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Role of Agent Update Cycle in Stability and Robustness of Second-Order Consensus Networks

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Abstract— **We consider the problems of asymptotic stability and robustness in large-scale second-order consensus networks and vehicle platoons in the discrete-time domain. First, we develop a graph-theoretic methodology to design the state feedback law for the second-order consensus networks and vehicle platoons in a discrete-time framework. We analyze the stability of such networks based on algebraic properties of the Laplacian matrices of underlying graphs and each vehicle's update cycle (also known as the time step). We further provide a necessary and sufficient condition of stability of a linear second-order consensus network in the discrete-time domain. Moreover, we evaluate the robustness of the consensus networks by employing the expected value of the steady-state dispersion of the state of the entire network, also known as squared $\mathcal{H}_2$-norm, as a performance measure. We show the connection between performance measures with respect to network size, connectivity, and the update cycle. The main contribution of this work is that we provide a formal framework to quantify the relation between scaling performance measures and restrictions of the vehicles' update cycles. Specifically, we show that denser networks (i.e., networks with more communications/edges) require faster agents (i.e., smaller update cycles) to outperform or achieve the same level of robustness as sparse networks (i.e., networks with fewer communications/edges).**

I. Introduction

A multi-agent system consists of multiple interacting autonomous agents to accomplish a mutual goal via a process of collaboration, feedback, and iteration [1]–[4]. Multi-agent systems have received enormous attention in recent years because of the ubiquity of complex dynamical networks in real-world applications such as smart power grids [5], vehicle platooning [6], aerial drone display [7], social networks [8], high-speed satellite internet [9], and Internet of Things (IoT) [10]. Moreover, the distributed systems have been extensively studied in the controls community due to their wide range of applications, from robotics [11]–[13] to biological and ecological networks [14]–[16].

One critical challenge in multi-agent systems is a communication protocol used for information exchange; each agent can share its states while obeying this protocol. All agents can reach an agreement by designing an appropriate interconnection topology where agents are limited to receive and interact with their neighbors. A well-known approach to investigate the behavior of multi-agent dynamical systems consensus is algebraic graph theory. Some necessary and sufficient algebraic conditions to achieve consensus for first-order and second-order systems in continuous time have been provided in [17]–[21]. Furthermore, in [22]–[26], an alternative framework of a consensus problem is studied in discrete time.

Previous studies mainly focus on identical underlying graphs for the first and second integrator; the first and second integrator share the same information flow between neighbors. In this paper, we construct a different structure of interrelation topology in a second-order dynamical system. Roughly speaking, we consider the case that agents exchange their states throughout different routes. In this case, the underlying graph for the agents' positions is different from the underlying graph for the agents' velocities, see Figs. 1 and 2. Specifically, two directly connected agents via position topology may not be directly connected via velocity topology. Upon this interconnection protocol, we investigate vehicle platoons models that have received growing attention in the past few decades due to their potential contribution to road transportation.

The primary goal of vehicle platoons is to make all vehicles in the group reach and remain at a common moving speed while maintaining a certain distance from their predecessors. Several challenging issues arise from this area of research: for instance, the role of delayed agents (i.e., long update cycle) and external disturbances on the stability and performance of the entire network. In [27], [28], the authors consider the robustness control which the system is subjected to stochastic disturbances. The disturbances can grow up and propagate along with information interchanging within vehicles. Article [29] studied the stability of a vehicle platoon network with a ring coupling graph and path graph in the presence of time delays. Spacing policy in [30], [31] specifies the desired distance between vehicles such that guarantee all vehicles asymptotic tracking a group of heterogeneous mobiles.

Due to the significant effect of external disturbances on consensus behavior, a tremendous amount of researches are dealing with the stochastic force. The performance of vehicle platoons in which each agent has to maintain a certain agreement, such as common velocity, steering angle, or inter-vehicular spacing, is deteriorated by exogenous stochastic disturbances. In [32], the performance measurement in terms of $\mathcal{H}_2$-norm, which captures the notion of coherence, is

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studied and shows the connection between performance measures scale of a multi-dimensional vehicular formation dynamical network and system size. Another performance measure that quantifies the expected value of the steady-state dispersion has been investigated in [33].

