AREA-MINIMIZING RULED GRAPHS AND THE BERNSTEIN PROBLEM IN THE HEISENBERG GROUP

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ABSTRACT. In this paper, we give a necessary and sufficient condition for a graphical strip in the Heisenberg group $\mathbb{H}$ to be area-minimizing in the slab $\{-1 < x < 1\}$. We show that our condition is necessary by introducing a family of deformations of graphical strips based on varying a vertical curve. We show that it is sufficient by showing that strips satisfying the condition have monotone epigraphs. We use this condition to show that any area-minimizing ruled entire intrinsic graph in the Heisenberg group is a vertical plane and to find a boundary curve that admits uncountably many fillings by area-minimizing surfaces.

1. INTRODUCTION

Bernstein's theorem states that an entire area-minimizing codimension–1 graph in $\mathbb{R}^n$ is a plane when $n \leqslant 8$. Analogues of Bernstein's theorem hold for many classes of surfaces in the Heisenberg group $\mathbb{H}$. For example, area-minimizing intrinsic graphs of entire C^2 functions are vertical planes [BASCV07], as are area-minimizing intrinsic graphs of entire C^1 functions [GR15] and locally Lipschitz functions [NSC19]. The problem of determining which classes of graphs satisfy an analogue of Bernstein's theorem is known as the Bernstein problem; for a survey of the Bernstein problem in $\mathbb{H}$, see [SCV20]. A key difficulty in proving analogues of Bernstein's theorem in $\mathbb{H}$ is finding appropriate classes of variations of surfaces. In this paper, we introduce new variations of ruled surfaces and use them to solve the Bernstein problem for ruled entire intrinsic graphs.

We briefly recall some terminology. Let $\mathbb{H}$ be the three-dimensional Heisenberg group; we identify $\mathbb{H}$ with $\mathbb{R}^3$ and give it the multiplication

$$(x, y, z) \cdot (x', y', z') = \left(x + x', y + y', z + z' + \frac{xy' - yx'}{2}\right). \tag{1}$$

Let $X = (1, 0, 0)$, $Y = (0, 1, 0)$, and $Z = (0, 0, 1)$; since a nilpotent Lie group can be identified with its Lie algebra via the exponential map, we also refer to the corresponding left-invariant vector fields as $X_{(x,y,z)} = (1, 0, -\frac{y}{2})$, $Y_{(x,y,z)} = (0, 1, \frac{x}{2})$, and $Z_{(x,y,z)} = (0, 0, 1)$. These fields generate 1–parameter subgroups and we denote the elements of these subgroups by $X^t = (t, 0, 0)$, $Y^t = (0, t, 0)$, and $Z^t = (0, 0, t)$ for $t \in \mathbb{R}$. Let $V_0 = \{(x, 0, z) \in \mathbb{H}\}$ be the xz–plane. For a subset $U \subset V_0$ and a function $f : U \to \mathbb{R}$, we define the intrinsic graph of f to be the set

$$\Gamma_f = \{uY^{f(u)} : u \in U\}.$$

If $U = V_0$, we call f an *entire* intrinsic graph. There is a corresponding intrinsic projection $\Pi: \mathbb{H} \to V_0$ given by $\Pi(p) = pY^{-y(p)}$, where $y(p)$ is the y–coordinate of p.

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For any $p \in \mathbb{H}$, we call the plane spanned by X_p and Y_p at p the *horizontal plane* centered at p. Let $\mathsf{A} = \{(x, y, 0) : x, y \in \mathbb{R}\}$ be the horizontal plane centered at $\mathbf{0}$. For any $p \in \mathbb{H}$ and any $v \in \mathsf{A}$, let $\langle v \rangle$ be the span of v; this is a one-parameter subgroup of $\mathbb{H}$ and we call the coset $p\langle v \rangle$ a *horizontal line*.

One of the main steps in the proofs of the analogues of Bernstein's theorem for C^2, C^1, and Lipschitz graphs is to calculate the first variation of the area and show that a graph that is area-stationary (i.e., a critical point of the area) is a ruled surface. A *ruled surface* in $\mathbb{H}$ is a surface $\Sigma \subset \mathbb{H}$ that can be written as a union of horizontal line segments (*rulings*) with endpoints in $\partial\Sigma$. One then calculates the second variation of the area of Σ and shows that if Σ is a ruled entire intrinsic graph which is area-stable (i.e., a second-order minimum of area), then Σ is a vertical plane.

This approach works well for surfaces that are smooth or regular with respect to the Riemannian structure on $\mathbb{H}$, but there are important classes of surfaces in $\mathbb{H}$ that are regular with respect to the sub-Riemannian structure but not the Riemannian structure. Once such class is defined in terms of the intrinsic gradient. Given a continuous function $f\colon U \to \mathbb{R}$, we define ∇_f to be the vector field $\nabla_f = X - fZ$ on U. This is the push-forward of X under the intrinsic projection from Γ_f to V_0, i.e., $\nabla_f = (\Pi|_{\Gamma_f})_*(X)$. When f is C^1, we can define the *intrinsic gradient* of f as $\nabla_f f$; when f is merely C^0, one can define $\nabla_f f$ distributionally (see Section 2). If $\nabla_f f$ can be represented by a continuous function, we say that f is of *class* $C^1_{\mathbb{H}}$ or $f \in C^1_{\mathbb{H}}(U)$. We call the intrinsic graphs of $C^1_{\mathbb{H}}$ functions $C^1_{\mathbb{H}}$ *graphs*. These are important examples of *regular surfaces*; that is, they are the level sets of functions ϕ such that the horizontal gradient $\nabla_{\mathbb{H}}\phi = (X\phi, Y\phi)$ is continuous and nonvanishing.

The $C^1_{\mathbb{H}}$ condition bounds how quickly f varies in the horizontal direction, but f can vary very rapidly in non-horizontal directions. For example, Kirchheim and Serra Cassano constructed $C^1_{\mathbb{H}}$ graphs that have Hausdorff dimension 2.5 with respect to the Euclidean metric [KSC04]. This makes it difficult to apply standard variational methods to such surfaces, because perturbing a $C^1_{\mathbb{H}}$ graph generally changes the set of horizontal directions. Indeed, a perturbation of a $C^1_{\mathbb{H}}$ graph need not be $C^1_{\mathbb{H}}$, and may even have infinite perimeter.

This even affects relatively simple graphs, like the broken plane depicted in Figure 1. For $u \geqslant 0$, the broken plane BP_u is a surface made up of two vertical half-planes with slope $\pm u$ (the slope of a vertical half-plane is the slope of its projection to the xy–plane) connected by two wedges in the xy–plane. For any u, there is a function $b\colon V_0 \to \mathbb{R}$ such that $\Gamma_b = \mathrm{BP}_u$ and b is intrinsic Lipschitz, that is, $\|\nabla_b b\|_\infty \leqslant u$. This bounds how quickly b varies in horizontal directions, but b can still vary rapidly in vertical directions. In particular, $\partial_z b$ goes to infinity near $\mathbf{0}$, so if $f \in C^\infty_c(V_0)$ is nonzero at $\mathbf{0}$, then

$$\nabla_{b+f}[b+f] = (X - (b+f)Z)b + \nabla_{b+f} f = \nabla_b b - f\partial_z b + \nabla_{b+f} f.$$

Since $\nabla_b b$ and $\nabla_{b+f} f$ are bounded, this goes to infinity near $\mathbf{0}$, which means that $b+f$ is not intrinsic Lipschitz. This makes it difficult to decrease the area of BP_u by a smooth deformation. Indeed, Nicolussi Golo and Serra Cassano showed [NSC19] that BP_u is area-stable in the sense that $\partial_\varepsilon[\mathrm{area}(\Gamma_{b+\varepsilon f})] = 0$ and $\partial^2_\varepsilon[\mathrm{area}(\Gamma_{b+\varepsilon f})] \geqslant 0$ for any $f \in C^\infty_c(V_0)$ and asked whether BP_u is area-minimizing.

One approach to questions like this is to consider variations of surfaces through intrinsic Lipschitz graphs. For example, [Gol18] and [You20] studied variations through contact diffeomorphisms. These variations preserve the horizontal distribution on $\mathbb{H}$ and send intrinsic Lipschitz graphs to intrinsic Lipschitz graphs, but there are many

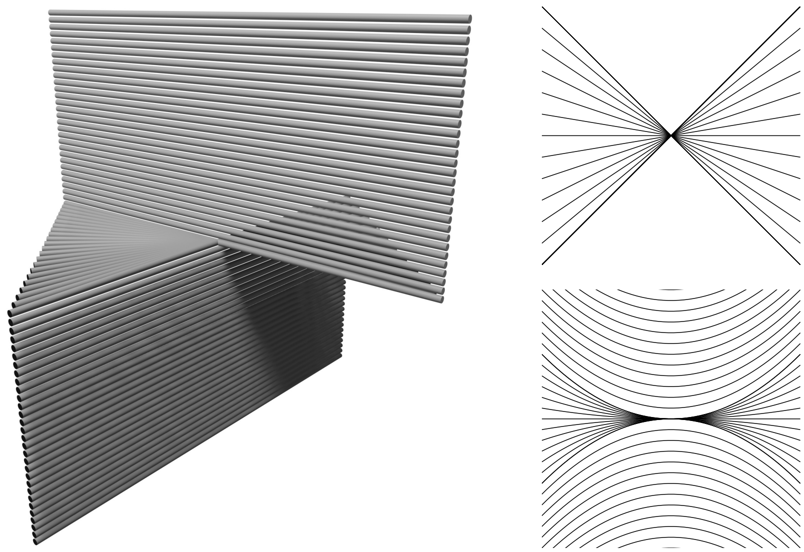

FIGURE 1. The left figure shows the broken plane $\mathrm{BP}_1 \subset \mathbb{H}$, which is made up of two half-planes connected by two wedges. (See Section 4 for the definition of BP_u.) While the half-planes are foliated by horizontal lines, the horizontal lines that make up the wedges all intersect at the origin. The right figures show the projection of the horizontal lines in BP_1 to the abelianization A and the projection along cosets of $\langle Y \rangle$ to the xz–plane V_0. Horizontal lines project to parabolas in V_0; the two half planes project to two families of parallel parabolas, while the wedges project to a family of parabolas through $\mathbf{0}$.

examples of surfaces that are area-stable with respect to contact variations but not smooth variations; Nicolussi Golo [Gol18] showed, for instance, that every ruled $C^1_{\mathbb{H}}$ graph is area-stable with respect to contact variations.

The issue is that there are too few contact variations to recognize area-minimizing surfaces. Because contact variations preserve the horizontal distribution, they send horizontal curves to horizontal curves and do not affect the horizontal connectivity of a surface. That is, if $f\colon \mathbb{H} \to \mathbb{H}$ is a contact diffeomorphism and p and q are connected by a horizontal curve in Σ, then $f(p)$ and $f(q)$ are connected by a horizontal curve in $f(\Sigma)$. Consequently, a surface that is area-minimizing with respect to contact variations can still have competitors with smaller area but different horizontal connectivity. Indeed, Figure 2 shows examples of competitors for BP_u with smaller area, but with a different pattern of horizontal curves. Finding a family of variations of non-smooth surfaces in $\mathbb{H}$ that is rich enough to recognize area-minimizing surfaces would be an important step in the study of perimeter-minimizers in $\mathbb{H}$.

In this paper, we introduce a family of variations of ruled surfaces and use it to characterize area-minimizing graphical strips. A *graphical strip* is an intrinsic graph that is ruled by horizontal lines through the z–axis (see Section 3). For example, the broken plane BP_u is a graphical strip. Previous work with graphical strips, for instance [NSC19], [GR15], and [DGNP09], mostly focused on the case of entire graphical strips, showing that, under some regularity conditions, an entire area-minimizing graphical strip is a vertical plane. Here, we will consider the case of graphical strips with boundary, especially graphical strips that project to a vertical strip in V_0.

We prove the following. Let $K = \{(x,0,z) \in V_0 : -1 \leqslant x \leqslant 1\}$ and let $U = \{(x,y,z) \in \mathbb{H} : -1 < x < 1\}$. Let Σ be a graphical strip over K. Then Σ is an intrinsic graph, so we can define its epigraph Σ^+ (Section 2.3). We say that Σ is area-minimizing in U if Σ^+ is perimeter-minimizing in U.

Theorem 1.1. *Let Σ be a graphical strip over K. Then Σ is area-minimizing in U if and only if there is a function $\sigma\colon \mathbb{R} \to \mathbb{R}$ such that $-2 \leqslant \frac{\sigma(t)-\sigma(s)}{t-s} < 2$ for all $s < t$ and*

$$\Sigma = \bigcup_{z\in\mathbb{R}} [(-1,-\sigma(z),z),(1,\sigma(z),z)], \tag{2}$$

where $[p_1,p_2]$ denotes the line segment from p_1 to p_2.

In particular, if Σ is area-minimizing, then there is exactly one ruling of Σ through any point on the z–axis. Therefore, BP_u is not area-minimizing for $u > 0$.

When there are $s < t$ such that $\frac{\sigma(t)-\sigma(s)}{t-s} \geqslant 2$, the surface Σ is not an intrinsic graph. When $\sigma'(t) = -2$ for all t, Σ is area-minimizing, but it is one of uncountably many area-minimizing surfaces with the same boundary.

Theorem 1.2. *Let $\gamma = \{(-1,2z,z) : z \in \mathbb{R}\} \cup \{(1,-2z,z) : z \in \mathbb{R}\}$. Any two points $(-1,2z_1,z_1)$ and $(1,-2z_2,z_2)$ are connected by a horizontal line. Let $\rho\colon \mathbb{R} \to \mathbb{R}$ be a surjective continuous increasing function and let*

$$\Sigma_\rho = \bigcup_{z\in\mathbb{R}} [(-1,2z,z),(1,-2\rho(z),\rho(z))].$$

Then Σ_ρ is an area-minimizing surface with $\partial\Sigma_\rho = \gamma$.

Pauls gave an example of a closed curve that admits two different fillings by ruled surfaces in [Pau04], showing that the Heisenberg minimal surface equation with Dirichlet boundary conditions can have multiple solutions, but the ruled surfaces he constructed are not area-minimizing [CHY07].

The proof of Theorem 1.1 relies on constructing deformations of graphical strips through piecewise-ruled surfaces and computing the second variation of the area under such deformations (Proposition 3.5). We construct these deformations as follows. Since Σ is an intrinsic graph over K which is symmetric around the z–axis, the boundary of Σ consists of two intrinsic graphs $X\gamma_\alpha$ and $X^{-1}\gamma_{-\alpha}$, where $\alpha\colon \mathbb{R} \to \mathbb{R}$ is continuous and $\gamma_\alpha := \{(0,y,z) : y = \alpha(z)\}$ is the graph of α in the yz–plane.

Cutting Σ along the z–axis produces two symmetric halves, one bounded by $\langle Z\rangle$ and $X\gamma_\alpha$ and one bounded by $\langle Z\rangle$ and $X^{-1}\gamma_{-\alpha}$. Given a function $\tau \in C_c^\infty(\mathbb{R})$, we construct a surface $S_{\alpha,\tau}$ consisting of two ruled surfaces, one bounded by γ_τ and $X\gamma_\alpha$ and one bounded by γ_τ and $X^{-1}\gamma_{-\alpha}$. In Section 3, we prove Theorem 1.1 by expanding area $S_{\alpha,\lambda\tau}$ to second order in λ.

In Sections 4–5, we apply Theorem 1.1 to show that Bernstein's theorem holds for the class of ruled intrinsic graphs.

Theorem 1.3. *An entire area-minimizing ruled intrinsic graph in $\mathbb{H}$ is a vertical plane.*

We prove Theorem 1.3 by showing that if Γ is a ruled intrinsic graph, then its scaling limit is a plane or a broken plane (Section 4). Scaling limits of perimeter-minimizing sets are perimeter-minimizing (Section 5), so if Γ satisfies Theorem 1.3, then its scaling limit is a plane. It then follows that Γ itself is a plane.

