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Learning-based Sensor Selection with Guaranteed Performance Bounds

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Abstract— **In this paper, we consider the problem of sensor selection for discrete-time linear dynamical networks. We develop a framework to design a sparse sensor schedule for a given large-scale linear system with guaranteed performance bounds using a learning-based algorithm. To sparsify the sensors in both time and space, we build our combinatorial optimization problems based on the notion of systemic controllability/observability metrics for linear dynamical networks with three properties: monotonicity, convexity, and homogeneity with respect to the controllability/observability Gramian matrix of the network. These combinatorial optimizations are inherently intractable and NP-hard. However, solving a continuous relaxation for each optimization is considered best practice. This is achievable since we constructed the objective based on the systemic metrics, which are convex. Furthermore, by leveraging recent advances in sparsification literature and regret minimization, we then round the fractional solution obtained by the continuous optimization to achieve a $(1+\epsilon)$ approximation sparse schedule that chooses on average a constant number of sensors at each time, to approximate all types of systemic metrics (cf. Table I).**

I. INTRODUCTION

Scientists and researchers have always been intrigued by the challenge of controlling and estimating large-scale complex networks such as power networks [1], social networks, and biological and genetic regulatory networks [2]–[4]. More recently, development of algorithms and the availability of portable computer storage as well as high-performance processors have sparked a new surge of interest within the control community to study and analyze these complex dynamical networks. Although classical concepts of observability and controllability are almost axiomatic now, there are still numerous ambiguities in the network controllability and observability. For example, the dependence of different measures of controllability or observability on the location of actuators and sensors in the large networks is not completely known. Given the increasingly large-scale nature of the problems presented by attempts to control and estimate these networks, the need to estimate the overall state of the system and to control it using a small subset of available sensors and actuators is inevitable. Moreover, a fully actuated or sensed network control system might not be practical, require unreasonable cost, or use too much energy. The need to have a small set of actuators/sensors might be crucial in some applications, such as multi-agent robotic networks, because of limited battery resources, communication bandwidth, and computation capacity [5] or power grid state estimation due to increasing cost of monitoring of the network for systemic failures, etc.

All in all, it is beneficial to maintain a few actuators and sensors; however, in this sparsification, we need to preserve most of the control or estimation performance. The problem of sparsifying actuators/sensors such that the resulting system is controllable/observable is NP-hard [6], [7]. Therefore, different algorithms have been developed to solve this problem [8]–[10]. The authors of [11] developed a framework to find a time-varying sparse actuator/sensor schedule while ensuring the sparse system has a controllability/observability performance that closely resembles that of the original system (i.e., fully actuated/sensed system). Later, [12] designed a time-varying joint sensors/actuators selection strategy to choose an average constant number of sensors and actuators to approximate the Henkel singular values of the system . It is shown in [13] that the separation principle holds for the Linear-Quadratic-Gaussian (LQG) control problem. More recently in [14], the authors leverage balanced model reduction and greedy matrix QR pivoting to efficiently perform sensor and actuator selections that optimize observability and controllability. A sufficiency condition of static output feedback stabilizability is exploited to achieve the minimal set of sensors and control nodes needed to stabilize an unstable network in [15].

A key observation is the close connection between the problem of actuator/sensor sparsification and some classical mathematics and statistics such as matrix low rank approximation, outlier detection, active learning, and optimal experimental design. In recent years, experimental design, which first emerged sixty-five years ago [16], has been the subject of considerable study again.

We propose a framework to solve the time-varying sensor selection problem based on new advances in theoretical computer science and machine learning. We show that the problem can be solved using a polynomial-time learning-based algorithm known as *regret* minimization. Similar to [11], we first introduce a notion of systemic controllability/observability metrics for discrete-time linear dynamical networks. These metrics are defined such that they are monotone, convex, and homogeneous with respect to the controllability/observability Gramian matrix of the network. It is shown that several popular and widely used optimality criteria in experimental design, including A(verage), D(eterminant), T(race), E(igen), V(ariance) and G-optimality

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(cf. Table I), belong to this class of systemic measures. Determining the exact solution to optimize these criteria, except the T-optimality, is challenging. More specifically, for certain instances of D/E-optimality, the optimization is proven to be NP-hard [17].

We use the notion of systemic metric to maintain the controllability/observability performance of the system in the desired level while sparsifying the actuators/sensors in online learning and regret minimization. Although the combinatorial optimizations built upon the systemic metrics are NP-hard, the continuous relaxation is solvable since the metrics are defined to be convex. Then, to find the sparse actuator/sensor schedule, one should round the continuous solution (sometimes called fractional solution[1]) obtained from continuous relaxation to integer ones. We propose to use the promising learning-based algorithm from theoretical computer science, regret minimization, to develop a scalable framework to round the fractional solution and find a sparse sensor schedule. We show that the proposed polynomial-time algorithm is able to achieve $(1+\epsilon)$ approximation for all the optimality criteria discussed.

