CONVERGENCE OF CURVE SHORTENING FLOW TO TRANSLATING SOLITON

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Abstract. This paper concerns with the asymptotic behavior of complete non-compact convex curves embedded in $\mathbb{R}^2$ under the α-curve shortening flow for exponents $\alpha > \frac{1}{2}$. We show that any such curve having in addition its two ends asymptotic to two parallel lines, converges under α-curve shortening flow to the unique translating soliton whose ends are asymptotic to the same parallel lines. This is a new result even in the standard case $\alpha = 1$, and we prove for all exponents up to the critical case $\alpha > \frac{1}{2}$.

1. introduction

Given a positive constant α, we say that a one-parameter family of immersions $X : N \times [0,T] \mapsto \mathbb{R}^2$ is a convex complete solution of the α-curve shortening flow (α-CSF in abbreviation) if each image $M_t \coloneqq X(N \times \{t\})$ is a smooth convex complete curve and the following holds

$$\frac{\partial}{\partial t} X(p,t) = \bar{\kappa}^{\alpha}(p,t)\, n(p,t) \tag{1.1}$$

where $\bar{\kappa}(p,t)$ is the curvature of M_t at $X(p,t)$, and $n(p,t)$ is the unit normal vector pointing the convex hull of M_t. Throughout the paper, if we need a distinction in the parametrizations of the curvature, we use $\bar{\kappa} = \bar{\kappa}(p,t)$ for the parametrization as in (1.1) and we use $\kappa = \kappa(\theta,t)$, where θ denotes the angle between $n(p,t)$ and e_1.

In 1984 [15], Gage showed that the CSF ($\alpha = 1$) makes closed convex curves circular by showing the isoperimetric ratio of the solution curve converges to that of round circle provided the solution exists until its enclosed area becomes zero. Jointly with Hamilton, they established the improved result [16] that the solution exists until it shrinks to a point and smoothly converges to round circle after rescaling. Namely, closed convex solutions converge to *shrinking* solitons.

Regarding complete non-compact solutions, Ecker and Huisken [12] proved that asymptotically conical n-dimensional entire graphs in $\mathbb{R}^{n+1}$ which evolve by the mean curvature flow (a higher dimensional analogue to the CSF) converge to *expanding* solitons after rescaling.

In this paper, we study the convergence of the CSF to *translating* solitons. Our main result states as follows:

Theorem 1.1. *Assume that M_0 is a strictly convex smooth non-compact complete curve embedded in $\mathbb{R}^2$, and that its two ends are asymptotic to two parallel lines. Then, for given $\alpha > \frac{1}{2}$ the unique strictly convex complete solution of the α-CSF converges, as $t \to \infty$, locally smoothly to the unique translating soliton of the α-CSF which is asymptotic to the two lines.*

In the classical case $\alpha = 1$, the translating solitons are the Grim Reaper curves which are homothetic to the curve $\Gamma = \{(x_1, -\log\cos x_1) : x_1 \in (-\frac{\pi}{2}, \frac{\pi}{2})\}$ up to rotation. Thus, the Grim Reaper curves have two ends asymptotic to two parallel lines.

On the other hand, by the result in [5] a convex complete graph M_0 over an open interval $I \subset \mathbb{R}$ (either bounded or unbounded) remains as a convex complete graph M_t over I under the CSF for the all time. Therefore, the initial graph M_0 must be defined over a bounded interval in order to converge to a Grim Reaper curve. Namely, for the convergence to a Grim Reaper curve it is necessary to assume that the two ends of M_0 are asymptotic to two parallel lines.

However, it was revealed by Calabi in [4] that translating solitons to the $\frac{1}{3}$-CSF are the parabola $\Gamma = \{(x_1, x_1^2) : x_1 \in \mathbb{R}\}$ up to affine transforms. Namely, a translating soliton to the $\frac{1}{3}$-CSF is not contained in a strip. Therefore, an initial graph M_0 must be an entire graph to converge to a parabola. Naturally, the two cases $\alpha = 1$ and $\alpha = \frac{1}{3}$ would expect different types of proofs for the convergence to translating solutions.

In this work, we concentrate on the range of exponents $\alpha > \frac{1}{2}$, due to the result of Urbas [21] that translating solitons to the α-Gauss curvature flow (α-GCF) with $\alpha > \frac{1}{2}$ are contained in cylinders while those with $0 < \alpha \leq \frac{1}{2}$ are entire graphs. We recall that the GCF is also a higher dimensional analogue to the CSF.