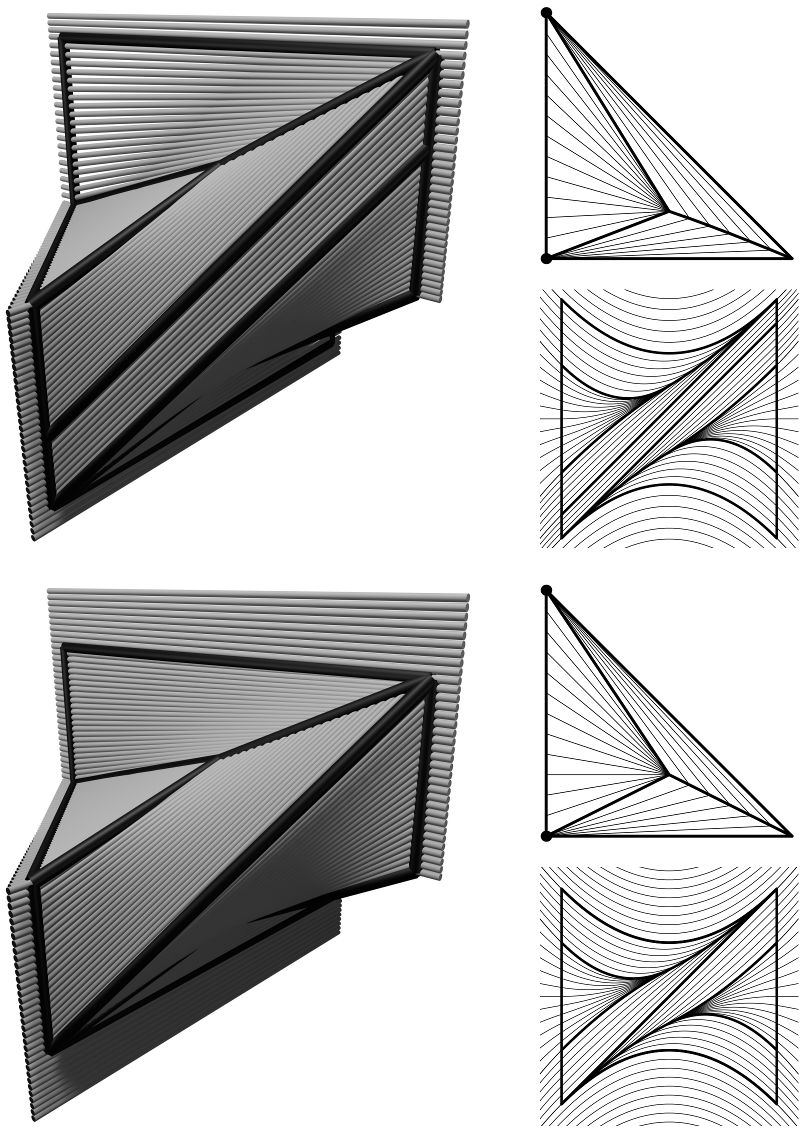

FIGURE 2. Conjectured area-minimizing (top) and energy-minimizing (bottom) competitors for BP_1. (See Section 7 for full definitions.) The figures on the right are the projections to A and V_0. For clarity, the projections to A only show the top half of each surface. Black lines on the surfaces correspond to thick lines in the projections; the dots mark the images of vertical black lines.

Finally, in Section 6, we prove Theorem 1.2, and in Section 7 we construct conjectural area-minimizing and energy-minimizing competitors for BP_u, seen in Figure 2. These competitors are each made up of two Z–graphs (graphs of an equation $z = f(x, y)$); one can show numerically that these surfaces have smaller area than BP_u, but we do not know whether they minimize the area or the energy.

The question of finding the broadest class of surfaces in $\mathbb{H}$ that satisfies Bernstein's theorem remains open. It is known that intrinsic Lipschitz graphs do not satisfy Bernstein's theorem; there are many area-minimizing entire intrinsic Lipschitz graphs that are not vertical planes. These can be highly singular; the first example of such a graph had a singularity along the x–axis [Pau06], and the examples in [NGR20] can be chosen to have a characteristic set (the set of points where the tangent plane to the surface is horizontal) whose closure has positive measure. In these examples, however, the surface fails to be $C^1_{\mathbb{H}}$ near the singularities, so the Bernstein problem for $C^1_{\mathbb{H}}$ graphs is an open question.

Remark 1.4. The 3D models used in the figures in this paper can be found in .obj format in the supplementary material for this paper. These files can be opened in Preview on Macs, Paint 3D on Windows, or any 3D modeling program. They can also be found at `https://cims.nyu.edu/~ryoung/ruledBernstein/`.

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2. Preliminaries and notation

2.1. **The Heisenberg group.** The Heisenberg group $\mathbb{H}$ is the 3–dimensional simply connected Lie group with Lie algebra

$$\mathfrak{h} := \langle X, Y, Z : [X,Y] = Z, [X,Z] = [Y,Z] = 0\rangle.$$

We identify $\mathbb{H}$ with $\mathfrak{h}$ via the Baker–Campbell–Hausdorff formula, i.e., $\mathbb{H} = \langle X, Y, Z\rangle \cong \mathbb{R}^3$ and

$$(x,y,z)\cdot(x',y',z') = \left(x+x', y+y', z+z' + \frac{xy'-yx'}{2}\right).$$

We use X, Y, and Z to denote the coordinate vectors of $\mathbb{R}^3$ and the corresponding left-invariant fields $X_{(x,y,z)} = (1,0,-\frac{y}{2})$, $Y_{(x,y,z)} = (0,1,\frac{x}{2})$, and $Z_{(x,y,z)} = (0,0,1)$. Let $x,y,z\colon \mathbb{H}\to\mathbb{R}$ be the coordinate functions of $\mathbb{H}$. Every vector $v\in\mathbb{H}$ generates a one-parameter subgroup of $\mathbb{H}$; we write $\langle v\rangle = \mathbb{R}v$ for this subgroup and define $v^t = tv$ for all $t\in\mathbb{R}$.

We equip $\mathbb{H}$ with the Korányi norm $\|(x,y,z)\|_{\mathsf{Kor}} = \sqrt[4]{(x^2+y^2)^2+z^2}$ and the corresponding left-invariant distance $d(p,q) = \|p^{-1}q\|_{\mathsf{Kor}}$. For $a,b\in\mathbb{R}\smallsetminus\{0\}$, let $s_{a,b}$ be the automorphism $s_{a,b}(x,y,z) = (ax,by,abz)$. We call automorphisms of the form $s_{t,t}$, $t>0$ *scaling automorphisms*; these satisfy $\|s_{t,t}(p)\|_{\mathsf{Kor}} = t\|p\|_{\mathsf{Kor}}$ and $d(s_{t,t}(p), s_{t,t}(q)) = td(p,q)$. For $p\in\mathbb{H}$ and $r>0$, let $B(p,r) := \{q\in\mathbb{H} : d(p,q) < r\}$ be the open ball of radius r around p. The tangent planes spanned by X and Y are called the *horizontal distribution*, and vectors in the horizontal distribution are called *horizontal vectors*. A curve $\gamma\colon I\to\mathbb{H}$ whose coordinates are Lipschitz and such that $\gamma'(t)$ is a horizontal vector for almost every t is called a *horizontal curve*.

2.2. **The perimeter measure.** The *sub-Riemannian perimeter* of a measurable subset $E\subset\mathbb{H}$ on an open set $\Omega\subset\mathbb{H}$ is given by

$$\mathrm{Per}_E(\Omega) = \sup\left\{\int_E \mathrm{div}_{\mathbb{H}}(aX+bY)\,\mathrm{d}\eta : a,b\in C^\infty_c(\Omega), a^2+b^2\leqslant 1\right\},$$

where the *horizontal divergence* $\mathrm{div}_{\mathbb{H}}$ is defined by $\mathrm{div}_{\mathbb{H}}(aX+bY) = Xa+Yb$ and η is Lebesgue measure on $\mathbb{H}$. This perimeter was introduced in [FSSC01] as a Heisenberg analogue of perimeter in Euclidean space. If $\mathrm{Per}_E(B(\mathbf{0},r))<\infty$ for every $r>0$, we say that E has *locally finite perimeter*. The perimeter is lower semicontinuous, i.e., if $E_1, E_2, \dots$ have locally finite perimeter and $\mathbf{1}_{E_i}$ converges locally in L_1 to $\mathbf{1}_E$, then E has locally finite perimeter and $\mathrm{Per}_E(\Omega) \leqslant \liminf_{i\to\infty} \mathrm{Per}_{E_i}(\Omega)$ [FSSC01, 2.12].

Let $E \triangle F := (E \smallsetminus F) \cup (F \smallsetminus E)$ be the symmetric difference operator. For $E \subset \mathbb{H}$ a measurable set and $\Omega \subset \mathbb{H}$ an open set, we say that E is a *perimeter minimizer* in Ω if for every $r>0$ and every measurable $F\subset \mathbb{H}$ such that satisfies $E \triangle F \Subset B(\mathbf{0},r)\cap\Omega$, we have $\mathrm{Per}_E(B(\mathbf{0},r)\cap\Omega) \leqslant \mathrm{Per}_F(B(\mathbf{0},r)\cap\Omega)$.

2.3. **Intrinsic graphs and intrinsic gradient.** For any $U\subset V_0$ and any function $f\colon U\to\mathbb{R}$, we define $\Gamma_f = \{uY^{f(u)} : u\in U\}$ to be the *intrinsic graph* of f and $\Gamma_f^+ = \{uY^t : t > f(u)\}$ to be the *epigraph* of f. The intrinsic graph Γ_f is parametrized by the function $\Psi_f\colon U\to\mathbb{H}$, $\Psi_f(u) = uY^{f(u)}$. In coordinates,

$$\Gamma_f = \left\{\left(x, f(x,0,z), z+\frac{1}{2}xf(x,0,z)\right) : (x,0,z)\in U\right\}.$$

We define $\Pi(p) = pY^{-y(p)}$,

$$\Pi(x,y,z) = \left(x,0,z-\frac{xy}{2}\right),$$

to be the intrinsic projection from $\mathbb{H}$ to V_0, so that $\Pi\circ\Psi_f = \mathrm{id}_U$.

Let ∇_f be the vector field $\nabla_f = X - fZ = (\Pi|_{\Gamma_f})_*(X)$ on U. We call this the *intrinsic gradient*; we are particularly interested in $\nabla_f f$, which determines the tangent plane to Γ_f. The derivative $\nabla_f f$ exists when f is C^1, but it can be defined distributionally when f is merely C^0. If $f\colon V_0\to\mathbb{R}$ is continuous, we say that $\nabla_f f$ exists in the sense of distributions if there is a function $\theta\in L_{\infty,\mathrm{loc}}$ such that for every $\psi\in C^1_c$,

$$\int_{V_0} \theta\psi\,\mathrm{d}\mu = \int_{V_0} -f\partial_x\psi + \frac{f^2}{2}\partial_z\psi\,\mathrm{d}\mu,$$

where μ is Lebesgue measure on V_0. If so, we write $\nabla_f f = \theta$. When f is C^1, this coincides with the previous definition. For U an open subset of V_0, we define $C^1_{\mathbb{H}}(U)$ to be the set of continuous functions $f\colon U\to\mathbb{R}$ such that $\nabla_f f$ is represented by a continuous function on U. This implies that Γ_f is a regular surface in the sense of [ASCV06].

One can also define a version of the Lipschitz condition adapted to the Heisenberg group. For $D\subset V_0$, we say that a function $f\colon D\to\mathbb{R}$ is *intrinsic Lipschitz* or that Γ_f is an *intrinsic Lipschitz graph* if there exists a $0<\lambda<1$ such that $|y(p)-y(q)|\leqslant\lambda d(p,q)$ for all $p,q\in\Gamma_f$. By Theorem 4.29 of [FSSC11], if U is an open set and $f\colon U\to\mathbb{R}$ is an intrinsic Lipschitz function, then f satisfies an intrinsic version of Rademacher's theorem in the sense that $\nabla_f f$ exists in the sense of distributions and $\|\nabla_f f\|_\infty$ is bounded by a function of λ. Conversely, if $f\in C^0(U)$ and $\nabla_f f\in L_\infty(U)$, then f is locally intrinsic Lipschitz [BCSC15].

Let $D\subset V_0$ and let $f\colon D\to\mathbb{R}$ be an intrinsic Lipschitz graph. The area formula [CMPSC14, Thm. 1.6] states that for any bounded open set $U\subset V_0$,

$$\mathrm{Per}_{\Gamma_f^+}(\Pi^{-1}(U)) = \int_U \sqrt{1+(\nabla_f f)^2}\,\mathrm{d}\mu.$$

More generally, we define

$$\operatorname{area}\Psi_f(W) := \int_W \sqrt{1+(\nabla_f f)^2}\,\mathrm{d}\mu. \tag{3}$$

for any measurable $W \subset D$. For a continuous function $f\colon V_0 \to \mathbb{R}$ and an open subset $\Omega \subset \mathbb{H}$, we say that Γ_f is *area-minimizing* in Ω if Γ_f^+ is perimeter-minimizing in $\Pi^{-1}(\Omega)$.

2.4. **Horizontal lines and characteristic curves.** Let $\pi\colon \mathbb{H} \to \mathsf{A}$, $\pi(x,y,z) = (x,y,0)$. A *horizontal line* is a coset of the form $p\langle V\rangle$, where $V \in \mathsf{A}$. The *slope* of a horizontal line L is the slope of $\pi(L)$, i.e., $\operatorname{slope}(p\langle aX+bY\rangle) = \frac{b}{a}$.

Let $p = (x_p, y_p, z_p) \in \mathbb{H}$ and $m \in \mathbb{R}$, and let $L = p\langle X + mY\rangle$ be the line of slope m through p. Let $\lambda(t) = p(X+mY)^{t-x_p}$ parametrize L. Then

$$\begin{aligned}\Pi(\lambda(x)) &= \Pi\left(p(X+mY)^{x-x_p}\right) = \Pi\left(x, y_p + m(x-x_p), z_p + \frac{x_p m(x-x_p) - y_p(x-x_p)}{2}\right)\\ &= \left(x, 0, z_p - \frac{x_p y_p}{2} - y_p(x-x_p) - \frac{m}{2}(x-x_p)^2\right)\\ &= \Pi(p) + \left(x - x_p, 0, -y_p(x-x_p) - \frac{m}{2}(x-x_p)^2\right). \end{aligned} \tag{4}$$

That is, $\Pi(L)$ is a parabola, and any parabola in V_0 is the projection of a unique horizontal line.

Given an open subset $U \subset V_0$ and a continuous function $f\colon U \to \mathbb{R}$, the *characteristic curves* of Γ_f are the integral curves of the vector field $\nabla_f = X - fZ$. By the Peano Existence Theorem, for every $p \in U$, there is a characteristic curve through p, but when f is not Lipschitz, this curve may not be unique. Note, however, that two characteristic curves that meet at a point p must have the same tangent vector at p.

The characteristic curves of Γ_f are projections of horizontal curves contained in Γ_f; if $f\colon V_0 \to \mathbb{R}$ is an intrinsic Lipschitz function and $\lambda\colon I \to V_0$ is a characteristic curve for Γ_f, then $\gamma = \Psi_f \circ \lambda$ is a horizontal curve in Γ_f; conversely, if $\gamma\colon I \to \Gamma_f$ is a horizontal curve such that $x(\gamma(t)) = t$ for all t, then $\Pi \circ \gamma$ is a characteristic curve.

Recall that if $\Sigma \subset \mathbb{H}$ is a surface, a *ruling* of Σ is a horizontal line segment (possibly infinite) that lies in Σ, with endpoints in $\partial\Sigma$. We say that Σ is a *ruled surface* if every point of Σ is contained in at least one ruling. Ruled surfaces need not be regular in the sense of [ASCV06]. For example, the broken plane in Figure 1 is a ruled surface but its tangent cone at the origin is not a plane.

When an intrinsic graph Γ_f is a ruled surface, we can say more about its characteristic curves. By (4), the projection of any ruling of Γ_f is a parabola. If R_1 and R_2 are distinct rulings of Γ_f such that $\Pi(R_1)$ and $\Pi(R_2)$ intersect at a point $p \in V_0$, then $\Pi(R_1)$ and $\Pi(R_2)$ must be tangent at p. It follows that $\Pi(R_1)$ and $\Pi(R_2)$ cannot cross. That is, the following lemma holds.

Lemma 2.1. *Let R_1 and R_2 be distinct rulings of Γ_f. Let $g_{R_i}\colon I_i \to \mathbb{R}$ be such that $\Pi(R_i) = \{(x,0,g_{R_i}(x)) : x \in I_i\}$. Then $g_{R_1}(x) \leqslant g_{R_2}(x)$ for all $x \in I_1 \cap I_2$ or $g_{R_1}(x) \geqslant g_{R_2}(x)$ for all $x \in I_1 \cap I_2$.*

3. Deformations of graphical strips

Let $D \subset V_0$. A *graphical strip* over D is an intrinsic graph Γ_f of a continuous function $f\colon D \to \mathbb{R}$ such that Γ_f is ruled and every ruling intersects the z–axis. (This definition differs slightly from the definitions found in [NSC19], [GR15], and [DGNP09], which assume that f is Lipschitz or C^2 with respect to the Euclidean structure on $\mathbb{R}^2$.)

A notable example of a graphical strip is the broken plane illustrated in Figure 1.