This paper provides some necessary and sufficient conditions to stabilize a discrete-time vehicle platoons model based on algebraic properties of the Laplacian matrices of underlying graphs and each vehicle's update cycle (a.k.a., the time step). Besides, this paper provides a quantitative method to evaluate the performance measure of vehicular formation dynamical systems in a discrete-time framework. We assess the robustness of the consensus networks by employing the expected value of the steady-state dispersion of the state of the entire network, i.e., squared $\mathcal{H}_2$-norm, as a performance measure. We show the connection between performance measures with respect to network size, connectivity, and the update cycle. Fundamental tradeoffs reveal the interplay between performance measures and restrictions of the vehicles' update cycles and are discussed in Section V.

In this conference paper, the proofs are omitted due to the space limitation.

II. Preliminaries

Throughout this paper, the $n \times n$ identity matrix is denoted by I_n, the $m \times n$ zero matrix by $\mathbf{0}_{m\times n}$, the $n \times n$ matrix of all ones by J_n, the transposition of matrix A by A^T, pseudo-inverse of matrix A by $A^{\dagger}$. All graphs are assumed to be finite, simple, and undirected. Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, w\}$ denote an undirected graph, where $\mathcal{V}$ is the set of nodes, $\mathcal{E} \subseteq \{(i,j) \mid i,j \in \mathcal{V}, i \neq j\}$ is the set of edges, and $w : \mathcal{E} \to \mathbb{R}_+$ is the weight function. An unweighted graph $\mathcal{G}$ is a graph with weight function $w(e) = 1$ for $e \in \mathcal{E}$. The neighbors of the i-th agent are denoted by $\mathcal{N}_i = \{i \in \mathcal{V} \mid (i,j) \in \mathcal{E},\ j \neq i\}$. The adjacency matrix $A = [a_{ij}]$ of graph $\mathcal{G}$ is defined by setting $a_{ij} = w(e)$ if $e = (i,j) \in \mathcal{E}$. The Laplacian matrix $\mathcal{L}$ of graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, w)$ is defined by:

$$\begin{cases} l_{ij} = -a_{ij} & \text{for } i \neq j, i,j \in \mathcal{V} \\ l_{ii} = -\sum\limits_{j=1, j\neq i}^{n} l_{ij} & \text{for } i \in \mathcal{V}. \end{cases} \tag{1}$$

The directed incidence matrix $\mathcal{D}$ of graph $\mathcal{G}$ is defined by:

$$\mathcal{D}_{ij} = \begin{cases} e & \text{for edge } j \text{ is } (k,i),\ k \text{ is the tail and } i \text{ is the head} \\ -e & \text{for edge } j \text{ is } (i,k),\ i \text{ is the tail and } k \text{ is the head} \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

III. Modeling and System Analysis

In this paper, we formulate two types of models, consensus networks and vehicle platoons. Both are second-order discrete-time dynamic models with differing involvement of absolute data in feedback law. Specifically, absolute position and absolute velocity are involved in the feedback law of a vehicle platoon model, while a consensus problem only considers the relative date. We begin with designing the feedback law based on the communication topologies. For the simplicity of notations, we denote the first integrator by x as position states, and second integrator by v as velocity states. It is assumed that a reduced-order model of these dynamical networks can be expressed using two state variables of each agent: i-th vehicle's position x_i and i-th vehicle's velocity v_i for $i \in \mathcal{V}$.

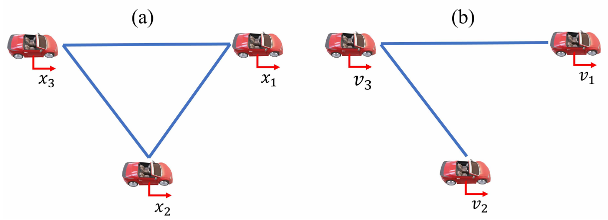

Fig. 1: Example of different underlying graphs with 3 agents: (a) position graph $\mathcal{G}_x$ (b) velocity graph $\mathcal{G}_v$.

