, sets of admissible almost complex structures such that d_+, d are defined on $C_+^{D_i}(H, J)$ and $C^{D_i}(H, J, f)$, respectively. Due to the *a priori* energy control, $\mathcal{J}_{\text{reg}}^{D_i}$ are open.

PROPOSITION 2.12. *Let $J \in \mathcal{J}_{\text{reg},+}(H)$ or $\mathcal{J}_{\text{reg}}(H, f, g)$, then d_+, d are differentials, respectively.*

Proof. We will show that $d_+^2 = 0$ on $C_+^{D_i}(H, J)$, the other cases are similar. Since we have an *a priori* energy bound, by compactness, there exists an open neighborhood $\mathcal{U} \subset \mathcal{J}(W)$ of J , such that $\mathcal{U} \subset \mathcal{J}_{\text{reg},+}^{D_i}(H)$. Since $\mathcal{J}_{\text{reg}}^{\leq 1}(H, f, g)$ is dense, we choose $J' \in \mathcal{U} \cap \mathcal{J}_{\text{reg}}^{\leq 1}(H, f, g)$. Let $d_{+,J'}$ be the differential defined using J' , then $d_{+,J'}^2 = 0$. We claim $d_+ = d_{+,J'}$. Let $J_s, s \in [0, 1]$ be a smooth path connecting J and J' in $\mathcal{U}$, we can consider the moduli space $\cup_s \mathcal{M}_{x,y,J_s}$ for $|x| - |y| = 1$. The regularity of each J_s implies that it is a compact manifold with boundary $\mathcal{M}_{x,y,J}$ and $\mathcal{M}_{x,y,J'}$, since other boundary will involve $\mathcal{M}_{x',y',J_s}$ with $|x'| - |y'| \leq 0$, which is empty by regularity of J_s . This proves the claim. $\square$

REMARK 2.13. Using $\mathcal{J}_{reg}$ instead of $\mathcal{J}_{reg}^{\leq 1}$ is important in the neck-stretching argument. The index room provided by the ADC condition only allows us to argue that moduli spaces $\mathcal{M}_{x,y}$ up to dimension zero stays completely in the symplectization of the boundary and are regular after neck-stretching. This forces us to use $J \in \mathcal{J}_{reg}$ instead of $\mathcal{J}_{reg}^{\leq 1}$, see Remark 3.13 for details.

Next we recall the continuation map when varying J . Given $J_1, J_2 \in \mathcal{J}_{reg}(\mathbf{H}, f, g)$ and homotopy $J_s \in \mathcal{J}(W)$ such that $J_s = J_2$ for $s < 0$ and $J_s = J_1$ for $s > 1$, then we can define the following moduli spaces.

- (1) For $x, y \in \mathcal{P}^*(\mathbf{H})$, $\mathcal{N}_{x,y}$ is defined to be the compactification of the following

$$\{u \mid \partial_s u + J_s(\partial_t u - X_{\mathbf{H}}) = 0, \lim_{s \rightarrow -\infty} u = x, \lim_{s \rightarrow \infty} u = y\}.$$

- (2) For $p \in \mathcal{C}(f)$, $y \in \mathcal{P}^*(\mathbf{H})$, $\mathcal{N}_{p,y}$ is defined to be the compactification of the following

$$\left\{ u : \mathbb{C} \rightarrow \widehat{W}, \begin{cases} \partial_s u + J_s(\partial_t u - X_{\mathbf{H}}) = 0, \partial_s \gamma + \nabla_g f = 0, \\ \gamma : (-\infty, 0] \rightarrow W \mid \lim_{s \rightarrow -\infty} \gamma = p, u(0) = \gamma(0), \lim_{s \rightarrow \infty} u = y \end{cases} \right\}.$$

Then for generic choice of J_s , the moduli space $\mathcal{N}_{a,b}$ is a compact manifold with boundary of dimension $|a| - |b|$ whenever $|a| - |b| \leq 1$. The boundary configuration of $\mathcal{N}_{x,y}$ implies the following cochain map

$$\Phi_{J_s} : C^*(\mathbf{H}, J_1, f) \rightarrow C^*(\mathbf{H}, J_2, f), \quad \begin{cases} y \mapsto \sum_{\substack{|a|-|y|=0, \\ a \in \mathcal{P}^*(\mathbf{H}) \cup \mathcal{C}(f)}} \# \mathcal{N}_{a,y} a, & y \in \mathcal{P}^*(\mathbf{H}), \\ q \mapsto q, & q \in \mathcal{C}(f). \end{cases}$$

Similarly for $J_1, J_2 \in \mathcal{J}_{reg,+}(\mathbf{H})$, for generic choice of J_s , we defined the following cochain map.

$$\Phi_{J_s,+} : C_+^*(\mathbf{H}, J_1, f) \rightarrow C_+^*(\mathbf{H}, J_2, f), \quad y \mapsto \sum_{\substack{|x|-|y|=0, \\ y \in \mathcal{P}^*(\mathbf{H})}} \# \mathcal{N}_{x,y} x.$$

Then we have the following standard result on the naturality of the construction.

PROPOSITION 2.14. *Cochain morphisms Φ_{J_s} and $\Phi_{J_s,+}$ up to homotopy are independent of the choice of J_s . Moreover, it is functorial with respect to concatenation of homotopies up to homotopy.*

This proposition follows from a standard homotopy argument, cf. [6]. Since we are not varying $\mathbf{H}$, the analytic setups are similar to Propositions 2.6 and 2.8. Hence we omit the proof. By Proposition 2.14, we may suppress J_s in Φ, Φ_+ . Another observation is that $\Phi(C^{D_i}(\mathbf{H}, J, f)) \subset C^{D_i}(\mathbf{H}, J, f)$, similarly for $C_+^{D_i}(\mathbf{H}, J)$, we use $\Phi^{D_i}, \Phi_+^{D_i}$ to denote the restrictions, they also satisfy Proposition 2.14. Similar to Proposition 2.12, we only need regularity of J_s for moduli spaces up to dimension 0 to get well-defined $\Phi^{D_i}, \Phi_+^{D_i}$. The key property we need in the neck-stretching is the following.

LEMMA 2.15. *Assume that we have a smooth family $J_s : [0, 1] \rightarrow \mathcal{J}_{reg}^{D_i}(\mathbf{H}, f, g)$ or $\mathcal{J}_{reg,+}^{D_i}(\mathbf{H})$, then $\Phi^{D_i} : C^{D_i}(\mathbf{H}, J_a, f) \rightarrow C^{D_i}(\mathbf{H}, J_b, f)$ or $\Phi_+^{D_i} : C_+^{D_i}(\mathbf{H}, J_a, f) \rightarrow C_+^{D_i}(\mathbf{H}, J_b, f)$ are identities (up to homotopy) for any $0 \leq a, b \leq 1$, respectively.*

Proof. We prove the C^{D_i} case, as the $C_+^{D_i}$ case is similar. It is sufficient to prove the following: for any $a \in [0, 1]$, there exists a $\delta > 0$ such that for every $|b - a| < \delta$, we have Φ^{D_i} is homotopic to identity from the chain complex using J_a to the one using J_b . Let $\rho(s)$ be a smooth function

such that $\rho(s) = 0$ for $s \leq 0$ and $\rho(s) = 1$ for $s \geq 1$. Then we have a family of homotopies of almost complex structures $J_{\epsilon,s} = J_{a+\epsilon\rho(s)}$ for ϵ in a neighborhood of 0. In the following, by a regular homotopy of almost complex structures, we mean a homotopy of almost complex structures such that moduli spaces up to dimension 0 in the definition of continuation map are cut out transversely. Then $J_{0,s} \equiv J_a$ is a regular homotopy of almost complex structures and the continuous map is identity corresponding to constant cylinders over periodic orbits. Since there is a universal energy bound when restricted on C^{D_i} , by compactness, there exists $\delta > 0$ such that if $|\epsilon| < \delta$, $J_{\epsilon,s}$ is a regular homotopy. It is clear that the trivial cylinders on periodic orbits contribute to the continuation map. But there are no other contribution, due to compactness and that $J_{0,s}$ has no other contributions to the continuation map. $\square$

REMARK 2.16. From Proposition 2.12, d is constant on each J_s . Lemma 2.15 says the obvious identification is natural, that is, it is the continuation map, which has the functorial property in Proposition 2.14.

In Proposition 2.12, we know that d_+ on $C_+^{D_i}$ is defined using $J \in \mathcal{J}_{\text{reg},+}^{D_i}(\mathbf{H})$. We still need to verify that $d_{+,0}$ is well defined on $C_+^{D_i}$ for later application. Following the proof of Proposition 2.12, let $\mathcal{U} \subset \mathcal{J}(W)$ be an open neighborhood of J that is contained in $\mathcal{J}_{\text{reg},+}^{D_i}(\mathbf{H})$. Pick any $J' \in \mathcal{U} \cap \mathcal{J}_{\text{reg}}(\mathbf{H}, f, g)$, then $d_{+,0,J'}$ is defined, and by Proposition 2.12 it is a cochain map using d_+ defined by J since d_+ is locally constant.

PROPOSITION 2.17. $d_{+,0}$ above is well defined on $C_+^{D_i}$ up to homotopy.

Proof. Let $J'' \neq J' \in \mathcal{U} \cap \mathcal{J}_{\text{reg}}(\mathbf{H}, f, g)$, then there exists a regular homotopy J_s connecting J'' and J' , and the continuation map gives the following relation

$$\Phi_{+,0} \circ d_{+,J''} + d_{+,0,J''} = d_{+,0,J'} \circ \Phi_+ + d_0 \circ \Phi_{+,0}.$$

By Lemma 2.15 and Proposition 2.14, Φ_+ is homotopic to identity. By the argument in Proposition 2.12, $d_{+,J'} = d_{+,J''} = d_+$. This implies that $d_{+,0,J''}$ and $d_{+,0,J'}$ are homotopic as cochain maps from $C_+^D(\mathbf{H}, J) \rightarrow C_0(f)[1]$. $\square$

The above discussion shows that, down to an action lower bound, we only need regularity for moduli spaces up to dimension 0 to define differentials or cochain maps up to homotopy.

3. Invariance of restriction map and persistence of vanishing

In this section, we prove Theorem A, the strategy of the proof is contained in the following picture, which we will explain in detail.

3.1. Morse description of the restriction $H^*(W) \rightarrow H^*(\partial W)$

In this section, we assume admissible Morse function f satisfies $\partial_r f > 0$ on $\partial W \times [1 - \epsilon, 1]$. Let h be a Morse function on $\partial W \times \{1 - \epsilon\}$. We pick Riemannian metrics g, g_∂ on W and $\partial W \times \{1 - \epsilon\}$, respectively. Then we can define the following moduli spaces

$$R_{p,q} := \left\{ \begin{array}{l} \gamma_1 : (-\infty, 0] \rightarrow \partial W \times \{1 - \epsilon\} \\ \gamma_2 : [0, \infty) \rightarrow W \end{array} \left| \begin{array}{l} \frac{d}{ds} \gamma_1 + \nabla_{g_\partial} h = 0, \frac{d}{ds} \gamma_2 + \nabla_g f = 0, \\ \lim_{s \rightarrow -\infty} \gamma_1 = p, \gamma_1(0) = \gamma_2(0), \lim_{s \rightarrow \infty} \gamma_2 = q \end{array} \right. \right\},$$

where $p \in \mathcal{C}(h), q \in \mathcal{C}(f)$. By adding broken flow lines in W and on $\partial W \times \{1 - \epsilon\}$, we have that $R_{p,q}$ admits a compactification $\mathcal{R}_{p,q}$. Then we have the following.

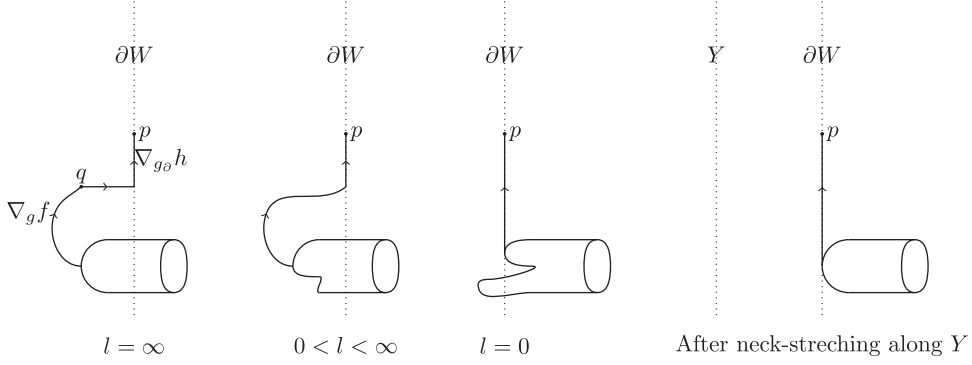


FIGURE 1. Pictorial proof of Theorem A.

PROPOSITION 3.1 [34, Proposition 2.2.8]. Given f, h as above, there exist generic metrics g, g_∂ , such that the unstable manifolds of $\nabla_g f$ intersect stable manifolds of $\nabla_{g_\partial} h$ transversely in W . Then $\mathcal{R}_{p,q}$ is a manifold with boundary of dimension $|p| - |q|$ if $|p| - |q| \leq 1$. When $\dim \mathcal{R}_{p,q} = 1$,

$$\partial \mathcal{R}_{p,q} = - \sum_{* \in \mathcal{C}(h)} \mathcal{M}_{p,*} \times \mathcal{R}_{*,q} + \sum_{* \in \mathcal{C}(f)} \mathcal{R}_{p,*} \times \mathcal{M}_{*,q}^\dagger$$

[†]Then counting $\mathcal{R}_{p,q}$ defines a cochain map $R : C(f) \rightarrow C(h)$ between Morse cochain complexes. And on cohomology, it is the restriction map $H^*(W) \rightarrow H^*(\partial W \times \{1 - \epsilon\}) = H^*(\partial W)$.

In the following, we fix (f, g, h, g_∂) such that Proposition 3.1 holds. Therefore on the complex level, the composition of $SH_+^*(W) \rightarrow H^{*+1}(W) \rightarrow H^{*+1}(\partial W)$ is given by the composition of $C_+(\mathbf{H}, J) \rightarrow C_0(f)[1] \rightarrow C(h)[1]$, with the map defined by counting the moduli space $\mathcal{R}_{p,q} \times \mathcal{M}_{q,y}$ for $y \in \mathcal{P}^*(\mathbf{H}), p \in \mathcal{C}(h), q \in \mathcal{C}(f)$, which is the $l = \infty$ part of Figure 1. Next we show that the middle $C_0(f)[1]$ can be bypassed. First, we define the following moduli space

$$P_{p,y} := \left\{ u : \mathbb{C} \rightarrow \widehat{W} \right. \left. \begin{array}{l} \partial_s u + J(\partial_t u - X_{\mathbf{H}}) = 0, \frac{d}{ds} \gamma + \nabla_{g_\partial} h = 0, \\ \gamma : (-\infty, 0] \rightarrow \partial W \times \{1 - \epsilon\} \left| \begin{array}{l} \lim_{s \rightarrow -\infty} \gamma = p, u(0) = \gamma(0), \lim_{s \rightarrow \infty} u = y \end{array} \right. \end{array} \right\} / \mathbb{R},$$

for $p \in \mathcal{C}(h), y \in \mathcal{P}^*(\mathbf{H})$, where the $\mathbb{R}$ -dilation acts on the u part. The equation makes sense, since on $\partial W \times \{1 - \epsilon\}$ we have $\mathbf{H} = 0$. Then we have a compactification $\mathcal{P}_{p,y}$ and when transversality holds, it defines a map from $C_+(\mathbf{H}, J) \rightarrow C(h)[1]$, which is the $l = 0$ part of Figure 1. In the following, we will show that it is homotopic to the composition $C_+(\mathbf{H}, J) \rightarrow C_0(f)[1] \rightarrow C(h)[1]$. To such purpose, we define the following moduli space for $p \in \mathcal{C}(h), y \in \mathcal{P}^*(\mathbf{H})$ involving finite time flow lines of $\nabla_g f$,

$$H_{p,y} := \left\{ \begin{array}{l} u : \mathbb{C} \rightarrow \widehat{W}, \\ l > 0, \\ \gamma_1 : (-\infty, 0] \rightarrow \partial W \times \{1 - \epsilon\}, \\ \gamma_2 : [0, l] \rightarrow W \end{array} \right. \left. \begin{array}{l} \partial_s u + J(\partial_t u - X_{\mathbf{H}}) = 0, \\ \frac{d}{ds} \gamma_1 + \nabla_{g_\partial} h = 0, \\ \frac{d}{ds} \gamma_2 + \nabla_g f = 0, \\ \lim_{s \rightarrow -\infty} \gamma_1 = p, \gamma_1(0) = \gamma_2(0), \\ u(0) = \gamma_2(l), \lim_{s \rightarrow \infty} u = y \end{array} \right\} / \mathbb{R},$$

[†]That is, the boundary orientation on $\mathcal{M}_{p,*} \times \mathcal{R}_{*,q}$ is the opposite of the product orientation and the boundary orientation on $\mathcal{R}_{p,*} \times \mathcal{M}_{*,q}$ is the product orientation.

where the $\mathbb{R}$ -dilation acts on the u part. In addition to configurations of breaking at an orbit in $\mathcal{P}^*(\mathbf{H})$ or a point in $\mathcal{C}(f)$, the compactification of $H_{p,y}$ also contains $\mathcal{R}_{p,q} \times \mathcal{M}_{q,y}$ corresponding to $l = \infty$, that is, a Morse breaking of the middle gradient flow line in W , and $\mathcal{P}_{p,y}$ corresponding to $l = 0$ on the nose. In particular, we have the following.

PROPOSITION 3.2. *$H_{p,y}$ has a compactification*

$$\mathcal{H}_{p,y} := H_{p,y} \cup_{q \in \mathcal{C}(f)} \mathcal{R}_{p,q} \times \mathcal{M}_{q,y} \cup_{q \in \mathcal{C}(f), x \in \mathcal{P}^*(\mathbf{H})} \mathcal{M}_{p,q} \times H_{q,x} \times \mathcal{M}_{x,y}.$$

For a generic choice of J , any p, y such that $|p| - |y| \leq 2$, we have $\mathcal{H}_{p,y}$ is a manifold with boundary of dimension $|p| - |y| - 1$. And

$$\partial \mathcal{H}_{p,y} = \sum_{q \in \mathcal{C}(f)} \mathcal{R}_{p,q} \times \mathcal{M}_{q,y} - \mathcal{P}_{p,y} + \sum_{x \in \mathcal{P}^*(\mathbf{H})} \mathcal{H}_{p,x} \times \mathcal{M}_{x,y} + \sum_{q \in \mathcal{C}(h)} \mathcal{M}_{p,q} \times \mathcal{H}_{q,y}.$$

To keep track of the regularity of almost complex structures, we introduce the following notations.

- (1) $\mathcal{J}_{\text{reg},P}^{\leq 1}(\mathbf{H}, h, g_\partial)$ is the set of regular admissible J for moduli spaces $\mathcal{P}_{p,y}$ of dimension up to 1. And $\mathcal{J}_{\text{reg},P}^{D_i}(\mathbf{H}, h, g_\partial)$ is the set of regular admissible J for moduli spaces $\mathcal{P}_{p,y}$ of dimension up to 0 and action down to $-D_i$.
- (2) $\mathcal{J}_{\text{reg},H}^{\leq 1}(\mathbf{H}, f, g, h, g_\partial)$ is the set of regular admissible J for moduli spaces $\mathcal{H}_{p,y}$ of dimension up to 1.

Then all of them are of second category, and $\mathcal{J}_{\text{reg}}^{D_i}$ are open. By looking at the potential boundary configurations, we have various relations among $\mathcal{J}_{\text{reg}}$, for example, $\mathcal{J}_{\text{reg},P}^{\leq 1}(\mathbf{H}, h, g_\partial) \subset \mathcal{J}_{\text{reg},+}^{D_i}(\mathbf{H})$, $\mathcal{J}_{\text{reg},H}^{\leq 1}(\mathbf{H}, f, g, h, g_\partial) \subset \mathcal{J}_{\text{reg}}^{D_i}(\mathbf{H}, f, g) \cap \mathcal{J}_{\text{reg},P}^{D_i}(\mathbf{H}, h, g_\partial)$.

An instant corollary of Proposition 3.2 is that if $J \in \mathcal{J}_{\text{reg}}^{\leq 1}(\mathbf{H}, f, g) \cap \mathcal{J}_{\text{reg},P}^{\leq 1}(\mathbf{H}, f, g) \cap \mathcal{J}_{\text{reg},H}^{\leq 1}(\mathbf{H}, f, g, h, g_\partial)$, then the composition of $C_+(\mathbf{H}, J) \rightarrow C_0(f)[1] \rightarrow C(h)[1]$ is homotopic to $\tilde{P}: C_+(\mathbf{H}, J) \rightarrow C(h)[1]$ defined by counting $\mathcal{P}_{p,y}$. The following proposition is in the same spirit of Proposition 2.17. Since P is defined on $C_+^{D_i}$ if $J \in \mathcal{J}_{\text{reg},P}^{D_i}(\mathbf{H}, h, g_\partial)$ and $d_{+,0}$ is defined if $J \in \mathcal{J}_{\text{reg},+}^{D_i}(\mathbf{H})$, the following proposition shows that they are the same up to homotopy for such J of low regularity.

PROPOSITION 3.3. *Let $J \in \mathcal{J}_{\text{reg},+}^{D_i}(\mathbf{H}) \cap \mathcal{J}_{\text{reg},P}^{D_i}(\mathbf{H}, h, g_\partial)$, then P is defined. In this case, $R \circ d_{+,0}$ is homotopic to P on $C_+^{D_i}(\mathbf{H}, J, f)$. P is compatible with continuation maps on $C_+^{D_i}$ up to homotopy, that is, the following is commutative up to homotopy,*

$$\begin{array}{ccc} C_+(\mathbf{H}, J_1) & \xrightarrow{P} & C(h)[1] \\ \downarrow \Phi_+ & & \downarrow = \\ C_+(\mathbf{H}, J_2) & \xrightarrow{P} & C(h)[1]. \end{array}$$

Proof. There exists an open neighborhood $\mathcal{U} \subset \mathcal{J}(W)$ of J contained in $\mathcal{J}_{\text{reg},+}^{D_i}(\mathbf{H}) \cap \mathcal{J}_{\text{reg},P}^{D_i}(\mathbf{H}, h, g_\partial)$. Then using $J' \in \mathcal{U} \cap \mathcal{J}_{\text{reg}}^{\leq 1}(\mathbf{H}, f, g) \cap \mathcal{J}_{\text{reg},P}^{\leq 1}(\mathbf{H}, f, g) \cap \mathcal{J}_{\text{reg},H}^{\leq 1}(\mathbf{H}, f, g, h, g_\partial)$, we have that P defines a cochain map homotopic to $R \circ d_{+,0,J'}$. By Proposition 2.17, $d_{+,0,J'}$ is well defined up to homotopy for different J' . Finally, similar to Proposition 2.12, d_+, P are locally constant. The compatibility with continuation map is standard, where the homotopy is defined by considering moduli spaces similar to $\mathcal{P}_{*,*}$ but with an s -dependent almost complex structure and without quotienting the $\mathbb{R}$ -action (which does not exist). $\square$

All the above discussions about naturality lead to the following proposition that will be used in the proof of Theorem A.

PROPOSITION 3.4. *Let $J^i \in \mathcal{J}_{reg,+}^{D_i}(\mathbf{H}) \cap \mathcal{J}_{reg,P}^{D_i}(\mathbf{H}, h, g_\partial)$, then the direct limit of the following computes $SH_+^*(W) \rightarrow H^{*+1}(\partial W)$:*

$$\left\{ H^*(C_+^{D_1}(\mathbf{H}, J_1)) \rightarrow H^*(C_+^{D_2}(\mathbf{H}, J_2)) \rightarrow \dots \right\} \rightarrow H^{*+1}(\partial W),$$

where the horizontal arrows in the bracket are continuation maps and maps $H^*(C_+^{D_i}(\mathbf{H}, J_i)) \rightarrow H^{*+1}(\partial W)$ are defined by P .

Proof. Pick a regular enough J , such that the direct limit of

$$\left\{ H^*(C_+^{D_1}(\mathbf{H}, J)) \rightarrow H^*(C_+^{D_2}(\mathbf{H}, J)) \rightarrow \dots \right\} \rightarrow H^{*+1}(\partial W)$$

computes $SH_+^*(W) \rightarrow H^{*+1}(\partial W)$. By Proposition 3.3 and functoriality of continuation maps, this diagram is isomorphic to the sequence in the claim by continuation maps. $\square$

3.2. Independence

Asymptotically dynamically convex (ADC) contact manifolds was introduced in [37]. Let (Y, ξ) be a $2n - 1$ dimensional contact manifold with a contact form α , then we define

$$\mathcal{P}^{<D}(\alpha) := \left\{ \gamma \mid \gamma \text{ is a contractible Reeb orbit and } \int_\gamma \alpha < D \right\}.$$

If $c_1(\xi) = 0$, then for any contractible nondegenerate Reeb orbit x , there is an associated Conley–Zehnder index $\mu_{CZ}(x) \in \mathbb{Z}$. The degree of x is defined to be $\deg(x) := \mu_{CZ}(x) + n - 3$.

DEFINITION 3.5. Let (Y, ξ) be a contact manifold. Y is called k -ADC if and only if there exists a sequence of contact form $\alpha_1 > \dots > \alpha_i > \dots$ and real numbers $D_1 < \dots < D_i < \dots \rightarrow \infty$, such that all (contractible) Reeb orbits in $\mathcal{P}^{<D_i}(\alpha_i)$ are nondegenerate and have degree greater than k . Y is called strongly k -ADC if, in addition, all Reeb orbits of α_i with period smaller than D_i are contractible. We will abbreviate (strongly) 0-ADC manifolds to (strongly) ADC manifolds.

REMARK 3.6. In general, when $c_1(\xi) = 0$, choosing a trivialization of $\det_{\mathbb{C}} \xi$ will assign a Conley–Zehnder index to every Reeb orbit. The Conley–Zehnder index of γ is independent of trivializations if γ is annihilated by $H^1(Y; \mathbb{Z}) = [Y, S^1]$ (which is the space of trivializations up to homotopy), that is, $\beta(\gamma) = 0$ for any $\beta \in H^1(Y; \mathbb{Z})$ in the pairing $H^1(Y; \mathbb{Z}) \otimes H_1(Y; \mathbb{Z}) \rightarrow \mathbb{Z}$. The reason of considering topologically simple filling W is to make sure a Reeb orbit γ contractible in W is assigned with a well-defined Conley–Zehnder index using only the boundary. From this point of view, one can consider a slight generalization of ADC manifolds and their topologically simple fillings, that is, $\mathcal{P}^{<D}$ now stands for orbits annihilated by $H^1(Y; \mathbb{Z})$ with period $< D$ and topologically simple filling now requires $H^1(W; \mathbb{Z}) \rightarrow H^1(Y; \mathbb{Z})$ is injective and $c_1(W) = 0$. Most results in the paper hold in such setting too.

EXAMPLE 3.7. Contact manifolds admitting flexible Weinstein fillings with vanishing first Chern class are ADC by the work of Lazarev [37] and are strongly ADC for subcritically fillable contact manifolds. Moreover, Lazarev showed that ADC is preserved under flexible surgery[†].

[†]Some extra conditions need to be satisfied when attaching a 2-handle, cf. [37, Theorem 3.17].

Cotangent bundle T^*M is ADC whenever $\dim M \geq 4$. Quantization bundles over a positive monotone symplectic manifold are ADC, whenever the degree of the bundle is not too big.

In the following, we introduce an analogous definition for filling, which will be used in the construction of ADC manifolds in § 6. Assume contact manifold (Y, ξ) has a symplectic filling W with $c_1(W) = 0$. Let x be a nondegenerate Reeb orbit, if x is contractible in W , then a canonical Conley–Zehnder index can be assigned.

DEFINITION 3.8. Let (W, λ) be a Liouville domain with $c_1(W) = 0$. W is called k -ADC if there exist positive functions on ∂W $f_1 > \dots > f_i > \dots$ and real numbers $D_1 < \dots < D_i < \dots \rightarrow \infty$, such that all contractible (in W) Reeb orbits of $(\partial W, f_i \lambda)$ with period smaller than D_i are nondegenerate and have degree greater than k . W is called strongly k -ADC if, in addition, all Reeb orbits of $f_i \lambda_i$ with period smaller than D_i are contractible in W . In particular, if W is ADC, then ∂W is also ADC.

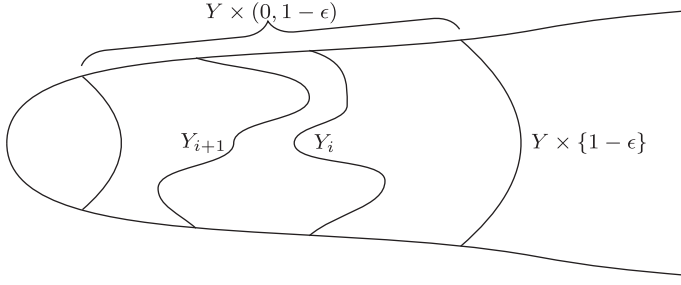
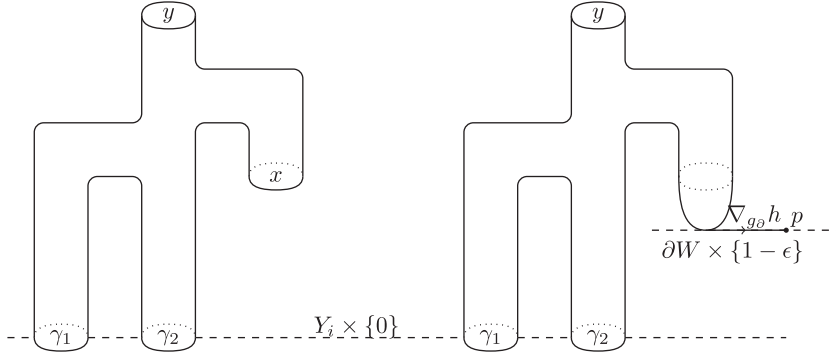
EXAMPLE 3.9. In § 6, we show that $V \times \mathbb{C}$ is always ADC for any Liouville domain V with $c_1(V) = 0$ and $\dim V > 0$. $V \times W$ is always ADC, given V, W are both ADC Liouville domains of dimension ≥ 4 . They provide more examples of ADC contact manifolds.

EXAMPLE 3.10. It is possible that ∂W is ADC but W is not ADC, the source of this discrepancy is that we can have Reeb orbits that are noncontractible in the boundary but are contractible in the filling and have low SFT degree. For example, S^1 is ADC, but the $\mathbb{D}$ disk is not ADC.

REMARK 3.11. Dynamical convexity was introduced in [31] on (S^3, ξ_{std}) as a substitute of the geometric convexity. For (S^{2n-1}, ξ_{std}) , a contact form is dynamical convex if the minimal Conley–Zehnder index is $n + 1$, as it is the case for convex hypersurfaces in $\mathbb{R}^{2n}$. Note that this is the lowest degree that is nontrivial in the cylindrical contact homology of (S^{2n-1}, ξ_{std}) . Following this idea, Abreu–Macarini [4] defined dynamical convex for a larger class of contact manifolds, as a property of contact forms. Although with similar names, Lazarev’s asymptotically dynamical convexity has a very different motivation, which is a generalization of index-positive in [18] and is related to the existence of nice contact forms introduced in [22]. It is clear that ADC is equivalent to $\sup_{D_1 < D_2 < \dots \rightarrow \infty} (\inf_{x \in \mathcal{P}^{< D_i}(\alpha_i), i \in \mathbb{N}} \deg(x)) > 0$. Such number is an invariant of the contact topology. A similar number was defined by McLean [46] and shown to be equal to twice the minimal discrepancy for a large class of isolated singularities, when it is nonnegative. In particular, the link is ADC if the singularity is terminal.

From the first glance of $\mathcal{P}_{p,y}$, $\mathcal{P}_{p,y}$ has some chance to be independent of the filling since both p, y only depend on the contact boundary. However, the curve u in $\mathcal{P}_{p,y}$ may rely on the filling. If we have the ADC property, a neck-stretching argument implies that $\mathcal{P}_{p,y}$ actually does not see the interior of W . Neck-stretching argument was used to show independence of $SH_+^*(W)$ in [18] for index-positive convex manifolds. It is easy to show $H^*(C_+^{D_i}(\mathbf{H}, J)) \rightarrow H^{*+1}(\partial W)$ is independent of filling for any D_i by neck-stretching. But we also need some naturality of the independence. In the case of independence of $SH_+^*(W)$, naturality was discussed carefully in [37, Proposition 3.8] for ADC manifolds. In the following, we give a simplified treatment. Since in our case $\mathbf{H}$ will not change and is already constant on W , we can bypass the Viterbo transfer map in the proof of [37, Proposition 3.8].

Let (Y, α) be an ADC contact manifold with two topologically simple fillings W_1, W_2 with fixed Hamiltonians $\mathbf{H}_1 = \mathbf{H}_2 = \mathbf{H}$ outside W_1, W_2 as in § 2. Note that W_1, W_2 both contain the negative end of symplectization $(Y \times (0, 1), d(r\alpha))$. Since Y is ADC, for every $i \in \mathbb{N}_+$, there

FIGURE 2. $Y_i \subset \widehat{W}_*$ FIGURE 3. Moduli spaces for the definition of $\mathcal{J}_{\text{reg}, SFT}^{D_i}(\mathbf{H}, h, g_\partial)$.

exist contact type surfaces $Y_i \subset Y \times (0, 1 - \epsilon) \subset W_1, W_2$, such that Y_i lies outside of Y_{i+1} and contractible Reeb orbits of contact form $r\alpha|_{Y_i}$ has the property that the degree of an orbit is greater than 0 if the period of the orbit is smaller than D_i , see Figure 2.

