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Stability and Robustness Analysis of Epidemic Networks with Multiple Time-Delays

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***Abstract*— Several sources of delay in an epidemic network might negatively affect the stability and robustness of the entire network. In this paper, a multi-delayed Susceptible-Infectious-Susceptible (SIS) model is applied on a metapopulation network, where the epidemic delays are categorized into local and global delays. While local delays result from intra-population lags such as symptom development duration or recovery period, global delays stem from inter-population lags, e.g., transition duration between subpopulations. The theoretical results for a network of subpopulations with identical linear SIS dynamics and different types of time-delay show that depending on the type of time-delay in the network, different eigenvalues of the underlying graph should be evaluated to obtain the feasible regions of stability. The delay-dependent stability of such epidemic networks has been analytically derived, which eliminates potentially expensive computations required by current algorithms. The effect of time-delay on the $\mathscr{H}_2$ norm-based performance of a class of epidemic networks with additive noise inputs and multiple delays is studied and the closed form of their performance measure is derived using the solution of delayed Lyapunov equations. As a case study, the theoretical findings are implemented on a network of United States' busiest airports.**

I. Introduction

In epidemiological models, a metapopulation consists of several interacting subpopulations with an underlying graph representing the map and strength of inter-population connections [1]. The prediction of epidemic progress in communities has been studied in [2], where a macro-modelling approach is used. The stochastic and deterministic metapopulation SIS dynamics are compared in [3], where the effect of variable disease transmission rates and interaction rates between the subpopulations is considered. A significant number of other research studies in this field have been accomplished; some focusing on the improvement of metapopulation models by introducing more epidemic compartments, e.g. Susceptible-Infected-Susceptible (SIS) [4], Susceptible-Infected-Removed (SIR), Susceptible-Exposed-Infected-Removed (SEIR) [5], and Susceptible-Infected-Quarantined-Susceptible (SIQS) [6].

The stability of single-delayed LTI systems has been investigated in [7], where an exact numerical approach to determine the regions of delay-dependent stability is presented. The same method is then extended for LTI systems with multiple time-delays in [8]. A frequency-domain stability analysis approach for LTI systems with two time-delays is developed by [9]. The delay-dependent α-stability and delay-independent asymptotic stability of multi-delayed LTI systems using Lyapunov-type stability criteria is studied in [10]. Moreover, the global stability of SIS ans SIR network models is investigated in several studies [11], [12].

The performance of noisy linear consensus networks has been investigated in [13], where a performance measure based on the $\mathscr{H}_2$ norm of system is introduced and established for consensus networks with different types of input noise. The $\mathscr{H}_2$ norm of delayed LTI systems by solving the delay Lyapunov equation is studied in [14], where a spectral discretization scheme is offered for systems with commensurate and non-commensurate time-delays. The performance of epidemic networks with single-delayed SIS dynamics has been studied in [15] and the effect of time-delay on the network performance is evaluated.

Although there is a strong literature on the networked epidemic models and multi-delayed systems with SIS dynamics, to the best of our knowledge, the investigation of explicit delay-dependent stability criteria and performance measure for this class of networks has not received sufficient attention. The importance of this problem comes from the fact that the computational complexity of the current algorithms for stability and performance analysis of such networks grows significantly with the increase of the network size. In this work, we aim to present efficient stability criteria and performance measure for a class of delayed epidemic networks with linear SIS dynamics. Our results show that local and global delays contribute to the network behaviour in different ways. We base our analysis on distinguishing the local and global delays and investigate their role in the epidemic progress. Our contributions are as below,

- Investigation of stability in epidemic networks with SIS dynamics and different types of time-delay, i.e., local delay, global delay, identical delays, and commensurate delays (Sections II and III).
- Performance analysis of delayed noisy networks with SIS dynamics and different time-delays (Section IV).

The simulation results for a network of United States' airports are presented in Section V, and the discussion around the key results is followed in Section VI.

II. Multi-delayed SIS network model

Let the undirected and weighted graph $\mathscr{G} = (\mathscr{V}, \mathscr{E}, w)$ represent an epidemic network with node set $\mathscr{V} = \{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$, where node i represents the i^{th} subpopulation in the network. $\mathscr{E} \subset \mathscr{V} \times \mathscr{V}$ denotes the edge set, which shows the connection between pairs of nodes in $\mathscr{V}$ with

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the corresponding weight of $w(e) = w_e = a_{ij} \in \mathbb{R}^+$ for all $e = \{i, j\} \in \mathscr{E}$. Note that the underlying graph is undirected; therefore we have, $a_{ij} = a_{ji} = w_e$. $\mathbf{w} \in \mathbb{R}^m$ is the vector of edge weights defined as $\mathbf{w} = [w_1,\ w_2,\ \ldots,\ w_m]$, where $m = |\mathscr{E}|$. The adjacency matrix of the underlying undirected graph, which has no self-loops, is then defined as $A = [a_{ij}]$, where $a_{ij} = 0$ for $i = j$. The vector of eigenvalues of the adjacency matrix is $\boldsymbol{\lambda} = [\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n]$, where $\lambda_1 < \lambda_2 < \cdots < \lambda_n$.