A. Second-order Consensus Network

The goal of a second-order consensus network is the states of all agents in the system reach certain agreement. A general second-order discrete-time version of second-order consensus (SOC) network without any feedback loops has the following form:

$$\begin{bmatrix} x(k+1) \\ v(k+1) \end{bmatrix} = \begin{bmatrix} I & \gamma I \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \gamma I \end{bmatrix} \bar{u}(k)$$

where γ is the update cycle and $\bar{u}(k)$ is a cooperative feedback of the network. This proposed feedback law can be expressed as certain virtual springs and dampers between the agents. Specifically, each agent is controlled based on the interconnection of relative position and relative velocity. The underlying graphs for position/velocity represent the access to relative position/velocity measurements. For example, given a position graph $\mathcal{G}_x = (\mathcal{V}_x, \mathcal{E}_x, w_x)$, if agent j is a neighbor of agent i, the system is able to measure the relative position $x_i - x_j$. To obtain relative position/velocity between vertices and their neighbors, we define r and q as relative position/velocity:

$$r := \mathcal{D}_x^T x \text{ and } q := \mathcal{D}_v^T v$$

where $x = [x_1, x_2, \cdots, x_n]^T$, $v = [v_1, v_2, \cdots, v_n]^T$, $\mathcal{D}_x$ and $\mathcal{D}_v$ are the corresponding incidence matrix with orientation on $\mathcal{G}_x$ and $\mathcal{G}_v$, respectively. The proposed feedback law with underlying graphs $\mathcal{G}_x = (\mathcal{V}_x, \mathcal{E}_x, F)$ and $\mathcal{G}_v = (\mathcal{V}_v, \mathcal{E}_v, G)$ is as follow:

$$\bar{u}(k) = -\mathcal{D}_x F\left(r(k)\right) - \mathcal{D}_v G\left(q(k)\right). \tag{3}$$

The interrelation feedback on agent i can be written as:

$$\bar{u}_i = -\sum_{j|(i,j)\in\mathcal{E}_x} f_{ij}(x_i - x_j) - \sum_{j|(i,j)\in\mathcal{E}_v} g_{ij}(v_i - v_j) \tag{4}$$

where $f_{ij} \in \mathcal{E}_x$ and $g_{ij} \in \mathcal{E}_v$ are nonnegative weight on edges between agent i and j, or they can be interpreted as virtual spring/damper constants between agent i and j. Let $\mathcal{L}_x$ and $\mathcal{L}_v$ be the corresponding Laplacian matrices of $\mathcal{G}_x = (\mathcal{V}_x, \mathcal{E}_x, F)$ and $\mathcal{G}_v = (\mathcal{V}_v, \mathcal{E}_v, G)$. From graph theory, the

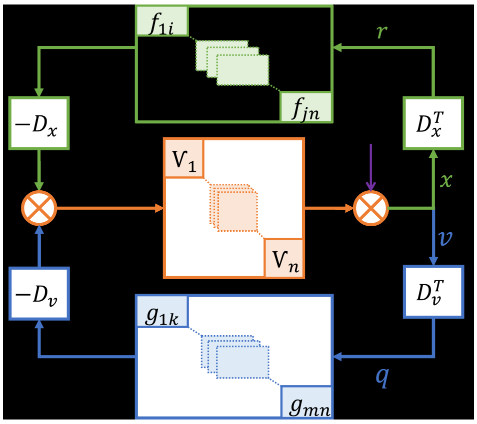

Fig. 2: Feedback law of the SOC networks in terms of block diagram. This network consists of n agents $V_1, \cdots, V_n$.The outputs from agent i are position x_i and velocity states v_i that are multiplied by the transposed incidence matrix of the underlying graph $\mathcal{G}_x$ and $\mathcal{G}_v$ to obtain relative position r and velocity q. Then multiplied these relative states by weights and incidence matrix of the underlying graphs to obtain the feedback $\bar{u}$.