Definition 3.1. Let $u \geqslant 0$. The *broken plane* BP_u consists of two vertical half-planes of slope $\pm u$ connected by two wedges in the xy–plane. (See Figure 1.) That is,

$$\mathrm{BP}_u := \{(x,-ux,z) : x \in \mathbb{R}, z \geqslant 0\} \cup \{(x,ux,z) : x \in \mathbb{R}, z \leqslant 0\} \cup \{(x,y,0) : |y| \leqslant u|x|\}. \tag{5}$$

As $u \to \infty$, this converges to the set

$$\mathrm{BP}_\infty := \langle Y, Z\rangle \cup \langle X, Y\rangle$$

We require $u \geqslant 0$ so that BP_u is an intrinsic graph. The upper half-plane $P_u^+ = \{(x,-ux,z) : x \in \mathbb{R}, z > 0\}$ has projection $\Pi(P_u^+) = \{(x,0,z) : z > \frac{u}{2}x^2\}$, while $P_u^- = \{(x,ux,z) : x \in \mathbb{R}, z < 0\}$ has projection $\Pi(P_u^-) = \{(x,0,z) : z < \frac{-u}{2}x^2\}$. When $0 \leqslant u < \infty$, these regions are disjoint and $\mathrm{BP}_u = \Gamma_{b_u}$, where

$$b_u(x,0,z) = \begin{cases} ux & z < -\frac{u}{2}x^2 \\ \frac{-2z}{x} & |z| \leqslant \frac{u}{2}x^2 \\ -ux & z > \frac{u}{2}x^2. \end{cases}$$

When $u < 0$, $\Pi(P_u^+)$ and $\Pi(P_u^-)$ overlap, so (5) defines a ruled surface, but not an intrinsic graph.

Since BP_u is an intrinsic graph, we can define its epigraph $\mathrm{BP}_u^+ = \Gamma_{b_u}^+$. This is the union of two quadrants in $\mathbb{H}$, one bounded by the xy–plane and the upper half-plane with slope $-u$ and one bounded by the xy–plane and the lower half-plane with slope u. As $u \to \infty$, these epigraphs converge to the set

$$\mathrm{BP}_\infty^+ := \{(x,y,z) \in \mathbb{H} : xz > 0\}.$$

In this paper, we mainly consider graphical strips Γ over $K = \{(x,0,z) \in V_0 : -1 \leqslant x \leqslant 1\}$, in which case Γ is symmetric around the z–axis (i.e., $s_{-1,-1}(\Gamma) = \Gamma$) and Γ is determined by the intersection $\Gamma \cap \{x = 1\}$. That is,

$$\Gamma = \bigcup_{p \in \Gamma \cap \{x=1\}} [p, s_{-1,-1}(p)],$$

where $[p_1,p_2]$ denotes the line segment from p_1 to p_2.

Let Γ_f be a graphical strip over K and let $\alpha\colon \mathbb{R} \to \mathbb{R}$, $\alpha(w) = f(1,0,w)$. Then

$$\Gamma_f \cap \{x = 1\} = X\gamma_\alpha,$$

where $\gamma_\alpha := \{(0,\alpha(z),z) : z \in \mathbb{R}\}$ is the graph of α in the yz–plane, and $\Gamma_f = S_\alpha$, where

$$\begin{aligned} S_\alpha &:= \bigcup_{w \in \mathbb{R}} \left[X^{-1}\cdot(0,-\alpha(w),w), X\cdot(0,\alpha(w),w)\right] \\ &= \bigcup_{w \in \mathbb{R}} \left[\left(-1,-\alpha(w),w+\frac{\alpha(w)}{2}\right),\left(1,\alpha(w),w+\frac{\alpha(w)}{2}\right)\right]. \end{aligned}$$

For $w \in \mathbb{R}$, let $\eta(w) := w + \frac{\alpha(w)}{2}$.

For example, for any $0 \leqslant u < \infty$, $\mathrm{BP}_u \cap \Pi^{-1}(K)$ is a graphical strip over K and $\mathrm{BP}_u \cap \Pi^{-1}(K) = S_\alpha$, where

$$\alpha(z) = b_u(1,0,z) = \begin{cases} u & z < -\frac{u}{2} \\ -2z & |z| \leqslant \frac{u}{2} \\ -u & z > \frac{u}{2}. \end{cases}$$

In general, for any continuous $\alpha\colon \mathbb{R} \to \mathbb{R}$, we can define S_α as above. This is a ruled surface bounded by $X\gamma_\alpha$ and $X^{-1}\gamma_{-\alpha}$, but it is not always an intrinsic graph. In Section 3.1, we will show the following lemma.

Lemma 3.2. *Let S_α be as above. Then S_α is a graphical strip over K if and only if $\frac{\alpha(w_2)-\alpha(w_1)}{w_2-w_1} \geqslant -2$ for all $w_1 < w_2$ and $\eta(\mathbb{R}) = \mathbb{R}$.*

Note in particular that if S_α is a graphical strip, then η is a non-decreasing function; η is increasing if and only if the rulings of S_α do not intersect.

The main goal of this section is to prove the following characterization of area-minimizing graphical strips over K. Let U be the interior of $\Pi^{-1}(K)$, i.e., $U = \{(x,y,z) \in \mathbb{H} : -1 < x < 1\}$.

Proposition 3.3. *Let S_α be a graphical strip over K. Then S_α is area-minimizing on U if and only if $\frac{\alpha(w_2)-\alpha(w_1)}{w_2-w_1} \geqslant -1$ for all $w_1 < w_2$.*

By the following lemma, this characterization is equivalent to the characterization in Theorem 1.1.

Lemma 3.4. *Let S_α be a graphical strip over K. Then $\frac{\alpha(w_2)-\alpha(w_1)}{w_2-w_1} \geqslant -1$ for all $w_1 < w_2$ if and only if there is a function $\sigma \colon \mathbb{R} \to \mathbb{R}$ such that $-2 \leqslant \frac{\sigma(z_2)-\sigma(z_1)}{z_2-z_1} < 2$ for all $z_1 < z_2$ and*

$$S_\alpha = \bigcup_{z\in\mathbb{R}} \big[(-1,-\sigma(z),z),(1,\sigma(z),z)\big]. \tag{6}$$

We will prove Lemma 3.4 in Section 3.1.

Proof of Theorem 1.1. The first part of the theorem follows from Proposition 3.3 and Lemma 3.4. For the second part, suppose that S_α is area-minimizing on U, so that it can be written as in (6). For any $z_0 \in \mathbb{R}$, any horizontal line through Z^{z_0} is contained in the plane $P = \{z = z_0\} \subset \mathbb{H}$. The intersection $P \cap S_\alpha$ is the segment $[(-1,-\sigma(z),z),(1,\sigma(z),z)]$, and this segment is the unique ruling through Z^{z_0}. If $u > 0$, then BP_u has infinitely many different rulings through the origin, so BP_u is not area-minimizing. □

There are two main ideas to the proof of Proposition 3.3. First, in Section 3.2, we will show that a perimeter-minimizing graphical strip satisfies $\frac{\alpha(w_2)-\alpha(w_1)}{w_2-w_1} \geqslant -1$ for all $w_1 < w_2$ by defining a family $S_{\alpha,\tau}$ of deformations of S_α. These deformations are parametrized by functions $\tau \in C_c^\infty(\mathbb{R})$. Each surface $S_{\alpha,\tau}$ is a union of a ruled graph over the strip $K^+ = [0,1] \times \{0\} \times \mathbb{R}$ and a ruled graph over the strip $K^- = [-1,0] \times \{0\} \times \mathbb{R}$, with common boundary γ_τ. We will show that the area of these deformations satisfies the following formula.

Proposition 3.5. *Let S_α be a graphical strip over K and suppose that $\alpha \colon \mathbb{R} \to \mathbb{R}$ is Lipschitz.*

Let $w_1 < w_2$ be such that $\eta^{-1}(\eta(w_i)) = \{w_i\}$ and let $D = \{(x,y,z) \in \mathbb{H} : \eta(w_1) \leqslant z \leqslant \eta(w_2)\}$. Let $\tau \in C_c^\infty([\eta(w_1),\eta(w_2)])$. Then there is a $C = C(\tau) > 0$ such that for all $\lambda \in (-C^{-1}, C^{-1})$,

$$\left|\mathrm{area}(D \cap S_{\alpha,\lambda\tau}) - \mathrm{area}(D \cap S_\alpha) - \lambda^2\, \mathrm{II}(\tau)\right| \leqslant C(w_2 - w_1)|\lambda|^3, \tag{7}$$

where

$$\mathrm{II}(\tau) = \int_{-\infty}^{\infty} \frac{\tau(\eta(w))^2}{(1+\alpha(w)^2)^{\frac{3}{2}}}(1+\alpha'(w))\,\mathrm{d}w$$

In Lemma 3.8, we conclude that if S_α is an area-minimizing graphical strip over K, then $\frac{\alpha(w_2)-\alpha(w_1)}{w_2-w_1} \geqslant -1$ for all $w_1 < w_2$.

Second, in Section 3.3, we show that monotone sets are perimeter-minimizing and conclude that S_α^+ is perimeter-minimizing when $\frac{\alpha(w_2)-\alpha(w_1)}{w_2-w_1} \geqslant -1$ for all $w_1 < w_2$. This is a consequence of the kinematic formula, which expresses the perimeter of a subset $E \subset \mathbb{H}$ in terms of an integral over the set of horizontal lines that intersect ∂E. When E is monotone, then almost every horizontal line intersects ∂E in at most one point. In this case, deformations of E increase the number of intersections and thus increase perimeter. Proposition 3.3 follows immediately from the results of Section 3.2 and Section 3.3.

3.1. **Graphical strips.**

In this section, we prove Lemma 3.2 and Lemma 3.4.

Proof of Lemma 3.2. By construction, every point in S_α is contained in a ruling and every ruling intersects the z–axis. In order for S_α to be a graphical strip, we need it to be an intrinsic graph over K.

Let $\Theta\colon [-1,1]\times\mathbb{R}\to\mathbb{H}$, $\Theta(x,w)=(x,x\alpha(w),\eta(w))$ parametrize S_α and let $p\colon [-1,1]\times\mathbb{R}\to K$ be the map

$$p(x,w)=\Pi(\Theta(x,w))=\Pi\left(x,x\alpha(w),\eta(w)\right)$$
$$=\left(x,0,\eta(w)-\frac{\alpha(w)}{2}x^2\right)=\left(x,0,\eta(w)(1-x^2)+wx^2\right).$$

Suppose that η is surjective and $\frac{\alpha(s)-\alpha(t)}{s-t}\geqslant -2$ for all $s<t$. We claim that $p|_{([-1,1]\smallsetminus\{0\})\times\mathbb{R}}$ is a homeomorphism. Let $h_x(w):=z(p(x,w))=\eta(w)(1-x^2)+wx^2$. Since η is nondecreasing, h_x is an increasing surjective function when $x\neq 0$. It follows that $p|_{([-1,1]\smallsetminus\{0\})\times\mathbb{R}}$ is a bijection. For any a and b such that $-1\leqslant a<b<0$ or $0<a<b\leqslant 1$ and any $c<d$,

$$p((a,b)\times(c,d))=\{(x,0,z):x\in(a,b),z\in(h_x(c),h_x(d))\}$$

is an open set, so $p|_{([-1,1]\smallsetminus\{0\})\times\mathbb{R}}$ is an open map. Therefore, p restricts to a homeomorphism from $([-1,1]\smallsetminus\{0\})\times\mathbb{R}$ to $K\smallsetminus\langle Z\rangle$.

Let $\mathbf{q}=(q_1,q_2):=(p|_{([-1,1]\smallsetminus\{0\})\times\mathbb{R}})^{-1}\colon K\smallsetminus\langle Z\rangle\to([-1,1]\smallsetminus\{0\})\times\mathbb{R}$. For $(x,z)\in K$, let

$$f_\alpha(x,z):=\begin{cases}x\alpha(q_2(x,z)) & x\neq 0\\ 0 & x=0.\end{cases}.$$

Then $f_\alpha(p(x,w))=x\alpha(w)$, so

$$\Psi_{f_\alpha}(p(x,w))=\Pi(\Theta(x,w))Y^{x\alpha(w)}=\Theta(x,w),$$

and $\Gamma_{f_\alpha}=S_\alpha$.

We claim that f_α is continuous. Since $\mathbf{q}$ is continuous, f_α is continuous on $K\smallsetminus\langle Z\rangle$. Let $z\in\mathbb{R}$ and let $w_1<w_2$ be such that $\eta(w_1)<z<\eta(w_2)$ and let $C=\max_{u\in[w_1,w_2]}|\alpha(u)|$. Let

$$D=p([-1,1]\times[w_1,w_2])=\{(x,0,z):x\in[-1,1],z\in[h_x(c),h_x(d)]\};$$

this contains a neighborhood of $(0,0,z)$, and if $v=(x,0,z)\in D$, then $|f_\alpha(v)|\leqslant Cx$. It follows that f_α is continuous at $(0,0,z)$ for any $z\in\mathbb{R}$.

Conversely, if there are $s<t$ such that $\frac{\alpha(s)-\alpha(t)}{s-t}<-2$, then $\eta(t)<\eta(s)$, and there is some $x>0$ such that $h_x(s)=h_x(t)$. Then $p(x,s)=p(x,t)$, but $y(\Theta(x,s))=x\alpha(s)\neq y(\Theta(x,t))$. Therefore, S_α is not an intrinsic graph. Likewise, if η is not surjective, then $\Pi(S_\alpha)$ is a strict subset of K. □

Proof of Lemma 3.4. Suppose that $\frac{\alpha(w_2)-\alpha(w_1)}{w_2-w_1}\geqslant -1$ for all $s<t$. Let $\eta(w):=w+\frac{\alpha(w)}{2}$; this is invertible and surjective by our hypothesis on α, and we define $\sigma(z)=\alpha(\eta^{-1}(z))$. When $z=\eta(w)$,

$$[(-1,-\alpha(w),\eta(w)),(1,\alpha(w),\eta(w))]=[(-1,-\sigma(z),z),(1,\sigma(z),z)],$$

so

$$S_\alpha=\bigcup_{z\in\mathbb{R}}[(-1,-\sigma(z),z),(1,\sigma(z),z)].$$

Let $z_1 < z_2$. We claim that $-2 \leqslant \frac{\sigma(z_2)-\sigma(z_1)}{z_2-z_1} < 2$. Let $w_i = \eta^{-1}(z_i)$ and $a_i = \alpha(w_i) = \sigma(z_i)$. Then $z_i = \eta(w_i) = w_i + \frac{a_i}{2}$. Since η is monotone, we have $w_1 < w_2$, so

$$w_2 - w_1 = z_2 - z_1 - \frac{a_2 - a_1}{2} > 0,$$

i.e., $a_2 - a_1 < 2(z_2 - z_1)$. Furthermore,

$$\frac{a_2 - a_1}{w_2 - w_1} = \frac{\alpha(w_2) - \alpha(w_1)}{w_2 - w_1} \geqslant -1,$$

so

$$a_2 - a_1 \geqslant w_1 - w_2 = z_1 - z_2 + \frac{1}{2}(a_2 - a_1),$$

so $a_2 - a_1 \geqslant -2(z_2 - z_1)$. Therefore, $-2 \leqslant \frac{\sigma(z_2)-\sigma(z_1)}{z_2-z_1} < 2$.

Conversely, suppose that

$$S_\alpha = \bigcup_{z\in\mathbb{R}} [(-1, -\sigma(z), z), (1, \sigma(z), z)] \tag{8}$$

for some σ such that

$$-2 \leqslant \frac{\sigma(z_2) - \sigma(z_1)}{z_2 - z_1} < 2 \tag{9}$$

for all $z_1 < z_2$. We claim that $\frac{\alpha(w_2)-\alpha(w_1)}{w_2-w_1} \geqslant -1$ for all $w_1 < w_2$.

We have

$$\{(x, y, z) \in S_\alpha : x = 1\} = \{(1, \sigma(z), z) : z \in \mathbb{R}\} = \{(1, \alpha(w), \eta(w)) : w \in \mathbb{R}\},$$

so $\sigma(\eta(w)) = \alpha(w)$ for all $w \in \mathbb{R}$. Let $w_1 < w_2$. Since η is nondecreasing, we have $\eta(w_1) \leqslant \eta(w_2)$. In fact $\eta(w_1) < \eta(w_2)$; otherwise, S_α would have two rulings with the same z–coordinate.