For an epidemic network with the described underlying graph, the nodes are representative of subpopulations, the edges show the map of connection between the subpopulations, and the edge weights indicate their connection strength. For this network, the state of the infectious subpopulations at time $t \geq 0$ is represented by the vector $\mathbf{p}(t) = [p_1(t),\ \ldots,\ p_n(t)]^\top$, where $p_i \in [0,1]$ is the marginal probability of subpopulation i being infectious at time t such that $p_i(t) = 1$ if the entire population of i is infectious and $p_i(t) = 0$ if it is completely susceptible. Therefore, $p_i(t)$ can be interpreted as the fraction of infectious individuals in subpopulation i at time t. The fraction of susceptible individuals in subpopulation i at time t is denoted by $s_i(t)$. Every subpopulation might experience a local delay $\tau_l \in \mathbb{R}^+$, which is caused by the considerable duration of recovery from the disease. There is also a global delay $\tau_g \in \mathbb{R}^+$, corresponding to the interconnection between subpopulations which accounts for the time it takes for infectious members of a subpopulation to travel to another subpopulation and start a potential connection with its susceptible members.

Assumption 1. *In this study we assume that the population of every subpopulation remains approximately constant over the course of epidemic. In other words, although the members can have inter-population travel, for every subpopulation the net of population change is close to zero.*

Considering a constant population size for each node i, we have $s_i(t) + p_i(t) = 1$; therefore, the fraction of susceptible individuals of i can be determined by $s_i(t) = 1 - p_i(t)$.

The approximated SIS dynamics of subpopulation i can now be described using the *mean-field approximation* model with multiple time delays and uncertainty as below,

$$\dot{p}_i(t) = -\delta p_i(t-\tau_l) + \beta\,(1 - p_i(t)) \sum_{j=1}^{n} a_{ij} p_j(t-\tau_g) + \xi_i(t); \quad t > \tau_m,$$
$$p_i(t) = \phi_i(t); \quad t \leq \tau_m. \tag{1}$$

where $\tau_m = \max[\tau_l, \tau_g]$, $\beta \in \mathbb{R}^+$ is the infection rate at which a subpopulation will get contaminated by its neighbors, and $\delta \in \mathbb{R}^+$ is the recovery rate of infectious individuals. a_{ij} is the ij^{th} component of the adjacency matrix of the coupling graph which is equivalent to the weight of edge $e = \{i, j\}$. The effect of uncertainties on the disease spread dynamics of subpopulation i is reflected through $\xi_i(t)$, which is modeled as an independent Gaussian white noise with zero mean. $\phi_i(t)$ is the history function of infection for node i.

The compact representation of the above multi-delayed

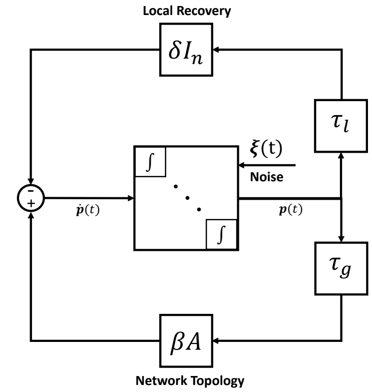

Fig. 1. Representation of an epidemic network with dynamics (3).

SIS model can now be expressed as below,

$$\dot{\mathbf{p}}(t) = -\delta \mathbf{p}(t-\tau_l) + \beta A \mathbf{p}(t-\tau_g) - \beta P(t) A \mathbf{p}(t-\tau_g) + \boldsymbol{\xi}(t); \quad t > \tau_m,$$
$$\mathbf{p}(t) = \phi(t); \quad t \leq \tau_m. \tag{2}$$

where $P(t) = \mathrm{diag}(\mathbf{p}(t))$, $\boldsymbol{\xi}(t)$ is the vector of independent Gaussian noise, and $\phi(t) = [\phi_1(t), \ldots, \phi_n(t)]$ is the vector of history functions.

As shown in the literature [11], this system always has a unique disease-free equilibrium $\mathbf{p}^*(t) = 0$. Linearizing this set of delayed differential equations around the equilibrium $\mathbf{p}^*(t) = 0$, we have

$$\dot{\mathbf{p}}(t) = -\delta \mathbf{p}(t-\tau_l) + \beta A \mathbf{p}(t-\tau_g) + \boldsymbol{\xi}(t); \quad t > \tau_m,$$
$$\mathbf{p}(t) = \phi(t); \quad t \leq \tau_m. \tag{3}$$

Fig. 1 illustrates a block diagram representation of linear network (3). The reproduction number of this network is determined by [16], [17],

$$R_0 \ = \ \frac{\beta}{\delta} \lambda_n, \tag{4}$$

which will be used later in the stability analysis of the network.

