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Eminence in Noisy Bilinear Networks

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***Abstract*— When measuring nodes' importance in a network, the interconnections and dynamics are often supposed to be perfectly known. In this paper, we consider networks of agents with both uncertain couplings and dynamics. The network uncertainty is modeled by structured additive stochastic disturbances on each agent's update dynamics and coupling weights. We then study how these uncertainties change the network's centralities. Disturbances on the couplings between agents result in bilinear dynamics, and classical centrality indices from linear network theory need to be redefined. To do that, we first show that, similarly to its linear counterpart, the squared $\mathcal{H}_2$ norm of bilinear systems measures the trace of the steady-state error covariance matrix subject to stochastic disturbances. This makes the $\mathcal{H}_2$ norm a natural candidate for a performance metric of the system. We propose a centrality index for the agents based on the $\mathcal{H}_2$ norm, and show how it depends on the network topology and the noise structure. Finally, we simulate a few graphs to illustrate how uncertainties on different couplings affect the agents' centrality rankings compared to a linearized model of the same system.**

I. Introduction

In the study of complex networks, the use of centrality indices is a standard method to rank the importance of their nodes and edges, and depending on the application, different indices become relevant [1]–[6]. In social networks, it is often important to be connected to as many different nodes as possible; hence the degree centrality is often used [7]. However, in an information network, one is often more interested in the nodes through which more information has to pass, requiring a different centrality definition.

When studying networks of dynamic systems, one is often interested in evaluating the system's performance under random disturbances. In [1] and [2], the authors define a performance metric from the covariance of the states, and rank a node based on how much a disturbance on its states affect the rest of the network. An important assumption often made when studying dynamical networks with uncertainties is the perfect knowledge of the interconnections [8]–[10]. In this case, the uncertainties on the agents' states propagate linearly to its neighbors, which allows the use of consolidated tools from linear system analysis.

However, the interconnection between agents is as much subject to the modeling process as the agents themselves. In population dynamics, for example, the movement of people between cities depends on time (e.g., day of the week, holidays, etc.) and highly volatile factors (e.g., gas and ticket prices, cultural events), which are usually best represented by stochastic models [11]. When modeling a network based on real measured data, estimations of those interconnections are uncertain, and they can change the importance of the nodes when compared to a network with no uncertainty.

Uncertainties on the interconnections between agents appear in the dynamic model as bilinear terms, meaning that the tools from the linear system analysis theory need to be carefully shown to still hold for bilinear systems. Particularly, the extension of the $\mathcal{H}_2$ norm for bilinear systems, commonly used in the context of model order reduction [12], can be used to build a lower bound for the controllability energy function of the system [13], [14]. Concepts such as transfer functions and impulse response can also be generalized for bilinear systems [15], allowing us to properly describe the input-output relationship of such systems. Similarly, bilinear dynamics are well studied in the stochastic system's literature. In [16], the authors compute the mean and second momentum for continuous bilinear stochastic systems and present conditions for their stability, some of which we use on this paper. In [17], the author looks at discrete bilinear stochastic systems, making similar analysis regarding its mean and covariance, and input-output representation.

This paper focus on networks of dynamic systems with uncertain interconnections. This particular type of uncertainty appears as a bilinear term in the system's dynamics, and we relate results on the covariance of a stochastic bilinear system to its $\mathcal{H}_2$ norm. We then propose a $\mathcal{H}_2$-based centrality index for the nodes and show how you can compute it by solving a Linear Matrix Equality (LME) that is a function of the system's matrices and of the interconnections' uncertainties.

II. Preliminary Definitions

A. Bilinear Dynamic Networks

Let $I_{n\times n}$ and $\mathbb{1}_{n\times n}$ be the $n\times n$ identity matrix and matrix of all ones, respectively. Let e_i be the elementary vector with all elements zero, except the i-th one, which is one, then $E_{ij}=e_ie_j^\top$ is called an elementary matrix and has all elements zero except for a one in position ij.