By (9) with $z_1 = \eta(w_1)$, $z_2 = \eta(w_2)$,

$$\begin{aligned}
\sigma\left(\eta(w_2)\right) - \sigma\left(\eta(w_1)\right) &\geqslant -2\left(\eta(w_2) - \eta(w_1)\right)\\
\alpha(w_2) - \alpha(w_1) &\geqslant -2\left(w_2 + \frac{\alpha(w_2)}{2} - w_1 - \frac{\alpha(w_1)}{2}\right)\\
2(\alpha(w_2) - \alpha(w_1)) &\geqslant -2(w_2 - w_1),
\end{aligned}$$

so $\frac{\alpha(w_2)-\alpha(w_1)}{w_2-w_1} \geqslant -1$, as desired. □

3.2. **Deforming ruled surfaces.** Recall that for any function $\tau\colon \mathbb{R} \to \mathbb{R}$, we define $\gamma_\tau := \{(0, y, z) : y = \tau(z)\}$ to be the graph of τ in the yz–plane, so that S_α has boundary

$$\partial S_\alpha = X^{-1}\gamma_\alpha \cup X\gamma_{-\alpha}.$$

In this section, for any sufficiently small τ, we construct a ruled graph $\Sigma_{\tau,\alpha}$ over $K^+ = [0, 1] \times \{0\} \times \mathbb{R}$ with boundary $\partial\Sigma_{\tau,\alpha} = \gamma_\tau \cup X\gamma_\alpha$. By gluing two such graphs together, we obtain a surface

$$S_{\alpha,\tau} := \Sigma_{\tau,\alpha} \cup s_{-1,-1}(\Sigma_{-\tau,\alpha})$$

which is a deformation of S_α.

First, we construct $\Sigma_{\tau,\alpha}$.

Lemma 3.6. *Let $\alpha, \tau\colon \mathbb{R} \to \mathbb{R}$ be continuous functions. Let $\eta(w) := w + \frac{\alpha(w)}{2}$, and suppose that $\frac{\alpha(s)-\alpha(t)}{s-t} \geqslant -2$ for all $s < t$ and $\eta(\mathbb{R}) = \mathbb{R}$. Suppose that $\mathrm{Lip}(\tau) < 2$. Then there is a unique ruled graph $\Sigma_{\tau,\alpha}$ with boundary $\partial\Sigma_{\tau,\alpha} = \gamma_\tau \cup X\gamma_\alpha$.*

Proof. We first construct a tilted coordinate system on the yz–plane. Let $P \subset \mathbb{H}$ be the yz–plane and let $Q = Y + \frac{Z}{2}$. Then $P = \langle Q, Z\rangle$. Since $\mathrm{Lip}(\tau) < 2$, we can write γ_τ as a P–graph. That is, any point $Z^z Y^{\tau(z)} \in \gamma_\tau$ can be written

$$Z^z Y^{\tau(z)} = Z^{z-\frac{\tau(z)}{2}} P^{\tau(z)} = Z^{\zeta(z)} P^{\tau(z)} \tag{10}$$

where $\zeta(z) := z - \frac{\tau(z)}{2}$. Then ζ is invertible, so, letting $w = \zeta(z)$,

$$\gamma_\tau = \{Z^w P^{\tau(\zeta^{-1}(w))} : w \in \mathbb{R}\}.$$

Let $\beta(w) := \tau(\zeta^{-1}(w))$.

Let $w \in \mathbb{R}$ and let $p = XZ^w Y^{\alpha(w)} \in X\gamma_\alpha$. For any $m \in \mathbb{R}$,

$$\begin{aligned} p \cdot (X + mY)^{-1} &= (1, \alpha(w), \eta(w))(-1, -m, 0) \\ &= \left(0, \alpha(w) - m, \eta(w) + \frac{\alpha(w) - m}{2}\right) = Z^{\eta(w)} P^{\alpha(w)-m}, \end{aligned} \tag{11}$$

which lies in γ_τ if and only if $m = \alpha(w) - \beta(\eta(w))$. In this case,

$$Z^{\eta(w)} P^{\alpha(w)-m} = Z^{\eta(w)} P^{\beta(\eta(w))} \overset{(10)}{=} Z^{\zeta^{-1}(\eta(w))} Y^{\tau(\zeta^{-1}(\eta(w)))} = Z^{h(w)} Y^{\tau(h(w))},$$

where $h(w) := \zeta^{-1}(\eta(w))$. Let

$$R_w := \left[XZ^w Y^{\alpha(w)}, Z^{\eta(w)} P^{\beta(\eta(w))}\right] = \left[XZ^w Y^{\alpha(w)}, Z^{h(w)} Y^{\tau(h(w))}\right] \tag{12}$$

be the segment connecting p to γ_τ. Since η is surjective, every point of γ_τ is the endpoint of some such segment. We call the union of these segments

$$\Sigma_{\tau,\alpha} := \bigcup_w R_w.$$

This is a surface ruled by the R_w and bounded by γ_τ and $X\gamma_\alpha$.

Let $\delta(w) = \alpha(w) - \beta(\eta(w))$, let

$$\lambda_w(x) := Z^{h(w)} Y^{\tau(h(w))} (X + \delta(w)Y)^x = XZ^w Y^{\alpha(w)} (X + \delta(w)Y)^{x-1}$$

parametrize R_w, and let $\Theta(x, w) := \lambda_w(x)$ parametrize $\Sigma_{\tau,\alpha}$. We claim that $\Sigma_{\tau,\alpha}$ is an intrinsic graph.

For each w, there is some quadratic g_w such that $\Pi(\lambda_w(x)) = (x, 0, g_w(x))$. We have

$$g_w(0) = z(\Pi(\lambda_w(0))) = z(\Pi(Z^{h(w)} Y^{\tau(h(w))})) = h(w),$$

$$g_w(1) = z(\Pi(\lambda_w(1))) = z(\Pi(XZ^w Y^{\alpha(w)})) = w,$$

and $g_w''(x) = -\operatorname{slope} R_w = -\delta(w)$, so

$$g_w(x) = (1-x)h(w) + xw + \frac{\delta(w)}{2} x(1-x).$$

Then $\Pi(\Theta(x, w)) = (x, 0, g_w(x))$. We claim that for every $x \in (0, 1]$, the map $w \mapsto g_w(x)$ is monotone increasing.

Suppose that there are $x_0 \in (0, 1]$ and $w_1 < w_2$ such that $g_{w_1}(x_0) = g_{w_2}(x_0)$. Let $p(x) = g_{w_2}(x) - g_{w_1}(x)$. Then $p(0) = h(w_2) - h(w_1) \geqslant 0$, $p(1) = w_2 - w_1 > 0$, and $p''(x) = 2c$ for all x, where $c = \frac{\delta(w_1)-\delta(w_2)}{2}$. Since p is quadratic, $p(0) \geqslant 0$, $p(1) > 0$, and $p(x_0) = 0$, it has a minimum at some point $r \in (0, 1)$ and $p(r) \leqslant 0$. Therefore, $p(x) = p(r) + c(x - r)^2$ and $c > 0$.

This implies

$$h(w_2) - h(w_1) = p(1) = p(r) + c(1 - r)^2.$$

Since h is nondecreasing, $p(r) < 0$, and $r \in (0,1)$,

$$0 \leqslant h(w_2) - h(w_1) \leqslant c(1-r)^2.$$

By equation (4), since λ_w is a line of slope $\delta(w)$ through $(0, \tau(h(w)), h(w))$, we can also write

$$g_w(x) = h(w) - \tau(h(w))x - \frac{\delta(w)}{2}x^2.$$

Then $g_w'(0) = -\tau(h(w))$, and

$$|\tau(h(w_2)) - \tau(h(w_1))| = |p'(0)| = 2c|1-r| > 2c(1-r)^2 \geqslant 2|h(w_2) - h(w_1)|,$$

which contradicts the assumption that $\operatorname{Lip}(\tau) < 2$.

Thus, for every $x \in (0,1]$, the map $w \mapsto g_w(x)$ is increasing, and the map $w \mapsto g_w(0) = h(w)$ is non-decreasing. Suppose that $p, q \in \Sigma_{\tau,\alpha}$ satisfy $\Pi(p) = \Pi(q)$. There are $x_1, x_2 \in [0,1]$ and $w_1, w_2 \in \mathbb{R}$ such that $p = \Theta(x_1, w_1)$ and $q = \Theta(x_2, w_2)$; since $\Pi(p) = \Pi(q)$, we have $x_1 = x_2$. By the above, if $x_1 \in (0,1]$, then $p = q$. Otherwise, if $x_1 = x_2 = 0$, then p and q lie on γ_τ, but Π is injective on γ_τ, so $p = q$. Therefore, Π is injective on $\Sigma_{\tau,\alpha}$, so $\Sigma_{\tau,\alpha}$ is an intrinsic graph. $\square$

Next, we calculate the area of $\Sigma_{\tau,\alpha}$ using the area formula (3).

Lemma 3.7. *Suppose that α is Lipschitz and that S_α is a graphical strip over K. Let $\eta(w) = w + \frac{\alpha(w)}{2}$, $\zeta(z) = z - \frac{\tau(z)}{2}$, $\beta(w) = \tau(\zeta^{-1}(w))$, and $\delta(w) = \alpha(w) - \beta(\eta(w))$ as above. By Lemma 3.2, $\alpha'(w) \geqslant -2$ for almost every w and η is nondecreasing.*

Let

$$\Theta(x,w) := XZ^w Y^{\alpha(w)}(X + \delta(w)Y)^{x-1}$$

parametrize $\Sigma_{\tau,\alpha}$. Let $a < b$. Then

$$\operatorname{area}\Theta([0,1]\times[a,b]) = \int_a^b \sqrt{1+\delta(w)^2}\left(1 + \frac{\alpha'(w)}{2}\right)\mathrm{d}w - \frac{1}{6}\int_{\delta(a)}^{\delta(b)} \sqrt{1+m^2}\,\mathrm{d}m. \tag{13}$$

Proof. By the area formula,

$$\begin{aligned}\operatorname{area}\Theta([0,1]\times[a,b]) &= \int_a^b \int_0^1 \sqrt{1+\delta(w)^2}\cdot\left|J[\Pi\circ\Theta](x,w)\right|\mathrm{d}x\,\mathrm{d}w \\ &= \int_a^b \sqrt{1+\delta(w)^2}\, j(w)\,\mathrm{d}w,\end{aligned}$$

where $j(w) = \int_0^1 \left|J[\Pi\circ\Theta](x,w)\right|\mathrm{d}x$.

Each ruling $\Theta([0,1]\times\{w\})$ is a line with slope $\delta(w)$ going through $XZ^wY^{\alpha(w)}$. Since $\Pi(XZ^wY^{\alpha(w)}) = (1,0,w)$, by (4),

$$\Pi(\Theta(x,w)) = \left(x, 0, w - \alpha(w)(x-1) - \frac{\delta(w)}{2}(x-1)^2\right)$$

Since $\Pi\circ\Theta$ is injective, for any $x \in (0,1)$, $w \mapsto z(\Pi(\Theta(x,w)))$ must be increasing. Then

$$J[\Pi\circ\Theta](x,w) = 1 + (1-x)\alpha'(w) - \frac{(1-x)^2}{2}\delta'(w) \geqslant 0,$$

and

$$j(w) = \int_0^1 1 + (1-x)\alpha'(w) - \frac{(1-x)^2}{2}\delta'(w)\,\mathrm{d}x = 1 + \frac{\alpha'(w)}{2} - \frac{\delta'(w)}{6}.$$

Therefore,

$$\begin{aligned}\operatorname{area}\Theta([0,1]\times[a,b]) &= \int_a^b \sqrt{1+\delta(w)^2}\left(1+\frac{\alpha'(w)}{2}-\frac{\delta'(w)}{6}\right)\mathrm{d}w\\ &= \int_a^b\int_0^1 \sqrt{1+\delta(w)^2}\left(1+\frac{\alpha'(w)}{2}\right)\mathrm{d}w - \int_{\delta(a)}^{\delta(b)}\sqrt{1+m^2}\,\mathrm{d}m,\end{aligned}$$

where we substitute $m=\delta(w)$ in the last equality. □

For τ such that $\operatorname{Lip}(\tau)<2$, let

$$S_{\alpha,\tau} := \Sigma_{\tau,\alpha}\cup s_{-1,-1}(\Sigma_{-\tau,\alpha}).$$

This is a deformation of S_α, and we will use Lemma 3.7 to calculate the second variation of its area.

Proof of Proposition 3.5. Let λ be small enough that $\operatorname{Lip}(\lambda\tau)<2$, so that $\Sigma_{\lambda\tau,\alpha}$ exists. Let $\zeta_\lambda(z)=z-\frac{\lambda\tau(z)}{2}$, let

$$\beta_\lambda(w)=\lambda\tau(\zeta_\lambda^{-1}(w)),$$

and let $\delta_\lambda(w)=\alpha(w)-\beta_\lambda(\eta(w))$, so that

$$\Theta_{\lambda\tau,\alpha}(x,w)=XZ^wY^{\alpha(w)}(X+\delta_\lambda(w)Y)^{x-1}$$

parametrizes $\Sigma_{\lambda\tau,\alpha}$ and $s_{-1,-1}\circ\Theta_{-\lambda\tau,\alpha}$ parametrizes $s_{-1,-1}(\Sigma_{-\lambda\tau,\alpha})$. Let

$$R_{w,\lambda} := \Theta_{\lambda\tau,\alpha}([0,1]\times\{w\}) = \left[Z^{\zeta_\lambda^{-1}(\eta(w))}Y^{\lambda\tau(\zeta_\lambda^{-1}(\eta(w)))}, XZ^wY^{\alpha(w)}\right].$$

Let $w_1<w_2$ be such that $\eta^{-1}(\eta(w_i))=\{w_i\}$ and let $D=\{(x,y,z)\in\mathbb{H}:\eta(w_1)\leqslant z\leqslant \eta(w_2)\}$. Then for $z\notin(\eta(w_1),\eta(w_2))$, we have $\zeta_\lambda(z)=z$, so $\zeta_\lambda^{-1}(z)=z$ and

$$\zeta_\lambda^{-1}((\eta(w_1),\eta(w_2)))\subset(\eta(w_1),\eta(w_2)).$$

We claim that for any $|\lambda|<1$,

$$S_{\alpha,\lambda\tau}\cap D=\Theta_{\lambda\tau,\alpha}([0,1]\times[w_1,w_2])\cup s_{-1,-1}(\Theta_{-\lambda\tau,\alpha}([0,1]\times[w_1,w_2])). \tag{14}$$

It suffices to show that $R_{w,\lambda}\subset D$ when $w\in[w_1,w_2]$ and $R_{w,\lambda}\cap D=\varnothing$ otherwise. The endpoints of $R_{w,\lambda}$ have z–coordinates

$$z\left(Z^{\zeta_\lambda^{-1}(\eta(w))}Y^{\lambda\tau(\zeta_\lambda^{-1}(\eta(w)))}\right)=\zeta_\lambda^{-1}(\eta(w))$$

and $z(XZ^wY^{\alpha(w)})=\eta(w)$. When $w\in[w_1,w_2]$, we have $\zeta_\lambda^{-1}(\eta(w))\in[\eta(w_1),\eta(w_2)]$ and $\eta(w)\in[\eta(w_1),\eta(w_2)]$, so $R_{w,\lambda}\subset D$. When $w\notin[w_1,w_2]$, we have $\zeta_\lambda^{-1}(\eta(w))=\eta(w)\notin[\eta(w_1),\eta(w_2)]$, so $R_{w,\lambda}$ is disjoint from D.

On the other hand, when $w\in[w_1,w_2]$, we have $\eta(w)\in[\eta(w_1),\eta(w_2)]$ and $\zeta_\lambda^{-1}(\eta(w))\in[\eta(w_1),\eta(w_2)]$, so both endpoints of $\Theta_{\lambda\tau,\alpha}([0,1]\times\{w\})$ lie in D. It follows that $\Theta_{\lambda\tau,\alpha}(x,w)\in D$ if and only if $w\in[w_1,w_2]$, which shows (14).

Therefore, by Lemma 3.7,

$$\begin{aligned}\operatorname{area}(S_{\alpha,\lambda\tau}\cap D) &= \operatorname{area}\Theta_{\lambda\tau,\alpha}([0,1]\times[w_1,w_2])+\operatorname{area}\Theta_{-\lambda\tau,\alpha}([0,1]\times[w_1,w_2])\\ &= \int_{w_1}^{w_2}\left(\sqrt{1+\delta_\lambda(w)^2}+\sqrt{1+\delta_{-\lambda}(w)^2}\right)\left(1+\frac{\alpha'(w)}{2}\right)\mathrm{d}w-\frac{1}{3}\int_{\delta(w_1)}^{\delta(w_2)}\sqrt{1+m^2}\,\mathrm{d}m.\end{aligned}$$

It remains to differentiate $\delta_\lambda(w)$. Let $z_w(\lambda):=\zeta_\lambda^{-1}(w)$. Then

$$\zeta_\lambda(z_w(\lambda))=z_w(\lambda)-\frac{\lambda}{2}\tau(z_w(\lambda))=w.$$

We differentiate both sides with respect to λ to find

$$z_w'(\lambda) - \frac{1}{2}\tau(z_w(\lambda)) + \frac{\lambda}{2}\tau'(z_w(\lambda))z_w'(\lambda) = 0.$$

Setting $\lambda = 0$ and noting that $z_w(0) = \zeta_0^{-1}(w) = w$, we find $z_w'(0) = \frac{1}{2}\tau(w)$. Likewise, let $b_w(\lambda) := \beta_\lambda(w) = \lambda\tau(z_w(\lambda))$. Then

$$b_w'(\lambda) = \tau(z_w(\lambda)) + \lambda\tau'(z_w(\lambda))z_w'(\lambda),$$

so $b_w'(0) = \tau(t)$ and $b_w''(0) = 2\tau'(z_w(0))z_w'(0) = \tau'(w)\tau(w)$, i.e.,

$$\beta_\lambda(w) = \lambda\tau(w) + \frac{\lambda^2}{2}\tau'(w)\tau(w) + O_\tau(\lambda^3)$$

for sufficiently small λ, where $O_\tau(\lambda^3)$ denotes an error term of magnitude at most $C(\tau)\lambda^3$.