III. Stability analysis

The goal to this section is to drive the regions of stability for the linear system (3) when there is no noise input in effect. The investigation of stability in multi-delayed LTI (MDLTI) systems is challenging due to the existence of infinitely many characteristic roots. One approach towards extracting the delay-dependant stability regions of such systems is the conversion of time delay domain such that the exponential terms in the characteristic equation are eliminated. This conversion allows us to evaluate the behaviour of system by its finite characteristic roots in an alternative domain. This approach is implemented by the Rekasius substitution defined below

$$e^{-\tau s} = \frac{1 - Ts}{1 + Ts}, \ \tau \in \mathbb{R}^+, \tag{5}$$

in which $s = j\omega$, $\omega \in \mathbb{R}^{+}$, and $T \in \mathbb{R}$ is called the agent parameter. Note that this is an exact substitution, not an approximation, with the following mapping condition,

$$\tau = \frac{2}{\omega}\left[\tan^{-1}(\omega T) \pm \ell\pi\right], \ \ell = 0,1,2,\ldots,\infty. \quad (6)$$

This equation represents an asymmetric mapping in which a given T will be mapped into infinitely many τ's for a specific ω. For the same ω and one τ, only a unique T can be found.

Now consider the network with delayed dynamics (3) and no input noise. We define its transfer function by

$$G(s) = \left(sI_n + \delta I_n e^{-\tau_l s} - \beta A e^{-\tau_g s}\right)^{-1}. \quad (7)$$

Note that the underlying graph is undirected; therefore, the eigenvalue decomposition of its symmetric adjacency matrix gives: $A = Q\Lambda Q^{\top}$, where Q and $\Lambda = \mathrm{diag}(\boldsymbol{\lambda})$ are the eigenvector and eigenvalue matrices of A. Equation (7) can then be expressed as

$$G(s) = Q\left(sI_n + \delta I_n e^{-\tau_l s} - \beta\Lambda e^{-\tau_g s}\right)^{-1} Q^{\top}. \quad (8)$$

Here, $sI_n + \delta I_n e^{-\tau_l s} - \beta\Lambda e^{-\tau_g s}$ is a diagonal matrix and its determinant is the product of its diagonal elements, which also gives the determinant of $G(s)$. The corresponding characteristic equation of system can then be expressed by

$$\begin{aligned}\chi(s) &= \det\left(sI_n + \delta I_n e^{-\tau_l s} - \beta\Lambda e^{-\tau_g s}\right)\\ &= \prod_{i=1}^{n}\left(s + \delta e^{-\tau_l s} - \beta\lambda_i e^{-\tau_g s}\right) = 0. \quad (9)\end{aligned}$$

A network with dynamics (3) (without noise) and characteristic equation (9) is asymptotically stable if all the roots of (9) are located on the left half of the complex plane. Given that the exponential terms in the characteristic equation of system result in having infinitely many characteristic roots, it is difficult to analyse the asymptotic stability of system in the time delay domain. The Rekasius substitution can now be applied to generate a polynomial-type characteristic equation, rather than an exponential one. Applying the Rekasius substitution of both time delays, τ_l and τ_g, on the characteristic equation (9), it will be converted to

$$\begin{aligned}\chi(s) = \prod_{i=1}^{n}\Big(&s(1+T_l s)(1+T_g s) + \delta(1-T_l s)(1+T_g s)\\ &- \beta\lambda_i(1-T_g s)(1+T_l s)\Big) = \prod_{i=1}^{n}\left(\sum_{k=0}^{3} c_{k,i}s^k\right) = 0 \quad (10)\end{aligned}$$

with the following polynomial coefficients,

$$\begin{aligned}c_{0,i} &= \delta - \beta\lambda_i,\\ c_{1,i} &= 1 + (\delta + \beta\lambda_i)(T_g - T_l),\\ c_{2,i} &= T_l + T_g - (\delta - \beta\lambda_i)T_l T_g,\\ c_{3,i} &= T_l T_g. \quad (11)\end{aligned}$$

These coefficients are functions of the agent parameters T_l and T_g. Note that only the imaginary spectra of (10) and (9) are identical. The degree transcendental characteristic equation (9) is now converted into a product of polynomials with degree of 3 without transcendentality and its purely imaginary characteristic roots coincide with those of (9) exactly.

Now, let us consider four different versions of network (3) (without noise) as below,

- **Network 1**: The local time delay is negligible compared to the global delay, i.e., $\tau_l = 0$.
- **Network 2**: The global time delay is negligible compared to the local delay, i.e., $\tau_g = 0$.
- **Network 3**: The global and local time delays are approximately identical, i.e., $\tau_g = \tau_l = \tau$.
- **Network 4**: The global and local time delays are commensurate, i.e., $\tau_g = a\tau_1$ and $\tau_l = b\tau_1$ for $\tau_1 > 0$ and $a,\ b \in \mathbb{Z}^{*}$.

We present the following theorems for their stability condition.