II. Preliminaries and Definitions

A. Mathematical Notations

Throughout the paper, discrete time index is denoted by k. The sets of real (integer), non-negative real (integer), and positive real (integer) numbers are represented by $\mathbb{R}$ ($\mathbb{Z}$), $\mathbb{R}_+$ ($\mathbb{Z}_+$) and $\mathbb{R}_{++}$ ($\mathbb{Z}_{++}$), respectively. The set of natural numbers $\{i \in \mathbb{Z}_{++} \ : \ i \leq n\}$ is denoted by $[n]$. Uppercase letters, such as A or B, stand for real-valued matrices, and lowercase letters denote vectors (e.g. b or c), except that T shows the total number of iterations in a regret minimization problem. For a square matrix X, $\det(X)$ and $\mathrm{Trace}(X)$ refer to the determinant and the summation of on-diagonal elements of X, respectively. $\mathbb{S}^n_+$ is the positive definite cone of n-by-n matrices. The n-by-n identity matrix is denoted by I. Notation $A \preceq B$ is equivalent to matrix $B - A$ being positive semi-definite. The transpose of matrix A is denoted by $A^\top$. The rank of matrix A is referred to by $\mathrm{rank}(A)$. Non-bold face letters are used for scalars and indices (*e.g.* j) and function names (*e.g.* $f(\cdot)$). Operator $\langle A, B\rangle \coloneqq \mathrm{Trace}(A^\top B)$ represents the inner product of two matrices A and B. $\{0,1\}^{m\times n}$ and $[0,1]^{m\times n}$ are the set of m-by-n matrices that their entries are only 0 and 1 and the set of m-by-n matrices that their entries are real numbers between 0 and 1 (inclusive), respectively. The symbol $\|\cdot\|$ denotes the Euclidean norm for vectors and the spectral norm for matrices. Finally, an actuator/sensor schedule is *sparse* if and only if on average a constant number of actuators/sensors, independent of the system dimension, are active each time.

[1]Fractional solution is the solution for the relaxation of an integer optimization in which the integrality constraints of each variable were removed/relaxed.

B. Linear Systems, Controllability and Observability

We start with a canonical linear discrete-time, time-invariant dynamics as follows

$$x(k+1) = Ax(k) + Bu(k), \text{ and } y(k) = Cx(k),$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $k \in \mathbb{Z}_+$. The state matrix A describes the underlying structure of the system and the interaction strength between the agents, matrix B identifies the nodes controlled by an outside controller, and output matrix C shows how output vector y relates to the state vector. One can rewrite the dynamics in the following form

$$x(k+1) = Ax(k) + \sum_{i\in[m]} b_i u_i(k), \tag{1}$$

$$y(k) = \sum_{j\in[p]} e_j c_j^\top x(k), \tag{2}$$

where b_i's are columns of matrix $B \in \mathbb{R}^{n\times m}$, $c_j^\top$'s are rows of matrix $C \in \mathbb{R}^{p\times n}$, and e_j's are the standard basis for $\mathbb{R}^p$. Then, the controllability and observability matrices at time t are respectively given by

$$\mathcal{R}(t) = \begin{bmatrix} B & AB & A^2B & \cdots & A^{t-1}B \end{bmatrix}, \tag{3}$$

and

$$\mathcal{O}(t) = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{t-1} \end{bmatrix}. \tag{4}$$

Assumption 1. *In this paper, we assume that integer number $t > 0$ is the time horizon to control or estimate (also known as the time-to-control or time-to-estimate).*

It is well-known that from a numerical standpoint it is better to characterize controllability and observability in terms of the Gramian matrices at time t defined as follows

$$\mathcal{W}(t) = \sum_{i=0}^{t-1} A^i BB^\top (A^i)^\top = \mathcal{R}(t)\mathcal{R}^\top(t), \text{ and} \tag{5}$$

$$\mathcal{X}(t) = \sum_{i=0}^{t-1} (A^i)^\top C^\top C A^i = \mathcal{O}^\top(t)\mathcal{O}(t). \tag{6}$$

When looking at time-varying schedules, we will consider linear system with time-varying input and output matrices $\mathfrak{B}(\cdot)$ and $\mathfrak{C}(\cdot)$ respectively.

$$x(k+1) = Ax(k) + \mathfrak{B}(k)u(k), \text{ and } y(k) = \mathfrak{C}(k)x(k). \tag{7}$$

For the above system, the controllability and Gramian matrices at time step t are defined as

$$\mathcal{R}_\star(t) = \left[\mathfrak{B}(t-1)\ A\mathfrak{B}(t-2)\ A^2\mathfrak{B}(t-3)\ \cdots\ A^{t-1}\mathfrak{B}(0)\right],$$

and $\mathcal{W}_\star(t) = \mathcal{R}_\star(t)\mathcal{R}_\star^\top(t)$, respectively. Furthermore, the respective Gramian and observability matrix at time step t are

$$\mathcal{X}_\star(t) = \mathcal{O}_\star^\top(t)\mathcal{O}_\star(t), \text{ and} \tag{8}$$

$$\mathcal{O}_\star(t) = \begin{bmatrix} \mathfrak{C}(t-1) \\ \mathfrak{C}(t-2)A \\ \mathfrak{C}(t-3)A^2 \\ \vdots \\ \mathfrak{C}(0)A^{t-1} \end{bmatrix}. \quad (9)$$

Assumption 2. *Through out the paper, we assume that the system* (1)-(2) *is an n-state minimal realization (i.e., it is reachable and observable).*

C. Systemic Controllability/Observability Metrics

Similar to the *systemic* notations introduced in [11], [18]–[20], we define various controllability/observability metrics. These measures are real-valued operators defined on the set of all linear dynamical systems derived from (7) and quantify various measures of the required energy in the system. All of the metrics depends on controllability/observability Gramian matrix of the system, which is a positive definite matrix. Therefore, one can define a systemic performance measure as an operator on the set of Gramian matrices of all controllable/observable systems over n agents, which we represent by $\mathbb{S}^n_+$.