Neck-stretching near Y_i is given by the following. Assume $Y_i \times [1 - \epsilon_i, 1 + \epsilon_i]_{r_i}$ does not intersect each other for some small ϵ_i . Assume $J|_{Y_i \times [1 - \epsilon_i, 1 + \epsilon_i]_{r_i}} = J_0$, where J_0 is independent of S^1 and r_i and $J_0(r_i \partial_{r_i}) = R_i, J_0 \xi_i = \xi_i$, where $\xi_i = \ker r\alpha|_{Y_i}$. Then we pick a family of diffeomorphism $\phi_R : [(1 - \epsilon_i)e^{1 - \frac{1}{R}}, (1 + \epsilon_i)e^{\frac{1}{R} - 1}] \rightarrow [1 - \epsilon_i, 1 + \epsilon_i]$ for $R \in (0, 1]$ such that $\phi_1 = \text{id}$ and ϕ_R near the boundary is linear with slope 1. Then the stretched almost complex structure $NS_{i,R}(J)$ is defined to be J outside $Y_i \times [1 - \epsilon_i, 1 + \epsilon_i]$ and is $(\phi_R \times \text{id})_* J_0$ on $Y_i \times [1 - \epsilon_i, 1 + \epsilon_i]$. Then $NS_{i,1}(J) = J$ and $NS_{i,0}(J)$ gives almost complex structures on the completions of the cobordism between Y and Y_i , the filling of Y_i and the symplectization $Y_i \times \mathbb{R}_+$. Since we need to stretch along different contact surfaces, we assume the $NS_{i,R}(J)$ have the property that $NS_{i,R}(J)$ will modify the almost complex structure near Y_{i+1} to a cylindrical almost complex structure for R from 1 to $\frac{1}{2}$ and for $R \leq \frac{1}{2}$, we only keep stretching along Y_i . We use $\mathcal{J}_{\text{reg}, SFT}^{D_i}(\mathbf{H}, h, g_\partial)$ to denote the set of admissible regular J , that is, almost complex structures satisfying Definition 2.3 on the completion of the cobordism between Y and Y_i and asymptotic (in a prescribed way as in the stretching process) to J_0 on the negative cylindrical end, such that the following two moduli spaces in Figure 3 up to dimension 0 with action of the positive end $\geq -D_i$ are cut out transversely. It is an open dense set.

PROPOSITION 3.12. *With setups above, there exist admissible $J_1^1, J_1^2, \dots, J_2^1, J_2^2, \dots$ on $\widehat{W}_1$ and $\widehat{W}_2$, respectively, and positive real numbers $\epsilon_1, \epsilon_2, \dots < \frac{1}{2}$ such that the following holds.*

(i) For $R < \epsilon_i$ and any R' , $NS_{i,R}(J_*^i), NS_{i+1,R'}(NS_{i,R}(J_*^i)) \in \mathcal{J}_{\text{reg},+}^{D_i}(\mathbf{H}) \cap \mathcal{J}_{\text{reg},P}^{D_i}(\mathbf{H}, h, g_\partial)$. Moreover, for such almost complex structures, all zero-dimensional

moduli spaces $\mathcal{M}_{x,y}$ and $\mathcal{P}_{p,y}$ are the same for both W_1, W_2 and contained outside Y_i for $x, y \in \mathcal{P}^*(\mathbf{H})$ with action $\geq -D_i$, $q \in C(h)$.

(ii) $J_*^{i+1} = NS_{i, \frac{\epsilon_i}{2}}(J_*^i)$ on W_*^i .

Proof. We prove the proposition by induction. We first choose a J^1 such that $NS_{1,0}(J^1) \in \mathcal{J}_{\text{reg}, SFT}^{D_1}(\mathbf{H}, h, g_\partial)$. Assume $\mathcal{M}_{x,y}$ is not contained outside Y_1 in the stretching process. Then a limit curve u outside Y_1 has one component by [18, Lemma 2.4][†]. Moreover, by the argument in [18, Lemma 2.4], u can only be asymptotic to Reeb orbits γ_j that are contractible in W_* on Y_1 with period smaller than D_1 . Since W_* is topological simple, γ_j is contractible in Y_1 . Note that $\text{ind}(u) = |y| - |x| - \sum(\mu_{CZ}(\gamma_j) + n - 3) < 1$ as $\mu_{CZ}(\gamma_j) + n - 3 > 0$. Since we need to quotient the $\mathbb{R}$ -action, the expected dimension of the moduli space of such u is negative. By our regularity assumption on $NS_{1,0}(J^1)$, u is cut out transversely, which contradicts its existence. Hence for R close to 0, using $NS_{1,R}(J^1)$, we have $\mathcal{M}_{x,y}$ lives outside Y_1 . Since $NS_{1,0}(J^1) \in \mathcal{J}_{\text{reg}, SFT}^{D_1}(\mathbf{H}, h, g_\partial)$ and every curve in $\mathcal{M}_{x,y}$ lives outside Y_1 , we have $NS_{1,R}(J^1) \in \mathcal{J}_{\text{reg}, +}^{D_1}(\mathbf{H})$ for R small. Similarly, we have the same property for $\mathcal{P}_{x,q}$ and $NS_{1,R}(J^1) \in \mathcal{J}_{\text{reg}, H}^{D_1}(\mathbf{H}, h, g_\partial)$ for R small. The argument can be applied to stretching on both Y_1, Y_2 and a compactness argument shows that for every $R' \in [0, 1]$, there exists $\epsilon_{R'} > 0$ and $\delta_{R'} > 0$ such that the same regular and outside property hold for $NS_{2,\delta}(NS_{1,\epsilon})(J^1)$ for $\epsilon < \epsilon_{R'}$ and $|\delta - R'| < \delta_{R'}$. Therefore Claim (i) holds for some ϵ_1 . Since $NS_{2,0}(NS_{1,0}(J^1))$ has the property that all curves in Figure 3 with $x, y \in C_+^{D_1}$ must be contained outside Y_1 , that is, curves in $\mathcal{M}_{x,y}, \mathcal{P}_{p,y}$ (which are viewed as limit curves after stretching), then we may assume ϵ_1 is small enough such that $NS_{2,0}(NS_{1, \frac{\epsilon_1}{2}}(J^1)) \in \mathcal{J}_{\text{reg}, SFT}^{D_1}(\mathbf{H}, h, g_\partial)$. Therefore we can perturb $NS_{1, \frac{\epsilon_1}{2}}(J^1)$ outside W_*^1 near orbits in $W_*^2 \setminus W_*^1$ to J^2 such that $NS_{2,0}(J^2) \in \mathcal{J}_{\text{reg}, SFT}^{D_2}(\mathbf{H}, h, g_\partial)$, this will not influence the previous regular property for periodic orbits with action down to $-D_1$ by the integrated maximum principle (Lemma 2.5). Then we can keep the induction going. It is clear the construction can be made on both W_1, W_2 yielding the same $\mathcal{M}_{x,y}$ and $\mathcal{P}_{p,y}$. $\square$

REMARK 3.13. From the proof above, it is clear that we cannot guarantee $NS_{i,R}(J^i)$ in $\mathcal{J}_{\text{reg}, +}^{\leq 1}(\mathbf{H})$ for all R small unless we assume 1-ADC. Moreover, there is no guarantee for $NS_{i,R}(J^i)$ in $\mathcal{J}_{\text{reg}}(\mathbf{H}, f, g)$, that is why we need Propositions 2.17 and 3.3.

Proof of Theorem A. Using the almost complex structures from Proposition 3.12, we have the following sequence for both fillings

$$\left\{ C_+^{D_1}(NS_{1, \frac{\epsilon_1}{2}} J^1) \rightarrow C_+^{D_2}(NS_{2, \frac{\epsilon_2}{2}} J^2) \rightarrow C_+^{D_3}(NS_{3, \frac{\epsilon_3}{2}} J^3) \dots \right\} \rightarrow C(h)[1], \quad (3.1)$$

where each complex and the map P to $C(h)[1]$ are identified with each other for both fillings. Therefore it suffices to show the continuation map $C_+^{D_i}(NS_{i, \frac{\epsilon_i}{2}} J^i) \rightarrow C_+^{D_{i+1}}(NS_{i+1, \frac{\epsilon_{i+1}}{2}} J^{i+1})$ is naturally identified. The continuation map is decomposed into continuation maps

$$\Phi : C_+^{D_i}(NS_{i, \frac{\epsilon_i}{2}} J^i) \rightarrow C_+^{D_i}(NS_{i+1, \frac{\epsilon_{i+1}}{2}}(NS_{i, \frac{\epsilon_i}{2}} J^i))$$

and

$$\Psi : C_+^{D_i}(NS_{i+1, \frac{\epsilon_{i+1}}{2}}(NS_{i, \frac{\epsilon_i}{2}} J^i)) \rightarrow C_+^{D_{i+1}}(NS_{i+1, \frac{\epsilon_{i+1}}{2}} J^{i+1}).$$

Then Φ is identity by Lemma 2.15 using homotopy $NS_{i+1,s}(NS_{i, \frac{\epsilon_i}{2}})(J^i)$ for $s \in [\frac{\epsilon_{i+1}}{2}, 1]$. Since J^{i+1} is the same as $NS_{i, \frac{\epsilon_i}{2}}(J^i)$ inside W^i , then the integrated maximum principle implies that Ψ is composition $C_+^{D_i}(NS_{i+1, \frac{\epsilon_{i+1}}{2}}(NS_{i, \frac{\epsilon_i}{2}} J^i)) \xrightarrow{\text{id}} C_+^{D_i}(NS_{i+1, \frac{\epsilon_{i+1}}{2}} J^{i+1}) \hookrightarrow$

[†]Note that our symplectic action has the opposite sign compared to [18, Proposition 9.17].

$C_+^{D_{i+1}}(NS_{i+1, \frac{\epsilon_{i+1}}{2}} J^{i+1})$, which is the inclusion, hence it is the same for both W_1, W_2 . Therefore continuation maps in (3.1) are inclusions, hence the whole diagram can be identified, and Proposition 3.4 implies the theorem. $\square$

Proof of Corollary B. If $SH^*(W) = 0$, then $1 \in \text{im } \delta_\partial$. Then by Theorem A, for any other topologically simple exact filling W' , we have $1 \in \text{im } \delta_\partial$. Then $SH^*(W') = 0$ and Theorem A implies the invariance of $H^*(W; \mathbb{Z}) \rightarrow H^*(Y; \mathbb{Z})$. $\square$

Using Theorem A, we derive the following obstruction to Weinstein fillability.

COROLLARY 3.14. *Let Y be a $2n - 1$ dimensional ADC contact manifold and $n \geq 3$. If Y admits a topologically simple exact filling W such that a nontrivial element of grading greater than n is in the image of $SH_+^*(W) \rightarrow H^{*+1}(Y)$, then Y does not admit Weinstein fillings.*

Proof. Assume otherwise that Y admits a Weinstein filling W' . Since $n \geq 3$ and W' is built from Y by attaching index $k \geq n \geq 3$ handles, we have $c_1(W') = 0$ and $\pi_1(Y) \rightarrow \pi_1(W')$ is isomorphism, that is, W' is topologically simple. Then $SH_+^*(W') \rightarrow H^{*+1}(W') \rightarrow H^{*+1}(Y)$ is isomorphic to $SH_+^*(W) \rightarrow H^{*+1}(Y)$. Since $H^*(W')$ is supported in degree $\leq n$, we arrive at a contradiction. $\square$

When the obstruction in Corollary 3.14 does not vanish for one contact manifold, it is easy to construct infinitely many obstructed examples by the following.

PROPOSITION 3.15. *Let Y be a contact manifold, such that conditions in Corollary 3.14 hold. For any ADC contact manifold Y' with a topologically simple exact filling W' , the contact connected sum $Y \# Y'$ is not Weinstein fillable.*

Proof. By assumption, the image of $SH_+^*(W) \rightarrow H^{*+1}(Y)$ contains an element α' of grading $k \geq n + 1$. That is, the image of $SH_+^*(W) \rightarrow H^{*+1}(W)$ contains an element α , which restricts to α' . That is, α is mapped to 0 in $SH^k(W)$. Note that k cannot be $2n - 1$, for otherwise it will imply that $H^{2n-1}(W; \mathbb{R}) \rightarrow H^{2n-1}(Y; \mathbb{R})$ is surjective, which contradicts the Stokes' theorem. By [15], if we view $\alpha \in H^k(W \natural W')$, then α is mapped to 0 in $SH^k(W \natural W')$. This implies that α is in the image of $SH_+^{k-1}(W \natural W') \rightarrow H^k(W \natural W')$. Since $n + 1 \leq k < 2n - 1$, α restricted to the boundary $Y \# Y'$ is represented by α' and nonzero. This implies that δ_∂ contains a nontrivial element of degree k for $W \natural W'$. By [37], $Y \natural Y'$ is ADC, and it is direct to check that $W \natural W'$ is topologically simple. Then by Corollary 3.14, $Y \# Y'$ is not Weinstein fillable. $\square$

3.3. Symplectic cohomology for covering spaces

Before proving Theorem E, we need to introduce symplectic cohomology for covering spaces. Let W be an exact domain and $\pi : \widehat{W} \rightarrow W$ a covering (not necessarily connected). The idea is lifting all geometric data to the covering space to define a Floer theory. We define the following two sets:

$$\begin{aligned} \widetilde{\mathcal{P}}^*(\mathbf{H}) &:= \{(x, a) | x \in \mathcal{P}^*(\mathbf{H}), a \in \widehat{\widehat{W}}, \pi(a) = x(0)\}, \\ \widetilde{\mathcal{C}}(f) &:= \{(p, a) | p \in \mathcal{C}(f), a \in \widehat{\widehat{W}}, \pi(a) = p\}. \end{aligned}$$

Then we can define $\mathcal{M}_{(x,a),(y,b)}$ for $(x, a), (y, b) \in \widetilde{\mathcal{P}}^*(\mathbf{H}), \widetilde{\mathcal{C}}(f)$ as follows, if $(x, a), (y, b) \in \widetilde{\mathcal{P}}^*(\mathbf{H})$,

$$\mathcal{M}_{(x,a),(y,b)} := \{u \in \mathcal{M}_{x,y} | u(s, 0) \text{ lifts to a path from } a, b \text{ in } \widehat{\widehat{W}}\},$$

other cases are similar. Hence for regular J , $\mathcal{M}_{(x,a),(y,b)}$ is always diffeomorphic to some connected components of $\mathcal{M}_{x,y}$ with the same orientation. In particular, by Proposition 2.8, $\mathcal{M}_{(x,a),(y,b)}$ is always a compact manifold with boundary of dimension $|x| - |y| - 1$ when $|x| - |y| \leq 2$ and $\partial\mathcal{M}_{(x,a),(y,b)} = \sum_{(z,c)} \mathcal{M}_{(x,a),(z,c)} \times \mathcal{M}_{(z,c),(y,b)}$. Then we can use them to define a cochain complex $\tilde{C}^*(\mathbf{H}, J, f)$. However, the generator set is infinite even with an action bound, and since we are trying to define the cohomology of the covering space, the cochain complex is the *direct product* in the fiber direction, that is, $\oplus_x \prod_a \mathbb{Z}\langle(x, a)\rangle$. The differential on a generator is again defined by

$$d(y, b) = \sum_{(x, a)} \#\mathcal{M}_{(x,a),(y,b)}(x, a).$$

The compactness of $\mathcal{M}_{x,y}$ implies that for any b , there are at most finitely many a such that $\mathcal{M}_{(x,a),(y,b)} \neq \emptyset$. Therefore the differential is well defined on the complex. We can similarly define $\tilde{C}_+^*(\mathbf{H}, J), \tilde{C}_0^*(f)$. By definition, the cohomology $H^*(\tilde{C}_0^*(f))$ is the cohomology of the universal cover $H^*(\tilde{W})$, since the cochain complex is dual to the Morse homology complex with $\mathbb{Z}[\pi_1(W)]$ local system, which computes the homology of the universal cover [29, §3.H]. Moreover, there is always a cochain morphism $C_{(+/0)}^*(\mathbf{H}, J, f) \rightarrow \tilde{C}_{(+/0)}^*(\mathbf{H}, J, f)$ defined by sending $x \rightarrow \prod_a(x, a)$ corresponding to the pull-back to the covering. We define $SH^*(\tilde{W}), SH_+^*(\tilde{W})$ to be the cohomology of $\tilde{C}^*(\mathbf{H}, J, f), \tilde{C}_+^*(\mathbf{H}, J)$. Then we have a long exact sequence,

$$\dots \rightarrow H^*(\tilde{W}) \rightarrow SH^*(\tilde{W}) \rightarrow SH_+^*(\tilde{W}) \rightarrow H^{*+1}(\tilde{W}) \rightarrow \dots$$

Moreover, the natural maps $SH_{(+/0)}^*(W) \rightarrow SH_{(+/0)}^*(\tilde{W})$ are compatible with the long exact sequence. We can lift everything discussed before to $\tilde{W}$, hence there is the following analogue of Theorem A.

PROPOSITION 3.16. *Under the same assumption as in Theorem A, let $\pi : \tilde{W} \rightarrow W$ be a covering, then $\tilde{Y} := \partial\tilde{W}$ is a covering of Y and $\tilde{\delta}_\partial : SH_+^*(\tilde{W}) \rightarrow H^{*+1}(\tilde{Y})$ is independent of the topologically simple exact filling W' and covering $\tilde{W}'$, as long as $\partial\tilde{W}' = \tilde{Y}$.*

THEOREM 3.17. *Assume Y is an ADC contact manifold. If W is a topologically simple exact filling of Y such that $\pi_1(Y) \rightarrow \pi_1(W)$ is isomorphism and $SH^*(W) = 0$, then for any other topological simple exact filling W' , we have $\pi_1(Y) \rightarrow \pi_1(W')$ is an isomorphism.*

Proof. Assume $\pi(Y) \rightarrow \pi(W')$ is only injective. Then the universal cover $\tilde{W}'$ restricted to boundary is $\pi_1(W')/\pi_1(Y)$ copies of the universal cover $\tilde{Y}$. Let $\tilde{W}$ be the universal cover of W . Since $\pi_1(Y) \rightarrow \pi_1(W)$ is an isomorphism, we have $\partial\tilde{W} = \tilde{Y}$. Since $SH_+^{-1}(\tilde{W}) \rightarrow H^0(\tilde{Y}) = \mathbb{Z}$ is surjective, $SH_+^{-1}(W) \rightarrow H^0(Y)$ is an isomorphism and the following commutative diagram

$$\begin{array}{ccccc} SH_+^{-1}(W) & \longrightarrow & H^0(W) & \longrightarrow & H^0(Y) \\ \downarrow & & \downarrow & & \downarrow \\ SH_+^{-1}(\tilde{W}) & \longrightarrow & H^0(\tilde{W}) & \longrightarrow & H^0(\tilde{Y}). \end{array}$$

Since $\pi_1(W')/\pi_1(Y)$ copies of the universal cover $\tilde{W}$ restricted to boundary is also $\pi_1(W')/\pi_1(Y)$ copies of $\tilde{Y}$, then by Proposition 3.16, we have that $SH_+^{-1}(\tilde{W}') \rightarrow H^0(\partial\tilde{W}') = \prod_{\pi_1(W')/\pi_1(Y)} \mathbb{Z}$ is surjective. However, $SH_+^{-1}(\tilde{W}') \rightarrow H^0(\partial\tilde{W}')$ factors through $H^0(\tilde{W}') = \mathbb{Z}$, which contradicts that the cardinality of $\pi_1(W')/\pi_1(Y)$ is greater than 1. $\square$

Proof of Theorem E. It is sufficient to prove the strongly ADC case, as the ADC case is proven in Theorem 3.17. Since $\pi_1(Y) \rightarrow \pi_1(W)$ is an isomorphism, we have $H_1(Y) \rightarrow H_1(W)$ is an isomorphism. By the universal coefficient theorem, we have $H^1(W) \rightarrow H^1(Y)$ is an isomorphism and $H^2(W) \rightarrow H^2(Y)$ is an isomorphism on the torsion part. By Corollary B, we have the same thing holds on W' . Then using the universal coefficient theorem again, we have $H_1(Y) \rightarrow H_1(W')$ is an isomorphism. Since $\pi_1(Y)$ is abelian, this implies that $\pi_1(Y) \rightarrow \pi_1(W')$ is injective, and then by Theorem 3.17, we have $\pi_1(Y) \rightarrow \pi_1(W')$ is an isomorphism. $\square$

3.4. Local systems

Albers–Frauenfelder–Oancea [5] found examples of cotangent bundles with vanishing symplectic cohomology after appropriately twisting by local systems. Since cotangent bundles are ADC, when base manifolds have dimension at least 4, we discuss the analogue of Theorem A for local systems and its applications to symplectic topology in this part. The discussion here works for general local systems; however, to get interesting applications, we will only consider the following special local systems. We use L_0W to denote the connected component of contractible loops in the free loop space LW and let R be a commutative ring.

DEFINITION 3.18. An admissible local system on a Liouville domain W is a flat R bundle over L_0W and is trivial on the constant loops.

Let $R^\times$ denote the group of units in R , then using parallel transportation, a flat R bundle over L_0W is represented by a class in $\text{Hom}(\pi_1(LW), R^\times)/R^\times$, where the $R^\times$ action is given by conjugation. We use Ω_0W to denote the space of contractible-based loops in W . Note that we have a short exact sequence of groups,

$$0 \rightarrow \pi_1(\Omega_0W) \rightarrow \pi_1(L_0W) \rightarrow \pi_1(W) \rightarrow 1$$

from the fibration $\pi : L_0W \rightarrow W$, where π is the evaluation map at the starting point. Moreover, we have

$$\pi_1(L_0W) \simeq \pi_1(\Omega_0W) \rtimes \pi_1(W),$$

where $\pi_1(W)$ acts on $\pi_1(\Omega_0W)$ by conjugation. As a consequence, we have

$$\text{Hom}(\pi_1(L_0W), R^\times) \simeq \text{Hom}_{\text{inv}}(\pi_2(W), R^\times) \times \text{Hom}(\pi_1(W), R^\times),$$

where $\text{Hom}_{\text{inv}}(\pi_2(W), R^\times) \subset \text{Hom}(\pi_2(W), R^\times)$ is the subgroup of $\pi_1(W)$ -invariant elements. The projection to the $\text{Hom}(\pi_1(W), R^\times)$ corresponds to the restriction on constant loops. Therefore an admissible local system is represented by an element of $\text{Hom}_{\text{inv}}(\pi_2(W), R^\times)/R^\times$, in other words, it can be represented by a trivial extension of a $\pi_1(W)$ -invariant local system on Ω_0W .

With an admissible local system ρ , we can define symplectic cohomology with local system ρ , where the cochain complex is generated by fibers $\rho_x \simeq R$ of the local system over $x \in \mathcal{P}^*(H)$ or constant loop $x \in \mathcal{C}(f)$. The differential now needs to take the parallel transportation into account, that is,

$$da = \sum_x \sum_{u \in \mathcal{M}_{x,y}} \text{sign}(u) \rho_u(a), \quad a \in \rho_y, \quad (3.2)$$

where $\mathcal{M}_{x,y}$ is a zero-dimensional moduli space, and ρ_u an R -module map $\rho_y \rightarrow \rho_x$, that is, the parallel transportation determined by u when we view u as a path in L_0W connecting y and x . We use $SH^*(W; \rho)$ to denote the cohomology, such theory shares all properties of $SH^*(W)$. In particular, we have the following proposition from the same argument of Theorem A.

PROPOSITION 3.19. *Let Y be an ADC manifold, then $\delta_{\partial,\rho} : SH_+^*(W; \rho) \rightarrow H^{*+1}(W; R) \rightarrow H^{*+1}(Y; R)$ is independent of topologically simple fillings W and ρ , as long as $\rho|_{L_0 Y}$ is fixed.*

To illustrate the necessity of the ADC condition, we consider the cotangent circle bundle $ST^*S^2 \simeq \mathbb{RP}^3$, then the minimal Conley–Zehnder index is 1, which makes it not ADC. By [5], there is a local system on $L_0 T^*S^2$, such that $SH^*(T^*S^2, \rho) = 0$. However, the local system on $L_0 \mathbb{RP}^3$ is trivial, hence it has a trivial extension to $L_0 T^*S^2$, whose symplectic cohomology is not zero. As a consequence, $\delta_{\partial,\rho}$ depends on ρ in this non-ADC case.

To make Proposition 3.19 useful as Theorem A, we will explain that $SH^*(W; \rho)$ is a ring and $H^*(W; R) \rightarrow SH^*(W; \rho)$ is a ring map. The product structure on symplectic cohomology is defined by counting holomorphic curves on 3-punctured spheres, with two positive puncture and one negative puncture, cf. [52]. We fix a trivialization of ρ on constant loops. Given any such curve u with positive ends asymptotic to x, y and negative end asymptotic to z , it determines a module morphism $\rho_u : \rho_x \otimes_R \rho_y \rightarrow \rho_z$ as follows:

$$\rho_x \otimes_R \rho_y \xrightarrow{\rho_{u_1} \otimes_R \rho_{u_2}} \rho_{x'} \otimes_R \rho_{y'} \xrightarrow{m} \rho_{x' \star y'} \xrightarrow{\rho_{u_3}} \rho_z, \quad (3.3)$$

where x', y' are the based loops with the same base point as the following picture shows, $x' \star y'$ is the concatenation of x' and y' , u_1, u_2, u_3 are the cylinders represented by the colored region in Figure 4 and $m : \rho_{x'} \otimes_R \rho_{y'} \rightarrow \rho_{x' \star y'}$ is the conical isomorphism in [5, Lemma 1] if we fix a

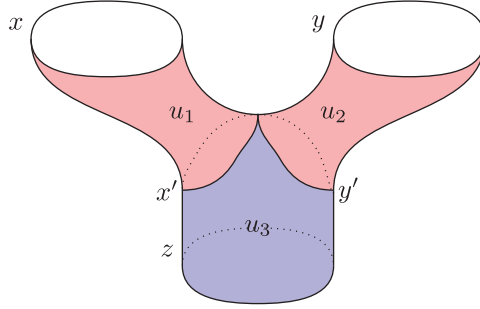


FIGURE 4 (colour online). Parallel transportation on pair of pants trivialization of ρ on constant loops.

PROPOSITION 3.20. *Given a pair-of-pants curve u , ρ_u in (3.3) is well defined and only depends on the homotopy class of u .*

Proof. Since the merging of x, y into $x' \star y'$ is not unique, the potential ambiguity follows from different ways of merging. More explicitly, if we have two choices of merged based loops x', y' and x'', y'' , then there are two cylinders u_x, u_y from x', y' to x'', y'' , respectively, and $u_x(s, 0) = u_y(s, 0), \forall s$. We can form a concatenation of cylinders $u_x \star u_y$ from $x' \star y'$ to $x'' \star y''$. Then it suffices to prove the following commutative diagram:

$$\begin{array}{ccc} \rho_{x'} \otimes_R \rho_{y'} & \xrightarrow{m} & \rho_{x' \star y'} \\ \downarrow \rho_{u_x} \otimes_R \rho_{u_y} & & \downarrow \rho_{u_x \star u_y} \\ \rho_{x''} \otimes_R \rho_{y''} & \xrightarrow{m} & \rho_{x'' \star y''} \end{array} \quad (3.4)$$

Let p be the common base point of x', y' , that is, $p = x'(0) = y'(0)$, we recall m from [5, Lemma 1],

$$m : \rho_{x'} \otimes \rho_{y'} \rightarrow \rho_p \otimes \rho_p \rightarrow \rho_p \rightarrow \rho_{x' \star y'},$$

where the first map is determined by two paths $\gamma_{x'}, \gamma_{y'}$ of based loops from x', y' to the constant loop p , the middle map is determined by the conical isomorphism $R \otimes_R R \rightarrow R$ since we fix a trivialization over constant loops and the last map is determined by the inverse of the concatenation of $\gamma_{x'}$ and $\gamma_{y'}$, which is a path from p to $x' \star y'$ in the based loop space. By the homotopy lifting property of $L_0 W \rightarrow W$, there exists a lift $\gamma_{x', x''}$ of $u_x(s, 0)$ extending $\gamma_{x'}$, similarly for y', y'' , then we have the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & \xrightarrow{m} & & & \\
 \rho_{x'} \otimes_R \rho_{y'} & \xrightarrow{\quad} & \rho_{x'}(0) \otimes_R \rho_{y'}(0) & \xrightarrow{\quad} & \rho_{x'}(0) & \xrightarrow{\quad} & \rho_{x' \star y'} \\
 \downarrow \rho_{\gamma_{x', x''}} \otimes_R \rho_{\gamma_{y', y''}} & & \downarrow & & \downarrow & & \downarrow \rho_{\gamma_{x', x''} \star \gamma_{y', y''}} \\
 \rho_{x''} \otimes_R \rho_{y''} & \xrightarrow{\quad} & \rho_{x''}(0) \otimes_R \rho_{y''}(0) & \xrightarrow{\quad} & \rho_{x''}(0) & \xrightarrow{\quad} & \rho_{x'' \star y''} \\
 & & & \xleftarrow{m} & & &
 \end{array}$$

$\gamma_{x', x''}$ may not be homotopic to u_x . But to prove (3.4), it suffices to prove the following commutative diagram

$$\begin{array}{ccc}
 \rho_{x''} \otimes_R \rho_{y''} & \xrightarrow{\quad} & \rho_{x'' \star y''} \\
 \downarrow \rho_v \otimes_R \rho_w & & \downarrow \rho_{v \star w} \\
 \rho_{x''} \otimes_R \rho_{y''} & \xrightarrow{\quad} & \rho_{x'' \star y''}
 \end{array} \tag{3.5}$$

where v and w are two loops in $L_0 W$ such that $v(s, 0) = w(s, 0)$ for $s \in S^1$. Note that the local system is trivial on W and we can modify v by elements in $\text{im}(\pi_1(W) \rightarrow \pi_1(L_0 W))$ from the inclusion of constant loops without changing ρ_v . Therefore we can assume $v(s, 0) = x''(0)$ for all $s \in S^1$. We may assume the same thing for w . Let $\gamma_{x''}, \gamma_{y''}$ denote the path in $\Omega_0 W$ connecting x'', y'' to $x''(0) = y''(0)$. Then it suffices to prove the following commutative diagram

$$\begin{array}{ccc}
 \rho_{x''}(0) \otimes_R \rho_{y''}(0) & \xrightarrow{\quad} & \rho_{x''}(0) \\
 \downarrow \rho_{\gamma_{x''} v \gamma_{x''}^{-1}} \otimes_R \rho_{\gamma_{y''} w \gamma_{y''}^{-1}} & & \downarrow \rho_{(\gamma_{x''} \star \gamma_{y''})(v \star w)(\gamma_{x''} \star \gamma_{y''})^{-1}} \\
 \rho_{x''}(0) \otimes_R \rho_{y''}(0) & \xrightarrow{\quad} & \rho_{x''}(0)
 \end{array} \tag{3.6}$$

Therefore it suffices to prove that $\rho_{\gamma_{x''} v \gamma_{x''}^{-1} \gamma_{y''} w \gamma_{y''}^{-1}} = \rho_{(\gamma_{x''} \star \gamma_{y''})(v \star w)(\gamma_{x''} \star \gamma_{y''})^{-1}}$. Therefore it is enough to verify that $\gamma_{x''} v \gamma_{x''}^{-1} \gamma_{y''} w \gamma_{y''}^{-1}$ is homotopic to $(\gamma_{x''} \star \gamma_{y''})(v \star w)(\gamma_{x''} \star \gamma_{y''})^{-1}$ as a loop in $L_0 W$. Note that $(\gamma_{x''} \star \gamma_{y''})(v \star w)(\gamma_{x''} \star \gamma_{y''})^{-1} = (\gamma_{x''} v \gamma_{x''}^{-1}) \star (\gamma_{y''} w \gamma_{y''}^{-1})$ and both $(\gamma_{x''} v \gamma_{x''}^{-1}), (\gamma_{y''} w \gamma_{y''}^{-1})$ represent a based loop of based loops in $\Omega \Omega_0 W$, where the base is the constant loop $x''(0)$. Then the claim follows from the same proof for the fact that second homotopy groups are abelian. Note that (3.4) also implies that ρ_u only depends on the homotopy class of u . $\square$

As a consequence of Proposition 3.20, we can count pair-of-pants moduli spaces twisted by ρ_u to define a ring structure on $SH^*(W; \rho)$. That $H^*(W; R) \rightarrow SH^*(W; \rho)$ is a ring map follows from the same argument for untwisted symplectic cohomology, see [52].

Proof of Theorem D. We use local systems with $R = \mathbb{C}$. The assumption implies that $\dim Q \geq 3$, in particular, $\pi_1(ST^*Q) \rightarrow \pi_1(T^*Q)$ is an isomorphism. We first assume that Q is oriented and spin. Following [5], we consider local systems on $L_0 W$ trivial on W induced from $\mathbb{Z}/p \subset \mathbb{C}^*$, which are represented by $\text{Hom}_{\text{inv}}(\pi_2(W), \mathbb{Z}/p) \simeq H_{\text{inv}}^2(\widetilde{W}, \mathbb{Z}/p)$, where $\widetilde{W}$ is the universal cover of W and $\pi_1(W)$ acts on the cohomology by deck transformation. By [5, Proposition 6, 9], $H^2(W) \xrightarrow{\pi^*} H_{\text{inv}}^2(\widetilde{W})$ contains nontrivial images if and only if the Hurewicz

map $\pi_2(W) \rightarrow H_2(W)$ is nonzero. In the case of (i), by [5, Theorem 1], there exists an admissible local system ρ such that $SH^*(T^*Q; \rho) = 0$. Moreover, such local system is represented by an element in the image of $H^2(T^*Q; \mathbb{Z}/p) \rightarrow H_{inv}^2(\widetilde{T^*Q}; \mathbb{Z}/p)$ for some prime p . Then the local system on L_0ST^*Q is represented by the restriction to $H_{inv}^2(\widetilde{ST^*Q}; \mathbb{Z}/p)$ and it is from an element of $H^2(ST^*Q; \mathbb{Z}/p)$. By assumption, $H^2(W; \mathbb{Z}/p) \rightarrow H^2(ST^*Q; \mathbb{Z}/p)$ is surjective, which implies that $\rho|_{L_0ST^*Q}$ extends to W . Hence the claim follows from Proposition 3.19. If Q is nonorientable or not spin, given a local system ρ , we have $SH^*(T^*Q, \rho \otimes \sigma^{TQ}) = H_{n-*}(LQ, \rho \otimes o_Q)$ [1, Chapter 3], where o_Q is the orientation local system on Q and σ^{TQ} is an admissible local system defined in [1, Chapter 2], which is transgressed from $H^2(Q, \mathbb{Z}/2)$ (in the oriented not spin case, it is transgressed from the second Stiefel–Whitney class). Hence by assumption, this local system σ^{TQ} can be extended to W . Hence the invariance follows similarly as we can find an admissible local system ρ such that $H_{n-*}(LQ, \rho \otimes o_Q) = 0$ by [5, Theorem 1].