Theorem 1. *For **Network 1**, if $R_0 < 1$, then the system is delay-independently stable.*

Theorem 2. *For **Network 2**, if $R_0 < 1$, then the delay-dependent stability is bounded above by $\bar{\tau}_l$, where*

$$\bar{\tau}_l = \frac{2}{\sqrt{\delta^2 - \beta^2\lambda_n^2}}\,tan^{-1}\left(\frac{\sqrt{\delta^2 - \beta^2\lambda_n^2}}{\delta + \beta\lambda_n}\right). \quad (12)$$

Theorem 3. *For **Network 3**, if $R_0 < 1$, then the delay-dependent stability of network is bounded above by $\bar{\tau}$, where*

$$\bar{\tau} = \frac{\pi}{2(\delta - \beta\lambda_1)}. \quad (13)$$

In **Network 4** with $\tau_g = a\tau_1$ and $\tau_l = b\tau_1$, the Rekasius substitution of the commensurate time-delays gives

$$e^{-\tau_g s} = \left(\frac{1 - T_1 s}{1 + T_1 s}\right)^{a},\ e^{-\tau_l s} = \left(\frac{1 - T_1 s}{1 + T_1 s}\right)^{b}, \quad (14)$$

which return the following characteristic equation,

$$\begin{aligned}\chi(s) = \prod_{i=1}^{n}\Big(&s(1+T_1 s)^{b+a} + \delta(1-T_1 s)^{b}(1+T_1 s)^{a}\\ &- \beta\lambda_i(1-T_1 s)^{a}(1+T_1 s)^{b}\Big) = \prod_{i=1}^{n}\sum_{k=0}^{a+b+1} q_{k,i}s^k = 0, \quad (15)\end{aligned}$$

where $q_{k,i} = q_{k,i}(T_1, \delta, \beta\lambda_i)$ is a polynomial of agent parameter T_1. This characteristic equation is a product of n polynomials of degree $a+b+1$. The Routh's array is then formed for every i, where its first column consists of rational functions of T_1. The real roots of these functions create the set of all the possible T_1's that might cause a change in the sign of elements in the first column of Routh's array. Every member of this set provides one pair of imaginary roots when plugged in (15). These roots coincide with those of the original characteristic equation in the domain of time delay. Given a T_1 and its associated frequency, the corresponding time delay can be found by (6) for $\ell = 0,1,2,\ldots,\infty$. The common range of τ_1 that results in asymptotic stability can then be obtained by applying this method for all the i's. This result provides all the feasible pairs of local and global commensurate delays, $(\tau_g,\ \tau_l)$, that guarantee system's stability. See [7] for more details on this methodology, which is known as the D-subdivision method.

IV. Performance analysis by delayed Lyapunov equation

In this section, we investigate the performance and robustness of **Networks 1-4** in the presence of input noise, where (3) is the governing dynamic. In this regard, a performance measure based on $\mathscr{H}_2$ norm of the system is adopted from [18], which is defined as

$$\rho \ := \ \|G\|_2^2. \tag{16}$$

This is a metric for performance loss when noise is affecting the network. The $\mathscr{H}_2$ norm can be found by its frequency domain definition as below

$$\|G\|_2^2 := \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{Tr}\left(G^*(j\omega)G(j\omega)\right)d\omega, \tag{17}$$

where $G^*(j\omega)$ denotes the complex conjugate transpose of $G(j\omega)$. It should be noted that ρ measures the performance loss of network; therefore, lower values of it are desired.

It is shown in the literature that for delayed systems with linear dynamics, the $\mathscr{H}_2$ norm can be found by the solution of delayed Lyapunov equations. In what follows, the performance measure of a general class of noisy and delayed linear networks is established using the general solution of delayed Lyapunov equation. It is then shown that the four studied networks (**Networks 1-4**) are special cases of this class of networks and their analytical performance measure can be found by this approach.

Consider an epidemic network with commensurate time-delays $\tau_a = a\tau_1$, and $\tau_b = b\tau_1$, which are integer multiples of some delay $\tau_1 > 0$. Without loss of generality, assume that the delays are ordered by $0 \le a < b$ for non-negative integers $a,\ b$. Assume that the network follows the following noisy linear dynamics

$$\begin{aligned}\dot{\mathbf{p}}(t) &= \sum_{k=0}^{b} A_k \mathbf{p}\left(t-\tau_k\right) + \boldsymbol{\xi}(t); \quad t > \tau_b,\\ \mathbf{p}(t) &= \phi(t); \quad t \le \tau_b,\end{aligned} \tag{18}$$

where vectors $\mathbf{p}(t)$, $\phi(t)$, and $\boldsymbol{\xi}(t)$ are defined as before. The matrices A_0, A_1, A_a, and A_b will be defined later with respect to the network type. Moreover, $A_i = 0_{n\times n}$ for all $i \in \{2,\dots,b-1\}/\{a\}$.