Definition 1 (Systemic Criteria)**.** *A Gramian-based metric $\rho : \mathbb{S}^n_+ \to \mathbb{R}$ is systemic if and only if it meets* Homogeneity *($\forall\, t > 1$, $\rho(tA) = t^{-1}\rho(A)$)*[2], Monotonicity *criteria (if $B \preceq A$, then $\rho(B) \succeq \rho(A)$) , and* convexity *as the following*

$$\rho\big(\alpha A + (1-\alpha)B\big) \le \alpha\rho(A) + (1-\alpha)\rho(B), \ \forall \alpha \in [0,1].$$

For many popular choices of ρ, including A(verage), D(eterminant), T(race), E(igen), V(ariance) and G-optimality, one can see that they satisfy the properties presented in Definition 1. They are listed in Table I. We note that similar criteria have been developed in [17], [21] for the purpose of experimental design.

Proposition 1. *For the given dynamics* (1)-(2) *with Gramian matrices $\mathcal{W}(t)$ and $\mathcal{X}(t)$, the metrics presented in Table I are systemic controllability as well as observability measures.*

Additionally, similar to [22], we define the respective transient controllability and observability functions of the system described by (1)-(2) at time step t as $L_c(x_0,t) = \min_{u,x_0} \sum_{i=-t+1}^{0} \|u(i)\|^2$, where $x(-t) = 0$ and $x(0) = x_0$, and $L_o(x_0,t) = \sum_{i=0}^{t-1} \|y(i)\|^2$, where $u = 0$.

Next, we assume that x_0 is randomly selected from a normal distribution with zero-mean and covariance matrix M, i.e., $x_0 \sim \mathcal{N}(0, M)$. Then, using [22, Theorem 2], one can obtain the expected values of these functions as

$$\mathbb{E}_{x_0}(L_c(x_0,t)) = \mathbb{E}_{x_0} \min_{u,x_0} \sum_{i=-t+1}^{0} \|u(i)\|^2 = \langle \mathcal{W}(t)^{-1}, M\rangle,$$

$$\text{and} \ \ \mathbb{E}_{x_0}(L_o(x_0,t)) = \mathbb{E}_{x_0} \sum_{i=0}^{t-1} \|y(i)\|^2 = \langle \mathcal{X}(t), M\rangle, \quad (10)$$

where $\mathcal{W}(t)$ and $\mathcal{X}(t)$ are controllability and observability Gramian matrices, respectively, and $M = \mathbb{E}(x_0 x_0^\top)$. These two functions play an important role in defining the regret for the system in the regret minimization algorithm that is introduced in Section IV-A.

In this paper, we will only focus on the observability problem and sensor scheduling due to space limitations. One can obtain actuator scheduling using the similar arguments.

III. Unweighted Sparse Sensor Schedule

Similar to [11], to synthesize a sparse approximation of the observability Gramian, we assume that the sensor/output strength only can take binary values (0 or 1). Given a time horizon $t \ge n$, our problem is to compute a sensor schedule $s_i(k)$'s where $s_i(k) \in \{0,1\}$ for the system (1)-(2), i.e.,

$$x(k+1) \ = \ A\,x(k) \ + \sum_{i\in[m]} b_i\,u_i(k), \quad (11)$$

$$y(k) \ = \ \sum_{i\in[p]} s_i(k) e_i c_i^\top x(k), \ k \in \mathbb{Z}_+. \quad (12)$$

The observability Gramian at time t for this schedule is given by

$$\mathcal{X}_s(t) \ = \ \sum_{k=0}^{t-1}\sum_{i\in[p]} s_i(k)(c_i^\top A^{t-k-1})^\top (c_i^\top A^{t-k-1}). \quad (13)$$

To obtain (13) we use the fact that $s_i^2(k) = s_i(k), \ \forall s_i(k) \in \{0,1\}$. Our goal is to reduce the number of active sensors on average d, where $d := \frac{\sum_{k=0}^{t-1}\sum_{i\in[p]} s_i(k)}{t}$, such that the system maintains in the desirable level of observability performance. This approximation will require horizon lengths that are potentially longer than the dimension of the state. Optimal sensor selection can now be formulated as a combinatorial optimization problem. We consider both static and dynamic sensor schedules, corresponding to time-invariant and time-varying output matrices.

1) *The static scheduling Problem*: In this case, all binary coefficients $s_i \in \{0,1\}^p$ for $k+1 \in [t]$ are identical, which means we keep the same schedule at every point in time for the whole time horizon t:

$$\min_{s\in\mathcal{S}_d} \quad \rho\bigg(\sum_{k=0}^{n-1}\sum_{i\in[p]} s_i (c_i^\top A^k)^\top (c_i^\top A^k)\bigg), \quad (14)$$

where $\mathcal{S}_d := \Big\{ s \in \{0,1\}^p, \ \sum_{i\in[p]} s_i \le d \Big\}$ and $s_i = s(i)$.

2) *The Time-varying Scheduling Problem*: In this case, the optimal dynamic strategy is given as

$$\min_{s\in\mathcal{S}_{d,t}} \quad \rho\bigg(\sum_{k=0}^{t-1}\sum_{i\in[p]} s_i(k)(c_i^\top A^{t-k-1})^\top (c_i^\top A^{t-k-1})\bigg), \quad (15)$$

[2]Normally, a function ρ is homogeneous if $\rho(tA) = t^{-n}\cdot\rho(A)$, where n is the degree of homogeneity. However, throughout this paper, when we say a metric is homogeneous, we mean it is homogeneous of degree 1.

TABLE I: Some important examples of systemic controllability/observability metrics.