For case (ii), if Q is spin, there exists some prime p such that $H^2(W; \mathbb{Z}/p) \rightarrow H^2(ST^*Q; \mathbb{Z}/p)$ is nonzero. Since when $\pi_1(Q) = 0$, any nontrivial element of $H^2(T^*Q; \mathbb{Z}/p) = H^2(ST^*Q; \mathbb{Z}/p)$ determines a local system with vanishing symplectic cohomology, then the claim follows. For not spin manifold, σ^{TQ} is the transgression of the second Stiefel–Whitney class, hence it can also be extended to W by our condition.

If $\chi(Q) = 0$, then we have a section $s : Q \rightarrow ST^*Q$ and $Q \xrightarrow{s} ST^*Q \rightarrow T^*Q$ is a homotopy equivalence. Therefore the invariance of $H^*(W; \mathbb{C}) \rightarrow H^*(ST^*Q; \mathbb{C})$ implies that $H^*(W; \mathbb{Q}) \rightarrow H^*(ST^*Q; \mathbb{Q}) \xrightarrow{s^*} H^*(Q; \mathbb{Q})$ is an isomorphism. In particular, the composition $Q \xrightarrow{s} ST^*Q \hookrightarrow W$ induces a rational homotopy equivalence. $\square$

REMARK 3.21. It is possible to get invariance of $\mathbb{Z}/p$ cohomology for $p > 2$ under the similar assumptions as in Theorem D. We can consider $\mathbb{Z}/p$ -local systems that are trivial on constant loops, such local system is classified by $\text{Hom}(\pi_1(L_0W), (\mathbb{Z}/p)^\times)/(\mathbb{Z}/p)^\times$. Since for any $g \neq 1 \in (\mathbb{Z}/p)^\times$, we have $1 - g \in (\mathbb{Z}/p)^\times$, then [5, Proposition 3] applies to get vanishing of symplectic cohomology.

3.5. Diffeomorphism type

In some situations, the invariance of cohomological information $H^*(W; \mathbb{Z}) \rightarrow H^*(Y; \mathbb{Z})$ would yield information about the diffeomorphism type via h-cobordism. Such argument was used extensively in [7]. In the following, we extract some of the topological conditions from [7] such that the h-cobordism argument can be used. In particular, combining with the results from previous sections, we get the uniqueness of diffeomorphism type for many cases.

DEFINITION 3.22. An oriented manifold W with boundary is good if the following conditions hold.

- (i) There exists W_0 diffeomorphic to W and is contained in the collar neighborhood of ∂W .
- (ii) There exists a locally closed manifold $V \subset \partial W$, such that $V \rightarrow \partial W \rightarrow W$ induces isomorphism on cohomology and the corresponding copy of V_0 on W_0 is homotopic to V in the collar neighborhood of ∂W .

EXAMPLE 3.23. The major source of good domains is the following.

- (i) $W = DT^*Q$ for manifold Q with $\chi(Q) = 0$. Then there exists a section $s : Q \rightarrow ST^*Q$. Then we may take $V = s(Q)$.
- (ii) $W = M \times \mathbb{D}$. Then V may be taken as $M \times \{1\}$.

PROPOSITION 3.24. *Let W be a good domain with boundary Y . If Y has a another filling W' such that $H^*(W') \rightarrow H^*(Y)$ is isomorphic to $H^*(W) \rightarrow H^*(Y)$, then $W' = W \cup H$, where H is a homology cobordism from Y to Y .*

Proof. In the collar neighborhood of Y in W' , we have a W_0 diffeomorphic to W . By deleting this W_0 , we have cobordism X from Y to Y . We claim $H^*(W') \rightarrow H^*(W_0)$ is an isomorphism. Let $F : V \times [0, 1]$ be the homotopy from V to V_0 in the collar neighborhood. This follows from the following commutative diagram

$$\begin{array}{ccccc} H^*(W') & \xrightarrow{F^*} & H^*(V \times [0, 1]) & \xrightarrow{\simeq} & H^*(V) \\ \downarrow & & & \searrow \simeq & \\ H^*(W_0) & \longrightarrow & H^*(\partial W_0) & \longrightarrow & H^*(V_0). \end{array}$$

By assumption, we have $H^*(W_0) \rightarrow H^*(\partial W_0) \rightarrow H^*(V_0)$ is an isomorphism and the first arrow $H^*(W') \xrightarrow{F^*} H^*(V \times [0, 1]) \rightarrow H^*(V)$ is identified with $H^*(W') \rightarrow H^*(Y) \rightarrow H^*(V)$, hence an isomorphism. Therefore, $H^*(W') \rightarrow H^*(W_0)$ is an isomorphism. Then by excision, we have $H^*(W', W_0) = H^*(X, \partial W_0) = 0$. Then the universal coefficient and Poincaré duality implies $H_*(X, \partial W_0) = H_*(X, \partial W') = 0$. Therefore X is a homology cobordism. $\square$

Once we know that Y is simply connected and the cobordism X in Proposition 3.24 is simply connected, the homology cobordism becomes h -cobordism by the relative Hurewicz theorem. Therefore, when $\dim X \geq 6$, we have X is diffeomorphic to $Y \times [0, 1]$. In particular, we can prove Theorem C.

Proof of Theorem C. Contact manifolds in the statement of Theorem C are ADC, where case (i) follows from Theorem 6.3. It follows from Corollary B and Proposition 3.24 that different fillings are differed by homology cobordisms. Note that by Theorem 3.17, we have W is simply connected. Then the homology-cobordism X from Proposition 3.24 is simply connected by the Van Kampens Theorem. In particular, X is an h -cobordism. Therefore the diffeomorphism type is unique. $\square$

4. Persistence of symplectic dilation

Symplectic cohomology is naturally equipped rich algebraic structures, in particular, $SH^*(W)$ is a BV algebra with a degree -1 BV operator Δ , cf. [58]. With such structure, symplectic dilation was introduced in [58] as an element x of $SH^1(W)$, such that $\Delta(x) = 1$.

EXAMPLE 4.1. If $SH^*(W) = 0$, then 0 is a symplectic dilation. If W admits a symplectic dilation and M be another Liouville domain, then $W \times M$ admits a symplectic dilation by [56, Example 2.6].

EXAMPLE 4.2 [58, Example 6.4]. T^*S^2 admits a dilation with field coefficient $\mathbf{k}$ if $\text{char}(\mathbf{k}) \neq 2$, and T^*S^n admits dilation for any coefficient when $n \geq 3$. $T^*\mathbb{CP}^n$ also admits dilation if $\text{char}(\mathbf{k}) \neq 2$. $T^*K(\pi, 1)$ does not admit dilation, in particular there is no dilation on T^*T^n .

More examples with symplectic dilations are constructed by Lefschetz fibrations, since by [58, Proposition 7.3], if the fiber F of a $2n$ -dimensional Lefschetz fibration $M \rightarrow \mathbb{C}$ contains a dilation for $n > 2$, then so does M . By repeatedly applying this result, Seidel [56, Example 2.13] showed that the Milnor fiber of a singularity of form $p = z_0^2 + z_1^2 + z_2^2 + q(z_3, \dots, z_n)$ admits symplectic dilations.

EXAMPLE 4.3. The link of $p = z_0^2 + z_1^2 + z_2^2 + z_3^{a_3} + \cdots + z_n^{a_n} = 0$ with $2 \leq a_i \in \mathbb{N}$ is ADC. The associated Milnor fiber provides examples with ADC boundaries admitting symplectic dilations. The contact boundary admits a Morse–Bott contact form whose generalized Conley–Zehnder indices were computed in [35, §5.3.1]. The Reeb orbits are completely classified by two natural numbers T, N . Let I_T be the maximal subset of $I := \{0, 1, \dots, n\}$ such that $\text{lcm}_{i \in I_T} a_i = T$ and I_T has at least two elements. There are also some restrictions on N , see [35] for detail. Then there is a $2(\#I_T - 2)$ -dimensional family of Reeb-orbits with period NT and generalized Conley–Zehnder index

$$2 \sum_{i \in I_T} \frac{NT}{a_i} + 2 \sum_{i \in I - I_T} \lfloor \frac{NT}{a_i} \rfloor + \#(I - I_T) - 2NT. \quad (4.1)$$

Then the minimal Conley–Zehnder index after a small perturbation is (4.1) minus $\#I_T - 2$. When T is even, the Conley–Zehnder index of a small perturbation is at least $NT + 2N(\#I_T - 3) + \#I - 2\#I_T + 2$, hence the degree is at least $NT + 2N(\#I_T - 3) + 2\#I - 2\#I_T - 2$ which is positive if $N > 1$. When $N = 1$, it is least $2\#I - 6 \geq 2$. When $T \geq 3$ is odd, the degree is at least $2N\#I_T + 3NT - 3 + 2\#I - 2\#I_T - 2 - 2NT \geq NT + 2\#I - 5 > 0$. Therefore such manifold is ADC.

Like the vanishing of symplectic cohomology, the existence of symplectic dilation is also preserved under the Viterbo transfer map. That is, let $V \subset W$ be a Liouville subdomain, then the Viterbo transfer map $SH^*(W) \rightarrow SH^*(V)$ preserves the BV structure. In particular, if W admits a symplectic dilation, so does V . Therefore, the existence of symplectic dilation may be viewed as an indication of the complexity of the Liouville domain, which is next to the vanishing of symplectic cohomology. The goal of this section is to prove for ADC contact manifolds that the existence of symplectic dilation is a property independent of the filling in many cases, hence measures the contact complexity. This is done by showing independence of a structure map as before, which also bears interests.

4.1. BV operator Δ

Similar to the discussion in §2, we will define Δ using $\mathbf{H}$. However, to make sure Lemma 2.5 can be applied, we need to consider two such functions $\mathbf{H}_+, \mathbf{H}_-$, such that the following holds.

- (1) $\mathbf{H}_+$ and $\mathbf{H}_-$ satisfy same conditions of $\mathbf{H}$ in §2, and share the same (a_i, b_i) and $D_i^+ \leq D_i^-$.
- (2) $\min_{t \in S^1} \mathbf{H}_-(t, x) \geq \max_{t \in S^1} \mathbf{H}_+(t, x)$ for all $x \in \widehat{W}$.
- (3) On each (a_i, b_i) , $\mathbf{H}_+ = f_+(r)$ and $\mathbf{H}_- = f_-(r)$ and $rf'_+(r) - f_+(r) \leq rf'_-(r) - f_-(r)$.

Roughly speaking, the requirements above ask that $\mathbf{H}_-$ grows faster than $\mathbf{H}_+$, for example $\mathbf{H}_-$ is a perturbation of $2r^2$ and $\mathbf{H}_+$ is a perturbation of r^2 . We fix such two functions and also fix a smooth decreasing function $\rho(s) : \mathbb{R} \rightarrow \mathbb{R}$, such that $\rho(s) = 1$ for $s < 0$ and $\rho(s) = 0$ for $s > 1$. Then we define

$$\mathbf{H}_{s,t}^\theta := \rho(s)\mathbf{H}_-(t + \theta) + (1 - \rho(s))\mathbf{H}_+(t).$$

Then for $s < 0$, we have $\mathbf{H}_{s,t}^\theta = \mathbf{H}_-(t + \theta)$ and for $s > 1$ we have $\mathbf{H}_{s,t}^\theta = \mathbf{H}_+(t)$. Moreover by construction, $\partial_s \mathbf{H}_{s,t}^\theta = \rho'(s)(\mathbf{H}_-(t + \theta) - \mathbf{H}_+(t)) \leq 0$. Moreover, on region (a_i, b_i) , $\mathbf{H}_{s,t}^\theta = \rho(s)f_-(r) + (1 - \rho(s))f_+(r)$. Then

$$\begin{aligned} \partial_s(r\partial_r \mathbf{H}_{s,t}^\theta - \mathbf{H}_{s,t}^\theta) &= \partial_s(r(\rho(s)f_-(r) + (1 - \rho(s))f_+(r))' - (\rho(s)f_-(r) + (1 - \rho(s))f_+(r))) \\ &= \rho'(s)(rf'_-(r) - f_-(r) - rf'_+(r) + f_+(r)) \leq 0. \end{aligned}$$

In particular, the conditions in Lemma 2.5 are satisfied for $\mathbf{H}_{s,t}^\theta, \forall \theta \in S^1$ on (a_i, b_i) .

REMARK 4.4. The extra complexity is due to that $\mathbf{H}$ depends on t . If we twist $\mathbf{H}$ to get $\mathbf{H}_{s,t}^\theta$, it is never true that $\partial_s \mathbf{H}_{s,t}^\theta \leq 0$. Hence Lemma 2.5 cannot be applied and the compactness proof fails. An alternative fix is using an autonomous Hamiltonian and the cascades moduli spaces to define the BV operator as in [11].

Let $\mathcal{J}_{s,\theta}(W)$ be the set of smooth families of admissible almost complex structures $J_{s,t}^\theta : \mathbb{R}_s \times S_\theta^1 \rightarrow \mathcal{J}(W)$, such that there exist $J_-, J_+ \in \mathcal{J}(W)$ with $J_{s,t}^\theta = J_{-,t+\theta}$ when $s < 0$ and $J_{s,t}^\theta = J_{+,t}$ when $s > 1$. Let $\mathcal{J}_{s,\theta,J_-,J_+}(W) \subset \mathcal{J}_{s,\theta}(W)$ be the set of families with positive ends given by J_+ and negative end given by J_- . Then for $J_+ \in \mathcal{J}_{\text{reg}}(\mathbf{H}_+, f, g)$, $J_- \in \mathcal{J}_{\text{reg}}(\mathbf{H}_-, f, g)$, for a generic choice of $J_{s,t}^\theta \in \mathcal{J}_{s,\theta,J_-,J_+}(W)$, we have the following moduli spaces.

(1) For $x \in \mathcal{P}^*(\mathbf{H}_-)$, $y \in \mathcal{P}^*(\mathbf{H}_+)$, $\mathcal{M}_{x,y}^\Delta$ is defined to be the compactification of the moduli space of solutions (u, θ) to the following:

$$\partial_s u + J_{s,t}^\theta (\partial_t u - X_{\mathbf{H}_{s,t}^\theta}) = 0, \quad \lim_{s \rightarrow -\infty} u = x(\cdot + \theta), \quad \lim_{s \rightarrow \infty} u = y. \quad (4.2)$$

(2) For $x \in \mathcal{P}^*(\mathbf{H}_+)$, $p \in \mathcal{C}(f)$, $\mathcal{M}_{p,x}^\Delta$ is defined to be the moduli space of solutions (u, θ, γ) to the following:

$$\partial_s u + J_{s,t}^\theta (\partial_t u - X_{\mathbf{H}_{s,t}^\theta}) = 0, \quad \frac{d}{ds} \gamma + \nabla_g f = 0, \quad \gamma(-\infty) = p, u(0) = \gamma(0), \quad \lim_{s \rightarrow \infty} u = x. \quad (4.3)$$

Since $\partial_s \mathbf{H}_{s,t}^\theta \leq 0$, any solution u to $\partial_s u + J_{s,t}^\theta (\partial_t u - X_{\mathbf{H}_{s,t}^\theta}) = 0$ will have the property that

$$\mathcal{A}_{\mathbf{H}_-}(u(-\infty)(\cdot - \theta)) - \mathcal{A}_{\mathbf{H}_+}(u(\infty)) \geq 0. \quad (4.4)$$

As a consequence, there is no $\mathcal{M}_{x,p}^\Delta$ for $p \in \mathcal{C}(f)$ and $x \in \mathcal{P}^*(\mathbf{H}_-)$. By the construction of $\mathbf{H}_{s,t}^\theta$, Lemma 2.5 can be applied to get compactness of $\mathcal{M}_{*,*}^\Delta$. Therefore we have the following with a similar proof to Propositions 2.6 and 2.8.

PROPOSITION 4.5. For a generic choice of $J_{s,t}^\theta$, we have $\mathcal{M}_{a,b}^\Delta$ is a compact manifold with boundary of dimension $|a| - |b| + 1$ when $|a| - |b| \leq 0$. And $\partial \mathcal{M}_{a,b}^\Delta := -\sum \mathcal{M}_{a,*}^\Delta \times \mathcal{M}_{*,b}^+ - \sum \mathcal{M}_{a,*}^- \times \mathcal{M}_{*,b}^\Delta$, where $\mathcal{M}_{*,*}^-, \mathcal{M}_{*,*}^+$ are moduli spaces associated to $\mathbf{H}_-$ and $\mathbf{H}_+$.

By this boundary configuration, Δ defines a cochain map $C(\mathbf{H}_+, J_+, f) \rightarrow C(\mathbf{H}_-, J_-, f)[-1]$. Moreover, Δ decomposes into Δ_+ and $\Delta_{+,0}$, which count $\mathcal{M}_{x,y}^\Delta$ and $\mathcal{M}_{p,x}^\Delta$, respectively. In particular, Δ_+ is a cochain map $C_+(\mathbf{H}_+, J_+) \rightarrow C_+(\mathbf{H}_-, J_-)[-1]$. By (4.4), Δ maps $C^{D_i^+}(\mathbf{H}_+, J_+, f)$ to $C^{D_i^-}(\mathbf{H}_-, J_-, f)$. Therefore we use $\Delta^{D_i^+}, \Delta_{+}^{D_i^+}$ and $\Delta_{+,0}^{D_i^+}$ to denote the restrictions, respectively. Lemma 2.5 implies that curves appearing in $\Delta^{D_i^+}$ are contained in W^i .

Let δ^- denote the connecting map $H^*(C_+(\mathbf{H}_-, J_-)) \rightarrow H^{*+1}(C_0(f))$. Then we can define a degree 1 map

$$\phi : \ker \Delta_+ \subset H^*(C_+(\mathbf{H}_+, J_+)) \rightarrow \text{coker } \delta^-, \quad x \mapsto \Delta_{+,0}(x) - d_{+,0}(b), \quad (4.5)$$

where $x \in C_+(\mathbf{H}_+, J_+)$, $b \in C_+(\mathbf{H}_-, J_-)$ such that $\Delta_+(x) = d_+(b)$.

PROPOSITION 4.6. ϕ is well defined. If ψ is a cochain map on $C = C_+ \oplus C_0$ that can be decomposed into $\psi_+ + \psi_0 + \psi_{+,0}$, such that $\Delta \circ \psi - \psi \circ \Delta = \eta \circ d - d \circ \eta$ for $\eta = \eta_+ + \eta_{+,0}$, then on cohomology we have $\phi \circ \psi_+ = \psi_0 \circ \phi$.

Proof. Note that we have $d^2 = 0$ and $\Delta \circ d + d \circ \Delta = 0$. When we write them using the zero and positive decomposition, we have the following formula.

$$d_0 \circ \Delta_{+,0} + d_{+,0} \circ \Delta_+(x) + \Delta_{+,0} \circ d_+(x) = 0, \quad (4.6)$$

$$d_0 \circ d_{+,0} + d_{+,0} \circ d_+ = 0. \quad (4.7)$$

We first show that $d_0 \circ \phi(x) = 0$, that is

$$\begin{aligned} d_0 \circ \phi(x) &= d_0 \circ \Delta_{+,0}(x) - d_0 \circ d_{+,0}(b) \\ &= -d_{+,0} \circ \Delta_+(x) - \Delta_{+,0} \circ d_+(x) - d_0 \circ d_{+,0}(b) \\ &= -d_{+,0} \circ d_+(b) - \Delta_{+,0} \circ d_+(x) - d_0 \circ d_{+,0}(b) \\ &= -\Delta_{+,0} \circ d_+(x). \end{aligned}$$

Since $d_+(x) = 0$, we have $\phi(x)$ is closed.

Now we consider $x' := x + d_+(y)$, then we can choose $b' := b + \Delta_+(y)$, then we have

$$\begin{aligned} \phi(x') - \phi(x) &= \Delta_{+,0} \circ d_+(y) - d_{+,0} \circ \Delta_+(y) \\ &= d_0 \circ \Delta_{+,0}(y). \end{aligned}$$

Hence the difference is exact. Finally, we consider $b' = b + c$ where $d_+(c) = 0$. Then we have the difference is $d_{+,0}(c)$ which is in the image of δ .

If we have such ψ and η , then we have the following:

$$\psi_+ \circ d_+ = d_+ \circ \psi_+, \quad (4.8)$$

$$\psi_0 \circ d_{+,0} + \psi_{+,0} \circ d_+ = d_0 \circ \psi_{+,0} + d_{+,0} \circ \psi_+, \quad (4.9)$$

$$\psi_+ \circ \Delta_+ - \Delta_+ \circ \psi_+ = d_+ \circ \eta_+ - \eta_+ \circ d_+, \quad (4.10)$$

$$\psi_0 \circ \Delta_{+,0} + \psi_{+,0} \circ \Delta_+ - \Delta_{+,0} \circ \psi_+ = d_0 \circ \eta_{+,0} + d_{+,0} \circ \eta_+ - \eta_{+,0} \circ d_+. \quad (4.11)$$

Then (4.8) implies that ψ_+ induces a map on $H^*(C_+)$. (4.9) implies that $\delta \circ \psi_+ = \psi_0 \circ \delta$ on cohomology. (4.10) implies that $\psi_+ \circ \Delta_+ = \Delta_+ \circ \psi_+$ on cohomology. Finally, for $x, b \in C_+$ with $\Delta_+(x) = d_+(b)$, $d_+(x) = 0$, we have

$$\begin{aligned} \Delta_+ \circ \psi_+(x) &\stackrel{(4.10)}{=} \psi_+ \circ \Delta_+(x) - d_+ \circ \eta_+(x) + \eta_+ \circ d_+(x) \\ &= \psi_+ \circ d_+(b) - d_+ \circ \eta_+(x) \\ &\stackrel{(4.8)}{=} d_+ \circ \psi_+(b) - d_+ \circ \eta_+(x). \end{aligned}$$

Therefore we have

$$\begin{aligned} \phi \circ \psi_+(x) &= \Delta_{+,0} \circ \psi_+(x) - d_{+,0} \circ \psi_+(b) + d_{+,0} \circ \eta_+(x) \\ &\stackrel{(4.11)}{=} \psi_0 \circ \Delta_{+,0}(x) + \psi_{+,0} \circ \Delta_+(x) - d_0 \circ \eta_{+,0}(x) - d_{+,0} \circ \psi_+(b) \\ &= \psi_0 \circ \Delta_{+,0}(x) + \psi_{+,0} \circ d_+(b) - d_{+,0} \circ \psi_+(b) - d_0 \circ \eta_{+,0}(x) \\ &\stackrel{(4.9)}{=} \psi_0 \circ \Delta_{+,0}(x) - \psi_0 \circ d_{+,0}(b) + d_0 \circ \psi_{+,0}(b) - d_0 \circ \eta_{+,0}(x) \\ &= \psi_0 \circ \phi(x) + d_0 \circ \psi_{+,0}(b) - d_0 \circ \eta_{+,0}(x). \end{aligned}$$

That is, $\phi \circ \psi_+ = \phi_0 \circ \phi$ on cohomology. $\square$

4.2. Continuation maps

There are two continuation maps we need to consider, one is the continuation from $\mathbf{H}_+$ to $\mathbf{H}_-$. The other is the continuation from homotopies of almost complex structures and its compatibility with Δ_+ and ϕ .

4.2.1. *Continuation map from $\mathbf{H}_+$ to $\mathbf{H}_-$.* Let $\mathbf{H}_s := \rho(s)\mathbf{H}_- + (1 - \rho(s))\mathbf{H}_+$, that is $\mathbf{H}_{s,t}^0$. In particular, Lemma 2.5 can be applied to $\mathbf{H}_s$ on (a_i, b_i) . Then we can consider the compactified moduli spaces of the following:

$$\{u : \mathbb{R} \times S^1 \rightarrow \widehat{W} \mid \partial_s u + J_s(\partial_t u - X_{\mathbf{H}_s}) = 0, \lim_{s \rightarrow -\infty} u = x, \lim_{s \rightarrow \infty} u = y\},$$

for $x \in \mathcal{P}(\mathbf{H}_-), y \in \mathcal{P}^*(\mathbf{H}_+)$.

$$\left\{ \begin{array}{l} u : \mathbb{C} \rightarrow \widehat{W}, \\ \gamma : (-\infty, 0] \rightarrow W \end{array} \middle| \begin{array}{l} \partial_s u + J_s(\partial_t u - X_{\mathbf{H}_s}) = 0, \partial_s \gamma + \nabla_g f = 0, \\ \lim_{s \rightarrow -\infty} \gamma = p, u(0) = \gamma(0), \lim_{s \rightarrow \infty} u = x \end{array} \right\}$$

for $x \in \mathcal{P}(\mathbf{H}_+), p \in \mathcal{C}(f)$. We denote them by $\mathcal{N}_{x,y}$ and $\mathcal{N}_{p,x}$. Then for $J_+ \in \mathcal{J}_{\text{reg}}(\mathbf{H}_+, f, g), J_- \in \mathcal{J}_{\text{reg}}(\mathbf{H}_-, f, g)$, there exists generic homotopy J_s from J_- to J_+ , so that the moduli spaces above are compact smooth manifolds with boundaries, when the expected dimension is ≤ 1 . In particular, it defines a cochain map $\Theta : C(\mathbf{H}_+, J_+, f) \rightarrow C(\mathbf{H}_-, J_-, f)$ by counting $\mathcal{N}_{x,*}$ when $x \in \mathcal{P}^*(\mathbf{H}_+)$ and is identity on $C_0(f)$. Θ respects the splitting into C_0 and C_+ . Moreover, Θ maps $C^{D_i^+}(\mathbf{H}_+, J_+, f) \rightarrow C^{D_i^-}(\mathbf{H}_-, J_-, f)$. We will show in § 4.6 that Θ is an isomorphism on cohomology. The following functoriality of the continuation map is also verified in § 4.6.

PROPOSITION 4.7. *Let $J_+^1, J_+^2 \in \mathcal{J}_{\text{reg}}^{D_i^+}(\mathbf{H}_+, f, g)$ and $J_-^1, J_-^2 \in \mathcal{J}_{\text{reg}}^{D_i^-}(\mathbf{H}_-, f, g)$, then the following diagram is commutative up to homotopy:*

$$\begin{array}{ccc} C_+^{D_i^+}(\mathbf{H}_+, J_+^1) & \longrightarrow & C_+^{D_i^-}(\mathbf{H}_-, J_-^1) \\ \downarrow & & \downarrow \\ C_+^{D_i^+}(\mathbf{H}_+, J_+^2) & \longrightarrow & C_+^{D_i^-}(\mathbf{H}_-, J_-^2), \end{array}$$

where the horizontal maps are continuation maps constructed above and the vertical maps are continuation maps in Proposition 2.14.

4.2.2. *Compatibility with homotopies of J .* Assume we have $J_{+,1}, J_{+,2} \in \mathcal{J}_{\text{reg}}(\mathbf{H}_+, f, g)$ and $J_{-,1}, J_{-,2} \in \mathcal{J}_{\text{reg}}(\mathbf{H}_-, f, g)$ and regular homotopies $J_{s,+}, J_{s,-}$ from $J_{+,2}, J_{-,2}$ to $J_{+,1}, J_{-,1}$, respectively, so that the continuation map in Proposition 2.14 is defined, and are denoted by Φ^+, Φ^- . Assume $J_{s,t,1}^\theta$ and $J_{s,t,2}^\theta$ are two homotopies with ends $J_{-,1}, J_{+,1}$ and $J_{-,2}, J_{+,2}$, respectively, such that Proposition 4.5 holds.

Following [1, §2.2.3], we consider a family of almost complex structures $J_{s,t}^{r,\theta}$ such that:

- (1) when $r \ll 0$, $J_{s,t}^{r,\theta} = J_{s-r,t+\theta,-}$ if $s < 0$ and is $J_{s+r,t,1}^\theta$ if $s \geq 0$. They patch up smoothly by our definition of J_s and $J_{s,t}^\theta$;
- (2) when $r \gg 0$, $J_{s,t}^{r,\theta} = J_{s-r,t,2}^\theta$ if $s < 0$ and is $J_{s+r,t,+}$ if $s \geq 0$;
- (3) for every r , when $s \gg 0$, $J_{s,t}^{r,\theta} = J_{+,1}$, when $s \ll 0$, $J_{s,t}^{r,\theta} = J_{t+\theta,-,2}$.

Then for $x \in \mathcal{P}^*(\mathbf{H}_-), y \in \mathcal{P}^*(\mathbf{H}_+)$, we can consider the moduli space of solutions (u, θ, r) to

$$\partial_s u + J_{s,t}^{\theta,r}(\partial_t u - X_{\mathbf{H}_{s,t}^\theta}) = 0, \quad \lim_{s \rightarrow -\infty} u = x(\cdot + \theta), \quad \lim_{s \rightarrow \infty} u = y. \quad (4.12)$$

Let $\mathcal{T}_{x,y}$ denote the compactification, we can similarly consider $\mathcal{T}_{p,x}$ for $x \in \mathcal{P}^*(\mathbf{H}_+)$, $p \in \mathcal{C}(f)$. Then for generic choice of $J_{s,t}^{r,\theta}$, $\mathcal{T}_{*,*}$ is a compact manifold with boundary if the expected dimension is ≤ 1 . Let Δ^1, Δ^2 be the BV operators defined using $J_{s,t,1}^\theta$ and $J_{s,t,2}^\theta$. Let $\eta : C(\mathbf{H}_+, J_{+,1}, f) \rightarrow C(\mathbf{H}_-, J_{-,2}, f)$ be the operator counting rigid points in $\mathcal{T}_{*,*}$. Then the boundary configuration leads to the following relation

$$\Phi^- \circ \Delta^1 - \Delta^2 \circ \Phi^+ = d_{J_{-,2}} \circ \eta - \eta \circ d_{J_{+,1}}.$$

Moreover, both Φ^+, Φ^- have splittings into $\Phi_+^\pm + \text{id}_{C_0} + \Phi_{+,0}^\pm$ and η has splitting into $\eta_+ + \eta_{+,0}$. That is, they are in the form that Proposition 4.6 can be applied.

The following propositions show that the structure we defined is the same BV operator defined in [58]. We will prove them in § 4.6.

PROPOSITION 4.8. *Using the identification in Proposition 2.10 to identify $SH^*(W)$ with $H^*(C(\mathbf{H}_+, J_+, f))$, then $\Theta^{-1} \circ \Delta : H^*(C(\mathbf{H}_+, J_+, f)) \rightarrow H^{*-1}(C(\mathbf{H}_+, J_+, f))$ is the BV operator on $SH^*(W)$.*

PROPOSITION 4.9. *$\Theta^{-1} \circ \Delta_+ : H^*(C_+(\mathbf{H}_+, J_+, f)) \rightarrow H^{*-1}(C_+(\mathbf{H}_+, J_+, f))$ and $\phi : \ker \Delta_+ \rightarrow \text{coker } \delta$ are invariants of exact domains up to exact symplectomorphisms using the identification in Proposition 2.10.*

Hence, from now on, we will proceed the construction using $\mathbf{H}_-, \mathbf{H}_+$.

PROPOSITION 4.10. *W admits a symplectic dilation if and only if $1 \in \text{im } \phi(\text{im}(SH(W) \rightarrow SH_+(W)) \cap \ker \Delta_+) = \text{im } \phi(\ker \delta \cap \ker \Delta_+)$.*

Proof. If $SH^*(W) = 0$, then $\text{coker } \delta = 0$, hence 1 is always in the image and there exists a dilation. In the following, we will consider the case when $SH^*(W) \neq 0$. Using Proposition 4.8, assume $x' \in C(\mathbf{H}_+, J_+, f)$ closed satisfies $\Delta([x']) = 1$ on cohomology, where 1 is represented by the unique local minimum of f . Since x' can be written as $x + y$, where $x \in C_+(\mathbf{H}_+, J_+)$, $y \in C_0(f)$, we have $\Delta_+(x) + \Delta_{+,0}(x) = 1 + d(b)$ for $b \in C_+(\mathbf{H}_-, J_-)$. Then we have $\Delta_+(x) = d_+(b)$, hence $\phi(x) = \Delta_{+,0}(x) - d_{+,0}(b) = 1$. Since Θ preserves 1, we have 1 is in the image of $\phi : \text{im}(SH(W) \rightarrow SH_+(W)) \cap \ker \Delta_+ \rightarrow \text{coker } \delta$.

