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The J-equation and the supercritical deformed Hermitian–Yang–Mills equation

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Abstract In this paper, we prove that for any Kähler metrics ω_0 and χ on M, there exists a Kähler metric $\omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi > 0$ satisfying the J-equation $\mathrm{tr}_{\omega_\varphi}\chi = c$ if and only if $(M, [\omega_0], [\chi])$ is uniformly J-stable. As a corollary, we find a sufficient condition for the existence of constant scalar curvature Kähler metrics with $c_1 < 0$. Using the same method, we also prove a similar result for the supercritical deformed Hermitian–Yang–Mills equation.

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1 Introduction

In this paper, our main goal is to prove the equivalence of the solvability of the J-equation and a notion of stability. Given Kähler metrics ω_0 and χ on M, the J-equation is defined as

$$\mathrm{tr}_{\omega_\varphi}\chi = c$$

for

$$\omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi > 0.$$

In general, the equivalence of the stability and the solvability of an equation is very common in geometry. One of the first results in this direction was the celebrated work by Donaldson–Uhlenbeck–Yau [17,46] on Hermitian–Yang–Mills connections. Inspired by the study of Hermitian–Yang–Mills connections, Donaldson proposed many questions, including the study of the J-equation using the moment map interpretation [18]. This was the first appearance of the J-equation in the literature.

Yau conjectured that the existence of a Fano Kähler–Einstein metric is also equivalent to some kind of stability [48]. Tian made this conjecture precise in the Fano Kähler–Einstein case, and it was called the K-stability condition [44]. It was generalized by Donaldson to the constant scalar curvature Kähler (cscK) problem in the projective case using a Riemann–Roch type formula to calculate the "Donaldson–Futaki invariants" on "test configurations" [19]. This conjecture has been proved by Chen–Donaldson–Sun [8–10] in the Fano Kähler–Einstein case. However, there is evidence that this conjecture may be wrong in the cscK case [1]. A folklore conjecture states that the uniform version of K-stability may be a correct substitution. More recently, the projective assumption in the definition of uniform K-stability was removed by Dervan–Ross [22] and, independently, by Sjöström Dyrefelt [36], using an intersection formula to replace the Riemann-Roch type formula. When restricted to special test configurations called "degeneration to normal cones," the uniform K-stability is reduced to Ross-Thomas's uniform slope K-stability [35].

It is easy to see that cscK metrics are critical points of the K-energy functional [11]

$$K(\varphi) = \int_M \log\left(\frac{\omega_\varphi^n}{\omega_0^n}\right)\frac{\omega_\varphi^n}{n!} + \mathcal{J}_{-\mathrm{Ric}(\omega_0)}(\varphi).$$

The $\mathcal{J}_\chi$ functional for any real smooth closed (1,1)-form χ is defined by

$$\mathcal{J}_\chi(\varphi) = \frac{1}{n!}\int_M \varphi \sum_{k=0}^{n-1} \chi \wedge \omega_0^k \wedge \omega_\varphi^{n-1-k} - \frac{1}{(n+1)!}\int_M c_0\varphi \sum_{k=0}^{n} \omega_0^k \wedge \omega_\varphi^{n-k},$$

where c_0 is the constant given by

$$\int_M \chi \wedge \frac{\omega_0^{n-1}}{(n-1)!} - c_0\frac{\omega_0^n}{n!} = 0.$$

When χ is a Kähler form, it is well known that the critical point of the $\mathcal{J}_\chi$ functional is exactly the solution of the J-equation. This result [11] was the second appearance of the J-equation in the literature. Following this formula and using the interpolation of the K-energy and the $\mathcal{J}_\chi$ functional, Chen-Cheng [5–7] proved that the existence of a cscK metric is equivalent to the geodesic stability of the K-energy functional. However, the relationship between the existence of cscK metrics and the uniform K-stability is still open.

When we replace the K-energy functional by the $\mathcal{J}_\chi$ functional for a Kähler form χ, the analogies of the K-stability and the slope stability conditions were proposed by Lejmi and Székelyhidi [30]. See also Sect. 6 of [22] for the extension to the non-projective case. The main theorem of this paper proves the equivalence between the existence of the critical point of the $\mathcal{J}_\chi$ functional, the solvability of the J-equation, the coerciveness of the $\mathcal{J}_\chi$ functional, and the uniform J-stability as well as the uniform slope J-stability.

Theorem 1.1 (Main Theorem) *Fix a Kähler manifold M^n with Kähler metrics χ and ω_0. Let $c_0 > 0$ be the constant such that*

$$\int_M \chi \wedge \frac{\omega_0^{n-1}}{(n-1)!} = c_0\int_M \frac{\omega_0^n}{n!}.$$

Then the following statements are equivalent:

(1) *There exists a unique smooth function φ up to a constant such that $\omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi > 0$ satisfies the J-equation*

$$\mathrm{tr}_{\omega_\varphi}\chi = c_0.$$

(2) *There exists a unique smooth function φ up to a constant such that $\omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi > 0$ satisfies the J-equation*

$$\chi \wedge \frac{\omega_\varphi^{n-1}}{(n-1)!} = c_0\frac{\omega_\varphi^n}{n!}.$$

(3) *There exists a unique smooth function* φ *up to a constant such that* φ *is the critical point of the* $\mathcal{J}_\chi$ *functional.*

(4) *The* $\mathcal{J}_\chi$ *functional is coercive; in other words, there exist a constant* $\epsilon_{1.1} > 0$ *and another constant* $C_{1.2}$ *such that*

$$\mathcal{J}_\chi(\varphi) \geq \epsilon_{1.1}\mathcal{J}_{\omega_0}(\varphi) - C_{1.2}.$$

(5) $(M, [\omega_0], [\chi])$ *is uniformly J-stable; in other words, there exists a constant* $\epsilon_{1.1} > 0$ *such that for all Kähler test configurations* $(\mathcal{X}, \Omega)$ *defined in Definition 2.10 of* [22]*, the numerical invariant* $J_{[\chi]}(\mathcal{X}, \Omega)$ *defined in Definition 6.3 of* [22] *satisfies*

$$J_{[\chi]}(\mathcal{X}, \Omega) \geq \epsilon_{1.1} J_{[\omega_0]}(\mathcal{X}, \Omega).$$

(6) $(M, [\omega_0], [\chi])$ *is uniformly slope J-stable; in other words, there exists a constant* $\epsilon_{1.1} > 0$ *such that for any analytic subvariety* V *of* M*, the degeneration to the normal cone* $(\mathcal{X}, \Omega)$ *defined in Example 2.11 (ii) of* [22] *satisfies*

$$J_{[\chi]}(\mathcal{X}, \Omega) \geq \epsilon_{1.1} J_{[\omega_0]}(\mathcal{X}, \Omega).$$

(7) *There exists a constant* $\epsilon_{1.1} > 0$ *such that*

$$\int_V (c_0 - (n-p)\epsilon_{1.1})\omega_0^p - p\chi \wedge \omega_0^{p-1} \geq 0$$

for all p*-dimensional analytic subvarieties* V *with* $p = 1, 2, ..., n$.

Remark 1.2 It is well known that there exists a constant $C_{1.3}$ depending on n such that the $\mathcal{J}_{\omega_0}$ functional

$$\int_0^1 \left(\int_M \varphi \left(\omega_0 \wedge \frac{\omega_{t\varphi}^{n-1}}{(n-1)!} - n\frac{\omega_{t\varphi}^n}{n!}\right)\right) dt = \int_0^1 \left(\sqrt{-1}\int_M \partial\varphi \wedge \bar{\partial}\varphi \wedge \frac{t\omega_{t\varphi}^{n-1}}{(n-1)!}\right) dt$$

and Aubin's I-functional

$$\int_M \varphi(\omega_0^n - \omega_\varphi^n) = \sqrt{-1}\int_M \partial\varphi \wedge \bar{\partial}\varphi \wedge \sum_{k=0}^{n-1} \omega_0^k \wedge \omega_\varphi^{n-k-1}$$

satisfy

$$C_{1.3}^{-1}\int_M \varphi(\omega_0^n - \omega_\varphi^n) \leq \mathcal{J}_{\omega_0}(\varphi) \leq C_{1.3}\int_M \varphi(\omega_0^n - \omega_\varphi^n).$$

For example, Collins and Székelyhidi used this fact in Definition 20 of [13] to replace $\mathcal{J}_{\omega_0}(\varphi)$ with $\int_M \varphi(\omega_0^n - \omega_\varphi^n)$ in the definition of the coerciveness, which was called "properness" in [13]. By (3) of [2], Aubin's I-functional can also be replaced by Aubin's J-functional in the definition of coerciveness. Accordingly, in the definition of uniform stability, the numerical invariant $J_{[\omega_0]}(\mathcal{X}, \Omega)$ can be replaced by the minimum norm of $(\mathcal{X}, \Omega)$ defined in Definition 2.18 of [22]. By (62) of [15], Aubin's J-functional can be further replaced by the d_1 distance in the definition of the coerciveness when φ is normalized such that the Aubin-Mabuchi energy of φ is 0.

Remark 1.3 By Proposition 2 of [11], if the solution of the J-equation exists, it is unique up to a constant. It is easy to see that (1) and (2) are equivalent. The equivalence of (2) and (3) follows from the formula

$$\frac{d\mathcal{J}_\chi}{dt} = \int_M \frac{\partial\varphi}{\partial t}(\chi \wedge \frac{\omega_\varphi^{n-1}}{(n-1)!} - c_0 \frac{\omega_\varphi^n}{n!}).$$

By Proposition 21 and Proposition 22 of [13] and Remark 1.2, (1) and (4) are equivalent. By Corollary 6.5 of [22], (4) implies (5). It is trivial that (5) implies (6). By [30], (6) implies (7) in the projective case if $\epsilon_{1.1}$ is replaced by 0. However, it is easy to see that this is also true in the non-projective case and for positive $\epsilon_{1.1}$. Thus, we only need to prove that (7) implies (1) in Theorem 1.1. We remark that there is a simpler proof showing that (1) implies (7). At each point x, after choosing local coordinates such that

$$\chi = \sqrt{-1}\sum_{i=1}^n dz^i \wedge d\bar{z}^i$$

and

$$\omega_\varphi = \sqrt{-1}\sum_{i=1}^n \lambda_i dz^i \wedge d\bar{z}^i$$

at x, for any $c > 0$, the condition

$$\begin{aligned} &c\omega_\varphi^p - p\chi \wedge \omega_\varphi^{p-1} \\ &= p! \sum_{i_1<...<i_p} \lambda_{i_1}...\lambda_{i_p} \left(c - \sum_{j=1}^p \frac{1}{\lambda_{i_j}}\right) \sqrt{-1}dz^{i_1} \wedge d\bar{z}^{i_1} \wedge ... \wedge \sqrt{-1}dz^{i_p} \wedge d\bar{z}^{i_p} \geq 0 \end{aligned}$$

is equivalent to $\sum_{j=1}^p \frac{1}{\lambda_{i_j}} \leq c$ for all $1 \leq i_1 < i_2... < i_p \leq n$. If (1) holds, then $\mathrm{tr}_{\omega_\varphi}\chi = c_0$ and the upper bound of λ_i imply that there exists a constant

$\epsilon_{1.1} > 0$ such that at each point, $\sum_{j=1}^{p} \frac{1}{\lambda_{i_j}} \leq c_0 - (n-p)\epsilon_{1.1}$, which is equivalent to

$$(c_0 - (n-p)\epsilon_{1.1})\omega_\varphi^p - p\chi \wedge \omega_\varphi^{p-1} \geq 0.$$

Condition (7) follows from the fact that

$$\int_V (c_0 - (n-p)\epsilon_{1.1})\omega_0^p - p\chi \wedge \omega_0^{p-1} = \int_V (c_0 - (n-p)\epsilon_{1.1})\omega_\varphi^p - p\chi \wedge \omega_\varphi^{p-1}.$$

Remark 1.4 Lejmi and Székelyhidi's original conjecture is that the solvability of

$$\mathrm{tr}_{\omega_\varphi}\chi = c_0$$

is equivalent to

$$\int_V c_0\omega_0^p - p\chi \wedge \omega_0^{p-1} > 0$$

for all p-dimensional analytic subvarieties V with $p = 1, 2, ..., n-1$ [30]. For technical reasons, we only prove the uniform version in this paper. When this paper was under review, this technical issue was solved by Datar and Pingali [21] in the projective case for the generalized Monge–Ampère equation which is more general than the J-equation. Moreover, Datar-Pingali's theorem includes the equivariant version. Later, Song [38] solved this technical issue for the J-equation without the projective assumption.

When $c_1(M) < 0$, we can choose χ as a Kähler form in $-c_1(M)$. Since $x \log x$ is bounded from below for any $x \in \mathbb{R}$, the entropy $\int_M \log(\frac{\omega_\varphi^n}{\omega_0^n})\frac{\omega_\varphi^n}{n!}$ is also bounded from below. So the coerciveness of the $\mathcal{J}_\chi$ functional implies the coerciveness of the K-energy functional. This observation appeared as Remark 2 of [11]. Using this observation, as a corollary of Theorem 1.3 of [6] and Theorem 1.1, we find a sufficient condition for the existence of constant scalar curvature Kähler metrics with $c_1 < 0$.

Corollary 1.5 *If $c_1(M) < 0$, and $\epsilon_{1.1} > 0$, then for any Kähler class $[\omega_0]$ such that*

$$\int_V \left(\left(\frac{-n[c_1(M)] \cdot [\omega_0]^{n-1}}{[\omega_0]^n} - (n-p)\epsilon_{1.1}\right)\omega_0^p - p\omega_0^{p-1} \wedge (-c_1(M)) \geq 0\right.$$

for all p-dimensional analytic subvarieties V with $p = 1, 2, ..., n$*, there exists a cscK metric in* $[\omega_0]$.

Remark 1.6 If there exists a metric $\omega_\varphi \in [\omega_0]$ such that $\mathrm{Ric}(\omega_\varphi) < 0$ and ω_φ has constant scalar curvature, then the condition above is necessary. In fact, in this case, the cscK equation

$$\mathrm{tr}_{\omega_\varphi}\left(-\frac{\mathrm{Ric}(\omega_\varphi)}{2\pi}\right) = -\frac{n[c_1(M)]\cdot[\omega_0]^{n-1}}{[\omega_0]^n}$$

implies that ω_φ and $-\frac{\mathrm{Ric}(\omega_\varphi)}{2\pi} \in -c_1(M)$ satisfy the J-equation.

In addition to its appearances in the moment map picture and the study of the cscK problem, the J-equation also features in mathematical physics. In fact, if ω_φ is positive, λ_i are the eigenvalues of ω_φ with respect to χ, and arccot is the inverse function of cot with range $(0, \pi)$, then using the observation of Collins–Jacob–Yau [12] that

$$\lim_{k\to\infty} k\sum_{i=1}^{n} \mathrm{arccot}(k\lambda_i) = \sum_{i=1}^{n}\frac{1}{\lambda_i},$$

the J-equation is exactly the limit of the deformed Hermitian–Yang–Mills equation

$$\sum_{i=1}^{n} \mathrm{arccot}(\lambda_i) = \theta_0,$$

where θ_0 is a constant. The deformed Hermitian–Yang–Mills equation is the mirror equation of Harvey-Lawson's special Lagrangian equation [26] and plays an important role in mathematical physics [31,32,41].

The most important case of the deformed Hermitian–Yang–Mills equation is the supercritical case, which means $\theta_0 \in (0, \pi)$. We remark that we use the function

$$\mathrm{arccot}(\lambda_i) = \frac{\pi}{2} - \arctan(\lambda_i)$$

to simplify the notations because the term $\frac{\pi}{2} - \arctan(\lambda_i)$ appears frequently in this paper. It is easy to see that the supercritical deformed Hermitian–Yang–Mills equation means that

$$\sum_{i=1}^{n} \arctan(\lambda_i) = \frac{n\pi}{2} - \theta_0 \in \left(\frac{(n-2)\pi}{2}, \frac{n\pi}{2}\right).$$

In the supercritical case, motivated by the J-equation, Collins–Jacob–Yau [12] conjectured that the solvability of the supercritical deformed Hermitian–Yang–Mills equation is also equivalent to a condition on integrals on analytic subvarieties. According to the results in [14], Collins–Jacob–Yau's condition can be understood as a notion of algebraic stability. In this paper, we prove the uniform version of their conjecture. We remark that in an earlier version of this paper, the author proved the result only in the range $(0, \frac{\pi}{4})$. When the earlier version was under review, inspired by the recent results in [27] and [43], the author made a key observation that the function $-\cot(\sum_{i=1}^{n} \operatorname{arccot}(\lambda_i))$ is convex and successfully extended the result to the whole supercritical range $(0, \pi)$.

Theorem 1.7 *Fix a Kähler manifold M^n with a Kähler metric χ and a real smooth closed (1,1)-form ω_0. Assume that there exists a constant $\theta_0 \in (0, \pi)$ such that*

$$\int_M (\mathrm{Re}(\omega_\varphi + \sqrt{-1}\chi)^n - \cot(\theta_0)\mathrm{Im}(\omega_\varphi + \sqrt{-1}\chi)^n) = 0.$$

Then the following statements are equivalent:

(1) *There exists a smooth function φ such that the corresponding eigenvalues λ_i satisfy the deformed Hermitian–Yang–Mills equation*

$$\sum_{i=1}^{n} \operatorname{arccot}(\lambda_i) = \theta_0.$$

(2) *For any smooth test family $\omega_{t,0}$, there exists a constant $\epsilon_{1.1} > 0$ independent of t, V such that for any $t \geq 0$ and any p-dimensional analytic subvariety V,*

$$\int_V (\mathrm{Re}(\omega_{t,0} + \sqrt{-1}\chi)^p - \cot(\theta_0)\mathrm{Im}(\omega_{t,0} + \sqrt{-1}\chi)^p) \geq (n-p)\epsilon_{1.1} \int_V \chi^p.$$

(3) *There exist a test family $\omega_{t,0}$ and a constant $\epsilon_{1.1} > 0$ independent of t, V such that for any $t \geq 0$ and any p-dimensional analytic subvariety V,*

$$\int_V (\mathrm{Re}(\omega_{t,0} + \sqrt{-1}\chi)^p - \cot(\theta_0)\mathrm{Im}(\omega_{t,0} + \sqrt{-1}\chi)^p) \geq (n-p)\epsilon_{1.1} \int_V \chi^p.$$

Here, we call a smooth family $\omega_{t,0}$, $t \in [0, \infty)$ of real closed (1,1)-forms a test family if and only if all the following conditions hold:

(A) *When $t = 0$, $\omega_{t,0} = \omega_0$.*

(B) *For all $s > t$, $\omega_s - \omega_t$ is positive definite.*

(C) *There exists a large enough number* $T \geq 0$ *such that for all* $t \geq T$, $\omega_{t,0} - \cot(\frac{\theta_0}{n})\chi$ *is positive definite.*

Remark 1.8 The main reason for introducing the test family is to correctly choose the branch of the arccot function such that we stay within the supercritical case. See Remark 1.10 below for the importance of the supercritical condition. In the special case when $\omega_0 > 0$, an important choice of the test family is $\omega_{t,0} = t\omega_0$. In general cases, $\omega_{t,0} = \omega_0 + t\chi$ is always a test family. However, more choices of test families are allowed.

Remark 1.9 Compared to Collins–Jacob–Yau's conjecture in [12], the main difference is that we require the uniform lower bound $(n-p)\epsilon_{1.1}\int_V \chi^p$.

Remark 1.10 The "easier" direction of Collins–Jacob–Yau's conjecture in [12] holds only in the supercritical case. To see this, consider

$$V = \{0\} \times (\mathbb{C}/(\mathbb{Z} \oplus \sqrt{-1}\mathbb{Z}))^p \subset M = (\mathbb{C}/(\mathbb{Z} \oplus \sqrt{-1}\mathbb{Z}))^{10},$$
$$\omega_0 = \chi = \sqrt{-1}\sum_{i=1}^{10} dz^i \wedge d\bar{z}^i,$$

and $\theta_0 = \frac{5\pi}{2} > \pi$. Then the deformed Hermitian–Yang–Mills equation

$$\sum_{i=1}^{10} \operatorname{arccot}(\lambda_i) = \theta_0$$

can be solved using $\varphi = 0$, but

$$\int_V (\mathrm{Re}(\omega_0 + \sqrt{-1}\chi)^p - \cot(\theta_0)\mathrm{Im}(\omega_0 + \sqrt{-1}\chi)^p)$$

changes sign when p varies in the set $\{1, 2, 3..., 10\}$. All known conjectures fail in the non-supercritical case except for very special θ_0 studied in [21,33].

Theorem 1.7 will be proved in Sect. 5 using the same method of the proof of Theorem 1.1.

Instead of Theorem 1.1, we will prove the following stronger statement by induction on the dimension of M:

Theorem 1.11 *Fix a Kähler manifold* M^n *with Kähler metrics* χ *and* ω_0. *Let* $c > 0$ *be a constant and* $f > -\frac{1}{2n}(\frac{1}{c})^{n-1}$ *be a smooth function satisfying*

$$\int_M f\frac{\chi^n}{n!} = c\int_M \frac{\omega_0^n}{n!} - \int_M \chi \wedge \frac{\omega_0^{n-1}}{(n-1)!} \geq 0.$$

Then there exists a Kähler metric $\omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi > 0$ *satisfying the equation*

$$\mathrm{tr}_{\omega_\varphi}\chi + f\frac{\chi^n}{\omega_\varphi^n} = c$$

and the inequality

$$c\omega_\varphi^{n-1} - (n-1)\chi \wedge \omega_\varphi^{n-2} > 0$$

if there exists a constant $\epsilon_{1.1} > 0$ *such that*

$$\int_V (c - (n-p)\epsilon_{1.1})\omega_0^p - p\omega_0^{p-1} \wedge \chi \geq 0$$

for all p*-dimensional analytic subvarieties* V *with* $p = 1, 2, ..., n$.

Remark 1.12 By Remark 1.3, Theorem 1.1 is a corollary of Theorem 1.11 by choosing $f = 0$.

Remark 1.13 When $n = 1$, Theorem 1.11 is trivial. When $n = 2$, Theorem 1.11 is the statement that Demailly-Paun's characterization [20] for $[c\omega_0 - \chi]$ being Kähler implies the solvability of the Calabi conjecture

$$(c\omega_\varphi - \chi)^2 = (cf + 1)\chi^2$$

by Yau [47]. In the toric case in which f is a non-negative constant, the equivariant version of Theorem 1.11 was proved by Collins and Székelyhidi [13]. In fact, the idea of using the equation

$$\mathrm{tr}_{\omega_\varphi}\chi + f\frac{\chi^n}{\omega_\varphi^n} = c$$

for induction originates from Collins–Székelyhidi's arguments [13]. In this paper, f is instead a function. It will be used in later steps to make sure that $\omega_\varphi^n \geq \frac{f}{c}\chi^n$ concentrates near a given analytic subvariety. To compensate for the mass concentration without changing the integral of f, we must allow f to be slightly negative at some points.

There are several steps involved in the proof of Theorem 1.11.

Step 1: Prove the following:

Theorem 1.14 *Fix a Kähler manifold M^n with Kähler metrics χ and ω_0. Let $c > 0$ be a constant and $f > -\frac{1}{2n}(\frac{1}{c})^{n-1}$ be a smooth function satisfying*

$$\int_M f\frac{\chi^n}{n!} = c\int_M \frac{\omega_0^n}{n!} - \int_M \chi \wedge \frac{\omega_0^{n-1}}{(n-1)!} \geq 0.$$

Then there exists a Kähler metric $\omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi > 0$ satisfying the equation

$$\mathrm{tr}_{\omega_\varphi}\chi + f\frac{\chi^n}{\omega_\varphi^n} = c$$

and the inequality

$$c\omega_\varphi^{n-1} - (n-1)\chi \wedge \omega_\varphi^{n-2} > 0$$

if

$$c\omega_0^{n-1} - (n-1)\chi \wedge \omega_0^{n-2} > 0.$$

We will use the continuity method to prove Theorem 1.14. The details will be provided in Sect. 2.

Remark 1.15 Let $\chi = \delta_{ij}$ and $\omega_\varphi = \lambda_i\delta_{ij}$. Then the equation

$$\mathrm{tr}_{\omega_\varphi}\chi + f\frac{\chi^n}{\omega_\varphi^n} = c$$

is equivalent to

$$\sum_{i=1}^n \frac{1}{\lambda_i} + \frac{f}{\prod_{i=1}^n \lambda_i} = c.$$

Remark 1.16 Suppose that

$$c\omega_\varphi^{n-1} - (n-1)\chi \wedge \omega_\varphi^{n-2} \geq 0$$

or, equivalently, $\sum_{i\neq k}\frac{1}{\lambda_i} \leq c$ for all $k = 1, 2, ..., n$. Then, as long as

$$\sum_{i=1}^n \frac{1}{\lambda_i} + \frac{f}{\prod_{i=1}^n \lambda_i} = c$$

for $f > -\frac{1}{2n}(\frac{1}{c})^{n-1}$, it is easy to see that $\sum_{i \neq k} \frac{1}{\lambda_i} < c$, which is equivalent to

$$c\omega_\varphi^{n-1} - (n-1)\chi \wedge \omega_\varphi^{n-2} > 0.$$

Remark 1.17 When $n = 2$, Theorem 1.14 is the Calabi conjecture solved by Yau [47]. When $f = 0$, Theorem 1.14 is a special case of Song and Weinkove's result [40]. When f is a constant times $\frac{\omega_0^n}{\chi^n}$, Theorem 1.14 was proved by Zheng [49].

Step 2: Prove the following:

Theorem 1.18 *Fix a Kähler manifold M^n with Kähler metrics χ and ω_0. Define $\Gamma_{\chi,c}$ as the set of ω satisfying*

$$c\omega^{n-1} - (n-1)\chi \wedge \omega^{n-2} > 0,$$

and let $\bar{\Gamma}_{\chi,c}$ be the closure of $\Gamma_{\chi,c}$. Suppose that for all $t > 0$, there exist a constant $c_t > 0$ and a smooth Kähler form $\omega_t \in [(1+t)\omega_0]$ satisfying $\omega_t \in \Gamma_{\chi,c}$ and

$$\mathrm{tr}_{\omega_t}\chi + c_t \frac{\chi^n}{\omega_t^n} = c.$$

Then there exist a constant $\epsilon_{1.4} > 0$ and a current $\omega_{1.5} \in [\omega_0 - \epsilon_{1.4}\chi]$ such that $\omega_{1.5} \in \bar{\Gamma}_{\chi,c}$ in the sense of Definition 3.3.

Remark 1.19 In general, we can take the wedge product of ω_φ only when φ is in C^2. Bedford-Taylor [4] proved that it can also be defined when φ is in L^∞. In our case, φ is unbounded, so it might be impossible to define $c\omega_{1.5}^{n-1} - (n-1)\chi \wedge \omega_{1.5}^{n-2}$. Therefore, we have to figure out the correct definition of $\omega_{1.5} \in \bar{\Gamma}_{\chi,c}$ without taking wedge products. This will be done in Definition 3.3.

Now let us sketch the proof here. It is analogous to the proof of Theorem 2.12 in Demailly-Paun's paper [20]. In fact, when $n = 1$, the proof of Theorem 1.18 is the same as the proof of Theorem 2.12 of [20]. Consider the diagonal Δ inside the product manifold $M \times M$. Cover it by finitely many open coordinate balls B_j. Since Δ is non-singular, we can assume that on B_j, $g_{j,k}$, $k = 1, 2, ..., 2n$, are coordinates and $\Delta = \{g_{j,k} = 0, 1 \leq k \leq n\}$. Assume that θ_j are smooth functions supported in B_j such that $\sum \theta_j^2 = 1$ in a neighborhood of Δ. For

$\epsilon_{1.6} > 0$, define

$$\psi_{\epsilon_{1.6}} = \log\left(\sum_j \theta_j^2 \sum_{k=1}^n |g_{j,k}|^2 + \epsilon_{1.6}^2\right).$$

Define

$$\chi_{M\times M} = \pi_1^*\chi + \pi_2^*\chi$$

and

$$\chi_{M\times M,\epsilon_{1.6},\epsilon_{1.7}} = \chi_{M\times M} + \epsilon_{1.7}\sqrt{-1}\partial\bar{\partial}\psi_{\epsilon_{1.6}}.$$

Let

$$f_{t,\epsilon_{1.6},\epsilon_{1.7}} = \frac{\chi^{2n}_{M\times M,\epsilon_{1.6},\epsilon_{1.7}}}{\chi^{2n}_{M\times M}} - 1 + \frac{c_t}{c^n} > \frac{\chi^{2n}_{M\times M,\epsilon_{1.6},\epsilon_{1.7}}}{\chi^{2n}_{M\times M}} - 1.$$

Then by Lemma 2.1 (ii) of [20], there exists a constant $\epsilon_{1.7} > 0$ such that for $\epsilon_{1.6}$ small enough,

$$\frac{\chi^{2n}_{M\times M,\epsilon_{1.6},\epsilon_{1.7}}}{\chi^{2n}_{M\times M}} - 1 > -\frac{1}{4n}\left(\frac{1}{(n+1)c}\right)^{2n-1}.$$

Now we consider $\omega_{0,M\times M,t} = \pi_1^*\omega_t + \frac{1}{c}\pi_2^*\chi$. By Theorem 1.14, there exists a Kähler metric $\omega_{t,\epsilon_{1.6},\epsilon_{1.7}} \in [\omega_{0,M\times M,t}]$ such that

$$\mathrm{tr}_{\omega_{t,\epsilon_{1.6},\epsilon_{1.7}}}\chi_{M\times M} + f_{t,\epsilon_{1.6},\epsilon_{1.7}}\frac{\chi^{2n}_{M\times M}}{\omega^{2n}_{t,\epsilon_{1.6},\epsilon_{1.7}}} = (n+1)c.$$

Using

$$\omega^{2n}_{t,\epsilon_{1.6},\epsilon_{1.7}} \geq f_{t,\epsilon_{1.6},\epsilon_{1.7}}\frac{\chi^{2n}_{M\times M}}{(n+1)c}$$

and the proof of Proposition 2.6 of [20], $\omega^n_{t,\epsilon_{1.6},\epsilon_{1.7}}$ looks like a positive multiple of $[\Delta]$ near the diagonal $\Delta \subset M \times M$. Now define $\Omega_{t,\epsilon_{1.6},\epsilon_{1.7}}$ by

$$\Omega_{t,\epsilon_{1.6},\epsilon_{1.7}} = \frac{c^{n-1}}{\int_M n\chi^n}(\pi_1)_*(\omega^n_{t,\epsilon_{1.6},\epsilon_{1.7}} \wedge \pi_2^*\chi).$$

Fix $\epsilon_{1.7}$ and let t and $\epsilon_{1.6}$ converge to 0. For small enough $\epsilon_{1.4}$, let $\omega_{1.5}$ be the weak limit of $\Omega_{t,\epsilon_{1.6},\epsilon_{1.7}} - \epsilon_{1.4}\chi$. Then we shall expect $\omega_{1.5} \in \bar{\Gamma}_{\chi,c}$ in the sense of Definition 3.3. Roughly speaking, in the proof of Theorem 2.12 of [20], the mass concentration near the diagonal provides the extra $\epsilon_{1.4}\chi$ while the current away from the diagonal is still positive. In our case, we perform a truncation cutting the positive multiple of $[\Delta]$ to get $\epsilon_{1.4}\chi$ while expecting that the remaining term $\omega_{1.5}$ is still in $\bar{\Gamma}_{\chi,c}$. The details will be provided in Sect. 3.