Optimality-criteria	Systemic Controllability/Observability Measure	Matrix Operator Form
A-optimality	Trace of the inverse	$\text{Trace}(\mathcal{P}^{-1}(t))/n$ [1]
D-optimality	The volume of the ellipsoid	$(\det(\mathcal{P}(t)))^{-1/n}$
T-optimality	Inverse of the trace	$n/\,\text{Trace}(\mathcal{P}(t))$
E-optimality	Inverse of the minimum eigenvalue	$1/\lambda_{\min}(\mathcal{P}(t))$
V-optimality	Average variance	$\frac{1}{n}\,\text{Trace}(\mathcal{Q}^{\top}(t)\mathcal{P}(t)^{-1}\mathcal{Q}(t))^{2}$
G-optimality	Maximum entry in the diagonal	$\max \operatorname{diag} \mathcal{Q}^{\top}(t)\mathcal{P}(t)^{-1}\mathcal{Q}(t)$

[1] $\mathcal{P}(t)$ can be any of the Gramian matrices $\mathcal{W}(t)$ or $\mathcal{X}(t)$.
[2] Based on what we chose for $\mathcal{P}(t)$, $\mathcal{Q}(t)$ can be either $\mathcal{R}(t)$ or $\mathcal{O}(t)$.

where $\mathcal{S}_{d,t} := \left\{ s \in \{0,1\}^{t\times p} : \quad \sum_{k=0}^{t-1}\sum_{i\in[p]} s_i(k) \leq td \right\}$, parameter d is the desired average number of active sensors at each time, t is the time horizon, p is the total number of sensors, and $s_i(k) := s(k+1, i)$.

The exact combinatorial optimization Problem (14) and (15) are intractable for most of ρ's and NP-hard optimization problems; however, it is straightforward to solve a continuous relaxation of these optimization problems because of the convexity property in Definition 1.

Remark 1. *The D-optimality $\rho_{\mathrm{D}} : \mathcal{X}(t) \to (\det \mathcal{X}(t))^{-\frac{1}{n}}$ does not satisfy convexity property in Definition 1. However, it is well-established practice to consider the negative* log-determinant *function* $\log \rho_{\mathrm{D}} : \mathcal{X}(t) \to -\frac{1}{n}\log\det\mathcal{X}(t)$, *which is convex (see Section 3 of [17]).*

Remark 2. *Exact solution for the optimization Problems* (14) *and* (15) *in T-optimality is trivial because, to maximize* $\text{Trace}\,\mathcal{X}(t)$, *by linearity, it suffices to pick the td distinct indices $(k,i) \in \mathcal{T}\times[p]$ (where $\mathcal{T} \subseteq [t-1]\cup\{0\}$) to maximize* $\text{Trace}(y_i(k)y_i^{\top}(k)) = \|y_i(k)\|^2$, *where $y_i(k) = (c_i^{\top}A^{t-k-1})^{\top}$.*

Assumption 3 (Continuous Relaxation). *We define the continuous relaxation of Problem* (15) *as*

$$\min_{\pi\in\mathcal{C}_{d,t}} \rho(\pi) = \min_{\pi\in\mathcal{C}_{d,t}} \rho\Big(\sum_{k=0}^{t-1}\sum_{i\in[p]} \pi_i(k) y_i(k) y_i^{\top}(k)\Big), \qquad (16)$$

where $\mathcal{C}_{d,t} := \left\{ \pi \in [0,1]^{t\times p} : \quad \sum_{k=0}^{t-1}\sum_{i\in[p]} \pi_i(k) \leq td \right\}$, $y_i(k) = (c_i^{\top}A^{t-k-1})^{\top}$, *and $\pi_i(k) := \pi(k+1,i)$. For any fixed $\mu \in (0,1)$, this problem can be solved with $(1+\mu)$-relative error (i.e., $\rho(\hat{\pi}) \leq (1+\mu)\min_{\pi\in\mathcal{C}_{d,t}}\rho(\pi)$) by a polynomial-time algorithm. The same arguments hold for Problem* (14)*, but due to limited space, it is not discussed here.*

To solve continuous relaxation, one can use a variety of standard methods for continuous optimizations such as *projected gradient descent* or *conic programming* [23]; however, we recommend to use *entropic mirror descent* [17], because it suits the geometry of our problem.

Assumption 4. *Without loss of generality, in the rest of this paper we assume that the weighted observability Gramian matrix*

$$\mathcal{X}_{\pi}(t) := \sum_{k=0}^{t-1}\sum_{i\in[p]} \pi_i(k)(c_i^{\top}A^{t-k-1})^{\top}(c_i^{\top}A^{t-k-1}), \qquad (17)$$

is invertible. If $\mathcal{X}_{\pi}(t)$ is singular instead, one can remove all $(c_i^{\top}A^{t-k-1})^{\top}$ that does not belong to the span of $\mathcal{X}_{\pi}(t)$ (Simply put $\pi_i(k)$ equal to zero) and project the rest of $(c_i^{\top}A^{t-k-1})^{\top}$ onto the linear space constructed by $\operatorname{rank}(\mathcal{X}_{\pi}(t))$. *The output would then be invertible in the projected low-dimensional space.*

In the rest of the paper, we discuss details of the rounding method. Since static scheduling Problem (14) is inherently a simple variation of time-varying scheduling Problem (15), from now on we will focus only on deriving the formulation for Problem (15).

IV. Rounding via swapping regret minimization

The following claim shows that if we use a systemic metric aligned with the properties discussed in Definition 1 to find an integral solution that performs $(1+\epsilon)$ close to the fractional solution obtained from a continuous optimization, we do not need any explicit information on the systemic metric.