Substituting this into the power series

$$\sqrt{1+(b-a)^2} = \sqrt{1+a^2} - \frac{a}{\sqrt{1+a^2}}b + \frac{1}{2(1+a^2)^{\frac{3}{2}}}b^2 + O(b^3),$$

and abbreviating $\alpha = \alpha(w)$, $z = \eta(w)$, $\tau = \tau(z)$, $\frac{d\tau}{dz} = \tau'(\eta(w))$, we find

$$\begin{aligned}
&\sqrt{1+(\beta_\lambda(\eta(w))-\alpha(w))^2} + \sqrt{1+(\beta_{-\lambda}(\eta(w))-\alpha(w))^2}\\
&\quad= \sqrt{1+\alpha^2} - \frac{\alpha}{\sqrt{1+\alpha^2}}\left(\lambda\tau + \frac{\lambda^2}{2}\frac{d\tau}{dz}\tau\right) + \frac{(\lambda\tau)^2}{2(1+\alpha^2)^{\frac{3}{2}}}\\
&\qquad + \sqrt{1+\alpha^2} - \frac{\alpha}{\sqrt{1+\alpha^2}}\left(-\lambda\tau + \frac{\lambda^2}{2}\frac{d\tau}{dz}\tau\right) + \frac{(\lambda\tau)^2}{2(1+\alpha^2)^{\frac{3}{2}}} + O_\tau(\lambda^3)\\
&\quad= 2\sqrt{1+\alpha^2} - \frac{\alpha}{\sqrt{1+\alpha^2}}\frac{d\tau}{dz}\tau\lambda^2 + \frac{\tau^2}{(1+\alpha^2)^{\frac{3}{2}}}\lambda^2 + O_\tau(\lambda^3).
\end{aligned}$$

Substituting this into (13), using the fact that $\frac{dz}{dw} = \frac{d\eta}{dw} = 1 + \frac{1}{2}\frac{d\alpha}{dw}$, and integrating by parts we find

$$\begin{aligned}
&\operatorname{area}(S_{\alpha,\lambda\tau}\cap D) - \operatorname{area}(S_{\alpha,0}\cap D)\\
&\quad= \lambda^2\int_{w_1}^{w_2}\left(-\frac{\alpha}{\sqrt{1+\alpha^2}}\tau\frac{d\tau}{dz} + \frac{\tau^2}{(1+\alpha^2)^{\frac{3}{2}}}\right)\left(1+\frac{1}{2}\frac{d\alpha}{dw}\right)dw + O_\tau(\lambda^3|w_2-w_1|)\\
&\quad= \lambda^2\int_{w_1}^{w_2} -\frac{\alpha}{\sqrt{1+\alpha^2}}\cdot\tau\frac{d\tau}{dz}\frac{dz}{dw} + \frac{\tau^2}{(1+\alpha^2)^{\frac{3}{2}}}\left(1+\frac{1}{2}\frac{d\alpha}{dw}\right)dw + O_\tau(\lambda^3|w_2-w_1|)\\
&\quad= \lambda^2\int_{w_1}^{w_2} \frac{1}{(1+\alpha^2)^{\frac{3}{2}}}\frac{d\alpha}{dw}\cdot\frac{\tau^2}{2} + \frac{\tau^2}{(1+\alpha^2)^{\frac{3}{2}}}\left(1+\frac{1}{2}\frac{d\alpha}{dw}\right)dw + O_\tau(\lambda^3|w_2-w_1|)\\
&\quad= \lambda^2\int_{w_1}^{w_2} \frac{\tau^2}{(1+\alpha^2)^{\frac{3}{2}}}\left(1+\frac{d\alpha}{dw}\right)dw + O_\tau(\lambda^3|w_2-w_1|).
\end{aligned}$$

This proves the proposition. □

Though the proposition deals with the case that α is Lipschitz, we can use it to find a necessary condition for S_α to be perimeter-minimizing in the general case.

Lemma 3.8. *Suppose that S_α is a graphical strip over K and S_α is area-minimizing in U. Then $\frac{\alpha(w_2)-\alpha(w_1)}{w_2-w_1} \geqslant -1$ for all $w_1 < w_2$.*

Proof. Note that $S_\alpha^+ \triangle S_{\alpha,\lambda\tau}^+$ is not compactly contained in U, so $S_{\alpha,\lambda\tau}$ is not a valid competitor for S_α in U. We fix this by scaling S_α slightly. This also lets us replace α by a Lipschitz function.

Let S_α be a graphical strip over K and suppose that S_α is perimeter-minimizing in U. Let $\varepsilon > 0$ and let $\hat{S} := s_{1+\varepsilon,1+\varepsilon}(S_\alpha)$. Then $\hat{S}$ is a graphical strip over $s_{1+\varepsilon,1+\varepsilon}(K)$, so $\hat{S} \cap U$ is a graphical strip over S. Let $f\colon K \to \mathbb{R}$ be the function such that $S_\alpha = \Gamma_f$. Then $\hat{S} = \Gamma_{\hat{f}}$, where $\hat{f}\colon s_{1+\varepsilon,1+\varepsilon}(K) \to \mathbb{R}$ is the function

$$\hat{f}_\varepsilon(p) = (1+\varepsilon) f\left(s_{1+\varepsilon,1+\varepsilon}^{-1}(p)\right).$$

Let $\hat{\alpha}_\varepsilon(w) = \hat{\alpha}(w) := \hat{f}_\varepsilon(1,w)$. Then $\hat{S} \cap U = S_{\hat{\alpha}}$, and $\hat{S}$ is perimeter-minimizing in $\hat{U} := s_{1+\varepsilon,1+\varepsilon}(U)$.

We claim that $\hat{\alpha}$ is Lipschitz. For $w \in \mathbb{R}$, let $p_w = XZ^wY^{\hat{\alpha}(w)}$. Then $p_w \in \hat{S}$ and there is a ruling $\hat{R}_w$ through p_w with slope $\hat{\alpha}(w)$. By (4),

$$\Pi(\hat{R}_w) = \{(x, 0, g_w(x)) : x \in [-1-\varepsilon, 1+\varepsilon]\},$$

where

$$g_w(x) = w - \hat{\alpha}(w)(x-1) - \frac{\hat{\alpha}(w)}{2}(x-1)^2.$$

Let $s < t$, so that $g_s(1) = s < t = g_t(1)$. By Lemma 2.1, $g_s(x) \leqslant g_t(x)$ for all $x \in [-1-\varepsilon, 1+\varepsilon]$. Then

$$g_t(0) - g_s(0) = t + \frac{\hat{\alpha}(t)}{2} - s - \frac{\hat{\alpha}(s)}{2} \geqslant 0,$$

so $\hat{\alpha}(t) - \hat{\alpha}(s) \geqslant -2(t-s)$ and

$$g_t(1+\varepsilon) - g_s(1+\varepsilon) = t - s - \left(\varepsilon + \frac{\varepsilon^2}{2}\right)(\hat{\alpha}(t) - \hat{\alpha}(s)) \geqslant 0,$$

i.e,

$$\hat{\alpha}(t) - \hat{\alpha}(s) \leqslant \left(\varepsilon + \frac{\varepsilon^2}{2}\right)^{-1}(t-s).$$

That is, $\hat{\alpha}$ is Lipschitz.

We claim that $\hat{\alpha}'(w) \geqslant -1$ for almost every $w \in \mathbb{R}$. Suppose by contradiction that $\{w : \hat{\alpha}'(w) < -1\}$ has positive measure. Let

$$\mathrm{II}(\tau) = \int_{-\infty}^{\infty} \frac{\tau(w)^2}{(1+\hat{\alpha}(w)^2)^{\frac{3}{2}}}\left(1+\hat{\alpha}'(w)\right) \mathrm{d}w;$$

we claim that there is a $\tau \in C_c^\infty$ such that $\mathrm{II}(\tau) < 0$.

Let $\hat{\eta}(w) := w + \frac{\hat{\alpha}(w)}{2}$. We consider two cases, depending on whether $\hat{\eta}$ is injective. If $\hat{\eta}$ is not injective, then there is some $z_0 \in \mathbb{R}$ such that $\hat{\eta}^{-1}(z_0) = [a,b]$ for some $a < b$. For any $w \in (a,b)$, we have $\hat{\eta}(w) = w + \frac{\hat{\alpha}(w)}{2} = z_0$, so $\hat{\alpha}(w) = 2z_0 - 2w$ and thus $\hat{\alpha}'(w) = -2$. Therefore,

$$\mathrm{II}(\mathbf{1}_{\{z_0\}}) = \int_{-\infty}^{\infty} \frac{\mathbf{1}_{\{z_0\}}(\hat{\eta}(w))}{(1+\hat{\alpha}(w)^2)^{\frac{3}{2}}}\left(1+\hat{\alpha}'(w)\right) \mathrm{d}w = \int_a^b -\frac{1}{(1+\hat{\alpha}(w)^2)^{\frac{3}{2}}}\,\mathrm{d}w < 0.$$

If $\hat{\eta}$ is injective, we let $w_0 \in \mathbb{R}$ be a Lebesgue point of $\hat{\alpha}'$ such that $\hat{\alpha}'(w_0) < -1$ and let $I_r = (\hat{\eta}(w_0 - r), \hat{\eta}(w_0 + r))$. Since $\hat{\alpha}$ is continuous, w_0 is also a Lebesgue point of $(1+\hat{\alpha}'(w))(1+\hat{\alpha}(w)^2)^{-\frac{3}{2}}$, and

$$\lim_{r\to 0} \frac{\mathrm{II}(\mathbf{1}_{I_r})}{2r} = \lim_{r\to 0} \frac{1}{2r}\int_{w_0-r}^{w_0+r} \frac{1}{(1+\hat{\alpha}(w)^2)^{\frac{3}{2}}}\left(1+\hat{\alpha}'(w)\right) \mathrm{d}w = \frac{1+\hat{\alpha}'(w_0)}{(1+\hat{\alpha}(w_0)^2)^{\frac{3}{2}}} < 0.$$

In either case, there is some bounded interval I (possibly $I = \{z_0\}$) such that $\mathrm{II}(\mathbf{1}_I) < 0$. Let $\tau_i \in C_c^\infty$ be a sequence of uniformly bounded functions with uniformly bounded supports such that $\tau_i \to \mathbf{1}_I$ pointwise. Then $\lim_{i\to\infty} \mathrm{II}(\tau_i) = \mathrm{II}(\mathbf{1}_I)$ by dominated convergence, so there is a τ_i such that $\mathrm{II}(\tau_i) < 0$. Let $F_\lambda = (\hat{S} \setminus U) \cup S_{\hat\alpha,\lambda\tau_i}$. Then F_λ is a deformation of $F_0 = \hat{S}$, and there is an $r > 0$ such that $F_\lambda^+ \triangle \hat{S}^+ \Subset \hat{U} \cap B(\mathbf{0}, r)$. By Proposition 3.5, there is a $C > 0$ such that

$$\operatorname{area}(B(\mathbf{0},r) \cap F_\lambda) \leqslant \operatorname{area}(B(\mathbf{0},r) \cap \hat{S}) + \lambda^2\,\mathrm{II}(\tau_i) + C\lambda^3,$$

so $\operatorname{area}(B(\mathbf{0},r) \cap F_\lambda) < \operatorname{area}(B(\mathbf{0},r) \cap \hat{S})$ when λ is sufficiently small. This contradicts the fact that $\hat{S}$ is area-minimizing on $\hat{U}$, so $\hat\alpha'(w) \geqslant -1$ for almost every $w \in \mathbb{R}$.

Thus, if S_α is area-minimizing in U, then $\frac{\hat\alpha_\varepsilon(w_2)-\hat\alpha_\varepsilon(w_1)}{w_2-w_1} \geqslant -1$ for all $\varepsilon > 0$ and all $w_2 > w_1$. The continuity of f implies that

$$\lim_{\varepsilon\to 0} \hat\alpha_\varepsilon(w) = \lim_{\varepsilon\to 0} (1+\varepsilon) f(s^{-1}_{1+\varepsilon,1+\varepsilon}(1,0,w)) = f(1,0,w) = \alpha(w),$$

so

$$\frac{\alpha(w_2)-\alpha(w_1)}{w_2-w_1} = \lim_{\varepsilon\to 0} \frac{\hat\alpha_\varepsilon(w_2)-\hat\alpha_\varepsilon(w_1)}{w_2-w_1} \geqslant -1.$$

This proves the lemma. □

3.3. **Minimality of graphical strips.** In this section, we show that monotone subsets of $\mathbb{H}$ are perimeter-minimizing and give a criterion for S_α^+ to be monotone. We first recall some definitions and formulas. Let $\mathscr{L}$ be the space of horizontal lines in $\mathbb{H}$, and for $U \subset \mathbb{H}$, let $\mathscr{L}(U)$ denote the set of horizontal lines that intersect U. Let $\mathscr{L}$ be the measure on $\mathscr{N}$ that is invariant under isometries of $\mathbb{H}$ (rotations around the z–axis, left-translations, and maps $s_{\pm1,\pm1}$). This measure is unique up to constants, and we normalize it so that $\mathscr{N}(\mathscr{L}(B(\mathbf{0},r))) = r^3$ for every $r > 0$.

The kinematic formula (see [Mon05] or equation (6.1) in [CKN11]) relates perimeter on lines to perimeter in $\mathbb{H}$ as follows. There is a constant $c > 0$ such that for any set $E \subset \mathbb{H}$ with locally finite perimeter and any open subset $W \subset \mathbb{H}$, $\operatorname{Per}_{E\cap L}(W \cap L) < \infty$ for $\mathscr{N}$–almost every line L and

$$\operatorname{Per}_E(W) = c\int_{\mathscr{L}} \operatorname{Per}_{E\cap L}(W\cap L)\,\mathrm{d}\mathscr{N}(L). \tag{15}$$

Here, $\operatorname{Per}_{E\cap L}$ is the perimeter of $E \cap L$ as a subset of L. If $\operatorname{Per}_{E\cap L}$ is finite, then there is a finite union of intervals $S \subset L$ such that $(E \triangle S) \cap L$ has measure zero and $\operatorname{Per}_{E\cap L}$ is the counting measure on ∂S.

We say that a subset $W \subset \mathbb{H}$ is convex if it is convex as a subset of $\mathbb{R}^3$. For any $g \in \mathbb{H}$, the map $h \mapsto gh$ is affine, so convexity is preserved by left-translation. Let $E \subset \mathbb{H}$ be a set with locally finite perimeter and let $W \subset \mathbb{H}$ be a convex open set. We say that E is *monotone* on W if for almost every $L \in \mathscr{L}$, we have $\operatorname{Per}_{E\cap L}(W \cap L) \leqslant 1$. That is, up to a set of measure zero, $E \cap W \cap L$ is one of $\varnothing$, $W \cap L$, or the intersection of $W \cap L$ with a ray. Monotonicity is preserved by left-translation.

Proposition 3.9. *Let $E \subset \mathbb{H}$ be a set with locally finite perimeter and let $W \subset \mathbb{H}$ be a convex open set such that E is monotone on W. Then E is perimeter-minimizing in W.*

Proof. Let $F \subset \mathbb{H}$ be a set with locally finite perimeter such that $E \triangle F \Subset W$. We claim that

$$\operatorname{Per}_{E\cap L}(W\cap L) \leqslant \operatorname{Per}_{F\cap L}(W\cap L) \tag{16}$$

for almost every line $L \in \mathscr{L}$.

For almost every line $L \in \mathscr{L}$, we have $\mathrm{Per}_{E\cap L}(W \cap L) \leqslant 1$. If $\mathrm{Per}_{E\cap L}(W \cap L) = 0$, then (16) holds, so we suppose that $\mathrm{Per}_{E\cap L}(W \cap L) = 1$. Then $W \cap \partial_{\mathscr{H}^1}(E \cap L)$ consists of a single point, say p, and there is a ray R based at p such that $E \cap W \cap L = R \cap W \cap L$, up to a null set.

Since W is convex, the intersection $W \cap L$ is an interval, with (up to null sets) one end of $W \cap L$ in E and the other end outside of E. Since $E \triangle F \Subset W$, we likewise have one end of $W \cap L$ in F and the other end outside of F. Therefore, $\mathrm{Per}_{F\cap L}(W \cap L) \geqslant 1 = \mathrm{Per}_{E\cap L}(W \cap L)$.