Step 3: Consider the set I of $t \geq 0$ such that there exist a constant $c_t \geq 0$ and a smooth Kähler form $\omega_t \in [(1+t)\omega_0]$ satisfying

$$(c\omega_t - (n-1)\chi) \wedge \omega_t^{n-2} > 0$$

and

$$\mathrm{tr}_{\omega_t}\chi + c_t \frac{\chi^n}{\omega_t^n} = c.$$

By Theorem 1.14, it suffices to show that $0 \in I$. When t is large enough, the condition of Theorem 1.14 is satisfied. So $t \in I$. It is easy to see that if $t \in I$, then for nearby t, the condition of Theorem 1.14 is also satisfied. So I is open. Again by Theorem 1.14, as long as $t \in I$, then for all $t' \geq t$, $t' \in I$. Thus, in order to prove the closedness of I, it suffices to show that if $t \in I$ for all $t > t_0$, then $t_0 \in I$. After replacing $(1+t_0)\omega_0$ with ω_0, we can, without loss of generality, assume that $t_0 = 0$. In particular, we can apply Theorem 1.18 to get $\omega_{1.5}$.

Let $\nu(x)$ be the Lelong number of $\omega_{1.5}$ at x. For $\epsilon_{1.8} > 0$ to be determined, let Y be the set

$$Y = \{x : \nu(x) \geq \epsilon_{1.8}\}.$$

By the result of Siu [37], Y is an analytic subvariety with dimension $p < n$. If we assume that Y is smooth, then by the induction hypothesis, we can apply Theorem 1.11 to Y to obtain a smooth function $\varphi_{1.9}$ on Y such that $\omega_{1.9} = \omega_0|_Y + \sqrt{-1}\partial\bar{\partial}\varphi_{1.9} \in \Gamma_{\chi|_Y, c-(n-p)\epsilon_{1.1}}$ satisfies

$$\mathrm{tr}_{\omega_{1.9}}(\chi|_Y) + c_{1.10}\frac{(\chi|_Y)^p}{\omega_{1.9}^p} = c - (n-p)\epsilon_{1.1}$$

on Y for the constant $c_{1.10}$ defined by

$$\int_Y c_{1.10}\frac{(\chi|_Y)^p}{p!} = (c-(n-p)\epsilon_{1.1})\int_Y \frac{(\omega_0|_Y)^p}{p!} - \int_Y (\chi|_Y) \wedge \frac{(\omega_0|_Y)^{p-1}}{(p-1)!} \geq 0.$$

This implies that

$$(c-(n-p)\epsilon_{1.1})\omega_{1.9}^{p} - p\chi|_Y \wedge \omega_{1.9}^{p-1} = c_{1.10}(\chi|_Y)^p \geq 0.$$

Then for large enough $C_{1.11}$,

$$\omega_{1.12} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_{1.12} = \omega_0 + \sqrt{-1}\partial\bar{\partial}(\mathrm{Proj}_Y^*\varphi_{1.9} + C_{1.11}d_\chi(.,Y)^2)$$

satisfies

$$\left(c - \frac{n-p}{2}\epsilon_{1.1}\right)\omega_{1.12}^{n-1} - (n-1)\chi \wedge \omega_{1.12}^{n-2} > 0$$

on a tubular neighborhood of Y, where Proj_Y means the projection on Y. By a generalization of the result of Błocki and Kołodziej [3], we can glue the smoothing of $\omega_{1.5}$ outside Y and $\omega_{1.12}$ near Y into $\omega_{1.13} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_{1.13}$ satisfying

$$c\omega_{1.13}^{n-1} - (n-1)\chi \wedge \omega_{1.13}^{n-2} > 0$$

on M. Then we are done by Theorem 1.14. In general, Y is singular, and we need to use Hironaka's desingularization theorem to resolve it.

The idea of Błocki and Kołodziej's gluing method can be illustrated by the following example: Consider $-\sqrt{-1}\sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$ on the torus $T^{2n} = \mathbb{C}^n/\mathbb{Z}^{2n}$. Then we choose finitely many points p_j on T^{2n}. The local potential near p_j is $-|z-p_j|^2$. Then

$$\sqrt{-1}\partial\bar{\partial}\max\{-|z-p_1|^2, -|z-p_2|^2, ...\}$$

is in the zero class. The main benefit of the shift from $[-\sqrt{-1}dz_i \wedge d\bar{z}_i]$ to the zero class is that $\max\{-|z-p_1|^2, -|z-p_2|^2, ...\}$ cannot be affected by the function $-|z-p_j|^2$ away from p_j so that we do not need to worry about the fact that $-|z-p_j|^2$ can be defined only locally.

Roughly speaking, in our case, the smoothing in different charts has only small differences away from Y as a result of the smallness of the Lelong number. It can be controlled if we shift the class from $[\omega_0 - \epsilon_{1.4}\chi]$ to $[\omega_0]$. On the other hand, the maximum must be achieved by $\varphi_{1.12}$ near Y because the potential functions of the smoothing of $\omega_{1.5}$ are close to $-\infty$ near Y. We remark that Błocki and Kołodziej [3] provided an example showing that $\epsilon_{1.8}$ converges to 0 if $\epsilon_{1.4}$ converges to 0. Therefore, we need to choose the small but non-zero constant $\epsilon_{1.8}$ corresponding to our small but non-zero constant $\epsilon_{1.4}$ obtained in Step 2. The details of Step 3 will be provided in Sect. 4.

2 The analysis part

In this section, we use the continuity method twice to prove Theorem 1.14. First of all, for $t \in [0, 1]$, define χ_t by

$$\chi_t = t\chi + (1-t)\frac{c}{n}\omega_0$$

and define $f_t \geq 0$ as the constant such that

$$\int_M f_t \frac{\chi_t^n}{n!} = c\int_M \frac{\omega_0^n}{n!} - \int_M \chi_t \wedge \frac{\omega_0^{n-1}}{(n-1)!} = t\left(c\int_M \frac{\omega_0^n}{n!} - \int_M \chi \wedge \frac{\omega_0^{n-1}}{(n-1)!}\right) \geq 0.$$

Now we consider the set I consisting of all $t \in [0, 1]$ such that there exists a Kähler metric $\omega_t = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_t > 0$ for smooth φ_t satisfying

$$\mathrm{tr}_{\omega_t}\chi_t + f_t\frac{\chi_t^n}{\omega_t^n} = c$$

and

$$c\omega_t^{n-1} - (n-1)\chi_t \wedge \omega_t^{n-2} > 0.$$

Then it is easy to see that $0 \in I$. We remark that the equation is the same as

$$c\frac{\omega_t^n}{n!} - \chi_t \wedge \frac{\omega_t^{n-1}}{(n-1)!} = f_t\frac{\chi_t^n}{n!}.$$

The linearization is

$$\frac{1}{(n-1)!}\left(c\omega_t^{n-1} - (n-1)\chi_t \wedge \omega_t^{n-2}\right) \wedge \sqrt{-1}\partial\bar{\partial}\frac{\partial\varphi_t}{\partial t} = \frac{\partial}{\partial t}\left(f_t\frac{\chi_t^n}{n!}\right) + \frac{\partial\chi_t}{\partial t} \wedge \frac{\omega_t^{n-1}}{(n-1)!}.$$

Assume that $t \in I$, then the left-hand side is a second-order elliptic equation on $\frac{\partial\varphi_t}{\partial t}$. On the other hand, our hypothesis regarding the integral implies that the integral of the right-hand side is 0. By standard elliptic theory and the implicit function theorem, I is open when we replace the smoothness assumption of φ with $C^{100,\alpha}$. However, standard elliptic regularity theory implies that any $C^{100,\alpha}$ solution is automatically smooth. So I is in fact open.

If we are able to show the closedness of I, then we have proved Theorem 1.14 for f replaced by f_1, where f_1 means f_t when $t = 1$. We can use another continuity path by fixing χ and ω_0 but choosing $\hat{f}_s = sf_1 + (1-s)f$. However, it is the same as before except that $\hat{f}_s > -\frac{1}{2n}(\frac{1}{c})^{n-1}$ is a function instead of a constant. Thus, we only need to prove the *a priori* estimate of ω_t by assuming that $f_t > -\frac{1}{2n}(\frac{1}{c})^{n-1}$ is a function because proving the openness and the estimate of $\hat{\omega}_s$ corresponding to $\hat{f}_s$ is similar to proving the statements for ω_t. We start with the following proposition which is analogous to Lemma 3.1 in Song-Weinkove's paper [40]:

Proposition 2.1 *Assume that* $t \in I$ *and* $\omega_t = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_t$ *is the corresponding solution. Then there exist constants* $C_{2.1}$ *and* $C_{2.2}$ *depending only on* c, ω_0, *the* C^∞*-norm of* χ_t *with respect to* ω_0, *and the* C^2*-norm of* $||f_t||$ *with respect to* ω_0 *such that*

$$\mathrm{tr}_{\chi_t}\omega_t \le C_{2.2}e^{C_{2.1}(\varphi_t - \inf \varphi_t)}.$$

Proof In local coordinates, $\chi_t = \sqrt{-1}\chi_{i\bar{j}}dz^i \wedge dz^{\bar{j}}$ and $\omega_t = \sqrt{-1}g_{i\bar{j}}dz^i \wedge dz^{\bar{j}}$. Fix any point x and choose a χ_t-normal coordinate such that $\chi_{i\bar{j}} = \delta_{i\bar{j}}$, $\chi_{i\bar{j},k} = 0$, $\chi_{i\bar{j},\bar{k}} = 0$, and $g_{i\bar{j}} = \lambda_i\delta_{i\bar{j}}$ at x, where the derivatives are all ordinary derivatives. Then the equation is

$$\sum_{i,j} g^{i\bar{j}}\chi_{i\bar{j}} + f_t\frac{\det \chi_{\alpha\bar{\beta}}}{\det g_{\alpha\bar{\beta}}} = c.$$

Define an operator $\tilde{\Delta}$ by

$$\tilde{\Delta}u = \sum_{i,l}\left(\sum_{j,k} g^{i\bar{j}}g^{k\bar{l}}\chi_{k\bar{j}} + f_t\frac{g^{i\bar{l}}\det \chi_{\alpha\bar{\beta}}}{\det g_{\alpha\bar{\beta}}}\right)u_{,i\bar{l}}.$$

Then it is easy to see that $\tilde{\Delta}$ is independent of the choice of local coordinates.

At x,

$$\tilde{\Delta}u = \sum_i\left(\frac{1}{\lambda_i^2} + f_t\frac{1}{\lambda_i}\frac{1}{\prod_{\alpha=1}^n \lambda_\alpha}\right)u_{,i\bar{i}}.$$

Since $\frac{1}{\lambda_\alpha} < c$ and $f_t > -\frac{1}{2n}(\frac{1}{c})^{n-1}$, it is easy to see that

$$\frac{1}{\lambda_i^2} + f_t\frac{1}{\lambda_i}\frac{1}{\prod_{\alpha=1}^n \lambda_\alpha} > 0$$

for all i. So $\tilde{\Delta}$ is a second-order elliptic operator.

Now we compute $\tilde{\Delta}(\log \operatorname{tr}_{\chi_t}\omega_t) = \tilde{\Delta}\left(\log\left(\sum_{i,j}\chi^{i\bar{j}}g_{i\bar{j}}\right)\right)$. It is equal to

$$\sum_k \left(\frac{\sum_i \left(g_{i\bar{i},k\bar{k}} + \left(\chi^{i\bar{i}}\right)_{,k\bar{k}} \lambda_i \right)}{\sum_i \lambda_i} - \frac{|\sum_i g_{i\bar{i},k}|^2}{\left(\sum_i \lambda_i\right)^2} \right) \left(\frac{1}{\lambda_k^2} + f_t \frac{1}{\lambda_k} \frac{1}{\prod_{\alpha=1}^n \lambda_\alpha} \right) \Bigg)$$

at x.

If we differentiate the equation

$$\sum_{i,j} g^{i\bar{j}}\chi_{i\bar{j}} + f_t \frac{\det \chi_{\alpha\bar{\beta}}}{\det g_{\alpha\bar{\beta}}} = c,$$

then we get

$$\sum_{i,j} g^{i\bar{j}}\chi_{i\bar{j},k} - \sum_{i,j,a,b} g^{i\bar{b}} g_{a\bar{b},k} g^{a\bar{j}}\chi_{i\bar{j}} + \frac{\det \chi_{\alpha\bar{\beta}}}{\det g_{\alpha\bar{\beta}}} \Bigg(f_{t,k}$$
$$+ f_t \sum_{i,j} \left(\chi^{i\bar{j}}\chi_{i\bar{j},k} - g^{i\bar{j}} g_{i\bar{j},k} \right) \Bigg) = 0.$$

So

$$\sum_k \Bigg(\sum_i \frac{1}{\lambda_i}\chi_{i\bar{i},k\bar{k}} - \sum_i \frac{1}{\lambda_i^2} g_{i\bar{i},k\bar{k}} + \sum_{i,j} \frac{1}{\lambda_i^2}\frac{1}{\lambda_j} \left(|g_{i\bar{j},k}|^2 + |g_{i\bar{j},\bar{k}}|^2 \right)$$
$$+ \frac{1}{\prod_\alpha \lambda_\alpha} \Bigg(f_t \left(|\sum_i \left(\frac{1}{\lambda_i} g_{i\bar{i},k}\right)|^2 + \sum_i \chi_{i\bar{i},k\bar{k}} + \sum_{i,j} \frac{1}{\lambda_i\lambda_j} |g_{i\bar{j},k}|^2 - \sum_i \frac{1}{\lambda_i} g_{i\bar{i},k\bar{k}} \right)$$
$$+ f_{t,k\bar{k}} - f_{t,\bar{k}} \sum_i \left(\frac{1}{\lambda_i} g_{i\bar{i},k} \right) - f_{t,k} \sum_i \left(\frac{1}{\lambda_i} g_{i\bar{i},\bar{k}} \right) \Bigg) \Bigg) = 0$$

at x.

By the Kähler condition, $g_{i\bar{i},k\bar{k}} = g_{k\bar{k},i\bar{i}}$, $g_{i\bar{j},k} = g_{k\bar{j},i}$, and $g_{i\bar{j},\bar{k}} = g_{i\bar{k},\bar{j}}$. Using the bounds $|\chi_{i\bar{i},k\bar{k}}| + |(\chi^{i\bar{i}})_{,k\bar{k}}| + |f_{t,k}| + |f_{t,\bar{k}}| + |f_{t,k\bar{k}}| + |f_t| + \frac{1}{\lambda_i} < C_{2.3}$ for all i, k, it is easy to see that by combining the previous equation with the expression for $\tilde{\Delta}(\log \operatorname{tr}_{\chi_t}\omega_t)$, we obtain

$$\tilde{\Delta}\left(\log \operatorname{tr}_{\chi} \omega_t\right) \geq -C_{2.4} - \sum_k \left(\frac{|\sum_i g_{i\bar{i},k}|^2}{\left(\sum_i \lambda_i\right)^2} \right) \left(\frac{1}{\lambda_k^2} + f_t \frac{1}{\lambda_k} \frac{1}{\prod_{\alpha=1}^n \lambda_\alpha} \right)$$
$$+ \frac{1}{\sum_i \lambda_i} \sum_k \left(\sum_{i,j} \frac{1}{\lambda_i^2} \frac{1}{\lambda_j} \left(|g_{i\bar{j},k}|^2 + |g_{i\bar{j},\bar{k}}|^2 \right) + \frac{1}{\prod_\alpha \lambda_\alpha} \left(f_t \left(|\sum_i \left(\frac{1}{\lambda_i} g_{i\bar{i},k} \right)|^2 \right.\right.\right.$$
$$\left.\left.\left. + \sum_{i,j} \frac{1}{\lambda_i \lambda_j} |g_{i\bar{j},k}|^2 \right) - f_{t,\bar{k}} \sum_i \left(\frac{1}{\lambda_i} g_{i\bar{i},k} \right) - f_{t,k} \sum_i \left(\frac{1}{\lambda_i} g_{i\bar{i},\bar{k}} \right) \right) \right).$$

We remark that

$$|\frac{1}{\prod_\alpha \lambda_\alpha} f_{t,\bar{k}} \sum_i \left(\frac{1}{\lambda_i} g_{i\bar{i},k} \right)| = |\frac{1}{\prod_\alpha \lambda_\alpha} f_{t,k} \sum_i \left(\frac{1}{\lambda_i} g_{i\bar{i},\bar{k}} \right)|$$
$$= |f_{t,\bar{k}}|| \sum_i \frac{1}{\prod_\alpha \lambda_\alpha} \left(\frac{1}{\lambda_i} g_{i\bar{i},k} \right)| \leq C_{2.5} \sum_i |\frac{1}{\lambda_i^2} g_{i\bar{i},k}|$$
$$\leq \frac{1}{4} \sum_i \frac{1}{\lambda_i^3} |g_{i\bar{i},k}|^2 + C_{2.6} \sum_i \frac{1}{\lambda_i} \leq \frac{1}{4} \sum_i \frac{1}{\lambda_i^3} |g_{i\bar{i},\bar{k}}|^2 + C_{2.7}$$

and

$$-\frac{f_t}{\prod_\alpha \lambda_\alpha} | \sum_i \left(\frac{1}{\lambda_i} g_{i\bar{i},k} \right) |^2 \leq -\frac{n f_t}{\prod_\alpha \lambda_\alpha} \sum_i \frac{1}{\lambda_i^2} |g_{i\bar{i},k}|^2 \leq \frac{1}{2} \sum_i \frac{1}{\lambda_i^3} |g_{i\bar{i},\bar{k}}|^2.$$

So

$$\tilde{\Delta}\left(\log \operatorname{tr}_{\chi_t} \omega_t\right) \geq -C_{2.8} - \sum_k \left(\frac{|\sum_i g_{i\bar{i},k}|^2}{\left(\sum_i \lambda_i\right)^2} \right) \left(\frac{1}{\lambda_k^2} + f_t \frac{1}{\lambda_k} \frac{1}{\prod_{\alpha=1}^n \lambda_\alpha} \right)$$
$$+ \frac{1}{\sum_i \lambda_i} \sum_k \left(\sum_{i,j} \frac{1}{\lambda_i^2} \frac{1}{\lambda_j} |g_{i\bar{j},k}|^2 + \frac{f_t}{\prod_\alpha \lambda_\alpha} \sum_{i,j} \frac{1}{\lambda_i \lambda_j} |g_{i\bar{j},k}|^2 \right).$$

We have used

$$\sum_{i,j} \frac{1}{\lambda_i^2} \frac{1}{\lambda_j} |g_{i\bar{j},\bar{k}}|^2 \geq \sum_i \frac{1}{\lambda_i^3} |g_{i\bar{i},\bar{k}}|^2$$

here.

By the Cauchy–Schwarz inequality and the fact that $g_{i\bar{i},k} = g_{k\bar{i},i}$,

$$
\begin{aligned}
&\sum_k \left(|\sum_i g_{i\bar{i},k}|^2 \right) \left(\frac{1}{\lambda_k^2} + f_t \frac{1}{\lambda_k} \frac{1}{\prod_{\alpha=1}^n \lambda_\alpha} \right) \\
&\quad \le \sum_{i,j,k} |g_{i\bar{i},k}||g_{j\bar{j},k}| \left(\frac{1}{\lambda_k^2} + f_t \frac{1}{\lambda_k} \frac{1}{\prod_{\alpha=1}^n \lambda_\alpha} \right) \\
&\quad \le \sum_{i,j} \sqrt{\sum_k |g_{i\bar{i},k}|^2 \left(\frac{1}{\lambda_k^2} + f_t \frac{1}{\lambda_k} \frac{1}{\prod_{\alpha=1}^n \lambda_\alpha} \right)} \sqrt{\sum_k |g_{j\bar{j},k}|^2 \left(\frac{1}{\lambda_k^2} + f_t \frac{1}{\lambda_k} \frac{1}{\prod_{\alpha=1}^n \lambda_\alpha} \right)} \\
&\quad = \left(\sum_i \sqrt{\sum_k |g_{i\bar{i},k}|^2 \left(\frac{1}{\lambda_k^2} + f_t \frac{1}{\lambda_k} \frac{1}{\prod_{\alpha=1}^n \lambda_\alpha} \right)} \right)^2 \\
&\quad \le \left(\sum_i \lambda_i \right) \sum_i \sum_k \frac{|g_{i\bar{i},k}|^2}{\lambda_i} \left(\frac{1}{\lambda_k^2} + f_t \frac{1}{\lambda_k} \frac{1}{\prod_{\alpha=1}^n \lambda_\alpha} \right) \\
&\quad = \left(\sum_i \lambda_i \right) \sum_{i,k} \frac{|g_{i\bar{k},k}|^2}{\lambda_k} \left(\frac{1}{\lambda_i^2} + f_t \frac{1}{\lambda_i} \frac{1}{\prod_{\alpha=1}^n \lambda_\alpha} \right) \\
&\quad \le \left(\sum_i \lambda_i \right) \sum_{i,j,k} \frac{|g_{i\bar{j},k}|^2}{\lambda_j} \left(\frac{1}{\lambda_i^2} + f_t \frac{1}{\lambda_i} \frac{1}{\prod_{\alpha=1}^n \lambda_\alpha} \right),
\end{aligned}
$$

so $\tilde{\Delta}(\log \operatorname{tr}_{\chi_t} \omega_t) \ge -C_{2.9}$ at x. However, since x is arbitrary and $\tilde{\Delta}$ is independent of the local coordinates, we see that $\tilde{\Delta}(\log \operatorname{tr}_{\chi_t} \omega_t) \ge -C_{2.9}$ on M.

If we choose $\epsilon_{2.10} < \frac{c}{2n}$ as a small constant such that

$$c\omega_0^{n-1} - (n-1)\chi \wedge \omega_0^{n-2} > 2\epsilon_{2.10}\omega_0^{n-1},$$

then

$$c\omega_0^{n-1} - (n-1)\chi_t \wedge \omega_0^{n-2} > 2\epsilon_{2.10}\omega_0^{n-1}$$

by the definition of χ_t. Choose $C_{2.1}$ as $\frac{2C_{2.9}}{\epsilon_{2.10}}$, so at the maximal point of $\log \operatorname{tr}_{\chi_t} \omega_t - C_{2.1}\varphi_t$,

$$-\tilde{\Delta}\varphi_t = -\sum_{i,l} \left(\sum_{j,k} g^{i\bar{j}} g^{k\bar{l}} \chi_{k\bar{j}} + f_t \frac{g^{i\bar{l}} \det \chi_{\alpha\bar{\beta}}}{\det g_{\alpha\bar{\beta}}} \right) \left(g_{i\bar{l}} - g^0_{i\bar{l}} \right) < \frac{\epsilon_{2.10}}{2}.$$

If

$$c - \sum_{i,l} \sum_{j,k} g^{i\bar{j}} g^{k\bar{l}} \chi_{k\bar{j}} (2g_{i\bar{l}} - g^0_{i\bar{l}}) < \epsilon_{2.10},$$

then by the proof of Lemma 3.1 of [40], $\mathrm{tr}_{\chi_t}\omega_t \leq C_{2.11}$. If

$$c - \sum_{i,l}\sum_{j,k} g^{i\bar{j}} g^{k\bar{l}} \chi_{k\bar{j}} \left(2g_{i\bar{l}} - g^0_{i\bar{l}}\right) \geq \epsilon_{2.10},$$

then

$$\begin{aligned} -\sum_{i,l} f_t \frac{g^{i\bar{l}} \det \chi_{\alpha\bar{\beta}}}{\det g_{\alpha\bar{\beta}}} \left(g_{i\bar{l}} - g^0_{i\bar{l}}\right) &< \frac{\epsilon_{2.10}}{2} + \sum_{i,l}\sum_{j,k} g^{i\bar{j}} g^{k\bar{l}} \chi_{k\bar{j}} \left(g_{i\bar{l}} - g^0_{i\bar{l}}\right) \\ &\leq -\frac{\epsilon_{2.10}}{2} + c - \sum_{i,j} g^{i\bar{j}} \chi_{i\bar{j}} = -\frac{\epsilon_{2.10}}{2} + f_t \frac{\det \chi_{\alpha\bar{\beta}}}{\det g_{\alpha\bar{\beta}}}, \end{aligned}$$

so

$$\frac{f_t}{\prod_\alpha \lambda_\alpha} \left(\sum_{i,l} g^{i\bar{l}} \left(g_{i\bar{l}} - g^0_{i\bar{l}}\right) + 1 \right) > \frac{\epsilon_{2.10}}{2}.$$

Using the fact that $\lambda_i > \frac{1}{c}$, the term $\sum_{i,l} g^{i\bar{l}}(g_{i\bar{l}} - g^0_{i\bar{l}}) + 1$ is bounded, so $\prod_\alpha \lambda_\alpha < C_{2.12}$. Using the lower bound on λ_i again, this implies the upper bound on λ_i, so $\mathrm{tr}_{\chi_t}\omega_t = \sum_i \lambda_i \leq C_{2.13}$ is also true.

In conclusion, we have proved that at the maximal point of the function $\log \mathrm{tr}_{\chi_t}\omega_t - C_{2.1}\varphi_t$, $\mathrm{tr}_{\chi_t}\omega_t$ is bounded by a constant $C_{2.2}$ in all cases, where $C_{2.2}$ is defined as the maximum of $C_{2.11}$ and $C_{2.13}$. So

$$\log \mathrm{tr}_{\chi_t}\omega_t - C_{2.1}\varphi_t \leq \log C_{2.2} - C_{2.1} \inf_M \varphi_t.$$

This completes the proof of the proposition. □

By adding a constant if necessary, we can without loss of generality assume that $\sup_M \varphi_t = 0$. Then we have the following C^0 estimate:

Proposition 2.2

$$||\varphi_t||_{C^0} \leq C_{2.14}.$$

Moreover, $C^{-1}_{2.15}\chi_t \leq \omega_t \leq C_{2.15}\chi_t$.

Proof Lemma 3.3 and Lemma 3.4 and Proposition 3.5 of [40] used only the inequality in Proposition 2.1. So they are still true in our case. □

Proposition 2.3 *I is closed.*

Proof First of all, we want to check the uniform ellipticity and the concavity for the Evans-Krylov estimate. The equation is

$$-g^{i\bar{j}}\chi_{i\bar{j}} - f_t \frac{\det \chi_{\alpha\bar{\beta}}}{\det g_{\alpha\bar{\beta}}} = -c.$$

If we view it as a function in terms of $g_{i\bar{j}}$, $\chi_{i\bar{j}}$, and f_t, then the partial derivative in the $g_{a\bar{b}}$ direction is

$$g^{i\bar{b}}g^{a\bar{j}}\chi_{i\bar{j}} + f_t \frac{\det \chi_{\alpha\bar{\beta}}}{\det g_{\alpha\bar{\beta}}} g^{a\bar{b}}.$$

At x, it is equal to

$$\left(\frac{1}{\lambda_a^2} + \frac{1}{\lambda_a}\frac{f_t}{\prod_i \lambda_i}\right)\delta_{a\bar{b}}.$$

It has a uniform upper bound and a uniform lower bound.