Claim 1. *We claim that to solve this rounding problem it suffices to find an integral solution $\hat{s} \in \mathcal{S}_{d,t}$ that satisfies*

$$\sum_{k=0}^{t-1}\sum_{i\in[p]} \hat{s}_i(k)(c_i^{\top}A^{t-k-1})^{\top}(c_i^{\top}A^{t-k-1}) \succeq$$

$$\kappa\cdot\sum_{k=0}^{t-1}\sum_{i\in[p]} \pi_i(k)(c_i^{\top}A^{t-k-1})^{\top}(c_i^{\top}A^{t-k-1}), \qquad (18)$$

for some constant $\kappa = 1 - O(\epsilon) > 0$.

The following definition gives us a useful tool that allows us, without loss of generality, to assume that the right-hand side matrix in (18) is identity matrix I.

Definition 2 (Whitening). *Given non-singular coordinate transformation* $\mathsf{T} \in \mathbb{R}^{n\times n}$

$$x(t) \mapsto \mathsf{T}x(t) := \overline{x}(t),$$

the transformed system realization of (1)-(2) *can be obtained by*

$$\overline{x}(k+1) \;=\; \underbrace{\mathsf{T}^{-1}A\mathsf{T}}_{\hat{A}}\,\overline{x}(k) \;+\; \underbrace{\mathsf{T}^{-1}B}_{\hat{B}}\,u(k), \qquad (19)$$

$$y(k) = \underbrace{C\mathsf{T}}_{\hat{C}} \overline{x}(k). \tag{20}$$

based on [24]. Furthermore, the observability Gramian matrix for the transformed system is

$$\mathcal{X}(t) \mapsto \mathsf{T}^\top \mathcal{X}(t)\mathsf{T} := \hat{\mathcal{X}}(t),$$

where $\mathcal{X}(t)$ is the observability Gramian matrix of the original system (1)-(2)*. If we define $\mathsf{T} := \mathcal{X}(t)^{-1/2}$, this change of coordinates converts the observability Gramian matrix of transformed system $\hat{\mathcal{X}}(t)$ to identity matrix I. This process is called* Whitening*, since it converts the covariance matrix of the given samples to the identity matrix.*

We then use this result later in our regret minimization to find a sparse unweighted sensor schedule.

Proposition 2. *Let us assume*

$$\hat{\mathcal{X}}_s(t) = \sum_{k=0}^{t-1}\sum_{i\in[p]} \hat{s}_i(k)(\hat{c}_i^\top \hat{A}^{t-k-1})^\top(\hat{c}_i^\top \hat{A}^{t-k-1}), \tag{21}$$

where $\hat{A}$ is given by (19) *and $\hat{c}_i^\top$'s are rows of matrix $\hat{C}$ given by* (20)*. Then we have $\hat{\mathcal{X}}_s(t) \succeq \kappa I$, if and only if* (18) *holds.*

Proposition 2 allows us to assume without loss of generality that $\mathcal{X}_\pi(t)$ is the identity matrix. The computational load imposed by matrix multiplications is not a bottleneck in the proposed algorithm described later.

Proposition 3. *The problem of showing*

$$\hat{\mathcal{X}}_s(t) \succeq \kappa I, \tag{22}$$

can be reduced to lower bounding the minimum eigenvalue of $\hat{\mathcal{X}}_s(t)$ or

$$\lambda_{\min}(\hat{\mathcal{X}}_s(t)) \geq \kappa. \tag{23}$$

We use this result to define our main problem in this section. The main theorem of this paper is Theorem 1 that guarantees that polynomial-time $(1+\epsilon)$ approximation exists. We show that the solution for the problem we define here together with the bound obtained from continuous relaxation can simply prove this theorem constructively. Moreover, the whole process leads us to establish a framework to find the unweighted sparse schedule with desirable bound. This problem is defined as follows:

Main Problem (From weighted to unweighted)**.** *Consider a given weighted sensor scheduling*

$$C_{d,t} = \Big\{\pi \in [0,1]^{t\times p} : \quad \sum_{k=0}^{t-1}\sum_{i\in[p]} \pi_i(k) \leq td\Big\}.$$

One can utilize this scheduling to obtain the transformed system realization described in Definition 2 with the observability Gramian matrix as

$$\hat{\mathcal{X}}_\pi(t) = \sum_{k=0}^{t-1}\sum_{i\in[p]} \pi_i(k) z_i(k) z_i(k)^\top = I,$$

where $z_i(k) = (\hat{c}_i^\top \hat{A}^{t-k-1})^\top$ and $\pi_i(k) = \pi(k+1, i)$. Then, the goal is to find unweighted sensor scheduling $\hat{s} \in \mathcal{S}_{d,t} = \{s \in \{0,1\}^{t\times p} : \quad \sum_{k=0}^{t-1}\sum_{i\in[p]} s_i(k) \leq td\}$, such that

$$\lambda_{\min}\left(\sum_{k=0}^{t-1}\sum_{i\in[p]} \hat{s}_i(k) z_i(k) z_i(k)^\top\right) \geq (1-3\epsilon) = \kappa, \tag{24}$$

where $\hat{s}_i(k) = \hat{s}(k+1, i)$. Putting $\kappa = (1-3\epsilon)$ is intentional. We will use this adjustment to prove Theorem 1.

In the remainder of this paper, we will illustrate how regret minimization algorithm can help us to solve this problem.