It is shown by [14, Th. 1] that the $\mathscr{H}_2$ norm of an exponentially stable system with delayed dynamics (18) can be found by

$$\|G\|_2^2 = \mathrm{Tr}\left(U_0\right), \tag{19}$$

where $U_0 = U(0)$ is the solution to the standard Lyapunov equations for the same system without delay. According to [14, Th. 4], the following equation has a unique solution $U_i \in \mathbb{R}^{n\times n}$ for $i = -b,\dots,b-1$ that provides the vector of U_0.

$$\left(M + Ne^{M_1^{-1}M_2\tau_1}\right)\begin{bmatrix}\mathrm{vec}\left(U_{b-1}\right)\\ \vdots \\ \mathrm{vec}\left(U_0\right)\\ \vdots \\ \mathrm{vec}\left(U_{-b}\right)\end{bmatrix} = \begin{bmatrix}-\mathrm{vec}\left(I\right)\\ \mathbf{0}_{(2b-1)n^2}\end{bmatrix}. \tag{20}$$

We define matrices M_1, M_2, M, and $N \in \mathbb{R}^{2bn^2\times 2bn^2}$ for system (18) as below

$$M = \begin{bmatrix} & A_0^\top\otimes I_n & A_1^\top\otimes I_n & \dots & A_b^\top\otimes I_n \\ I_{n^2} & & & & \\ & I_{n^2} & & & \\ & & \ddots & & \\ & & & I_{n^2} & \end{bmatrix},$$

$$N = \begin{bmatrix} I_n\otimes A_b^\top & \dots & I_n\otimes A_1^\top & I_n\otimes A_0^\top & \\ & -I_{n^2} & & & \\ & & -I_{n^2} & & \\ & & & \ddots & \\ & & & & -I_{n^2}\end{bmatrix},$$

$$M_1 = I_{2bn^2},$$

$$M_2 = \begin{bmatrix} A_0\otimes I_n & \dots & A_b\otimes I_n & & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ & & A_0\otimes I_n & & A_b\otimes I_n \\ -A_b\otimes I_n & \dots & -A_0\otimes I_n & & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ & & -A_b\otimes I_n & \dots & -A_0\otimes I_n\end{bmatrix}. \tag{21}$$

The components of matrix $Y = [Y_{ij}] = M + Ne^{M_1^{-1}M_2\tau_1}$ where $Y_{ij} \in \mathbb{R}^{n^2\times n^2}$ for $i,\ j \in \{1,2,\dots,2b\}$ can now be specified by the following expressions

$$Y_{ij} = \begin{cases} \left(I\otimes A_b^\top\right)S_{1j} + \left(I\otimes A_a^\top\right)S_{(b-a+1)j} & \\ +A_{j-b}^\top\otimes I & \text{if } i=1, j\in\{b+a,2b\},\\ \left(I\otimes A_b^\top\right)S_{1j} + \left(I\otimes A_a^\top\right)S_{(b-a+1)j} & \text{if } i=1, j\notin\{b+a,2b\},\\ I - S_{ij} & \text{if } i>1, j=i-1,\\ -S_{ij} & \text{if } i>1, j\neq i+1.\end{cases} \tag{22}$$

where $S = [S_{ij}] = e^{M_1^{-1}M_2\tau_1} = e^{M_2\tau_1}$. Equation (20) is a non-homogeneous linear equation with respect to the vector of delay Lyapunov matrices. Assuming that Y is invertibe, we have

$$X = Y^{-1} = \left(M + Ne^{M_1^{-1}M_2\tau_1}\right)^{-1}, \tag{23}$$

which then provides the solution to the linear equation (20).

Note that in order to compute the performance measure, it is sufficient to determine $\mathrm{vec}(U_0)$, see (19). Therefore, we only need to solve the following sub-set of equations in (20) associated with $\mathrm{vec}(U_0)$

$$\mathrm{vec}(U_0) = -X_V\ \ \mathrm{vec}\left(I\right), \tag{24}$$

in which $V = [(b-1)n^2+1 : bn^2, 1 : n^2]$. Using (19), the performance measure can now be computed by

$$\rho = -\operatorname{vec}(I)^\top X_V \ \operatorname{vec}(I). \tag{25}$$

In a more specific case, when the underlying graph is undirected and the epidemic rates are identical for all the subpopulations in the network, the presented results get even simpler by decoupling the dynamics of subpopulations using eigenvalue decomposition. This approach will lead into computing the performance measure of n subsystems each presenting only one mode of the original network. This would reduce the computational complexity of the original approach established in (25).

Let us consider the following dynamics for the i^{th} mode of the decoupled system, which is a single input, single output, single delay (SISOSD) subsystem with noise input $\xi_i(t)$,

$$\Sigma_i = \begin{cases} \dot{p}_i(t) = \alpha_i p_i(t) + \theta_i p_i(t-\tau) + \xi_i(t); \ t > \tau, \\ p_i(t) = \phi_i(t); \ t \le \tau, \end{cases} \tag{26}$$

where $p_i(t)$ is the fraction of infection in the subpopulation i and $\alpha_i, \ \theta_i \in \mathbb{R}$ will be specified with respect to the type of network.

Lemma 1. *For the SISOSD subsystem Σ_i with dynamics* (26)*, the performance measure defined by* (16) *is found as below,*

$$\rho_i \ = \ f(\alpha_i, \theta_i, \tau), \tag{27}$$

where

$$f(\alpha_i, \theta_i, \tau) \ := \ -\frac{\alpha_i z_i \sin(\tau z_i) - z_i^2 \cos(\tau z_i)}{2\left(\alpha_i^2 - \theta_i^2\right)\left(z_i \sin(\tau z_i) + \alpha_i \cos(\tau z_i) + \theta_i\right)},$$

and $z_i = Im\left(\sqrt{\alpha_i^2 - \theta_i^2}\right)$ if $|\alpha_i| < |\theta_i|$; and $z_i = j\sqrt{\alpha_i^2 - \theta_i^2}$ if $|\alpha_i| > |\theta_i|$.