The second-order derivative in the $g_{a\bar{b}}$ and $g_{c\bar{d}}$ direction is

$$-g^{i\bar{d}}g^{c\bar{b}}g^{a\bar{j}}\chi_{i\bar{j}} - g^{i\bar{b}}g^{a\bar{d}}g^{c\bar{j}}\chi_{i\bar{j}} - f_t \frac{\det \chi_{\alpha\bar{\beta}}}{\det g_{\alpha\bar{\beta}}} g^{a\bar{b}}g^{c\bar{d}} - f_t \frac{\det \chi_{\alpha\bar{\beta}}}{\det g_{\alpha\bar{\beta}}} g^{a\bar{d}}g^{c\bar{b}}.$$

At x, when taking the product with $w_{a\bar{b}}\overline{w_{c\bar{d}}}$ and summing a, b, c, and d for any matrix $w_{i\bar{j}}$, we get

$$\begin{aligned}
&-\sum_{a,b}\frac{1}{\lambda_a^2\lambda_b}|w_{a\bar{b}}|^2 - \sum_{a,b}\frac{1}{\lambda_b^2\lambda_a}|w_{a\bar{b}}|^2 - \frac{f_t}{\prod_i \lambda_i}\left(\sum_a \frac{w_{a\bar{a}}}{\lambda_a}\right)^2 - \frac{f_t}{\prod_i \lambda_i}\sum_{a,b}\frac{1}{\lambda_a\lambda_b}|w_{a\bar{b}}|^2 \\
&\le -\sum_{a,b}\frac{|w_{a\bar{b}}|^2}{\lambda_a^2\lambda_b} - \sum_{a,b}\frac{|w_{a\bar{b}}|^2}{\lambda_b^2\lambda_a} + \frac{1}{2n}\left(\frac{1}{c}\right)^{n-1}\frac{1}{\prod_i \lambda_i}\left(\sum_a \frac{w_{a\bar{a}}}{\lambda_a}\right)^2 + \frac{1}{2n}\sum_{a,b}\frac{|w_{a\bar{b}}|^2}{\lambda_a^2\lambda_b} \\
&\le -\sum_{a,b}\frac{|w_{a\bar{b}}|^2}{\lambda_b^2\lambda_a} + \frac{1}{2}\left(\frac{1}{c}\right)^{n-1}\frac{1}{\prod_i \lambda_i}\sum_a |\frac{w_{a\bar{a}}}{\lambda_a}|^2 \le -\sum_a \frac{|w_{a\bar{a}}|^2}{\lambda_a^3} + \frac{1}{2}\sum_a \frac{|w_{a\bar{a}}|^2}{\lambda_a^3} \le 0,
\end{aligned}$$

using the estimate $\lambda_i \ge \frac{1}{c}$ as well as the assumption that $f_t > -\frac{1}{2n}(\frac{1}{c})^{n-1}$.

Thus, if we replace the complex second derivatives with real second derivatives, the uniform ellipticity and concavity for the Evans-Krylov estimate [23,24,29,45] are satisfied. By checking Evans-Krylov's estimate carefully, it is easy to see that in our complex case, the estimate

$$[(\varphi_t)_{i\bar{j}}]_{C^\alpha} \le C_{2.16}$$

is still true.

By the standard elliptic estimate, $||\varphi_t||_{C^{101,\alpha}}$ is bounded. By the Arzela-Ascoli theorem, if $t_i \to t_\infty$ and $t_i \in I$, then a subsequence of φ_t converges to φ_{t_∞} in the $C^{100,\alpha}$-norm. By Remark 1.16,

$$c\omega_{t_\infty}^{n-1} - (n-1)\chi_t \wedge \omega_{t_\infty}^{n-2} > 0.$$

So by the standard elliptic regularity, φ_{t_∞} is smooth. So $t_\infty \in I$. □

3 Concentration of mass and its application

In this section, we prove Theorem 1.18. However, before doing that, we need to figure out the correct definition of $\omega \in \bar{\Gamma}_{\chi,c}$ when ω is only a current and it might be impossible to take wedge products.

Recall the following definition of the smoothing:

Definition 3.1 Fix a smooth non-negative function ρ supported in [0,1] such that

$$\int_0^1 \rho(t)t^{2n-1}\mathrm{Vol}(\partial B_1(0))dt = 1$$

and ρ is a positive constant near 0. For any $\delta > 0$, the smoothing φ_δ is defined by

$$\varphi_\delta(x) = \int_{\mathbb{C}^n} \varphi(x-y)\delta^{-2n}\rho\left(|\frac{y}{\delta}|\right) d\mathrm{Vol}_y.$$

We can define the smoothing of a current using a similar formula. It is easy to see that the smoothing commutes with the derivatives. So

$$\left(\sqrt{-1}\partial\bar{\partial}\varphi\right)_\delta = \sqrt{-1}\partial\bar{\partial}(\varphi_\delta).$$

Recall that $\sqrt{-1}\partial\bar{\partial}\varphi \geq 0$ if and only if $\sqrt{-1}\partial\bar{\partial}\varphi_\delta \geq 0$ for all $\delta > 0$. In analogy, we can define $\omega \in \bar{\Gamma}_{\chi,c}$ for a closed positive (1,1) current ω using smoothing. We remark that any closed positive (1,1) current can be written as $\sqrt{-1}\partial\bar{\partial}$ acting on a real function locally.

Definition 3.2 Suppose that χ is a Kähler form with constant coefficients on an open set $O \subset \mathbb{C}^n$. Then we say that $\sqrt{-1}\partial\bar{\partial}\varphi \in \bar{\Gamma}_{\chi,c}$ on O if for any $\delta > 0$, the smoothing φ_δ satisfies $\sqrt{-1}\partial\bar{\partial}\varphi_\delta \in \bar{\Gamma}_{\chi,c}$ on the set $O_\delta = \{x : B_\delta(x) \subset O\}$, which is, by definition, equivalent to

$$c\left(\sqrt{-1}\partial\bar{\partial}\varphi_\delta\right)^{n-1} - (n-1)\chi \wedge \left(\sqrt{-1}\partial\bar{\partial}\varphi_\delta\right)^{n-2} \geq 0.$$

We can also define this without assuming constant coefficients.

Definition 3.3 We say that $\omega \in \bar{\Gamma}_{\chi,c}$ if on any open subset O of any coordinate chart, for any Kähler form $\chi_0 \leq \chi$ with constant coefficients, $\omega \in \bar{\Gamma}_{\chi_0,c}$.

Remark 3.4 $\bar{\Gamma}_{\chi_0,c}$ is convex. So if ω is smooth, then $\omega \in \bar{\Gamma}_{\chi,c}$ on O pointwise if and only if it is true on O in the sense of Definition 3.3. A useful characterization for $\bar{\Gamma}_{\chi,c}$ is that a current $\sqrt{-1}\partial\bar{\partial}\varphi$ on $O \subset \mathbb{C}^n$ is in $\bar{\Gamma}_{\chi,c}$ in the sense of Definition 3.3 if and only if it is the weak limit of smooth forms $\sqrt{-1}\partial\bar{\partial}\varphi_i$ in $\bar{\Gamma}_{\chi,c_i}$ for $c_i \to c$. In particular, this characterization implies that Definition 3.3 is independent of the choice of the weight function ρ in the smoothing. Another useful corollary of this characterization is that if χ_1, χ_2 are Kähler forms on M and ω_1, ω_2 are closed positive currents on M such that $\frac{\chi_1}{c_1} - \frac{\chi_2}{c_2}$ is non-negative definitive and $\omega_2 - \omega_1$ is also a positive current, then $\omega_1 \in \bar{\Gamma}_{\chi_1,c_1}$ in the sense of Definition 3.3 implies that $\omega_2 \in \bar{\Gamma}_{\chi_2,c_2}$ in the sense of Definition 3.3.

For simplicity, for any positive definite Hermitian $n \times n$ matrices A and B, we define $P_B(A)$ as $\max_k(\sum_{j\neq k}\frac{1}{\lambda_j})$, where λ_j are the eigenvalues of $B^{-1}A$. As mentioned in Remark 1.3,

$$c\omega^{n-1} - (n-1)\chi \wedge \omega^{n-2} \geq 0,$$

$P_\chi(\omega) \leq c$, and $\omega \in \bar{\Gamma}_{\chi,c}$ are equivalent conditions.

For any $(n-1)$-dimensional subspace V of $\mathbb{C}^n$, there exists $U \in \mathbb{C}^{n\times(n-1)}$ such that $\bar{U}^T U = I_{n-1}$ and a basis of V consists of the columns of U. If we view A as a bilinear form, then the restriction $A|_V$ of A on V can be expressed as $\bar{U}^T AU$ in the basis of V which consists of the columns of U. Then $\mathrm{tr}(A|_V)^{-1} = \mathrm{tr}(\bar{U}^T AU)^{-1}$ depends only on A and V because a different choice of U just means multiplying U with a unitary matrix on the right. Choose U such that $(\bar{U}^T AU)_{i\bar{j}} = \lambda'_i\delta_{i\bar{j}}$ and $S \in \mathbb{C}^{n\times n}$ such that $\bar{S}^T S = I_n$ and $S_{i\bar{j}} = U_{i\bar{j}}$ for $i = 1, 2, ..., n$ and $j = 1, 2, ..., n-1$. Then by the Schur-Horn theorem, the diagonal $(\lambda'_1, ...\lambda'_{n-1}, (\bar{S}^T AS)_{n\bar{n}})$ of the matrix $\bar{S}^T AS$ lies in the convex hull of the vectors obtained by permuting the entries of $(\lambda_1, ..., \lambda_n)$, where λ_i are the eigenvalues of A. By the convexity of $\sum_{i=1}^{n-1}\frac{1}{\lambda'_i}$, it follows that

$$P_{I_n}(A) = \max_k \left(\sum_{j \neq k} \frac{1}{\lambda_j} \right) = \max_{U \in \mathbb{C}^{n\times(n-1)}, \bar{U}^T U = I_{n-1}, (\bar{U}^T AU)_{i\bar{j}} = \lambda'_i \delta_{i\bar{j}}} \sum_{i=1}^{n-1} \frac{1}{\lambda'_i}$$
$$= \max_{U \in \mathbb{C}^{n\times(n-1)}, \bar{U}^T U = I_{n-1}} \left(\mathrm{tr}(\bar{U}^T AU)^{-1} \right) = \max_{V^{n-1} \subset \mathbb{C}^n} \left(\mathrm{tr}(A|_V)^{-1} \right),$$

which is similar to the Courant–Fischer–Weyl min-max principle.

Now we need a lemma:

Lemma 3.5 *Suppose that* $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{n\times n}$, $C \in \mathbb{C}^{m\times n}$ *are matrices such that* $\begin{bmatrix} A & C \\ \bar{C}^T & B \end{bmatrix}$ *is Hermitian and positive definite. Then*

$$P_{I_m}(A - CB^{-1}\bar{C}^T) + \mathrm{tr}(B^{-1}) \leq P_{I_{m+n}} \left(\begin{bmatrix} A & C \\ \bar{C}^T & B \end{bmatrix} \right).$$

Proof It is easy to see that

$$\begin{bmatrix} I_m & -CB^{-1} \\ O & I_n \end{bmatrix} \begin{bmatrix} A & C \\ \bar{C}^T & B \end{bmatrix} \begin{bmatrix} I_m & O \\ -B^{-1}\bar{C}^T & I_n \end{bmatrix} = \begin{bmatrix} A - CB^{-1}\bar{C}^T & O \\ O & B \end{bmatrix},$$

so $A - CB^{-1}\bar{C}^T$ is also positive definite. By taking the inverse, we obtain

$$\begin{bmatrix} I_m & O \\ -B^{-1}\bar{C}^T & I_n \end{bmatrix} \begin{bmatrix} A - CB^{-1}\bar{C}^T & O \\ O & B \end{bmatrix}^{-1} \begin{bmatrix} I_m & -CB^{-1} \\ O & I_n \end{bmatrix} = \begin{bmatrix} A & C \\ \bar{C}^T & B \end{bmatrix}^{-1}.$$

After taking traces, the left-hand side equals

$$\mathrm{tr}\left(\left(A - CB^{-1}\bar{C}^T \right)^{-1} \right) + \mathrm{tr}(B^{-1}) + \mathrm{tr}\left(B^{-1}\bar{C}^T \left(A - CB^{-1}\bar{C}^T \right)^{-1} CB^{-1} \right).$$

Thus,

$$\mathrm{tr}\left((A - CB^{-1}\bar{C}^T)^{-1} \right) + \mathrm{tr}(B^{-1}) \leq \mathrm{tr}\left(\begin{bmatrix} A & C \\ \bar{C}^T & B \end{bmatrix}^{-1} \right).$$

Let $U \in \mathbb{C}^{m\times(m-1)}$ be the matrix realizing the maximum in

$$P_{I_m}\left(A - CB^{-1}\bar{C}^T \right) = \max_{U \in \mathbb{C}^{m\times(m-1)}, \bar{U}^T U = I_{m-1}} \left(\mathrm{tr}\left(\bar{U}^T \left(A - CB^{-1}\bar{C}^T \right) U \right)^{-1} \right).$$

Then

$$
\begin{aligned}
&P_{I_m}(A - CB^{-1}\bar{C}^T) + \operatorname{tr}(B^{-1}) \\
&\quad = \operatorname{tr}\left(\bar{U}^T\left(A - CB^{-1}\bar{C}^T\right)U)^{-1} + \operatorname{tr}(B^{-1}\right) \\
&\quad = \operatorname{tr}\left(\bar{U}^T AU - (\bar{U}^T C)B^{-1}(\bar{C}^T U)\right)^{-1} + \operatorname{tr}(B^{-1}) \\
&\quad \leq \operatorname{tr}\left(\begin{bmatrix} \bar{U}^T AU & \bar{U}^T C \\ \bar{C}^T U & B \end{bmatrix}^{-1}\right) \\
&\quad = \operatorname{tr}\left(\left(\begin{bmatrix} \bar{U}^T & O \\ O & I_n \end{bmatrix}\begin{bmatrix} A & C \\ \bar{C}^T & B \end{bmatrix}\begin{bmatrix} U & O \\ O & I_n \end{bmatrix}\right)^{-1}\right) \\
&\quad \leq P_{I_{m+n}}\left(\begin{bmatrix} A & C \\ \bar{C}^T & B \end{bmatrix}\right).
\end{aligned}
$$

This is the required estimate. □

Now we start the proof of Theorem 1.18. By assumption, for any $t > 0$, there exist a constant $c_t > 0$ and a Kähler metric $\omega_t \in [(1+t)\omega_0]$ satisfying

$$c\omega_t^{n-1} - (n-1)\chi \wedge \omega_t^{n-2} > 0$$

and

$$\operatorname{tr}_{\omega_t}\chi + c_t \frac{\chi^n}{\omega_t^n} = c.$$

Consider $\omega_{0,M\times M,t} = \pi_1^*\omega_t + \frac{1}{c}\pi_2^*\chi$ and $\chi_{M\times M} = \pi_1^*\chi + \pi_2^*\chi$. At each point, diagonalize them so that $\chi_{i\bar{j}} = \delta_{i\bar{j}}$ and $(\omega_t)_{i\bar{j}} = \lambda_{i,M}\delta_{i\bar{j}}$. Then the eigenvalues on the product manifold are $\lambda_{1,M}$, ...$\lambda_{n,M}$, $\frac{1}{c}$, ..., $\frac{1}{c}$. Their inverses are $\frac{1}{\lambda_{1,M}}$, ..., $\frac{1}{\lambda_{n,M}}$, c, ..., c. So the sum of them is at most $(n+1)c$ because $c_t > 0$. In particular, the sum of (2n-1) distinct elements among them is also at most $(n+1)c$. If we define $f_{t,\epsilon_{1.6},\epsilon_{1.7}}$ as in Sect. 1, then there exists a constant $\epsilon_{1.7} > 0$ such that for $\epsilon_{1.6}$ small enough, $f_{t,\epsilon_{1.6},\epsilon_{1.7}} > -\frac{1}{4n}(\frac{1}{(n+1)c})^{2n-1}$. So we can apply Theorem 1.14 to get $\omega_{t,\epsilon_{1.6},\epsilon_{1.7}} \in [\omega_{0,M\times M,t}]$ such that $P_{\chi_{M\times M}}(\omega_{t,\epsilon_{1.6},\epsilon_{1.7}}) < (n+1)c$ and

$$\operatorname{tr}_{\omega_{t,\epsilon_{1.6},\epsilon_{1.7}}}\chi_{M\times M} + f_{t,\epsilon_{1.6},\epsilon_{1.7}}\frac{\chi_{M\times M}^{2n}}{\omega_{t,\epsilon_{1.6},\epsilon_{1.7}}^{2n}} = (n+1)c.$$

For each point (x_1, x_2), we assume that $z_1^{(1)}$, ..., $z_n^{(1)}$ are the local coordinates on $M \times \{x_2\}$ and that $z_1^{(2)}$, ..., $z_n^{(2)}$ are the local coordinates on $\{x_1\} \times M$. Then we can express $\omega_{t,\epsilon_{1.6},\epsilon_{1.7}}$ as

$$\omega_{t,\epsilon_{1.6},\epsilon_{1.7}} = \omega^{(1)}_{t,\epsilon_{1.6},\epsilon_{1.7}} + \omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}} + \omega^{(1,2)}_{t,\epsilon_{1.6},\epsilon_{1.7}} + \omega^{(2,1)}_{t,\epsilon_{1.6},\epsilon_{1.7}},$$

where

$$\begin{aligned}
\omega^{(1)}_{t,\epsilon_{1.6},\epsilon_{1.7}} &= \sum_{i,j=1}^{n} \sqrt{-1}\omega^{(1)}_{t,\epsilon_{1.6},\epsilon_{1.7},i\bar{j}} dz^{(1)}_i \wedge d\bar{z}^{(1)}_j,\\
\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}} &= \sum_{i,j=1}^{n} \sqrt{-1}\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7},i\bar{j}} dz^{(2)}_i \wedge d\bar{z}^{(2)}_j,\\
\omega^{(1,2)}_{t,\epsilon_{1.6},\epsilon_{1.7}} &= \sum_{i,j=1}^{n} \sqrt{-1}\omega^{(1,2)}_{t,\epsilon_{1.6},\epsilon_{1.7},i\bar{j}} dz^{(1)}_i \wedge d\bar{z}^{(2)}_j,\\
\omega^{(2,1)}_{t,\epsilon_{1.6},\epsilon_{1.7}} &= \overline{\omega^{(1,2)}_{t,\epsilon_{1.6},\epsilon_{1.7}}}.
\end{aligned}$$

After changing the definition of $z^{(2)}_i$ if necessary, we can assume that

$$\pi_2^*\chi = \sqrt{-1}\sum_{i=1}^{n} dz^{(2)}_i \wedge d\bar{z}^{(2)}_i$$

and

$$\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}} = \sqrt{-1}\sum_{i=1}^{n} \lambda_i dz^{(2)}_i \wedge d\bar{z}^{(2)}_i$$

at (x_1, x_2).

Now consider $\Omega_{t,\epsilon_{1.6},\epsilon_{1.7}}$ defined as

$$\begin{aligned}
&\Omega_{t,\epsilon_{1.6},\epsilon_{1.7}}\\
&\quad= \frac{c^{n-1}}{\int_M n\chi^n}(\pi_1)_*\left(\omega^n_{t,\epsilon_{1.6},\epsilon_{1.7}} \wedge \pi_2^*\chi\right)\\
&\quad= \frac{c^{n-1}}{\int_M n\chi^n}(\pi_1)_*\left(n\omega^{(1)}_{t,\epsilon_{1.6},\epsilon_{1.7}} \wedge \left(\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}}\right)^{n-1} \wedge \pi_2^*\chi\right)\\
&\quad+ \frac{c^{n-1}}{\int_M n\chi^n}(\pi_1)_*(n(n-1)\omega^{(1,2)}_{t,\epsilon_{1.6},\epsilon_{1.7}} \wedge \omega^{(2,1)}_{t,\epsilon_{1.6},\epsilon_{1.7}} \wedge (\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}})^{n-2} \wedge \pi_2^*\chi).
\end{aligned}$$

At (x_1, x_2),

$$n\omega^{(1)}_{t,\epsilon_{1.6},\epsilon_{1.7}} \wedge \left(\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}}\right)^{n-1} \wedge \pi_2^*\chi$$
$$= \sum_{i,j=1}^{n} \sqrt{-1}\omega^{(1)}_{t,\epsilon_{1.6},\epsilon_{1.7},i\bar{j}} dz_i^{(1)} \wedge d\bar{z}_j^{(1)} \wedge \left(\sum_{\alpha=1}^{n} \frac{1}{\lambda_\alpha}\right)\left(\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}}\right)^{n},$$

and

$$n(n-1)\omega^{(1,2)}_{t,\epsilon_{1.6},\epsilon_{1.7}} \wedge \omega^{(2,1)}_{t,\epsilon_{1.6},\epsilon_{1.7}} \wedge \left(\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}}\right)^{n-2} \wedge \pi_2^*\chi$$
$$= -\sum_{i,j,k=1}^{n} \sqrt{-1}\omega^{(1,2)}_{t,\epsilon_{1.6},\epsilon_{1.7},i\bar{k}} \overline{\omega^{(1,2)}_{t,\epsilon_{1.6},\epsilon_{1.7},j\bar{k}}} dz_i^{(1)} \wedge d\bar{z}_j^{(1)} \wedge \frac{1}{\lambda_k}\left(\sum_{\alpha\neq k} \frac{1}{\lambda_\alpha}\right)\left(\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}}\right)^{n}$$
$$\geq -\sum_{i,j,k=1}^{n} \sqrt{-1}\omega^{(1,2)}_{t,\epsilon_{1.6},\epsilon_{1.7},i\bar{k}} \overline{\omega^{(1,2)}_{t,\epsilon_{1.6},\epsilon_{1.7},j\bar{k}}} dz_i^{(1)} \wedge d\bar{z}_j^{(1)} \wedge \frac{1}{\lambda_k}\left(\sum_{\alpha=1}^{n} \frac{1}{\lambda_\alpha}\right)\left(\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}}\right)^{n}.$$

By Lemma 3.5,

$$P_{\pi_1^*\chi}\left(\sqrt{-1}\sum_{i,j=1}^{n}\left(\left(\omega^{(1)}_{t,\epsilon_{1.6},\epsilon_{1.7},i\bar{j}} - \sum_{k=1}^{n}\frac{1}{\lambda_k}\omega^{(1,2)}_{t,\epsilon_{1.6},\epsilon_{1.7},i\bar{k}}\overline{\omega^{(1,2)}_{t,\epsilon_{1.6},\epsilon_{1.7},j\bar{k}}}\right) dz_i^{(1)} \wedge d\bar{z}_j^{(1)}\right)\right)$$
$$\leq P_{\chi_{M\times M}}\left(\omega_{t,\epsilon_{1.6},\epsilon_{1.7}}\right) - \mathrm{tr}_{\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}}}(\pi_2^*\chi)$$
$$\leq (n+1)c - \mathrm{tr}_{\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}}}(\pi_2^*\chi).$$

We remark that there is an abuse of notation here by identifying $M \times \{x_2\}$ or $\{x_1\} \times M$ with M. For example, $\pi_1^*\chi$ on $M \times \{x_2\}$ is identified as χ on M.

If we view

$$\frac{c^{n-1}}{\int_M n\chi^n}\left(\mathrm{tr}_{\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}}}\pi_2^*\chi\right)\left(\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}}\right)^{n}$$

as a measure on $\{x_1\} \times M$, then it is easy to see that

$$\frac{c^{n-1}}{\int_M n\chi^n}\int_{\{x_1\}\times M}\left(\mathrm{tr}_{\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}}}\pi_2^*\chi\right)\left(\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}}\right)^{n}$$
$$= \frac{c^{n-1}}{\int_M \chi^n}\int_{\{x_1\}\times M}(\pi_2^*\chi) \wedge \left(\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}}\right)^{n-1}$$
$$= \frac{c^{n-1}}{\int_M \chi^n}\int_M \chi \wedge \left(\frac{\chi}{c}\right)^{n-1}$$
$$= 1.$$

By the monotonicity and convexity of P_χ,

$$\begin{aligned}
P_\chi\left(\Omega_{t,\epsilon_{1.6},\epsilon_{1.7}}\right) &\le (n+1)c - \frac{c^{n-1}}{\int_M n\chi^n}\int_{\{x_1\}\times M}\left(\mathrm{tr}_{\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}}}\pi_2^*\chi\right)^2\left(\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}}\right)^n \\
&\le (n+1)c - \frac{c^{n-1}}{\int_M n\chi^n}\frac{\left(\int_{\{x_1\}\times M}\left(\mathrm{tr}_{\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}}}\pi_2^*\chi\right)\left(\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}}\right)^n\right)^2}{\int_{\{x_1\}\times M}\left(\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}}\right)^n} \\
&= (n+1)c - \frac{nc^{n-1}}{\int_M \chi^n}\frac{\left(\int_M\left(\frac{\chi}{c}\right)^{n-1}\wedge\chi\right)^2}{\int_M\left(\frac{\chi}{c}\right)^n} \\
&= c.
\end{aligned}$$

To get better estimates, we need to study the weak limit Θ of a subsequence of $\omega^n_{t,\epsilon_{1.6},\epsilon_{1.7}}$ and the weak limit Θ' of a subsequence of $\omega^{n-1}_{t,\epsilon_{1.6},\epsilon_{1.7}}$ when t and $\epsilon_{1.6}$ converge to 0 and $\epsilon_{1.7}$ is fixed. Then both Θ and Θ' are closed positive currents. Let $\xi_{\epsilon_{3.2}}$ be a non-decreasing family of functions with values in [0,1] that equal to 1 on Δ and are supported in the region such that the $\chi_{M\times M}$-distance to the diagonal Δ is smaller than $\epsilon_{3.2}$. Then $\xi_{\epsilon_{3.2}}\Theta$ has a weak limit $1_\Delta\Theta$ when $\epsilon_{3.2}$ goes to 0. By the Skoda–El Mir extension theorem (Theorem III.2.3 of [16]), $1_\Delta\Theta$ is a closed positive current. A similar statement is true for $1_\Delta\Theta'$. By the support theorem (Corollary III.2.14 of [16]),

$$1_\Delta\Theta = \epsilon_{3.1}[\Delta]$$

for a non-negative constant $\epsilon_{3.1}$ (assuming that M is connected; otherwise we consider each component separately). On the other hand, the support theorem (Theorem III.2.10 of [16]) implies that

$$1_\Delta\Theta' = 0.$$

Up to here, we have not used the equation

$$\mathrm{tr}_{\omega_{t,\epsilon_{1.6},\epsilon_{1.7}}}\chi_{M\times M} + f_{t,\epsilon_{1.6},\epsilon_{1.7}}\frac{\chi^{2n}_{M\times M}}{\omega^{2n}_{t,\epsilon_{1.6},\epsilon_{1.7}}} = (n+1)c.$$

By this equation, $\omega^{2n}_{t,\epsilon_{1.6},\epsilon_{1.7}} \ge \frac{f_{t,\epsilon_{1.6},\epsilon_{1.7}}}{(n+1)c}\chi^{2n}_{M\times M}$. So as in Proposition 2.6 of [20], it is easy to see that the constant $\epsilon_{3.1} > 0$.

Let $\epsilon_{1.4} = \frac{\epsilon_{3.1}}{4}\frac{c^{n-1}}{\int_M n\chi^n}$ and let $\omega_{1.5}$ be the weak limit of a subsequence of $\Omega_{t,\epsilon_{1.6},\epsilon_{1.7}} - \epsilon_{1.4}\chi$ when t and $\epsilon_{1.6}$ converge to 0 and $\epsilon_{1.7}$ is fixed. Then by Remark 3.4, it suffices to show that for any constant $\epsilon_{3.3} > 0$, $\omega_{1.5} \in \bar{\Gamma}_{\chi,c+(n-1)\epsilon_{3.3}}$ in the sense of Definition 3.3. In other words, on any open subset O of any coordinate chart, for any Kähler form $\chi_0 \leq \chi$ with constant coefficients, for any $\delta > 0$, we need to show that $P_{\chi_0}((\omega_{1.5})_\delta) \leq c+(n-1)\epsilon_{3.3}$, where $(\omega_{1.5})_\delta$ is the smoothing of $\omega_{1.5}$ as in Definition 3.1. Since this is a local problem, we can assume that $\frac{\chi}{2} \leq \chi_0 \leq \chi$ on O by shrinking O if necessary.

For any point $x_1 \in O$, let $\rho_{x_1,\delta}(y) = \rho(\frac{|x_1-y|}{\delta})$ in local coordinates, where ρ is the function used in Definition 3.1. Then since the weak limit of $\xi_{\epsilon_{3.2}}\Theta'$ is 0 and Θ' itself is the limit of $\omega^{n-1}_{t,\epsilon_{1.6},\epsilon_{1.7}}$, we see that

$$\begin{aligned}
&\lim_{\epsilon_{3.2}\to 0}\lim_{t,\epsilon_{1.6}\to 0}\int_{M\times M}\left(\rho_{x_1,\delta}\circ\pi_1\right)\cdot\xi_{\epsilon_{3.2}}\cdot\pi_1^*\chi\wedge\omega^{n-1}_{t,\epsilon_{1.6},\epsilon_{1.7}}\wedge\pi_2^*\chi\wedge\pi_1^*\eta\\
&=\lim_{\epsilon_{3.2}\to 0}\lim_{t,\epsilon_{1.6}\to 0}\int_{M\times M}\left(\rho_{x_1,\delta}\circ\pi_1\right)\cdot\xi_{\epsilon_{3.2}}\cdot\pi_1^*\chi\wedge\left(\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}}\right)^{n-1}\wedge\pi_2^*\chi\wedge\pi_1^*\eta\\
&=0
\end{aligned}$$

for any $(n-1,n-1)$-form η with constant coefficients on $O \subset \mathbb{C}^n$. Therefore, for sufficiently small $\epsilon_{3.2}$, t, and $\epsilon_{1.6}$ depending on M, χ, ω_0, Θ, O, χ_0, δ, x_1, $\epsilon_{1.7}$, and $\epsilon_{3.3}$,

$$\frac{c^{n-1}}{\int_M n\chi^n}\left((\pi_1)_*\left(\xi_{\epsilon_{3.2}}\cdot\pi_1^*\chi\wedge\left(\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}}\right)^{n-1}\wedge\pi_2^*\chi\right)\right)_\delta-\epsilon_{1.4}\epsilon_{3.3}\chi_0$$

is negative definite at x_1.