A. Regret Minimization

In this section, we explain how, by using a learning-based iterative process (a.k.a. regret minimization), we can solve *Main Problem*.

We start with arbitrary unweighted sensor scheduling $\hat{s}^{(0)} \in \mathcal{S}_{d,t}$. We define the *action space* of a regret minimization as $\Upsilon_{n\times n} := \{M \in \mathbb{S}_+^n : \ \mathrm{Trace}(M) = 1\}$. Specifically, action space $\Upsilon_{n\times n}$ contains the covariance matrix of the zero-mean Gaussian initial state $x_0 \in \mathbb{R}^n$ with $\mathbb{E}(\|x_0\|^2) = 1$. Hence, at each iteration ℓ, the player picks an initial state $x_0{}^{(\ell)} \sim \mathcal{N}(0, M^{(\ell)})$ where $M^{(\ell)} \in \Upsilon_{n\times n}$ for the coordinate-transformed dynamics described in (19)-(20) with sensor scheduling $\hat{s}^{(\ell)} \in \mathcal{S}_{d,t}$ and transient Gramian $\hat{\mathcal{X}}_s^{(\ell)}(t)$, i.e.,

$$\hat{\mathcal{X}}_s^{(\ell)}(t) = \sum_{k=0}^{t-1}\sum_{i\in[p]} \hat{s}_i^{(\ell)}(k) z_i(k) z_i(k)^\top, \tag{25}$$

where $z_i(k) = (\hat{c}_i^\top \hat{A}^{t-k-1})^\top$. The player selects $x_0^{(\ell)}$ to minimize a cost at each iteration. The cost/loss at each iteration ℓ is reflected by observing the transient energy of the system over the time interval 0 to $t-1$ as follows,

$$\mathbb{E}(L_o(x_0{}^{(\ell)}, t)) = \mathbb{E}\sum_{i=0}^{t-1} \|y(i)\|^2,$$

where $u = 0$. Based on (10), this cost can be calculated by

$$\mathbb{E}(L_o(x_0{}^{(\ell)}, t)) = \langle \hat{\mathcal{X}}_s(t)^{(\ell)}, M^{(\ell)}\rangle. \tag{26}$$

Let us define elementary t-by-p matrices $J_{k,i}$ which is one at indices $(k+1, i)$ and zero elsewhere. At each iteration, the adversary updates the sensor schedule by swapping active sensors in time and place (i.e., $\hat{s}^{(\ell)} = \hat{s}^{(\ell-1)} - J_{k,i} + J_{v,j}$, where

$$(k,i) \in \mathcal{M} = \{(m,n) : \hat{s}(m+1,n)^{(\ell-1)} = 1\}, \tag{27}$$

$\hat{s}(m+1,n)^{(\ell-1)}$ is the common notation for the $(m+1,n)^{\text{th}}$ entry of the matrix schedule $\hat{s}^{(\ell-1)}$, and $(v,j) \in (([t-1] \cup \{0\}) \times [p]) \setminus \mathcal{M})$, to maximize the loss (26). Let us consider T iterations of this game. The goal of the player is to minimize their *regret*, defined as

$$\mathrm{Regret}(\{M^{(\ell)}\}_{\ell=0}^{T-1}) := \min_{\mathfrak{U}\in\Upsilon_{n\times n}} \sum_{\ell=0}^{T-1} \langle \hat{\mathcal{X}}_s^{(\ell)}(t), M^{(\ell)} - \mathfrak{U}\rangle, \tag{28}$$

which is the difference between cumulative losses with the single optimal action $\mathfrak{U}$ when all the previous loss matrices (or equivalently, observability matrices $\{\hat{\mathcal{X}}_s^{(\ell)}(t)\}_{\ell=0}^{T-1}$) are known. We devote the next subsection to a popular strategy to minimize regret of the player.

B. Strategy to Minimize Regret

One of the popular strategies to minimize regret is *follow the regularized leader* or FTRL [25]. In fact, FTRL determines how to choose action $M^{(\ell)}$ (or equivalently initial state $x_0^{(\ell)}$) for the player at each round ℓ (where $\ell+1 \in [T]$). Based on this strategy:

$$M^{(\ell)} = \operatorname*{argmin}_{\mathcal{Z}\in\Upsilon_{n\times n}} \left\{ \Omega_{\Xi}(M^{(\ell-1)}, \mathcal{Z}) + \eta \, \langle \hat{\mathcal{X}}_s^{(\ell-1)}(t), \mathcal{Z}\rangle \right\}, \tag{29}$$

where $\eta > 0$ is learning rate, $\Xi : \mathbb{R}^{n\times n} \to \mathbb{R}$ is the regularizer function, and so-called Bregman divergence function associated with Ξ is $\Omega_{\Xi}(X,Y) = \Xi(Y) - \Xi(X) - \langle \nabla\Xi(X), Y - X\rangle$.

Although the matrix entropy $\Xi(M) = \langle M, \log M - I\rangle$ is likely the most famous choice for Ξ [26], in this paper, we decide to use $l_{1/2}$-regularizer defined as $\Xi(M) = -2\,\mathrm{Trace}(M^{1/2})$, which is first introduced in [27]. The reason is this matrix generalization leads to a better algorithm for similar problems such as graph sparsification [27] and online eigenvector problems [28]. The closed form strategy resulted by $l_{1/2}$-regularizer was obtained in [17] (see its appendix) as

$$M^{(\ell)} = \left(f^{(\ell)} I + \eta H^{(0)} + \eta \sum_{j=0}^{\ell-1} \hat{\mathcal{X}}_s^{(j)}(t) \right)^{-2}, \ \forall\, \ell \in \mathbb{Z}_{++}, \tag{30}$$

where $\eta > 0$ and $H^{(0)}$ is a positive semidefinite matrix such that $(f^{(0)} I + \eta H^{(0)}) \succ 0$. Moreover, $f^{(\ell)} \in \mathbb{R}$ is the unique constant such that $M^{(\ell)} \succ 0$ and $\mathrm{Trace}(M^{(\ell)}) = 1$. In the next subsection, by leveraging the learning approach discussed here, we solve *Main Problem*.