Define $\mathcal{V}\subset\mathbb{N}$ as a set of n vertices (i.e. $|\mathcal{V}|=n$), each one representing a dynamic system, all with $\tilde{n}$ states and $\tilde{o}$ outputs. Set $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ to be the set of m edges (i.e. $|\mathcal{E}|=m$), representing the interconnections between the systems. A linear graph $\mathcal{G}$ is the triple $\mathcal{G}=(\mathcal{V},\mathcal{E},w)$ where the function $w:\mathcal{E}\to\mathbb{R}^{\tilde{n}\times\tilde{o}}$ is the edge weight function. We define the adjacency matrix $A\in\mathbb{R}^{m\tilde{n}\times m\tilde{o}}$ of the graph as the block matrix with $w(i,j)$ in block ij if there is an edge from node i to node j and a $\tilde{n}\times\tilde{o}$ matrix of all zeros otherwise.

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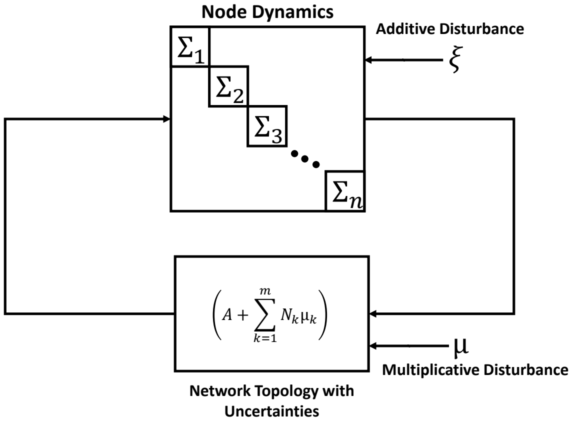


Fig. 1: Block diagram of (6), illustrating the relationship between SISO (Single Input, Single Output) node dynamics and the network topology for additive and multiplicative disturbances.

The edges connect the outputs of a node to the inputs of another. Generically we can describe the dynamics of a node i as

$$\Sigma_i : \begin{cases} \dot{x}_i &= f(x_i) + \sum_{j \to i} w(j,i) y_j \\ y_i &= g(x_i), \end{cases} \tag{1}$$

where $x_i \in \mathbb{R}^{\tilde{n}}$, $y_i \in \mathbb{R}^{\tilde{o}}$, and $j \to i$ denotes an edge that points from node j to node i. From this we can define additive and multiplicative disturbances as below.

Definition 1. *An* additive disturbance *at a node i of a graph consist of a vector of independently sampled Gaussian disturbances $\xi_i \in \mathbb{R}^{\tilde{m}}$, and a matrix $B_i \in \mathbb{R}^{\tilde{n} \times \tilde{m}}$. The dynamics of the system with additive disturbances is*

$$\Sigma_i : \begin{cases} \dot{x}_i &= f(x_i) + \sum_{j \to i} w(j,i) y_j + B_i \xi_i \\ y_i &= g(x_i). \end{cases} \tag{2}$$

Definition 2. *A* multiplicative disturbance *at an edge (i, j) of a graph is a matrix $\mu_{ij} \in \mathbb{R}^{\tilde{n} \times \tilde{o}}$ where each of its elements are independently sampled Gaussian disturbances with zero mean. The dynamics of a system with multiplicative disturbances is*

$$\Sigma_i : \begin{cases} \dot{x}_i &= f(x_i) + \sum_{j \to i} (w(j,i) + \mu_{ji}) y_j \\ y_i &= g(x_i). \end{cases} \tag{3}$$

We then formally define a bilinear digraph, or a graph under stochastic node and link disturbances as follows.