On the other hand, if $\phi(x) = 1$ for $x \in C_+(\mathbf{H}_+, J_+)$ and $[x] \in \text{im}(H^*(C(\mathbf{H}_+, J_+, f)) \rightarrow H^*(C_+(\mathbf{H}_+, J_+))) \cap \ker \Delta_+$, that is, there exists $y \in C_0(f)$, such that $x + y$ is closed, we have $\Delta_{+,0}(x) - d_{+,0}(b) = 1 + c$, where $c \in C_0(f)$ is closed and $[c] \in \text{im } \delta^-$ and $d_+(b) = \Delta_+(x)$. Since $[c] \in \text{im } \delta^{-1}$, $[c]$ in $H^*(C(\mathbf{H}_-, J_-, f))$ is zero. Therefore we have $\Delta(x + y) = \Delta(x) = \Delta_{+,0}(x) + \Delta_+(x) = d(b) + 1 + c$, which is 1 in cohomology. Therefore $x + y$ is a dilation. $\square$

By composing with the restriction map $H^*(W) \rightarrow H^*(\partial W)$, we define a map

$$\Delta_\partial : \ker \Delta_+ \rightarrow \text{coker } \delta_\partial. \quad (4.13)$$

And this is the second structure map we are interested in.

4.3. Shrinking the gradient flow

In this subsection, we will apply the same idea in § 3 to Δ_∂ and rewrite it without using the Morse function f . As in § 3, we choose a Morse function h on $\partial W \times \{1 - \epsilon\}$ along with a metric g_∂ , so that Proposition 3.1 holds. On the complex level, the composition of $\ker \Delta_+ \rightarrow \text{coker } \delta \rightarrow \text{coker } \delta_\partial$ is represented by counting the moduli space $\mathcal{R}_{p,q} \times \mathcal{M}_{q,x}^\Delta$ and $\mathcal{R}_{p,q} \times \mathcal{M}_{q,b}^-$ for $x \in \mathcal{P}^*(\mathbf{H}_+)$, $b \in \mathcal{P}^*(\mathbf{H}_-)$, $q \in \mathcal{C}(f)$, $p \in \mathcal{C}(h)$. In particular, if we define

$\Delta_{+,0,\partial}(x)$ by counting $\mathcal{R}_{p,q} \times \mathcal{M}_{q,x}^\Delta$ and $d_{+,0,\partial}$ by counting $\mathcal{R}_{p,q} \times \mathcal{M}_{q,b}^-$, then we define a map $\Delta_+^{-1}(\text{im } d_+) \cap \ker d_+ \rightarrow \text{coker } \delta_\partial$ representing Δ_∂ by

$$x \mapsto \Delta_{+,0,\partial}(x) - d_{+,0,\partial}(b), \quad (4.14)$$

for $d_+(b) = \Delta_+(x)$. To bypass the middle $C^{*-1}(f)$, we define $\mathcal{P}_{p,x}^\Delta$ to be the compactification of the following moduli space

$$\left\{ \begin{array}{l} u : \mathbb{C} \rightarrow \widehat{W}, \\ \theta \in S^1, \\ \gamma : (-\infty, 0] \rightarrow \partial W \times \{1 - \epsilon\}. \end{array} \left| \begin{array}{l} \partial_s u + J_{s,t}^\theta (\partial_t u - X_{\mathbf{H}_{s,t}^\theta}) = 0, \frac{d}{ds} \gamma + \nabla_{g_\partial} h = 0, \\ \lim_{s \rightarrow -\infty} \gamma = p, u(0) = \gamma(0), \lim_{s \rightarrow \infty} u = x \end{array} \right. \right\},$$

for $p \in \mathcal{C}(h), x \in \mathcal{P}^*(\mathbf{H}_+)$. The space makes sense, since we push ∂W into interior, where $\mathbf{H}_{s,t}^\theta = 0$. Then $\mathcal{P}_{p,x}^\Delta$ defines a map from $P^\Delta : C_+^*(\mathbf{H}_+, J_+) \rightarrow C^{*-1}(h)$. To show it defines the same thing as (4.14), we define $\mathcal{H}_{p,x}^\Delta$ to be the compactification of the following moduli space for $p \in \mathcal{C}(h), x \in \mathcal{P}^*(\mathbf{H}_+)$,

$$\left\{ \begin{array}{l} u : \mathbb{C} \rightarrow \widehat{W}, \\ \theta \in S^1, \\ l \in \mathbb{R}_+, \\ \gamma_1 : (-\infty, 0] \rightarrow \partial W \times \{1 - \epsilon\}, \\ \gamma_2 : [0, l] \rightarrow W. \end{array} \left| \begin{array}{l} \partial_s u + J_{s,t}^\theta (\partial_t u - X_{\mathbf{H}_{s,t}^\theta}) = 0, \\ \frac{d}{ds} \gamma_1 + \nabla_{g_\partial} h = 0, \frac{d}{ds} \gamma_2 + \nabla_g f = 0, \\ \lim_{s \rightarrow -\infty} \gamma_1 = p, \gamma_1(0) = \gamma_2(0), \\ \gamma_2(l) = u(0), \lim_{s \rightarrow \infty} u = x. \end{array} \right. \right\}.$$

Then we have the following regularity property.

PROPOSITION 4.11. *For generic choices of J_- , J_+ and $J_{s,t}^\theta$, for any x, p with $|p| - |x| \leq 0$, $\mathcal{H}_{p,x}^\Delta$ is a manifold with boundary of dimension $|p| - |x| + 1$ and*

$$\begin{aligned} \partial \mathcal{H}_{p,x}^\Delta &= -\mathcal{P}_{p,x}^\Delta + \sum_{q \in \mathcal{C}(f)} \mathcal{R}_{p,q} \times \mathcal{M}_{q,x}^\Delta + \sum_{y \in \mathcal{P}^*(\mathbf{H}_-)} \mathcal{H}_{p,y} \times \mathcal{M}_{y,x}^\Delta \\ &\quad + \sum_{q \in \mathcal{C}(h)} \mathcal{M}_{p,q} \times \mathcal{H}_{q,x}^\Delta - \sum_{y \in \mathcal{P}^*(\mathbf{H}_+)} \mathcal{H}_{p,y} \times \mathcal{M}_{y,x}. \end{aligned}$$

PROPOSITION 4.12. *For generic choices of J_- , J_+ and $J_{s,t}^\theta$ as in Proposition 4.11, let $x \in \ker d_+ \subset C_+^*(\mathbf{H}_+, J_+)$ such that $\Delta_+(x) = d_+(b)$, then $P^\Delta(x) - P(b)$ represents the same class in $\text{coker}(H^{*-2}(C_+(\mathbf{H}_-, J_-)) \rightarrow H^{*-1}(\partial W))$ as (4.14). Hence on cohomology, Δ_∂ in (4.14) equals to the difference of counting $\mathcal{P}_{x,q}^\Delta, \mathcal{P}_{b,q}$.*

Proof. By Proposition 3.2 and 4.11, we have $P^\Delta(x) - P(b) - \Delta_{+,0,\partial}(x) + d_{+,0,\partial}(b)$ as the following:

$$\begin{aligned} &\sum \left(\sum \# \mathcal{H}_{p,y} \# \mathcal{M}_{y,x}^\Delta + \sum \# \mathcal{M}_{p,q} \# \mathcal{H}_{q,x}^\Delta - \sum \# \mathcal{H}_{p,y} \# \mathcal{M}_{y,x} \right. \\ &\quad \left. - \sum \# \mathcal{H}_{p,y} \# \mathcal{M}_{y,b} - \sum \# \mathcal{M}_{p,q} \# \mathcal{H}_{q,b} \right) p. \end{aligned}$$

It is $H \circ \Delta_+(x) + d_{\partial W} \circ H^\Delta(x) - H^\Delta \circ d_+(x) - H \circ d_+(b) - d_{\partial W} \circ H(b)$, where H, H^Δ are defined by counting $\mathcal{H}$ and $\mathcal{H}^\Delta$ and $d_{\partial W}$ is the Morse differential on ∂W . Since $d_+(x) = 0$ and $\Delta_+(x) = d_+(b)$, the above term is $d_{\partial W} \circ H^\Delta(x) - d_{\partial W} \circ H(b)$, which is exact. Hence the claim is proven. $\square$

4.4. Naturality

To keep track of the regularity of almost complex structures, we introduce the following.

- (1) $\mathcal{J}_{\text{reg},\Delta}^{\leq 1}(f, g, J_-, J_+) \subset \mathcal{J}_{s,\theta,J_-,J_+}(W)$ with ends $J_- \in \mathcal{J}_{\text{reg}}(\mathbf{H}_-, f, g), J_+ \in \mathcal{J}_{\text{reg}}(\mathbf{H}_+, f, g)$ is the set of regular almost complex structures for moduli spaces $\mathcal{M}^\Delta$ of dimension up to 1. $\mathcal{J}_{\text{reg},\Delta,J_-,J_+}^{D_i^+}(f, g) \subset \mathcal{J}_{s,\theta,J_-,J_+}(W)$ with ends $J_- \in \mathcal{J}_{\text{reg}}^{D_i^-}(\mathbf{H}_-, f, g), J_+ \in \mathcal{J}_{\text{reg}}^{D_i^+}(\mathbf{H}_+, f, g)$ is the set of regular almost complex structures for $\mathcal{M}_{*,x}^\Delta$ of dimension up to 0 and action of x down to $-D_i^+$. Similarly for $\Delta_+^{D_i}$.
- (2) $\mathcal{J}_{\text{reg},P\Delta}^{\leq 1}(h, g_\partial, J_-, J_+)$ with ends $J_- \in \mathcal{J}_{\text{reg},+}(\mathbf{H}_-) \cap \mathcal{J}_{\text{reg},P}(\mathbf{H}_-, h, g_\partial)$ and $J_+ \in \mathcal{J}_{\text{reg},+}(\mathbf{H}_+)$ for moduli space $\mathcal{P}_{p,x}^\Delta$ of dimension up to 1. $\mathcal{J}_{\text{reg},P\Delta}^{D_i^+}(h, g_\partial, J_-, J_+)$ is the set of regular admissible homotopy $J_{s,t}^\theta$ with ends $J_- \in \mathcal{J}_{\text{reg},+}^{D_i^-}(\mathbf{H}_-) \cap \mathcal{J}_{\text{reg},P}^{D_i^-}(\mathbf{H}_-, h, g_\partial), J_+ \in \mathcal{J}_{\text{reg},+}^{D_i^+}(\mathbf{H}_+)$ for moduli spaces $\mathcal{P}_{p,x}^\Delta$ of dimension up to 0 and action of x down to $-D_i^+$.
- (3) $\mathcal{J}_{\text{reg},H\Delta}^{\leq 1}(f, g, h, g_\partial, J_-, J_+) \subset \mathcal{J}_{s,\theta,J_-,J_+}(W)$ with ends J_- in the intersection $\mathcal{J}_{\text{reg},+}(\mathbf{H}_-) \cap \mathcal{J}_{\text{reg},H}(\mathbf{H}_-, f, g, h, g_\partial) \cap \mathcal{J}_{\text{reg},P}(\mathbf{H}_-, h, g_\partial), J_+ \in \mathcal{J}_{\text{reg},+}(\mathbf{H}_+)$ is the set of regular $J_{s,t}^\theta$ for moduli spaces $\mathcal{H}_{p,x}^\Delta$ of dimension up to 1.
- (4) $\mathcal{J}_{\text{reg},N,+}^{\leq 1}(J_-, J_+) \subset \mathcal{J}_{s,\theta,J_-,J_+}(W)$ with ends $J_- \in \mathcal{J}_{\text{reg},+}(\mathbf{H}_-), J_+ \in \mathcal{J}_{\text{reg},+}(\mathbf{H}_+)$ is the set of regular J_s for moduli spaces $\mathcal{N}_{x,y}$ in §4.2.1 of dimension up to 1. $\mathcal{J}_{\text{reg},N,+}^{D_i^+}(J_-, J_+) \subset \mathcal{J}_{s,\theta,J_-,J_+}(W)$ with ends $J_- \in \mathcal{J}_{\text{reg},+}^{D_i^-}(\mathbf{H}_-), J_+ \in \mathcal{J}_{\text{reg},+}^{D_i^+}(\mathbf{H}_+)$ is the set of regular J_s for moduli spaces $\mathcal{N}_{x,y}$ up to dimension 0 with action of y down to $-D_i^+$.

As usual, $\mathcal{J}_{\text{reg}}$ is of second category, and $\mathcal{J}_{\text{reg}}^D$ is open dense.

PROPOSITION 4.13. *We summarize naturality properties in the following.*

- (i) For $J_{s,t}^\theta \in \mathcal{J}_{\text{reg},\Delta}^{D_i^+}(f, g, J_-, J_+)$ with $J_- \in \mathcal{J}_{\text{reg}}^{D_i^-}(\mathbf{H}_-, f, g), J_+ \in \mathcal{J}_{\text{reg}}^{D_i^+}(\mathbf{H}_+, f, g)$, $\Delta_+^{D_i^+}$ is a well-defined cochain map. Similarly for $\Delta_+^{D_i^+}$.
- (ii) For $J_{s,t}^\theta \in \mathcal{J}_{\text{reg},\Delta,+}^{D_i^+}(J_-, J_+)$ with $J_- \in \mathcal{J}_{\text{reg},+}^{D_i^-}(\mathbf{H}_-), J_+ \in \mathcal{J}_{\text{reg},+}^{D_i^+}(\mathbf{H}_+)$, ϕ is well defined on $\ker \Delta_+^{D_i^+}$.
- (iii) For $J_{s,t}^\theta \in \mathcal{J}_{\text{reg},\Delta,+}^{D_i^+}(J_-, J_+) \cap \mathcal{J}_{\text{reg},P\Delta}^{D_i^+}(h, g_\partial, J_-, J_+)$ for $J_- \in \mathcal{J}_{\text{reg},+}^{D_i^-}(\mathbf{H}_-) \cap \mathcal{J}_{\text{reg},P}^{D_i^-}(\mathbf{H}_-, h, g_\partial), J_+ \in \mathcal{J}_{\text{reg},+}^{D_i^+}(\mathbf{H}_+)$, Δ_∂ is same as counting the difference between $\mathcal{P}^\Delta$ and $\mathcal{P}$, when restricted to $C_+^{D_i^+}$.

Proof. For (i), there exists a neighborhood $\mathcal{U}$ of $J_{s,t}^\theta \in \mathcal{J}_{s,\theta,J_-,J_+}(W)$ contained in $\mathcal{J}_{\text{reg},\Delta,J_-,J_+}^{D_i^+}(f, g)$. Then $\Delta_+^{D_i^+}$ is locally constant by the proof of Proposition 2.12. The claim follows from that $\Delta_+^{D_i^+}$ is a cochain map for more regular $J_{s,t}^\theta$ nearby.

For (ii), we have Δ_+ and d_+ are locally constant with respect to the almost complex structures. Since $d_{+,0}$ is defined using two nearby $J'', J' \in \mathcal{J}_{\text{reg}}^{D_i^-}(\mathbf{H}_-, f, g)$ of J_- , then there are two homotopy $J_{s,t}^{\theta''} \in \mathcal{J}_{\text{reg},\Delta}^{\leq 1}(f, g, J'', J_+), J_{s,t}^{\theta'} \in \mathcal{J}_{\text{reg},\Delta}^{\leq 1}(f, g, J', J_+)$ close to $J_{s,t}^\theta$ in $\mathcal{J}_{s,\theta}(W)$. It is sufficient to prove for $x \in C_+^{D_i^+}(\mathbf{H}_+)$ that $d_+(x) = 0, \Delta_+(x) = d_+(b)$ for $b \in C_+^{D_i^-}(\mathbf{H}_-)$, we have $\Delta_{+,0,J',J_+}(x) - d_{+,0,J'}(b)$ and $\Delta_{+,0,J'',J_+}(x) - d_{+,0,J''}(b)$ are differed by exact form. Since we have a continuation map $\Phi = \Phi_+ + \Phi_{+,0} + \text{id}_{C_0(f)} : C^{D_i^-}(\mathbf{H}_-, J', f) \rightarrow C^{D_i^-}(\mathbf{H}_-, J'', f)$,

such that Φ_+ is identity by the proof of Lemma 2.15, then we have

$$d_{+,0,J''}(b) - d_{+,0,J'}(b) = d_0 \circ \Phi_{+,0}(b) - \Phi_{+,0} \circ d_+(b) = d_0 \circ \Phi_{+,0}(b) - \Phi_{+,0} \circ \Delta_+(x).$$

On the other hand, by § 4.2.2, we have

$$\Delta_{+,0,J'',J_+}(x) + \Phi_{+,0} \circ \Delta_+(x) - \Delta_{+,0,J',J_+}(x) = d_0 \circ \eta_{+,0}(x) + d_{+,0} \circ \eta_+(x).$$

We claim η_+ could be chosen to be zero. By assumption, the almost complex structure $J_{s,t}^{\theta,r}$ in the construction of η can be chosen such that for every r , $J_{s,t}^{\theta,r} \in \mathcal{J}_{\text{reg},\Delta,+}^{D_+^+}(J_-, J_+)$. As a consequence, $\eta_+ = 0$. Combining them together proves the claim.

For (iii), $P, P^\Delta, \Delta_+, d_+$ are all locally constant, for a nearby $J_{s,t}^\theta$, we have (the perturbed) ϕ defines the same thing as counting $\mathcal{P}, \mathcal{P}^\Delta$ by Proposition 4.12. By claim (ii), ϕ is independent of the perturbation. This finish the proof. $\square$

Combining Propositions 4.7, 4.9 and 4.13 and the compatibility of continuation maps with Δ in § 4.2.2, we have the following.

PROPOSITION 4.14. *Let $J_-^i \in \mathcal{J}_{\text{reg},+}^{D_i^-}(\mathbf{H}_-) \cap \mathcal{J}_{\text{reg},P}^{D_i^-}(\mathbf{H}_-, h, g_\partial)$, $J_+^i \in \mathcal{J}_{\text{reg},+}^{D_+^+}(\mathbf{H}_+)$, $J_{s,t}^{\theta,i} \in \mathcal{J}_{\text{reg},\Delta,+}^{D_+^+}(J_-^i, J_+^i) \cap \mathcal{J}_{\text{reg},P^\Delta}^{D_+^+}(h, f_\partial, J_-^i, J_+^i)$ and $J_{s,t}^i \in \mathcal{J}_{\text{reg},N,+}^{D_+^+}(J_-^i, J_+^i)$, then we have the following:*

$$\begin{array}{ccccccc} H^*(C_+^{D_+^+}(\mathbf{H}_+, J_+^1)) & \longrightarrow & H^*(C_+^{D_+^+}(\mathbf{H}_+, J_+^2)) & \longrightarrow & \dots & H^*(C_+^{D_+^+}(\mathbf{H}_+, J_+^1)) & \succcurlyeq H^*(C_+^{D_+^+}(\mathbf{H}_+, J_+^2)) \succcurlyeq \dots \\ \downarrow \Delta_+ & & \downarrow \Delta_+ & & & \downarrow \Theta_+ & \downarrow \Theta_+ \\ H^{*-1}(C_+^{D_+^-}(\mathbf{H}_-, J_-^1)) & \succcurlyeq & H^{*-1}(C_+^{D_+^-}(\mathbf{H}_-, J_-^2)) & \succcurlyeq & \dots & H^*(C_+^{D_+^-}(\mathbf{H}_-, J_-^1)) & \succcurlyeq H^*(C_+^{D_+^-}(\mathbf{H}_-, J_-^2)) \succcurlyeq \dots \end{array}$$

and $\lim_{\rightarrow} H^*(C_+^{D_+^+}(\mathbf{H}_+, J_+^i)) \xrightarrow{\lim \Delta_+} \lim_{\rightarrow} H^*(C_+^{D_+^-}(\mathbf{H}_-, J_-^i)) \xrightarrow{(\lim \Theta_+)^{-1}} \lim_{\rightarrow} H^*(C_+^{D_+^+}(\mathbf{H}_+, J_+^i))$ computes $\Delta_+ : SH_+^*(W) \rightarrow SH_+^{*-1}(W)$. Similarly for Δ_∂ , which can either be computed using $\Delta_{+,0}, d_{+,0}$ or P^Δ, P .

4.5. Independence

The following statement follows from the same proof of Proposition 3.12.

PROPOSITION 4.15. *With the same setup in Proposition 3.12, there exist admissible $J_{\pm,1}^1, J_{\pm,1}^2, \dots, J_{\pm,2}^1, J_{\pm,2}^2, \dots$ on $\widehat{W}_1$ and $\widehat{W}_2$, respectively, and admissible homotopies $J_{s,1}^1, J_{s,1}^2, \dots, J_{s,2}^1, J_{s,2}^2, \dots$ with ends $J_{\pm,*}^i$ and admissible S^1 families of homotopies $J_{s,t,1}^{\theta,1}, J_{s,t,1}^{\theta,2}, \dots, J_{s,t,2}^{\theta,1}, J_{s,t,2}^{\theta,2}, \dots$ with ends $J_{\pm,*}^i$ and positive real number $\epsilon_1, \epsilon_2, \dots$ such that the following holds.*

(i) For $R < \epsilon_i$ and any R' , we have

$$\begin{aligned} NS_{i,R}(J_{\pm,*}^i), NS_{i+1,R'}(NS_{i,R}(J_{\pm,*}^i)) &\in \mathcal{J}_{\text{reg},+}^{D_i^\pm}(\mathbf{H}_\pm) \cap \mathcal{J}_{\text{reg},P}^{D_i^\pm}(\mathbf{H}_\pm, h, g_\partial), \\ NS_{i,R}(J_{s,t,*}^{\theta,i}), NS_{i+1,R'}(NS_{i,R}(J_{s,t,*}^{\theta,i})) &\in \mathcal{J}_{\text{reg},P^\Delta}^{D_+^+}(h, g_\partial) \cap \mathcal{J}_{\text{reg},\Delta,+}^{D_+^+}, \\ NS_{i,R}(J_{s,*}^i), NS_{i+1,R'}(NS_{i,R}(J_{s,*}^i)) &\in \mathcal{J}_{\text{reg},N,+}^{D_+^+}, \end{aligned}$$

such that all zero-dimensional $\mathcal{M}_{x,y}, \mathcal{P}_{p,x}, \mathcal{M}_{x,y}^\Delta$ and $\mathcal{P}_{p,x}^\Delta, \mathcal{N}_{x,y}$ are the same for both W_1, W_2 and contained outside Y_i for $x, y \in C_+^{D_i^\pm}, p \in C(h)$.

(ii) $J_{s,*}^{i+1} = NS_{i, \frac{\epsilon_i}{2}}(J_{s,*}^i)$ on W_*^i . $J_{s,t,*}^{\theta,i+1} = NS_{i, \frac{\epsilon_i}{2}}(J_{s,t,*}^{\theta,i})$ on W_*^i . Note that they imply that $J_{\pm,*}^{i+1} = NS_{i, \frac{\epsilon_i}{2}}(J_{\pm,*}^i)$ on W_*^i .

Now, we are ready to prove Theorem G.

Proof of Theorem G. By Proposition 4.15, we have the following two diagrams for both W_1, W_2 .

$$\begin{array}{ccccc}
H^*(C_+^{D_1^+}(\mathbf{H}_+, NS_{1, \frac{\epsilon_1}{2}} J_{+,*}^1)) & \longrightarrow & H^*(C_+^{D_2^+}(\mathbf{H}_+, NS_{2, \frac{\epsilon_2}{2}} J_{+,*}^2)) & \longrightarrow & \dots \\
\downarrow \Delta_+ & & \downarrow \Delta_+ & & \\
H^{*-1}(C_+^{D_1^-}(\mathbf{H}_-, NS_{1, \frac{\epsilon_1}{2}} J_{-,*}^1)) & \longrightarrow & H^{*-1}(C_+^{D_2^-}(\mathbf{H}_-, NS_{2, \frac{\epsilon_2}{2}} J_{-,*}^2)) & \longrightarrow & \dots \\
\\
H^*(C_+^{D_1^+}(\mathbf{H}_+, NS_{1, \frac{\epsilon_1}{2}} J_{+,*}^1)) & \longrightarrow & H^*(C_+^{D_2^+}(\mathbf{H}_+, NS_{2, \frac{\epsilon_2}{2}} J_{+,*}^2)) & \longrightarrow & \dots \\
\downarrow \Theta_+ & & \downarrow \Theta_+ & & \\
H^*(C_+^{D_1^-}(\mathbf{H}_-, NS_{1, \frac{\epsilon_1}{2}} J_{-,*}^1)) & \longrightarrow & H^*(C_+^{D_2^-}(\mathbf{H}_-, NS_{2, \frac{\epsilon_2}{2}} J_{-,*}^2)) & \longrightarrow & \dots
\end{array}$$

where Θ_+ is continuation defined using $NS_{i, \frac{\epsilon_i}{2}}(J_{s,*}^i)$ and Δ_+ is the positive BV operator defined by $NS_{i, \frac{\epsilon_i}{2}}(J_{s,*}^{\theta,i})$. In particular, they are identified on W^1, W^2 . The horizontal arrows are continuation maps. As proved in A, they are represented by inclusions and hence are also independent of fillings. Moreover, P, P^Δ on them are independent of the filling. That is, the diagrams can be identified for W_1, W_2 . Proposition 4.14 implies the claim. Note that the diagram contains the part used in the proof of Theorem A. Hence the proof gives a (nonconical) isomorphism Γ , identifying $\delta_\partial, \Delta_+$ and Δ_∂ . $\square$

REMARK 4.16. It is a more delicate question to give a ‘natural’ isomorphism in Theorems A and G. Nevertheless, the isomorphism can be chosen so that it matches up all the structures.

The following corollary will imply Corollary H in the introduction.

COROLLARY 4.17. *Let Y be an ADC manifold with a topologically simple filling admitting a symplectic dilation, then for any other topologically simple filling W' such that $H^2(W') \rightarrow H^2(Y)$ is injective, we have W' also admits a dilation. In particular, it holds for W' being Weinstein of dimension at least 6.*

Proof. We assume $SH^*(W) \neq 0$, for otherwise, it is proven by Corollary B. By Proposition 4.10, W admits a dilation that implies there exists $x \in \ker \delta \cap \ker \Delta_+ \subset SH_+^*(W)$, such that $\phi(x) = 1$. In particular, $\Delta_\partial(x) = 1$. By Theorem G, we have an identification and $\Delta_\partial(x) = 1$ for W' . In particular, $\phi(x) = 1$ on W' , because in degree 0, we have both $\text{coker } \delta_\partial$ and $\text{coker } \delta$ spanned by 1 for both W and W' . We only need to prove x is from $SH^*(W')$. Since $\delta_\partial(x) = 0$ on W , we have $\delta_\partial(x) = 0$ on W' . Since $H^2(W') \rightarrow H^2(Y)$ is injective, then $\delta(x) = 0$ on W' . Then by Proposition 4.10, we have that W' admits a dilation. $\square$

The existence of symplectic dilations puts strong restrictions on the symplectic topology. Let W^{2n} be a Liouville domain with $c_1(W) = 0$ and $n > 1$, and assume W admits a dilation. Seidel [56] showed that there are at most finitely many Lagrangian spheres in W^{2n} that are pairwise disjoint. Given Corollary 4.17, we can ask the following natural question.

QUESTION 4.18. Let W be an ADC manifold with a symplectic dilation. Is there a universal upper bound for the maximal collections of pairwise disjoint Lagrangian spheres for all Weinstein fillings of ∂W ?

Similar to Corollary 3.14, we have following obstruction to Weinstein fillings.

COROLLARY 4.19. *Let Y be a $2n - 1$ dimensional ADC contact manifold and $n \geq 3$. If Y admits a topologically simple exact filling W and $\text{im } \Delta_\partial$ contains an element of grading greater than n , then Y does not admit Weinstein filling.*

Proof. Since Δ_∂ factors through $\text{coker } \delta = \text{coker}(SH_+^*(W) \rightarrow H^{*+1}(W))$, $\text{im } \Delta_\partial$ cannot contain an element of grading greater than n if W is Weinstein. By Theorem G, since Δ_∂ is independent of topologically simple exact fillings, in particular Weinstein fillings, the claim follows. $\square$

We apply Corollaries 3.14 and 4.19 in § 6 to construct many exactly fillable, but not Weinstein fillable manifolds. Although Corollaries 3.14 and 4.19 use the topology of W , those obstructions as contact invariants of the ADC boundary are not topological. In particular, we will show that they are very different from the topological criterion in [13] by proving Theorem I.

4.6. $\Delta_+, \phi, \Delta_\partial$ as invariants

In this subsection, we will explain that Δ_+ and ϕ are invariants of the exact domains up exact symplectomorphisms. It is clear that Δ_+, ϕ can be defined on Hamiltonians with finite slope and C^2 -small on W . Using Proposition 4.6, they commute with continuation maps, hence the direct limit of them define $\Delta_+ : SH_+^*(W) \rightarrow SH_+^{*-1}(W)$ and $\phi : \ker \Delta_+ \rightarrow \text{coker } \delta$. The following proposition implies that Δ_+, ϕ are invariants of the exact domains.

PROPOSITION 4.20. *Let $V \subset W$ be a subdomain, then Viterbo transfer maps perverse Δ_+ and ϕ . In particular, we have the following commutative diagram,*

$$\begin{array}{ccc} \ker \Delta_+ \subset SH_+^*(W) & \longrightarrow & \text{coker } \delta_W \\ \downarrow & & \downarrow \\ \ker \Delta_+ \subset SH_+^*(V) & \longrightarrow & \text{coker } \delta_V, \end{array}$$

where $\text{coker } \delta_W \rightarrow \text{coker } \delta_V$ is induced by

$$\begin{array}{ccc} SH_+^*(W) & \longrightarrow & H^{*+1}(W) \\ \downarrow & & \downarrow \\ SH_+^*(V) & \longrightarrow & H^{*+1}(V). \end{array}$$

Since the BV operator is the second term in the differential of the S^1 -equivariant symplectic (co)chain complexes [67], this proposition follows from the functoriality of positive S^1 -equivariant symplectic cohomology, which was proven in [28]. In fact, it is sufficient to consider the approximation ES^1 by S^3 in the S^1 -equivariant symplectic (co)homology. Using the fact that Viterbo transfer $SH_+^*(W \cup \partial W \times (1, r)) \rightarrow SH_+^*(W)$ is an isomorphism and exactly symplectomorphic exact domains have nested embeddings into each other [17, Proposition 11.8], Proposition 4.20 implies that Δ_+ and ϕ are invariants of exact domains up to exact symplectomorphisms.

Next we verify that Δ_+, ϕ defined on $H^*(C(\mathbf{H}_+, J_+))$ match with Δ_+, ϕ on $SH_+^*(W)$ naturally.

Proof of Proposition 4.7, 4.8, 4.9. We first prove Propositions 4.8 and 4.9. We need to relate the structure maps defined using $\mathbf{H}_\pm$ to those defined as limits of nondegenerate linear Hamiltonians. Let $H_\pm$ be the nondegenerate perturbation to $\mathbf{H}_\pm$ in the proof of Proposition 2.10. Let $H_{\pm,s}$ be the decreasing homotopies between them. Then Δ, Δ_+, Φ are defined similarly using $H_\pm$ as the integrated maximum principle holds for the moduli spaces. We pick two sequences of Hamiltonians with finite slope $H_\pm^{D_i^\pm}$ such that the following holds.

- (i) $H_\pm^{D_i^\pm} = H_\pm$ on W^i .
- (ii) $H_\pm^{D_i^\pm} = f'_\pm(c_i)r$ for $r \geq c_i$, where $c_i \in (a_i, b_i)$.
- (iii) $H_\pm^{D_i^\pm} \leq H_\pm^{D_{i+1}^\pm}$.
- (iv) $H_+^{D_i^+} \leq H_-^{D_i^-}$.

Then by the continuation maps on finite slope Hamiltonians and the compatibility with Δ following [1, §2.2.3], we have the following commutative diagram.

$$\begin{array}{ccccccc}
 H^*(C_{(+)}^{D_1^+}(H_+)) & \longrightarrow & H^*(C_{(+)}^{D_2^+}(H_+)) & \longrightarrow & \dots & \longrightarrow & H^*(C_{(+)}(H_+)) \\
 \uparrow & & \uparrow & & & & \uparrow \\
 H^*(C_{(+)}(H_+^{D_1^+})) & \longrightarrow & H^*(C_{(+)}(H_+^{D_2^+})) & \longrightarrow & \dots & \longrightarrow & \varinjlim H^*(C_{(+)}(H_+^{D_i^+})) \\
 \downarrow \Delta_{(+)} & & \downarrow \Delta_{(+)} & & & & \downarrow \Delta_{(+)} \\
 H^{*-1}(C_{(+)}(H_-^{D_1^-})) & \longrightarrow & H^{*-1}(C_{(+)}(H_-^{D_2^-})) & \longrightarrow & \dots & \longrightarrow & \varinjlim H^{*-1}(C_{(+)}(H_-^{D_i^-})) \\
 \downarrow \Delta_{(+)} & & \downarrow \Delta_{(+)} & & & & \downarrow \Delta_{(+)} \\
 H^{*-1}(C_{(+)}^{D_1^-}(H_-)) & \longrightarrow & H^{*-1}(C_{(+)}^{D_2^-}(H_-)) & \longrightarrow & \dots & \longrightarrow & H^{*-1}(C_{(+)}(H_-))
 \end{array}$$

The unmarked arrows are continuation maps, and the horizontal arrows on top and bottom row are inclusions, which are continuation maps induced by the trivial homotopy. By Proposition 4.6, the diagram also induces an commutative diagram of ϕ for different pairs of Hamiltonians $H_\pm, H^{D_i^\pm}$. This proves Propositions 4.8 and 4.9 for $H_\pm$.