Similarly, using the fact that $1_\Delta\Theta = \epsilon_{3.1}[\Delta]$, for sufficiently small $\epsilon_{3.2}$, t, and $\epsilon_{1.6}$ depending on M, χ, ω_0, Θ, O, χ_0, δ, x_1, $\epsilon_{1.7}$, and $\epsilon_{3.3}$,

$$\frac{c^{n-1}}{\int_M n\chi^n}\left((\pi_1)_*\left(\xi_{\epsilon_{3.2}}\cdot\omega^n_{t,\epsilon_{1.6},\epsilon_{1.7}}\wedge\pi_2^*\chi\right)\right)_\delta-3\epsilon_{1.4}\chi_0$$

is positive definite at x_1.

For any Kähler form ω restricted to the first n coordinates of $M\times M$, after choosing good coordinates, assume that

$$\pi_1^*\chi=\sqrt{-1}\sum_{i=1}^n dz_i^{(1)}\wedge d\bar{z}_i^{(1)}$$

and

$$\omega=\sqrt{-1}\sum_{i=1}^n\lambda_i dz_i^{(1)}\wedge d\bar{z}_i^{(1)}.$$

We define the truncation $T_{\frac{\pi_1^*\chi}{\epsilon_{3.3}}}(\omega)$ by

$$\left(T_{\frac{\pi_1^*\chi}{\epsilon_{3.3}}}(\omega)\right)_{i\bar{j}} = \sqrt{-1}\sum_{i=1}^{n} \min\{\lambda_i, \frac{1}{\epsilon_{3.3}}\} dz_i^{(1)} \wedge d\bar{z}_i^{(1)}.$$

We remark that the truncation is independent of the choice of local coordinates. Now consider the truncation $\Omega_{t,\epsilon_{1.6},\epsilon_{1.7}}^{(\frac{\pi_1^*\chi}{\epsilon_{3.3}})}$ defined as

$$\frac{c^{n-1}}{\int_M n\chi^n}(\pi_1)_*\left(T_{\frac{\pi_1^*\chi}{\epsilon_{3.3}}}\left(\tilde{\omega}^{(1)}_{t,\epsilon_{1.6},\epsilon_{1.7}}\right) \wedge \left(\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}}\right)^{n-1} \wedge \pi_2^*\chi\right),$$

where the (1,1)-form $\tilde{\omega}^{(1)}_{t,\epsilon_{1.6},\epsilon_{1.7}}$ is defined by

$$\begin{aligned}
&n\omega^{(1)}_{t,\epsilon_{1.6},\epsilon_{1.7}} \wedge \left(\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}}\right)^{n-1} \wedge \pi_2^*\chi \\
&\quad + n(n-1)\omega^{(1,2)}_{t,\epsilon_{1.6},\epsilon_{1.7}} \wedge \omega^{(2,1)}_{t,\epsilon_{1.6},\epsilon_{1.7}} \wedge \left(\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}}\right)^{n-2} \wedge \pi_2^*\chi \\
&\quad = \tilde{\omega}^{(1)}_{t,\epsilon_{1.6},\epsilon_{1.7}} \wedge \left(\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}}\right)^{n-1} \wedge \pi_2^*\chi.
\end{aligned}$$

Then

$$\begin{aligned}
&\Omega_{t,\epsilon_{1.6},\epsilon_{1.7}} - \Omega_{t,\epsilon_{1.6},\epsilon_{1.7}}^{\left(\frac{\pi_1^*\chi}{\epsilon_{3.3}}\right)} \\
&\quad \geq \frac{c^{n-1}}{\int_M n\chi^n}(\pi_1)_*\left(\xi_{\epsilon_{3.2}}\left(\tilde{\omega}^{(1)}_{t,\epsilon_{1.6},\epsilon_{1.7}} - \frac{\pi_1^*\chi}{\epsilon_{3.3}}\right) \wedge \left(\omega^{(2)}_{t,\epsilon_{1.6},\epsilon_{1.7}}\right)^{n-1} \wedge \pi_2^*\chi\right).
\end{aligned}$$

So

$$\left(\Omega_{t,\epsilon_{1.6},\epsilon_{1.7}}\right)_\delta - \left(\Omega_{t,\epsilon_{1.6},\epsilon_{1.7}}^{\left(\frac{\pi_1^*\chi}{\epsilon_{3.3}}\right)}\right)_\delta - 2\epsilon_{1.4}\chi_0$$

is positive definite at x_1.

It is easy to see that

$$P_{\pi_1^*\chi}\left(T_{\frac{\pi_1^*\chi}{\epsilon_{3.3}}}(\omega)\right) - P_{\pi_1^*\chi}(\omega) \leq (n-1)\epsilon_{3.3}$$

for any (1,1)-form ω on the first n coordinates of $M \times M$. So using the estimate of $P_\chi(\Omega_{t,\epsilon_{1.6},\epsilon_{1.7}})$, it is easy to see that

$$P_\chi\left(\Omega_{t,\epsilon_{1.6},\epsilon_{1.7}}^{\left(\frac{\pi_1^*\chi}{\epsilon_{3.3}}\right)}\right) \le c + (n-1)\epsilon_{3.3}.$$

By monotonicity of P_χ, and the property that $\frac{1}{2}\chi \le \chi_0 \le \chi$, we see that

$$P_{\chi_0}\left(\left(\Omega_{t,\epsilon_{1.6},\epsilon_{1.7}}\right)_\delta - \epsilon_{1.4}\chi\right) \le c + (n-1)\epsilon_{3.3}$$

at x_1. This completes the proof of Theorem 1.18.

4 Regularization

In this section, we prove Theorem 1.11. By Remark 1.13, the $n = 1$ and $n = 2$ cases have been proved. By induction, we can assume that Theorem 1.11 has been proved in dimensions $1, 2, ..., n-1$. By Sect. 1, we can in addition assume that the conditions for Theorem 1.18 are satisfied. So by Theorem 1.18, there exist a constant $\epsilon_{1.4} > 0$ and a current $\omega_{1.5} \in [\omega_0 - \epsilon_{1.4}\chi]$ such that $\omega_{1.5} \in \bar{\Gamma}_{\chi,c}$ in the sense of Definition 3.3.

Pick a small enough constant $\epsilon_{4.1} < \frac{1}{10000}$ such that

$$\omega_0 - 100\epsilon_{4.1}\omega_0 \ge (1+\epsilon_{4.1})^2(\omega_0 - \epsilon_{1.4}\chi).$$

Then there exists a current $\omega_{4.2} = \omega_0 - 100\epsilon_{4.1}\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_{4.2} \in \bar{\Gamma}_{\chi,\frac{c}{(1+\epsilon_{4.1})^2}}$ in the sense of Definition 3.3 by Remark 3.4.

Now we pick a finite number of coordinate balls $B_{2r}(x_i)$ such that $B_r(x_i)$ is a cover of M. Moreover, we require that

$$\chi_0^i < \chi < (1+\epsilon_{4.1})\chi_0^i$$

on $B_{2r}(x_i)$ for Kähler forms χ_0^i with constant coefficients. We also assume that

$$(1-\epsilon_{4.1})\sqrt{-1}\partial\bar{\partial}|z|^2 \le \omega_0 \le (1+\epsilon_{4.1})\sqrt{-1}\partial\bar{\partial}|z|^2$$

on $B_{2r}(x_i)$. Let $\varphi_{\omega_0}^i$ be a potential such that $\sqrt{-1}\partial\bar{\partial}\varphi_{\omega_0}^i = \omega_0$ on $B_{2r}(x_i)$. Then we also assume that

$$|\varphi_{\omega_0}^i - |z|^2| \le \epsilon_{4.1}r^2.$$

Let φ^i_δ be the smoothing of $\varphi_{4.2} + (1 - 100\epsilon_{4.1})\varphi^i_{\omega_0}$. When $\delta < \frac{r}{5}$, this is well defined on $\overline{B_{\frac{9}{5}r}(x_i)}$. By assumption, it is easy to see that

$$\frac{c}{(1+\epsilon_{4.1})^2}\left(\sqrt{-1}\partial\bar{\partial}\varphi^i_\delta\right)^{n-1} - (n-1)\chi^i_0 \wedge \left(\sqrt{-1}\partial\bar{\partial}\varphi^i_\delta\right)^{n-2} \geq 0.$$

So

$$\frac{c}{1+\epsilon_{4.1}}\left(\sqrt{-1}\partial\bar{\partial}\varphi^i_\delta\right)^{n-1} - (n-1)\chi \wedge \left(\sqrt{-1}\partial\bar{\partial}\varphi^i_\delta\right)^{n-2} > 0.$$

Now define the function $\varphi^i_{4.3}$ from $\overline{B_{\frac{9}{5}r}(x_i)}$ to $\mathbb{R}$ as $\varphi^i_\delta - \varphi^i_{\omega_0}$ so that we can study the regularized maximum $\varphi_{4.4}$ of $\varphi^i_{4.3}$.

Recall the definition of the regularized maximum in Lemma I.5.18 of [16]. Let θ be a nonnegative smooth function on $\mathbb{R}$ with support in $[-1, 1]$ such that $\int_{\mathbb{R}} \theta(h)dh = 1$ and $\int_{\mathbb{R}} h\theta(h)dh = 0$. Let η_i be positive numbers smaller than $\frac{\epsilon_{4.1}r^2}{3}$. Then the regularized maximum $\varphi_{4.4}$ of $\varphi^i_{4.3}$ is defined by

$$\varphi_{4.4} = M_\eta\left(\varphi^1_{4.3}, ...\varphi^I_{4.3}\right) = \int_{\mathbb{R}^I} \max_{i=1}^{I}\{\varphi^i_{4.3} + h_i\}\prod_{i=1}^{I}\theta\left(\frac{h_i}{\eta_i}\right)dh_1...dh_I,$$

where I is the number of points x_i, and $\varphi^i_{4.3}$ is defined as $-\infty$ outside $\overline{B_{\frac{9}{5}r}(x_i)}$.

Our goal is to show that for any $x \in M$,

$$\epsilon_{4.1}r^2 + \max_{\left\{i: x\in \overline{B_{\frac{9}{5}r}(x_i)}\setminus B_{\frac{8}{5}r}(x_i)\right\}} \varphi^i_{4.3}(x) < \max_{\left\{i: x\in \overline{B_r(x_i)}\right\}} \varphi^i_{4.3}(x).$$

If this is true, then the maximum will never be achieved by the function $\varphi^i_{4.3}+h_i$ outside $B_{\frac{8}{5}r}(x_i)$. So by Lemma I.5.18 (c) of [16], we can discard the function $\varphi^i_{4.3} + h_i$ outside $B_{\frac{8}{5}r}(x_i)$. Thus, without loss of generality, we can assume that all $\varphi^i_{4.3}$ under consideration are smooth. It follows that the function $\varphi_{4.4}$ is smooth by Lemma I.5.18 (a) of [16]. Moreover, $\omega_{4.4} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_{4.4} > 0$ by Lemma I.5.18 (e) of [16].

Now we claim that $c\omega^{n-1}_{4.4} - (n-1)\chi \wedge \omega^{n-2}_{4.4} > 0$. In fact,

$$\frac{\partial^2\varphi_{4.4}}{\partial z^j\partial\bar{z}^k} = \sum_{a,b=1}^{I}\frac{\partial^2 M_\eta}{\partial\varphi^a_{4.3}\partial\varphi^b_{4.3}}\frac{\partial\varphi^a_{4.3}}{\partial z^j}\frac{\partial\varphi^b_{4.3}}{\partial\bar{z}^k} + \sum_{a=1}^{I}\frac{\partial M_\eta}{\partial\varphi^a_{4.3}}\frac{\partial^2\varphi^a_{4.3}}{\partial z^j\partial\bar{z}^k}.$$

Since $M_\eta(\varphi^1_{4.3}+\xi,...\varphi^I_{4.3}+\xi) = M_\eta(\varphi^1_{4.3},...\varphi^I_{4.3})+\xi$ for any constant ξ by Lemma I.5.18 (d) of [16], it follows that $\sum_{a=1}^I \frac{\partial M_\eta}{\partial \varphi^a_{4.3}} = 1$. So the term $\sum_{a=1}^I \frac{\partial M_\eta}{\partial \varphi^a_{4.3}} \frac{\partial^2 \varphi^a_{4.3}}{\partial z^j \partial \bar{z}^k}$ is a weighted average of $\frac{\partial^2 \varphi^a_{4.3}}{\partial z^j \partial \bar{z}^k}$. On the other hand, M_η is convex by Lemma I.5.18 (a) of [16]. So by the monotonicity and convexity of P_χ, $P_\chi(\omega_{4.4}) < \frac{c}{1+\epsilon_{4.1}}$. Thus, we are done if

$$\epsilon_{4.1}r^2 + \max_{\left\{i: x\in \overline{B_{\frac{9}{5}r}(x_i)}\setminus B_{\frac{8}{5}r}(x_i)\right\}} \varphi^i_{4.3}(x) < \max_{\left\{i: x\in \overline{B_r(x_i)}\right\}} \varphi^i_{4.3}(x).$$

In general,

$$\epsilon_{4.1}r^2 + \max_{\left\{i: x\in \overline{B_{\frac{9}{5}r}(x_i)}\setminus B_{\frac{8}{5}r}(x_i)\right\}} \varphi^i_{4.3}(x) < \max_{\left\{i: x\in \overline{B_r(x_i)}\right\}} \varphi^i_{4.3}(x)$$

is not true, so $\omega_{4.4}$ may not be smooth. However, using the proof of the results of Błocki and Kołodziej [3], this is in fact true if the Lelong number is small enough. The details constitute the rest of this section.

It is easy to see that if $x \in \overline{B_{\frac{9}{5}r}(x_i)} \cap \overline{B_{\frac{9}{5}r}(x_j)}$ and $\delta < \frac{r}{10}$, $B^i_{\frac{\delta}{2}}(x) \subset B^j_\delta(x)$, where $B^i_{\frac{\delta}{2}}(x)$ means the coordinate ball with center x and radius $\frac{\delta}{2}$ using the coordinates corresponding to x_i, and the meaning for $B^j_\delta(x)$ is similar. For any $\delta < \frac{r}{20}$ and $x \in \overline{B_{\frac{9}{5}r}(x_i)}$, we define $\hat{\varphi}^i_\delta$ by

$$\hat{\varphi}^i_\delta(x) = \sup_{B^i_\delta(x)} \left(\varphi_{4.2} + (1-100\epsilon_{4.1})\varphi^i_{\omega_0}\right)$$

and $\nu^i(x,\delta)$ by

$$\nu^i(x,\delta) = \frac{\hat{\varphi}^i_{\frac{r}{16}}(x) - \hat{\varphi}^i_\delta(x)}{\log\left(\frac{r}{16}\right) - \log\delta}.$$

Then $\nu^i(x,\delta)$ is monotonically non-decreasing in δ. Recall that the Lelong number is defined by

$$\nu^i(x) = \lim_{\delta\to 0} \nu^i(x,\delta).$$

It is independent of i and can instead be denoted as $\nu(x)$. Recall the definition of ρ in Definition 3.1. Let

$$\epsilon_{4.5} = \frac{\epsilon_{4.1} r^2}{5\left(\int_0^1 \log\left(\frac{1}{t}\right) \mathrm{Vol}(\partial B_1(0)) t^{2n-1}\rho(t)dt + \log 2 + \frac{3^{2n-1}}{2^{2n-3}}\log 2\right)}.$$

Then by the result of Siu [37], the set $Y = \{x : \nu(x) \geq \epsilon_{4.5}\}$ is an analytic subvariety.

For simplicity, we assume that Y is smooth. The singular case will be addressed at the end of this section.

Since Y is smooth by our assumption, as in the outline of the proof in Sect. 1, there exists a smooth function $\varphi_{1.12}$ in a neighborhood O of Y such that

$$\left(c - \frac{n-p}{2}\epsilon_{1.1}\right)\omega_{1.12}^{n-1} - (n-1)\chi \wedge \omega_{1.12}^{n-2} > 0$$

on O. Now we pick smaller neighborhoods O' and O'' such that $\overline{O'} \subset O$ and $\overline{O''} \subset O'$. We need to prove the following proposition:

Proposition 4.1 *(1) For small enough $\delta < \frac{r}{20}$, if*

$$\max_{\left\{i : x \in \overline{B_{\frac{9}{5}r}(x_i)}\right\}} \nu^i(x, \delta) \leq 2\epsilon_{4.5},$$

then

$$\sup_{\overline{O'}} \varphi_{1.12} + 3\epsilon_{4.5}\log\delta + \epsilon_{4.1} r^2 < \max_{\left\{i : x \in \overline{B_{\frac{9}{5}r}(x_i)}\right\}} (\varphi_\delta^i(x) - \varphi_{\omega_0}^i(x)).$$

(2) For small enough $\delta < \frac{r}{20}$, if

$$\inf_{\overline{O'}} \varphi_{1.12} + 3\epsilon_{4.5}\log\delta - \epsilon_{4.1} r^2 \leq \max_{\left\{i : x \in \overline{B_{\frac{9}{5}r}(x_i)}\right\}} \left(\varphi_\delta^i(x) - \varphi_{\omega_0}^i(x)\right),$$

then

$$\max_{\left\{i : x \in \overline{B_{\frac{9}{5}r}(x_i)}\right\}} \nu^i(x, \delta) < 4\epsilon_{4.5}.$$

(3) For small enough $\delta < \frac{r}{20}$, *if*

$$\max_{\left\{i: x \in \overline{B_{\frac{9}{5}r}(x_i)}\right\}} \nu^i(x,\delta) \leq 4\epsilon_{4.5}.$$

then

$$\max_{\left\{i: x \in \overline{B_{\frac{9}{5}r}(x_i)} \setminus B_{\frac{8}{5}r}(x_i)\right\}} \left(\varphi^i_\delta(x) - \varphi^i_{\omega_0}(x)\right) + \epsilon_{4.1} r^2 < \max_{\left\{i: x \in \overline{B_r(x_i)}\right\}} \left(\varphi^i_\delta(x) - \varphi^i_{\omega_0}(x)\right).$$

If Proposition 4.1 is true, for small enough δ, we can define $\varphi_{1.13}$ as the regularized maximum of $\varphi_{1.12}(x) + 3\epsilon_{4.5} \log \delta$ on $\overline{O'}$ and $\varphi^i_\delta - \varphi^i_{\omega_0}$ on $\overline{B_{\frac{9}{5}r}(x_i)}$. Since $\nu(x) < \epsilon_{4.5}$ for $x \notin Y$, for small enough δ, $\max_{\{i: x \in \overline{B_{\frac{9}{5}r}(x_i)}\}} \nu^i(x,\delta) \leq 2\epsilon_{4.5}$ for all $x \notin O''$. So by Proposition 4.1 (1), we do not need to worry about the discontinuity near the boundary of $\overline{O'}$. By Proposition 4.1 (2) and (3), there is also no need to worry about the discontinuity near the boundary of $\overline{B_{\frac{9}{5}r}(x_i)}$. In conclusion, $\varphi_{1.13}$ will be smooth and satisfy

$$c\omega^{n-1}_{1.13} - (n-1)\chi \wedge \omega^{n-2}_{1.13} > 0$$

on M as long as Y is smooth and Proposition 4.1 is true.

In order to prove Proposition 4.1, we need the following lemma of Błocki and Kołodziej [3].

Lemma 4.2 *For any* $\delta < \frac{r}{20}$ *and* $x \in \overline{B_{\frac{9}{5}r}(x_i)}$, *the following estimates hold:*

(1) $0 \leq \hat{\varphi}^i_\delta - \hat{\varphi}^i_{\frac{\delta}{a}} \leq \nu^i(x,\delta) \log a$ *for all* $a \geq 1$, *and*

(2) $0 \leq \hat{\varphi}^i_\delta - \varphi^i_\delta \leq \nu^i(x,\delta) \left(\int_0^1 \log\left(\frac{1}{t}\right) \mathrm{Vol}\left(\partial B_1(0)\right) t^{2n-1} \rho(t) dt + \frac{3^{2n-1}}{2^{2n-3}} \log 2\right)$.

Proof For readers' convenience, we almost line by line copy the paper [3] here:

(1) The estimate $0 \leq \hat{\varphi}^i_\delta - \hat{\varphi}^i_{\frac{\delta}{a}} \leq \nu^i(x,\delta) \log a$ follows from the logarithmical convexity of $\hat{\varphi}^i_\delta$ and the definition of $\nu^i(x,\delta)$.

(2) Define another regularization $\tilde{\varphi}^i_\delta$ by

$$\tilde{\varphi}^i_\delta(x) = \frac{1}{\mathrm{Vol}(\partial B_\delta(x))} \int_{\partial B_\delta(x)} \left(\varphi_{4.2} + (1 - 100\epsilon_{4.1})\varphi^i_{\omega_0}\right) d\mathrm{Vol}.$$

Then by the Poisson kernel for subharmonic functions [3] and the estimate in (1),

$$\hat{\varphi}^i_{t\delta}(x) - \tilde{\varphi}^i_{t\delta}(x) \leq \frac{3^{2n-1}}{2^{2n-2}} \left(\hat{\varphi}^i_{t\delta} - \hat{\varphi}^i_{t\delta/2} \right) \leq \left(\frac{3^{2n-1}}{2^{2n-2}} \log 2 \right) \nu^i(x, t\delta)$$

for all $t \in (0, 1]$. By monotonicity,

$$\hat{\varphi}^i_{t\delta}(x) - \tilde{\varphi}^i_{t\delta}(x) \leq \left(\frac{3^{2n-1}}{2^{2n-2}} \log 2 \right) \nu^i(x, t\delta) \leq \left(\frac{3^{2n-1}}{2^{2n-2}} \log 2 \right) \nu^i(x, \delta).$$

If we define

$$\tilde{\rho}(t) = \mathrm{Vol}(\partial B_1(0)) t^{2n-1} \rho(t),$$

then $\int_0^1 \tilde{\rho}(t) = 1$. So

$$\tilde{\varphi}^i_\delta - \varphi^i_\delta = \int_0^1 \left(\tilde{\varphi}^i_\delta - \tilde{\varphi}^i_{t\delta} \right) \tilde{\rho}(t) dt \leq \int_0^1 \left(\hat{\varphi}^i_\delta - \hat{\varphi}^i_{t\delta} \right) \tilde{\rho}(t) dt + \left(\frac{3^{2n-1}}{2^{2n-2}} \log 2 \right) \nu^i(x, \delta).$$

By the estimate in (1) again,

$$\hat{\varphi}^i_\delta - \hat{\varphi}^i_{t\delta} \leq \nu^i(x, \delta) \log \left(\frac{1}{t} \right).$$

The other side of inequality $0 \leq \hat{\varphi}^i_\delta - \varphi^i_\delta$ is trivial. □

It is easy to see that there exists a constant $C_{4.6}$ such that for any $\delta < \frac{r}{20}$ and $x \in \overline{B_{\frac{9}{5}r}(x_i)}$, $\nu^i(x, \delta) < C_{4.6}$. Now we are ready to prove Proposition 4.1.

(1) Suppose that $\delta < \frac{r}{20}$, $x \in \overline{B_{\frac{9}{5}r}(x_i)}$, and

$$\nu^i(x, \delta) = \frac{\hat{\varphi}^i_{\frac{r}{16}}(x) - \hat{\varphi}^i_\delta(x)}{\log \left(\frac{r}{16} \right) - \log \delta} \leq 2\epsilon_{4.5}.$$

Then

$$\hat{\varphi}^i_\delta(x) \geq \hat{\varphi}^i_{\frac{r}{16}}(x) + 2\epsilon_{4.5} \left(\log \delta - \log \left(\frac{r}{16} \right) \right) \geq -C_{4.7} + 2\epsilon_{4.5} \log \delta.$$

By Lemma 4.2 (2),

$$\varphi^i_\delta(x) \geq -C_{4.8} + 2\epsilon_{4.5} \log \delta.$$

It is easy to see that for δ small enough,

$$\sup_{\overline{O'}} \varphi_{1.12} + 3\epsilon_{4.5}\log\delta + \epsilon_{4.1}r^2 < \varphi^i_\delta(x) - \varphi^i_{\omega_0}(x)$$

because $\varphi^i_{\omega_0}$ is uniformly bounded on $\overline{B_{\frac{9}{5}r}(x_i)}$.

(2) Suppose that $\delta < \frac{r}{20}$, $x \in \overline{B_{\frac{9}{5}r}(x_i)}$, and

$$\inf_{\overline{O'}} \varphi_{1.12} + 3\epsilon_{4.5}\log\delta - \epsilon_{4.1}r^2 \leq \varphi^i_\delta(x) - \varphi^i_{\omega_0}(x).$$

Then as before

$$\hat{\varphi}^i_\delta(x) \geq \varphi^i_\delta(x) \geq -C_{4.9} + 3\epsilon_{4.5}\log\delta.$$

By Lemma 4.2 (1) and the definition of $\hat{\varphi}^i_{\frac{\delta}{2}}(x)$,

$$\sup_{B^i_{\frac{\delta}{2}}(x)} \varphi_{4.2} \geq -C_{4.10} + 3\epsilon_{4.5}\log\delta.$$

If $x \in \overline{B_{\frac{9}{5}r}(x_j)}$, then $B^i_{\frac{\delta}{2}}(x) \subset B^j_\delta(x)$ and therefore

$$\sup_{B^j_\delta(x)} \varphi_{4.2} \geq \sup_{B^i_{\frac{\delta}{2}}(x)} \varphi_{4.2} \geq -C_{4.10} + 3\epsilon_{4.5}\log\delta.$$

By the definition of $\hat{\varphi}^j_\delta(x)$ and $\nu^j(x,\delta)$, it is easy to see that $\nu^j(x,\delta) < 4\epsilon_{4.5}$ if δ is small enough.

(3) Suppose that $\delta < \frac{r}{20}$, $x \in (\overline{B_{\frac{9}{5}r}(x_i)} \setminus B_{\frac{8}{5}r}(x_i)) \cap \overline{B_r(x_j)}$, and

$$\max_{\{i : x \in \overline{B_{\frac{9}{5}r}(x_i)}\}} \nu^i(x,\delta) \leq 4\epsilon_{4.5}.$$

Then

$$\hat{\varphi}^i_{\frac{\delta}{2}}(x) - \varphi^i_{\omega_0}(x) \leq \sup_{B^i_{\frac{\delta}{2}}(x)} \varphi_{4.2} + 2\epsilon_{4.1}r^2 + (2r+\delta)^2 - (2r)^2 - 100\epsilon_{4.1}\left(\frac{7}{5}r\right)^2,$$

and

$$\sup_{B^j_\delta(x)} \varphi_{4.2} \leq \hat{\varphi}^j_\delta(x) - \varphi^j_{\omega_0}(x) + 2\epsilon_{4.1}r^2 + (2r+\delta)^2 - (2r)^2 + 100\epsilon_{4.1}\left(\frac{6}{5}r\right)^2.$$

By Lemma 4.2 (1),

$$\hat{\varphi}^i_\delta - \hat{\varphi}^i_{\frac{\delta}{2}} \leq \nu^i(x,\delta)\log 2 \leq 4\epsilon_{4.5}\log 2.$$

By Lemma 4.2 (2), $\varphi^i_\delta \leq \hat{\varphi}^i_\delta$, and

$$\hat{\varphi}^j_\delta - \varphi^j_\delta \leq 4\epsilon_{4.5}\left(\int_0^1 \log\left(\frac{1}{t}\right)\mathrm{Vol}\left(\partial B_1(0)\right)t^{2n-1}\rho(t)dt + \frac{3^{2n-1}}{2^{2n-3}}\log 2\right).$$

Since $\sup_{B^i_{\frac{\delta}{2}}(x)}\varphi_{4.2} \leq \sup_{B^j_\delta(x)}\varphi_{4.2}$, by summing everything together, for δ small enough, $\varphi^i_\delta(x) - \varphi^i_{\omega_0}(x) + \epsilon_{4.1}r^2 < \varphi^j_\delta(x) - \varphi^j_{\omega_0}(x)$. We are done if Y is smooth.

In general, Y is singular. By Hironaka's desingularization theorem, there exists a blow-up $\tilde{M}$ of M obtained by a sequence of blow-ups with smooth centers such that the proper transform $\tilde{Y}$ of Y is smooth. Without loss of generality, assume that we only need to blow up once. Let π be the projection of $\tilde{M}$ on M. Let E be the exceptional divisor. Let s be the defining section of E. Let h be any smooth metric on the line bundle $[E]$, so $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|s|^2_h = [E]+\omega_{4.11}$ by the Poincaré-Lelong equation. Then it is well known that the smooth (1,1)-form $\omega_{4.11} \in -[E]$ on $\tilde{M}$ and $\omega_{4.11} > -C_{4.12}\pi^*\omega_0$. For example, see Lemma 3.5 of [20] for the explanation. Define $\omega_{4.13} = C_{4.12}\pi^*\omega_0 + \omega_{4.11}$. Then $\omega_{4.13}$ is a Kähler form on $\tilde{M}$.