The network performance measure can ultimately be found by $\rho = \sum_{i=1}^n \rho_i$. Note that to compute the performance measure using (25), we need to take the inverse of two n^2 by n^2 matrices and do matrix multiplication which could be significantly expensive for large networks. On the other hand, if the network is undirected with identical node dynamics, the performance measure computation includes only scalar inversion and summation over n parameters, which can be done much faster. With the presented approach, it is now possible to find a closed form for the performance measure of networks introduced earlier.

Network 1: The dynamics of this network can be represented by equation (18) where $a = 0$ and $b = 1$. Therefore, we have $\tau_0 = \tau_a = \tau_l = 0$, $\tau_1 = \tau_b = \tau_g$, $A_0 = A_a = -\delta I_n$, and $A_1 = A_b = \beta A$. Y is now a $2n^2$ by $2n^2$ matrix which could break into 4 sub-matrices/blocks of size $n^2 \times n^2$ as below

$$\begin{aligned} Y_{11} &= \left(I \otimes A_1^\top\right) S_{11} + \left(I \otimes A_0^\top\right) S_{21} + A_0^\top \otimes I, \\ Y_{12} &= \left(I \otimes A_1^\top\right) S_{12} + \left(I \otimes A_0^\top\right) S_{22} + A_1^\top \otimes I, \\ Y_{21} &= I - S_{21}, \\ Y_{22} &= -S_{22}. \end{aligned} \tag{28}$$

The performance measure can then be found by

$$\rho = -\operatorname{vec}(I)^\top \left[Y_{11}^{-1} + \left(Y_{11}^{-1} Y_{12}\right)\left(Y_{22} - Y_{21} Y_{11}^{-1} Y_{12}\right)^{-1} \left(Y_{21} Y_{11}^{-1}\right)\right] \operatorname{vec}(I). \tag{29}$$

A simpler version of this result can be obtained by the implementation of **Lemma 1**, where $\alpha_i = -\delta, \ \theta_i = \beta\lambda_i, \ \tau = \tau_g$ for the i^{th} mode of system. According to **Theorem 1**, **Network 1** is stable for any $R_0 \le 1$, which is equivalent to $|\beta\lambda_i| < |\delta|$ for all $i \in \mathscr{V}$. This condition is equivalent to $|\alpha_i| > |\theta_i|$ and therefore, $z_i = j\sqrt{\delta^2 - \beta^2\lambda_i^2}$. The corresponding performance measure of mode i can now be found by $\rho_i = f(-\delta, \beta\lambda_i, \tau_g))$ in (27). The overall performance of the network, defined in (16), is then determined by,

$$\rho \ = \ \sum_{i=1}^n f(-\delta, \beta\lambda_i, \tau_g). \tag{30}$$

Network 2: The dynamics of this network can be represented by equation (18), where $a = 0$ and $b = 1$, $\tau_0 = \tau_a = \tau_g = 0$, $\tau_1 = \tau_b = \tau_l$, $A_0 = A_a = \beta A$, and $A_1 = A_b = -\delta I_n$. The performance measure of this network is found by the same approach as (29) where the matrix Y is defined in (28).

When this network is associated with an undirected graph we can use **Lemma 1** to compute the performance measure analytically. In this regard, we define $\alpha_i = \beta\lambda_i, \ \theta_i = -\delta, \ \tau = \tau_l$ for the i^{th} mode of system. The condition $|\beta\lambda_i| < |\delta|$ is a necessary condition for the stability of system which is equivalent to $|\alpha_i| < |\theta_i|$. As a result, we have $z_i = \operatorname{Im}\left(\sqrt{\beta^2\lambda_i^2 - \delta^2}\right)$. The resulting performance measure for mode i, $\rho_i = f(\beta\lambda_i, -\delta, \tau_l)$, can be determined by substituting the above parameters in (27). The performance of the network can then be computed by,

$$\rho \ = \ \sum_{i=1}^n f(\beta\lambda_i, -\delta, \tau_l). \tag{31}$$

Network 3: The dynamics of this network can be represented by equation (18) where $a = 0$, $b = 1$, $\tau_0 = 0$, $\tau_1 = \tau_l = \tau_g$, $A_0 = A_a = 0_{n\times n}$, and $A_1 = A_b = \beta A - \delta I_n$. The performance measure of this network can be found by the same approach as (29) where the matrix Y is defined in (28).