feedback law in (3) can be rewritten as:

$$\bar{u}(k) = -\mathcal{L}_x x(k) - \mathcal{L}_v v(k). \tag{5}$$

The state space of this second-order discrete-time consensus system, where all agents collaborate based on the sharing information from their neighbors, can be rewritten as:

$$\begin{bmatrix} x(k+1) \\ v(k+1) \end{bmatrix} = \begin{bmatrix} I & \gamma I \\ -\gamma\mathcal{L}_x & I-\gamma\mathcal{L}_v \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \gamma I \end{bmatrix} \xi(k) \tag{6}$$

where ξ is the external input. Fig. 2 depicts the feedback law of the complex network systems in terms of block diagram. The output from the agents x and v are position/velocity states, which are multiplied by the incidence matrix transposition of the underlying graph $\mathcal{G}_x$ and $\mathcal{G}_v$ to obtain relative position/velocity r and q. Then, we multiplied these relative states by weights and the incidence matrix of the underlying graphs to obtain the feedback $\bar{u}$.

B. Vehicle Platoons

Next, we extend the idea of cooperative feedback in SOC networks to formulate a vehicle platoon model. We assume all vehicles in the platoons have access to their own states; that is, x_i and v_i are available.

Consider having all vehicles in the platoon follow a desired trajectory with driving at the desired speed of v^d while maintaining certain spacing Δ between each other. The desired trajectory x^d for i-th vehicle is:

$$x_i^d := \gamma v^d + i\Delta. \tag{7}$$

The position/velocity deviations from desired trajectory of agent i are defined as

$$\tilde{x}_i := x_i - x_i^d, \text{ and } \tilde{v}_i := v_i - v_i^d.$$

The interrelation feedback of i-th vehicle that satisfies the goal that follows the desired trajectory x_i^d at a desired moving speed of v^d while keep distance from predecessor is:

$$\bar{u}_i = -\sum_{j|(i,j)\in\mathcal{E}_x} f_{ij}(\tilde{x}_i-\tilde{x}_j) - \sum_{j|(i,j)\in\mathcal{E}_v} g_{ij}(\tilde{v}_i-\tilde{v}_j) - \tilde{x}_i - \tilde{v}_i. \tag{8}$$

The vehicle platoons system in state space can be written as:

$$\begin{bmatrix} \tilde{x}(k+1) \\ \tilde{v}(k+1) \end{bmatrix} = \begin{bmatrix} I & \gamma I \\ -\gamma(I+\mathcal{L}_x) & I-\gamma(I+\mathcal{L}_v) \end{bmatrix} \begin{bmatrix} \tilde{x}(k) \\ \tilde{v}(k) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \gamma I \end{bmatrix} \xi(k), \tag{9}$$

where $\xi(.)$ is external input.

C. Stability Analysis

It is shown that in continuous-time SOC networks with the same connected underlying graphs for position and velocity (i.e., $\mathcal{G}_x = \mathcal{G}_v$), all agents in the system will asymptotically reach an agreement. However, we investigate these dynamical networks in the discrete-time framework where the update cycle γ plays an important role. The SOC networks (6) and vehicle platoons (9) can be unstable or fragile with inappropriate choices for the update cycle (i.e., γ). Therefore, in this paper, we first investigate the upper bound of the update cycle γ to have marginally stable systems. To this end, the following classic and well-known result is used.

Proposition 1: A discrete-time LTI system is marginally stable if and only if the largest eigenvalue of state matrix A or the largest magnitude of the poles of the transfer function is on a unit disk of $\mathbb{C}$.

The next lemma presents necessary and sufficient conditions for stability of SOC networks.

Lemma 1: Discrete-time second-order consensus network (6) with the same underlying graph but different scalar weights, i.e. $\mathcal{L}_x = \zeta\mathcal{L}_v$, where $\zeta \in \mathbb{R}_+$ is marginally stable if and only if

$$\begin{cases} 0 < \gamma \leq \frac{\lambda_n^{(v)}}{\lambda_n^{(x)}}, \text{ where } \left(\lambda_n^{(v)}\right)^2 - 4\lambda_n^{(x)} < 0 \\ 0 < \gamma \leq \frac{4}{\lambda_n^{(v)} + \sqrt{\left(\lambda_n^{(v)}\right)^2 - 4\lambda_n^{(x)}}}, \text{ otherwise} \end{cases} \tag{10}$$

where $\lambda_n^{(x)}$ and $\lambda_n^{(v)}$ are the largest eigenvalues of $\mathcal{L}_x$ and $\mathcal{L}_v$, respectively.