Lemma 4.3 *Let $C_{4.14} = \frac{6n}{\epsilon_{1.1}}$. Then for all small enough t and q-dimensional analytic subvarieties V of $\tilde{M}$, as long as $q < n$,*

$$\int_V \left(c - \frac{n-q}{3n}\epsilon_{1.1}\right)\left((1+C_{4.14}t)\pi^*\omega_0 + C_{4.14}t^2\omega_{4.13}\right)^q$$
$$\geq \int_V q\left((1+C_{4.14}t)\pi^*\omega_0 + C_{4.14}t^2\omega_{4.13}\right)^{q-1}\wedge\left(\pi^*\chi + t^2\omega_{4.13}\right).$$

Proof By assumption,

$$\int_V \left(c - \frac{\epsilon_{1.1}}{3}\right)\pi^*\omega_0^q - q\pi^*\omega_0^{q-1}\wedge\pi^*\chi = \int_{\pi(V)}\left(c - \frac{\epsilon_{1.1}}{3}\right)\omega_0^q - q\omega_0^{q-1}\wedge\chi \geq 0.$$

So

$$\int_V \left(c - \frac{\epsilon_{1.1}}{3}\right)\left((1+C_{4.14}t)\pi^*\omega_0\right)^q - q\left((1+C_{4.14}t)\pi^*\omega_0\right)^{q-1}\wedge\left((1+C_{4.14}t)\pi^*\chi\right) \geq 0.$$

It suffices to show that

$$\int_V \left(c - \frac{\epsilon_{1.1}}{3}\right) \left((1 + C_{4.14}t)\pi^*\omega_0 + C_{4.14}t^2\omega_{4.13}\right)^q$$
$$- q\left((1 + C_{4.14}t)\pi^*\omega_0 + C_{4.14}t^2\omega_{4.13}\right)^{q-1} \wedge \left(\pi^*\chi + t^2\omega_{4.13}\right)$$
$$\geq \int_V \left(c - \frac{\epsilon_{1.1}}{3}\right) \left((1 + C_{4.14}t)\pi^*\omega_0\right)^q$$
$$- q\left((1 + C_{4.14}t)\pi^*\omega_0\right)^{q-1} \wedge \left((1 + C_{4.14}t)\pi^*\chi\right).$$

Since it depends only on the cohomology classes, we want to replace ω_0 with a better representative in its cohomology class. We remark that $\pi(E)$ is smooth by assumption. So we can apply Theorem 1.11 to $\pi(E)$. As in Sect. 1, there exists a smooth function $\varphi_{4.15}$ on a neighborhood $O_{4.16}$ of $\pi(E)$ in M such that $\omega_{4.15} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_{4.15}$ satisfies

$$\left(c - \frac{\epsilon_{1.1}}{2}\right)\omega_{4.15}^{n-1} - (n-1)\chi \wedge \omega_{4.15}^{n-2} > 0$$

on $O_{4.16}$. Define $\varphi_{4.17} = \pi_* \frac{\log|s|_h^2}{4\pi C_{4.12}}$ on $M \setminus \pi(E)$. Recall the definition of the regularized maximum in Lemma I.5.18 of [16]. For large enough $C_{4.18}$, let $\varphi_{4.19}$ be the regularized maximum of $\varphi_{4.17} + C_{4.18}$ and $\varphi_{4.15}$. Then $\varphi_{4.19}$ is smooth on M and $\omega_{4.19} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_{4.19} > 0$ on M. Moreover, there exists a smaller neighborhood $O_{4.20}$ of $\pi(E)$ such that $\varphi_{4.19} = \varphi_{4.15}$ on $O_{4.20} \subset O_{4.16}$.

After replacing ω_0 by $\omega_{4.19}$, it suffices to show that

$$\left(c - \frac{\epsilon_{1.1}}{3}\right)\sum_{i=1}^{q} \frac{q!}{i!(q-i)!}\left((1 + C_{4.14}t)\pi^*\omega_{4.19}\right)^{q-i} \wedge \left(C_{4.14}t^2\omega_{4.13}\right)^i$$
$$- q\sum_{i=1}^{q} \frac{(q-1)!}{(q-i)!(i-1)!}\left((1 + C_{4.14}t)\pi^*\omega_{4.19}\right)^{q-i} \wedge C_{4.14}^{i-1}(t^2\omega_{4.13})^i$$
$$- q\sum_{i=1}^{q-1} \frac{(q-1)!}{i!(q-1-i)!}\left((1 + C_{4.14}t)\pi^*\omega_{4.19}\right)^{q-1-i}$$
$$\wedge \left(C_{4.14}t^2\omega_{4.13}\right)^i \wedge (1 + C_{4.14}t)\pi^*\chi$$
$$+ q\left((1 + C_{4.14}t)\pi^*\omega_{4.19} + C_{4.14}t^2\omega_{4.13}\right)^{q-1} \wedge C_{4.14}t\pi^*\chi$$
$$\geq 0.$$

By definition of $C_{4.14}$,

$$q\frac{(q-1)!}{(q-i)!(i-1)!} < \frac{\epsilon_{1.1}}{6}\frac{q!}{i!(q-i)!}C_{4.14}$$

for all $i = 1, 2, ..., q$. So we can combine the first term and the second term. If the point is inside $\pi^{-1}(O_{4.20})$, then for all $i = 1, 2, ..., q-1$,

$$\left(c - \frac{\epsilon_{1.1}}{2}\right)\left(\pi^*\omega_{4.19}\right)^{q-i} \geq (q-i)\left(\pi^*\omega_{4.19}\right)^{q-1-i} \wedge \pi^*\chi$$

because

$$\left(c - \frac{\epsilon_{1.1}}{2}\right)\omega_{4.19}^{n-1} \geq (n-1)\omega_{4.19}^{n-2} \wedge \chi$$

on $O_{4.20}$. So the sum of the first three terms is non-negative if $i = 1, 2, ..., q-1$. Therefore we are done because the $i = q$ term and the fourth term are non-negative. If the point is outside $\pi^{-1}(O_{4.20})$, then there exists a constant $C_{4.21}$ such that

$$C_{4.21}\pi^*\chi > \omega_{4.13} > C_{4.21}^{-1}\pi^*\chi$$

and

$$C_{4.21}\pi^*\chi > \pi^*\omega_{4.19} > C_{4.21}^{-1}\pi^*\chi$$

on $\overline{\tilde{M} \setminus \pi^{-1}(O_{4.20})}$. The only first-order term in t is

$$q\pi^*\omega_{4.19}^{q-1} \wedge C_{4.14}t\pi^*\chi.$$

Since it is positive, for small enough t, we also get the required inequality. □

Now we pick $t > 0$ such that t satisfies Lemma 4.3 and

$$\frac{c}{1 + C_{4.14}t + C_{4.12}C_{4.14}t^2} > \max\left\{c - \frac{\epsilon_{1.1}}{4n}, \frac{c}{1+\epsilon_{4.1}}\right\}.$$

We apply Theorem 1.11 to the lower-dimensional smooth manifold $\tilde{Y}$ with the Kähler forms $(1 + C_{4.14}t)\pi^*\omega_0 + C_{4.14}t^2\omega_{4.13}$ and $\pi^*\chi + t^2\omega_{4.13}$. As in Sect. 1, there exists a smooth function $\varphi_{4.22}$ on a neighborhood of $\tilde{Y}$ in $\tilde{M}$ such that

$$\omega_{4.22} = (1 + C_{4.14}t)\pi^*\omega_0 + C_{4.14}t^2\omega_{4.13} + \sqrt{-1}\partial\bar{\partial}\varphi_{4.22}$$

satisfies

$$\left(c-\frac{\epsilon_{1.1}}{4n}\right)\omega_{4.22}^{n-1}-(n-1)\left(\pi^*\chi+t^2\omega_{4.13}\right)\wedge\omega_{4.22}^{n-2}>0$$

near $\tilde{Y}$. Similarly, let $\varphi_{4.23}$ be the potential function near E. Then for a large enough constant $C_{4.24}$, we define $\varphi_{4.25}$ as the regularized maximum of $\varphi_{4.23}$ and $\varphi_{4.22}+C_{4.24}^{-1}\pi^*\varphi_{4.17}+C_{4.24}$ and define $\omega_{4.25}$ by

$$\omega_{4.25}=(1+C_{4.14}t)\pi^*\omega_0+C_{4.14}t^2\omega_{4.13}+\sqrt{-1}\partial\bar{\partial}\varphi_{4.25}.$$

Then

$$\left(c-\frac{\epsilon_{1.1}}{4n}\right)\omega_{4.25}^{n-1}-(n-1)\left(\pi^*\chi+t^2\omega_{4.13}\right)\wedge\omega_{4.25}^{n-2}>0$$

on a neighborhood O of $\tilde{Y}\cup E$ in $\tilde{M}$. Since $t^2\omega_{4.13}>0$, it is easy to see that

$$\left(c-\frac{\epsilon_{1.1}}{4n}\right)(\pi_*\omega_{4.25})^{n-1}-(n-1)\chi\wedge(\pi_*\omega_{4.25})^{n-2}>0$$

on $\pi(O\setminus E)$. Now we choose neighborhoods O' and O'' of $Y\cup\pi(E)$ in M such that $\overline{O'}\subset\pi(O)$ and $\overline{O''}\subset O'$. Then as before, for small enough δ, we can define $\varphi_{4.26}$ as the regularized maximum of $\pi_*\varphi_{4.25}+3\epsilon_{4.5}\log\delta$ on $\overline{O'}\setminus\pi(E)$ and $\varphi_\delta^i-\varphi_{\omega_0}^i$ on $\overline{B_{\frac{9}{5}r}(x_i)}$. Then $\varphi_{4.26}$ is smooth and bounded on $M\setminus\pi(E)$. Moreover, for

$$\begin{aligned}\omega_{4.26}&=(1+C_{4.14}t)\omega_0+C_{4.14}t^2\pi_*\omega_{4.13}+\sqrt{-1}\partial\bar{\partial}\varphi_{4.26}\\&=(1+C_{4.14}t+C_{4.12}C_{4.14}t^2)\omega_0+C_{4.14}t^2\pi_*\omega_{4.11}+\sqrt{-1}\partial\bar{\partial}\varphi_{4.26},\end{aligned}$$

it is easy to see that

$$\left(\max\left\{c-\frac{\epsilon_{1.1}}{4n},\frac{c}{1+\epsilon_{4.1}}\right\}\right)\omega_{4.26}^{n-1}-(n-1)\chi\wedge\omega_{4.26}^{n-2}>0$$

on $M\setminus\pi(E)$ because $C_{4.14}t\omega_0+C_{4.14}t^2\pi_*\omega_{4.13}>0$. Now we define

$$\omega_{4.27}=\frac{\omega_{4.26}}{1+C_{4.14}t+C_{4.12}C_{4.14}t^2}=\omega_0+\frac{C_{4.14}t^2\pi_*\omega_{4.11}+\sqrt{-1}\partial\bar{\partial}\varphi_{4.26}}{1+C_{4.14}t+C_{4.12}C_{4.14}t^2},$$

so by the choice of t,

$$c\omega_{4.27}^{n-1}-(n-1)\chi\wedge\omega_{4.27}^{n-2}>0$$

on $M \setminus \pi(E)$. For a large enough constant $C_{4.28}$, define $\varphi_{4.29}$ as the regularized maximum of

$$\frac{\frac{C_{4.14}t^2}{2\pi}\pi_* \log |s|_h^2 + \varphi_{4.26}}{1 + C_{4.14}t + C_{4.12}C_{4.14}t^2} + C_{4.28}$$

and $\varphi_{4.15}$. Then $\varphi_{4.29}$ is smooth on M, and $\omega_{4.29} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_{4.29}$ satisfies

$$c\omega_{4.29}^{n-1} - (n-1)\chi \wedge \omega_{4.29}^{n-2} > 0$$

on M. We are done.

5 Deformed Hermitian–Yang–Mills equation

In this section, we prove Theorem 1.7. As in the J-equation case, for simplicity, we define the following notations:

Definition 5.1 Define $P : \mathbb{R}^n \to (0, (n-1)\pi)$ and $Q : \mathbb{R}^n \to (0, n\pi)$ by

$$P(\lambda_1, ...\lambda_n) = \max_{i=1}^{n} \left(\sum_{k \neq i} \operatorname{arccot}(\lambda_k) \right)$$

and

$$Q(\lambda_1, ...\lambda_n) = \sum_{k=1}^{n} \operatorname{arccot}(\lambda_k).$$

Let $0 < \theta_0 < \Theta_0 < \pi$ be any constants. Define $\Gamma_{\theta_0,\Theta_0}$ to be the subset of $\mathbb{R}^n$ such that P is smaller than θ_0 and Q is smaller than Θ_0. Its closure is denoted by $\bar{\Gamma}_{\theta_0,\Theta_0}$.

Let A, B be Hermitian matrices. Assume that A is positive definite. Then $P_A(B)$ is defined as $P(\lambda_1, ...\lambda_n)$, where λ_i are the eigenvalues of the matrix $A^{-1}B$. The function $Q_A(B)$ is defined as $Q(\lambda_1, ...\lambda_n)$. The set $\Gamma_{A,\theta_0,\Theta_0}$ is defined as the set of all matrices B such that $(\lambda_1, ..., \lambda_n) \in \Gamma_{\theta_0,\Theta_0}$. Its closure is denoted by $\bar{\Gamma}_{A,\theta_0,\Theta_0}$.

If (1) of Theorem 1.7 holds, then for any smooth test family $\omega_{t,0}$, we can define another family $\omega_{t,\varphi} = \omega_{t,0} + \sqrt{-1}\partial\bar{\partial}\varphi$. Then $P_\chi(\omega_{t,\varphi}) < P_\chi(\omega_\varphi) \leq \theta_0$. So by Lemma 8.2 of [12],

$$\frac{d}{dt}\int_V \left(\mathrm{Re}\left(\omega_{t,\varphi}+\sqrt{-1}\chi\right)^p-\cot(\theta_0)\mathrm{Im}\left(\omega_{t,\varphi}+\sqrt{-1}\chi\right)^p\right)$$
$$=\int_V p\left(\mathrm{Re}\left(\omega_{t,\varphi}+\sqrt{-1}\chi\right)^{p-1}-\cot(\theta_0)\mathrm{Im}\left(\omega_{t,\varphi}+\sqrt{-1}\chi\right)^{p-1}\right)\wedge\frac{d}{dt}\omega_{t,\varphi}\geq 0.$$

By Lemma 8.2 of [12], we also know that there exists a constant $\epsilon_{1.1}>0$ such that for any point $x\in M$ and any p-dimensional vector space $V_x\subset T_xM$, the restriction of the form

$$\mathrm{Re}\left(\omega_{t,0}+\sqrt{-1}\chi\right)^p-\cot(\theta_0)\mathrm{Im}\left(\omega_{t,0}+\sqrt{-1}\chi\right)^p-(n-p)\epsilon_{1.1}\chi^p$$

on V_x is positive. Then we get (2) of Theorem 1.7 using the fact that

$$\int_V \left(\mathrm{Re}\left(\omega_{t,0}+\sqrt{-1}\chi\right)^p-\cot(\theta_0)\mathrm{Im}\left(\omega_{t,0}+\sqrt{-1}\chi\right)^p\right)$$
$$=\int_V \left(\mathrm{Re}\left(\omega_{t,\varphi}+\sqrt{-1}\chi\right)^p-\cot(\theta_0)\mathrm{Im}\left(\omega_{t,\varphi}+\sqrt{-1}\chi\right)^p\right).$$

It is trivial that (2) of Theorem 1.7 implies (3) of Theorem 1.7. On the other hand, as long as the following proposition holds, then (3) of Theorem 1.7 implies (1) of Theorem 1.7 by choosing the function f as 0 and choosing an arbitrary $\Theta_0\in(\theta_0,\pi)$.

Proposition 5.2 *Fix a Kähler manifold M^n with a Kähler metric χ and a real smooth closed (1,1)-form ω_0. Let $\theta_0\in(0,\pi)$ be a constant, and let $\Theta_0\in(\theta_0,\pi)$ be another constant. Then there exists a constant $\epsilon_{5.1}>0$ depending only on n,θ_0,Θ_0 such that the following statement holds.*

Assume the following: (1) When $n\geq 4$, $f>-\epsilon_{5.1}$ is a smooth function satisfying

$$\int_M f\chi^n=\int_M \left(\mathrm{Re}\left(\omega_\varphi+\sqrt{-1}\chi\right)^n-\cot(\theta_0)\mathrm{Im}\left(\omega_\varphi+\sqrt{-1}\chi\right)^n\right)\geq 0.$$

(2) When $n=1,2,3$, $f\geq 0$ is a constant satisfying

$$\int_M f\chi^n=\int_M \left(\mathrm{Re}\left(\omega_\varphi+\sqrt{-1}\chi\right)^n-\cot(\theta_0)\mathrm{Im}\left(\omega_\varphi+\sqrt{-1}\chi\right)^n\right)\geq 0.$$

(3) There exists a test family $\omega_{t,0}$ and a constant $\epsilon_{1.1}>0$ independent of t,V such that for any $t\geq 0$ and any p-dimensional analytic subvariety V,

$$\int_V \left(\mathrm{Re}\left(\omega_{t,0}+\sqrt{-1}\chi\right)^p-\cot(\theta_0)\mathrm{Im}\left(\omega_{t,0}+\sqrt{-1}\chi\right)^p\right)\geq (n-p)\epsilon_{1.1}\int_V\chi^p.$$

Then there exists a smooth function φ satisfying

$$\mathrm{Re}\left(\omega_\varphi + \sqrt{-1}\chi\right)^n - \cot(\theta_0)\mathrm{Im}\left(\omega_\varphi + \sqrt{-1}\chi\right)^n - f\chi^n = 0,$$

where $\omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi \in \Gamma_{\chi,\theta_0,\Theta_0}$.

Remark 5.3 Proposition 5.2 is similar to Theorem 1.11 and the results in [33]. In fact, the equation

$$\mathrm{Re}\left(\omega_\varphi + \sqrt{-1}\chi\right)^n - \cot(\theta_0)\mathrm{Im}\left(\omega_\varphi + \sqrt{-1}\chi\right)^n - f\chi^n = 0$$

originates from [33].

Remark 5.4 The proof of Proposition 5.2 relies on Lemma 5.6, which requires f to be a constant when $n = 1, 2, 3$. However, sometimes we can prove Proposition 5.2 directly without using Lemma 5.6. In fact, when $n = 1$, it is trivial that Proposition 5.2 holds for non-constant f. When $n = 2$, Proposition 5.2 holds for non-constant f using the observation of Jacob and Yau in [28]. When $n = 3$, the methods used in [34] may be useful, but it is still open whether Proposition 5.2 holds for non-constant f. Nevertheless, this does not affect the proof of Theorem 1.7.

When $n = 1$, Proposition 5.2 is trivial. In higher dimensions, we will prove it by induction on the dimension n of M. As the first step, inspired by the work of Collins–Jacob–Yau [12], we state the following proposition in analogy with Theorem 1.14:

Proposition 5.5 *Fix a Kähler manifold M^n with a Kähler metric χ and a real smooth closed (1,1)-form ω_0. Let $\theta_0 \in (0,\pi)$ be a constant, and let $\Theta_0 \in (\theta_0,\pi)$ be another constant. Then there exists a constant $\epsilon_{5.1} > 0$ depending only on n, θ_0, Θ_0 such that the following statement holds.*

Assume the following: (1) When $n \geq 4$, $f > -\epsilon_{5.1}$ is a smooth function satisfying

$$\int_M f\chi^n = \int_M \left(\mathrm{Re}\left(\omega_\varphi + \sqrt{-1}\chi\right)^n - \cot(\theta_0)\mathrm{Im}\left(\omega_\varphi + \sqrt{-1}\chi\right)^n\right) \geq 0.$$

(2) When $n = 1, 2, 3$, $f \geq 0$ is a constant satisfying

$$\int_M f\chi^n = \int_M \left(\mathrm{Re}\left(\omega_\varphi + \sqrt{-1}\chi\right)^n - \cot(\theta_0)\mathrm{Im}\left(\omega_\varphi + \sqrt{-1}\chi\right)^n\right) \geq 0.$$

(3) $\omega_0 \in \Gamma_{\chi,\theta_0,\Theta_0}$.

Then there exists a smooth function φ *satisfying*

$$\mathrm{Re}\left(\omega_\varphi + \sqrt{-1}\chi\right)^n - \cot(\theta_0)\mathrm{Im}\left(\omega_\varphi + \sqrt{-1}\chi\right)^n - f\chi^n = 0,$$

where $\omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi \in \Gamma_{\chi,\theta_0,\Theta_0}$.

We will use the continuity method to prove Proposition 5.5. We first choose the path

$$\omega_{1,s} = s\left(\omega_0 - \cot(\theta_0)\chi\right) + \cot\left(\frac{\theta_0}{2n}\right)\chi$$

and $f_{1,s}$ as the constant satisfying

$$\int_M f_{1,s}\chi^n = \int_M \left(\mathrm{Re}\left(\omega_{1,s} + \sqrt{-1}\chi\right)^n - \cot(\theta_0)\mathrm{Im}\left(\omega_{1,s} + \sqrt{-1}\chi\right)^n\right)$$

for $s \in [0, 1]$. Then, since $\omega_{1,s} \geq \cot(\frac{\theta_0}{2n})\chi$, we see that $P_\chi(\omega_{1,s}) \leq Q_\chi(\omega_{1,s}) \leq \frac{\theta_0}{2}$ and $f_{1,s} \geq 0$. Let I_1 be the set of s such that there exists a smooth function φ_s satisfying $\omega_{1,\varphi_s,s} = \omega_{1,s} + \sqrt{-1}\partial\bar{\partial}\varphi_s \in \Gamma_{\chi,\theta_0,\Theta_0}$ and

$$\mathrm{Re}\left(\omega_{1,\varphi_s,s} + \sqrt{-1}\chi\right)^n - \cot(\theta_0)\mathrm{Im}\left(\omega_{1,\varphi_s,s} + \sqrt{-1}\chi\right)^n - f_{1,s}\chi^n = 0.$$

Then $0 \in I_1$. The openness of I_1 follows from the condition on the integral of $f_{1,s}$, the implicit function theorem, and the standard elliptic estimates. If we can prove the *a priori* estimate, then we can take a weak limit. The limit is still in the region $\Gamma_{\chi,\theta_0,\Theta_0}$ by Lemma 5.6 (5) below. So we get the closedness of I_1 and therefore prove that $1 \in I_1$ assuming the *a priori* estimate.

Then we choose the path

$$\omega_{2,s} = \omega_0 + s\chi$$

and $f_{2,s}$ as the constant satisfying

$$\int_M f_{2,s}\chi^n = \int_M \left(\mathrm{Re}\left(\omega_{2,s} + \sqrt{-1}\chi\right)^n - \cot(\theta_0)\mathrm{Im}\left(\omega_{2,s} + \sqrt{-1}\chi\right)^n\right)$$

for $s \in [0, \cot\left(\frac{\theta_0}{2n}\right) - \cot(\theta_0)]$. Then we see that

$$P_\chi(\omega_{2,s}) \leq P_\chi(\omega_0) < \theta_0$$

and

$$Q_\chi(\omega_{2,s}) \leq Q_\chi(\omega_0) < \Theta_0.$$

So

$$\frac{\partial f_{2,s}}{\partial s}\int_M \chi^n = \int_M n\left(\mathrm{Re}\left(\omega_{2,s}+\sqrt{-1}\chi\right)^{n-1} - \cot(\theta_0)\mathrm{Im}\left(\omega_{2,s}+\sqrt{-1}\chi\right)^{n-1}\right)\wedge\chi \geq 0$$

by Lemma 8.2 of [12]. So $f_{2,s} \geq 0$ because $f_{2,s}$ is non-negative when $s = 0$ by the assumption in Proposition 5.5. We remark that we move forward when proving $f_{2,s} \geq 0$ but move backward when solving $\omega_{2,\varphi_s,s} = \omega_{2,s} + \sqrt{-1}\partial\bar{\partial}\varphi_s \in \Gamma_{\chi,\theta_0,\Theta_0}$ satisfying the equation

$$\mathrm{Re}\left(\omega_{2,\varphi_s,s}+\sqrt{-1}\chi\right)^n - \cot(\theta_0)\mathrm{Im}\left(\omega_{2,\varphi_s,s}+\sqrt{-1}\chi\right)^n - f_{2,s}\chi^n = 0.$$

Finally, we fix ω_0 and choose the path

$$f_{3,s} = sf + (1-s)\frac{\int_M f\chi^n}{\int_M \chi^n}$$

for $s \in [0,1]$. We omit the arguments for the second path and the third path because they are similar to the first path.

Therefore, we only need to prove the *a priori* estimate along the paths. This will be achieved by Székelyhidi's estimates in [42]. In order to apply Székelyhidi's estimates in [42], we need the following lemma:

Lemma 5.6 *For any $0 < \theta_0 < \Theta_0 < \pi$, there exist a constant $\epsilon_{5.1} > 0$ depending only on n, θ_0, Θ_0 and constants $C_{5.2} > 0, \epsilon_{5.3} > 0$ depending only on n, Θ_0 such that the following holds:*

Assume that f is a parameter such that $f \geq 0$ when $n = 1, 2, 3$ and such that $f \geq -\epsilon_{5.1}$ when $n \geq 4$. Then the function $F : \bar{\Gamma}_{\theta_0,\Theta_0} \to \mathbb{R}$ defined by

$$F(\lambda_1, ...\lambda_n) = \frac{\mathrm{Re}\prod_{k=1}^n(\lambda_k+\sqrt{-1})}{\mathrm{Im}\prod_{k=1}^n(\lambda_k+\sqrt{-1})} - \frac{f}{\mathrm{Im}\prod_{k=1}^n(\lambda_k+\sqrt{-1})} - \cot(\theta_0)$$

satisfies the following properties:

(1) $\frac{1}{\operatorname{Im}\prod_{k=1}^{n}(\lambda_k+\sqrt{-1})} \leq \frac{1}{C_{5.2}}$.

(2) $|\frac{\partial}{\partial\lambda_i}\frac{1}{\operatorname{Im}\prod_{k=1}^{n}(\lambda_k+\sqrt{-1})}| \leq \frac{1}{\sqrt{C_{5.2}}}\sqrt{\frac{\prod_{k=1}^{n}(1+\lambda_k^2)}{(\operatorname{Im}\prod_{k=1}^{n}(\lambda_k+\sqrt{-1}))^3}}\frac{1}{\sqrt{1+\lambda_i^2}}$.

(3) $\frac{\partial F}{\partial\lambda_i} > 0$.

(4) When $n \geq 4$, *for any real numbers* u_i,

$$\sum_{i,j=1}^{n}\frac{\partial^2 F}{\partial\lambda_i\partial\lambda_j}u_i u_j \leq -\epsilon_{5.3}\frac{\prod_{k=1}^{n}(1+\lambda_k^2)}{\left(\operatorname{Im}\prod_{k=1}^{n}\left(\lambda_k+\sqrt{-1}\right)\right)^3}\sum_{i=1}^{n}\frac{u_i^2}{1+\lambda_i^2}.$$

When $n = 1, 2, 3$, *for any real numbers* u_i, $\sum_{i,j=1}^{n}\frac{\partial^2 F}{\partial\lambda_i\partial\lambda_j}u_i u_j \leq 0$.

(5) If $\lambda \in \bar{\Gamma}_{\theta_0,\Theta_0}$, *and* $F(\lambda) = 0$, *then* $\lambda \in \Gamma_{\theta_0,\Theta_0}$.

(6) For any $\lambda \in \Gamma_{\theta_0,\Theta_0}$, *the set*

$$\left\{\lambda' \in \Gamma_{\theta_0,\Theta_0} : F(\lambda') = 0, \lambda_i' \geq \lambda_i, \text{ for all } i = 1, 2, 3, ..., n\right\}$$

is bounded, where the bound depends on $n, \theta_0, \Theta_0, \lambda, |f|$.

(7) $\bar{\Gamma}_{\theta_0,\Theta_0}$ *is convex.*

(8) $\frac{\partial}{\partial\lambda_i}F(\lambda) \leq \frac{\partial}{\partial\lambda_j}F(\lambda)$ *if* $\lambda_i \geq \lambda_j$.

(9) For any positive definite Hermitian matrix A, the function $F_A : \bar{\Gamma}_{A,\theta_0,\Theta_0} \to \mathbb{R}^n$ *is concave, where* $F_A(B) = F(\lambda_1, ...\lambda_n)$ *if* λ_i *are the eigenvalues of* $A^{-1}B$.

(10) For any positive definite Hermitian matrix A, the set $\bar{\Gamma}_{A,\theta_0,\Theta_0}$ *is convex.*

Proof When $n = 1$, $F(\lambda_1) = \lambda_1 - f - \cot(\theta_0)$. So all the properties are trivial. So we assume that $n \geq 2$.

For simplicity, define $\theta_i = \operatorname{arccot}(\lambda_i)$. Then it is easy to see that

$$\sin(\theta_i) = \frac{1}{\sqrt{1+\lambda_i^2}}, \cos(\theta_i) = \frac{\lambda_i}{\sqrt{1+\lambda_i^2}},$$

$$\operatorname{Re}\prod_{k=1}^{n}(\lambda_k+\sqrt{-1}) = \cos\left(\sum_{k=1}^{n}\theta_k\right)\prod_{k=1}^{n}\sqrt{1+\lambda_k^2},$$

and

$$\operatorname{Im}\prod_{k=1}^{n}(\lambda_k+\sqrt{-1}) = \sin\left(\sum_{k=1}^{n}\theta_k\right)\prod_{k=1}^{n}\sqrt{1+\lambda_k^2}.$$

(1) First of all, there exists a constant $C_{5.4} > 0$ depending only on Θ_0 such that $\sin(x) \geq C_{5.4}$ as long as $\pi > \Theta_0 \geq x \geq \frac{\Theta_0}{2} > 0$. Moreover, it is easy to see that there exist constants $C_{5.5} > 0, C_{5.6} > 0$ depending only on Θ_0 such that $\sin(x) \geq C_{5.5}x$ for all $x \in (0, \Theta_0)$ and $\tan(x) \leq C_{5.6}x$ for all $x \in (0, \frac{\Theta_0}{2})$.