When the underlying graph is undirected, we have: $\alpha_i = 0, \ \theta_i = \beta\lambda_i - \delta$ for the i^{th} mode of system. Having $z_i = \delta - \beta\lambda_i$, the corresponding performance measure of mode i, $\rho_i = f(0, \beta\lambda_i - \delta, \tau)$, can be found by substituting the above parameters in (27), which gives the following simple form,

$$f(0, \beta\lambda_i - \delta, \tau) \ = \ -\frac{\cos\left(\left(\beta\lambda_i - \delta\right)\tau\right)}{2\left(\beta\lambda_i - \delta\right)\left(1 + \sin\left(\left(\beta\lambda_i - \delta\right)\tau\right)\right)}. \tag{32}$$

The network performance can then be derived by,

$$\rho \ = \ \sum_{i=1}^n f(0, \beta\lambda_i - \delta, \tau). \tag{33}$$

Network 4: Assuming that $\tau_g < \tau_l$ the dynamics of this network can be represented by equation (18) where $\tau_a = \tau_g = a\tau_1$, $\tau_b = \tau_l = b\tau_1$, $A_0 = 0_{n\times n}$, $A_1 = 0_{n\times n}$, $A_a = \beta A$, and

$A_b = -\delta I_n$. The matrices (21)-(22) can now be formed to obtain the network performance measure by (25).

Note that for the studied networks, the condition $\beta\lambda_i < \delta$ for all $i \in \mathscr{V}$ must be satisfied in order to avoid unstabilities; and therefore, be able to obtain the network performance. Table I represents the stability criteria and performance measure of **Networks 1-3** found in closed form.

V. Case Study

As a case of study, we consider a selected group of 15 states in the United States as the representative of an epidemic network within the country, where every state is a subpopulation in the network. The states are ranked based on the number of passengers traveled to or from the states by air transportation in a certain period of time. The top 15 busiest airports and their corresponding states are then selected to generate the epidemic network, see Fig. 2. Without loss of generality, the interstate transportation is limited to air transportation. Assume the a virus first arrives to the United States by 3 of these airports and infects 2 percent of the entire metapopulation while spreading through the rest of the states by air transportation. Please note that this scenario is just for the purpose of illustration and does not affect the generality of the studied epidemic problem.

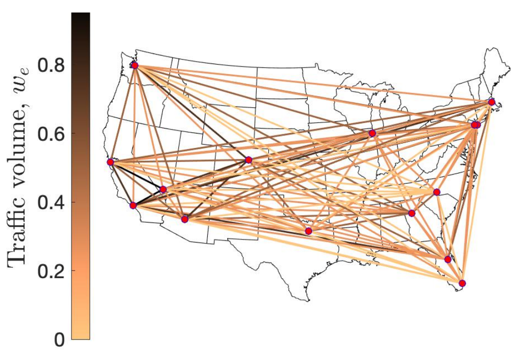

Fig. 2. The network of United States' 15 busiest airports and their air traffic volume specified by the edge colors.

For this network, all the simulations are based on the linear epidemic SIS model (3). The United States' air traffic data used in this study can be found in [19]. The range of the eigenvalues of the underlying graph is: $[\lambda_1 = -1.582, 6.641 = \lambda_{15}]$. For **Network 4**, assume that $a = 1$ and $b = 2$.

The Average Infection Size (AIS), $\bar{p} = \frac{1}{n}\sum_{i=1}^{n} p_i(t)$, of **Networks 1-4** is presented in Figs. 3(a), (b), (c), and (d), respectively. Note that having AIS outside the physically meaningful range $[0,1]$ in some cases is due to the linearization of dynamics.

Fig. 3(a) indicates that when $R_0 < 1$, **Network 1** is delay-independently stable, which corroborates the result of *Theorem 1*. Additionally, the infection size of networks experiencing a longer global delay take more time to converge to a zero AIS. This is due to the fact that a larger global delay, i.e., longer transition time between subpopulations, results in a slower progress in the spread of disease within the network.

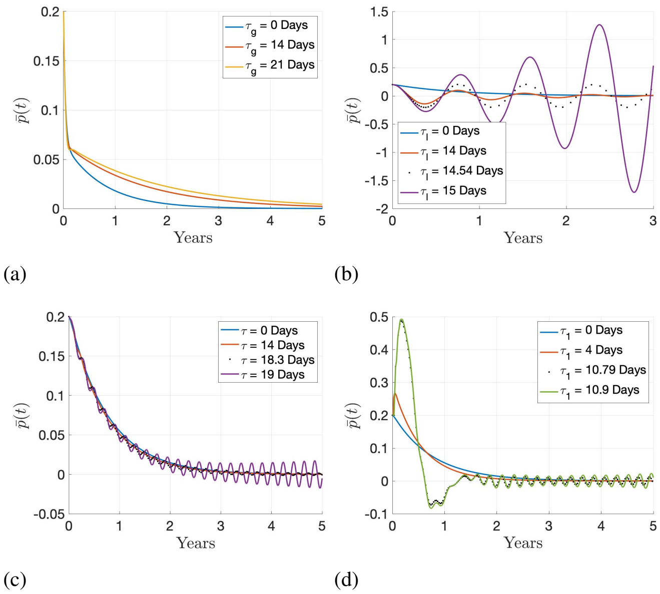

Fig. 3. The AIS of (a) **Network 1**, (b) **Network 2**, (c) **Network 3**, (d) **Network 4** with respect to different time delays. The dotted black curves show the marginally stable behaviour when the critical time-delay (found in section III) is applied.