C. Swapping Regret algorithm

In this section, we first state two fundamental lemmas and illustrate how one can use the result of these lemmas to build an iterative solver for the *Main Problem*. Then, we present our main theorem and show how the tools we have developed so far can be combined to obtain the proof for Theorem 1 constructively.

We denote the contribution of swapping the sensors in Gramian transient at each iteration ℓ by $N^{(\ell)}(t)$ and define it as $N^{(\ell)}(t) = \hat{\mathcal{X}}_s^{(\ell)}(t) - \hat{\mathcal{X}}_s^{(\ell-1)}(t)$, where $\hat{\mathcal{X}}_s^{(\ell)}(t)$ is defined in (25). This contribution can also be expressed equivalently as

$$\begin{aligned} N^{(\ell)}(t) = {} & (\hat{c}_{j^{(\ell)}}^\top \hat{A}^{t-v^{(\ell)}-1})^\top (\hat{c}_{j^{(\ell)}}^\top \hat{A}^{t-v^{(\ell)}-1}) \\ & - (\hat{c}_{i^{(\ell)}}^\top \hat{A}^{t-k^{(\ell)}-1})^\top (\hat{c}_{i^{(\ell)}}^\top \hat{A}^{t-k^{(\ell)}-1}), \end{aligned} \tag{31}$$

where $(k^{(\ell)}, i^{(\ell)}) \in \mathcal{M}$ are the indices for the sensor getting deactivated, $(v^{(\ell)}, j^{(\ell)}) \in (([t-1] \cup \{0\}) \times [p]) \backslash \mathcal{M}$ are the indices for the sensor getting activated at iteration ℓ, and $\mathcal{M}$ is defined as (27). It is obvious that $N^{(\ell)}(t)$ is a rank-2 matrix.

Lemma 1. *Assume in this game, strategies $\{M^{(\ell)}\}_{\ell=0}^{T-1} \in \Upsilon_{n\times n}$ taken by the player are defined according to the $l_{1/2}$ strategy with some learning rate $\eta > 0$ and $H^{(0)} = \hat{\mathcal{X}}_s^{(0)}(t)$. If $\eta\langle (M^{(\ell)})^{1/2}, (\hat{c}_{i^{(\ell)}}^\top \hat{A}^{t-k^{(\ell)}-1})^\top (\hat{c}_{i^{(\ell)}}^\top \hat{A}^{t-k^{(\ell)}-1})\rangle < 1/2$ for all ℓ, then for any $\mathfrak{U} \in \Upsilon_{n\times n}$ we have*

$$\begin{aligned} & \langle \hat{\mathcal{X}}_s^{(0)}(t) + \sum_{\ell=0}^{T-1} N^{(\ell)}(t), \mathfrak{U}\rangle = \lambda_{\min}(\hat{\mathcal{X}}_s^{(T)}(t)) \geq \\ & \sum_{\ell=0}^{T-1} \Bigg(- \frac{\langle M^{(\ell)}, (\hat{c}_{j^{(\ell)}}^\top \hat{A}^{t-v^{(\ell)}-1})^\top (\hat{c}_{j^{(\ell)}}^\top \hat{A}^{t-v^{(\ell)}-1})\rangle}{1 + 2\eta\langle (M^{(\ell)})^{1/2}, (\hat{c}_{j^{(\ell)}}^\top \hat{A}^{t-v^{(\ell)}-1})^\top (\hat{c}_{j^{(\ell)}}^\top \hat{A}^{t-v^{(\ell)}-1})\rangle} \\ & \quad + \frac{\langle M^{(\ell)}, (\hat{c}_{i^{(\ell)}}^\top \hat{A}^{t-k^{(\ell)}-1})^\top (\hat{c}_{i^{(\ell)}}^\top \hat{A}^{t-k^{(\ell)}-1})\rangle}{1 - 2\eta\langle (M^{(\ell)})^{1/2}, (\hat{c}_{i^{(\ell)}}^\top \hat{A}^{t-k^{(\ell)}-1})^\top (\hat{c}_{i^{(\ell)}}^\top \hat{A}^{t-k^{(\ell)}-1})\rangle} \Bigg) \\ & \quad + \frac{2\sqrt{n}}{\eta}. \end{aligned} \tag{32}$$

In (32), the equality holds since $\mathfrak{U}$ can be chosen arbitrary matrix from $\Upsilon_{n\times n}$. Lemma 1 presents the lower bound for the final transient Gramian matrix of the sparse system. The complementary step is to find indices $\{(v^{(\ell)}, j^{(\ell)}), (k^{(\ell)}, i^{(\ell)})\}$ to make the right-hand side of (32) as small as possible. The following lemma leads us to establish an iterative process toward this end.