Lemma 2: Discrete-time vehicle platoon system (9) with the same underlying graph but different scalar weights, i.e. $\mathcal{L}_x = \zeta\mathcal{L}_v$, where $\zeta \in \mathbb{R}_+$ is stable if and only if

$$\begin{cases} 0 < \gamma \leq \frac{\lambda_n^{(v)}+1}{\lambda_n^{(x)}+1}, \text{ where } (\lambda_n^{(v)}+1)^2 - 4(\lambda_n^{(x)}+1) < 0 \\ 0 < \gamma \leq \frac{4}{\lambda_n^{(v)}+1+\sqrt{(\lambda_n^{(v)}+1)^2 - 4(\lambda_n^{(x)}+1)}}, \text{ otherwise} \end{cases} \tag{11}$$

where $\lambda_n^{(x)}$ and $\lambda_n^{(v)}$ are the largest eigenvalues of $\mathcal{L}_x$ and $\mathcal{L}_v$, respectively.

Remark 1: Due to the fact that the upper bound of time step γ only depends on the largest eigenvalue of Laplacian matrix, one can relate this to degree of agents. For example, given a SOC network (6) where $\mathcal{L}_x = \mathcal{L}_v$, one is always the upper bound of time step when the number of agents $n \leq 4$. In addition, Anderson and Morley [34] claimed that $\lambda_n \leq \max(d_i + d_j|(i,j) \in \mathcal{E})$, thus we know that when the number of agents $n > 4$, for an unweighted ring or path graph, the system is always stable while setting $\gamma \leq 1$.

IV. Performance Measurements

In this paper, we adopt the $\mathcal{H}_2$-norm of the system (from the disturbance input to output) as a scalar performance measure. Due to the fact that the state matrix of the dynamical system is constructed with interconnection graphs, the performance measure can be calculated as a function of the eigenvalues of the Laplacian matrices of corresponding underlying graphs. We consider two cases of networks system: SOC networks (6) and vehicle platoons (9) with pre-defined desired trajectory. We begin with a general discrete time LTI system subjects to persistent stochastic disturbances (white noise) with zero mean, resulting in the states fluctuate around the equilibrium:

$$\begin{cases} z(k+1) \ = \ A\, z(k) \ + \ B\, \xi(k) \\ y(k) \ = \ C\, z(k) \end{cases}$$

where $z(k) = [x_1, \cdots, x_n, v_1, \cdots, v_n]^T$, $\xi(k)$ is an exogenous uncorrelated white stochastic process with zero-mean and identity covariance matrix. $y(k)$ is the performance output of the networks. Matrix C is the output matrix of the network and takes the following structural constraint

$$C = C_{Q_x} \oplus C_{Q_v}$$

where C_{Q_x} and $C_{Q_v} \in \mathbb{R}^{n\times n}$ are the output graph for position and velocity, respectively.

Definition 1: Suppose that C is an output graph of the system. The steady-state variance of the performance output of the network is considered as the performance measure

$$\rho_{\mathrm{ss}}(A, C) = \lim_{k\to\infty} \mathbb{E}\left[y(k)^T y(k)\right]. \tag{12}$$

Since A is not necessarily Hurwitz, the states may not have finite steady-state variances. However, unstable modes of the system should not be observable from the output $y(k)$ in order to guarantee the performance measure is well-defined.

This performance index (12) is equivalent to (squared) $\mathcal{H}_2$-norm of the network, which measures the expected output value of the system subjected to a stochastic perturbation defined as follow:

$$\begin{aligned} \|\mathcal{H}\|_2^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Trace}(H(e^{j\omega})^T H(e^{j\omega}))d\omega \\ &= \mathrm{Trace}(C\mathcal{W}_c C^T) = \mathrm{Trace}(B^T\mathcal{W}_o B). \end{aligned} \tag{13}$$

Without any exogenous input, all agents in the SOC networks (6) converge to a common state, while all vehicles in the vehicle platoon (9) follow their desired trajectory and desired velocity with the values of the performance measures equal to zero.