Definition 3. *A* bilinear digraph *is a quintet $\mathcal{G} := (\mathcal{V}, \mathcal{E}, w, \mathcal{E}_a, \mathcal{V}_a)$ where $\mathcal{V} = \mathbb{N}_{\leq n}$ (that is, the set of natural numbers smaller or equal to n), for some $n \in \mathbb{N}$, and is called a node set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is called an edge set, $\mathcal{V}_a \subseteq \mathcal{V}$ is called disturbed node set, $\mathcal{E}_a \subseteq \mathcal{E}$ is called disturbed edge set, and $w : \mathcal{E} \to \mathbb{R}$ is a edge weight function.*

For the rest of the paper we consider a SISO scalar system as the dynamics of the nodes ($\tilde{n} = 1$, $\tilde{o} = 1$ and $\tilde{m} = 1$). More complex linear dynamics can be considered by properly stacking the matrices of the systems, specifically, if the nodes all have the same order, number of inputs and output we can easily compose them by using the Kronecker product and obtain the network dynamics. We also assume that every node and edge are under independent disturbances, that is, $\mathcal{V}_a = \mathcal{V}$ and $\mathcal{E}_a = \mathcal{E}$. For the purposes of this paper, the particular case where some node or edge is not disturbed can be dealt with by making $\xi_i = 0$ or $\mu_{ij} = 0$. The dynamics of the nodes, then, simplifies to

$$\Sigma_i : \begin{cases} \dot{x}_i &= d_i x_i + \sum_{j \to i} (w(j,i) + \mu_{ji}) x_j + b_i \xi_i \\ y_i &= x_i, \end{cases} \tag{4}$$

for d_i, b_i, μ_{ji} and $\xi_i \in \mathbb{R}$ and $w : \mathcal{E} \to \mathbb{R}$. Without loss of generality we assume $b_i = 1$. With this, the network dynamics is given by

$$\begin{aligned} \dot{x} = & \left(\overbrace{D + \underbrace{\sum_{i=1}^{n} \sum_{i \to j} w(i,j) E_{ji}}_{A}}^{N_0} \right) x \\ & + \sum_{i=1}^{n} \left(\sum_{i \to j} E_{ji} \mu_{ij} x + e_i \xi_i \right), \end{aligned} \tag{5}$$

where $x = [x_1,\ x_2,\ \ldots,\ x_n]^\top$, $D = \mathrm{diag}(d_1,\ d_2,\ \ldots,\ d_n)$ and A is the adjacency matrix of the network. Consider a one to one mapping from $\mathcal{E}$ to $\mathbb{N}_{\leq m}$, then we can rewrite (5) as

$$\dot{x} = N_0 x + \sum_{k=1}^{m} N_k \mu_k x + B\xi, \tag{6}$$

where $N_0 = D + A$, $N_k = E_{j_k i_k}$, $B = [e_1,\ e_2,\ \ldots, e_n] = I_{n \times n}$, and $\xi = [\xi_1,\ \xi_2,\ \ldots,\ \xi_n]^\top$.

We bring attention to the fact that the presented dynamics is a specific case of the general bilinear system

$$\dot{x} = N_0 x + \sum_{k=1}^{\overline{m}} N_k \overline{u}_k x + \overline{B}\overline{u}, \tag{7}$$

where $\overline{u} = [\mu^\top,\ \xi^\top]^\top$, $\overline{m} = m + n$, $N_k = 0$ for $k > m$ and $\overline{B} = [0_{n \times m},\ I_{n \times n}]$. In Fig. 1 we show the block diagram of the relationship between the node dynamics and the network topology. Finally, we assume throughout this paper that for any considered system, the following assumption holds:

Assumption 1. *Matrices N_0 and*

$$\left(N_0 \otimes I + I \otimes N_0^\top + \sum_k N_k \otimes N_k^\top \right)$$

are Hurwitz, that is, both have eigenvalues with strictly negative real parts.

In [18] and [19], the authors make a similar assumption, presented below.

Assumption 2. *The matrix N_0 is stable and for two numbers α and β, which satisfy the inequality $\|e^{N_0 t}\| \leq \beta e^{-\alpha t}$ for all $t > 0$, we have $\sqrt{\sum_{k \in \mathcal{E}_a} \|N_k N_k^\top\|} < \sqrt{2\alpha}/\beta$.*

It can be verified that Assumption 2 implies Assumption

1, but the converse is not true. It was known for some time that for Gaussian inputs, Assumption 1, tighter than 2, is sufficient for evaluating the covariance of the states (see proof of Theorem 1). Recently in [12] the author showed that even in the deterministic case Assumption 1 is sufficient for the proper definition of the $\mathcal{H}_2$-norm, improving previous results, like from [19].