Therefore to prove the claim, it is sufficient to prove commutativity of the following two diagrams (and the $+$ version).

$$\begin{array}{ccc}
 H^*(C(H_+)) & \longrightarrow & H^*(C(\mathbf{H}_+)) \\
 \downarrow & & \downarrow \Theta \\
 H^*(C(H_-)) & \longrightarrow & H^*(C(\mathbf{H}_-))
 \end{array}
 \quad
 \begin{array}{ccc}
 H^*(C(H_+)) & \longrightarrow & H^*(C(\mathbf{H}_+)) \\
 \downarrow \Delta & & \downarrow \Delta \\
 H^{*-1}(C(H_-)) & \longrightarrow & H^{*-1}(C(\mathbf{H}_-))
 \end{array}$$

where the unmarked arrows are continuation maps, and the horizontal ones are those in Proposition 2.10. They can be shown using a homotopy argument. The only new thing we need to verify is compactness and regularity for moduli spaces. Since it involves degenerate Hamiltonians $\mathbf{H}$, the moduli spaces have some cascades part. For the first diagram, let $\mathbf{H}_s = \rho(s)\mathbf{H}_- + (1 - \rho(s))\mathbf{H}_+$, $H_s = \rho(s)H_- + (1 - \rho(s))H_+$, they are used to define continuation maps Θ and $H^*(C(H_+)) \rightarrow H^*(C(H_-))$. Then we define a smooth homotopy of homotopy

$H_{s,t}^a$ in the following. Let χ be the increasing function such that $\chi(a) = a$ for $a \gg 0$ and $\chi(a) = 2$ near $a = 1$.

- (i) For $a > 1$, $H_{s,t}^a = H_{+,s-\chi(a)}$ for $s > 0$ and $H_{s,t}^a = \mathbf{H}_{s+\chi(a)}$ for $s \leq 0$.
- (ii) For $a < 0$, $H_{s,t}^a = H_{s-\chi(1-a)}$ for $s > 0$ and $H_{s,t}^a = H_{-,s+\chi(1-a)}$ for $s \leq 0$.
- (iii) For $0 \leq a \leq 1$, $H_{s,t}^a = \rho(a)H_{s,t}^0 + (1 - \rho(a))H_{s,t}^1$.

It is clear that we have $\partial_s H_{s,t}^a \leq 0$. Moreover, on (a_i, b_i) , we have the following three cases.

- (i) For $a > 1$, $H_{s,t}^a = f_+(r)$ when $s > 0$ and $H_{s,t}^a = \rho(s + \chi(a))f_-(r) + (1 - \rho(s + \chi(a)))f_+(r)$ when $s \leq 0$. Then $\partial_s(r\partial_r H_{s,t}^a - H_{s,t}^a) \leq 0$.
- (ii) For $a < 0$, $H_{s,t}^a = f_-(r)$ when $s \leq 0$ and $H_{s,t}^a = \rho(s - \chi(1 - a))f_-(r) + (1 - \rho(s - \chi(1 - a)))f_+(r)$ when $s > 0$. Then $\partial_s(r\partial_r H_{s,t}^a - H_{s,t}^a) \leq 0$.
- (iii) For $0 \leq a \leq 1$, $H_{s,t}^a = \rho(a)(\rho(s - 2)f_-(r) + (1 - \rho(s - 2))f_+(r)) + (1 - \rho(a))(\rho(s + 2)f_-(r) + (1 - \rho(s + 2))f_+(r))$, then $\partial_s(r\partial_r H_{s,t}^a - H_{s,t}^a) = (\rho(a)\rho'(s - 2) + (1 - \rho(a))\rho'(s + 2))(rf'_-(r) - f_-(r) - rf'_+(r) - f_+(r)) \leq 0$.

In particular, Lemma 2.5 can be applied to get compactness. The regularity is standard, except when $a \gg 0$ and breaks at W . In principle, it is a Morse–Bott breaking at W . However, in our special case, for $a \gg 0$, solutions near such breaking is isomorphic to solutions to the equation using $H_{+,s}$ shifted by $\chi(a)$. Hence such type of breaking is again a boundary corresponding to composing with Θ on $C_0(\mathbf{H}_+)$, which is identity. The proof for compatibility with Δ is the same. Since all the maps respect the $0, +$ splitting, the proof above also yields the identification on Δ_+ and ϕ by Proposition 4.6.

To prove Proposition 4.7, we can view $H_{\pm,s} \equiv \mathbf{H}_{\pm}$. Then the above construction yields a homotopy of Hamiltonians $H_{s,t}^a$, such that Lemma 2.5 can be applied. Then Proposition 4.7 follows from the standard homotopy argument. $\square$

5. Uniruledness

A variety is uniruled if and only if it is covered by a family of rational curves. Similarly, an affine variety is uniruled if it is covered by a family of rational curves possibly with punctures. In the symplectic setup, one can replace rational curve by pseudo-holomorphic rational curves. The uniruledness for Liouville domains was introduced in [45].

DEFINITION 5.1 [45, §2]. A $d\lambda$ -compatible almost complex structure J on W is convex if and only if there is a function ϕ such that:

- (i) ϕ attains its maximum on ∂W and ∂W is a regular level set,
- (ii) $\lambda \circ J = d\phi$ near ∂W .

This is more general than the cylindrical convex almost complex structure used in Definition 2.3, where $\phi = r$ near the boundary. A maximal principle still holds for holomorphic curves near ∂W using the function ϕ . Proposition 5.3 relates a general convex almost complex structure with a cylindrical convex almost complex structure.

DEFINITION 5.2 [45, Definition 2.2]. Let $k > 0$ be an integer and $\Lambda > 0$ a real number. We say that a Liouville domain (W, λ) is (k, Λ) -uniruled if, for every convex almost complex structure J on W and every point $p \in W^0$ where J is integrable on a neighborhood of p , there is a proper J -holomorphic map $u : S \rightarrow W^0$ to the interior W^0 of W passing through this point. We require that S is a genus 0 Riemann surface, the rank of $H_1(S; \mathbb{Q})$ is at most $k - 1$ and the area of u is at most Λ .

PROPOSITION 5.3. *Let Y be a contact manifold with a contact form α , and there is a function ϕ on $Y \times [1, 3]_r$, such that $\partial_r \phi > 0$ and $Y \times \{3\}$ is a level set. Assume J is $d(r\alpha)$ compatible almost complex structure, such that $r\alpha \circ J = d\phi$ on $Y \times [1, 2]$. Then there exists an extension of $d(r\alpha)$ compatible almost complex structure $\tilde{J}$, such that $r\alpha \circ \tilde{J} = d\phi$ on $Y \times [1, 3]$.*

Proof. The Liouville vector field on $Y \times [1, 3]$ is $r\partial_r$. Since $\partial_r \phi > 0$, every level set of ϕ is of contact type. On each level surface $\phi^{-1}(a)$, let ξ_a denote the contact structure $\ker \alpha \cap T\phi^{-1}(a)$ and E_x be the $d(r\alpha)$ complement of ξ_a for $x \in \phi^{-1}(a)$. Then J being compatible with $d(r\alpha)$ and $r\alpha \circ J = d\phi$ determines a complex structure on E_x and $J|_{\xi_a}$ is a $d(r\alpha)$ compatible almost complex structure, and those two descriptions are equivalent. It is clear we can extend J to $\tilde{J}$ maintaining such properties. $\square$

McLean proved that the symplectic uniruledness is equivalent to the algebraic uniruledness for affine varieties. In particular, the algebraic uniruledness is rather a symplectic property. In the following, we use the constructions in §§ 3 and 4 to prove that uniruledness is implied by the existence of symplectic dilation, which is, of course, a symplectic property. Unlike the results in other sections, we do not assume $c_1 = 0$ in this section. Hence all gradings should be understood as $\mathbb{Z}/2$ gradings.

THEOREM 5.4. *Let R be a ring and W a Liouville domain such that $SH^*(W; R) = 0$ or there exists a symplectic dilation in $SH^1(W; R)$, then W is $(1, \Lambda)$ -uniruled for some Λ .*

REMARK 5.5. For Liouville domain with vanishing symplectic cohomology, a version of uniruledness different from Definition 5.2 was obtained, see [5]. However, to obtain Theorem 5.10, we need to achieve the more refined version of uniruledness in Definition 5.2.

Proof. We write $W_\delta = W \setminus \partial W \times (1 - \delta, 1]$. Then by [45, Theorem 2.3], there exists a $\Lambda > 0$ such that W is $(1, \Lambda)$ uniruled if and only if W_δ is $(1, \Lambda')$ uniruled for some $\Lambda' > 0$. Assume $SH^*(W; R) = 0$, then $1 \in \text{im}(SH_+^{-1}(W; R) \rightarrow H^0(W; R))$. That is, $1 \in \text{im}(H^{-1}(C_+(\mathbf{H}, J)) \rightarrow H^0(C_0(f)))$, in particular this means that for some $i > 0$, $1 \in \text{im}(H^{-1}(C_+^{D_i}(\mathbf{H}, J)) \rightarrow H^0(C_0(f)))$. Since 1 is generated by the unique local minimum point m of f , for a generic almost complex structure J , we have a dimension 0 compact manifold $\mathcal{M}_{m,x} \neq \emptyset$ for some $x \in \mathcal{P}^*(\mathbf{H})$ with grading $|x| = -1$ and action $\geq -D_i$. Since m is a minimum, an element of $\mathcal{M}_{m,x}$ is simply a solution to the following:

$$u : \mathbb{C} \rightarrow \widehat{W}, \quad \partial_s u + J(\partial_t u - X_{\mathbf{H}}) = 0, \quad u(0) = m, \quad \lim_{s \rightarrow \infty} u = x. \quad (5.1)$$

For any convex almost complex structure J_δ on W_δ , assume that ϕ is the function in Definition 5.1, and we may assume $\phi|_{\partial W_\delta} = 1 - \delta$. Then we can extend ϕ to $\widehat{\phi}$ on $\widehat{W}$ such that $\partial_r \widehat{\phi} > 0$ on $\partial W \times [1 - \delta, \infty)$ and $\widehat{\phi} = r$ when $r \geq 1$. By Proposition 5.3, J_δ can be extended to a $J \in \mathcal{J}(W)$. We may assume $J \in \mathcal{J}_{\text{reg}}^{\leq 1}(\mathbf{H}, f, g)$, since we perturb J near $\mathcal{P}^*(\mathbf{H})$ to achieve transversality. Therefore we have a curve u solving (5.1) and when restricted to W_δ it is a J_δ -curve. Such curve has an energy bound by D_i . Let S denote the connected component of 0 of $u^{-1}(W_\delta^0)$. We claim $H_1(S; \mathbb{Q})$ must be rank 0. We can find a small $0 < \epsilon < \delta$, such that $S' \subset u^{-1}(W \setminus \partial W \times (1 - \epsilon, 1])$ is the connected compact Riemann surface containing 0 with boundary and $u(\partial S') \subset \partial W \times \{1 - \epsilon\}$ and J is cylindrical convex on $\partial W \times \{1 - \epsilon\}$. Then S' is a disk, for otherwise, there is a domain $D \subset \mathbb{C}$ diffeomorphic to a disk, such that $u|_D$ solves (5.1) and $u(\partial D) \subset \partial W \times \{1 - \epsilon\}$, this contradicts Lemma 2.5. Note that $S \subset S'$. If $H_1(S; \mathbb{Q})$ is not rank 0, then there is a loop $\gamma \subset S$ bounding a disk $D \subset S'$ and D is not contained in S . Since we have $\widehat{\phi} \circ u|_\gamma < 1 - \delta$ and $\max \widehat{\phi} \circ u|_{D \setminus S} \geq 1 - \delta$, which contradicts the maximal

principle for holomorphic curves, $u|_S$ is the J_δ -curve we are looking for in W_δ passing through m .

Next, we need to show such construction can be applied to any m with a universal energy bound D_i . What we need is a universal D_i , such that $1 \in \text{im}(H^0(C_+^{D_i}(\mathbf{H}, J, f)) \rightarrow H^{-1}(C_0(f)))$ for any admissible Morse function f . This can be seen from Proposition 2.10, since there is a D^i , such that $1 \in \text{im}(H^{-1}(C_+^{D^i}(H)) \rightarrow H^0(C_0(H)))$, where H is the perturbation of $\mathbf{H}$. Therefore W_δ is $(1, D_i)$ -uniruled.

Next, we assume $SH^*(W; R) \neq 0$ and admits a dilation. Then by Proposition 4.10, we have $1 \in \text{im } \phi$. By the same argument as above, there is a D_i^+ , such that

$$\phi^{D_i^+} : \ker \Delta_+^{D_i^+} \subset H^1(C_+^{D_i^+}(\mathbf{H}_+, J_+)) \rightarrow H^0(C_0(f)) / \text{im } \delta^{D_i^-}$$

contains 1 in the image. Since $SH^*(W; R) \neq 0$, we have $\text{im } \delta^{D_i^-}$ does not contain 1. Therefore we have $x \in C_+^{D_i^+}(\mathbf{H}_+, J_+)$, $b \in C_+^{D_i^-}(\mathbf{H}_-, J_-)$, such that $\Delta_{+,0}(x) + d_{+,0}(b) = m + c$ for $c \in C_0^*(f)$ is a cochain representing class in $\text{im } \delta^{D_i^-}$, where m is the minimum point of f .[†] Similar to the argument above, since m is a minimum, for generic J_+, J_- and $J_{s,t}^\theta$, either there is a $x \in \mathcal{P}^*(\mathbf{H}_+)$ and $\theta \in S^1$ and a solution to

$$\partial_s u + J_{s,t}^\theta (\partial_t u - X_{\mathbf{H}_{s,t}^\theta}) = 0, u(0) = m, \lim_{s \rightarrow \infty} u = x;$$

or there is a $b \in \mathcal{P}^*(\mathbf{H}_-)$ and a solution to

$$\partial_s u + J_- (\partial_t u - X_{\mathbf{H}_-}) = 0, u(0) = m, \lim_{s \rightarrow \infty} u = b. \quad (5.2)$$

We may assume $J_\pm, J_{s,t}^\theta$ restrict to W_δ is J_δ as before. Then in either case, when u restricted to the preimage of W_δ^0 is a J_δ -curve passing through m , with energy bound D_i^- , note that integrated maximal principle can also be applied to $r = 1 - \epsilon$ for $\mathbf{H}_{s,t}^\theta$, since $\mathbf{H}_{s,t}^\theta \equiv 0$ there and $\partial_s \mathbf{H}_{s,t}^\theta \leq 0$. Hence the component of $u^{-1}(W_\delta^0)$ containing 0 has rank 0 in rational first homology in either case as before. Then by the functoriality in § 4.6, we can change f as before to find curves passing through any point with a universal energy bound. $\square$

REMARK 5.6. If one considers the holomorphic planes with one marked point in the completion, then virtual dimension of such space is given by $\mu_{CZ}(\gamma) + n - 1$, when the plane is asymptotic to a Reeb orbit γ . To hope the evaluation map at the marked point covers every point in W , we need $\mu_{CZ}(\gamma) \geq n + 1$. If we expect the evaluation map to behave like a covering map, then we need $\mu_{CZ}(\gamma) = n + 1$. In many cases, the vanishing of symplectic cohomology is due to a Reeb orbit of index $n + 1$. Note that a nondegenerate Reeb orbit of index $n + 1$ can be perturbed into two nondegenerate Hamiltonian orbits with indices $n + 1, n + 2$. In our grading convention, the index $n + 1$ Hamiltonian orbits is of grading -1 , which is often responsible for the vanishing. This is the case for the standard symplectic ball. On the other hand, when we consider dilation $x \in SH^1(W)$, the associated Reeb orbit is of index $n - 1$ or $n - 2$, hence the uniruledness should not be provided by such Reeb orbits. However, the $b \in C_+^{-1}(\mathbf{H}_-, J_-)$ in the proof of Theorem 5.4 provides the Reeb orbit of the right index. This suggests that the uniruledness should be from a solution to (5.2). In fact, this is the case for T^*S^n . We will investigate them in more detail in [69].

REMARK 5.7. On the other hand, 1-uniruledness does not imply the existence of symplectic dilation. For example, we take a smooth cubic hypersurface V in $\mathbb{CP}^6$ and W is the complement of the intersection of V with a generic hyperplane. Then W is an affine variety and W does not admit a symplectic dilation by degree reasons, see [56, Example 2.7]. However, by [33, Corollary 5.4], W is 1-uniruled.

[†]When $c_1(W) = 0$, we have $c = 0$ by degree reasons.

REMARK 5.8. In fact, the vanishing of symplectic cohomology and existence of symplectic dilation are the first two simplest conditions implying uniruledness in a whole family. The next one is whether 1 is in the image of the map $\Delta^2 : \ker \Delta \rightarrow \operatorname{coker} \Delta$, which is defined using Δ and the homotopy operator in the $(\Delta)^2 = 0$ relation and is with a degree shift by 3. One way of packaging all the homotopic relations is using the S^1 -equivariant theory as in [67], then the spectral sequence from the u -filtration (on any S^1 -cochain complex) inducing maps $\Delta^{i+1} : \ker \Delta^i \rightarrow \operatorname{coker} \Delta^i$ with Δ^1 is the BV operator considered here. We say W admits a k -dilation if and only if $1 \in \operatorname{im} \Delta^k$. In fact, the counterexample in Remark 5.7 satisfies that $1 \in \operatorname{im} \Delta^2$, hence it has a 2-dilation but no 1-dilation. Every such structure has a related map ϕ^i defined on a subspace of $SH_+^*(W)$ to a quotient space of $H^*(W)$ generalizing ϕ in (4.5). And there is a boundary version for all i in a similar way to $\Delta_\delta, \delta_\partial$. These structures have similar property to $\delta_\partial, \Delta_\partial$ in §§ 3 and 4. That is, they are independent of fillings for ADC manifolds and 1 being in the image of any of them implies uniruledness. They can be used to develop more obstructions to Weinstein fillability. Moreover, they give an infinity hierarchy on the complexity of symplectic manifolds, and also an infinity hierarchy on the complexity of ADC contact manifolds. Details of such construction will appear in the sequel paper [69]. The existence of k -dilation for some k is equivalent to the existence of cyclic dilation for $h = 1$, which is defined independently by Li [38], where the open string implication of cyclic dilations is studied.

REMARK 5.9. From the proof of Theorem 5.4, it is clear that $1 \in \operatorname{im} \Delta_\partial$ would also imply uniruledness. By Proposition 4.10, $1 \in \operatorname{im} \Delta_\partial$ is potentially weaker than existence of dilation.

Using Theorem 5.4, nonuniruledness is an implication of nonexistence of symplectic dilation in any coefficient. Since log Kodaira dimension provides an obstruction to uniruledness, we have the following.

THEOREM 5.10. *Let W be an affine variety. In either of the following cases, we have $SH^*(W; R) \neq 0$ and there is no symplectic dilation using any coefficient ring R .*

- (i) *The log Kodaira dimension is not $-\infty$.*
- (ii) *W admits a projective variety V compactification, such that V is not uniruled, for example, the Kodaira dimension of V is not $-\infty$.*

Proof. By [45, Theorem 2.5, Lemma 7.1], if W does not have log Kodaira dimension $-\infty$, W cannot be $(1, \Lambda)$ -uniruled for any Λ . Then by Theorem 5.4, we have $SH^*(W, R) \neq 0$ and there is no symplectic dilation for any coefficient ring R .

Let V be projective compactification of W . If we have $SH^*(W, R) = 0$ or there is a symplectic dilation, then W is $(1, \Lambda)$ -uniruled. By [45, Theorem 2.5], W is 1-algebraically uniruled. Hence V is algebraically uniruled and the Kodaira dimension of V is $-\infty$, hence it is a contradiction. $\square$

Now, Corollary M follows from Theorem 5.10 directly. Symplectic exotic $\mathbb{C}^n$ was constructed in [2, 44, 57] for all $n \geq 3$. The proofs of exoticity were based on nonvanishing of symplectic cohomology for some coefficient, but the mechanisms for nonvanishing were very different. In view of Corollary M, the exoticity can also be explained by the failure of uniruledness.

EXAMPLE 5.11. Seidel–Smith [57] proved that product $M^m, m \geq 2$, is an exotic $\mathbb{C}^{2m}$ for Ramanujam surface M [51] by showing that M contains an essential Lagrangian, hence $SH^*(M) \neq 0$ and M^m is exotic. Since M has log Kodaira dimension 2, we have M^m has log

Kodaira dimension $2m$. Therefore by Theorem 5.10, $SH^*(M^m) \neq 0$ and $SH^*(M^m)$ does not admit symplectic dilations. This reproves Seidel–Smith’s result.

REMARK 5.12. There are many algebraic exotic $\mathbb{C}^n$ with log Kodaira dimension $-\infty$, for example, $M \times \mathbb{C}$ for Ramanujam surface M . However, $M \times \mathbb{C}$ is symplectically standard by h-principle, since it is subcritical and diffeomorphic to $\mathbb{C}^3$. The atomic exotic $\mathbb{C}^4$ considered by McLean [44] from Kaliman modification also has negative log Kodaira dimension. Therefore log-Kodaira dimension being nonnegative is not the criterion of being symplectic exotic.

By Corollary M, one can look for symplectic exotic $\mathbb{C}^n$ in complex exotic $\mathbb{C}^n$. They exist in abundance when $n \geq 3$, see [66].

Nonvanishing of symplectic cohomology also implies nondisplaceability [32]. Hence we have the following.

EXAMPLE 5.13. Let V be a projective variety with nonnegative Kodaira dimension. Let $W = V \setminus D$ be an affine variety for a divisor D . Then $SH^*(W) \neq 0$, and W is not displaceable. That is, when we view W as a Liouville domain, there is no Hamiltonian $F \in C_c^\infty(S^1 \times \widehat{W})$ such that the generated time one Hamiltonian diffeomorphism ϕ_F displaces W , that is, $\phi_F(W) \cap W = \emptyset$. This holds, in particular, if V is Calabi–Yau, since the Kodaira dimension is 0.

As showed in §§3 and 4, the vanishing of symplectic cohomology and the existence of symplectic dilations are properties independent of fillings for ADC contact manifolds. A natural question is whether uniruledness shares the similar property.

QUESTION 5.14. Is uniruledness independent of the (topologically simply) filling for ADC contact manifold?

6. Constructions of ADC manifolds

As explained in Example 3.7, ADC contact manifolds exist in abundance. Moreover, a flexible handle attachment does not change the ADC property [37]. In this section, we provide two other simple constructions of ADC manifolds, which provides many examples, where Corollaries 3.14 and 4.19 can be applied. In particular, we will prove Theorem I.

6.1. Product with the complex plane

In this subsection, we show that the boundary of $V \times \mathbb{C}$ is 0-ADC for any Liouville domain V such that $c_1(V) = 0$ and $\dim V > 0$. Moreover by [48], $V \times \mathbb{C}$ has vanishing symplectic cohomology. Since there are examples of $V \times \mathbb{C}$ such that it cannot be a Weinstein domain, such construction proves many nonflexible examples where results from §3 can be applied. In the following, we fix the symplectic form on $\mathbb{C}$ by $d(r^2 d\theta)$.

DEFINITION 6.1. Let V be a connected manifold with nonempty boundary, the Morse dimension $\dim_M V$ is defined to be the minimum of the max index of an admissible Morse function on V , then $\dim_M V \leq \dim V - 1$.

We also introduce the notion of tamed asymptotically dynamically convex contact manifold, which is ADC and α_i does not collapse to 0. This will be important for our discussion on nonexact fillings.

DEFINITION 6.2. (Y, ξ) is k -TADC if there exist contact forms $\alpha_1 > \alpha_2 > \dots$, positive numbers $D_1 < D_2 < \dots \rightarrow \infty$ and a contact form α , such that $\alpha_i > \alpha$ and all elements of $\mathcal{P}^{<D_i}(\alpha_i)$ are nondegenerate and have degree greater than k . We have similar definition for exact domains.

Basic examples of (0-)TADC manifolds are index-positive contact manifolds, for example, cotangent bundles when dimension of the base is at least 4. The other case is index-positive in the Morse–Bott sense, for example, Example 4.3. That they are TADC follows from that up to action D , there exist small perturbations into nondegenerate contact forms with prescribed Conley–Zehnder indexes [8, Lemmas 2.3 and 2.4], and the perturbation can be made arbitrarily small so that the perturbed contact form does not collapse to 0. The following result will provide some more examples of TADC manifolds.

THEOREM 6.3. *Let V be a $2n$ -dimensional Liouville domain such that $c_1(V) = 0$ and $n > 0$. We define Y to be the contact boundary of $V \times \mathbb{C}$, then we have the following.*

- (i) Y is 0-ADC.
- (ii) *If ∂V supports a nondegenerate contact form such that all Reeb orbits are contractible in V , then Y is strongly 0-ADC.*

In fact, Y is (strongly) $(2n - \dim_M V - 1)$ -ADC. If V is k -TADC for some k , then Y is TADC.

We will prove the theorem by setting up the following propositions.

PROPOSITION 6.4. *Assume Y is a contact manifold with a contact form α and V is a Liouville domain with a Liouville form λ_V . Let f be a strictly positive function on V , then $\alpha_f := f\alpha + \lambda_V$ is a contact form on $Y \times V$ if and only if $d(\frac{1}{f}\lambda_V)$ is a symplectic form on V . The Reeb vector field is given by $R_f := \frac{1}{f}R_\alpha + \widetilde{X_{\frac{1}{f}}}$. Here $\widetilde{X_{\frac{1}{f}}} - X_{\frac{1}{f}} \in \langle R_\alpha \rangle$ and $\alpha_f(\widetilde{X_{\frac{1}{f}}}) = 0$ and $X_{\frac{1}{f}}$ is the Hamiltonian vector field for $\frac{1}{f}$ using the symplectic form $d(\frac{1}{f}\lambda_V)$.*

Proof. Assume $\dim Y = 2m + 1$ and $\dim V = 2n$. To show α_f is a contact form, it is sufficient to show $\frac{1}{f}\alpha_f$ is a contact form. Note that we have

$$\begin{aligned} \frac{1}{f}\alpha_f \wedge \left(d\left(\frac{1}{f}\alpha_f\right)\right)^{m+n} &= \left(\alpha + \frac{1}{f}\lambda_V\right) \wedge \left(d\alpha + d\left(\frac{1}{f}\lambda_V\right)\right)^{m+n} \\ &= \binom{m+n}{m} \left(\alpha + \frac{1}{f}\lambda_V\right) \wedge (d\alpha)^m \wedge \left(d\left(\frac{1}{f}\lambda_V\right)\right)^n \\ &= \binom{m+n}{m} \alpha \wedge (d\alpha)^m \wedge \left(d\left(\frac{1}{f}\lambda_V\right)\right)^n. \end{aligned}$$

Hence α_f is a contact form if and only if $d(\frac{1}{f}\lambda)$ is nondegenerate. Note that by construction, we have $\alpha_f(R_f) = 1$. Moreover, we have

$$\begin{aligned} \iota_{R_f} d(\alpha_f) &= \iota_{R_f} d\left(f\left(\alpha + \frac{1}{f}\lambda_V\right)\right) \\ &= \iota_{R_f} df \wedge \left(\alpha + \frac{1}{f}\lambda_V\right) + \iota_{R_f} f d\left(\alpha + \frac{1}{f}\lambda_V\right). \end{aligned}$$

Since $\alpha_f(\widetilde{X_{\frac{1}{f}}}) = 0$, we have $\iota_{R_f}(\alpha + \frac{1}{f}\lambda_V) = \frac{1}{f}$. Since $\iota_{R_f}df = 0$, $\iota_{R_f}d\alpha = 0$ and $\iota_{R_f}d(\frac{1}{f}\lambda_V) = -d\frac{1}{f}$ by definition, we have $\iota_{R_f}d(\alpha_f) = -\frac{df}{f} - fd\frac{1}{f} = 0$. Hence R_f is the associated Reeb vector. $\square$

REMARK 6.5. Note that $Y \times V$ can be viewed as a hypersurface $r = f$ in the Liouville cobordism $Y \times \mathbb{R}_+ \times V$ with Liouville form $r\alpha + \lambda_V$. Then this hypersurface is of contact type if the Liouville vector $r\partial_r + X_{\lambda_V}$ is transverse to the surface and pointing out, that is $(r\partial_r + X_{\lambda_V})(r - f) = f - X_{\lambda_V}f > 0$.

REMARK 6.6. Proposition 6.4 can be viewed as a generalization of the computation carried out for prequantization bundles. Let γ be a Reeb trajectory on Y , then over $\gamma \times V$ we have a connection given by $\alpha + \frac{1}{f}\lambda_V$ such that the curvature is the symplectic form $d(\frac{1}{f}\lambda_V)$. The contact structure is given by the horizontal subspace direct sum with the contact structure of Y , the splitting holds in the symplectic sense.

We will divide the boundary of $V \times \mathbb{C}$ into two parts, one of them is diffeomorphic to $\partial V \times \mathbb{D}$ and the other is diffeomorphic to $V \times S^1$, where $\mathbb{D} \subset \mathbb{C}$ is a disk. In the following, we discuss the Reeb dynamics on the $\partial V \times \mathbb{D}$ part.

PROPOSITION 6.7. Let (Y, ξ) be a contact manifold with a contact form α . We fix a small $\epsilon > 0$. Assume g is a smooth function on $[0, 1]$ such that the following holds.

- (i) $g(x) = x$ for x near 0, $g(1 - \epsilon) = 1$.
- (ii) $g(x)$ is increasing and $g''(x) > 0$ unless $g(x) = x$.

Then for $\rho \in \mathbb{R}_+$, $(2 - g(\rho r^2))\alpha + r^2 d\theta$ is a contact form on $Y \times \mathbb{D}_{\sqrt{1-\epsilon/\rho}}$. And Reeb orbits of it are of the form $(\gamma(At), re^{Bit+\theta_0})$, where γ is a Reeb orbit of R_α on Y and

$$A = \frac{1}{2 - g(\rho r^2) + \rho r^2 g'(\rho r^2)}, \quad B = \frac{\rho g'(\rho r^2)}{2 - g(\rho r^2) + \rho r^2 g'(\rho r^2)}.$$

If γ is nondegenerate of period D , then we have the following.

- (i) When $r = 0$, then the Reeb orbit $(\gamma(\frac{1}{2}t), 0, 0)$ is nondegenerate if and only if $\frac{D\rho}{2\pi} \notin \mathbb{N}$ and the Conley–Zehnder index is given by $\mu_{CZ}(\gamma) + 2\lfloor \frac{D\rho}{2\pi} \rfloor + 1$.
- (ii) When ρr^2 is in the domain where $g(x) \neq x$, then $(\gamma(At), re^{Bit+\theta_0})$ is an orbit if and only if $D\rho g'(\rho r^2) \in 2\pi\mathbb{N}$. It is a Morse–Bott S^1 -family of orbits with period $\frac{D}{A}$, and the generalized Conley–Zehnder index is given by $\mu_{CZ}(\gamma) + \frac{D\rho g'(\rho r^2)}{\pi} + \frac{1}{2}$.

Proof. Note that $(2 - g(\rho r^2)) - \frac{r}{2}\partial_r(2 - g(\rho r^2)) = 2 - g(\rho r^2) + \rho r^2 g'(\rho r^2) > 0$ on $Y \times \mathbb{D}_{\sqrt{1-\epsilon/\rho}}$. Then by Remark 6.5, $\alpha_g := (2 - g(\rho r^2))\alpha + r^2 d\theta$ is a contact form. By Proposition 6.4, the Reeb vector field is given by

$$R := \frac{1}{2 - g(\rho r^2)} R_\alpha + \widetilde{X}_{1/(2-g(\rho r^2))}.$$

By a direct computation, we have

$$\widetilde{X}_{1/(2-g(\rho r^2))} = \frac{\rho g'(\rho r^2)}{2 - g(\rho r^2) + \rho r^2 g'(\rho r^2)} \partial_\theta + \frac{-\rho r^2 g'(\rho r^2)}{(2 - g(\rho r^2) + \rho r^2 g'(\rho r^2))(2 - g(\rho r^2))} R_\alpha.$$

Therefore the Reeb vector is

$$R = \frac{1}{2 - g(\rho r^2) + \rho r^2 g'(\rho r^2)} R_\alpha + \frac{\rho g'(\rho r^2)}{2 - g(\rho r^2) + \rho r^2 g'(\rho r^2)} \partial_\theta = AR_\alpha + B\partial_\theta.$$

Hence all Reeb orbits are in the form prescribed in the proposition.

As explained in Remark 6.6, the contact structure on $Y \times \mathbb{D}_{\sqrt{1-\epsilon/\rho}}$ is the direct sum of ξ with kernel of $\alpha + \frac{r^2}{2-g(\rho r^2)}d\theta$ in the space $\langle R_\alpha \rangle \oplus T\mathbb{D}_{\sqrt{1-\epsilon/\rho}}$. Since the Reeb vector is tangent to $Y \times S_r^1$ and the R_α component is a constant, we know that Reeb flow of R preserves ξ and the splitting. Therefore when computing Conley–Zehnder index, it is sufficient to figure out the Conley–Zehnder index of the horizontal direction. In the $r = 0$ case, the linearized return map is given by the linearized return map of γ direct sum with the rotation by $e^{D\rho i}$ in the horizontal direction. Therefore it is nondegenerate if and only if $\frac{D\rho}{2\pi} \notin \mathbb{N}$ and the Conley–Zehnder index in the horizontal direction is $1 + 2\lfloor \frac{D\rho}{2\pi} \rfloor$.