Remark 2: If a stable discrete-time LTI system is controllable, there is a unique solution $\mathcal{W}_c$ such that $A\mathcal{W}_c A^T - \mathcal{W}_c = -BB^T$, where $\mathcal{W}_c$ is positive definite and given by $\mathcal{W}_c = \sum_{k=0}^{\infty} A^k B^k B^T (A^T)^k$. Similarly, if the stable system is observable, there is a unique solution $\mathcal{W}_o$ such that $A^T\mathcal{W}_o A - \mathcal{W}_o = -C^T C$, where $\mathcal{W}_o$ is positive definite and given by $\mathcal{W}_o = \sum_{k=0}^{\infty}(A^k)^T C^T C A^k$.

We next evaluate the performance measures with respect to their corresponding output graph.

Theorem 1: Given SOC network (6) with the same underlying graphs for position and velocity but different scalar weights, i.e. $\mathcal{L}_x = \zeta\mathcal{L}_v$ where $\zeta \in \mathbb{R}_+$, the performance measure of the system can be quantified as:

$$\rho_{\mathrm{ss}}(A; M_n \oplus 0) = \sum_{i=2}^{n} \frac{\gamma\lambda_i^{(v)} - \gamma^2\lambda_i^{(x)} - 2}{\lambda_i^{(x)}\mathcal{M}} \tag{14}$$

$$\rho_{\mathrm{ss}}(A; 0 \oplus M_n) = \sum_{i=2}^{n} \frac{-2}{\mathcal{M}} \tag{15}$$

$$\rho_{\mathrm{ss}}(A; M_n \oplus M_n) = \sum_{i=2}^{n} \frac{\gamma\lambda_i^{(v)} - (\gamma^2 - 2\gamma + 2)\lambda_i^{(x)} - 2}{\lambda_i^{(x)}\mathcal{M}} \tag{16}$$

where

$$\mathcal{M} = \gamma^3\left(\lambda_i^{(x)}\right)^2 + 2\gamma\left(\lambda_i^{(v)}\right)^2 + 4\gamma\lambda_i^{(x)} - 3\gamma^2\lambda_i^{(x)}\lambda_i^{(v)} - 4\lambda_i^{(v)}.$$

Theorem 2: Given vehicle platoon (9) with the same underlying graphs for position and velocity but different scalar weights, i.e., $\mathcal{L}_x = \zeta\mathcal{L}_v$ where $\zeta \in \mathbb{R}_+$, the performance measure of the system can be quantified as:

$$\rho_{\mathrm{ss}}(A; M_n \oplus 0) = \sum_{i=2}^{n} \frac{\gamma(\lambda_i^{(v)} + 1) - \gamma^2(\lambda_i^{(x)} + 1) - 2}{(\lambda_i^{(x)} + 1)\mathcal{M}} \tag{17}$$

$$\rho_{\mathrm{ss}}(A; 0 \oplus M_n) = \sum_{i=2}^{n} \frac{-2}{\mathcal{M}} \tag{18}$$

$$\rho_{\mathrm{ss}}(A; M_n \oplus M_n) = \sum_{i=2}^{n} \frac{\gamma(\lambda_i^{(v)} + 1) - (\gamma^2 - 2\gamma + 2)(\lambda_i^{(x)} + 1) - 2}{(\lambda_i^{(x)} + 1)\mathcal{M}} \tag{19}$$

where $\mathcal{M} = \gamma^3\left(\lambda_i^{(x)} + 1\right)^2 + 2\gamma\left(\lambda_i^{(v)} + 1\right)^2 + 4\gamma(\lambda_i^{(x)} + 1) - 3\gamma^2(\lambda_i^{(x)} + 1)(\lambda_i^{(v)} + 1) - 4(\lambda_i^{(v)} + 1)$.

V. Numerical Examples

In this section, we consider several numerical examples to demonstrate our theoretical results.