B. The $\mathcal{H}_2$-norm of Bilinear Systems

This section presents general bilinear system results that will be useful when analysing bilinear networks. In [18] we discussed how one can compute the $\mathcal{H}_2$ norm of a bilinear system and how it compares with the very well known $\mathcal{H}_2$ norm of linear systems.

For linear systems, the $\mathcal{H}_2$ norm is the $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain and it can be characterized as the energy of the impulse response of deterministic systems, and proportional to the covariance of the states when subject to white noise inputs [20].

For bilinear system, the $\mathcal{H}_2$ norm is defined as

$$\|\Sigma\|_{\mathcal{H}_2} = \left(\sum_{i=1}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \operatorname{trace}(H_i^\top(jw_1, \ldots, jw_i) \times H_i(jw_1, \ldots, jw_i)) dw_1 \ldots dw_i \right)^{1/2}, \tag{8}$$

where the H_is are the multivariable Laplace transforms of the Volterra kernels of the bilinear system [15], [18], [21].

When defined this way, the $\mathcal{H}_2$ norm is shown to be a lower bound for the controlability energy functional of the bilinear system [13], [14]. Furthermore, based on results from the literature [16], [17] we collect the following result for bilinear system subject to white noise inputs:

Theorem 1. *For a bilinear stochastic differential equation (SDE) as in* (7) *that satisfies Assumption 1, we can say that*

$$\lim_{t\to\infty} \mathbb{E}(x(t)) = 0, \tag{9}$$

and

$$\lim_{t\to\infty} \operatorname{Cov}(x(t)) = P, \tag{10}$$

where P is the solution of the generalized Lyapunov Equation

$$N_0 P + P N_0^\top + \sum_{k=1}^{m} N_k P N_k^\top + BB^\top = 0. \tag{11}$$

We ommit the proof for brevity since this is a well known result in the stochastic systems literature (see [16], [17], but one can easily verify (9) by taking the expected value of the integral form of the bilinear stochastic differential equation. From there we can conclude that the covariance matrix converges to the second momentum as times goes to infinity, which results in (10) and (11). Through some manipulation of the equations for the second momentum, one can also verify that Assumption 2 implies 1.

Based on this theorem, and in results from [19], that show that for a bilinear system (7) with output $y = x$

$$\|\Sigma\|_{\mathcal{H}_2}^2 = \operatorname{trace}(P), \tag{12}$$

next we develop a $\mathcal{H}_2$ norm based centrality measure to evaluate the importance of the nodes of a bilinear network subject to additive and multiplicative Gaussian disturbances.

III. $\mathcal{H}_2$-Based Centrality Index for Bilinear Dynamic Networks

For a network, a centrality index evaluates the importance of its component with regard to some performance measure. Degree centrality, for example, classifies a node with respect to how many direct connections it has, which might be a relevant in some applications, but insufficient in others.

In [1], the authors explore a centrality based on the $\mathcal{H}_2$ norm as a performance measure. This centrality evaluates how much each node disturbance contributes for the covariance of the system around a consensus and can be used to minimize the effect of noise in the final system's state. In this section we explore a similar idea, since the $\mathcal{H}_2$ norm of a bilinear system measures its steady state covariance. We propose an index to evaluate the effect of node disturbances in the network by derivating the performance metric with respect to the covariance of said disturbance. While the resulting metric is too complex to compute for arbitrary disturbances, it simplifies to a new generalized Lyapunov equation for additive ones.