Now we study two cases. If $\sum_{k=1}^{n} \theta_k \geq \frac{\Theta_0}{2}$, then

$$\operatorname{Im} \prod_{k=1}^{n} \left(\lambda_k + \sqrt{-1}\right) = \sin\left(\sum_{k=1}^{n} \theta_k\right) \prod_{k=1}^{n} \sqrt{1+\lambda_k^2} \geq C_{5.4}.$$

If $\sum_{k=1}^{n} \theta_k \leq \frac{\Theta_0}{2}$, then $\lambda_i \geq \cot(\frac{\Theta_0}{2}) > 0$ for all $i = 1, 2, 3, ..., n$. So

$$\operatorname{Im} \prod_{k=1}^{n} \left(\lambda_k + \sqrt{-1}\right) \geq C_{5.5} \left(\sum_{k=1}^{n} \theta_k\right) \prod_{k=1}^{n} \lambda_k \geq \frac{C_{5.5}}{C_{5.6}} \left(\sum_{k=1}^{n} \frac{1}{\lambda_k}\right) \prod_{k=1}^{n} \lambda_k \geq \frac{nC_{5.5}}{C_{5.6}} \cot^{n-1}\left(\frac{\Theta_0}{2}\right).$$

So we can choose $C_{5.2}$ as $\min\{\frac{nC_{5.5}}{C_{5.6}} \cot^{n-1}(\frac{\Theta_0}{2}), C_{5.4}\}$.

(2)

$$\begin{aligned} &\left|\frac{\partial}{\partial\lambda_i} \frac{1}{\operatorname{Im} \prod_{k=1}^{n} \left(\lambda_k + \sqrt{-1}\right)}\right| = \frac{\operatorname{Im} \prod_{k\neq i} \left(\lambda_k + \sqrt{-1}\right)}{\left(\operatorname{Im} \prod_{k=1}^{n} \left(\lambda_k + \sqrt{-1}\right)\right)^2} \leq \frac{\sqrt{\prod_{k\neq i} (1+\lambda_k^2)}}{\left(\operatorname{Im} \prod_{k=1}^{n} \left(\lambda_k + \sqrt{-1}\right)\right)^2} \\ &= \frac{\sqrt{\prod_{k=1} (1+\lambda_k^2)}}{\left(\operatorname{Im} \prod_{k=1}^{n} \left(\lambda_k + \sqrt{-1}\right)\right)^2} \frac{1}{\sqrt{1+\lambda_i^2}} \leq \frac{1}{\sqrt{C_{5.2}}} \sqrt{\frac{\prod_{k=1}^{n} \left(1+\lambda_k^2\right)}{\left(\operatorname{Im} \prod_{k=1}^{n} \left(\lambda_k + \sqrt{-1}\right)\right)^3}} \frac{1}{\sqrt{1+\lambda_i^2}}. \end{aligned}$$

(3)

$$\begin{aligned} \frac{\partial F}{\partial\lambda_i} &= \frac{1}{\sin^2\left(\sum_{k=1}^{n} \theta_k\right)} \frac{1}{1+\lambda_i^2} + \frac{f \operatorname{Im} \prod_{k\neq i} (\lambda_k + \sqrt{-1})}{\left(\operatorname{Im} \prod_{k=1}^{n} \left(\lambda_k + \sqrt{-1}\right)\right)^2} \\ &= \frac{1}{\sin^2\left(\sum_{k=1}^{n} \theta_k\right)} \frac{1}{1+\lambda_i^2} + \frac{1}{\sin^2\left(\sum_{k=1}^{n} \theta_k\right)} \frac{1}{1+\lambda_i^2} \frac{f \sin\left(\sum_{k\neq i} \theta_k\right)}{\prod_{k\neq i} \sqrt{1+\lambda_k^2}}. \end{aligned}$$

Therefore, if $\epsilon_{5.1} < 1$, then $\frac{\partial F}{\partial\lambda_i} > 0$.

(4)

$$\begin{aligned} \frac{\partial^2 F}{\partial\lambda_i^2} &= \frac{2\cos\left(\sum_{k=1}^{n} \theta_k\right)}{\sin^3\left(\sum_{k=1}^{n} \theta_k\right)} \frac{1}{\left(1+\lambda_i^2\right)^2} + \frac{1}{\sin^2\left(\sum_{k=1}^{n} \theta_k\right)} \frac{-2\lambda_i}{\left(1+\lambda_i^2\right)^2} - 2\frac{f\left(\operatorname{Im} \prod_{k\neq i} \left(\lambda_k + \sqrt{-1}\right)\right)^2}{z\left(\operatorname{Im} \prod_{k=1}^{n} \left(\lambda_k + \sqrt{-1}\right)\right)^3} \\ &= \frac{2\cot\left(\sum_{k=1}^{n} \theta_k\right) - 2\lambda_i}{\sin^2\left(\sum_{k=1}^{n} \theta_k\right)\left(1+\lambda_i^2\right)^2} - \frac{f \prod_{k=1}^{n} (1+\lambda_k^2)}{(1+\lambda_i^2)\left(\operatorname{Im} \prod_{k=1}^{n} \left(\lambda_k + \sqrt{-1}\right)\right)^3} \cdot 2\sin^2\left(\sum_{k\neq i} \theta_k\right). \end{aligned}$$

When $i \neq j$, then

$$\frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j} = \frac{2\cot\left(\sum_{k=1}^n \theta_k\right)}{\sin^2\left(\sum_{k=1}^n \theta_k\right)} \frac{1}{\left(1+\lambda_i^2\right)\left(1+\lambda_j^2\right)}$$
$$+ \frac{f\left(\operatorname{Im}\prod_{k\neq i,j}\left(\lambda_k+\sqrt{-1}\right)\right)\left(\operatorname{Im}\prod_{k=1}^n\left(\lambda_k+\sqrt{-1}\right)\right)}{\left(\operatorname{Im}\prod_{k=1}^n\left(\lambda_k+\sqrt{-1}\right)\right)^3}$$
$$- \frac{2f\left(\operatorname{Im}\prod_{k\neq i}\left(\lambda_k+\sqrt{-1}\right)\right)\left(\operatorname{Im}\prod_{k\neq j}\left(\lambda_k+\sqrt{-1}\right)\right)}{\left(\operatorname{Im}\prod_{k=1}^n\left(\lambda_k+\sqrt{-1}\right)\right)^3}.$$

Using

$$\operatorname{Im}\prod_{k\neq j}\left(\lambda_k+\sqrt{-1}\right) = \lambda_i \operatorname{Im}\prod_{k\neq i,j}\left(\lambda_k+\sqrt{-1}\right) + \operatorname{Re}\prod_{k\neq i,j}\left(\lambda_k+\sqrt{-1}\right),$$

$$\operatorname{Im}\prod_{k\neq i}\left(\lambda_k+\sqrt{-1}\right) = \lambda_j \operatorname{Im}\prod_{k\neq i,j}\left(\lambda_k+\sqrt{-1}\right) + \operatorname{Re}\prod_{k\neq i,j}\left(\lambda_k+\sqrt{-1}\right),$$

and

$$\operatorname{Im}\prod_{k=1}^n\left(\lambda_k+\sqrt{-1}\right) = \left(\lambda_i\lambda_j - 1\right)\operatorname{Im}\prod_{k\neq i,j}\left(\lambda_k+\sqrt{-1}\right) + \left(\lambda_i+\lambda_j\right)\operatorname{Re}\prod_{k\neq i,j}\left(\lambda_k+\sqrt{-1}\right),$$

it is easy to see that

$$\left(\operatorname{Im}\prod_{k\neq i,j}\left(\lambda_k+\sqrt{-1}\right)\right)\left(\operatorname{Im}\prod_{k=1}^n\left(\lambda_k+\sqrt{-1}\right)\right) - \left(\operatorname{Im}\prod_{k\neq i}\left(\lambda_k+\sqrt{-1}\right)\right)\left(\operatorname{Im}\prod_{k\neq j}\left(\lambda_k+\sqrt{-1}\right)\right)$$
$$= -\left(\operatorname{Im}\prod_{k\neq i,j}\left(\lambda_k+\sqrt{-1}\right)\right)^2 - \left(\operatorname{Re}\prod_{k\neq i,j}\left(\lambda_k+\sqrt{-1}\right)\right)^2$$
$$= -\frac{\prod_{k=1}^n\left(1+\lambda_k^2\right)}{\left(1+\lambda_i^2\right)\left(1+\lambda_j^2\right)} = -\prod_{k=1}^n\left(1+\lambda_k^2\right)\frac{\sin(\theta_i)}{\sqrt{1+\lambda_i^2}}\frac{\sin(\theta_j)}{\sqrt{1+\lambda_j^2}}.$$

Thus,

$$\begin{aligned}\sum_{i,j=1}^{n}\frac{\partial^2 F}{\partial\lambda_i\partial\lambda_j}u_iu_j &= \frac{-2}{\sin^2\left(\sum_{k=1}^{n}\theta_k\right)}\left(-\cot\left(\sum_{k=1}^{n}\theta_k\right)\sum_{i,j=1}^{n}\frac{u_iu_j}{(1+\lambda_i^2)(1+\lambda_j^2)}+\sum_{i=1}^{n}\frac{\lambda_iu_i^2}{\left(1+\lambda_i^2\right)^2}\right)\\ &\quad-\frac{f\prod_{k=1}^{n}\left(1+\lambda_k^2\right)}{\left(\operatorname{Im}\prod_{k=1}^{n}\left(\lambda_k+\sqrt{-1}\right)\right)^3}\cdot\left(\sum_{i,j=1}^{n}\sin\left(\sum_{k\neq i}\theta_k\right)\sin\left(\sum_{k\neq j}\theta_k\right)\frac{u_i}{\sqrt{1+\lambda_i^2}}\frac{u_j}{\sqrt{1+\lambda_j^2}}\right.\\ &\quad\left.+\sum_{i=1}^{n}\sin^2\left(\sum_{k\neq i}\theta_k\right)\frac{u_i^2}{1+\lambda_i^2}+\sum_{i=1}^{n}\sum_{j\neq i}\sin(\theta_i)\sin(\theta_j)\frac{u_i}{\sqrt{1+\lambda_i^2}}\frac{u_j}{\sqrt{1+\lambda_j^2}}\right).\end{aligned}$$

Without loss of generality, assume that $\lambda_1 \geq \lambda_2 ... \geq \lambda_n$. When $n \geq 4$, we first claim that there exists a constant $\epsilon_{5.7}$ depending only on n, Θ_0 such that

$$\begin{aligned}&\frac{-2}{\sin^2\left(\sum_{k=1}^{n}\theta_k\right)}\left(-\cot\left(\sum_{k=1}^{n}\theta_k\right)\sum_{i,j=1}^{n}\frac{u_iu_j}{\left(1+\lambda_i^2\right)\left(1+\lambda_j^2\right)}+\sum_{i=1}^{n}\frac{\lambda_iu_i^2}{(1+\lambda_i^2)^2}\right)\\ &\leq -\epsilon_{5.7}\frac{\prod_{k=1}^{n}\left(1+\lambda_k^2\right)}{\left(\operatorname{Im}\prod_{k=1}^{n}\left(\lambda_k+\sqrt{-1}\right)\right)^3}\sum_{i=1}^{n}\frac{u_i^2}{1+\lambda_i^2}.\end{aligned}$$

This claim is equivalent to

$$-\cot\left(\sum_{k=1}^{n}\theta_k\right)\left(\sum_{i=1}^{n}\frac{u_i}{1+\lambda_i^2}\right)^2+\sum_{i=1}^{n}\frac{\lambda_iu_i^2}{\left(1+\lambda_i^2\right)^2}\geq\frac{\epsilon_{5.7}}{2\operatorname{Im}\prod_{k=1}^{n}\left(\lambda_k+\sqrt{-1}\right)}\sum_{i=1}^{n}\frac{u_i^2}{1+\lambda_i^2}.$$

We divide it into several cases.

In the first case, $\theta_1 \leq ... \leq \theta_n \leq \frac{\Theta_0}{2} < \frac{\pi}{2}$ and $\cot(\sum_{k=1}^{n}\theta_k) \leq \frac{1}{2n}\cot(\frac{\Theta_0}{2})$, then

$$-\cot\left(\sum_{k=1}^{n}\theta_k\right)\left(\sum_{i=1}^{n}\frac{u_i}{1+\lambda_i^2}\right)^2\geq-\frac{1}{2n}\cot\left(\frac{\Theta_0}{2}\right)\sum_{i=1}^{n}\frac{\lambda_iu_i^2}{(1+\lambda_i^2)^2}\sum_{i=1}^{n}\frac{1}{\lambda_i}\geq-\frac{1}{2}\sum_{i=1}^{n}\frac{\lambda_iu_i^2}{(1+\lambda_i^2)^2}.$$

Since $\sin(\sum_{k=1}^{n}\theta_k) \geq C_{5.4}$,

$$\frac{1}{2\operatorname{Im}\prod_{k=1}^{n}\left(\lambda_k+\sqrt{-1}\right)}\sum_{i=1}^{n}\frac{u_i^2}{1+\lambda_i^2}\leq\frac{1}{2C_{5.4}\prod_{k=1}^{n}\sqrt{1+\lambda_k^2}}\sum_{i=1}^{n}\frac{u_i^2}{1+\lambda_i^2}$$

$$\leq \frac{1}{2C_{5.4}} \sum_{i=1}^{n} \frac{\lambda_i u_i^2}{\lambda_i^2(1+\lambda_i^2)} \leq \frac{\cot^2\left(\frac{\Theta_0}{2}\right)+1}{2C_{5.4}\cot^2\left(\frac{\Theta_0}{2}\right)} \sum_{i=1}^{n} \frac{\lambda_i u_i^2}{\left(1+\lambda_i^2\right)^2}.$$

So we get the required estimate if $\epsilon_{5.7} < \frac{C_{5.4}\cot^2\left(\frac{\Theta_0}{2}\right)}{\cot^2\left(\frac{\Theta_0}{2}\right)+1}$.

In the second case, $\theta_1 \leq ... \leq \theta_n \leq \frac{\Theta_0}{2} < \frac{\pi}{2}$ and $\cot(\sum_{k=1}^{n}\theta_k) > \frac{1}{2n}\cot(\frac{\Theta_0}{2})$, then

$$\sum_{k=1}^{n}\theta_k < \operatorname{arccot}\left(\frac{1}{2n}\cot\left(\frac{\Theta_0}{2}\right)\right) < \frac{\pi}{2}.$$

So

$$-\cot\left(\sum_{k=1}^{n}\theta_k\right)\left(\sum_{i=1}^{n}\frac{u_i}{1+\lambda_i^2}\right)^2 \geq -\cot\left(\sum_{k=1}^{n}\theta_k\right)\sum_{i=1}^{n}\frac{\lambda_i u_i^2}{\left(1+\lambda_i^2\right)^2}\sum_{i=1}^{n}\frac{1}{\lambda_i}.$$

If $\alpha, \beta > 0$ and $\alpha+\beta < \frac{\pi}{2}$, then

$$\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)} > \tan(\alpha)+\tan(\beta).$$

So

$$\begin{aligned}
1 - \cot\left(\sum_{k=1}^{n}\theta_k\right)\sum_{i=1}^{n}\frac{1}{\lambda_i} &= 1 - \cot\left(\sum_{k=1}^{n}\theta_k\right)\sum_{i=1}^{n}\tan(\theta_i) \\
&> 1 - \cot\left(\sum_{k=1}^{n}\theta_k\right)\left(\tan(\theta_n) + \tan\left(\sum_{i=1}^{n-1}\theta_i\right)\right) = \tan(\theta_n)\tan\left(\sum_{i=1}^{n-1}\theta_i\right) \\
&\geq \tan(\theta_n)\sum_{i=1}^{n-1}\tan(\theta_i) = \frac{1}{\lambda_n}\sum_{i=1}^{n-1}\frac{1}{\lambda_i} \geq \frac{1}{\lambda_n\lambda_{n-1}}.
\end{aligned}$$

As in (1), we know that

$$\frac{\operatorname{Im}\prod_{k=1}^{n}\left(\lambda_k+\sqrt{-1}\right)}{\prod_{k=1}^{n}\sqrt{1+\lambda_k^2}} = \sin\left(\sum_{k=1}^{n}\theta_k\right) \geq C_{5.5}\sum_{k=1}^{n}\theta_k \geq \frac{C_{5.5}}{C_{5.6}}\sum_{k=1}^{n}\frac{1}{\lambda_k} \geq \frac{C_{5.5}}{C_{5.6}\lambda_n}.$$

$$\operatorname{Im}\prod_{k=1}^{n}\left(\lambda_k+\sqrt{-1}\right)\geq\frac{C_{5.5}}{C_{5.6}\lambda_n}\prod_{k=1}^{n}\sqrt{1+\lambda_k^2}\geq\frac{C_{5.5}}{C_{5.6}}\prod_{k=1}^{n-1}\sqrt{1+\lambda_k^2}$$
$$\geq\frac{C_{5.5}}{C_{5.6}}\lambda_1\lambda_{n-2}\lambda_{n-1}\geq\frac{C_{5.5}}{C_{5.6}}\lambda_1\lambda_{n-1}\lambda_n$$

using the assumption that $n\geq 4$.

Therefore,

$$\frac{1}{2\operatorname{Im}\prod_{k=1}^{n}\left(\lambda_k+\sqrt{-1}\right)}\sum_{i=1}^{n}\frac{u_i^2}{1+\lambda_i^2}\leq\frac{C_{5.6}}{2C_{5.5}}\frac{1}{\lambda_n\lambda_{n-1}}\sum_{i=1}^{n}\frac{\lambda_i u_i^2}{\lambda_i^2\left(1+\lambda_i^2\right)}$$
$$\leq\frac{C_{5.6}\left(\cot^2\left(\frac{\Theta_0}{2}\right)+1\right)}{2C_{5.5}\cot^2\left(\frac{\Theta_0}{2}\right)\lambda_n\lambda_{n-1}}\sum_{i=1}^{n}\frac{\lambda_i u_i^2}{\left(1+\lambda_i^2\right)^2}.$$

So we get the required estimate if $\frac{C_{5.6}\epsilon_{5.7}\left(\cot^2\left(\frac{\Theta_0}{2}\right)+1\right)}{2C_{5.5}\cot^2\left(\frac{\Theta_0}{2}\right)}\leq 1$.

In the third case, $\theta_n>\frac{\Theta_0}{2}$. So $\sum_{k=1}^{n-1}\theta_k<\frac{\Theta_0}{2}<\frac{\pi}{2}$. As in the second case,

$$-\cot\left(\sum_{k=1}^{n-1}\theta_k\right)\left(\sum_{i=1}^{n-1}\frac{u_i}{1+\lambda_i^2}\right)^2+\sum_{i=1}^{n-1}\frac{\lambda_i u_i^2}{\left(1+\lambda_i^2\right)^2}\geq\frac{1}{\lambda_{n-1}\lambda_{n-2}}\sum_{i=1}^{n-1}\frac{\lambda_i u_i^2}{\left(1+\lambda_i^2\right)^2}.$$

We already know that $\sin(\sum_{k=1}^{n}\theta_k)\geq C_{5.4}$, so

$$\frac{1}{2\operatorname{Im}\prod_{k=1}^{n}\left(\lambda_k+\sqrt{-1}\right)}\sum_{i=1}^{n-1}\frac{u_i^2}{1+\lambda_i^2}\leq\frac{1}{2C_{5.4}\prod_{k=1}^{n}\sqrt{1+\lambda_k^2}}\sum_{i=1}^{n-1}\frac{u_i^2}{1+\lambda_i^2}$$
$$\leq\frac{1}{2C_{5.4}\lambda_{n-1}\lambda_{n-2}\lambda_1}\sum_{i=1}^{n-1}\frac{u_i^2}{1+\lambda_i^2}\leq\frac{1}{2C_{5.4}\lambda_{n-1}\lambda_{n-2}}\sum_{i=1}^{n-1}\frac{\lambda_i u_i^2}{\lambda_i^2(1+\lambda_i^2)}$$
$$\leq\frac{\cot^2\left(\frac{\Theta_0}{2}\right)+1}{2C_{5.4}\cot^2\left(\frac{\Theta_0}{2}\right)\lambda_{n-1}\lambda_{n-2}}\sum_{i=1}^{n-1}\frac{\lambda_i u_i^2}{\left(1+\lambda_i^2\right)^2}.$$

On the other hand,

$$
\begin{aligned}
&-\cot\left(\sum_{k=1}^{n}\theta_k\right)\left(\sum_{i=1}^{n}\frac{u_i}{1+\lambda_i^2}\right)^2+\cot\left(\sum_{k=1}^{n-1}\theta_k\right)\left(\sum_{i=1}^{n-1}\frac{u_i}{1+\lambda_i^2}\right)^2+\frac{\lambda_n u_n^2}{(1+\lambda_n^2)^2}\\
&=-2\cot\left(\sum_{k=1}^{n}\theta_k\right)\left(\sum_{i=1}^{n-1}\frac{u_i}{1+\lambda_i^2}\right)\frac{u_n}{1+\lambda_n^2}+\left(\cot\left(\sum_{k=1}^{n-1}\theta_k\right)-\cot\left(\sum_{k=1}^{n}\theta_k\right)\right)\left(\sum_{i=1}^{n-1}\frac{u_i}{1+\lambda_i^2}\right)^2\\
&+\frac{\left(\lambda_n-\cot\left(\sum_{k=1}^{n}\theta_k\right)\right)u_n^2}{(1+\lambda_n^2)^2}\\
&\geq\left(\lambda_n-\cot\left(\sum_{k=1}^{n}\theta_k\right)-\frac{\cot^2\left(\sum_{k=1}^{n}\theta_k\right)}{\cot\left(\sum_{k=1}^{n-1}\theta_k\right)-\cot\left(\sum_{k=1}^{n}\theta_k\right)}\right)\frac{u_n^2}{(1+\lambda_n^2)^2}\\
&=\left(\lambda_n-\frac{\lambda_n\cot\left(\sum_{k=1}^{n-1}\theta_k\right)-1}{\cot\left(\sum_{k=1}^{n-1}\theta_k\right)+\lambda_n}-\frac{\left(\frac{\lambda_n\cot\left(\sum_{k=1}^{n-1}\theta_k\right)-1}{\cot\left(\sum_{k=1}^{n-1}\theta_k\right)+\lambda_n}\right)^2}{\cot\left(\sum_{k=1}^{n-1}\theta_k\right)-\frac{\lambda_n\cot\left(\sum_{k=1}^{n-1}\theta_k\right)-1}{\cot\left(\sum_{k=1}^{n-1}\theta_k\right)+\lambda_n}}\right)\frac{u_n^2}{(1+\lambda_n^2)^2}\\
&=\left(\frac{\lambda_n^2+1}{\cot\left(\sum_{k=1}^{n-1}\theta_k\right)+\lambda_n}-\frac{\left(\lambda_n\cot\left(\sum_{k=1}^{n-1}\theta_k\right)-1\right)^2}{\left(\cot\left(\sum_{k=1}^{n-1}\theta_k\right)+\lambda_n\right)\left(\cot^2\left(\sum_{k=1}^{n-1}\theta_k\right)+1\right)}\right)\frac{u_n^2}{(1+\lambda_n^2)^2}\\
&=\frac{\cot^2\left(\sum_{k=1}^{n-1}\theta_k\right)+\lambda_n^2+2\lambda_n\cot\left(\sum_{k=1}^{n-1}\theta_k\right)}{\left(\cot\left(\sum_{k=1}^{n-1}\theta_k\right)+\lambda_n\right)\left(\cot^2\left(\sum_{k=1}^{n-1}\theta_k\right)+1\right)}\frac{u_n^2}{(1+\lambda_n^2)^2}
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{\lambda_n+\cot\left(\sum_{k=1}^{n-1}\theta_k\right)}{\cot^2\left(\sum_{k=1}^{n-1}\theta_k\right)+1}\frac{u_n^2}{(1+\lambda_n^2)^2}=\sin^2\left(\sum_{k=1}^{n-1}\theta_k\right)\left(\frac{\cos(\theta_n)}{\sin(\theta_n)}+\cot\left(\sum_{k=1}^{n-1}\theta_k\right)\right)\frac{u_n^2}{(1+\lambda_n^2)^2}\\
&=\frac{\sin\left(\sum_{k=1}^{n-1}\theta_k\right)\sin\left(\sum_{k=1}^{n}\theta_k\right)}{\sin(\theta_n)}\frac{u_n^2}{(1+\lambda_n^2)^2}\geq\frac{C_{5.5}}{C_{5.6}\lambda_{n-1}}C_{5.4}\frac{u_n^2}{1+\lambda_n^2}\frac{1}{\sqrt{1+\lambda_n^2}}.
\end{aligned}
$$

It is easy to see that

$$
\frac{1}{2\mathrm{Im}\prod_{k=1}^{n}(\lambda_k+\sqrt{-1})}\frac{u_n^2}{1+\lambda_n^2}\leq\frac{1}{2C_{5.4}\prod_{k=1}^{n}\sqrt{1+\lambda_k^2}}\frac{u_n^2}{1+\lambda_n^2}\leq\frac{1}{2C_{5.4}\lambda_{n-1}}\frac{u_n^2}{1+\lambda_n^2}\frac{1}{\sqrt{1+\lambda_n^2}}.
$$

Therefore, as long as $\frac{\epsilon_{5.7}(\cot^2(\frac{\Theta_0}{2})+1)}{2C_{5.4}\cot^2(\frac{\Theta_0}{2})}<1$ and $\frac{\epsilon_{5.7}}{2C_{5.4}}<\frac{C_{5.5}}{C_{5.6}}C_{5.4}$, we get the required estimate.

Thus, when $n \geq 4$, we have proved that

$$\frac{-2}{\sin^2\left(\sum_{k=1}^n \theta_k\right)}\left(-\cot\left(\sum_{k=1}^n \theta_k\right)\sum_{i,j=1}^n \frac{u_i u_j}{(1+\lambda_i^2)(1+\lambda_j^2)}+\sum_{i=1}^n \frac{\lambda_i u_i^2}{\left(1+\lambda_i^2\right)^2}\right)$$
$$\leq -\epsilon_{5.7}\frac{\prod_{k=1}^n(1+\lambda_k^2)}{\left(\operatorname{Im}\prod_{k=1}^n\left(\lambda_k+\sqrt{-1}\right)\right)^3}\sum_{i=1}^n \frac{u_i^2}{1+\lambda_i^2}.$$

When $n = 2, 3$, a similar argument implies that

$$\frac{-2}{\sin^2(\sum_{k=1}^n \theta_k)}\left(-\cot\left(\sum_{k=1}^n \theta_k\right)\sum_{i,j=1}^n \frac{u_i u_j}{(1+\lambda_i^2)(1+\lambda_j^2)}+\sum_{i=1}^n \frac{\lambda_i u_i^2}{(1+\lambda_i^2)^2}\right)\leq 0.$$

Compared to Theorem 1.1 of [43], the main improvement is that we choose the variable $\hat{\Theta}$ in Theorem 1.1 of [43] as $(n-1)\frac{\pi}{2}$, and we also have a better estimate $-\epsilon_{5.7}\frac{\prod_{k=1}^n(1+\lambda_k^2)}{(\operatorname{Im}\prod_{k=1}^n(\lambda_k+\sqrt{-1}))^3}\sum_{i=1}^n \frac{u_i^2}{1+\lambda_i^2}$, which will be used to deal with terms involving f when $n \geq 4$.

The next goal is to prove that when $n \geq 3$, the matrix corresponding to

$$\sum_{i=1}^n \sin^2\left(\sum_{k\neq i}\theta_k\right)v_i^2+\sum_{i=1}^n\sum_{j\neq i}\sin(\theta_i)\sin(\theta_j)v_i v_j$$

is positive definite. When $\theta_1 = \theta_2 = \ldots = \theta_n$ is sufficiently small, it is easy to see that $\sin^2(\sum_{k\neq i}\theta_k) > \sin^2(\theta_i)$. So the matrix is positive definite. Since the space $\bar{\Gamma}_{\theta_0,\Theta_0}$ is path connected, it suffices to show that the determinant of the matrix is positive on $\bar{\Gamma}_{\theta_0,\Theta_0}$.

Without loss of generality, assume that $\theta_n \geq \theta_{n-1}\ldots \geq \theta_1$. When $i \neq n$,

$$\theta_i \leq \theta_n < \sum_{k\neq i}\theta_k \leq \Theta_0 - \theta_i < \pi - \theta_i,$$

so $\sin(\theta_i) < \sin\left(\sum_{k\neq i}\theta_k\right)$. When $\sin^2\left(\sum_{k=1}^{n-1}\theta_k\right) \neq \sin^2(\theta_n)$, let A be the complex diagonal matrix such that

$$A_{ii} = \sqrt{\sin^2\left(\sum_{k\neq i}\theta_k\right) - \sin^2(\theta_i)},$$

and define

$$B = \left(\frac{\sin(\theta_1)}{A_{11}}, \ldots, \frac{\sin(\theta_n)}{A_{nn}} \right).$$

Then we need to compute $\det A^T(I + B^T B)A$. By elementary linear algebra,

$$\begin{aligned}
&\det A^T(I + B^T B)A = (\det A)^2(1 + BB^T) \\
&= \prod_{i=1}^{n} \left(\sin^2 \left(\sum_{k \neq i} \theta_k \right) - \sin^2(\theta_i) \right) \left(1 + \sum_{i=1}^{n} \frac{\sin^2(\theta_i)}{\sin^2 \left(\sum_{k \neq i} \theta_k \right) - \sin^2(\theta_i)} \right) \\
&= \prod_{i=1}^{n} \left(\sin^2 \left(\sum_{k \neq i} \theta_k \right) - \sin^2(\theta_i) \right) + \sum_{i=1}^{n} \sin^2(\theta_i) \prod_{j \neq i} \left(\sin^2 \left(\sum_{k \neq j} \theta_k \right) - \sin^2(\theta_j) \right).
\end{aligned}$$

By continuity, this equation also holds when $\sin^2(\sum_{k=1}^{n-1} \theta_k) = \sin^2(\theta_n)$.