Example 1: Assume that a discrete-time vehicle platoon model (9) has five vehicles, and both underlying graphs $\mathcal{G}_x$ and $\mathcal{G}_v$ are star graphs with $F = 2$ and $G = 1.5$, where F is the scalar weights of $\mathcal{G}_x$, and G is the scalar weights of $\mathcal{G}_v$, respectively. From Lemma 2, direct computation shows that the upper bound of time step $\gamma \simeq 0.2895$. We then simulate the dynamic model (9) with random initial states in three cases: $\gamma = 0.2885$, $\gamma = 0.2895$, $\gamma = 0.2905$. Fig. 3 demonstrates that when the update cycle is slightly smaller that the upper bound; $\gamma = 0.2885$, the system is stable. Velocity output converge asymptotically to desired velocity while all vehicles keeping spacing $\Delta = 1$ between each others. The system is marginally stable where $\gamma = 0.2895$, which is equal to upper bound. Both position and velocity keep fluctuating around its equilibrium. When the update cycle is slightly larger than the upper bound; $\gamma = 0.2905$, the

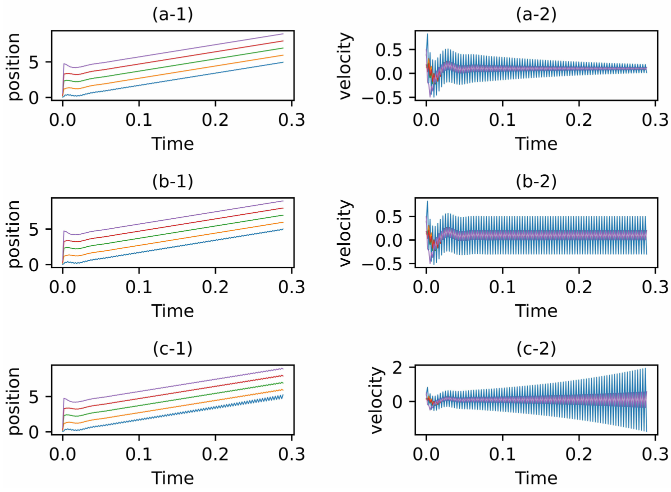

Fig. 3: Example of different time step γ in a discrete-time vehicle platoon dynamic (9) consisting of $n = 5$ vehicles with setting $\delta = 1$ and desired velocity $\bar{v} = 0.1$. In subplots (a-1) and (a-2), the update cycle is $\gamma = 0.2885$; according to Lemma 2 the system is stable, and vehicles' velocity converge to v^d while all vehicles keeping spacing $\delta = 1$ from each others. In subplots (b-1) and (b-2), the update cycle is $\gamma = 0.2895$; based on Lemma 2 the system is marginally stable, both position and velocity output keep fluctuating around its equilibrium. In subplots (c-1) and (c-2), the update cycle is $\gamma = 0.2905$, based on Lemma 2 the system is unstable, both position output and velocity output diverge.

system is unstable. Both position output and velocity output diverge.

Example 2: Next, we evaluate the relationship between connectivity and the performance measures. We consider the performance measures by three different structures of output matrices of a vehicle platoon networks (9) consisting of twenty agents with their underlying graphs constructed by the Erdős–Rényi model [35]. In the Erdős–Rényi model, two parameters are determined to create random networks, where n is the number of nodes and p is the probability of connecting edges between each two nodes. As shown in Fig. 4, the values of the performance measure grow in association with the time step.

Remark 3: For continuous-time consensus network, it is shown that the more connected the networks, the more coherent or better performance measures the system has (see [33], [36], [37] and references therein). As a result, these networks are more capable of reducing the influence of stochastic disturbance by increasing the number of connections/edges. However, in the discrete-time domain, as demonstrated in the subplots of Fig. 4, depending on value of the update cycle adding edges can improve or worsen the performance measure. Specifically, for small value of the update cycle, adding edges improves the performance loss; however, for the large update cycle, increasing the connectivity of the underlying graph can result larger performance loss. This implies a fundamental tradeoff among connectivity of the networks, performance measure, and limitations or restrictions of time step in the discrete-time framework. For large value of $\mathcal{H}_2$-norm, one can see from Fig. 4 that denser networks require faster agents (i.e., smaller update cycles) to achieve the same level of robustness as sparse networks. Moreover, the subplots in Fig. 4 show that the values of performance measure dramatically increase where the time step γ is approaching the condition of marginally stable discussed in Section III-C.