A. Node Centrality in Noisy Bilinear Networks

To evaluate the effect of additive noise in a bilinear network with dynamics (6) subject to Gaussian white noise inputs, define the following performance measure:

$$\rho_{\text{ss}} := \lim_{t\to\infty} \operatorname{trace}\left(\operatorname{Cov}(x(t))\right) = \operatorname{trace}(P) = \|\Sigma\|_{\mathcal{H}_2}^2. \tag{13}$$

This measure evaluates how much the system oscillates around the equilibrium in steady state. As such, networks with larger ρ_{ss} would perform badly when subject to noisy inputs. With this consider the following definition

Definition 4. *For bilinear network* (7)*, assume $\bar{u}_k \sim N(0, \sigma_k^2)$. Then, the centrality index associated with such input is given by*

$$\eta_k := \frac{\partial \rho_{ss}}{\partial \sigma_k^2}. \tag{14}$$

Notice that, here, σ represents the standart deviation of the inputs, and not singular values. Quantity η_k measures the direct influence of the covariance of a given input on the covariance of the state. For linear networks this index depends only on the edge weights and on the topology of the network. For additive inputs in bilinear networks we can state the following theorem:

Theorem 2. *For a bilinear network with dynamics* (6)*, where $\mu_k \sim N(0, \sigma_k^2)$ and $\xi_k \sim N(0, \sigma_{m+k}^2)$ the centrality index* (14) *for $k > m$ can be computed as*

$$\eta_k = \operatorname{trace}(\overline{P}_k), \tag{15}$$

where $\overline{P}_k$ is the solution of the generalized Lyapunov equa-

tion

$$N_0\overline{P}_k + \overline{P}_k N_0^\top + \sum_{q=1}^{m} \sigma_q^2 (N_q \overline{P}_k N_q^\top) + b_k b_k^\top = 0, \quad (16)$$

where b_k is the k-th column of $\overline{B}$, or the $(k-m)$-th column of B.

The proof is omitted for the brevity of the paper. For a sketch of the proof, first notice that for a matrix B and a vector of linear inputs ξ as defined above, we can write

$$B\xi = \sum_{k=1}^{n} b_k \xi_k = \sum_{k=1}^{n} b_k \sigma_{m+k} \xi_0 = B\Delta\bar{\xi}, \quad (17)$$

where $\xi_0 \sim N(0,1)$, $\bar{\xi} \in \mathbb{R}^{n\times m}$ has all its elements equal to ξ_0, and Δ is the diagonal matrix with σ_{k+m} for k between 1 and n as its diagonal entries. From this we point that the Gramian P can be computed as a function of the variance of the inputs, σ_k and the results of the theorem follow from doing such computation. Since the change on the systems matrices occur only on B for the additive disturbances, they do not affect Assumptions 1 or 2.

Remark 1. If $N_k = 0$ for all $k > 0$, then the steady state covariance matrix of the states of the system is given by

$$\mathrm{Cov}(x) = \int_0^\infty \mathrm{e}^{N_0 t} B B^\top \mathrm{e}^{N_0^\top t} dt,$$

that is the Gramian of the linear system. The performance metric, then, is given by the trace of the Gramian, which is the $\mathcal{H}_2$ norm of the linear system. This means that for the case of no disturbances on the edges, our centrality index coincides with the one specified in [1] for linear systems. For the rest of this paper, whenever we write "linear centrality" or "centrality of the linear network" we are referring for the centrality index defined as in [1].

IV. Simulations and Comparisons

In this section we simulate two different graphs and investigate how uncertainties in the edges change the node centralities. Our hope is to make it clearer when the knowledge of the uncertainties of the edges can help on centrality-based decision making, and when it is unecessary.

A. Tree Graph with 20 Nodes

For the first few simulations we randomly generated the 20 node tree graph presented in Fig. 2. All edges are undirected with unitary weight. We simulate system (7) with

$$N_0 = -L - \frac{1}{n}\mathbb{1}_{n\times n}, \quad (18)$$

where L is the Laplacian matrix of the graph, and $N_k = E_{i_k j_k} + E_{j_k i_k}$ for all $(i_k, j_k) \in \mathcal{E}$.