By (15),

$$\mathrm{Per}_E(W) = c \int_{\mathscr{L}} \mathrm{Per}_{E\cap L}(W \cap L)\, \mathrm{d}\mathscr{N}(L) \leqslant \mathrm{Per}_F(W),$$

so E is perimeter-minimizing on W. □

Finally, we apply this to S_α.

Lemma 3.10. *Let S_α be a graphical strip over K and suppose that $\frac{\alpha(t)-\alpha(s)}{t-s} \geqslant -1$ for all $s < t$. Then S_α^+ is perimeter-minimizing on $U = \{-1 < x < 1\} \subset \mathbb{H}$.*

Proof. We claim that S_α^+ is monotone on U. By Lemma 3.4,

$$S_\alpha = \bigcup_{z\in\mathbb{R}} [(-1, -\sigma(z), z), (1, \sigma(z), z)]$$

for some Lipschitz function $\sigma\colon \mathbb{R} \to \mathbb{R}$ such that $-2 \leqslant \sigma'(z) < 2$ for all z.

For $x \in [-1, 1]$, $z \in \mathbb{R}$, let $\Phi(x, z) = (x, x\sigma(z), z)$ parametrize S_α. It suffices to show that any horizontal line that does not contain a ruling of S_α intersects $\Phi((-1, 1) \times \mathbb{R})$ at most once.

Let $x_1, x_2 \in (-1, 1)$ and $z_1 < z_2$. The horizontal plane P_1 centered at $\Phi(x_1, z_1)$ is the set

$$\begin{aligned} P_1 &= \left\{(x_1, x_1\sigma(z_1), z_1) \cdot (x - x_1, y - x_1\sigma(z_1), 0) : x, y \in \mathbb{R}\right\} \\ &= \left\{\left(x, y, z_1 + \frac{x_1 y - x x_1 \sigma(z_1)}{2}\right) : x, y \in \mathbb{R}\right\}, \end{aligned}$$

so $\Phi(x_2, z_2) \in P_1$ if and only if

$$z_2 = z_1 + \frac{x_1 x_2 \sigma(z_2) - x_2 x_1 \sigma(z_1)}{2},$$

i.e.,

$$x_1 x_2 \frac{\sigma(z_2) - \sigma(z_1)}{z_2 - z_1} = 2.$$

Since $\frac{|\sigma(z_2)-\sigma(z_1)|}{z_2-z_1} \leqslant 2$ and $|x_1 x_2| < 1$, this is impossible. Therefore, if L intersects $\Phi((-1, 1) \times \mathbb{R})$ at two points, then those two points have the same z–coordinate and thus lie on the same ruling of S_α. Therefore, S_α^+ is monotone on U and thus perimeter-minimizing on U. □

Lemma 3.8 and Lemma 3.10 prove the two directions of Proposition 3.3.

4. Scaling limits of ruled graphs

In the last section, we classified area-minimizing graphical strips. In this section and the next section, we will use the classification of area-minimizing graphical strips to classify entire area-minimizing ruled intrinsic graphs. We first show that the scaling limit of an entire ruled intrinsic graph is a graphical strip; in fact, it is a broken plane (Definition 3.1).

Lemma 4.1. *Let $f\colon V_0 \to \mathbb{R}$ be a continuous function such that Γ_f is a ruled surface. There are $u \in [0,\infty]$ and $\theta \in \mathbb{R}$ such that if $\mathrm{rot}_\theta\colon \mathbb{H} \to \mathbb{H}$ is rotation by θ around the z–axis, then $s_{t,t}^{-1}(\Gamma_f^+) \to \mathrm{rot}_\theta(\mathrm{BP}_u)$ as $t \to \infty$. Furthermore, if $u = 0$, then Γ_f is a vertical plane.*

We write $E_i \to E$ if E_i converges locally to E, that is, for every compact set $K \subset \mathbb{H}$, we have $\mathscr{H}^4((E_i \triangle E) \cap K) \to 0$.

We prove Lemma 4.1 by analyzing the characteristic curves of Γ_f. As noted in Section 2.4, every ruling of Γ_f projects to a parabola in V_0, and since Γ_f is entire, these parabolas cover V_0.

For each ruling R, let $m(R)$ be the slope of R, let $w(R) = (0, w_y(R), w_z(R)) \in W_0$ be the point where R intersects the yz–plane, and let $\lambda_R(t) = w(R)(X + m(R)Y)^t$ parametrize R. Let $g_R(t) = z(\Pi(\lambda_R(t)))$ so that $\Pi(R)$ is the graph $\{(x, 0, g_R(x)) \in V_0 : x \in \mathbb{R}\}$; by (4),

$$g_R(x) = w_z(R) - w_y(R)x - \frac{m(R)}{2}x^2. \tag{17}$$

We first prove some lemmas describing Γ_f.

Lemma 4.2. *Let R_1 and R_2 be distinct rulings of Γ_f and suppose that $w_z(R_1) < w_z(R_2)$. Then $g_{R_1}(x) \leqslant g_{R_2}(x)$ for all $x \in \mathbb{R}$ and $m(R_1) \geqslant m(R_2)$.*

Proof. We have $g_{R_1}(0) = w_z(R_1) < w_z(R_2) = g_{R_2}(t)$, so by Lemma 2.1, $g_{R_1}(t) \leqslant g_{R_2}(t)$ for all t. By (17), this implies $m(R_1) \geqslant m(R_2)$. □

For $p \in \mathbb{H}$, $m \in \mathbb{R}$, let $L_{p,m}$ be the horizontal line of slope m through p.

Lemma 4.3. *Let R_1 and R_2 be distinct rulings of Γ_f that intersect at a point $p \in \mathbb{H}$; suppose that $m(R_1) < m(R_2)$. Then for every $m \in [m(R_1), m(R_2)]$, $L_{p,m}$ is a ruling of Γ_f.*

Proof. After a translation, we may suppose that $p = \mathbf{0}$, so that $R_i = L_{\mathbf{0},m(R_i)}$ and $\Pi(R_i)$ is the graph of the function $g_i(x) = -\frac{m(R_i)}{2}x^2$. Let $m \in (m(R_1), m(R_2))$, $q = (1, 0, -\frac{m}{2})$, and let M be the ruling through $\Psi_f(q)$. By Lemma 2.1, $g_{R_2}(x) \leqslant g_M(x) \leqslant g_{R_1}(x)$, and since the graphs of g_{R_1} and g_{R_1} are tangent to the x–axis at 0, so is the graph of g_M. Since $g_M(1) = -\frac{m}{2}$, we have $g_M(x) = -\frac{m}{2}x^2$, so $M = L_{\mathbf{0},m}$. □

Combining these two lemmas, we get the following characterization of the rulings of Γ_f.

Lemma 4.4. *Let $f\colon V_0 \to \mathbb{R}$ be a continuous function such that Γ_f is a ruled surface. Let*

$$\mathscr{R} = \{(z, m) \in \mathbb{R}^2 : L_{\Psi_f(Z^z),m} \subset \Gamma_f\}$$

and for $z \in \mathbb{R}$, let

$$\sigma(z) = \sup\{m : L_{\Psi_f(Z^z),m} \subset \Gamma_f\}.$$

Then σ is nondecreasing and $\mathscr{R} = \{(z, m) \in \mathbb{R}^2 : \sigma(z^-) \leqslant m \leqslant \sigma(z)\}$, where $\sigma(z^-) = \lim_{t \to z^-} \sigma(t)$.

Proof. Since Γ_f is an intrinsic graph, no ruling of Γ_f is a coset of $\langle Y \rangle$, i.e., every ruling of Γ_f has finite slope. For $p \in \Gamma_f$, let $M_p = \{m : L_{p,m} \subset \Gamma_f\}$. Then M_p is nonempty. Since Γ_f is closed, any sequence of rulings through p has a subsequence that converges to a ruling through p, so M_p is compact. Therefore, $\sigma(z) = \max M_{\Psi_f(Z^z)}$ and $(z, \sigma(z)) \in \mathscr{R}$ for all $z \in \mathbb{R}$. Let $z_1 < z_2$. Then $(z_1, \sigma(z_1)), (z_2, \sigma(z_2)) \in \mathscr{R}$, so by Lemma 4.2, $\sigma(z_1) \leqslant \sigma(z_2)$, i.e., σ is nondecreasing.

For every $z \in \mathbb{R}$, the sequence of rulings $L_{\Psi_f(Z^{z-\frac{1}{n}}),\sigma(z-\frac{1}{n})}$ converges to $L_{\Psi_f(Z^z),\sigma(z^-)}$, so $L_{\Psi_f(Z^z),\sigma(z^-)}$ is a ruling of Γ_f. By Lemma 4.3, we have $(z, m) \in \mathscr{R}$ for all $m \in [\sigma(z^-), \sigma(z)]$, so $\{(z, m) \in \mathbb{R}^2 : \sigma(z^-) \leqslant m \leqslant \sigma(z)\} \subset \mathscr{R}$.

Conversely, suppose that $(z,m)\in\mathbb{R}$. On one hand, $m\leqslant\sigma(z)$ by definition. On the other hand, if $(z',m')\in\mathscr{R}$ and $z'<z$, then $m'\leqslant m$ by Lemma 4.2, so $\sigma(z^-)\leqslant m$. Therefore, $\mathscr{R}=\{(z,m)\in\mathbb{R}^2:\sigma(z^-)\leqslant m\leqslant\sigma(z)\}$, as desired. □

Let $m_\infty=\lim_{t\to\infty}\sigma(t)$ and $m_{-\infty}=\lim_{t\to-\infty}\sigma(t)$. These limits determine the scaling limit of Γ_f.

Lemma 4.5. *For $t>0$, let $f_t(p)=t^{-1}f(s_{t,t}(p))$ so that $\Gamma_{f_t}=s_{t,t}^{-1}(\Gamma_f)$. For $x\neq 0$, let*

$$F(x,0,z)=\begin{cases} m_{-\infty}x & \text{if } m_{-\infty}<\infty \text{ and } z\leqslant -\frac{m_{-\infty}}{2}x^2\\ -\frac{2z}{x} & \text{if } -\frac{m_{-\infty}}{2}x^2<z<-\frac{m_\infty}{2}x^2\\ m_\infty x & \text{if } m_\infty>-\infty \text{ and } z\geqslant -\frac{m_\infty}{2}x^2.\end{cases}$$

Then f_t converges to F pointwise almost everywhere on V_0 as $t\to\infty$.

Proof. Let $x_0,z_0\in\mathbb{R}$. We first consider the case that $x_0\neq 0$ and $z_0\in(-\frac{m_{-\infty}}{2}x_0^2,-\frac{m_\infty}{2}x_0^2)$. Let R_1, R_2 be rulings of Γ_f such that $-\frac{m(R_1)}{2}x_0^2<z_0<-\frac{m(R_2)}{2}x_0^2$. Then

$$g_{R_1}(tx_0)=-\frac{m(R_1)}{2}(tx_0)^2+O(t)<t^2z_0$$

when t is sufficiently large; likewise, $g_{R_2}(tx_0)>t^2z_0$ when t is sufficiently large. That is, there is some $t_0>0$ such that $s_{t,t}(x_0,0,z_0)=(tx_0,0,t^2z_0)$ is between $\Pi(R_1)$ and $\Pi(R_2)$ when $t>t_0$.

For each $t>0$, let S_t be a ruling of Γ_f such that $(tx_0,0,t^2z_0)\in\Pi(S_t)$. When $t>t_0$, we have $g_{R_1}(tx_0)<t^2z_0<g_{R_2}(tx_0)$ and thus $g_{R_1}\leqslant g_{S_t}\leqslant g_{R_2}$ on all of $\mathbb{R}$. We will use this inequality to bound the coefficients of g_{S_t}.

Let $C>0$ be such that $|g_{R_1}(x)|<C$ and $|g_{R_2}(x)|<C$ for all $x\in[-1,1]$. Then $|g_{S_t}(x)|<C$ for all $x\in[-1,1]$. By (17), this implies $|w_z(S_t)|=|g_{S_t}(0)|<C$ and

$$|w_y(S_t)|=\left|\frac{g_{S_t}(-1)-g_{S_t}(1)}{2}\right|<C.$$

By (17) with $x=tx_0$,

$$g_{S_t}(tx_0)+\frac{m(S_t)}{2}(tx_0)^2=w_z(S_t)-w_y(S_t)tx_0,$$

so

$$\left|t^2z_0+\frac{m(S_t)}{2}(tx_0)^2\right|\leqslant C+C|tx_0|,$$

and

$$\left|\frac{m(S_t)}{2}+\frac{z_0}{x_0^2}\right|\leqslant\frac{C}{|tx_0|^2}+\frac{C}{|tx_0|}.$$

That is, $\lim_{t\to\infty}m(S_t)=-\frac{2z_0}{x_0^2}$. Since the graph of g_{S_t} is a characteristic curve, g_{S_t} satisfies the differential equation $g'_{S_t}(x)=-f(x,0,g_{S_t}(x))$. Therefore,

$$\begin{aligned}\lim_{t\to\infty}f_t(x_0,0,z_0)&=\lim_{t\to\infty}t^{-1}f(tx_0,0,t^2z_0)=\lim_{t\to\infty}-t^{-1}g'_{S_t}(tx_0)\\ &=\lim_{t\to\infty}-t^{-1}\left(-m(S_t)tx_0-w_y(S_t)\right)=\lim_{t\to\infty}m(S_t)x_0=-\frac{2z_0}{x_0},\end{aligned}$$

as desired.

Next, we consider the case that $m_\infty > -\infty$ and $\frac{z_0}{x_0^2} > -\frac{m_\infty}{2}$. Let $\varepsilon \in (0,1)$ and let R_1 be a ruling of Γ_f such that $m_\infty \leqslant m(R_1) < m_\infty + \varepsilon$. Then there is a t_0 such that

$$-\frac{m_\infty+\varepsilon}{2}t^2x_0^2 < g_{R_1}(tx_0) < t^2 z_0 \tag{18}$$

when $t > t_0$.

Let $t > t_0$ and let S_t be a ruling of Γ_f such that $(tx_0, 0, t^2z_0) \in \Pi(S_t)$. By Lemma 2.1, $g_{R_1}(x) \leqslant g_{S_t}(x)$ for all $x \in \mathbb{R}$, and by Lemma 4.2, $m(R_1) \geqslant m(S_t) \geqslant m_\infty$. Let $h(x) = g_{S_t}(x) - g_{R_1}(x)$. We have $h(x) \geqslant 0$ for all x and $h''(x) = m(R_1) - m(S_t) \in [0,\varepsilon]$. Since h is a quadratic, for any $w \in \mathbb{R}$,

$$0 \leqslant h(tx_0+w) = h(tx_0) + h'(tx_0)w + \frac{1}{2}h''(tx_0)w^2 \leqslant h(tx_0) + h'(tx_0)w + \frac{\varepsilon}{2}w^2.$$

Letting $w = -\varepsilon^{-1}h'(tx_0)$, this implies

$$h(tx_0) - \frac{h'(tx_0)^2}{2\varepsilon} \geqslant 0$$

and thus $|h'(tx_0)| \leqslant \sqrt{2\varepsilon h(tx_0)} = \sqrt{2\varepsilon(t^2z_0 - g_{R_1}(tx_0))}$. By (18),

$$|h'(tx_0)| \leqslant \sqrt{2\varepsilon t^2\left(z_0 + \frac{m_\infty+\varepsilon}{2}x_0^2\right)} \leqslant C|t|\sqrt{\varepsilon},$$

where $C = \sqrt{2z_0 + (m_\infty+1)x_0^2}$.

Therefore,

$$\left|f_t(x_0,0,z_0) - \frac{-g'_{R_1}(tx_0)}{t}\right| = \left|\frac{-g'_{S_t}(tx_0)}{t} - \frac{-g'_{R_1}(tx_0)}{t}\right| = |t|^{-1}|h'(tx_0)| \leqslant C\sqrt{\varepsilon}.$$

When $t > \max\{t_0, \varepsilon^{-1}|w_y(R_1)|\}$,

$$\begin{aligned}
&\left|f_t(x_0,0,z_0) - m_\infty x_0\right| \\
&\quad\leqslant \left|f_t(x_0,0,z_0) - \frac{-g'_{R_1}(tx_0)}{t}\right| + \left|\frac{-g'_{R_1}(tx_0)}{t} - m(R_1)x_0\right| + |m(R_1)-m_\infty||x_0| \\
&\quad\leqslant C\sqrt{\varepsilon} + \frac{|w_y(R_1)|}{t} + \varepsilon|x_0| \\
&\quad\leqslant C\sqrt{\varepsilon} + \varepsilon(1+|x_0|).
\end{aligned}$$

Since ε was arbitrary, $\lim_{t\to\infty} f_t(x_0,0,z_0) = m_\infty x_0$, as desired. The case that $z_0 < -\frac{m_{-\infty}}{2}x_0^2$ is similar. □

Finally, we prove Lemma 4.1.