When ρr^2 is in the domain where $g(x) \neq x$, then periodic orbits are in the form of $(\gamma(At), re^{Bit+\theta_0})$, hence they always come in an S^1 family for $\theta_0 \in S^1$. To verify that it is a Morse–Bott in the sense of [8, Definition 1.7], note that the linearized return map ϕ is the direct sum of the linearized return map of γ and the return map on the horizontal direction is given by

$$\phi_T = \begin{bmatrix} 1 & 0 \\ T \frac{d}{dr} \frac{\rho g'(\rho r^2)}{2 - g(\rho r^2) + \rho r^2 g'(\rho r^2)} & 1 \end{bmatrix},$$

where we use $\partial_r, \partial_\theta$ as the basis and T is the period $D\rho g'(\rho r^2)$. Note that we have

$$\frac{d}{dr} \frac{\rho g'(\rho r^2)}{2 - g(\rho r^2) + \rho r^2 g'(\rho r^2)} = \frac{2\rho r g''(\rho r^2)(2 - g(\rho r^2)) + 2\rho^2 r (g'(\rho r^2))^2}{(2 - g(\rho r^2) + \rho r^2 g'(\rho r^2))^2} > 0.$$

Therefore $\ker(\phi_T - \text{id})$ is spanned by ∂_θ . Hence the 1-eigenspace of the total linearized return map is spanned by ∂_θ and R_α , which is the tangent space of $\text{im } \gamma \times S_r^1$. Moreover, $d\alpha_g = (2 - g(\rho r^2))d\alpha - 2\rho r g'(\rho r^2)dr \wedge \alpha + 2rdr \wedge d\theta$ is rank 0 on $\text{im } \gamma \times S_r^1$. Therefore such Reeb orbits are of Morse–Bott type. To compute the generalized Maslov index, it is enough to compute the generalized Maslov index in the horizontal direction. Note that the linearized map is given by

$$\phi_t := \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ Ct & 1 \end{bmatrix}, \quad 0 \leq t \leq T = \frac{D\rho g'(\rho r^2)}{2\pi},$$

where $C = \frac{2\rho r g''(\rho r^2)(2-g(\rho r^2))+2\rho^2 r (g'(\rho r^2))^2}{(2-g(\rho r^2)+\rho r^2 g'(\rho r^2))^2} > 0$. Therefore the generalized Conley–Zehnder index [54] is given by $\frac{D\rho g'(\rho r^2)}{\pi}$ from the rotation plus the generalized Conley–Zehnder index of

$$\begin{bmatrix} 1 & 0 \\ Ct & 1 \end{bmatrix}, \quad 0 \leq t \leq T.$$

The latter is a symplectic shear. Since $C > 0$, the index is given by $\frac{1}{2}$. Then the total generalized Conley–Zehnder index is given by $\mu_{CZ}(\gamma) + \frac{D\rho g'(\rho r^2)}{\pi} + \frac{1}{2}$. $\square$

On the complement $V \times S^1$, we will use the following lemma to guarantee all Reeb orbits are either over critical points or have very large period.

LEMMA 6.8. *Let (V, ω) be a compact symplectic manifold possibly with boundary. For all $C > 0$, there exists $\epsilon \in \mathbb{R}^+$, such that for all functions H with $|\mathrm{d}H|_{C^0} < \epsilon$ and all symplectic forms ω' such that $|\omega' - \omega|_{C^1} < \epsilon$, we have all nonconstant periodic orbits of Hamiltonian vector field X_H using symplectic form ω' have period at least C .*

Proof. Let g be a Riemannian metric on V . We can find a cover of V by finitely many Darboux charts $\{U_i\}$, such that there exists $\delta > 0$ so that every δ -ball is contained in one of the Darboux charts. Moreover, for smaller enough ϵ' , we can assume that each U_i is a Darboux chart of ω' for $|\omega' - \omega|_{C^1} < \epsilon'$. Using the standard metric on each $U_i \subset \mathbb{R}^{2n}$ by viewing U_i as a ω Darboux chart, we have a C^0 norm on $\Omega^1(V)$. Since ω' is C^1 close to ω , we know that the induced C^0 norm on $\Omega^1(V)$ are uniformly equivalent for any ω' nearby and is equivalent to the C^0 norm we use to state the lemma. If $|\mathrm{d}H|_{C^0} < \frac{\delta}{C}$, then any trajectory $\phi(t)$ of X_H has the property that $\mathrm{d}(\phi(0), \phi(C)) < \delta$. By [6, Proposition 6.1.5], on each Darboux chart, if $|\mathrm{d}H|_{C^0} < \frac{2\pi}{C}$, then any C -periodic orbit in the chart is a constant. Therefore the claim holds for $\epsilon = \min(\frac{\delta}{C}, \frac{2\pi}{C}, \epsilon')$. $\square$

Proof of Theorem 6.3. Since we have $c_1(V) = 0$, then $c_1(V \times \mathbb{C}) = 0$. Hence $c_1(Y) = 0$, that is, the Conley–Zehnder index is well defined in $\mathbb{Z}$ for contractible orbits on Y . Now we pick a Liouville form λ on V such that the associated Reeb dynamic on ∂V is nondegenerate. Then there exist positive real numbers converging to infinity $D_1 < D_2 < \dots$ and integers $N_1, N_2, \dots$, such that all periodic orbits of R_λ contractible in V with period smaller than D_i have Conley–Zehnder index greater than N_i . We consider orbits contractible in V , since (γ, p) for $\gamma \in \mathcal{L}\partial V, p \in \mathbb{C}$ is homotopically trivial on $Y := \partial(V \times \mathbb{C})$ if γ is contractible in V .

We view Y as the hypersurface $Y_{\rho, f, g}$ in $\widehat{V} \times \mathbb{C}$ such that $Y_{\rho, f, g}^1 := Y_{\rho, f, g} \cap (\partial V \times [1, \infty)_t \times \mathbb{C})$ is given by $t = 2 - g(\rho r^2)$ for big $\rho \in \mathbb{R}_+$ and function g as in Proposition 6.7, and $Y_{\rho, f, g}^2 := Y_{\rho, f, g} \cap (V \times \mathbb{C})$ is given by $r^2 = \frac{f}{\rho}$ for a Morse function f on V such that near the boundary $f = g^{-1}(2 - t)$, where t is the collar coordinate smaller than 1. To see $Y_{\rho, f, g}$ is of contact type, it is sufficient to prove that the Liouville vector field $X_\lambda + \frac{1}{2}r\partial_r$ is transverse to Y_ρ and points out. On $Y_{\rho, f, g}^1$, this is verified in Proposition 6.7. On $Y_{\rho, f, g}^2$, we have $(X_\lambda + \frac{1}{2}r\partial_r)(r^2 - \frac{f}{\rho}) = r^2 - \frac{1}{\rho}X_\lambda f = \frac{1}{\rho}(f - X_\lambda f)$. Note that if $g'(1 - \epsilon)$ is very big, $f'(1) = -\frac{1}{g'(1 - \epsilon)}$ is very small, hence we can choose f such that $X_\lambda f$ is very small and $f \geq 1 - \epsilon$. Hence $Y_{\rho, f, g}$ is of contact type. And for $\rho_1 > \rho_2$, there exists such f_1, g_1, f_2, g_2 so that the domain between Y_{ρ_1, f_1, g_1} and Y_{ρ_2, f_2, g_2} is a Liouville cobordism. Using the Liouville vector field, we have the induced contact form on Y_{ρ_1, f_1, g_1} is smaller than the induced contact form on Y_{ρ_2, f_2, g_2} .

On $Y_{\rho, f, g}^1$, by Proposition 6.7, the Reeb orbits of type (i) of action bounded by D_i have Conley–Zehnder indices greater than $N_i + 2\lfloor \frac{D_i \rho}{2\pi} \rfloor + 1$. Since $A < 1$ in Proposition 6.7, Reeb orbits of type (ii) of action bounded by D_i is of Morse–Bott type in S^1 -families with generalized Conley–Zehnder indices greater than $N_i + \lfloor \frac{D_i \rho}{\pi} \rfloor + \frac{1}{2}$, since $g' \geq 1$. On $Y_{\rho, f, g}^1$, it is possible that there are periodic orbits not described by (i) and (ii) in Proposition 6.7. That is, those orbits with ρr^2 in the domain where $g(x) = x$. Then we have $A = \frac{1}{2}$ and $B = \frac{\rho}{2}$. Then if we pick ρ such that $|\gamma| \rho \notin 2\pi\mathbb{N}$ for all Reeb orbits γ on $(\partial V, \lambda_V)$ with period smaller than D_i , then there are no Reeb orbits with period smaller than $2D_i$ on this domain.

$Y_{\rho, f, g}^2$ is contactomorphic to $V \times S^1$ with contact form $\lambda + \frac{\rho}{f}d\theta$. Then by Proposition 6.4, the Reeb vector is given by $\frac{\rho}{f}\partial_\theta + \widetilde{X_{\frac{\rho}{f}}}$. Therefore there are two types of Reeb orbits. One is of form $(p, \phi_k(t))$ where p is a critical point of $\frac{\rho}{f}$ and $\phi_k(t)$ is a reparametrization of $e^{ikt}, t \in [0, 2\pi]$. The other type is γ such that $\pi \circ \gamma$ is a nonconstant periodic orbit of $X_{\frac{\rho}{f}}$ for the projection $\pi : V \times S^1 \rightarrow V$. When p is a nondegenerate critical point of $\frac{\rho}{f}$, we have $\mu_{CZ}(p, \phi_k(t)) = n - \mathrm{ind} p + 2k$. If we choose f such that critical points of $\frac{1}{f}$ have max index $\dim_M V$, then we have the degree

of $(p, \phi_k(t))$ is at least $n - \dim_M V + 2k + n + 1 - 3 = 2n - \dim_M V + 2k - 2 \geq 2n - \dim_M V$. Since $\dim_M V \leq 2n - 1$, those orbits have positive degree. For the other type of periodic orbits, periodic orbits of $X_{\frac{\rho}{f}}$ using symplectic form $d(\frac{\rho}{f}\lambda)$ is the same as periodic orbits of $X_{\frac{1}{f}}$ using symplectic form $d(\frac{1}{f}\lambda)$. Since we can choose g and f such that $\frac{1}{f} - \frac{1}{1-\epsilon}$ is C^2 small, therefore $d(\frac{1}{f}\lambda)$ is C^1 close to $\frac{1}{1-\epsilon}d\lambda$. This can be achieved by choosing g such that $g'(1-\epsilon)$ is very large and $g''(1-\epsilon)$ is very small. Therefore by Lemma 6.8, we can assume all nonconstant periodic orbits of $X_{\frac{\rho}{f}}$ are of periods greater than D_i .

Therefore we can pick big enough ρ_i and f_i, g_i , such that all contractible Reeb orbits on Y_{ρ_i, f_i, g_i} with action smaller than D_i are either nondegenerate with positive degrees or Morse–Bott in S^1 -families with generalized Conley–Zehnder indices greater than $4 - n$ (or any fixed large number). By [8, Lemma 2.3, 2.4], there exist a very small perturbation of the contact form on Y_{ρ_i, f_i, g_i} , such that all contractible Reeb orbits with periods smaller than D_i have positive degree. For all $j > i$, we have a similar construction by choosing $\rho_j > \rho_i$, such that the contact form on Y_{ρ_j, f_j, g_j} is smaller than that on Y_{ρ_i, f_i, g_i} . This finishes the proof of Theorem for ADC case.

In the TADC case, since we do not need to use arbitrarily large ρ to lift the Conley–Zehnder index on $Y_{\rho, f, g}^1$, $Y_{\rho, f, g}$ can be chosen without shrinking to $V \times \{0\}$. $\square$

REMARK 6.9. By [16], every subcritical domain splits into the product of a Weinstein domain with $\mathbb{C}$. Theorem 6.3 implies that all $2n$ -dimensional subcritical domains are $(n - 2)$ -ADC, which is a special case of [37, Corollary 4.1].

COROLLARY 6.10. *Let V be a Liouville domain, such that $c_1(V) = 0$. Let $Y = \partial(V \times \mathbb{C})$. Then for any topologically simple exact filling W of Y , we have a ring isomorphism $\phi: H^*(W) \rightarrow H^*(V \times \mathbb{C})$, such that the following commutes*

$$\begin{array}{ccc} H^*(W) & \longrightarrow & H^*(Y) \\ \downarrow \phi & \nearrow & \\ H^*(V \times \mathbb{C}) & & \end{array}$$

Proof. First note that Y is always connected and $c_1(Y) = 0$. Then by Corollary B and Theorem 6.3 and that $SH^*(V \times \mathbb{C}) = 0$, we have a group morphism $\phi: H^*(W) \rightarrow H^*(V \times \mathbb{C})$ such that the diagram commutes. Since the composition $H^*(V \times \mathbb{C}) \rightarrow H^*(Y) \rightarrow H^*(V \times S^1)$ is injective, $H^*(V \times \mathbb{C}) \rightarrow H^*(Y)$ is injective and is a ring map, and $H^*(W) \rightarrow H^*(Y)$ is a ring map. They force ϕ to be a ring isomorphism. $\square$

If V is k -TADC for some k , then $\partial(V \times \mathbb{C})$ is TADC. Hence by Remark 8.13, the exactness assumption in Corollary 6.10 can be dropped. As another instant application of Corollary 6.10, we have the following.

COROLLARY 6.11. *If V^{2n} is a Liouville domain, such that $n \geq 2$ and $c_1(V) = 0$, then $\partial(V \times \mathbb{C})$ is not Weinstein fillable if one of the following conditions hold.*

- (i) *There exists $k > n + 1$, such that $H^k(V) \neq 0$.*
- (ii) *There exists $k < n$, such that $H^k(V \times \mathbb{C}) \rightarrow H^k(\partial(V \times \mathbb{C}))$ is not isomorphism.*

COROLLARY 6.12. *If Y is constructed by attaching subcritical (for 2-handles, conditions in [37, Theorem 3.17] need to hold) or flexible handles onto $\partial(V \times \mathbb{C})$ in Corollary 6.11, such that $c_1(Y) = 0$, then Y cannot be filled by a Weinstein domain.*

Proof. In other words, Y is the boundary of $(V \times \mathbb{C}) \cup W$, where W is a flexible Weinstein cobordism from $\partial(V \times \mathbb{C})$ to Y . Since $\dim W = 2n + 2 \geq 6$, $c_1(Y) = 0$ implies $c_1(W) = 0$. Hence $c_1(\partial(V \times \mathbb{C})) = 0$. Since $H^*(V \times \mathbb{C}) \rightarrow H^*(\partial(V \times \mathbb{C}))$ is injective, we have $c_1(V \times \mathbb{C}) = c_1(V) = 0$. Therefore by Theorem 6.3, $\partial(V \times \mathbb{C})$ is ADC. Then by [37, Theorem 3.15, 3.17 and 3.18], Y is ADC and $SH^*((V \times \mathbb{C}) \cup W) = 0$ by [9]. Therefore $SH_+^*((V \times \mathbb{C}) \cup W) \rightarrow H^{*+1}((V \times \mathbb{C}) \cup W)$ is an isomorphism. If $H^k(V) \neq 0$ for $k > n + 1$, then $H^k((V \times \mathbb{C}) \cup W) \neq 0$. Hence Y is not Weinstein fillable by Corollary B. If $H^k(V \times \mathbb{C}) \rightarrow H^k(\partial(V \times \mathbb{C}))$ is not isomorphism for $k < n$, then it is not surjective since it is always injective. Assume k is the smallest among all such integers. Then we have the following exact sequence

$$\begin{aligned} \dots \rightarrow H^k(V \times \mathbb{C}, \partial(V \times \mathbb{C})) \rightarrow H^k(V \times \mathbb{C}) \hookrightarrow H^k(\partial(V \times \mathbb{C})) \\ \rightarrow H^{k+1}(V \times \mathbb{C}, \partial(V \times \mathbb{C})) \rightarrow \dots \end{aligned}$$

Since k is the first failure of surjectivity, we have $H^k(V \times \mathbb{C}, \partial(V \times \mathbb{C})) = 0$ and $H^{k+1}(V \times \mathbb{C}, \partial(V \times \mathbb{C})) \neq 0$. Since we have $H^*((V \times \mathbb{C}) \cup W, W) = H^*(V \times \mathbb{C}, \partial(V \times \mathbb{C}))$, there is another long exact sequence

$$\begin{aligned} \dots \rightarrow H^k(V \times \mathbb{C}, \partial(V \times \mathbb{C})) \rightarrow H^k((V \times \mathbb{C}) \cup W) \rightarrow H^k(W) \rightarrow \\ H^{k+1}(V \times \mathbb{C}, \partial(V \times \mathbb{C})) \rightarrow H^{k+1}((V \times \mathbb{C}) \cup W) \rightarrow H^{k+1}(W) \rightarrow \dots \end{aligned}$$

Therefore either $H^k((V \times \mathbb{C}) \cup W) \rightarrow H^k(W)$ is not surjective or $H^{k+1}((V \times \mathbb{C}) \cup W) \rightarrow H^{k+1}(W)$ is not injective. Since $H^k(W) \rightarrow H^k(Y)$ is isomorphism and $H^{k+1}(W) \rightarrow H^{k+1}(Y)$ is injective, by $k < n$ and W being Weinstein, either $\delta_\partial : SH_+^{k-1}((V \times \mathbb{C}) \cup W) \rightarrow H^k(Y)$ is not surjective or $\delta_\partial : SH_+^k((V \times \mathbb{C}) \cup W) \rightarrow H^{k+1}(Y)$ is not injective. Then by Corollary B and Theorem A, Y cannot be filled by a Weinstein domain, otherwise δ_∂ should be an isomorphism on SH_+^{k-1} and injective on SH_+^k . $\square$

Corollary 6.12 provides many exactly fillable but not Weinstein fillable manifolds. Exactly fillable but not Weinstein fillable manifolds were found in [12] for dimension 3 and in [13] for higher dimensions. Our construction makes very few topological requirements and roughly shows that most Liouville but not Weinstein manifolds give rise to exactly fillable but not Weinstein fillable contact manifolds after a product with $\mathbb{C}$ and attaching subcritical or flexible handles.

REMARK 6.13. In case (i) of Corollaries 6.11 and 6.12, the obstruction in Corollary 3.14 does not vanish. Using Proposition 3.15, we have many examples of not Weinstein fillable manifolds, whose symplectic cohomology is nonzero for a filling.

EXAMPLE 6.14. Liouville but not Weinstein domains were first constructed in [42] for dimension 4, and higher dimensional examples were constructed in [26, 40]. All such examples are of a diffeomorphism type $M \times [0, 1]$, where the diffeomorphism type of M is the following.

- (i) $ST^*\Sigma_g$, where Σ_g is a genus $g \geq 2$ surface [42].
- (ii) T^n bundle over T^{n-1} for any $n \geq 1$ [26, 40]. In fact, such examples are ADC, as the contact boundaries are hypertight, meaning that there is a contact form with all Reeb orbits noncontractible.

Those Liouville domains have vanishing first Chern class. Then by attaching Weinstein handles to them without changing the first Chern class, or taking products among them yield many Liouville domains V^{2n} such that $c_1(V) = 0$ and $H^k(V) \neq 0$ for some $k > n + 1$. Therefore the boundary of $V \times \mathbb{C}$ admits no Weinstein filling by Corollary 6.11. One may keep attaching

subcritical or flexible handles to it preserving the ADC property, then the boundary is not Weinstein fillable by Corollary 6.12.

Given an almost contact manifold, the existence of an almost Weinstein filling is purely homotopy theoretical, this was solved by Bowden–Crowley–Stipsicz in [13]. The obstruction was phrased using bordism theory. Bowden–Crowley–Stipsicz used them to construct exactly fillable but not almost Weinstein fillable contact manifolds in all dimensions ≥ 5 . In the following, we show that in dimension $4k + 3, k \geq 1$, our construction yields examples where the homotopy obstruction vanishes, while the obstruction in Corollary 3.14 does not vanish. Hence the obstruction is symplectic in nature.

We first recall the criterion of almost Weinstein fillability from [13]. An almost contact structure on a closed oriented $2n + 1$ manifold Y is a reduction of the structure group of TY to $U(n)$. Then an almost contact structure defines a map $\zeta : Y \rightarrow BU(n) \rightarrow BU$. The n th Postnikov factorization of ζ consists of a space B_ζ^{n-1} and maps

$$Y \xrightarrow{\bar{\zeta}} B_\zeta^{n-1} \xrightarrow{\eta_\zeta^{n-1}} BU,$$

such that η_ζ^{n-1} is a fibration, $\zeta = \eta_\zeta^{n-1} \circ \bar{\zeta}$, the map $\bar{\zeta}$ is an n -equivalence, that is, $\bar{\zeta}$ induces isomorphism on π_j for $j < n$ and surjective map on π_n , and the map η_ζ^{n-1} is a coequivalence, that is, it induces isomorphism on π_j for $j > n$ and injective map on π_n . Then the pair $(Y, \bar{\zeta})$ defines a bordism class $[Y, \bar{\zeta}]$ in $\Omega_{2n+1}(B_\zeta^{n-1}; \eta_\zeta^{n-1})$. For the definition of this bordism group, see [13, §2.1].

THEOREM 6.15 [13, Theorem 1.2]. *A closed almost contact manifold Y of dimension $2n + 1 \geq 5$ is almost Weinstein fillable if and only if $[Y, \bar{\zeta}] = 0 \in \Omega_{2n+1}(B_\zeta^{n-1}; \eta_\zeta^{n-1})$.*

Proof of Theorem I. By [40], there exists an exact domain V diffeomorphic to $M \times [0, 1]$, such that M is a T^{n+2} bundle over T^{n+1} for $n \geq 0$. Moreover, the contact structure on each component of ∂V is a trivial complex bundle, since the contact structure is an invariant distribution on a Lie group. Hence the complex structure of TV on ∂V is trivial. Since $M \times \{0\} \rightarrow V$ is a homotopy equivalence, we have the complex structure on TV is trivial. Let Y' denote the contact boundary of $V \times \mathbb{C}^n$. Then by Theorem 6.3, Y' is ADC and Y' is diffeomorphic to $M \times S^{2n}$. Assume now $n \geq 1$. Let $S = \{*\} \times S^{2n}$ be a sphere on Y' . The $T(V \times \mathbb{C}^n)|_S$ is a trivial complex bundle, that is, the structure map $S \rightarrow BU(2n + 2)$ is trivial. Let ξ be the contact structure on Y' . Since $BU(2n + 1) \rightarrow BU(2n + 2)$ induces isomorphism on π_{2n} , we have $\xi|_S$ is trivial $\mathbb{C}^{2n+1}$ bundle. Therefore there exists a homotopy of monomorphisms $F_s : TS \rightarrow TY'$, such that F_0 is the inclusion and F_1 is an inclusion to ξ and isotropic. Hence by the h-principle of isotropic embedding [17, Theorem 7.11], S is homotopic to an isotropy sphere S' with trivial $(TS')^\perp / TS'$. Then we can attach a $2n + 1$ subcritical handle to S' . The resulted contact manifold is Y with filling $W := V \times \mathbb{C}^n \cup H_{2n+1}$.

By [37, Theorem 3.15], Y is 0-ADC. By [15], we have $SH^*(W) = 0$. Since $H^{2n+3}(V) = \mathbb{Z}$, we have Y does not admit Weinstein filling by Corollary 6.12. We claim Y admits almost Weinstein fillings. Since the structure map $V \times \mathbb{C}^n \rightarrow BU$ is trivial, the structure map $W \rightarrow BU$ is represented by a class $\pi_{2n+1}(BU) = 0$. Therefore the structure map $W \rightarrow BU$ is trivial. Assume that the $2n + 1$ th Postnikov factorization of the trivial structure map $\zeta : W \rightarrow BU$ is given by

$$W \xrightarrow{\bar{\zeta}} B^{2n} \xrightarrow{\eta^{2n}} BU.$$

We claim that $Y \hookrightarrow W \xrightarrow{\bar{\zeta}} B^{2n} \xrightarrow{\eta^{2n}} BU$ gives the $2n + 1$ th of Postnikov factorization of the trivial structure map $Y \rightarrow BU$. Then Y is almost Weinstein fillable by Theorem 6.15 once the

claim is proven, as the $[W, \bar{\zeta}]$ is a bordism to $\emptyset$. By definition, it is sufficient to prove $Y \hookrightarrow W \xrightarrow{\bar{\zeta}} B^{2n}$ is a $2n + 1$ equivalence. Since $W \rightarrow BU$ is trivial, we know that $\pi_{2n+1}(B^{2n}) = 0$. Therefore $Y \hookrightarrow W \xrightarrow{\bar{\zeta}} B^{2n}$ induces epimorphism on π_{2n+1} . Since $\bar{\zeta}$ is a $2n + 1$ equivalence, it suffices to show that $\pi_*(Y) \rightarrow \pi_*(W)$ is an isomorphism for $* \leq 2n$. We consider the universal cover $\widetilde{W}$ of W . Since V is a $K(\pi, 1)$, we have $\widetilde{W}$ is constructed by attaching $|\pi|$ copies $2n + 1$ -handles to the boundary of $\mathbb{R}^{2n+3} \times B^{2n+1}$, where $B^{2n+1} \subset \mathbb{R}^{2n+1}$ is the unit ball. Then the boundary $\tilde{Y} := \partial \widetilde{W}$ is connected. Since $\widetilde{W}$ can be viewed as attaching $|\pi|$ copies $2n + 3$ handles and one $2n + 1$ -handle to $\tilde{Y}$, then $\pi_*(\tilde{Y}) \rightarrow \pi_*(\widetilde{W})$ is isomorphism for $* \leq 2n - 1$. In particular, $\pi_1(\tilde{Y}) = 0$, that is, $\tilde{Y}$ is the universal cover of Y . Therefore it is sufficient to prove that $\pi_{2n}(\tilde{Y}) \rightarrow \pi_{2n}(\widetilde{W})$ is an isomorphism. We will use the relative Hurewicz theorem to prove the claim. First, we have the following commutative diagram,

$$\begin{array}{ccc} H_{2n+1}(\widetilde{W}) & \xrightarrow{\quad} & H_{2n+1}(\widetilde{W}, \tilde{Y}) = \mathbb{Z} \\ \downarrow & \swarrow & \\ H_{2n+1}(\widetilde{W}, \widetilde{W} \setminus (\mathbb{R}^{2n+3} \times B^{2n+1})) & & \end{array}$$

By excision, we have $H_{2n+1}(\widetilde{W}, \widetilde{W} \setminus (\mathbb{R}^{2n+3} \times B^{2n+1})) = H_{2n+1}(\mathbb{R}^{2n+3} \times B^{2n+1}, \mathbb{R}^{2n+3} \times S^{2n}) = \mathbb{Z}$, and the map from $H_{2n+1}(\widetilde{W}, \tilde{Y})$ is an isomorphism. Note that $H_{2n+1}(\widetilde{W}) \rightarrow H_{2n+1}(\widetilde{W}, \widetilde{W} \setminus (\mathbb{R}^{2n+3} \times B^{2n+1}))$ is surjective, since any $2n + 1$ handle in $\widetilde{W}$ attached to $\mathbb{R}^{2n+3} \times S^{2n}$ along with $\{*\} \times B^{2n+1}$ gives a chain that is mapped to the generator of $H_{2n+1}(\widetilde{W}, \widetilde{W} \setminus (\mathbb{R}^{2n+3} \times B^{2n+1}))$. Hence $H_{2n+1}(\widetilde{W}) \rightarrow H_{2n+1}(\widetilde{W}, \tilde{Y})$ is surjective. By the homology long exact sequence for $\widetilde{W}, \tilde{Y}$, we have $H_{2n}(\tilde{Y}) \rightarrow H_{2n}(\widetilde{W})$ is an isomorphism. Note that $(\widetilde{W}, \tilde{Y})$ is $2n - 1$ connected and $\tilde{Y}$ is simply connected, then by the relative Hurewicz theorem, $\pi_{2n}(\widetilde{W}, \tilde{Y}) = 0$. That is, $(\widetilde{W}, \tilde{Y})$ is $2n$ connected. Then the relative Hurewicz theorem implies that $\pi_{2n+1}(\widetilde{W}, \tilde{Y}) \rightarrow H_{2n+1}(\widetilde{W}, \tilde{Y}) = \mathbb{Z}$ is an isomorphism. As we have seen, there is an $S^{2n+1} \in \widetilde{W}$ that is mapped to the generator in $H_{2n+1}(\widetilde{W}, \tilde{Y})$. Therefore $\pi_{2n+1}(\widetilde{W}) \rightarrow \pi_{2n+1}(\widetilde{W}, \tilde{Y})$ is surjective, hence $\pi_{2n}(\widetilde{W}) \rightarrow \pi_{2n}(\tilde{Y})$ is an isomorphism. Since the obstruction in Corollary 3.14 does not vanish for Y , then by Proposition 3.15, there are infinity many exactly fillable, almost Weinstein fillable but not Weinstein fillable manifolds. $\square$

REMARK 6.16. In dimension $4n + 1$ for $n \geq 2$, one may consider V_1, V_2 from Example 6.14, such that $\dim V_1 + \dim V_2 = 2n + 4$. In particular, $V_1 \times V_2 \simeq M_1 \times M_2 \times [0, 1]^2$. Then we consider Y to the boundary of W , which is constructed by attaching a $2n$ -handle to $V_1 \times V_2 \times \mathbb{C}^{n-1}$. Most of the arguments in Theorem I that go through, in particular, Y is exactly fillable but not Weinstein fillable. The only issue is showing that the $2n$ th Postnikov factorization of W gives a bordism to $\emptyset$. In fact, one can show that everything boils down to whether the class in $\pi_{2n}(BU)$ classifying the structure map $W \rightarrow BU$ is trivial.

REMARK 6.17. In dimension 5, manifold $V \times \mathbb{C}$ always admits Morse function of index at most 3, hence Corollary 6.11 cannot give obstructions to Weinstein fillings. In dimension 3, $V \times \mathbb{C}$ is always a subcritical Weinstein domain.

6.2. Product of ADC manifolds

In the following, we will show the product of two ADC domains is again ADC. In particular, such construction provides many examples where Corollary 4.19 can be applied. Before stating the precise theorem, we first prove a proposition on the Reeb orbits on the boundary of general

products. In the following, we define a function for a fixed $N \in \mathbb{N}$.

$$g_N(x) = 2 - \frac{1}{N(2-x)}, \quad x \in \left[1, 2 - \frac{1}{N}\right].$$

Then g_N takes values in $[1, 2 - \frac{1}{N}]$ and $g_N^{-1} = g_N$. Let V and W be two Liouville domains with Liouville forms λ_V, λ_W that are nondegenerate as contact forms, such that $c_1(V) = c_1(W) = 0$. Then we define $V \times_N W$ to be the following subset of $\widehat{V} \times \widehat{W}$,

$$\begin{aligned} V \times W \cup (\partial V \times \{r_V < f_W\}) \times W \cup V \times (\partial W \times \{r_W < f_V\}) \cup \partial V \\ \times \partial W \times \{(r_V, r_W) | r_V < g_N(r_W)\}, \end{aligned}$$

where we have:

- (1) f_V, f_W are Morse functions on V and W , such that near the collar of V and W they are g_N and $\frac{1}{f_V}, \frac{1}{f_W}$ realize the Morse dimensions of V, W , respectively.
- (2) $2 - \frac{1}{N} \leq |f_V|, |f_W| \leq 2$ and $|df_V|_{C^1}, |df_W|_{C^1} \leq \frac{1}{N}$.

PROPOSITION 6.18. *Assume V, W as above and $\dim V, \dim W \geq 4$. For $D > 0$, for $N \in \mathbb{N}$ big enough, the contractible Reeb orbits of $V \times_N W$ of period at most D are of the following three types.*

- (i) *Nondegenerate orbit (p, γ) for $p \in \mathcal{C}(\frac{1}{f_V})$ and $\gamma \in \mathcal{P}^{<D}(W)$ with index $\frac{1}{2} \dim V - \text{ind } p + \mu_{CZ}(\gamma)$.*
- (ii) *Nondegenerate orbit (γ, p) for $p \in \mathcal{C}(\frac{1}{f_W})$ and $\gamma \in \mathcal{P}^{<D}(V)$ with index $\frac{1}{2} \dim W - \text{ind } p + \mu_{CZ}(\gamma)$.*
- (iii) *Morse–Bott orbits $(\gamma_1(t), \gamma_2(t+t_0))$ in an S^1 family for γ_1, γ_2 in $\mathcal{P}^{<D}(V), \mathcal{P}^{<D}(W)$, respectively, up to reparametrization. The generalized Conley–Zehnder index is $\mu_{CZ}(\gamma_1) + \mu_{CZ}(\gamma_2) + \frac{1}{2}$.*

Proof. The boundary of $V \times_N W$ is given by three components $(\partial V \times \{r_V = f_W\}) \times W \cup V \times (\partial W \times \{r_W = f_V\}) \cup \partial V \times \partial W \times \{(r_V, r_W) | r_V = g_N(r_W)\}$. Using Remark 6.5, we have the Liouville vector of $\widehat{V} \times \widehat{W}$ is transverse to the boundary. Hence $\partial(V \times_N W)$ is indeed of contact type. By Proposition 6.4 and Lemma 6.8, for N big enough, the Reeb orbits on $V \times (\partial W \times \{r_W = f_V\})$ and $(\partial V \times \{r_V = f_W\}) \times W$ are those described in (i) and (ii).