For a first analysis of the network, we compute the centrality without edge disturbances (linear case) and with identical disturbances in all edges (homogeneous edge disturbances). To compute the worst disturbance that can be applied to all edges and still satisfy Assumption 2, we calculate

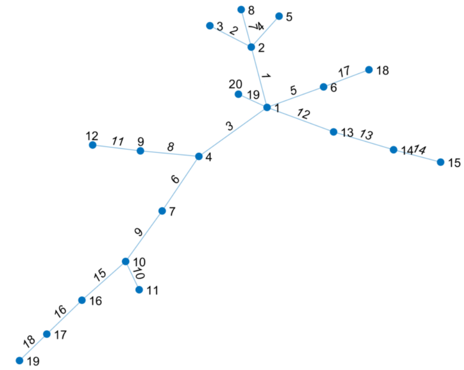


Fig. 2: Tree graph used in the first set of simulations. The numbers are the node and edge labels. All edges have unitary weight.

$$\sigma_{\max} = \sqrt{\frac{2\alpha}{\beta^2 \sum_{k=1}^{m} \|N_k N_k^\top\|}}, \quad (19)$$

for $\alpha = -\lambda_{\max}(N_0) = 1/n$ and $\beta = 1$. The centralities, obtained by simulating the network with homogeneous edge disturbances and $\sigma = \sigma_{\max}$, are presented in Fig. 3a. While there are some noticeable changes in the relative centrality of the nodes (nodes 6, 16, etc) those changes are so small that there is no noticeable change in their order.

Next we increase the magnitude of the disturbances enough to break Assumption 2 but still barely respect Assumption 1. The resulting centralities are presented in Fig. 3b. Notice that the effect of the edge disturbances is much more evident in this simulation, bringing a question about the conservativeness of Assumption 2.

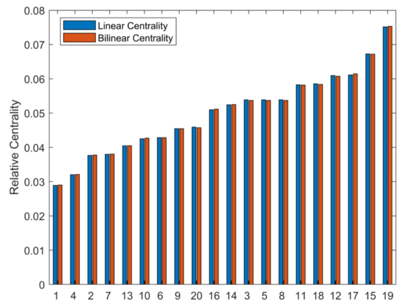
Furthermore, we can see from Fig. 3b that the nodes with the largest increase in relative centrality are 19, 17, 16 and 10, all in the far end of the longest branch of the tree.

To investigate how different edge disturbances change the centrality of the nodes of the network we conducted two simulations where only one edge was disturbed. First we disturbed edge 3, which connects nodes 4 and 1, which are the ones with the smallest linear centrality and farthest away from the long branches of the network. The resulting centralities are presented in Fig. 3c, where σ is selected to respect Assumption 2.

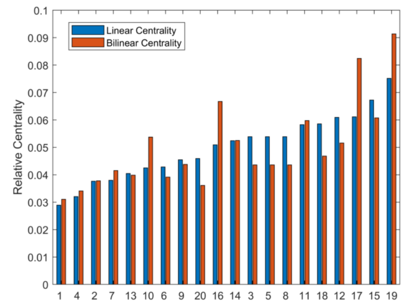
Similarly, we conduct the exact same experiment, but disturbing edge 14 instead. We chose this edge because it is the last one of the second largest branch of the network. By simulating a single disturbance on this edge we can evaluate if it is not an important edge for computing the centralities, or if it was simply overshadowed by other edges in the homogeneous case in Fig. 3b. The results of the simulation are presented in Fig. 3d.

We can see on both Figs. 3c and 3d that the nodes directly connected by the disturbed edge are the ones with the largest increase in the centrality, but it is clear that disturbing edge 14 had a much larger impact on the network than disturbing edge 3, even if both were disturbed by the same σ.

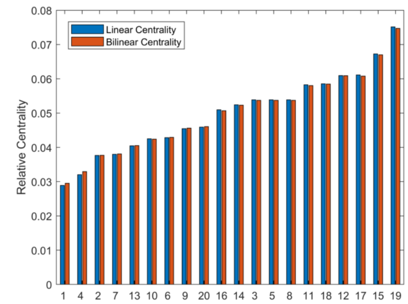
We now look back to Assumption 2 and how conservative it is. While for homogeneous perturbations in Fig. 3a there



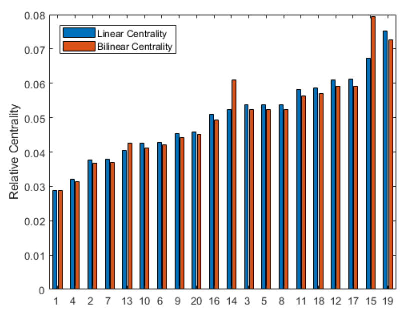
(a) *Bilinear centralities* for homogeneous perturbations on the edges with $\sigma > 0$ satisfying Assumption 2.