Lemma 2. *Let $\hat{s}^{(\ell)} \in \mathcal{S}_{d,t}$ be the sparse sensor schedule at iteration ℓ, but $\lambda_{\min}(\hat{\mathcal{X}}_s^{(\ell)}(t)) \not\succeq 1 - 3\epsilon$. Moreover, let us assume the action the player pick at iteration ℓ is given by $M^{(\ell)} = (f^{(\ell)} I + \eta \hat{\mathcal{X}}_s^{(\ell-1)}(t))^{-2}$, where $f^{(\ell)} \in \mathbb{R}$ is a unique number that guarantees $M^{(\ell)} \succeq 0$ and $\mathrm{Trace}(M^{(\ell)}) = 1$. Then, these statements hold:*

$$\tau := \min_{(k^{(\ell)}, i^{(\ell)}) \in \mathcal{G}} \frac{\langle M^{(\ell)}, z_{i^{(\ell)}}(k^{(\ell)}) z_{i^{(\ell)}}(k^{(\ell)})^\top\rangle}{1 - 2\eta\langle (M^{(\ell)})^{1/2}, z_{i^{(\ell)}}(k^{(\ell)}) z_{i^{(\ell)}}(k^{(\ell)})^\top\rangle} \leq \frac{1-\epsilon}{td}, \tag{33}$$

$$\max_{\substack{\mathcal{T} \subseteq [t-1] \cup \{0\} \\ (v^{(\ell)}, j^{(\ell)}) \in (\mathcal{T} \times [p]) \backslash \mathcal{M}}} \frac{\langle M^{(\ell)}, z_{j^{(\ell)}}(v^{(\ell)}) z_{j^{(\ell)}}(v^{(\ell)})^\top\rangle}{1 + 2\eta\langle (M^{(\ell)})^{1/2}, z_{j^{(\ell)}}(v^{(\ell)}) z_{j^{(\ell)}}(v^{(\ell)})^\top\rangle} \geq \tau + \frac{\epsilon}{td}. \tag{34}$$

where $z_\alpha(\beta) := (\hat{c}_\alpha^\top \hat{A}^{t-\beta-1})^\top$ for all $(\beta, \alpha) \in ([t-1] \cup \{0\}) \times [p]$ and $\mathcal{G} = \{(x,y) \in \mathcal{M} : 2\eta\langle (M^{(\ell)})^{1/2}, (\hat{c}_y^\top \hat{A}^{t-x-1})^\top (\hat{c}_y^\top \hat{A}^{t-x-1})\rangle < 1\}$.

Remark 3. *One can show that setting $\epsilon \in (0, 1/6]$, $td \geq 5n/\epsilon^2$, and $\eta = \sqrt{n}/\epsilon$ guarantees that we can always find a pair of indices $(k,i) \in \mathcal{M}$ such that the denominator of* (33) *is positive.*

By (33) and (34) becoming true, for (32), one can say

$$-\lambda_{\min}(\hat{\mathcal{X}}_s^{(T)}(t)) \leq \sum_{\ell=0}^{T-1} -\frac{\epsilon}{td} + \frac{2\sqrt{n}}{\eta}. \tag{35}$$

Let assume that the parameters of the game were chosen such that they are aligned with the conditions stated in Remark 3. Therefore, considering $\eta = \sqrt{n}/\epsilon$, the right-hand side of inequality (35) is reduced to $-\frac{T\epsilon}{td} + 2\epsilon$. Moreover, if we take $T \geq td/\epsilon$, the following inequality holds

$$\lambda_{\min}(\hat{\mathcal{X}}_s^{(T)}(t)) \geq 1 - 2\epsilon. \tag{36}$$

Claim 2. *Lemma 2 offers a way to solve the* Main Problem *iteratively.*

Theorem 1. *Assume $\epsilon \in (0, 1/6]$, $d \in [5n/(t\epsilon^2), p]$, and $\rho : \mathbb{S}_n^+ \to \mathbb{R}$ is an systematic metric that satisfies all the properties in Definition 1. Then, there exists a polynomial-time algorithm that computes a schedule $\hat{s}$ that satisfies*

$$\exists\ \hat{s} \in \mathcal{S}_{d,t} : \rho(\hat{s}) \leq (1 + 8\epsilon) \cdot \min_{s \in \mathcal{S}_{d,t}} \rho(s).$$

Remark 4. *Theorem 1 requires $td \geq \Omega(n/\epsilon^2)$ in order to achieve $(1 + \epsilon)$ approximation.*

Based on this remark, we can see the bigger d and t we get, the more accurate approximation we achieve.

The results presented in [11]- [12] are based on the spectral approximation of Gramians of the fully actuated/sensed systems using randomized and deterministic greedy-based algorithms. However, in this work, we consider a fundamentally different problem which is exploiting a learning-based algorithm to find a near optimal unweighted sensor schedule, and we compare the result with an optimal solution rather than the performance of the fully sensed system. Interestingly for both cases, the lower bound on the performance loss ϵ is linearly proportional to $1/\sqrt{d}$ for large d.

V. Concluding Remarks

In this paper, we have investigated designing an unweighted sparse sensor schedule for discrete-time linear dynamical networks where sparsification is performed in time and place. We show how the recent advances in theoretical computer science can be exploited to choose a relatively small number of sensors at each time. Specifically, we show that applying a learning-based algorithm known as regret minimization achieves near-optimal deterministic approximation guarantees for sensor sparsification. This learning-based algorithm is fed with an approximate continuous solution available from a convex optimization. All the algorithms introduced in this paper can be modified for the case where we are dealing with an actuator scheduling. We have attempted to put the dual tools for actuator scheduling alongside the sensor scheduling materials. Therefore, one can easily derive the algorithms for the actuator schedule problem. A potential future direction is to extend the results presented in this paper to uncontrollable and unobservable systems.

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