Proof of Lemma 4.1. For $-\infty \leqslant M \leqslant m \leqslant \infty$, let

$$F_{m,M}(x,0,z) = \begin{cases} mx & \text{if } m<\infty \text{ and } z \leqslant -\frac{m}{2}x^2 \\ -\frac{2z}{x} & \text{if } -\frac{m}{2}x^2 < z < -\frac{M}{2}x^2 \\ Mx & \text{if } M > -\infty \text{ and } z \geqslant -\frac{M}{2}x^2. \end{cases}$$

By Lemma 4.5, f_t converges pointwise almost everywhere to $F := F_{m_{-\infty},m_\infty}$, so $\mathbf{1}_{\Gamma^+_{f_t}}$ converges weakly to $\mathbf{1}_{\Gamma^+_F}$. We claim that for any m, M there are θ and u such that, up to a set of measure zero, $\Gamma^+_{F_{m,M}} = \mathrm{rot}_\theta(\mathrm{BP}^+_u)$.

Suppose that m and M are both finite. Then Γ_F contains the intrinsic graph of $a_M(x,0,z) = Mx$ over the set $\{z \leqslant -Mx^2\}$. This is the upper vertical half-plane through $\mathbf{0}$

of slope M, bounded by the xy–plane. Likewise, Γ_F contains the lower half-plane through $\mathbf{0}$ of slope m, bounded by the xy–plane. Finally, Γ_F contains the graph of $h(x,0,z) = -\frac{2z}{x}$ over the set $\{-\frac{m}{2}x^2 < z < -\frac{M}{2}x^2\}$; this is the union of the horizontal lines through the origin with slopes between m and M. It follows that, up to a set of measure 0, $\Gamma^+_{F_{m,M}}$ consists of two quadrants of $\mathbb{H}$, one above the xy–plane, bounded by the vertical plane through $\mathbf{0}$ of slope M, and one below the xy–plane, bounded by the vertical plane through $\mathbf{0}$ of slope m.

Thus, if $m_{-\infty}$ and m_∞ are finite, then Γ^+_F is a union of two quadrants. If $m_{-\infty}$ or m_∞ are infinite, then F is a limit of $F_{m,M}$'s. As $m \to m_{-\infty}$ and $M \to m_\infty$, $\Gamma^+_{F_{m,M}}$ converges to a union of two quadrants with slopes $m_{-\infty}$ and m_∞, so Γ^+_F is again a union of two quadrants. Any such union can be written as $\mathrm{rot}_\theta(\mathrm{BP}^+_u)$ for some $u \in [0,\infty]$ and $\theta \in \mathbb{R}$, so $\Gamma^+_{f_t} \to \mathrm{rot}_\theta(\mathrm{BP}^+_u)$ as desired.

Finally, suppose that $u = 0$, so that $\mathrm{rot}_\theta(\mathrm{BP}_u)$ is a vertical plane V. In this case, $m_\infty = m_{-\infty}$; let $m = m_\infty$. Then, by Lemma 4.4, every ruling of Γ_f has slope m. By (4), if L_1 and L_2 are two horizontal lines with slope m and $w_y(L_1) \neq w_y(L_2)$, then $\Pi(L_1)$ and $\Pi(L_2)$ intersect transversely, so L_1 and L_2 cannot both be rulings of the same intrinsic graph. It follows that any two rulings $R_1, R_2 \subset \Gamma_f$ have the same projection to the xy–plane, i.e.,

$$\pi(R_1) = \pi(R_2) = \left\{(x,y,0) \in \mathbb{H} : y = mx + w_y(R_1)\right\}.$$

Therefore, $\Gamma_f = \{(x,y,z) \in \mathbb{H} : y = mx + w_y(R_1)\}$ is a vertical plane. □

5. PROOF OF THEOREM 1.3

Now we prove Theorem 1.3. We will need the following closure result for perimeter-minimizing subsets of $\mathbb{H}$. This statement and proof is based on Theorem 21.14 of [Mag12].

Proposition 5.1. *Let $A \subset \mathbb{H}$ be an open set and let $E_1, E_2, \dots \subset \mathbb{H}$ be a sequence of perimeter minimizers in A such that $E_i \cap A \to E \cap A$, where* $\mathrm{Per}_E(U) < \infty$ *for every $U \Subset A$. Then E is a perimeter minimizer in A.*

We will need the following metric characterization of perimeter. For a measurable set $E \subset \mathbb{H}$, let $E^{(1)}$ be the set of density points of E, i.e,

$$E^{(1)} := \left\{ p \in \mathbb{H} : \lim_{r \to 0^+} \frac{\mathcal{H}^4(B(p,r) \cap E)}{\mathcal{H}^4(B(p,r))} = 1 \right\},$$

and let $E^{(0)} = (\mathbb{H} \smallsetminus E)^{(1)}$. Let $\partial_{\mathcal{H}^4} E$ be the measure-theoretic boundary of E, i.e.

$$\partial_{\mathcal{H}^4} E := \mathbb{H} \smallsetminus \left(E^{(1)} \cup E^{(0)}\right).$$

Franchi, Serapioni, and Serra Cassano [FSSC01] showed that there is a left-invariant homogeneous metric d_∞ and a corresponding spherical Hausdorff measure $\mathscr{S}^3_\infty$ such that for any set $E \subset \mathbb{H}$ with locally finite perimeter,

$$\mathrm{Per}_E(U) = \mathscr{S}^3_\infty(U \cap \partial_{\mathcal{H}^4} E).$$

Proof of Proposition 5.1. Let $F \subset \mathbb{H}$ be a set such that $E \triangle F \Subset A$ and $\mathrm{Per}_F(U) < \infty$ for every $U \Subset A$. We claim that $\mathrm{Per}_E(A) \leq \mathrm{Per}_F(A)$.

Our first goal is to find a set G such that $E \triangle F \Subset G \Subset A$ and such that

$$\mathscr{S}^3_\infty\left(\partial_{\mathcal{H}^4} F \cap \partial G\right) = 0 \tag{19}$$

and

$$\liminf_{i \to \infty} \mathscr{S}^3_\infty\left(\left(E^{(1)} \triangle E_i^{(1)}\right) \cap \partial G\right) = 0. \tag{20}$$

By the compactness of $E \triangle F$, there are finitely many points p_j and radii r_j such that $E \triangle F \Subset \bigcup_j B(p_j, r_j)$ and $B(p_j, 2r_j) \Subset A$ for all j.

We claim that there is a $t \in [1,2]$ such that $G_t = \bigcup_j B(p_j, tr_j)$ satisfies (19) and (20). Since $\mathscr{S}^3_\infty(\partial_{\mathscr{H}^4} F \cap G_2) < \infty$, G_t satisfies (19) for all but countably many $t \in [1,2]$. Furthermore, the coarea formula implies that

$$\int_1^2 \mathscr{S}^3_\infty\left(\left(E^{(1)} \triangle E_i^{(1)}\right) \cap \partial B(p_j, tr_j)\right) \mathrm{d}t \lesssim r_j^{-1} \mathscr{H}^4\left(\left(E^{(1)} \triangle E_i^{(1)}\right) \cap B(p_j, 2r_j)\right).$$

By Fatou's lemma,

$$\begin{aligned}\int_1^2 \liminf_{i\to\infty} & \mathscr{S}^3_\infty\left(\left(E^{(1)} \triangle E_i^{(1)}\right) \cap \partial G_t\right) \mathrm{d}t \\ &\leq \liminf_{i\to\infty} \int_1^2 \sum_j \mathscr{S}^3_\infty\left(\left(E^{(1)} \triangle E_i^{(1)}\right) \cap \partial B(p_j, tr_j)\right) \mathrm{d}t \\ &\lesssim \liminf_{i\to\infty} \sum_j r_j^{-1} \mathscr{H}^4\left(\left(E^{(1)} \triangle E_i^{(1)}\right) \cap B(p_j, 2r_j)\right) = 0,\end{aligned}$$

so G_t satisfies (20) for almost every $t \in [1,2]$. Let $G = G_t$ for some t such that (19) and (20) are satisfied.

Let $\overline{G}$ be the closure of G, and for each i, let $F_i = (F \cap G) \cup (E_i \setminus G)$. To bound $\partial_{\mathscr{H}^4} F_i$, note that $\partial_{\mathscr{H}^4} F_i \cap G = \partial_{\mathscr{H}^4} F \cap G$ and $\partial_{\mathscr{H}^4} F_i \setminus \overline{G} = \partial_{\mathscr{H}^4} E_i \setminus \overline{G}$. If $p \in \partial_{\mathscr{H}^4} F_i \cap \partial G$, then either $p \in \partial_{\mathscr{H}^4} F$, $p \in \partial_{\mathscr{H}^4} E_i$, or $p \in E_i^{(1)} \triangle F^{(1)}$. Therefore,

$$\partial_{\mathscr{H}^4} F_i \subset \left(\partial_{\mathscr{H}^4} F \cap \overline{G}\right) \cup \left(\partial_{\mathscr{H}^4} E_i \setminus G\right) \cup \left(\left(E_i^{(1)} \triangle F^{(1)}\right) \cap \partial G\right). \tag{21}$$

Let U be an open set such that $G \Subset U \Subset A$. Since E_i is a perimeter minimizer and $E_i \triangle F_i \Subset U$,

$$\mathrm{Per}_{E_i}(U) \leq \mathrm{Per}_{F_i}(U) = \mathscr{S}^3_\infty(U \cap \partial_{\mathscr{H}^4} F_i).$$

By (21),

$$\mathrm{Per}_{E_i}(U) \leq \mathrm{Per}_F(\overline{G}) + \mathrm{Per}_{E_i}(U \setminus G) + \mathscr{S}^3_\infty\left(\left(E^{(1)} \triangle E_i^{(1)}\right) \cap \partial G\right).$$

Subtracting $\mathrm{Per}_{E_i}(U \setminus G)$ from both sides,

$$\mathrm{Per}_{E_i}(G) \leq \mathrm{Per}_F(\overline{G}) + \mathscr{S}^3_\infty\left(\left(E^{(1)} \triangle E_i^{(1)}\right) \cap \partial G\right).$$

By the lower semicontinuity of perimeter and equations (19)–(20),

$$\mathrm{Per}_E(G) \leq \liminf_{i\to\infty} \mathrm{Per}_{E_i}(G) \leq \mathrm{Per}_F(\overline{G}) = \mathrm{Per}_F(G),$$

as desired. □

Finally, we prove Theorem 1.3.

Proof of Theorem 1.3. Let Γ be an entire area-minimizing ruled intrinsic graph and let Γ^+ be its epigraph, so that Γ^+ is perimeter-minimizing. By Lemma 4.1, there are θ and $u \in [0,\infty]$ such that $s_{t,t}^{-1}(\Gamma^+) \to \mathrm{rot}_\theta(\mathrm{BP}_u)$ as $t \to \infty$. By Proposition 5.1, $\mathrm{rot}_\theta(\mathrm{BP}_u^+)$ is perimeter-minimizing; we claim that $u = 0$.

By Theorem 1.1, BP^+_u is not perimeter-minimizing when $u \in (0,\infty)$. We claim that BP^+_∞ is not perimeter-minimizing. Recall that $\mathrm{BP}^+_\infty = \{(x,y,z) \in \mathbb{H} : xz > 0\}$; one calculates that

$$\begin{aligned} Y^{-t}\,\mathrm{BP}^+_\infty &= \left\{\left(x, y-t, z+\frac{tx}{2}\right) \in \mathbb{H} : xz > 0\right\} \\ &= \left\{(x,y,z) \in \mathbb{H} : x\left(z - \frac{tx}{2}\right) > 0\right\}. \end{aligned}$$

That is, $\partial(Y^{-t}\,\mathrm{BP}^+_\infty)$ consists of the plane $\{x = 0\}$ and the plane $\{z = \frac{tx}{2}\}$. As $t \to \infty$, these planes grow closer together. In fact, $\mathcal{H}^4(\mathrm{BP}^+_\infty \cap B(Y^t,1)) \to 0$, $\mathcal{H}^3(\mathrm{BP}^+_\infty \cap \partial B(Y^t,1)) \to 0$, and $\mathrm{Per}_{\mathrm{BP}^+_\infty}(B(Y^t,1)) \gtrsim 1$. Let $C_t = \mathrm{BP}^+_\infty \setminus B(Y^t,1)$; then

$$\mathrm{Per}_{C_t}(B(Y^t,2)) = \mathrm{Per}_{\mathrm{BP}^+_\infty}(B(Y^t,2) \setminus B(Y^t,1)) + \mathcal{H}^3(\mathrm{BP}^+_\infty \cap \partial B(Y^t,1)),$$

and $\mathrm{Per}_{C_t}(B(Y^t,2)) < \mathrm{Per}_{\mathrm{BP}^+_\infty}(B(Y^t,2))$ when t is large. Thus BP^+_∞ is not area-minimizing.

The only remaining possibility is that $s_{t,t}(\Gamma^+) \to \mathrm{rot}_\theta(\mathrm{BP}^+_0)$. By the second part of Lemma 4.1, this implies that Γ is a vertical plane. $\square$

6. Non-uniqueness of area-minimizers

In this section, we consider the boundary case of Theorem 1.1, where $\sigma(z) = -2z$. By Theorem 1.1, the surface

$$\Sigma = \bigcup_{z\in\mathbb{R}} [(-1,2z,z),(1,-2z,z)]$$

is area-minimizing, but Theorem 1.2 asserts that there are uncountably many area-minimizing surfaces with the same boundary. In this section, we will prove Theorem 1.2.

First, note that for all $z_1, z_2 \in \mathbb{R}$,

$$(-1,2z_1,z_1)^{-1}(1,-2z_2,z_2) = (1,-2z_1,-z_1)(1,-2z_2,z_2) = (2,-2z_1-2z_2,0), \tag{22}$$

so any two points $(-1,2z_1,z_1)$ and $(1,-2z_2,z_2)$ are connected by a horizontal line. For any surjective continuous increasing function $\rho\colon \mathbb{R} \to \mathbb{R}$, let Σ_ρ be the ruled surface

$$\Sigma_\rho = \bigcup_{z\in\mathbb{R}} [(-1,2z,z),(1,-2\rho(z),\rho(z))].$$

Let

$$F_\rho(x,z) = (-1,2z,z)\cdot(1,-z-\rho(z),0)^{x+1}$$

parametrize Σ_ρ and let $U = \{(x,y,z) \in \mathbb{H} : -1 < x < 1\}$. We claim that Σ_ρ is an intrinsic graph whose epigraph is monotone in U.

Lemma 6.1. *Let a_1,a_2,b_1,b_2 be such that $b_1 < b_2$ and $a_1 < a_2$. For $i = 1,2$, let $R_i = ((-1,2a_i,a_i),(1,-2b_i,b_i))$. There are no horizontal lines from R_1 to R_2.*

Proof. Our argument is based on the hyperboloid lemma proved in [CK10, Lemma 2.4], but we give a self-contained proof.

Let $\pi\colon \mathbb{H} \to \mathbb{R}^2$, $\pi(x,y,z) = (x,y)$ be the projection to the xy–plane. Then $\pi(R_i) = ((-1,2a_i),(1,-2b_i))$, and since $b_1 < b_2$ and $a_1 < a_2$, the projections $\pi(R_1)$ and $\pi(R_2)$ intersect at some point $p = (x_0,y_0)$. Let $q_i = (x_0,y_0,z_i) \in R_i$ be the point such that $\pi(q_i) = p$.

Let $V_i = (1,-a_i-b_i,0)$, $s_0 = -1-x_0$ and $t_0 = 1-x_0$ so that $R_i = \{q_i V_i^r : r \in (s_0,t_0)\}$. Suppose that $q_1 V_1^s$ is connected to $q_2 V_2^t$ by a horizontal segment. Recalling that V_1^s is just alternate notation for sV_1, we have $q_1 \cdot sV_1 \cdot W = q_2 \cdot tV_2$ for some $W \in \mathsf{A}$. Projecting to $\mathbb{R}^2$, we see that

$$\pi(q_1 \cdot sV_1 \cdot W) = p + sV_1 + W = p + tV_2,$$

i.e., $W = tV_2 - sV_1$ and

$$sV_1 \cdot (tV_2 - sV_1) \cdot (-tV_2) = (-q_1) \cdot q_2 = (z_2 - z_1)Z.$$

By the Baker–Campbell–Hausdorff formula, $V \cdot W = V + W + \frac{1}{2}[V,W]$, so

$$sV_1 \cdot (tV_2 - sV_1) \cdot (-tV_2) = \frac{1}{2}[sV_1, tV_2 - sV_1] + \frac{1}{2}[tV_2, -tV_2] = \frac{st}{2}[V_1, V_2].$$

We calculate that $[V_1, V_2] = wZ$, where $w = \frac{1}{2}(a_1 - a_2 + b_1 - b_2) > 0$, so $z_2 - z_1 = wst$.