Therefore, when $\sin^2(\sum_{k=1}^{n-1} \theta_k) \geq \sin^2(\theta_n)$, we already get the required inequality. We only need to prove that

$$\sum_{i=1}^{n} \frac{\sin^2(\theta_i)}{\sin^2 \left(\sum_{k \neq i} \theta_k \right) - \sin^2(\theta_i)} < -1$$

when $\sin^2(\sum_{k=1}^{n-1} \theta_k) < \sin^2(\theta_n)$. In this case, $\sum_{k=1}^{n-1} \theta_k < \theta_n$.

Now we want to study the function

$$G(\alpha, \beta) = \frac{\sin^2(\beta)}{\sin^2(\alpha - \beta) - \sin^2(\beta)} = \frac{-2\sin^2(\beta)}{\cos(2\alpha - 2\beta) - \cos(2\beta)} = \frac{\sin^2(\beta)}{\sin(\alpha)\sin(\alpha - 2\beta)}$$

for any $0 < \beta < \frac{\alpha}{2} < \frac{\pi}{2}$. Then

$$\frac{\partial G}{\partial \beta} = \frac{2\sin(\beta)\cos(\beta)\sin(\alpha - 2\beta) + 2\sin^2(\beta)\cos(\alpha - 2\beta)}{\sin(\alpha)\sin^2(\alpha - 2\beta)} = \frac{2\sin(\beta)\sin(\alpha - \beta)}{\sin(\alpha)\sin^2(\alpha - 2\beta)} > 0,$$

so

$$\begin{aligned}
\frac{\partial}{\partial \beta} \log \left(\frac{\partial G}{\partial \beta} \right) &= \cot(\beta) - \cot(\alpha - \beta) + 4\cot(\alpha - 2\beta) \\
&= \cot(\beta) - \cot(\alpha - \beta) + 4\frac{\cot(\beta)\cot(\alpha - \beta) + 1}{\cot(\beta) - \cot(\alpha - \beta)} \\
&= \frac{(\cot(\beta) + \cot(\alpha - \beta))^2 + 4}{\cot(\beta) - \cot(\alpha - \beta)}.
\end{aligned}$$

Since $0 < \beta < \alpha - \beta < \pi$, we know that $\cot(\beta) - \cot(\alpha - \beta) > 0$, so $\frac{\partial^2 G}{\partial \beta^2} > 0$. Therefore, when we replace θ_1 by 0 and replace θ_{n-1} by $\theta_{n-1} + \theta_1$, we see that $\sum_{i=1}^n \frac{\sin^2(\theta_i)}{\sin^2(\sum_{k \neq i} \theta_k) - \sin^2(\theta_i)}$ strictly increases. We can repeat the process to prove that

$$\sum_{i=1}^n \frac{\sin^2(\theta_i)}{\sin^2\left(\sum_{k \neq i} \theta_k\right) - \sin^2(\theta_i)} < \frac{\sin^2(\theta_n)}{\sin^2\left(\sum_{k=1}^{n-1} \theta_k\right) - \sin^2(\theta_n)} + \frac{\sin^2\left(\sum_{k=1}^{n-1} \theta_k\right)}{\sin^2(\theta_n) - \sin^2\left(\sum_{k=1}^{n-1} \theta_k\right)} = -1.$$

This is the required inequality.

Thus, when $n \geq 3$, we have proved that

$$\sum_{i,j=1}^n \frac{\partial^2}{\partial \lambda_i \partial \lambda_j} \left(\frac{1}{\operatorname{Im} \prod_{k=1}^n (\lambda_k + \sqrt{-1})} \right) u_i u_j \geq 0.$$

When $n = 2$, it is also true because

$$\sum_{i=1}^n \sin^2 \left(\sum_{k \neq i} \theta_k \right) v_i^2 + \sum_{i=1}^n \sum_{j \neq i} \sin(\theta_i) \sin(\theta_j) v_i v_j \geq 0$$

by the Cauchy–Schwarz inequality.

Therefore, when $f \geq 0$, we get (4) as long as $\epsilon_{5.3} < \epsilon_{5.7}$. When $-\epsilon_{5.1} \leq f < 0$ and $n \geq 4$, using the bound that $|\sin(\theta_i)| \leq 1$ and $|\sin(\sum_{k \neq i} \theta_k)| \leq 1$, it is easy to see that

$$\begin{aligned}
&- \frac{f \prod_{k=1}^n (1 + \lambda_k^2)}{\left(\operatorname{Im} \prod_{k=1}^n \left(\lambda_k + \sqrt{-1}\right)\right)^3} \cdot \left(\sum_{i,j=1}^n \sin \left(\sum_{k \neq i} \theta_k \right) \sin \left(\sum_{k \neq j} \theta_k \right) \frac{u_i}{\sqrt{1 + \lambda_i^2}} \frac{u_j}{\sqrt{1 + \lambda_j^2}} \right. \\
&\quad \left. + \sum_{i=1}^n \sin^2 \left(\sum_{k \neq i} \theta_k \right) \frac{u_i^2}{1 + \lambda_i^2} + \sum_{i=1}^n \sum_{j \neq i} \sin(\theta_i) \sin(\theta_j) \frac{u_i}{\sqrt{1 + \lambda_i^2}} \frac{u_j}{\sqrt{1 + \lambda_j^2}} \right) \\
&\quad \leq \frac{3n\epsilon_{5.1} \prod_{k=1}^n (1 + \lambda_k^2)}{\left(\operatorname{Im} \prod_{k=1}^n \left(\lambda_k + \sqrt{-1}\right)\right)^3} \sum_{i=1}^n \frac{u_i^2}{1 + \lambda_i^2}.
\end{aligned}$$

Therefore, as long as $\epsilon_{5.1} < \frac{\epsilon_{5.7}}{6n}$ and $\epsilon_{5.3} < \frac{\epsilon_{5.7}}{2}$, we get the required estimate.

(5) Suppose that $\lambda \in \bar{\Gamma}_{\theta_0, \Theta_0}$ and $F(\lambda) = 0$. For any $i = 1, 2, 3, ...n$, using (3), we see that $F(\lambda)$ is strictly smaller than the limit of F when we fix λ_k for $k \neq i$ and let λ_i go to infinity. Using a similar argument to (1), we see that

$$\operatorname{Im} \prod_{k=1}^n (\lambda_k + \sqrt{-1}) \geq \min\{C_{5.4}, \frac{(n-1)C_{5.5}}{C_{5.6}} \cot^{n-2} \left(\frac{\Theta_0}{2} \right)\} \lambda_i.$$

So the limit of F is $\cot(\sum_{k\neq i}\operatorname{arccot}(\lambda_k))-\cot(\theta_0)$. So $\sum_{k\neq i}\operatorname{arccot}(\lambda_k)<\theta_0$.

Moreover, using the fact that

$$0=\cot\left(\sum_{k=1}^{n}\operatorname{arccot}(\lambda_k)\right)-\frac{f}{\operatorname{Im}\prod_{k=1}^{n}(\lambda_k+\sqrt{-1})}-\cot(\theta_0)$$
$$\leq\cot\left(\sum_{k=1}^{n}\operatorname{arccot}(\lambda_k)\right)-\cot(\theta_0)+\frac{\epsilon_{5.1}}{C_{5.2}},$$

we see that $\sum_{k=1}^{n}\operatorname{arccot}(\lambda_k)<\Theta_0$ as long as $\epsilon_{5.1}<C_{5.2}(\cot(\theta_0)-\cot(\Theta_0))$.

Thus, $\lambda\in\Gamma_{\theta_0,\Theta_0}$.

(6) Let λ be any element in $\Gamma_{\theta_0,\Theta_0}$. Let λ' be any element in $\Gamma_{\theta_0,\Theta_0}$ such that $F(\lambda')=0$ and $\lambda'_k\geq\lambda_k$ for all $k=1,2,3,...,n$.

Then for any $i=1,2,3,...n$, $\sum_{k\neq i}\operatorname{arccot}(\lambda_k)<\theta_0$. If

$$\lambda'_i>\cot\left(\frac{\theta_0-\sum_{k\neq i}\operatorname{arccot}(\lambda_k)}{2}\right)$$

and

$$\min\left\{C_{5.4},\frac{(n-1)C_{5.5}}{C_{5.6}}\cot^{n-2}\left(\frac{\Theta_0}{2}\right)\right\}\lambda'_i>\frac{|f|}{\cot\left(\frac{\theta_0+\sum_{k\neq i}\operatorname{arccot}(\lambda_k)}{2}\right)-\cot(\theta_0)},$$

we get a direct contradiction to the estimate that

$$0\geq\cot\left(\sum_{k=1}^{n}\operatorname{arccot}(\lambda'_k)\right)-\cot(\theta_0)-\frac{|f|}{\min\left\{C_{5.4},\frac{(n-1)C_{5.5}}{C_{5.6}}\cot^{n-2}\left(\frac{\Theta_0}{2}\right)\right\}\lambda'_i}.$$

(7) Fix any $\lambda\in\bar{\Gamma}_{\theta_0,\Theta_0}$. Consider the set C of $\lambda'\in\bar{\Gamma}_{\theta_0,\Theta_0}$ such that $t\lambda+(1-t)\lambda'\in\bar{\Gamma}_{\theta_0,\Theta_0}$ for all $t\in[0,1]$. It is easy to see that C is a closed set in the relative topology on $\bar{\Gamma}_{\theta_0,\Theta_0}$. Now if λ' is in this set, for any $\lambda''\in\bar{\Gamma}_{\theta_0,\Theta_0}$ sufficiently close to λ', there exist $\pi>\Theta'_0>\Theta_0>\theta'_0>\theta_0$ such that $t\lambda+(1-t)\lambda''\in\bar{\Gamma}_{\theta'_0,\Theta'_0}$. By (4) applied to the set $\bar{\Gamma}_{\theta'_0,\Theta'_0}$ and the case $f=0$, we see that $\cot(\sum_{k=1}^{n}\operatorname{arccot}(t\lambda_k+(1-t)\lambda''_k))$ is a concave function. So it is at least Θ_0. Similar arguments can be applied to $\cot(\sum_{k\neq i}\operatorname{arccot}(t\lambda_k+(1-t)\lambda''_k))$ for any $i=1,2,3,...,n$. So we see that λ'' is in C. In other words, C is also open in the relative topology. Since $\bar{\Gamma}_{\theta_0,\Theta_0}$ is connected, and $\lambda\in C$, we see that $\bar{\Gamma}_{\theta_0,\Theta_0}=C$. So $\bar{\Gamma}_{\theta_0,\Theta_0}$ is convex because for any $\lambda,\lambda'\in\bar{\Gamma}_{\theta_0,\Theta_0}$, $t\lambda+(1-t)\lambda'\in\bar{\Gamma}_{\theta_0,\Theta_0}$.

(8) This follows from the concavity of

$$F\left(\lambda_1, ...\lambda_{i-1}, t\lambda_i + (1-t)\lambda_j, \lambda_{i+1}, ..., \lambda_{j-1}, t\lambda_j + (1-t)\lambda_i, \lambda_{j+1}, ..., \lambda_n\right)$$
$$= F\left(\lambda_1, ...\lambda_{i-1}, t\lambda_j + (1-t)\lambda_i, \lambda_{i+1}, ..., \lambda_{j-1}, t\lambda_i + (1-t)\lambda_j, \lambda_{j+1}, ..., \lambda_n\right)$$

(9) This follows from (4), (8), and the result in [39], which was also used as Equation (66) in [42].

(10) This is similar to (7). □

As a corollary, we get the required *a priori* estimate:

Corollary 5.7 *Let M^n be a Kähler manifold with a Kähler metric χ and a real smooth closed (1,1)-form ω_0. Let $\theta_0 \in (0, \pi)$ be a constant, and let $\Theta_0 \in (\theta_0, \pi)$ be another constant. Then there exists a constant $\epsilon_{5.1} > 0$ depending only on n, θ_0, Θ_0 such that the following statement holds.*

Assume the following: (1) When $n \geq 4$, $f > -\epsilon_{5.1}$ is a smooth function.

(2) When $n = 1, 2, 3$, $f \geq 0$ is a constant.

(3) $\omega_0 \in \Gamma_{\chi,\theta_0,\Theta_0}$.

Assume that φ is a smooth function satisfying $\sup_M \varphi = 0$, $\omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar\partial\varphi \in \Gamma_{\chi,\theta_0,\Theta_0}$, and

$$\mathrm{Re}(\omega_\varphi + \sqrt{-1}\chi)^n - \cot(\theta_0)\mathrm{Im}(\omega_\varphi + \sqrt{-1}\chi)^n - f\chi^n = 0.$$

Then for any $k \in \mathbb{N}$, any $\alpha \in (0, 1)$, there exists a constant $C_{5.8}$ depending only on M, n, χ, $||\omega_0||_{C^\infty(\chi)}$, θ_0, Θ_0, $||f||_{C^\infty(\chi)}$, k, α, $\max_{x\in M}(\theta_0 - P_\chi(\omega_0)(x))$, and $\max_{x\in M}(\Theta_0 - Q_\chi(\omega_0)(x))$ such that

$$||\varphi||_{C^{k,\alpha}(\chi)} \leq C_{5.8}.$$

Proof Compared to Székelyhidi's conditions in [42], there are two major differences. First, F also depends on f. Second, Γ does not contain the positive orthant. However, we will show that his results still survive without many changes.

Székelyhidi's C^0 estimate relies on a variant of the Alexandroff-Bakelman-Pucci maximum principle similar to Lemma 9.2 of [25]. Clearly, it does not take derivatives of f. So Székelyhidi's C^0 estimate is still true.

The next step is to prove that

$$|\sqrt{-1}\partial\bar\partial\varphi|_\chi \leq C_{5.9}(1 + \sup_M |\nabla\varphi|^2_\chi).$$

We will use the same notations as in [42] except that the letter f in [42] is replaced by F, the letter F is replaced by F_χ, and the letter u is replaced by φ.

It is easy to see that (78) of [42] still holds. Now we differentiate the equation $F_\chi(f, \omega_\varphi) = 0$. We see that

$$F_\chi^{ij} g_{i\bar{j}1} + F_\chi^f f_1 = 0,$$

and

$$F_\chi^{pq,rs} g_{p\bar{q}1} g_{r\bar{s}\bar{1}} + F_\chi^{kk} g_{k\bar{k}1\bar{1}} + F_\chi^{kk,f} g_{k\bar{k}1} f_{\bar{1}} + F_\chi^{kk,f} g_{k\bar{k}\bar{1}} f_1 + F_\chi^f f_{1\bar{1}} = 0$$

because $F_\chi^{ff} = 0$. Since $|F_\chi^f| \leq \frac{1}{C_{5.2}}$ by Lemma 5.6 (1), the term $F_\chi^f f_{1\bar{1}}$ is bounded. So the only additional term in (85) of [42] is $-C_0\lambda_1^{-1}|F_\chi^{kk,f} g_{k\bar{k}1}|$ on the right-hand side. Instead of (94) of [42], we get

$$F_\chi^{kk} g_{k\bar{k}p} + F_\chi^f f_p = 0.$$

Since $|F_\chi^f f_p|$ is bounded, the estimate in (95) still holds. So the only additional term in (99) and (104) of [42] is $-C_0\lambda_1^{-1}|F_\chi^{kk,f} g_{k\bar{k}1}|$ on the right-hand side. Case 1 in [42] will not happen if λ_1 is large enough. The additional term in (120) of [42] is also $-C_0\lambda_1^{-1}|F_\chi^{kk,f} g_{k\bar{k}1}|$. However, recall that (67) of [42] is

$$-F_\chi^{ij,rs} g_{i\bar{j}1} g_{r\bar{s}\bar{1}} \geq -F_{ij} g_{i\bar{i}1} g_{j\bar{j}\bar{1}} - \sum_{i>1} \frac{F_1 - F_i}{\lambda_1 - \lambda_i} |g_{i\bar{1}1}|.$$

(We remark that the letter f in [42] is replaced by F and that the letter F is replaced by F_χ.) The term $-F_{ij} g_{i\bar{i}1} g_{j\bar{j}\bar{1}}$ was thrown away. However, by Lemma 5.6 (2), Lemma 5.6 (4), and the Cauchy–Schwarz inequality,

$$-F_{ij} g_{i\bar{i}1} g_{j\bar{j}\bar{1}} \geq C_0 |F_\chi^{kk,f} g_{k\bar{k}1}| - C_{5.10}.$$

So Székelyhidi's estimate

$$|\sqrt{-1}\partial\bar{\partial}\varphi|_\chi \leq C_{5.9} \left(1 + \sup_M |\nabla\varphi|_\chi^2\right)$$

still holds.

Székelyhidi used the property that Γ contains the positive orthant to prove the C^2 estimate [42]. We do not have this property. However, we can use Proposition 5.1 of [12] to achieve this.

Evans-Krylov's estimate requires the uniform ellipticity and concavity of $F_\chi(f, .)$. These conditions follows from Lemma 5.6 (3) and Lemma 5.6 (9). The higher-order estimate follows from standard elliptic theories. □

The analog of Theorem 1.18 is the following:

Proposition 5.8 *Fix a Kähler manifold M^n with a Kähler metric χ and a test family $\omega_{t,0}$ of real closed (1,1)-forms. Suppose that for all $t > 0$, there exist a constant $c_t > 0$ and a smooth function φ_t such that $\omega_t = \omega_{t,0} + \sqrt{-1}\partial\bar{\partial}\varphi_t \in \Gamma_{\chi,\theta_0,\Theta_0}$ satisfies*

$$\mathrm{Re}\left(\omega_t + \sqrt{-1}\chi\right)^n - \cot(\theta_0)\mathrm{Im}\left(\omega_t + \sqrt{-1}\chi\right)^n - c_t\chi^n = 0.$$

Then there exist a constant $\epsilon_{5.11} > 0$ and a current $\omega_{5.12} \in [\omega_0 - \epsilon_{5.11}\chi]$ such that $\omega_{5.12} \in \bar{\Gamma}_{\chi,\theta_0,\Theta_0}$ in the sense of Definition 5.10.

The definition of the sum ω of a real smooth closed (1,1)-form and a closed positive (1,1)-current being in $\bar{\Gamma}_{\chi,\theta_0,\Theta_0}$ is similar to Definition 3.3.

Definition 5.9 Suppose that χ_0 is a Kähler form with constant coefficients on an open set $O \subset \mathbb{C}^n$ and that ω is the sum of a real smooth closed (1,1)-form and a closed positive (1,1)-current. Then we say that $\omega \in \bar{\Gamma}_{\chi_0,\theta_0,\Theta_0}$ on O if for any $\delta > 0$, the smoothing ω_δ satisfies $\omega_\delta \in \bar{\Gamma}_{\chi_0,\theta_0,\Theta_0}$ on the set $O_\delta = \{x : B_\delta(x) \subset O\}$.

Definition 5.10 We say that $\omega \in \bar{\Gamma}_{\chi,\theta_0,\Theta_0}$ if for any $\epsilon_{5.13} > 0$ and $\epsilon_{5.14} > 0$ satisfying

$$(1 + \epsilon_{5.14})\left(\cot(\theta_0) + \epsilon_{5.13}\right) > \cot(\theta_0),$$

on any open subset O of any coordinate chart, for any Kähler form χ_0 with constant coefficients satisfying

$$(1 + \epsilon_{5.14})\chi_0 \geq \chi \geq \chi_0,$$

we have

$$\omega + \epsilon_{5.13}\chi \in \bar{\Gamma}_{\chi_0,\theta_0,\Theta_0}.$$

Remark 5.11 By Lemma 5.6 (10), when ω is smooth, it is easy to see that $\omega \in \bar{\Gamma}_{\chi,\theta_0,\Theta_0}$ in the sense of Definition 5.10 if and only if $\omega \in \bar{\Gamma}_{\chi,\theta_0,\Theta_0}$ in the usual sense. Another key property is that the condition $\omega \in \bar{\Gamma}_{\chi,\theta_0,\Theta_0}$ in the sense of Definition 5.10 is preserved under weak limits in the sense of currents.

The analog of Lemma 3.5 is the following:

Lemma 5.12 *Suppose that A is a $p \times p$ Hermitian matrix, B is a diagonal $q \times q$ Hermitian matrix with $B_{ii} = \lambda_i$, C is a $p \times q$ complex matrix, and D is another diagonal $q \times q$ matrix such that*

$$D_{ii} = \frac{\operatorname{Im} \prod_{k \neq i} (\lambda_k + \sqrt{-1})}{\operatorname{Im} \prod_{k=1}^{q} (\lambda_k + \sqrt{-1})}.$$

Suppose that

$$Q_{I_{p+q}} \left(\begin{bmatrix} A & C \\ \bar{C}^T & B \end{bmatrix} \right) < \pi.$$

Then D is well defined. Moreover,

$$Q_{I_p} \left(A - C D \bar{C}^T \right) + Q_{I_q}(B) \leq Q_{I_{p+q}} \left(\begin{bmatrix} A & C \\ \bar{C}^T & B \end{bmatrix} \right)$$

and

$$P_{I_p} \left(A - C D \bar{C}^T \right) + Q_{I_q}(B) \leq P_{I_{p+q}} \left(\begin{bmatrix} A & C \\ \bar{C}^T & B \end{bmatrix} \right).$$

Proof Define E, F to be diagonal $q \times q$ matrices such that

$$E_{ii} = \frac{\lambda_i}{1 + \lambda_i^2}, \; F_{ii} = \frac{1}{1 + \lambda_i^2}.$$

We first claim that

$$Q_{I_p + C F \bar{C}^T}(A - C E \bar{C}^T) + Q_{I_q}(B) = Q_{I_{p+q}} \left(\begin{bmatrix} A & C \\ \bar{C}^T & B \end{bmatrix} \right).$$

In fact, using

$$\begin{aligned} &\begin{bmatrix} A - C(B + \sqrt{-1} I_q)^{-1} \bar{C}^T + \sqrt{-1} I_p & O \\ O & B + \sqrt{-1} I_q \end{bmatrix} \\ &= \begin{bmatrix} I_p & -C(B + \sqrt{-1} I_q)^{-1} \\ O & I_q \end{bmatrix} \begin{bmatrix} A + \sqrt{-1} I_p & C \\ \bar{C}^T & B + \sqrt{-1} I_q \end{bmatrix} \begin{bmatrix} I_p & O \\ -(B + \sqrt{-1} I_q)^{-1} \bar{C}^T & I_q \end{bmatrix}, \end{aligned}$$

it is easy to see that

$$\begin{aligned}&\det\left(\begin{bmatrix} A - CE\bar{C}^T & O \\ O & B\end{bmatrix} + \sqrt{-1}\begin{bmatrix} I_p + CF\bar{C}^T & O \\ O & I_q\end{bmatrix}\right)\\ &= \det\left(\begin{bmatrix} A - C(B+\sqrt{-1}I_q)^{-1}\bar{C}^T + \sqrt{-1}I_p & O \\ O & B+\sqrt{-1}I_q\end{bmatrix}\right)\\ &= \det\left(\begin{bmatrix} A + \sqrt{-1}I_p & C \\ \bar{C}^T & B+\sqrt{-1}I_q\end{bmatrix}\right) = \det\left(\begin{bmatrix} A & C \\ \bar{C}^T & B\end{bmatrix} + \sqrt{-1}\begin{bmatrix} I_p & O \\ O & I_q\end{bmatrix}\right).\end{aligned}$$

It follows that

$$Q_{I_p+CF\bar{C}^T}(A - CE\bar{C}^T) + Q_{I_q}(B) \equiv Q_{I_{p+q}}(\begin{bmatrix} A & C \\ \bar{C}^T & B\end{bmatrix}) \mod 2\pi.$$

This is also true when we replace A by $A + tI_p$ and replace B by $B + tI_q$ for $t \geq 0$. However, when t is large enough, all the quantities are close to 0. So there is no multiple of 2π there. By continuity, there is also no multiple of 2π when $t = 0$.

As a corollary of the claim, we see that $Q_{I_q}(B) < \pi$. This implies that

$$\operatorname{Im}\prod_{k=1}^{q}(\lambda_k + \sqrt{-1}) > 0.$$

So D is well-defined.

Moreover,

$$D_{ii} - E_{ii} = \frac{-\operatorname{Re}\prod_{k=1}^{q}(\lambda_k + \sqrt{-1})}{(1+\lambda_i^2)\left(\operatorname{Im}\prod_{k=1}^{q}(\lambda_k + \sqrt{-1})\right)} = -\cot(Q_{I_q}(B))F_{ii}.$$

Now we write $A - CE\bar{C}^T$ as a_{ij} and $CF\bar{C}^T$ as b_{ij}. Define

$$a = \sqrt{-1}\sum_{i,j=1}^{p} a_{ij}dz^i \wedge d\bar{z}^j, b = \sqrt{-1}\sum_{i,j=1}^{p} b_{ij}dz^i \wedge d\bar{z}^j, c = \sqrt{-1}\sum_{i=1}^{p} dz^i \wedge d\bar{z}^i.$$

Then $Q_{c+b}(a) < \pi - Q_{I_q}(B) < \pi$. Now we define

$$I = \left\{t \in [0,1] \text{ such that } Q_{c_s}(a_s) \leq Q_{c+b}(a) \text{ for all } s \in [0,t]\right\},$$

where $a_s = a + s\cot(Q_{I_q}(B))b$ and $c_s = c + b - sb$. If $b = 0$, then it is trivial that $I = [0, 1]$. So we only need to consider the case when $b \neq 0$. It is also trivial that $0 \in I$ and that I is closed. Now we assume that $t \in I$. Then

$$\begin{aligned}
&\frac{d}{ds}|_{s=t}\cot(Q_{c_s}(a_s)) = \frac{d}{ds}|_{s=t}\frac{\mathrm{Re}(a_s+\sqrt{-1}c_s)^p}{\mathrm{Im}(a_s+\sqrt{-1}c_s)^p}\\
&= \frac{p(\mathrm{Re}(a_s+\sqrt{-1}c_s)^{p-1}\wedge\cot(Q_{I_q}(B))b + \mathrm{Im}(a_s+\sqrt{-1}c_s)^{p-1}\wedge b)}{\mathrm{Im}(a_s+\sqrt{-1}c_s)^p}\\
&- \frac{\mathrm{Re}(a_s+\sqrt{-1}c_s)^p}{\mathrm{Im}(a_s+\sqrt{-1}c_s)^p}\frac{p\left(-\mathrm{Re}(a_s+\sqrt{-1}c_s)^{p-1}\wedge b + \mathrm{Im}(a_s+\sqrt{-1}c_s)^{p-1}\wedge\cot(Q_{I_q}(B))b\right)}{\mathrm{Im}(a_s+\sqrt{-1}c_s)^p}\\
&= p(\cot(Q_{I_q}(B)) + \cot(Q_{c_s}(a_s)))\cdot\\
&\frac{\left(\mathrm{Re}(a_s+\sqrt{-1}c_s)^{p-1} - \cot(Q_{I_q}(B)+Q_{c_s}(a_s))\mathrm{Im}(a_s+\sqrt{-1}c_s)^{p-1}\right)\wedge b}{\mathrm{Im}(a_s+\sqrt{-1}c_s)^p} > 0
\end{aligned}$$

by Lemma 8.2 of [12]. So I is open. It must be $[0, 1]$. So $1 \in I$. It follows that

$$Q_{I_p}(A - CD\bar{C}^T) = Q_{c_1}(a_1) \leq Q_{c+b}(a) = Q_{I_p+CF\bar{C}^T}(A - CE\bar{C}^T).$$

Thus, we have proved that

$$Q_{I_p}(A - CD\bar{C}^T) + Q_{I_q}(B) \leq Q_{I_{p+q}}\left(\begin{bmatrix} A & C \\ \bar{C}^T & B \end{bmatrix}\right) < \pi.$$

Using a similar argument to the paragraph before Lemma 3.5, by the Schur-Horn theorem and the convexity of $-\cot(\sum_{i=1}^{p-1}\mathrm{arccot}(\lambda_i'))$,

$$P_{I_p}(A - CD\bar{C}^T) = \max_{U\in\mathbb{C}^{p\times(p-1)},\bar{U}^TU=I_{p-1}} Q_{I_{p-1}}\left(\bar{U}^T(A - CD\bar{C}^T\right)U).$$

This is also a generalization of the celebrated Courant–Fischer–Weyl min-max principle.