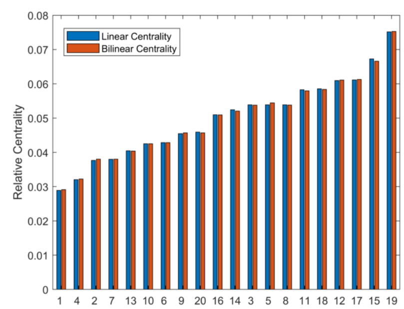
(b) *Bilinear centralities* for homogeneous perturbations on the edges with $\sigma > 0$ satisfying Assumption 1 but not 2.



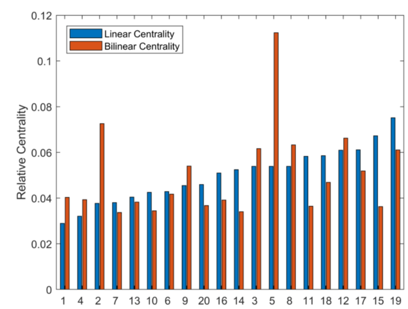
(c) *Bilinear centralities* for single multiplicative perturbation on edge 3 with $\sigma > 0$ satisfying Assumption 2



(d) *Bilinear centralities* for single multiplicative perturbation on edge 14 with $\sigma > 0$ satisfying Assumption 2



(e) *Bilinear centralities* for unbalanced edge perturbations with $\sigma > 0$ satisfying Assumption 2



(f) *Bilinear centralities* for unbalanced edge perturbations with $\sigma > 0$ satisfying Assumption 1 but not 2

Fig. 3: $\mathcal{H}_2$-based centrality for the nodes in the tree graph given by Fig. 2. *Linear centralities* are computed for unperturbed edges, and *Bilinear centralities* are computed for the specific disturbance of each figure. Nodes are ordered according to the linear centralities.

is very little difference between the linear and bilinear centralities, we investigate further by simulating unbalanced perturbations on all edges. To do that we generate a random vector $r \in \mathbb{R}^m$ and calculate a constant $p \in \mathbb{R}$ such that Assumption 2 is respected for $N_k = \sigma_k E_{i_k j_k}$, with $\sigma = pr$. In Fig. 3e, we present the result of this simulation and in Fig. 3f we simulate the same scenario but for a larger value of p that respects Assumption 1 instead of Assumption 2.

We can observe from Figs. 3e and 3f that barely any change was observed in the nodes centralities for disturbances that respect Assumption 2, even if the disturbances are unbalanced. This means that while the topology of the edge disturbances is important, as evident from Figs. 3c and 3d, the magnitude of those disturbances are just as important.

B. Barabási-Albert Graph with 20 nodes

In the next set of simulations we take a look at a randomly generated undirected Barabási-Albert graph with 20 nodes, four initial connected nodes ($m_0 = 4$) and each new node connected to two existing nodes ($m = 2$). The resulting graph is presented in Fig. 4. Similarly to the tree graph, we simulate the system for N_0 and N_k as in (18).

We evaluate again the effect of homogeneous disturbances to verify the robustness of our previous results in different underlying graphs. The results are presented in Figs. 5a and 5b, where one can see that, similarly to Figs. 3a and 3b, the nodes centrality change noticeably only for a value of σ that respects Assumption 1, but breaks Assumption 2. Differently from the simulation for tree graphs, however, we can notice that the bilinear dynamics homogenized the node centralities, a behaviour possibly explained by the difference in topology between a tree graph and a connected Barabási-Albert graph with many loops, as the one we generated.