It remains to study the Reeb orbits on $\partial V \times \partial W \times \{(r_V, r_W) | r_V = g_N(r_W)\}$. It can be either viewed as graph of function on $\widehat{W}$ or $\widehat{V}$. We choose to view it as graph on $\widehat{W}$. Then by Proposition 6.4, the Reeb vector field is given by

$$\frac{1}{g_N(r_W) - g'_N(r_W)r_W} R_V + \frac{-g'_N(r_W)}{g_N(r_W) - g'_N(r_W)r_W} R_W.$$

Note that $g'_N(r_W) = \frac{r_V - 2}{2 - r_W}$, we write the Reeb orbit in a more symmetric way as below:

$$\frac{r_W - 2}{2r_V r_W - 2r_V - 2r_W} R_V + \frac{r_V - 2}{2r_V r_W - 2r_V - 2r_W} R_W. \quad (6.1)$$

Here the coefficient varies in $[\frac{1}{2N}, \frac{1}{2}]$. Therefore the Reeb orbit of (6.1) are in the form of

$$\left(\gamma_1 \left(\frac{r_W - 2}{2r_V r_W - 2r_V - 2r_W} t \right), \gamma_2 \left(\frac{r_V - 2}{2r_V r_W - 2r_V - 2r_W} t \right) \right),$$

where γ_1, γ_2 are Reeb orbits of ∂V and ∂W with periods D_1 and D_2 such that $T := D_1 / \frac{r_W - 2}{2r_V r_W - 2r_V - 2r_W} = D_2 / \frac{r_V - 2}{2r_V r_W - 2r_V - 2r_W} \geq 2D_1, 2D_2$ is the period. By Remark 6.6, the contact restructure is given by $\xi_V \oplus \xi_W \oplus \langle \partial_{r_W}, -r_W R_V + r_V R_W \rangle$. The linearized return map

preserves the decomposition. Therefore the linearized return maps on ξ_V and ξ_W do not have 1 as eigenvalue by nondegenerate assumptions on V and W . By taking derivative of (6.1), the linearized return map on the remaining part is given by

$$\begin{bmatrix} 1, & 0 \\ \frac{4-2r_V}{(2r_V r_W - 2r_V - 2r_W)^2} T, & 1 \end{bmatrix}. \quad (6.2)$$

Let ϕ_T denote the linearized return map on the whole tangent space, then $\ker(\phi_T - \text{id})$ is spanned by the Reeb vector field (6.1) and $-r_W R_V + r_V R_W$, which is the 1-eigenspace of (6.2). Equivalently, $\ker(\phi_T - \text{id})$ is spanned by R_V and R_W , which is the tangent of $\text{im } \gamma_1 \times \text{im } \gamma_2$. Moreover, $d(r_V \lambda_V + r_W \lambda_W)$ is rank 0 on the tangent of $\text{im } \gamma_1 \times \text{im } \gamma_2$. Hence such Reeb orbits are of Morse–Bott type. To compute the generalized Conley–Zehnder index, since we have the decomposition of contact structure, it is sufficient to compute the contribution in the $\langle \partial_{r_W}, -r_W R_V + r_V R_W \rangle$ direction. Since $\dim V, \dim W \geq 4$, here the complex trivialization of the tangent bundles over the Reeb orbits splits into trivializations of ξ_V and ξ_W used in defining Conley–Zehnder index for V and W and a trivial complex bundle spanned by ∂_{r_W} and $-r_W R_V + r_V R_W$. The total generalized Conley–Zehnder index is the generalized Conley–Zehnder index of

$$\begin{bmatrix} 1, & 0 \\ \frac{4-2r_V}{(2r_V r_W - 2r_V - 2r_W)^2} t, & 1 \end{bmatrix},$$

which is $\frac{1}{2}$ plus $\mu_{CZ}(\gamma_1) + \mu_{CZ}(\gamma_2)$. $\square$

THEOREM 6.19. *Let V, W be two Liouville domains of dimension ≥ 4 , respectively, such that $c_1(V) = c_1(W) = 0$. Assume V is p -ADC and W is q -ADC, then we have $V \times W$ is $\min\{p+q+4, p+\dim W - \dim_M W, q+\dim V - \dim_M V\}$ -ADC. In particular, if V, W are both ADC, then $V \times W$ is ADC. If V, W are both TADC, then $V \times W$ is TADC.*

Proof. Assume the dimensions of V and W are $2n, 2m$, respectively. Since V, W are ADC, there exist positive functions $f_1 > f_2 > \dots > f_k > \dots$, $g_1 > g_2 > \dots > g_k > \dots$ and real numbers $D_1 < D_2 < \dots < \dots < D_k < \dots$ converge to ∞ , such that all orbits in $\mathcal{P}^{<D_i}(f_i \lambda_V|_{\partial V})$ are nondegenerate and have Conley–Zehnder indices at least $p-n+4$ and all orbits in $\mathcal{P}^{<D_i}(g_i \lambda_W|_{\partial W})$ are nondegenerate and have Conley–Zehnder indices at least $q-m+4$. Let $V_i := \{r_V \leq f_i\}$ and $W_i := \{r_W \leq g_i\}$, then for $N \in \mathbb{N}$ big enough $V_i \times_N W_i \subset \widehat{V} \times \widehat{W}$ is close to $\{r_V \leq 2f_i\} \times \{r_W \leq 2g_i\}$ and the Reeb orbits of period smaller than D_i are decied by Proposition 6.18. Using [8, Lemma 2.3, 2.4], there exist a very small perturbation $(V \times W)_i$ of $V_i \times_N W_i$, such that all Reeb orbits in (iii) of Proposition 6.18 are perturbed into nondegenerate orbits with index $\mu_{CZ}(\gamma_1) + \mu_{CZ}(\gamma_2), \mu_{CZ}(\gamma_1) + \mu_{CZ}(\gamma_2) + 1$. Then we have all elements in $\mathcal{P}^{<D_i}((V \times W)_i)$ are nondegenerate with Conley–Zehnder indices at least $\min\{p+q-n-m+8, p-n+4-\dim_M W+m, q-m+4-\dim_M V+n\}$. Hence the degree is at least $\min\{p+q+5, p+2m-\dim_M W+1, q+2n-\dim_M V+1\}$. Since for $j > i$, we can choose N in the construction big enough such that $(V \times W)_j \subset (V \times W)_i$, which implies that the contact form on $(V \times W)_j$ is smaller than that on $(V \times W)_i$, hence $\partial(V \times W)$ is $\min\{p+q+4, p+\dim W - \dim_M W, q+\dim V - \dim_M V\}$ -ADC. The TADC case is similar. $(V \times W)_i$ will not collapse, because V_i, W_i do not collapse. $\square$

The following theorem provides more examples of exactly fillable, but not Weinstein fillable contact manifolds using the obstruction in Corollary 4.19.

COROLLARY 6.20. *Let V^{2n} be the Liouville manifold in (ii) of Example 6.14, W^{2m} be any ADC Liouville domain with dilations, see Example 4.3. If $n - 1 > m$, then $\partial(V \times W)$ cannot be filled by Weinstein domains.*

Proof. By [48, 56], $\Delta(a \otimes b) = \pm \Delta(a) \otimes b \pm a \otimes \Delta(b)$ on $SH^*(V \times W) = SH^*(V) \otimes SH^*(W)$. Since those V are hypertight, that is, they allow a Reeb flow without contractible Reeb orbits, then V is automatically ADC. Since we only use contractible orbits to generate symplectic cohomology, we have $SH^*(V) = H^*(V)$. We may assume $SH^*(W) \neq 0$, otherwise Corollary 3.14 can be applied to obtain the conclusion. Let $\alpha \in SH^{2n-1}(V)$ be a generator of $H^{2n-1}(V)$ and $\beta \in SH^1(W)$ a dilation. Then we have $\Delta(\alpha \otimes \beta) = \pm \alpha \otimes 1 \in \text{im}(H^{2n-1}(V \times W) \rightarrow SH^{2n-1}(V \times W))$. Similar to the proof of Proposition 4.10, if we view $\alpha \otimes 1 \in H^{2n-1}(V \times W)$, then it represents a class in the image of $\phi : \text{im}(SH^*(V \times W) \rightarrow SH_+^*(V \times W)) \cap \ker \Delta_+ \rightarrow \text{coker } \delta$. $H^*(V \times W) \rightarrow H^*(\partial(V \times W))$ is injective and $2n - 1 > n + m$. To show $\text{im } \Delta_\partial$ contains a class in grading $2n - 1$, it is enough to show that $\alpha \otimes 1$ is not in $\text{im } \delta$. Since V is hypertight, the Künneth formula implies that $SH_+^*(V \times W) = H^*(V) \otimes SH_+^*(W)$, and $\delta(\alpha \otimes \beta) = \pm \alpha \otimes \delta_W(\beta)$. Hence if $\alpha \otimes 1$ is in the image of δ , then $1 \in \text{im } \delta_W$, which implies $SH^*(W) = 0$, contradicting the assumption. $\square$

7. Obstructions to exact cobordisms

A natural question in symplectic geometry is understanding the ‘size’ or ‘complexity’ of exact domains. One simple way of comparing complexity is by asking if one exact domain can be embedded into another. Due to the Viterbo transfer map, the vanishing of symplectic cohomology and existence of symplectic dilation are two levels of complexity, which are in fact the first two of the infinity many structures in [69]. For example, one cannot embed an exact domain W with $SH^*(W) \neq 0$ to a flexible Weinstein domain. In particular, flexible Weinstein domain does not contain closed exact Lagrangians. On the other hand, one can always embed the standard ball into any domain. Moreover, one cannot embed an exact domain W with no dilation into T^*S^n . In particular, T^*S^n contains no exact tori.

The same question can be asked for contact manifolds, and the comparison is based on existence of symplectic cobordism. By putting different adjectives in front of the symplectic cobordism, we get several comparisons in different flavors. In this section, we will restrict to exact cobordisms. Similar to the discussion above, one can use the functoriality of contact invariants like contact homology and symplectic field theory [22] to study this problem. For example, an overtwisted contact manifold has vanishing contact homology [65], hence there is no exact cobordism from a contact manifold with nonvanishing contact homology to an overtwisted one. In particular, there is no cobordism from $\emptyset$. A more general obstruction in a similar spirit using the full SFT was constructed in [36]. However, unlike symplectic cohomology, such invariants are difficult to define [?, 50] and notoriously hard to compute.

Since we have shown that for ADC manifolds, the vanishing of symplectic cohomology and existence of symplectic dilation is independent of the filling, hence can be understood as contact invariants. In particular, we can use them to define obstructions to the existence of exact cobordisms to some ADC manifolds. Such obstructions using symplectic cohomology are relatively easy to define and compute.

THEOREM 7.1. *Let Y be an ADC contact manifold with a topologically simple exact filling W . Let V be an exact domain with $c_1(V) = 0$. If one of the following holds, then there is no exact cobordism U from ∂V to Y , such that $c_1(U) = 0$ and $H^1(V) \oplus H^1(U) \rightarrow H^1(\partial V)$, $(a, b) \mapsto a|_{\partial V} - b|_{\partial V}$ is surjective and $\pi_1(Y) \rightarrow \pi_1(U \cup V)$ is injective.*

- (i) If $1 \in \text{im } \delta_\partial$ for W , and $1 \notin \text{im } \delta_\partial$ for V .
- (ii) If $1 \in \text{im } \Delta_\partial$ for W , and $1 \notin \text{im } \Delta_\partial$ for V .

Proof. If there is a such cobordism, since $H^1(V) \oplus H^1(U) \rightarrow H^1(\partial V)$ is surjective, we have $H^2(U \cup V) \rightarrow H^2(U) \oplus H^2(V)$ is injective. Since $c_1(V) = c_1(U) = 0$, we have $c_1(U \cup V) = 0$. Therefore $U \cup V$ is a topologically simple filling. Since $1 \in \text{im } \delta_\partial$ for W , then $SH^*(W) = 0$. In particular, by Corollary B, we have $SH^*(U \cup V) = 0$. As $1 \notin \text{im } \delta_\partial$ for V implies that $SH^*(V) \neq 0$, we have a contradiction by that the Viterbo transfer map preserves the unit, hence there is no such U .

When $1 \in \text{im } \Delta_\partial$ for W , we know that $\phi : \ker \Delta_+ \rightarrow \text{coker } \delta$ contains 1 in the image for W . Assume there is such a cobordism U . By Theorem G, we have that $1 \in \text{im } \Delta_\partial, \text{im } \phi$ for $U \cup V$. If $1 \notin \text{im } \Delta_\partial$ for V , we know that $1 \notin \text{im } \phi$ for V . The Viterbo transfer map commutes with Δ_+, δ and ϕ by Proposition 4.20, hence the Viterbo transfer map induces the following commutative diagram

$$\begin{array}{ccc}
 \ker \Delta_+ \subset SH_+^*(U \cup V) & \xrightarrow{\phi} & \text{coker}(SH_+^{*-2}(U \cup V) \rightarrow H^{*-1}(U \cup V)) \\
 \downarrow \text{Viterbo transfer} & & \downarrow \text{Viterbo transfer} \\
 \ker \Delta_+ \subset SH_+^*(V) & \xrightarrow{\phi} & \text{coker}(SH_+^{*-2}(V) \rightarrow H^{*-1}(V))
 \end{array}$$

Hence we have a contradiction. $\square$

Proof of Corollary J. Assume otherwise, there is a Weinstein cobordism U , then $U \cup V$ is another Weinstein filling of Y . In particular, $\pi_1(Y) \rightarrow \pi_1(U \cup V)$ is an isomorphism and $c_1(U \cup V) = 0$. Then we reach a contradiction by the same argument in the proof of Theorem 7.1 for the first two conditions. The third one follows from Corollary H. $\square$

As a direct application of Corollary J, if an ADC Weinstein domain V of dimension ≥ 6 admits a dilation and $c_1(V) = 0$, then not only V does not contain $K(\pi, 1)$ as closed exact Lagrangian, there is no Weinstein cobordism from $S^*K(\pi, 1)$ to ∂V , since $T^*K(\pi, 1)$ has no dilation.

On the other hand, the existence of almost Weinstein cobordism is purely homotopical and was studied in [14]. In particular, Bowden–Crowley–Stipsicz [14, Theorem 1.2] showed that for dimension $2n - 1 \geq 5$, there exists an almost contact manifold $(M_{\max}, \phi_{\max})$ such that for any almost contact manifold (M, ϕ) there is an almost Weinstein cobordism from (M, ϕ) to $(M_{\max}, \phi_{\max})$. The maximal element also exists when restricted to the class of almost contact manifolds with vanishing first Chern class. In the latter case, when dimension is 5 or 7, the maximal element can be chosen as the standard $(S^{2n-1}, \phi_{\text{std}})$. If we take the maximal element $(M_{\max}, \phi_\phi)$ for the class of contact manifolds with vanishing Chern class, since there is a Weinstein cobordism from $\emptyset$ to $(M_{\max}, \phi_{\max})$, in particular, in the homotopy class of $(M_{\max}, \phi_{\max})$, there is a contact manifold (M, ξ) admitting a flexible filling. Then for any contact manifold Y with vanishing first Chern class and a Weinstein filling V such that $SH^*(V) \neq 0$, there is an almost Weinstein cobordism from Y to M , but there is no Weinstein cobordism by Corollary J.

There are also many examples with nonvanishing symplectic cohomology. For example, one takes the Milnor fiber W of the singularity $z_1^9 + z_2^2 + z_3^2 + z_4^2 = 0$, then ∂W as an almost contact manifold is (S^5, ϕ_{std}) . By [56, Example 2.13], we have $SH^*(W) \neq 0$ and admits a dilation and ∂W is ADC by [61, Lemma 4.2]. Hence by Corollary J, for any Weinstein domain V with $c_1(V) = 0$ and no dilation, there is an almost Weinstein cobordism from ∂V to ∂W , but there is no Weinstein cobordism from ∂V to ∂W . In higher dimensions, let W be an ADC Weinstein domain with nonvanishing symplectic cohomology and a symplectic

dilation. Let V be an exotic $\mathbb{C}^n$ with nonnegative log Kodaira dimension, in particular V has no symplectic dilations and the almost contact structure on ∂V is the standard almost contact structure on S^{2n-1} . Then there is no Weinstein cobordism from ∂V to ∂W , but there is an almost Weinstein cobordism from $\partial V = (S^{2n-1}, \phi_{\text{std}})$ to ∂W as W is almost Weinstein fillable.

8. Nonexact fillings

In this section, we consider a strong filling W of a contact manifold Y , such that $c_1(W) = 0$. Floer cohomology is still well defined using the Novikov field $\Lambda = \{\sum_{i=1}^{\infty} q_i T^{\lambda_i} \mid q_i \in \mathbb{Q}, \lambda_i \in \mathbb{R}, \lim \lambda_i = \infty\}$ with a $\mathbb{Z}$ -grading, cf. [30], similarly for symplectic cohomology [53]. In particular, sphere bubbles can be avoided by dimension reasons. In this section, we show that positive symplectic cohomology is still defined to the extent that constructions in §§ 2 and 3 can be generalized to strong fillings to finish the proof of Theorem F.

8.1. Positive symplectic cohomology

In the nonexact case, the action functional (2.1) is not well defined. In particular, there is no action separation of the symplectic cohomology into zero length part and positive length part. However, positive symplectic cohomology can still be defined due to the following lemma by Bourgeois–Oancea [10, proof of Proposition 5]. We first recall it from [18]. In this section, by strong filling of a contact manifold (Y, α) , we mean a symplectic manifold (W, ω) with collar neighborhood of the boundary symplectomorphic to $(Y \times [1 - \delta, 1], d(r\alpha))$ for some $\delta > 0$. The completion $\widehat{W}$ is defined to be $W \cup Y \times (1, \infty)$, with a symplectic form $\widehat{\omega}$ such that $\widehat{\omega}|_W = \omega, \widehat{\omega}|_{Y \times (1, \infty)} = d(r\alpha)$.

LEMMA 8.1 [18, Lemma 2.3]. *Let $H = h(r)$ be a Hamiltonian on the symplectization $(Y \times \mathbb{R}_+, d(r\alpha))$. Let $u : \mathbb{R}_- \times S^1 \rightarrow Y \times \mathbb{R}_+$ be a finite energy solution to the Floer equation and $r_0 := \lim_{s \rightarrow -\infty} r \circ u(s, t)$. Assume $h''(r_0) > 0$, J is cylindrical convex on a neighborhood of $u(\mathbb{R}_- \times S^1)$, then either there exists (s_0, t_0) such that $r \circ u(s_0, t_0) > r_0$ or $r \circ u \equiv r_0$.*

The combination of Lemmas 2.5 and 8.1 yields the following.

PROPOSITION 8.2. *Consider a Hamiltonian $H : \widehat{W} \rightarrow \mathbb{R}$, such that $H = 0$ on $r \leq r_0 > 1$ and $H = h(r)$ for $r \geq r_0$ such that when $h'(r) \in \mathcal{S}$ we have $h''(r) > 0$, and assume J is cylindrical convex near all nonconstant periodic orbits of X_H , then there is no Floer solution u , such that $\lim_{s \rightarrow -\infty} u \in \mathcal{P}^*(H)$ and $\lim_{s \rightarrow \infty} u \in W \cup Y \times (1, r_0)$.*

Motivated by Proposition 8.2, instead of using the perturbed Hamiltonian $\mathbf{H}$ in § 2, we will use the autonomous Hamiltonian before perturbation, denote the Hamiltonian by $\mathbf{F}$. Then it has the following property.

- (1) $\mathbf{F} = 0$ on W .
- (2) $\mathbf{F} = h(r)$ on $\partial W \times (1, \infty)$ with $h''(r) > 0$.
- (3) The nonconstant periodic orbits of $\mathbf{F}$ are contained in levels $1 < r_1 < r_2 < \dots$ we will use W^i to denote the domain inside $r = r^i$.

Then by Proposition 8.2, there are no Floer cylinders with negative end asymptotic to a nonconstant orbit and positive end asymptotic to a point in W . In fact, the following holds.

PROPOSITION 8.3. *Using $\mathbf{F}$ as the Hamiltonian, if there is a nontrivial Floer cylinder u with the negative end asymptotic to an orbit inside $r = r_i$ and with the positive end asymptotic to an orbit inside $r = r_j$, then $i < j$.*

Proof. If $i > j$, then the combination of Lemmas 2.5 and 8.1 yields a contradiction. If $i = j$, then by Lemmas 2.5 and 8.1, we have u is contained in $r = r_i$. Then the energy of u must be zero, contradicting that u is nontrivial. $\square$

Using $\mathbf{F}$ to define symplectic cohomology will result in a more Morse–Bott situation than $\mathbf{H}$. We will use cascades to deal with degenerate S^1 -family orbits, such construction was studied in [10, 11] and more degenerate Morse–Bott cases were studied in [8, 19]. We will follow their constructions.

8.1.1. *Notations and setups.* We first fix the following notations.

(1) First note that every Reeb orbit of R_α corresponds to an S^1 -family of periodic orbits of $X_{\mathbf{F}}$. The set of nonconstant (contractible) periodic orbits of $\mathbf{F}$ has a decomposition into the union $\bigcup_{x \in \mathcal{P}(\alpha)} S_x$, each S_x can be viewed as an embedded circle in $\widehat{W}$. For each S_x , we fix a metric g_x and a Morse function f_x with a unique local maximum $\hat{x}$ and a unique local minimum $\check{x}$. With a little abuse of language, we denote $\mathcal{P}^*(\mathbf{F}) := \bigcup_x \{\hat{x}, \check{x}\}$.

(2) For each $x \in \mathcal{P}(\alpha)$, we fix a disk $u_x : \mathbb{D} \rightarrow Y$ extending one $\gamma \in S_x$ up to homotopy, which gives extension to any element in S_x by rotation.

(3) $\Lambda := \{\sum_{i=1}^{\infty} q_i T^{\lambda_i} \mid q_i \in \mathbb{Q}, \lambda_i \in \mathbb{R}, \lim \lambda_i = \infty\}$ is the Novikov field and we define the Novikov ring $\Lambda_0 := \{\sum_{i=1}^{\infty} q_i T^{\lambda_i} \mid q_i \in \mathbb{Q}, \lambda_i \geq 0, \lim \lambda_i = \infty\}$. Note that Λ is the fraction field of the integral domain Λ_0 . In particular, Λ is a flat Λ_0 -module.

(4) We formally define $\mathcal{A}_{\mathbf{F}}(T^a \hat{x}) = \mathcal{A}_{\mathbf{F}}(T^a \check{x}) = a - \int_{S^1} \gamma^* \widehat{\lambda} + \int_{S^1} \mathbf{F} \circ \gamma dt$, for $\gamma \in S_x$. And $\mathcal{A}_{\mathbf{F}}(T^a p) = a$ for $p \in \mathcal{C}(f)$, where f is an admissible Morse function on W .

8.1.2. *Moduli spaces and cochain complexes.* In view of Proposition 8.2, we need to modify the definition of admissible almost complex structures to the following. We first fix $\epsilon_i > 0$ such that $r_i + \epsilon_i < r_{i+1} - \epsilon_{i+1}$ and $\epsilon_0 > 0$ such that $1 + \epsilon_0 < r_1 - \epsilon_1$.

DEFINITION 8.4. A time-dependent almost complex structure $J : S^1 \rightarrow \text{End}(T\widehat{W})$ is admissible if and only if the following holds.

- (i) J is compatible with $\widehat{\omega}$ on $\widehat{W}$.
- (ii) J is cylindrical convex on $\partial W \times (r_i - \epsilon_i, r_i + \epsilon_i)$ and on $(1 - \epsilon_0, 1 + \epsilon_0)$.
- (iii) J is S^1 -independent on W .

The class of admissible almost complex structure is again denoted by $\mathcal{J}(W)$

Let J be an admissible almost complex structure. First note that for any $t \in S^1$, any nontrivial closed J_t -holomorphic curve must be contained in W^0 , where in particular J_t does not depend t . This is because the curve cannot be contained outside W by exactness there and then we apply Lemma 2.5 to $r = 1 - \epsilon_0$. We will be interested in the following uncompactified moduli spaces.

(1) $S_A(J) := \{u : \mathbb{CP}^1 \rightarrow \widehat{W} \mid \bar{\partial}_J u = 0, [u] = A, u \text{ is simple}\}$ for $A \in H_2(W)$. It has an evaluation map $ev_0(u) = u(0)$.

(2)

$$M_{n,A}(J) := \left\{ (u_1, \dots, u_n) \left| \begin{array}{l} u_i : \mathbb{CP}^1 \rightarrow W, \bar{\partial}_J u_i = 0, \sum [u_i] = A, \\ u_i \text{ is simple, and } u_i(\mathbb{CP}^1) \neq u_j(\mathbb{CP}^1), \\ u_i(\infty) = u_{i+1}(0) \end{array} \right. \right\}.$$

It is equipped with two evaluation maps $ev_0((u_i)) = u_1(0)$ and $ev_\infty((u_i)) = u_n(\infty)$.

(3)

$$M_{S_x, S_y, A}(J) := \left\{ u : \mathbb{R} \times S^1 \rightarrow \widehat{W} \left| \begin{array}{l} \partial_s u + J(\partial_t - X_F) = 0, \\ \lim_{s \rightarrow -\infty} u \in S_x, \lim_{s \rightarrow +\infty} u \in S_y, [u \# (-u_y) \# u_x] = A \end{array} \right. \right\}.$$

It is equipped with evaluation maps ev_-, ev_+ to S_x, S_y , respectively

(4) $B_{S_x, A}(J) := \{u : \mathbb{C} \rightarrow \widehat{W} | \partial_s u + J(\partial_t - X_F) = 0, u(0) \in W^0, \lim_{s \rightarrow +\infty} u \in S_x, [u \# (-u_x)] = A\}$. It is equipped with evaluations maps ev_+, ev_0 to S_x, W , respectively.

The energy $E(u) := \frac{1}{2} \int_{\mathbb{R} \times S^1} ||du - X_F dt||^2 ds \wedge dt$ of $u \in M_{S_x, S_y, A}$ is given by $\omega(A) + \mathcal{A}_F(S_x) - \mathcal{A}_F(S_y)$. The energy for $u \in B_{S_x, A}$ is given by $\omega(A) - \mathcal{A}_F(S_x)$.

To study the regularization of (1)–(4), we need to consider the following universal moduli spaces.

(1) Let $\mathbf{x} := (x_1, \dots, x_{n+1} \in \mathcal{P}(\alpha))$, $\mathbf{A} := (A_1, A_2, \dots, A_n \in H_2(W))$ and $B \in H_2(W)$, we define $\mathcal{U}_{\mathbf{x}, \mathbf{A}, B}^l$ to be the set of $(u_1, \dots, u_n, u, p, J)$ such that:

- (i) J is a C^l admissible almost complex structure;
- (ii) $u_i \in M_{S_{x_i}, S_{x_{i+1}}, A_i}(J)$;
- (iii) $u \in S_B(J)$;
- (iv) p is a point on domains of u_i .

Then $\mathcal{U}_{\mathbf{x}, \mathbf{A}, B}^l$ is equipped with an evaluation map

$$EV := ev_p \times ev_0(u) \times \prod_i ev_-(u_i) \times ev_+(u_i) \in \widehat{W} \times \widehat{W} \times S_{x_1} \times (S_{x_2})^2 \times \dots \times (S_{x_n})^2 \times S_{x_{n+1}}.$$

(2) $\mathbf{x} := (x_1, \dots, x_n \in \mathcal{P}(\alpha))$, $\mathbf{A} := (A_1, A_2, \dots, A_n \in H_2(W))$ and $B \in H_2(W)$, we define $\mathcal{U}_{\mathbf{x}, \mathbf{A}, B, k}^l$ be the set of $(u_1, \dots, u_n, u, J)$ such that:

- (i) J is a C^l admissible almost complex structure;
- (ii) $u_i \in M_{S_{x_{i-1}}, S_{x_i}, A_i}(J)$ if $i > 1$;
- (iii) $u_1 \in B_{S_{x_1}, A_1}(J)$;
- (iv) $u \in M_{k, B}(J)$;

Then we define EV on $\mathcal{U}_{\mathbf{x}, \mathbf{A}, B, k}^l$ by

$$\begin{aligned} & ev_0(u) \times ev_\infty(u) \times ev_0(u_1) \times ev_+(u_1) \times \prod_{i>1} (ev_-(u_i) \times ev_+(u_i)) \\ & \in \widehat{W} \times \widehat{W} \times \widehat{W} \times (S_{x_1})^2 \times \dots \times (S_{x_{n-1}})^2 \times S_{x_n}. \end{aligned}$$

(3) Let $\mathbf{x} := (x_1, \dots, x_n \in \mathcal{P}(\alpha))$, $\mathbf{A} = (A_1, A_2, \dots, A_n \in H_2(W))$ and $B \in H_2(W)$, we define $\mathcal{V}_{\mathbf{x}, \mathbf{A}, B}^l$ be the set of $(u_1, \dots, u_n, u, p, J)$ such that:

- (i) J is a C^l admissible almost complex structure;
- (ii) $u_i \in M_{S_{x_{i-1}}, S_{x_i}, A_i}(J)$ for $i > 1$;
- (iii) $u_1 \in B_{S_{x_1}, A_1}(J)$;
- (iv) $u \in S_B(J)$;
- (v) p is a point on domains of u_i .

Then we define EV on $\mathcal{V}_{x,A,B}^l$ by

$$\begin{aligned} & ev_p \times ev_0(u) \times ev_0(u_1) \times ev_+(u_1) \times \prod_{i>1} (ev_-(u_i) \times ev_+(u_i)) \\ & \in \widehat{W} \times \widehat{W} \times \widehat{W} \times (S_{x_1})^2 \times \dots \times (S_{x_{n-1}})^2 \times S_{x_n}. \end{aligned}$$

The point of considering such moduli spaces is that they are cut out transversely, moreover the evaluation map is a submersion. This will allow us to perturb J only without changing (f, g) and (f_x, g_x) . The following proposition is derived from the same argument in [43, §3.4].

PROPOSITION 8.5. *The three universal moduli spaces considered above are cut out transversely as Banach manifolds and EV are submersive.*

Proof (Sketch). Note that by Proposition 8.3, for $x = (x_1, \dots, x_n)$ in the universal moduli spaces, the r -coordinate must be strictly decreasing. Therefore the objects we consider in the universal moduli spaces are somewhere injective (not just component-wise, but as a whole object). Then the claim follows from the proof of [43, Proposition 3.4.2] and the proof of [11, Proposition 3.5][†]. $\square$

The cochain complex model we will use is the cascades construction [10, 11], in particular the cochain complex will be generated by $\mathcal{C}(f)$ and $\mathcal{P}^*(F)$ as before using the Novikov field Λ . We recall the cascades moduli spaces first. Let $\bar{x}$ denote an element in $\{\hat{x}, \check{x}\}$.

- (1) $M_{\bar{x}, \bar{y}, A, m}$ denotes the set of m -cascades, that is, tuples

$$(u_1, l_{1,2}, u_2, \dots, l_{m-1,m}, u_m),$$

where:

- (i) $u_i \in M_{S_{x_i}, S_{x_{i+1}}, A_i} / \mathbb{R}$ with $x_1 = x, x_{m+1} = y$, and $\sum A_i = A$;
 - (ii) $l_{i,i+1} > 0$ and $\phi_{x_{i+1}, l_{i,i+1}} ev_+(u_i) = ev_-(u_{i+1})$, where $\phi_{x_{i+1}, t}$ is the negative gradient flow on $S_{x_{i+1}}$;
 - (iii) $\lim_{t \rightarrow -\infty} \phi_{x_1, t} ev_-(u_1) = \bar{x}$ and $\lim_{t \rightarrow \infty} \phi_{x_{m+1}, t} ev_+(u_m) = \bar{y}$.
- (2) For $p \in \mathcal{C}(f)$, $M_{p,x,A,m}$ is defined similarly, except the first curve u_1 is in $B_{S_{x_1}, A_1}$ with $u_1(0)$ in the stable manifold of p .

For every $x \in \mathcal{P}(\alpha)$, we can assign the gradings $|\hat{x}| := n - \mu_{CZ}(x)$ and $|\check{x}| := n - 1 - \mu_{CZ}(x)$. As in §2, we fix a Morse–Smale pair (f, g) and $\mathcal{M}_{p,q}$ is a Morse moduli spaces for $p, q \in \mathcal{C}(f)$. Let us denote $\mathcal{M}_{\bar{x}, \bar{y}, A} = \cup_{m \geq 0} M_{\bar{x}, \bar{y}, A, m}$ and $\mathcal{M}_{p, \bar{x}, A} = \cup_{m \geq 0} M_{p, \bar{x}, A, m}$. From Proposition 8.5, we have the following transversality and compactness result.

PROPOSITION 8.6. *There exists a second category set $\mathcal{J}_{reg}(f, g)$ such that $\mathcal{M}_{\bar{x}/p, \bar{y}, A}$ is compact smooth manifold of dimension $|\bar{x}/q| - |\bar{y}| - 1$ whenever it is ≤ 0 .*

Proof. On every S_x , there is submanifold $H_x := \{(x, \phi_t(x)) | t > 0\} \subset S_x \times S_x$, where ϕ_t is the negative gradient flow on S_x . Let S_p, U_p be the stable manifold and unstable manifold of $p \in \mathcal{C}(f), \mathcal{C}(f_x)$. Since EV on universal moduli spaces are submersive, then the following space are Banach manifolds.

- (i) $EV_{\mathcal{U}_{x,A,B}}^{-1}(\Delta_{\widehat{W}} \times S_{x_1} \times \prod_{i>1} H_{x_i} \text{ (or } \Delta_{S_{x_i}}) \times U_{\bar{x}_{n+1}}).$

[†]Reference [11] did not claim the evaluation maps to S_x are submersive, but the proof of [11, Proposition 3.5] implies the fact. In the universal moduli space, the surjectivity of the linearized operator holds without using the tangent of S_x .

- (ii) $EV_{\mathcal{U}_{\mathbf{x}, \mathbf{A}, B, k}^l}^{-1}(S_p \times \Delta_{\widehat{W}} \times \prod_{i < n} H_{x_i}(\text{ or } \Delta_{S_{x_i}}) \times U_{\overline{x}_n})$, for $p \in \mathcal{C}(f)$.
- (iii) $EV_{\mathcal{V}_{\mathbf{x}, \mathbf{A}, B}^l}^{-1}(\Delta_{\widehat{W}} \times S_p \times \prod_{i < n} H_{x_i}(\text{ or } \Delta_{S_{x_i}}) \times U_{\overline{x}_n})$, for $p \in \mathcal{C}(f)$.

Then we can pick any J in the regular value of the projections of above spaces to the space of C^l admissible almost complex structure. Then transversality on $\mathcal{M}_{*,*,A}$ is verified. It is sufficient to prove compactness. First by a dimension argument, it cannot have a Morse breaking, or a fiber product breaking at S_x when $|\overline{x}/p| - |\overline{y}| \leq 1$, by the transversality of the preimage of $\Delta_{S_{x_i}}$ in (i)–(iii) above. For the general case in the compactification, we still need to consider the following.

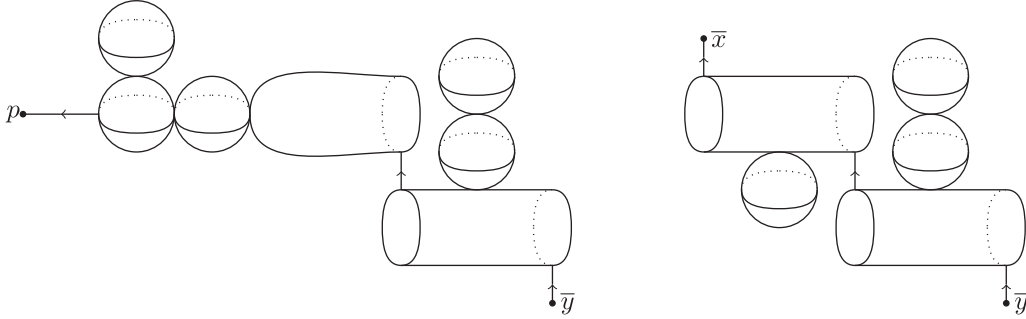


FIGURE 5. The general curves, the arrows stand for gradient flow.