Let $U \in \mathbb{C}^{p\times(p-1)}$ be the matrix realizing $\max_{U\in\mathbb{C}^{p\times(p-1)},\bar{U}^T U=I_{p-1}} Q_{I_{p-1}} (\bar{U}^T(A - CD\bar{C}^T)U)$, then

$$\begin{aligned}
&P_{I_p}(A - CD\bar{C}^T) + Q_{I_q}(B)\\
&= Q_{I_{p-1}}\left(\bar{U}^T(A - CD\bar{C}^T)U\right) + Q_{I_q}(B)\\
&= Q_{I_{p-1}}\left(\bar{U}^T AU - (\bar{U}^T C)D(\bar{C}^T U)\right) + Q_{I_q}(B)\\
&\le Q_{I_{p+q-1}}\left(\begin{bmatrix}\bar{U}^T AU & \bar{U}^T C\\ \bar{C}^T U & B\end{bmatrix}^{-1}\right)\\
&= Q_{I_{p+q-1}}\left(\begin{bmatrix}\bar{U}^T & O\\ O & I_n\end{bmatrix}\begin{bmatrix}A & C\\ \bar{C}^T & B\end{bmatrix}\begin{bmatrix}U & O\\ O & I_n\end{bmatrix}\right)\\
&\le P_{I_{p+q}}\left(\begin{bmatrix}A & C\\ \bar{C}^T & B\end{bmatrix}\right).
\end{aligned}$$

□

Now we need to prove the following:

Proposition 5.13 *Let $\chi_{M\times M} = \pi_1^*\chi + \pi_2^*\chi$ be a Kähler form on $M \times M$. Let $C_{5.15}$, $\theta_{5.16}$, $\Theta_{5.17}$, $\Theta_{5.18}$, and $\Theta_{5.19}$ be constants depending only on n, θ_0, Θ_0 such that*

$$\theta_{5.16} = \theta_0 + n\operatorname{arccot}(C_{5.15}) < \Theta_{5.17} = \Theta_{5.18} + n\operatorname{arccot}(C_{5.15}) < \Theta_{5.19} = \Theta_0 + n\operatorname{arccot}(C_{5.15}) < \pi.$$

Suppose that

$$\omega_{5.20} = \pi_1^*\omega_0 + C_{5.15}\pi_2^*\chi + \sqrt{-1}\partial\bar{\partial}\varphi_{5.20}$$

is a real smooth closed (1,1)-form on $M\times M$ such that $\omega_{5.20} \in \Gamma_{\chi_{M\times M},\theta_{5.16},\Theta_{5.17}}$. Define $\omega_{5.21}$ by

$$\omega_{5.21} = \frac{\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^k\frac{n!}{(n-2k)!(2k+1)!}(\pi_1)_*(\omega_{5.20}^{n-2k}\wedge\pi_2^*\chi^{2k+1})}{\int_M \operatorname{Im}(C_{5.15}\chi + \sqrt{-1}\chi)^n}.$$

Then $\omega_{5.21} \in \Gamma_{\chi,\theta_0,\Theta_0}$.

Remark 5.14 Proposition 5.13 also holds when $\Theta_{5.17} = \Theta_{5.19}$. However, we instead require $\Theta_{5.17} < \Theta_{5.19}$ to make sure that if $\omega \in \Gamma_{\chi,\theta_{5.16},\Theta_{5.17}}$, then $\frac{1}{\cot(Q_\chi(\omega))-\cot(\Theta_{5.19})}$ changes only a little when we perform the truncation as in Sect. 3.

Proof As in Sect. 3, at each $p=(p_1,p_2)\in M\times M$, let $z_i^{(1)}$ be the local coordinates on $M\times\{p_2\}$ and $z_i^{(2)}$ be the local coordinates on $\{p_1\}\times M$. Then

$$\omega_{5.20}=\omega^{(1)}+\omega^{(2)}+\omega^{(1,2)}+\omega^{(2,1)},$$

where

$$\omega^{(1)}=\sum_{i,j=1}^{n}\sqrt{-1}\omega_{i\bar{j}}^{(1)}dz_i^{(1)}\wedge d\bar{z}_j^{(1)},\ \omega^{(2)}=\sum_{i,j=1}^{n}\sqrt{-1}\omega_{i\bar{j}}^{(2)}dz_i^{(2)}\wedge d\bar{z}_j^{(2)},$$

$$\omega^{(1,2)}=\sum_{i,j=1}^{n}\sqrt{-1}\omega_{i\bar{j}}^{(1,2)}dz_i^{(1)}\wedge d\bar{z}_j^{(2)},$$

and $\omega^{(2,1)}=\overline{\omega^{(1,2)}}$. After changing the definition of $z_i^{(2)}$ if necessary, we can assume that

$$\pi_2^*\chi=\sqrt{-1}\sum_{i=1}^{n}dz_i^{(2)}\wedge d\bar{z}_i^{(2)}$$

and

$$\omega^{(2)}=\sqrt{-1}\sum_{i=1}^{n}\lambda_i dz_i^{(2)}\wedge d\bar{z}_i^{(2)}$$

at p.

Then

$$\omega_{5.21}=\frac{\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^k(\pi_1)_*\hat{\omega}_k}{\int_M \mathrm{Im}(C_{5.15}\chi+\sqrt{-1}\chi)^n},$$

where $\hat{\omega}_k$ equals

$$\begin{aligned}&\frac{n!}{(n-2k-1)!(2k+1)!}\omega^{(1)}\wedge(\omega^{(2)})^{n-2k-1}\wedge\pi_2^*\chi^{2k+1}\\&+\frac{n!}{(n-2k-2)!(2k+1)!}\omega^{(1,2)}\wedge\omega^{(2,1)}\wedge(\omega^{(2)})^{n-2k-2}\wedge\pi_2^*\chi^{2k+1}\\&=\sum_{i,j=1}^{n}\frac{\sqrt{-1}\omega_{i\bar{j}}^{(1)}dz_i^{(1)}\wedge d\bar{z}_j^{(1)}}{(2k+1)!}\wedge\left(\sum_{\alpha_1,\ldots\alpha_{2k+1}\text{distinct}}\frac{1}{\lambda_{\alpha_1}\ldots\lambda_{\alpha_{2k+1}}}\right)(\omega^{(2)})^n\end{aligned}$$

$$-\sum_{i,j,l=1}^{n}\frac{\sqrt{-1}\omega^{(1,2)}_{i\bar l}\overline{\omega^{(1,2)}_{j\bar l}}dz^{(1)}_i\wedge d\bar z^{(1)}_j}{(2k+1)!}\wedge\sum_{\alpha_1,\dots\alpha_{2k+1},l\text{distinct}}\frac{1}{\lambda_l\lambda_{\alpha_1}\dots\lambda_{\alpha_{2k+1}}}(\omega^{(2)})^n.$$

So $\omega_{5.21}=(\pi_1)_*\omega_{5.22}$, where $\omega_{5.22}$ equals

$$\frac{\mathrm{Im}(\omega^{(2)}+\sqrt{-1}\pi_2^*\chi)^n}{\int_{\{p_1\}\times M}\mathrm{Im}(\omega^{(2)}+\sqrt{-1}\pi_2^*\chi)^n}\wedge\sum_{i,j,l=1}^{n}(\omega^{(1)}_{i\bar j}-\omega^{(1,2)}_{i\bar l}\frac{\mathrm{Im}\prod_{k\neq l}(\lambda_k+\sqrt{-1})}{\mathrm{Im}\prod_{k=1}^n(\lambda_k+\sqrt{-1})}\overline{\omega^{(1,2)}_{j\bar l}})dz^{(1)}_i\wedge d\bar z^{(1)}_j.$$

By Lemma 5.12,

$$Q_{\pi_1^*\chi}\left(\sum_{i,j,l=1}^{n}\left(\omega^{(1)}_{i\bar j}-\omega^{(1,2)}_{i\bar l}\frac{\mathrm{Im}\prod_{k\neq l}\left(\lambda_k+\sqrt{-1}\right)}{\mathrm{Im}\prod_{k=1}^n\left(\lambda_k+\sqrt{-1}\right)}\overline{\omega^{(1,2)}_{j\bar l}}\right)dz^{(1)}_i\wedge d\bar z^{(1)}_j\right)<\Theta_{5.17}-Q_{\pi_2^*\chi}(\omega^{(2)}).$$

Now we consider the function $\frac{1}{\cot(Q_\chi(\omega))-\cot(\Theta_{5.19})}$ for $\omega\in\Gamma_{\chi,\theta_{5.16},\Theta_{5.17}}$. Since

$$D^2\left(\frac{1}{\cot(Q_\chi(\omega))-\cot(\Theta_{5.19})}\right)=\frac{-D^2\cot(Q_\chi(\omega))}{\left(\cot(Q_\chi(\omega))-\cot(\Theta_{5.19})\right)^2}+\frac{2D\cot(Q_\chi(\omega))\otimes D\cot(Q_\chi(\omega))}{\left(\cot(Q_\chi(\omega))-\cot(\Theta_{5.19})\right)^3}$$

and $\cot(Q_\chi(\omega))$ is concave by Lemma 5.6 (9), we see that $\frac{1}{\cot(Q_\chi(\omega))-\cot(\Theta_{5.19})}$ is convex on $\Gamma_{\chi,\theta_{5.16},\Theta_{5.17}}$.

So

$$\begin{aligned}
&\frac{1}{\cot\left(Q_\chi(\omega_{5.21})\right)-\cot(\Theta_{5.19})}\\
&<\int_{\{p_1\}\times M}\frac{1}{\cot\left(\Theta_{5.19}-Q_{\pi_2^*\chi}(\omega^{(2)})\right)-\cot(\Theta_{5.19})}\frac{\mathrm{Im}\left(\omega^{(2)}+\sqrt{-1}\pi_2^*\chi\right)^n}{\int_{\{p_1\}\times M}\mathrm{Im}\left(\omega^{(2)}+\sqrt{-1}\pi_2^*\chi\right)^n}\\
&=\frac{\int_{\{p_1\}\times M}\left(\mathrm{Re}\left(\omega^{(2)}+\sqrt{-1}\pi_2^*\chi\right)^n-\cot(\Theta_{5.19})\mathrm{Im}\left(\omega^{(2)}+\sqrt{-1}\pi_2^*\chi\right)^n\right)}{(1+\cot^2(\Theta_{5.19}))\int_{\{p_1\}\times M}\mathrm{Im}\left(\omega^{(2)}+\sqrt{-1}\pi_2^*\chi\right)^n}\\
&=\frac{\mathrm{Re}\left(C_{5.15}+\sqrt{-1}\right)^n-\cot(\Theta_{5.19})\mathrm{Im}\left(C_{5.15}+\sqrt{-1}\right)^n}{(1+\cot^2(\Theta_{5.19}))\mathrm{Im}(C_{5.15}+\sqrt{-1})^n}\\
&=\frac{1}{\cot(\Theta_0)-\cot(\Theta_{5.19})}.
\end{aligned}$$

By a similar calculation, $\frac{1}{\cot(P_\chi(\omega_{5.21}))-\cot(\Theta_{5.19})}<\frac{1}{\cot(\theta_0)-\cot(\Theta_{5.19})}$. By the convexity of the set $\Gamma_{\chi,\theta_{5.16},\Theta_{5.17}}$, we also know that $\omega_{5.21}\in\Gamma_{\chi,\theta_{5.16},\Theta_{5.17}}$. It follows that $\omega_{5.21}\in\Gamma_{\chi,\theta_0,\Theta_0}$. ☐

We use the same method as in Sect. 3 to prove Proposition 5.8. The equation

$$\mathrm{tr}_{\omega_{t,\epsilon_{1.6},\epsilon_{1.7}}}\chi_{M\times M}+f_{t,\epsilon_{1.6},\epsilon_{1.7}}\frac{\chi^{2n}_{M\times M}}{\omega^{2n}_{t,\epsilon_{1.6},\epsilon_{1.7}}}=(n+1)c$$

is replaced by

$$\mathrm{Re}\left(\omega_{t,\epsilon_{1.6},\epsilon_{1.7}}+\sqrt{-1}\chi_{M\times M}\right)^{2n}-\cot(\theta_{5.16})\mathrm{Im}\left(\omega_{t,\epsilon_{1.6},\epsilon_{1.7}}+\sqrt{-1}\chi_{M\times M}\right)^{2n}=f_{t,\epsilon_{1.6},\epsilon_{1.7}}\chi^n$$

for $\omega_{t,\epsilon_{1.6},\epsilon_{1.7}}\in[\pi_1^*\omega_{t,0}+C_{5.15}\pi_2^*\chi]$, $\chi_{M\times M}=\pi_1^*\chi+\pi_2^*\chi$, and $f_{t,\epsilon_{1.6},\epsilon_{1.7}}$ similar to how it was defined before. It is easy to see that there exists a constant $C_{5.23}>0$ depending only on n, θ_0, Θ_0, and $\Theta_{5.18}$ such that

$$\mathrm{Re}\prod_{k=1}^{2n}(\lambda_k+\sqrt{-1})-\cot(\theta_{5.16})\mathrm{Im}\prod_{k=1}^{2n}(\lambda_k+\sqrt{-1})\le C_{5.23}\prod_{k=1}^{2n}(\lambda_k-\cot(\Theta_{5.19}))$$

for all $\lambda\in\Gamma_{\theta_{5.16},\Theta_{5.17}}$. We also know that $\omega-\cot(\Theta_{5.19})\chi$ must be a Kähler form for all $\omega\in\Gamma_{\chi,\theta_{5.16},\Theta_{5.17}}$. Combining these facts with Proposition 5.5, Remark 5.11, Proposition 5.13 and Remark 5.14, we can prove Proposition 5.8 using a similar method to that in Sect. 3.

With all these preparations, we can prove Proposition 5.2 and, as a corollary, Theorem 1.7. We prove it by induction on the dimension n of M. When $n=1$, it is trivial. So we assume that it has been proved for all lower dimensions and then try to prove it. Define I to be the set of $t\in[0,\infty)$ such that there exists a smooth function φ_t and a constant $c_t\ge 0$ satisfying $\omega_{t,\varphi_t}=\omega_{t,0}+\sqrt{-1}\partial\bar\partial\varphi_t\in\Gamma_{\chi,\theta_0,\Theta_0}$ and

$$\mathrm{Re}(\omega_{t,\varphi_t}+\sqrt{-1}\chi)^n-\cot(\theta_0)\mathrm{Im}(\omega_{t,\varphi_t}+\sqrt{-1}\chi)^n-c_t\chi^n=0.$$

By (C) of the definition of the test family and Proposition 5.5, I is non-empty. We also know that I is open by Proposition 5.5. In fact, the space $\Gamma_{\chi,\theta_0,\Theta_0}$ is open, and the condition $c_t\ge 0$ is ensured by the third assumption of Proposition 5.2 applied to $V=M$. To show the closeness, assume that $t_k\in I$ is a sequence converging to t_∞. Then by the monotonicity of P_χ and Q_χ, the third assumption of Proposition 5.2 applied to $V=M$, and Proposition 5.5, we know that $t\in I$ for all $t>t_\infty$. We need to show that $t_\infty\in I$. Without loss of generality, assume that $t_\infty=0$. By Proposition 5.8, there exist a constant $\epsilon_{5.11}>0$ and a current $\omega_{5.12}=\omega_0-\epsilon_{5.11}\chi+\sqrt{-1}\partial\bar\partial\varphi_{5.12}$ such that $\omega_{5.12}\in\bar\Gamma_{\chi,\theta_0,\Theta_0}$ in the sense of Definition 5.10. By Proposition 5.5, it suffices to find a form $\omega_{5.24}=\omega_0+\sqrt{-1}\partial\bar\partial\varphi_{5.24}$ in $\Gamma_{\chi,\theta_0,\Theta_0}$. We essentially follow the procedure in Sect. 4, with minor adjustments to deal with the problem that ω_0 is no longer Kähler.

Let $\epsilon_{5.13} = \min\{\frac{\epsilon_{5.11}}{3}, \frac{1}{100}\}$ to get the corresponding $\epsilon_{5.14}$. By choosing $\epsilon_{5.14}$ small enough, we can also assume that as long as $(1+\epsilon_{5.14})\chi_0 \geq \chi \geq \chi_0$ and a real smooth closed (1,1)-form $\omega \in \bar{\Gamma}_{\chi_0,\theta_0,\Theta_0}$, then $\omega + \epsilon_{5.13}\chi \in \Gamma_{\chi,\theta_0,\Theta_0}$. We can also assume that $\epsilon_{5.14} < \frac{1}{100}$. Then, as in Sect. 4, there exist a finite number of coordinate balls $B_{2r}(x_i)$ such that $B_r(x_i)$ is a cover of M. Moreover, let $\varphi^i_{\omega_0}, \varphi^i_\chi$ be potentials such that $\sqrt{-1}\partial\bar\partial\varphi^i_{\omega_0} = \omega_0$ and $\sqrt{-1}\partial\bar\partial\varphi^i_\chi = \chi$ on $B_{2r}(x_i)$. Then we require that

$$|\varphi^i_\chi - |z|^2| \leq \frac{\epsilon_{5.13} r^2}{100(1+|\cot(\theta_0)|)}$$

and

$$\sqrt{-1}\partial\bar\partial|z|^2 \leq \chi \leq (1+\epsilon_{5.14})\sqrt{-1}\partial\bar\partial|z|^2$$

on $B_{2r}(x_i)$. By the uniform continuity of $\varphi^i_{\omega_0}$, there exists a constant $\epsilon_{5.25} < \frac{r}{5}$ such that $|\varphi^i_{\omega_0}(x) - \varphi^i_{\omega_0}(y)| \leq \frac{\epsilon_{5.13} r^2}{100}$ for all $x \in \overline{B_{\frac{9r}{5}}(x_i)}$ such that $|x-y| \leq \epsilon_{5.25}$.

As in Sect. 4, we take $\delta < \frac{\epsilon_{5.25}\epsilon_{5.13}}{100(1+|\cot(\theta_0)|)}$, let φ^i_δ be the smoothing of $\varphi^i_{\omega_0} - 2\epsilon_{5.13}\varphi^i_\chi + \varphi_{5.12}$, and let $\varphi^i_{5.26} = \varphi^i_\delta - \varphi^i_{\omega_0} + \epsilon_{5.13}\varphi^i_\chi$. Since $\omega_{5.12} \in \bar{\Gamma}_{\chi,\theta_0,\Theta_0}$ in the sense of Definition 5.10, we know that $\sqrt{-1}\partial\bar\partial\varphi^i_\delta \in \bar{\Gamma}_{\chi_0,\theta_0,\Theta_0}$ on $\overline{B_{\frac{9r}{5}}(x_i)}$ by Definition 5.10 and the monotonicity of $\bar{\Gamma}_{\chi_0,\theta_0,\Theta_0}$. This implies that

$$\omega_0 + \sqrt{-1}\partial\bar\partial\varphi^i_{5.26} = \sqrt{-1}\partial\bar\partial\varphi^i_\delta + \epsilon_{5.13}\chi \in \Gamma_{\chi,\theta_0,\Theta_0}$$

on $\overline{B_{\frac{9r}{5}}(x_i)}$. We also know that

$$\omega_0 - 3\epsilon_{5.13}\chi + \sqrt{-1}\partial\bar\partial\varphi_{5.12} - \cot(\theta_0)\chi \geq \omega_{5.12} - \cot(\theta_0)\chi$$

is a positive current because $\omega_{5.12} \in \bar{\Gamma}_{\chi,\theta_0,\Theta_0}$ in the sense of Definition 5.10.

As in Sect. 4, we pick a small enough number

$$\epsilon_{5.27} = \frac{\epsilon_{5.13} r^2}{100(\int_0^1 \log\left(\frac{1}{t}\right)\mathrm{Vol}(\partial B_1(0))t^{2n-1}\rho(t)dt + \log 2 + \frac{3^{2n-1}}{2^{2n-3}}\log 2)},$$

where ρ is the function in Definition 3.1. Then we consider the set Y in which the Lelong number of $\varphi_{5.12}$ is at least $\epsilon_{5.27}$. By Siu's work [37], Y is an analytic subvariety.

If we can find a smooth function $\varphi_{5.28}$ near Y such that $\omega_0+\sqrt{-1}\partial\bar\partial\varphi_{5.28}\in\Gamma_{\chi,\theta_0,\Theta_0}$, then using the methods in Sect. 4, as long as δ is small enough, the regularized maximum of $\varphi_{5.28}+3\epsilon_{5.27}\log\delta$ with $\varphi^i_{5.26}$ provides the required smooth function $\varphi_{5.24}$ on M.

Therefore, we only need to find $\varphi_{5.28}$. By Hironaka's desingularization theorem, there exists a blow-up $\tilde M$ of M obtained by a sequence of blow-ups with smooth centers such that the proper transform $\tilde Y$ of Y is smooth. Without loss of generality, assume that we only need to blow up once because otherwise we just repeat the process. Let π be the projection from $\tilde M$ to M. Let E be the exceptional divisor. Let s be the defining section of E. Let h be any smooth metric on the line bundle $[E]$. Then $\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log|s|_h^2=[E]+\omega_{5.29}$ by the Poincaré-Lelong equation. Further, there exists a constant $C_{5.30}$ such that $\omega_{5.31}=\omega_{5.29}+C_{5.30}\pi^*\chi>0$.

Let $\omega_{2,0}$ be $\omega_{t,0}$ when $t=2$, and let $\omega_{1,0}$ be $\omega_{t,0}$ when $t=1$. Then there exists a constant $\epsilon_{5.32}>0$ such that $\omega_{2,0}-\omega_{1,0}\geq\epsilon_{5.32}\chi$. Further, there exists a smooth function $\varphi_{5.33}$ on M such that $\omega_{1,0}+\sqrt{-1}\partial\bar\partial\varphi_{5.33}\in\Gamma_{\chi,\theta_0,\Theta_0}$ on M. This implies that

$$\begin{aligned}&\int_V\left(\mathrm{Re}\left(\omega_{t,0}-\epsilon_{5.32}\chi+\sqrt{-1}\chi\right)^p-\cot(\theta_0)\mathrm{Im}\left(\omega_{t,0}-\epsilon_{5.32}\chi+\sqrt{-1}\chi\right)^p\right)\\&\quad\geq\int_V\left(\mathrm{Re}\left(\omega_{1,0}+\sqrt{-1}\chi\right)^p-\cot(\theta_0)\mathrm{Im}\left(\omega_{1,0}+\sqrt{-1}\chi\right)^p\right)\geq(n-p)\epsilon_{1.1}\int_V\chi^p\end{aligned}$$

for all $t\geq 2$ and all p-dimensional analytic subvarieties V of M. By choosing $\epsilon_{5.32}$ small enough, using the fact that $\omega_{t,0}$ is bounded with respect to χ for all $t\in[0,2]$, we can also assume that

$$\int_V\left(\mathrm{Re}\left(\omega_{t,0}-\epsilon_{5.32}\chi+\sqrt{-1}\chi\right)^p-\cot(\theta_0)\mathrm{Im}\left(\omega_{t,0}-\epsilon_{5.32}\chi+\sqrt{-1}\chi\right)^p\right)\geq(n-p)\frac{\epsilon_{1.1}}{2}\int_V\chi^p.$$

for all $t\geq 0$ and all p-dimensional analytic subvarieties V of M.

Now we want to find constants $0<\epsilon_{5.34}<1$ and $C_{5.35}$ independent of t and consider the Kähler form $\pi^*\chi+\epsilon_{5.34}\omega_{5.31}$ on $\tilde M$ and the test family

$$\pi^*\omega_{t,0}-\epsilon_{5.32}\pi^*\chi+(t+C_{5.35})\epsilon_{5.34}\omega_{5.31}$$

on $\tilde M$. We know that $\pi(E)$ is smooth. So by the induction hypothesis, as in Sect. 4, we can find a smooth function $\varphi_{5.36}$ on M such that $\omega_{5.36}=\omega_0+\sqrt{-1}\partial\bar\partial\varphi_{5.36}$ satisfies $\omega_{5.36}\in\Gamma_{\chi,\theta_0,\Theta_0}$ on a neighborhood $U_{5.37}$ of $\pi(E)$. By shrinking $U_{5.37}$ and replacing $\epsilon_{5.32}$ if necessary, we can assume that there exists a constant $\epsilon_{5.38}>0$ such that $\omega_{5.36}-\epsilon_{5.32}\chi\in\Gamma_{\chi,\theta_0-\epsilon_{5.38},\Theta_0}$ on $U_{5.37}$. By the compactness of M, there exists a constant $\epsilon_{5.39} > 0$ such that

$$\omega_{1,0} + \sqrt{-1}\partial\bar{\partial}\varphi_{5.33} \in \Gamma_{\chi,\theta_0-\epsilon_{5.39},\Theta_0}$$

on M.

Then we required that $C_{5.35} > \cot\left(\frac{\epsilon_{5.38}}{n}\right)$ and $C_{5.35} > \cot\left(\frac{\epsilon_{5.39}}{n}\right)$. By Lemma 8.2 of [12], when $t \geq 2$,

$$\begin{aligned}
&\operatorname{Im}\left(e^{-\sqrt{-1}\theta_0}\left(\pi^*\omega_{t,5.33} + (t + C_{5.35})\epsilon_{5.34}\omega_{5.31} + \sqrt{-1}\left(\pi^*\chi + \epsilon_{5.34}\omega_{5.31}\right)\right)^q\right)\\
&\quad - \operatorname{Im}\left(e^{-\sqrt{-1}\theta_0}\left(\pi^*\omega_{t,5.33} + \sqrt{-1}\pi^*\chi\right)^q\right)\\
&\quad = \operatorname{Im}\left(e^{-\sqrt{-1}\theta_0}\sum_{k=0}^{q-1}\frac{q!}{k!(q-k)!}\left(\pi^*\omega_{t,5.33} + \sqrt{-1}\pi^*\chi\right)^k \wedge \left(\left(t + C_{5.35} + \sqrt{-1}\right)\epsilon_{5.34}\omega_{5.31}\right)^{q-k}\right)\\
&\quad \leq \operatorname{Im}\left(e^{-\sqrt{-1}\theta_0}\left(t + C_{5.35} + \sqrt{-1}\right)^q \epsilon_{5.34}^q\omega_{5.31}^q\right),
\end{aligned}$$

where $\omega_{t,5.33} = \omega_{t,0} - \epsilon_{5.32}\chi + \sqrt{-1}\partial\bar{\partial}\varphi_{5.33}$. So

$$\begin{aligned}
&\int_V \operatorname{Im}\left(e^{-\sqrt{-1}\theta_0}\left(\pi^*\omega_{t,0} - \epsilon_{5.32}\pi^*\chi + (t + C_{5.35})\epsilon_{5.34}\omega_{5.31} + \sqrt{-1}\left(\pi^*\chi + \epsilon_{5.34}\omega_{5.31}\right)\right)^q\right)\\
&\quad \leq \int_V \operatorname{Im}(e^{-\sqrt{-1}\theta_0}(\pi^*\omega_{t,0} - \epsilon_{5.32}\pi^*\chi + \sqrt{-1}\pi^*\chi)^q)\\
&\quad + \int_V \operatorname{Im}\left(e^{-\sqrt{-1}\theta_0}\left(t + C_{5.35} + \sqrt{-1}\right)^q \epsilon_{5.34}^q\omega_{5.31}^q\right)\\
&\quad \leq -\sin(\theta_0)(n-q)\frac{\epsilon_{1.1}}{2}\int_V \pi^*\chi^q + \operatorname{Im}\left(e^{-\sqrt{-1}\theta_0}\left(t + C_{5.35} + \sqrt{-1}\right)^q\right)\int_V (\epsilon_{5.34}\omega_{5.31})^q\\
&\quad \leq -\epsilon_{5.40}\int_V \left(\pi^*\chi + \epsilon_{5.34}\omega_{5.31}\right)^q
\end{aligned}$$

for any q-dimensional analytic subvariety V of $\tilde{M}$ and a constant $\epsilon_{5.40}$ independent of t and V.

On the other hands, for $t \in [0, 2]$, we get a similar estimate within $U_{5.37}$. As for the set $\tilde{M}\setminus\pi^{-1}(U_{5.37})$, we know that all the forms $\pi^*\omega_{5.36}, \pi^*\chi$, and $\omega_{5.29}$ are bounded using the norm defined by $\pi^*\chi + \epsilon_{5.34}\omega_{5.31}$. Moreover, $\pi^*\chi$ is also bounded below by a positive constant multiple of $\pi^*\chi + \epsilon_{5.34}\omega_{5.31}$. So if $\epsilon_{5.34}$ is small enough, then the Kähler form $\pi^*\chi + \epsilon_{5.34}\omega_{5.31}$ and the test family $\pi^*\omega_{t,0} - \epsilon_{5.32}\pi^*\chi + (t + C_{5.35})\epsilon_{5.34}\omega_{5.31}$ on $\tilde{M}$ satisfy the assumption of Proposition 5.2. Since $\tilde{Y}$ is smooth, by the induction hypothesis and the arguments in Sect. 1, there exists a smooth function $\varphi_{5.41}$ on $\tilde{M}$ such that

$$\pi^*\omega_0 - \epsilon_{5.32}\pi^*\chi + C_{5.35}\epsilon_{5.34}\omega_{5.31} + \sqrt{-1}\partial\bar{\partial}\varphi_{5.41} \in \Gamma_{\pi^*\chi+\epsilon_{5.34}\omega_{5.31},\theta_0,\Theta_0}$$

on a neighborhood $U_{5.42}$ of $\tilde{Y}$. By a similar argument to that in the proof of Lemma 5.12, this implies that

$$\pi^*\omega_0 - \epsilon_{5.32}\pi^*\chi + (C_{5.35} - \cot(\theta_0))\,\epsilon_{5.34}\omega_{5.31} + \sqrt{-1}\partial\bar{\partial}\varphi_{5.41} \in \Gamma_{\pi^*\chi,\theta_0,\Theta_0}$$

on $U_{5.42} \setminus \pi^{-1}(E)$. So by choosing $(C_{5.35} - \cot(\theta_0))\epsilon_{5.34}C_{5.30} < \epsilon_{5.32}$, we see that

$$\pi^*\omega_0 + \sqrt{-1}\partial\bar{\partial}\left(\varphi_{5.41} + (C_{5.35} - \cot(\theta_0))\,\epsilon_{5.34}\frac{\sqrt{-1}}{2\pi}\log|s|_h^2\right) \in \Gamma_{\pi^*\chi,\theta_0,\Theta_0}$$

on $U_{5.42} \setminus \pi^{-1}(E)$. Finally, we choose a large enough constant $C_{5.43}$ and define $\varphi_{5.28}$ as the regularized maximum of $\pi_*(\varphi_{5.41} + (C_{5.35} - \cot(\theta_0))\epsilon_{5.34}\frac{\sqrt{-1}}{2\pi}\log|s|_h^2)$ with $\varphi_{5.36} - C_{5.43}$.

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