The range of delay-dependent stability of **Network 2** for $R_0 < 1$ is a function of local delay threshold (12) found by *Theorem 2*, which is $\overline{\tau}_l = 14.54$ days for the simulated network. In Fig. 3(b), the AIS of network for $\tau_l = \overline{\tau}_l$ is shown by dotted black line. When $\tau_l = \overline{\tau}_l$, the system is marginally stable and for any local delay higher than that, it is unstable.

The AIS of **Network 3** when $R_0 < 1$ is shown in Fig. 3(c). As discussed in *Theorem 3*, for $R_0 < 1$, the range of delay-dependent stability of network is found by (13) which is $\overline{\tau} = 18.3$ days for the simulated network (shown by dotted black line). The marginal stability happens at $\tau = 18.3$ days, as expected by theoretical results. Fig. 3(d) represents the epidemic trend for **Network 4**.

The performance measure of **Networks 1-4** with respect to time-delay is shown in Figs. 4(a), (b), (c), and (d), respectively. The dashed red line in Figs. 4(a-c) indicates the result of analytic delay thresholds, which is exactly in line with the tangent of performance curves found in section IV. According to Fig. 4(a), increasing the global time-delay would improve network robustness against input noises, which results from the fact that a larger global delay would slow the spread of disease down. On the other hand, when there is only a local delay involved (Fig. 4(b)), it is desired to decrease the delay as much has possible to avoid performance loss caused by longer recovery periods. In Fig. 4(c) for **Network 3**, the presence of global delay along with local delay has reduced the scale of performance loss in compared to Fig. 4(b), which results from the positive effect of longer global delays on network performance. When the network is experiencing a local delay twice as long as its global delay, Fig. 4(d), it is expected to see a higher performance measure as well as lower range of time-delay

TABLE I
THE ANALYTIC RANGE OF STABILITY AND PERFORMANCE MEASURE FOR **NETWORKS 1-3**

	Network 1	**Network 2**	**Network 3**
Stability criteria	$\tau_g > 0$	$0 \leq \tau_l \leq \frac{2}{\sqrt{\delta^2-\beta^2\lambda_n^2}} \tan^{-1}\left(\frac{\sqrt{\delta^2-\beta^2\lambda_n^2}}{\delta+\beta\lambda_n}\right)$	$0 \leq \tau \leq \frac{\pi}{2(\delta-\beta\lambda_1)}$
Performance measure	$\rho = \sum_{i=1}^n f(-\delta, \beta\lambda_i, \tau_g)$ $z_i = j\sqrt{\delta^2 - \beta^2\lambda_i^2}$	$\rho = \sum_{i=1}^n f(\beta\lambda_i, -\delta, \tau_l)$ $z_i = \text{Im}\left(\sqrt{\beta^2\lambda_i^2 - \delta^2}\right)$	$\rho = \sum_{i=1}^n f(0, \beta\lambda_i - \delta, \tau)$ $z_i = \delta - \beta\lambda_i$

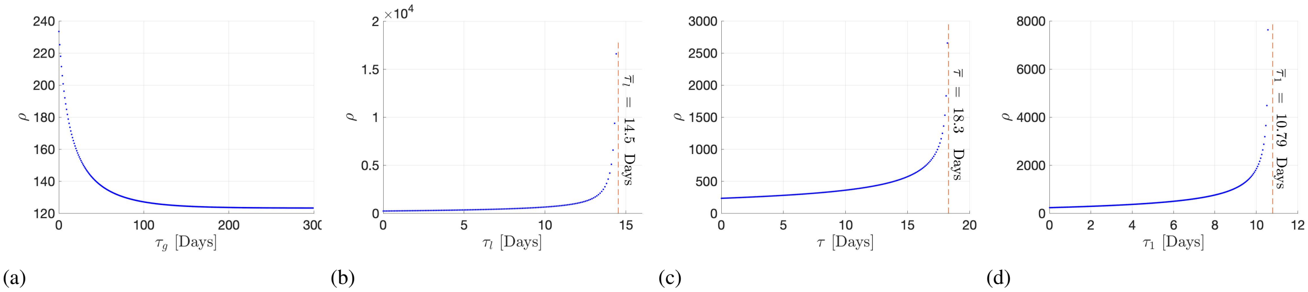

Fig. 4. The $\mathscr{H}_2$ norm of network Fig. 2 with respect to the number of days past from epidemic onset. (a) **Network 1**. (b) **Network 2**. (c) **Network 3**. (d) **Network 4**.

to keep the system stable, because longer local delay would impair the performance.

VI. DISCUSSIONS

In this study, the multi-delayed SIS dynamics of an epidemic network when it is experiencing (i) only global delay, (ii) only local delay, (iii) identical global and local delays, and (iv) commensurate delays, is established. The delay-dependent stability analysis for every scenario is then accomplished to determine the necessary and sufficient conditions of stability and find a closed-form stability criteria regardless of network size. Furthermore, a performance measure based on network's $\mathscr{H}_2$ norm is adopted to investigate the effect of additive noise and time-delay on the network performance. The simulation results on the network of United States' 15 busiest airports corroborate the analytical findings.

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