Note that for the figure in the left, the link of spheres connecting Morse trajectory and Floer cylinder should be understood as Floer cylinders, hence they are J -curves without quotienting S^1 . We will not emphasize the difference, since we will rule out such configuration altogether. Note that the existence of curves above in the compactification of $\mathcal{M}_{*,*,A}$ will lead to existence of one of the following configurations.

The single sphere bubble above is simple and the link of spheres in the left is in $M_{n,A}(J)$. To see that we can reduce to the above three case, if we have a curve in left of Figure 5, then the bubble tree in the left can be reduced to a simple stable map by [43, Proposition 6.1.2]. We can pick out the link of spheres containing the two marked points and stabilize it by collapse unstable constant spheres. If the two marked points are on nonconstant sphere(s), then we get the first case in Figure 6. If the two marked points are on a constant sphere, then we get the third case in Figure 6. The other cases in Figure 5 are similar.

By construction, the moduli space in Figure 6 are cut out transversely. The expected dimensions after module the reparametrization action are $|p| - |\overline{y}| - 3 - 2(k - 1)$, $|\overline{x}| - |\overline{y}| - 3$, $|p| - |\overline{y}| - 3$, respectively, where k is the number of spheres. All of them are negative. Hence $\mathcal{M}_{*,*,A}$ is compact when expected dimension ≤ 0 (also holds for dimension ≤ 1 when adding

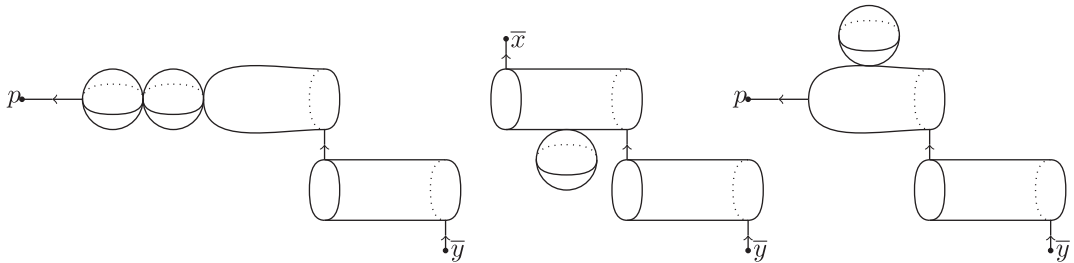


FIGURE 6. Special components.

Morse breaking and fiber product breaking). The proof of boosting C^l almost complex structure to C^∞ follows from the same argument in [30, Theorem 5.1]. $\square$

Then for $J \in \mathcal{J}^{\text{reg}}(f, g)$, we can define cochain complexes $C(\mathbf{F}, J, f), C_+(\mathbf{F}, J), C_0(f)$ as free Λ -module generated by both $\mathcal{P}^*(\mathbf{F})$ and $\mathcal{C}(f)$, only $\mathcal{P}^*(\mathbf{F})$, or only $C_0(f)$, respectively. And the differential is defined to be

$$dy := \sum_{\dim \mathcal{M}_{x,y,A}=0} \# \mathcal{M}_{x,y,A} T^{\omega(A)} x.$$

PROPOSITION 8.7. *d is a differential.*

Proof. Since the differential increases action $\mathcal{A}_{\mathbf{F}}$ and preserve the length filtration from r_i by Proposition 8.3, it is sufficient to verify $d^2 = 0$ on orbits inside W^i within a bounded action window. In such case, we have compactness and [11, Theorem 3.7] can be applied to verify $d^2 = 0$. $\square$

As usual, $C(\mathbf{F}, J, f), C_+(\mathbf{F}, J), C_0(f)$ form a short exact sequence, which induces a long exact sequence. We also have subcomplexes $C^{r_i}, C_+^{r_i}$ generated by orbits inside W^i . By definition, C, C_+ are direct limits of them. On the other hand, we have cochain complexes by action truncation. In particular, we will consider $C^{r_i, \geq 0}$ and $C_+^{r_i, \geq 0}$, that is, the complex only with elements of nonnegative action inside W^i . $C^{r_i, \geq 0}$ and $C_+^{r_i, \geq 0}$ are Λ_0 modules, and we have $C^{r_i, \geq 0} \otimes_{\Lambda_0} \Lambda = C^{r_i}$ and $C_+^{r_i, \geq 0} \otimes_{\Lambda_0} \Lambda = C_+^{r_i}$. Since Λ is a flat Λ_0 -module, the relation pass to cohomology. We may consider even smaller cochain complex $C_+^{r_i, [0, j]}$ of elements with action in $[0, j]$. On such complex, we have an energy control, hence neck-stretching can be applied.

On the other hand, like the exact case, symplectic cohomology can also be defined as a direct limit using nondegenerate Hamiltonian with finite slope when sphere bubbles can be avoided (for example, $c_1 = 0$), see [53].

PROPOSITION 8.8. *There is an isomorphism $SH^*(W; \Lambda) \rightarrow H^*(C(\mathbf{H}, J, f))$ such that the following diagram commutes*

$$\begin{array}{ccc} QH^*(W) & \longrightarrow & SH^*(W; \Lambda) \\ \downarrow & & \downarrow \\ H^*(C_0(f)) & \longrightarrow & H^*(C(\mathbf{F}, J, f)) \end{array}$$

Proof (Sketch). Similar to Proposition 2.10, we first perturb $\mathbf{F}$ to F , such that on W , F is a C^2 -small time-independent Hamiltonian but $F = \mathbf{F}$ on the cylindrical end. Then there is a cascade continuation map [10, §2] from the cochain complex of F to the cochain complex of $\mathbf{F}$ preserving the r_i filtration. As in Proposition 2.10, it is quasi-isomorphism as it induces isomorphism on the first page of the spectral sequence (which is convergent). Then similar to the discussion in §4.6, we first get F_i with finite slope, which is the same as F on W^i but linear afterward. Then cohomology of $C(F)$ can be written as direct limit of $H^*(C(F_i))$. The last step is perturbing F_i into nondegenerate Hamiltonians, then using the cascade continuation map, we have that the direct limit using nondegenerate Hamiltonians with finite slop is the same as $\varinjlim_i H^*(C(F_i))$. Since the inclusion from $C_0(f) \rightarrow C(\mathbf{F}, J, f)$ can also be viewed as continuation map from a homotopy of zero slope truncation of $\mathbf{F}$ to $\mathbf{F}$ by Lemma 2.5, then all

of the construction above is compatible with continuation maps from the 0-length part[†]. This finishes the proof. $\square$

From here, we give an *ad hoc* definition of $SH_+^*(W; \Lambda)$ motivated from Proposition 8.8,

$$SH_+^*(W) = \text{coker}(QH^*(W) \rightarrow SH^*(W; \Lambda)) \oplus \ker(QH^{*+1}(W) \rightarrow SH^{*+1}(W; \Lambda)).$$

Since Λ is field, Proposition 8.8 implies that there is a (noncanonical) isomorphism $SH_+^*(W) \rightarrow H^*(C_+(\mathbf{F}, J))$. We do not claim this definition is good in a functorial way, but it is sufficient for our application.

8.2. Proof of the independence

Similarly, we can define moduli spaces $\mathcal{P}_{p,\bar{x},A}$ and $\mathcal{H}_{p,\bar{x},A}$ as in § 3.1. Combining with the proofs in Propositions 8.5 and 8.7, § 3.1 can be generalized to strong fillings with vanishing first Chern class with the same statements. Hence we will use the same terminology, in particular we have various regular sets of almost complex structures. The following Proposition follows from the same proof of Proposition 3.4.

PROPOSITION 8.9. *Let $J^{i,j} \in \mathcal{J}_{reg,+}^{r_i,[0,j]}(\mathbf{F}) \cap \mathcal{J}_{reg,P}^{r_i,[0,j]}(\mathbf{F}, h, g_\partial)$, then we have a commutative diagram*

$$\begin{array}{ccccc} H^*(C_+^{r_1,[0,1]}(J^{1,1})) & \longrightarrow & H^*(C_+^{r_2,[0,1]}(J^{2,1})) & \longrightarrow & \dots \longrightarrow H^{*+1}(\partial W) \times \Lambda^{[0,1]} \\ \uparrow & & \uparrow & & \uparrow \\ H^*(C_+^{r_1,[0,2]}(J^{1,2})) & \longrightarrow & H^*(C_+^{r_2,[0,2]}(J^{2,2})) & \longrightarrow & \dots \longrightarrow H^{*+1}(\partial W) \times \Lambda^{[0,2]} \\ \uparrow & & \uparrow & & \uparrow \\ \vdots & & \vdots & & \vdots \end{array}$$

where vertical and horizontal arrows are continuation maps except those mapped to the last column and $\Lambda^{[0,j]}$ consists of elements in Λ with action in $[0, j]$. The arrow to the last column is defined by counting $\mathcal{P}_{*,*}$. Then $\lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty}$ of the diagram computes $SH_+^{\geq 0}(W) \rightarrow H^{*+1}(\partial W; \Lambda_0)$.

Now let Y be a TADC contact manifold with two strong fillings W_1, W_2 , we assume α_i in Definition 6.2 is represented by a nested sequences $Y_i \in Y \times [1, R]$. Then we view $W_* \cup Y \times [1, R]$ as the new W_* , in particular $\mathbf{F}$ is zero on this new W_* . Then we can stretch Y_i in the same way as in § 3.2.

PROPOSITION 8.10. *With the set-up above, there exist admissible $J_1^1, J_1^2, \dots$ and $J_2^1, J_2^2, \dots$ on $\widehat{W}_1$ and $\widehat{W}_2$, respectively, and positive real numbers $\epsilon_{i,j}, i, j \in \mathbb{N}_+$ such that the following holds.*

- (i) We have $\epsilon_{i,j} > \epsilon_{i,j+1}$. For $R < \epsilon_{i,j}$ and any R' , $NS_{i,R}(J_*^i), NS_{i+1,R'}(NS_{i,R}(J_*^i)) \in \mathcal{J}_{reg,+}^{r_i,[0,j]} \cap \mathcal{J}_{reg,P}^{r_i,[0,j]}(h, g_\partial)$, such that all zero-dimensional $\mathcal{M}_{x,y,A}$ and $\mathcal{P}_{p,x,A}$ are the same for both W_1, W_2 and contained outside Y_i for $x, y \in C_+^{r_i}, p \in C(h)$ with action change at most j .
- (ii) For every i , there exists $i_j \in \mathbb{N}_+$ such that $J_*^{i+1} = NS_{i, \frac{\epsilon_{i,i_j}}{2}}(J_*^i)$ on W_*^i .

[†]Note that for a general nondegenerate Hamiltonian H with finite slope, Proposition 8.2 may not apply and the 0-length part may not be a subcomplex.

Proof. The proof is exactly the same as Proposition 3.12. We start with J^1 such that $NS_{1,0}(J^1) \in \mathcal{J}_{\text{reg},SFT}^{r_1}(\mathbf{F}, h, g_\partial)$. The threshold $\epsilon_{i,j}$ will be dependent on j , because we need compactness to apply neck-stretching. $\epsilon_{i,j+1}$ is smaller than $\epsilon_{i,j}$, as $\mathcal{J}_{\text{reg}}^{r_i, [0, j+1]} \subset \mathcal{J}_{\text{reg}}^{r_i, [0, j]}$. To prove the second property, note that all curves in Figure 3 for $x, y \in C_+^{r_i}$ have a universal bound depending only on i . Therefore there exists i_j , such that $NS_{i+1,0}(NS_{i, \frac{\epsilon_{i,j}}{2}})(J^i) \in \mathcal{J}_{\text{reg},SFT}^{r_i}(\mathbf{F}, h, g_\partial)$, since the related moduli spaces are contained outside Y_i by the first property. Hence we can choose J^{i+1} such that $NS_{i+1,0}(J^{i+1}) \in \mathcal{J}_{\text{reg},SFT}^{r_{i+1}}(\mathbf{F}, h, g_\partial)$ and $J^{i+1} = NS_{i, \frac{\epsilon_{i,j}}{2}}(J^i)$ on W^i as in Proposition 3.12. $\square$

THEOREM 8.11. *Let Y be a TADC manifold, then $SH_+^*(W; \Lambda) \rightarrow H^{*+1}(Y; \Lambda)$ is independent of the filling.*

Proof. Using the almost complex structures in Proposition 8.10, by Proposition 8.9, we have that the part where $j \geq i_j$ of the following diagram computes $SH_+^{\geq 0}(W) \rightarrow H^{*+1}(Y; \Lambda_0)$,

$$\begin{array}{ccccc}
H^*(C_+^{r_1, [0,1]}(NS_{1, \frac{\epsilon_{1,1}}{2}} J_*^1)) & \longrightarrow & H^*(C_+^{r_2, [0,1]}(NS_{2, \frac{\epsilon_{2,1}}{2}} J_*^2)) & \longrightarrow & \dots \longrightarrow H^{*+1}(\partial W) \times \Lambda^{[0,1]} \\
\uparrow & & \uparrow & & \uparrow \\
H^*(C_+^{r_1, [0,2]}(NS_{1, \frac{\epsilon_{1,2}}{2}} J_*^1)) & \longrightarrow & H^*(C_+^{r_2, [0,2]}(NS_{2, \frac{\epsilon_{2,2}}{2}} J_*^2)) & \longrightarrow & \dots \longrightarrow H^{*+1}(\partial W) \times \Lambda^{[0,2]} \\
\uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots
\end{array}$$

By the same argument in the proof of Theorem A, the horizontal continuation maps are inclusions for $j \geq i_j$. The vertical continuation map can be decomposed into continuation maps $C_+^{r_i, [0, j+1]}(NS_{i, \frac{\epsilon_{i,j+1}}{2}} J_*^i) \rightarrow C_+^{r_i, [0, j]}(NS_{i, \frac{\epsilon_{i,j}}{2}} J_*^i)$ and $C_+^{r_i, [0, j]}(NS_{i, \frac{\epsilon_{i,j}}{2}} J_*^i) \rightarrow C_+^{r_i, [0, j]}(NS_{i, \frac{\epsilon_{i,j}}{2}} J_*^i)$. The former map from the trivial homotopy of almost complex structure is the obvious quotient, the latter map is homotopic to identity by Lemma 2.15. Therefore the part where $j \geq i_j$ of the diagram is identified for both W_1, W_2 . Hence $SH_+^{\geq 0}(W) \rightarrow H^{*+1}(Y; \Lambda_0)$ is independent of the filling. Since Λ is a flat Λ_0 module, the claim follows from tensoring Λ . $\square$

COROLLARY 8.12. *If Y is a TADC contact manifold, then $SH^*(W; \Lambda) = 0$ is a property independent of topologically simple strong fillings.*

Proof. Since $SH^*(W; \Lambda) = 0$ is equivalent to $QH^0(W) \rightarrow SH^0(W; \Lambda)$ maps 1 to 0 since it a unital ring map [53, Corollary 14], which is equivalent to $1 \in \text{im}(SH_+^{-1}(W; \Lambda) \rightarrow H^0(W; \Lambda) \rightarrow H^0(Y; \Lambda))$, which by Theorem 8.11 is independent of such fillings. $\square$

Proof of Theorem F. By Theorem 8.11 and Corollary 8.12, the map $H^*(W; \Lambda) \rightarrow H^*(Y; \Lambda)$ is independent of the filling. Hence for any other topologically simple strong filling W' , we have $H^2(W'; \mathbb{Q}) \rightarrow H^2(Y; \mathbb{Q})$ is injective and $H^1(W'; \mathbb{Q}) \rightarrow H^1(Y; \mathbb{Q})$ is surjective. They imply that there exists one form β on W' , such that $\omega = d\beta$ and $\beta = \alpha$ near Y . That is W' is exact. $\square$

REMARK 8.13. Another similar case is for a TADC $Y := \partial(V \times \mathbb{C})$. Then by the same argument for any topologically simple strong filling W , we have $H^2(W; \mathbb{Q}) \rightarrow H^2(Y; \mathbb{Q})$ is surjective. Therefore the symplectic form ω must be exact. It may not be true that there is a primitive restricted to a contact form. But such exactness is enough to rule out all sphere bubbles, hence the invariance can be lifted to $\mathbb{Z}$ coefficient by Corollary B.

Theorem 8.11 also holds for symplectic cohomology with local systems (using Novikov field over $\mathbb{C}$). As a combination of twisted version of Theorems 8.11 and D, we have the following.

THEOREM 8.14. *Assume Q is in Theorem D and $\dim Q \geq 4$. Then any strong filling W of ST^*Q with vanishing first Chern class that satisfies other topological conditions in Theorem D is exact.*

REMARK 8.15. For general ADC manifolds, Y_i may collapse to zero. Since nonexact filling W does not contain $Y \times (0, 1)$, we can only seek room in $Y \times (1, \infty)$. In particular, we need to accommodate the increasing of $\text{supp } \mathbf{F}$. However, in addition to the problem in Remark 2.11, it seems to be very difficult to arrange homotopies of Hamiltonians, so that integrated maximum principle can be applied to exclude contributions from the 0-length generators to the positive length generators. Requiring $\text{supp } \mathbf{F}$ converges to ∞ indicates that it might be better to define the theory on the SFT level, as in SFT, the vanishing of contributions of the zero length generators to the positive length generators is automatic. In particular, Theorem 8.11 should generalize to ADC manifolds by considering the theory defined using SFT, see Remark 1.7.

REMARK 8.16. For ADC but not TADC contact manifolds, it is still possible to get some information about the strong filling. For example, if we know that 1 is killed by Hamiltonian orbits with action at least $-D$ in the exact filling, then possibly after rescaling, we have a contact hypersurface Y' inside the strong filling such that Reeb orbits with period smaller than D are nondegenerate and have positive degree. Then the argument in Theorem 8.11 implies that the symplectic cohomology of the strong filling with vanishing first Chern class is zero as we have found the primitive of 1. In general, our argument proves that $\text{im}(SH_+^*(W; \Lambda) \rightarrow H^{*+1}(Y; \Lambda))$ is an invariant, while the invariance of the map cannot be obtained.

Appendix A. Orientations

A.1. Coherent orientations in §§ 2, 3 and 8

Following [1, §1.4], for every nonconstant periodic orbit $x \in \mathcal{P}^*(\mathbf{H})$, one can associate an orientation line o_x , which is the determinant line bundle of the following operator

$$D_x : W^{1,p}(\mathbb{C}, \mathbb{C}^n) \rightarrow L^p(\mathbb{C}, \mathbb{C}^n), \quad X \mapsto \partial_s X + I(\partial_t X - B \cdot X), \quad (\text{A.1})$$

where $\mathbb{C}$ is equipped with the negative cylindrical end $\mathbb{R} \times S^1 \rightarrow \mathbb{C}, (s, t) \mapsto e^{-2\pi(s+it)}$, I is the complex structure on $\mathbb{C}^n$ and $B = \Psi'(t) \cdot \Psi(t)^{-1}$ when $s \ll 0$, here $\Psi(t)$ is the path in $Sp(2n)$ determined by the linearization of the Hamiltonian flow of $X_{\mathbf{H}}$ around x and a trivialization of $x^*(TW)$. By [1, Proposition 1.4.10], determinant bundles of D_x using different choices of trivializations are conically isomorphic. Hence o_x is well defined. Moreover, we have $\text{ind } D_x = |x|$.

On the other hand, if we equip $\mathbb{C}$ with the positive cylindrical end [1, §1.4.3], then (A.1) induces another determinant line o_x^+ . Note that in this case, (A.1) is the linearization of the equation of B_x (2.5), therefore we have a conical isomorphism $\langle \partial_s \rangle \otimes \det B_x = o_x^+$. By the gluing property of such determination bundles [58, Lemma 1.4.5], we have a canonical isomorphism $o_x^+ \otimes o_x = \det \mathbb{C}^n$.

Let u be a solution to the Floer equation with negative end asymptotic to $x \in \mathcal{P}^*(\mathbf{H})$ and positive end asymptotic to $y \in \mathcal{P}^*(\mathbf{H})$. After choosing a trivialization of u^*TM , the linearized operator of the Floer equation is in the form of

$$D_u : W^{1,p}(\mathbb{R} \times S^1, \mathbb{C}^n) \rightarrow L^p(\mathbb{R} \times S^1, \mathbb{C}^n), \quad X \mapsto \partial_s X + I(\partial_t X - B \cdot X), \quad (\text{A.2})$$

where asymptotics of B are determined by the linearization of the Hamiltonian flow of X_H near two ends as before. Let o_u denote the determinant line. When $\mathcal{M}_{x,y}$ is a manifold, they form a continuous bundle $o_{x,y}$. Then with a regular J in Proposition 2.8, the gluing property of such bundles [58, Lemma 1.4.5] yields the following structures.

- (1) Canonical isomorphisms $\rho_{x,y} : o_{x,y} \otimes o_y \rightarrow o_x$, $\rho_{x,y,z} : o_{x,y} \otimes o_{y,z} \rightarrow o_{x,z}$ on $\mathcal{M}_{x,y} \times \mathcal{M}_{y,z} \subset \mathcal{M}_{x,z}$ such that $\rho_{x,z} \circ \rho_{x,y,z} = \rho_{x,y} \circ \rho_{y,z}$ on $o_{x,y} \otimes o_{y,z} \otimes o_z$.
- (2) $o_{x,y} = \det(\langle \partial_s \rangle \oplus T\mathcal{M}_{x,y})$ and the $\rho_{x,y,z}$ is induced by a map $(\langle \partial_{s_1} \rangle \oplus T\mathcal{M}_{x,y}) \oplus (\langle \partial_{s_2} \rangle \oplus T\mathcal{M}_{y,z}) \rightarrow (\langle \partial_s \rangle \oplus T\mathcal{M}_{x,z})$ with the property that $\partial_{s_1} + \partial_{s_2}$ is mapped to ∂_s and $\partial_{s_2} - \partial_{s_1}$ is mapped to the out normal vector of $T\mathcal{M}_{x,z}$, cf. [1, Lemma 1.5.7].
- (3) Canonical isomorphism $\rho_{x,y}^+ : o_x^+ \otimes o_{x,y} \rightarrow o_y^+$ over $\mathcal{B}_x \times \mathcal{M}_{x,y} \subset \mathcal{B}_y$, which is induced by a map $\langle \partial_{s_1} \rangle \oplus T\mathcal{B}_x \oplus \langle \partial_{s_2} \rangle \oplus T\mathcal{M}_{x,y} \rightarrow \langle \partial_s \rangle \oplus T\mathcal{B}_y$ with $\partial_{s_1} + \partial_{s_2}$ mapped to ∂_s and $\partial_{s_2} - \partial_{s_1}$ mapped the out normal vector.

If we fix orientations on o_x , then there are induced orientations on $o_{x,y}$ and o_x^+ , which determine orientations of $\mathcal{M}_{x,y}$ and $\mathcal{B}_x$ by quotienting out the $\mathbb{R}$ factor from the left.

Next we orient the Morse theory part. For a Morse function f , let S_p, U_p denote the stable and unstable manifold of $\nabla_g f$. Then there is a conical isomorphism $T_p U_p \oplus T_p S_p = T_p W$. Moreover, we have $\langle \partial_s \rangle \oplus T\mathcal{M}_{p,q} = T(S_p \cap U_q)$, the latter at p has a natural isomorphism to TS_p/TS_q . Therefore if we fix orientations for every S_p , then there is an induced orientation on $\mathcal{M}_{p,q}$. When m is the unique local minimum, we orient S_m such that the induced orientation U_m coincides with orientation of W , this guarantees the identity is generated by m . Since $\mathcal{M}_{p,y}$ is the fiber product $S_p \times_W \mathcal{B}_y$, we orient $\mathcal{M}_{p,y}$, such that the isomorphism $T\Delta_W \oplus T\mathcal{M}_{p,y} \rightarrow TS_p \oplus T\mathcal{B}_y$ preserves the orientation (it is actually twisted by $(-1)^{\dim S_p \times \dim W}$ for general fiber product).

REMARK A.1. Our convention is from the following consideration: if we view f as a Hamiltonian, we can assign two line bundles o_p, o_p^+ as before. Then there are conical isomorphism $o_p = \det S_p$ and $o_p^+ = \det U_p$, because D_p is the linearization of an equation whose solution corresponds to the stable manifold S_p . Similarly for o_p^+ . Then the gluing of determinant bundle gives an isomorphism $o_p^+ \otimes o_p = \det \mathbb{C}^n$ corresponding to $T_p U_p \oplus T_p S_p = T_p W$. Moreover, $o_{p,q} = \det(TS_p/TS_q)$ and the gluing map $\rho_{p,q} : o_{p,q} \otimes o_q \rightarrow o_p$ is induced from the obvious map. As for the orientation convention for fiber products, it is different from the one used in [19] by a sign twisting for general fiber products, but they coincide in the special case considered here since $\dim W = 2n$. Our fiber product orientation rule also satisfies associativity.

Therefore we have oriented all $\mathcal{M}_{*,*}$, the following proposition shows that orientations are coherent in the sense that they imply $d^2 = 0$.

PROPOSITION A.2. For 1-dimensional $\mathcal{M}_{*,*}$, with orientations above, we have $\partial \mathcal{M}_{*,*} = \sum \mathcal{M}_{*,*} \times \mathcal{M}_{*,*}$.

Proof. For $x, y, z \in \mathcal{P}^*(H)$, by property (3), we have an orientation-preserving map $(\langle \partial_{s_1} \rangle \oplus T\mathcal{M}_{x,y}) \oplus (\langle \partial_{s_2} \rangle \oplus T\mathcal{M}_{y,z}) \rightarrow (\langle \partial_s \rangle \oplus T\mathcal{M}_{x,z})$ over $\mathcal{M}_{x,y} \times \mathcal{M}_{y,z}$ with the property that $\partial_{s_1} + \partial_{s_2}$ is mapped to ∂_s and $\partial_{s_2} - \partial_{s_1}$ is mapped to the out normal vector of $T\mathcal{M}_{x,z}$. Hence the product orientation $\mathcal{M}_{x,y} \times \mathcal{M}_{y,z}$ is the boundary orientation. The situation for $\mathcal{M}_{p,q} \times \mathcal{M}_{q,r}$ for $p, q, r \in \mathcal{C}(f)$ is similar. Next we consider $\mathcal{M}_{p,x} \times \mathcal{M}_{x,y}$ for $p \in \mathcal{C}(f), x, y \in \mathcal{P}^*(H)$. Then by property (3), we have an orientation-preserving map $\langle \partial_{s_1} \rangle \oplus T\mathcal{B}_x \oplus \langle \partial_{s_2} \rangle \oplus T\mathcal{M}_{x,y} \rightarrow \langle \partial_s \rangle \oplus T\mathcal{B}_y$ with $\partial_{s_1} + \partial_{s_2}$ mapped to ∂_s and $\partial_{s_2} - \partial_{s_1}$ mapped the out normal vector. Then by our fiber product orientation rule, the product orientation $\mathcal{M}_{p,x} \times \mathcal{M}_{x,y}$ is the boundary orientation. The situation for $\mathcal{M}_{p,q} \times \mathcal{M}_{q,x}$ for $p, q \in \mathcal{C}(f), x \in \mathcal{P}^*(H)$ is the same. $\square$

To orient other moduli spaces in §§ 2 and 3, we need to assign orientation to S_p for every $p \in \mathcal{C}(h)$. Then for $\mathcal{R}_{p,q}$, we have an isomorphism $T\mathcal{R}_{p,q} = TS_p/TS_q$ at the intersection point on Y . Therefore orientations of S_p for $p \in \mathcal{C}(f) \cup \mathcal{C}(h)$ determines orientation of $\mathcal{R}_{p,q}$. The orientation of $\mathcal{P}_{p,y}$ is determined by the fiber product rule. For $\mathcal{H}_{p,y}$, the orientation is determined by fiber product rule and the orientation on the Morse part is given by $\langle \partial_t \rangle \oplus S_p$ for $p \in \mathcal{C}(h)$. Then by the same argument in Proposition A.2, we have the following.

PROPOSITION A.3. *For moduli spaces up to dimension 1, we have the induced orientations above satisfying the following.*

- (i) $\partial \mathcal{R}_{*,*} = \sum \mathcal{M}_{*,*} \times \mathcal{R}_{*,*} - \sum \mathcal{R}_{*,*} \times \mathcal{M}_{*,*}$.
- (ii) $\partial \mathcal{P}_{*,*} = \sum \mathcal{M}_{*,*} \times \mathcal{P}_{*,*} + \sum \mathcal{P}_{*,*} \times \mathcal{M}_{*,*}$.
- (iii) $\partial \mathcal{H}_{*,*} = \sum \mathcal{M}_{*,*} \times \mathcal{H}_{*,*} + \sum \mathcal{H}_{*,*} \times \mathcal{M}_{*,*} + \sum \mathcal{R}_{*,*} \times \mathcal{M}_{*,*} - \sum \mathcal{P}_{*,*}$.

The orientations for the continuation map moduli spaces in Proposition 2.10 and § 2.2 is done similarly, but there is no $\mathbb{R}$ factor needed to be quotiented.

For the orientation in § 8, by [11, Theorem 3.7], the moduli space encountered in § 8 has a one-to-one correspondence to the moduli space using nondegenerate Hamiltonians nearby. Hence we can use the orientation from a perturbed Hamiltonian, and then we have coherent orientations from Propositions A.2 and A.3. Alternative approaches to orient the cascades moduli space directly can be found in [11, §4.4], but they require a twisting depending on the number of cascades [11, Proposition 3.9].

A.2. Coherent orientations in § 4

We follow the orientation convention for the BV operator in [58]. We consider a solution (u, θ) to equation (4.2), that is,

$$\partial_s u + J_{s,t}^\theta (\partial_t u - X_{H_{s,t}^\theta}) = 0, \quad \lim_{s \rightarrow -\infty} u = x(\cdot + \theta), \quad \lim_{s \rightarrow \infty} u = y, \quad x \in \mathcal{P}^*(H_-), y \in \mathcal{P}^*(H_+).$$

If transversality for $\mathcal{M}_{x,y}^\Delta$ holds, let D_u denote the linearization of the equation above at u , then we have a short exact sequence

$$0 \rightarrow T_{(u,\theta)} \mathcal{M}_{x,y}^\Delta \rightarrow \ker D_u \oplus TS^1 \rightarrow \text{coker } D_u \rightarrow 0.$$

Since $\det D_u = o_{x,y}$, the short exact sequence induces an isomorphism $\det T\mathcal{M}_{x,y}^\Delta = o_{x,y} \otimes TS^1$. Hence given orientations of o_x, o_y and S^1 , we have induced orientation on $\mathcal{M}_{x,y}^\Delta$. Then we can similarly orient for $\mathcal{M}_{p,y}^\Delta, p \in \mathcal{C}(f), y \in \mathcal{P}^*(H_+), \mathcal{P}_{*,*}^\Delta$ and $\mathcal{H}_{*,*}^\Delta$ by tensoring TS^1 from the right. One last type is $\mathcal{T}_{*,*}$ in the construction of homotopy in § 4.2.2, similar to the BV operator case, for every solution (u, θ, r) to (4.12), there is a short exact sequence

$$0 \rightarrow T_{(u,\theta,r)} \mathcal{T}_{x,y} \rightarrow \ker D_u \oplus TS^1 \oplus T\mathbb{R} \rightarrow \text{coker } D_u \rightarrow 0,$$

which yields an isomorphism $\det T\mathcal{T}_{x,y} = o_{x,y} \otimes \det S^1 \otimes \det \mathbb{R}$ for $x \in \mathcal{P}^*(H_-)$ and $y \in \mathcal{P}^*(H_+)$. Hence $\mathcal{T}_{x,y}$ is oriented, and the orientation for $\mathcal{T}_{p,y}$ is similar. Then we have the following result with the same proof of Proposition A.2.

PROPOSITION A.4. *For moduli spaces up to dimension 1, the orientations above satisfy the following.*

- (i) $\partial \mathcal{M}_{*,*}^\Delta = -\sum \mathcal{M}_{*,*}^\Delta \times \mathcal{M}_{*,*} - \sum \mathcal{M}_{*,*} \times \mathcal{M}_{*,*}^\Delta$.
- (ii) $\partial \mathcal{T}_{*,*}^\Delta = -\sum \mathcal{M}_{*,*} \times \mathcal{T}_{*,*}^\Delta + \sum \mathcal{T}_{*,*} \times \mathcal{M}_{*,*} + \sum \mathcal{N}_{*,*} \times \mathcal{M}_{*,*}^\Delta - \sum \mathcal{M}_{*,*}^\Delta \times \mathcal{N}_{*,*}$.
- (iii) $\partial \mathcal{P}_{*,*}^\Delta = -\sum \mathcal{M}_{*,*} \times \mathcal{P}_{*,*}^\Delta - \sum \mathcal{P}_{*,*}^\Delta \times \mathcal{M}_{*,*} - \sum \mathcal{P}_{*,*} \times \mathcal{M}_{*,*}^\Delta$.
- (iv) $\partial \mathcal{H}_{*,*}^\Delta = \sum \mathcal{R}_{*,*} \times \mathcal{M}_{*,*}^\Delta - \mathcal{P}_{*,*}^\Delta + \sum \mathcal{H}_{*,*} \times \mathcal{M}_{*,*}^\Delta + \sum \mathcal{M}_{*,*} \times \mathcal{H}_{*,*}^\Delta - \sum \mathcal{H}_{*,*}^\Delta \times \mathcal{M}_{*,*}$.

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