By (22), there is a horizontal line from $q_1 V_1^{s_0} = (-1, 2a_1, a_1)$ to $q_2 V_2^{t_0} = (1, -2b_2, b_2)$, so $z_2 - z_1 = ws_0 t_0$. Therefore, $p_1 V_1^s$ is connected to $p_2 V_2^t$ by a horizontal line if and only if $st = s_0 t_0$. But there are no $s, t \in (s_0, t_0)$ such that $st = s_0 t_0$, so there are no horizontal lines connecting R_1 and R_2. $\square$

This implies that Σ_ρ is an intrinsic graph, since any coset of $\langle Y \rangle$ intersects Σ_ρ at most once. We claim that Σ_ρ is an intrinsic graph over $K = [-1,1] \times \{0\} \times \mathbb{R} \subset V_0$. By (4),

$$\Pi([(-1, 2z_1, z_1)(1, -2z_2, z_2)]) = \{(x, 0, g_{z_1,z_2}(x)) : x \in \mathbb{R}\}$$

where

$$\begin{aligned} g_{z_1,z_2}(x) = 2z_1 - 2z_1(x+1) + \frac{z_1 + z_2}{2}(x+1)^2 &= \frac{z_1}{2}\left((x+1)^2 - 4(x+1) + 4\right) + \frac{z_2}{2}(x+1)^2 \\ &= \frac{z_1}{2}(x-1)^2 + \frac{z_2}{2}(x+1)^2. \end{aligned} \tag{23}$$

Therefore, $\Pi(F_\rho(x,z)) = (x, 0, g_{z,\rho(z)}(x))$. Since $\rho(z)$ is surjective, $z \mapsto g_{z,\rho(z)}(x)$ is surjective for any $x \in [-1,1]$. We conclude that $\Pi(\Sigma_\rho) = K$.

By Lemma 6.1, the epigraph Σ_ρ^+ is monotone in U. Proposition 3.9 implies that Σ_ρ is area-minimizing in U. This proves Theorem 1.2.

Finally, we calculate the area of Σ_ρ; since Σ_ρ is area-minimizing in U, the area of $F_\rho([-1,1] \times [a,b])$ should depend only on a, b, $\rho(a)$ and $\rho(b)$. Suppose that ρ is Lipschitz. By the area formula (3), for any $a < b$,

$$\begin{aligned} \operatorname{area} F_\rho([-1,1] \times [a,b]) &= \int_a^b \int_{-1}^1 \sqrt{1 + (z + \rho(z))^2} \cdot \left| J[\Pi \circ F_\rho](x,z) \right| \mathrm{d}x\, \mathrm{d}z \\ &= \int_a^b \sqrt{1 + (z + \rho(z))^2}\, j(z)\, \mathrm{d}z, \end{aligned}$$

where $j(z) = \int_{-1}^1 \left| J[\Pi \circ F_\rho](x,z) \right| \mathrm{d}x$. By (23), we have

$$j(z) = \int_{-1}^1 \partial_z [g_{z,\rho(z)}(x)]\, \mathrm{d}x = \int_{-1}^1 \frac{1}{2}(x-1)^2 + \frac{\rho'(z)}{2}(x+1)^2\, \mathrm{d}x = \frac{4}{3}(1 + \rho'(z)),$$

so, by substitution,

$$\operatorname{area} F_\rho([-1,1] \times [a,b]) = \frac{4}{3} \int_a^b \sqrt{1 + (z + \rho(z))^2}(1 + \rho'(z))\, \mathrm{d}z = \frac{4}{3} \int_{a+\rho(a)}^{b+\rho(b)} \sqrt{1 + m^2}\, \mathrm{d}m.$$

7. Conjectural minimizers

In this section, we construct the surfaces shown in Figure 2, which we conjecture to be area-minimizing or energy-minimizing competitors for BP_u. We first construct the family of conjectural energy minimizers. Given an intrinsic Lipschitz graph Γ_f such that f is defined on an open subset $U \subset V_0$, we define the *intrinsic Dirichlet energy* of f on U by $E_U(f) = \frac{1}{2} \int_U (\nabla_f f)^2\, \mathrm{d}\mu$, where μ is Lebesgue measure on V_0. We say that f

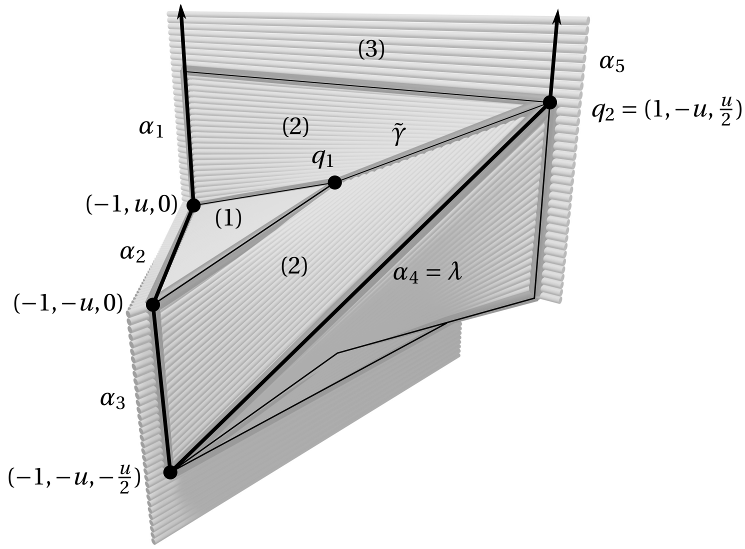

FIGURE 3. A schematic of Σ_u^h with important features labeled.

is *energy-minimizing* on U if, for any $r > 0$, we have $E_{U\cap B(\mathbf{0},r)}(f) \leqslant E_{U\cap B(\mathbf{0},r)}(g)$ for any intrinsic Lipschitz function g such that $g - f \in C_c^0(U \cap B(\mathbf{0}, r))$.

An intrinsic Lipschitz graph is *harmonic* if it is a critical point of the energy with respect to contact deformations, which are smooth deformations of $\mathbb{H}$ that preserve the horizontal distribution. These graphs were studied in [You20]. Harmonic graphs can often be written as unions of horizontal lines which meet along horizontal curves. These graphs satisfy a slope condition; if two horizontal segments intersect on a horizontal curve, then the slope of the curve is the average of the slopes of the horizontal segments.

Let $W_u := \mathrm{BP}_u \cap \{(x, y, z) : |x| \leqslant 1\}$ and let $\partial_\pm W_u = \mathrm{BP}_u \cap \{x = \pm 1\}$ be the two curves bounding W_u. We will construct a harmonic intrinsic graph Σ_u^h which is bounded by ∂W_u and ruled away from two singular curves; we conjecture that Σ_u^h is an energy-minimizing filling of ∂W_u.

The idea behind the construction of Σ_u^h is that there is a horizontal segment $\lambda = [(-1, -u, -\frac{u}{2}), (1, -u, \frac{u}{2})]$ connecting $\partial_- W_u$ to $\partial_+ W_u$. This separates ∂W_u into two parts: one consisting of λ and the upper portions of $\partial_\pm W_u$ and one consisting of λ and the lower portions of $\partial_\pm W_u$. We will fill each part by a harmonic intrinsic graph.

Let $\alpha = \alpha(u)$ be the curve marked by a thick line in Figure 3. This is the curve made up of five segments $\alpha_1, \dots, \alpha_5$,

$$\alpha = (-1, u, \infty) \xrightarrow{\alpha_1} (-1, u, 0) \xrightarrow{\alpha_2} (-1, -u, 0) \xrightarrow{\alpha_3} \left(-1, -u, -\frac{u}{2}\right) \xrightarrow{\alpha_4} \left(1, -u, \frac{u}{2}\right) \xrightarrow{\alpha_5} (1, -u, \infty).$$

(That is, $\alpha = \alpha_1 \cup \cdots \cup \alpha_5$, where α_1 is the vertical ray pointing upward from $(-1, u, 0)$, $\alpha_2 = [(-1, u, 0), (-1, -u, 0)]$, and so on.) Here, α_1, α_3, and α_5 are vertical, α_4 is horizontal, and α_2 is neither.

The formula for the graph filling α is complicated, so we describe it in terms of its horizontal foliation. Let $q_1 = (0, -\frac{u}{2}, \frac{u}{4})$. For any $y \in [-u, u]$, we have $q_1 \cdot (-1, y + \frac{u}{2}, 0) = (-1, y, 0)$, so there is a horizontal line from q_1 to any point in α_2. Let $q_2 = (1, -u, \frac{u}{2})$; then

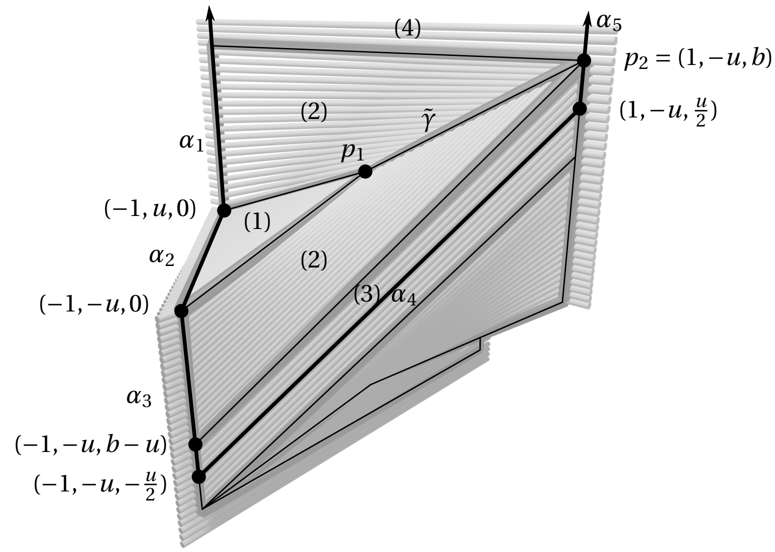

FIGURE 4. A schematic of Σ_u, with important features labeled.

$q_1 \cdot (1, -\frac{u}{2}, 0) = q_2$, so q_1 and q_2 are connected by a horizontal segment. The segment $[q_1, q_2]$ will be the characteristic set of Σ_u^h.

We write Σ_u^h as a union of three families of horizontal segments, labeled (1), (2), and (3) in Figure 3.

(1) Segments connecting q_1 to the points of α_2.
(2) For each point $p \in [q_1, q_2]$, segments connecting p to a point on α_1 and a point on α_3.
(3) Segments with slope $-u$ connecting $(-1, u, z)$ to $(1, -u, z)$ for $z \in [\frac{u}{2}, \infty)$.

For each $t \in [0,1]$, the second family contains two horizontal segments connecting $q_1 \cdot (1, -\frac{u}{2}, 0)^t$ to α_1 and α_3. The projections of these segments to A connect $(t, -\frac{u}{2}(t+1))$ to $(-1, \pm u)$, so they have slope $-\frac{u}{2} \pm \frac{u}{t+1}$, while the characteristic nexus of Σ_u^h has slope $-\frac{u}{2}$. That is, Σ_u^h satisfies the slope condition in [You20]. By Remark 3.10 of [You20], Σ_u is an intrinsic Lipschitz graph, so Σ_u^h is a harmonic graph. The full surface in Figure 2 is a union $\Sigma_u^h \cup s_{-1,1}(\Sigma_u^h)$ of two copies of Σ_u^h, and we conjecture that this is the minimal-energy surface filling ∂W_u.

Note that for any $u > 0$, we can write $\Sigma_u^h = s_{1,u}(\Sigma_1^h)$. Stretch automorphisms send energy minimizers to energy minimizers [You20, 3.2], so proving that Σ_1^h minimizes energy would imply that all of the Σ_u^h's minimize energy.

Now we construct conjectural area-minimizing surfaces Σ_u filling ∂W_u. These look similar to the harmonic surfaces constructed above; the main differences are that the characteristic set is the horizontal lift of a hyperbola rather than a horizontal line segment and that Σ_u contains a family of horizontal segments of slope 0, all parallel to α_4 (labeled (3) in Figure 4).

We construct Σ_u out of area-minimizing Z–graphs and vertical rectangles. As described in [CHY07], an area-minimizing Z–graph Σ can be written as a union of horizontal line segments. These segments can meet along horizontal curves in Σ as long as

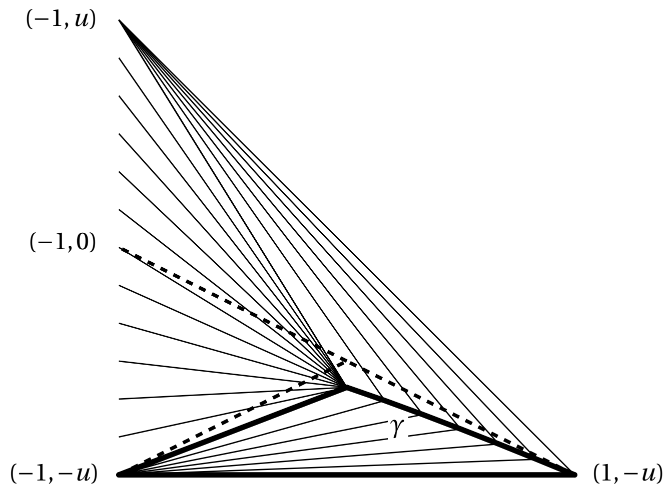

FIGURE 5. The projections to A of the horizontal segments that make up Σ_u.

the tangent line to the horizontal curve bisects the angle between the horizontal line segments. Such singularities form the characteristic set of the surface.

Figure 5 shows the projection to the xy–plane A of the horizontal segments that make up Σ_u. The characteristic set of Σ_u, marked by one of the thick lines in Figure 5, is a lift of a segment of the hyperbola in A with foci at $(-1,\pm u)$ that passes through the point $(1,-u)$. We call this hyperbola γ; one can calculate that γ satisfies the equation

$$\frac{y^2}{(\sqrt{u^2+1}-1)^2}-\frac{(x+1)^2}{u^2-(\sqrt{u^2+1}-1)^2}=1.$$

Let

$$a=-\left(\sqrt{u^2+1}-1\right)\sqrt{\frac{1}{u^2-(\sqrt{u^2+1}-1)^2}+1}$$

so that $(0,a)\in\gamma$. Let $L=[(-1,0),(1,-u)]$ be the long dashed line in Figure 5. The segment of γ from $(0,a)$ to $(1,-u)$ lies below L, so $a<-\frac{u}{2}$.

We construct Σ_u by lifting the lines in Figure 5 to $\mathbb{H}$. Let $p_1=(0,a,\frac{a}{2})$. Then for any $y\in[-u,u]$, we have $p_1\cdot(-1,y-a,0)=(-1,y,0)$, so the lines on the left of Figure 5 lift to horizontal lines from the points of α_2 to p_1. Let $\tilde\gamma$ be the lift of γ that passes through $p_1=(0,a,\frac{a}{2})$, and let b be such that $p_2=(1,-u,b)\in\tilde\gamma$. The concatenation of $[(-1,-u,0),p_1]$, $\tilde\gamma$, and $[p_2,p_2X^{-2}]$ is a horizontal curve in Σ_u that connects $(-1,-u,0)$ to $p_2X^{-2}=(-1,-u,b-u)$ and projects to the thick curve in Figure 5. The area of this curve is equal to $u-b$, and the curve is contained in the triangle with vertices $(-1,-u),(1,-u)$, and $(0,-\frac{u}{2})$ (marked by dashed lines in Figure 5), so $0<u-b<\frac{u}{2}$ and $\frac{u}{2}<b<u$. That is, the endpoint of $\tilde\gamma$ lies in the interior of α_3, as illustrated in Figure 4.

The horizontal segments in Σ_u can be divided into four families, labeled (1), (2), (3), and (4) in Figure 4:

(1) Segments connecting p_1 to the points of α_2.
(2) For each point $p\in\tilde\gamma$, segments connecting p to a point on α_1 and a point on α_3. The projections of these segments to A connect points on the hyperbola γ to its two foci.

(3) Segments with slope 0 connecting $(-1,-u,z-u)$ to $(1,-u,z)$ for $z \in [\frac{u}{2}, b]$.
(4) Segments with slope $-u$ connecting $(-1,u,z)$ to $(1,-u,z)$ for $z \in [b,\infty)$.

Since the tangent line to a hyperbola at a point bisects the lines from the point to the foci of the hyperbola, this surface satisfies the angle condition in Section 7 of [CHY07] and is thus an area-minimizer. The full surface in Figure 2 is a union $\Sigma_u \cup s_{-1,1}(\Sigma_u)$ of two copies of Σ_u, and we conjecture that this is the area-minimizing surface filling ∂W_u.

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Availability of data and material The 3D models used in the figures in this paper can be found in .obj format in the supplementary material for this paper.

Code availability Not applicable

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