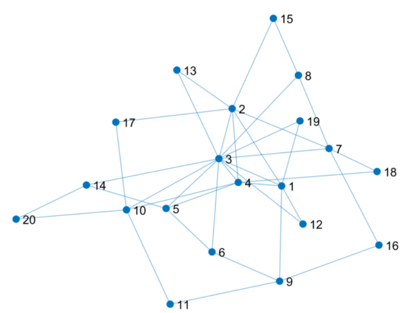
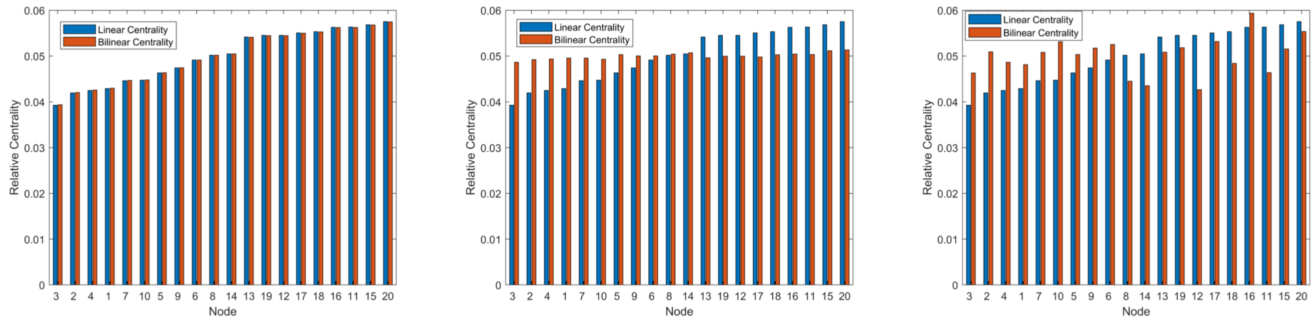


Fig. 4: 20-node Barabási-Albert graph, generated with $m_0 = 4$, $m = 2$. All edges have unitary weights.

In the final simulation we use unbalanced disturbances for a σ that respects Assumption 1 but is large enough to break Assumption 2, our goal is to see if the balancing effect that we observed for homogeneous disturbances is also



(a) (b) (c)

Fig. 5: $\mathcal{H}_2$-based centrality values of the nodes in the Barabási-Albert graph given by Fig. 4. In the Figs. *Linear centralities* are computed for unperturbed edges, and *Bilinear centralities* are computed for the specific disturbance of each figure. Nodes are ordered according to the linear centralities. **(a)** *Bilinear centralities* computed for homogeneous perturbations on the edges with $\sigma > 0$ satisfying Assumption 2. **(b)** *Bilinear centralities* computed for homogeneous perturbations on the edges with $\sigma > 0$ satisfying Assumption 1 but not 2. **(c)** *Bilinear centralities* computed for unbalanced edge perturbations with $\sigma > 0$ satisfying Assumption 1 but not 2.

perceptible for unbalanced disturbances. In Fig. 5c, we can see that for unbalanced disturbances, the order of the nodes has nothing to do with the order of the nodes for the linear case. Even if the results in Fig. 5c are much less balanced than the ones in Fig. 5b, the changes in centrality are still well distributed among the nodes, instead of concentrating on a few as in Fig. 3f.

V. Conclusions

In this paper, we introduced the notion of node centrality for bilinear dynamical networks with uncertain edges/couplings and nodes/dynamics. We showed how the coupling uncertainties appear as bilinear terms in the entire network dynamics. We proposed an $\mathcal{H}_2$-based metric to evaluate the centrality of the nodes. We showed that this centrality measure can be explicitly expressed as a function of the solution of the generalized Lyapunov equation.

We conducted simulations aiming to study the different effects multiplicative disturbances have on the network: homogeneous disturbances; unbalanced disturbances; single edge disturbances. We presented the simulations for two different network topologies and compared, for each simulation scenario, the effects of respecting or not Assumption 2 (that is, the difference of having a dominant linear dynamics or not). For relatively small homogeneous edge disturbances (respecting Assumption 2), the centrality of the nodes did not change significantly, but for values that approached the conditions in Assumption 1 the difference was consistently clearer. These simulations give us insights of when the linear hypothesis might hold and when we should be more careful when considering the uncertainties on the edges.

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