In Fig. 4, we can see that the value of performance loss grows as the update cycle increases. Moreover, where the update cycle tends to zero (i.e., $\gamma \to 0$), the value of performance measure can be approximated by the continuous-time counterpart.

VI. Concluding Remarks

This paper investigates the distributed consensus and vehicle platoons control problems by introducing a graph-theoretic methodology to design the feedback law of these systems in a discrete-time framework. The stability of this class of dynamical networks can be evaluated by the specific structure of the underlying graphs and the update cycle (the time step) of autonomous agents where a necessary and sufficient condition is presented. Furthermore, we investigated the robustness and performance of cooperative control approaches in discrete-time vehicle platoons using algebraic graph theory. We use a $\mathcal{H}_2$-based metric as a macroscopic performance measure capturing the notion of coherence [32]. This performance measure quantifies the expected values of output dispersion of the linear consensus networks subjected to stochastic disturbances. The measure is monotonically increasing as the network size enlarges or the connectivity of the underlying graph reduces. We observed a fundamental tradeoff between the graph connectivity and the update cycle.

A potential future direction is to cast the feedback design problem as a convex optimization problem to improve both stability and robustness of second-order consensus networks at the same time, similar to [38].

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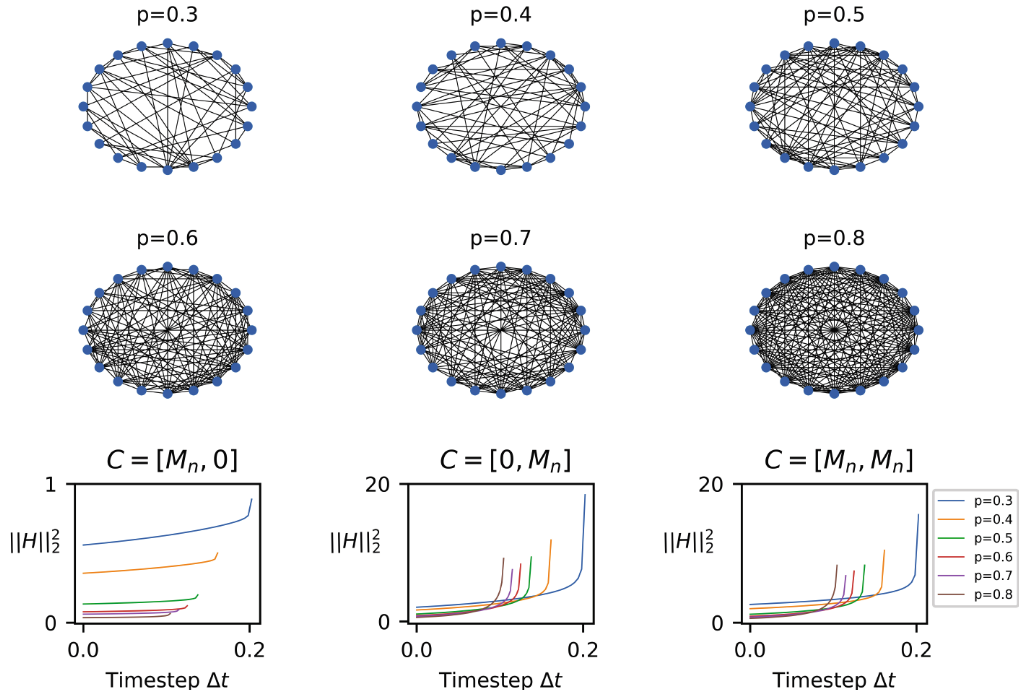

Fig. 4: Top six subplots show the networks consisting of 20 agents where their interconnection topologies constructed by Erdős–Rényi model with ascending connected probability. The three bottom subplots show three measurements associated with different output matrices. For small value of update cycle $\Delta t = \gamma$ (i.e., agile agents), the values of the performance measure decrease as the probability of connectivity increases. However, this does not hold for slow agents (i.e., large update cycles)

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