We treat the α-CSF with $\alpha < 1$ as a fast diffusion type equation and Proposition 3.2, the asymptotic property of the ends of M_t, follows from this consideration. Then, the condition $\alpha > \frac{1}{2}$ yields a sharp lower bound of curvature decay which is needed to prove convergence of solutions to the translating solitons.

However, we will also derive upper bounds for the curvature and its derivatives for $\alpha > \frac{1}{3}$ which are independent from the shape of the ends of M_t. This $\alpha = \frac{1}{3} = \frac{1}{1+2}$ is also a critical exponent which is due to the fact that in this case the equation is invariant under affine transformations. By the work [4] of Calabi, the shrinkers, expanders, and translators to the $\frac{1}{n+2}$-GCF are ellipsoids, hyperboloids, and paraboloids, respectively. Namely, the $\frac{1}{n+2}$-GCF has infinitely many different solitons, but they all are equivalent up to affine transformations.

Recently, Andrews-Guan-Ni [1] showed the convergence of closed solutions of the α-GCF to shrinking solitons for $\alpha > \frac{1}{n+2}$, and Brendle-Choi-Daskalopoulos [3] obtained the uniqueness of closed shrinkers for $\alpha > \frac{1}{n+2}$. In this regard, the upper bounds for the curvature and its derivatives for $\alpha > \frac{1}{3}$ in this paper could be helpful in studying the convergence of entire graph solutions to the translating solitons for $\frac{1}{3} < \alpha \leq \frac{1}{2}$.

Remark 1.2 (*Local convergence*). In Theorem 1.1, the term "locally smoothly converges" indicates that, for instance, if the two ends for the initial curve M_0 are asymptotic to $\{x_1 = -1\}$ and $\{x_1 = 1\}$ then after translating the solution as $\{M_t - h(t)\, e_2\}$ so that it contains the origin, it smoothly converges to the soliton on $[-1+\delta, 1-\delta] \times \mathbb{R}$, for every small $\delta > 0$. For more details, see the theorem 2.3.

Remark 1.3 (Uniqueness of the limit). Given a complete convex CSF solution M_t defined in a slab region, and any sequence $t_i \to +\infty$, then the sequence of flows $M_t^i =: M_{t-t_i} - x_{\text{tip}}(t_i)$, where $x_{\text{tip}}(t)$ denotes the tip of M_{t_i}, sub-converges to an eternal solution. By applying the Harnack inequality one can then show that the limiting eternal solution is a Grim Reaper. See Hamilton [18], Polden [20], Altschuler-Grayson [2]. Our result shows that the limit is *uniquely determined by the asymptotic slab.*

Remark 1.4 (*Translating solitons of $\alpha > 1$ contain flat lines*). Given $\alpha > 1$, the C^1 convex translating solitons have two half lines and the solitons are not of C^∞ class. See [21]. For example, given $\alpha > 1$ there exists a convex even function $f : [-1, 1] \to \mathbb{R}$ such that

- f is smooth strictly convex on $(-1, 1)$, and $|Df|(x_1) \to +\infty$ as $|x_1| \to 1$,
- $\Gamma \coloneqq \{(\pm 1, x_2) : x_2 \geq f(1)\} \cup \{(x_1, f(x_1)) : |x_1| < 1\}$ is a translating soliton to the α-CSF.

In the higher dimensional case of the GCF, the evolution of surfaces with flat sides has been studied as a free-boundary problem which is also motivated from the wearing precess of stones [14, 17, 9, 10]. In particular the works [9, 10], treat the GCF as a slow diffusion of a similar nature as that appearing in the the Porous medium equation. Similarly, the α-CSF with $\alpha > 1$ sufficiently large is a slow diffusion equation. This can be seen from the evolution equation the speed κ^α which given in (2.14). Thus, in this case too one may consider weakly convex initial data with flat lines and study its evolution. However, in this work we consider only strictly convex and complete initial data and we show that the solution converges to a weakly convex C^1 translator with flat lines.

In addition, it was recently discovered in [6] that translating solitons to the GCF in $\mathbb{R}^3$ have flat sides if their asymptotic cylinders at infinity have flat sides. Namely, translating solitons to nonlinear flows may have flat sides, arising from slow diffusion at infinity.

Discussion on the Proof: The key idea of the paper is to utilize the monotonicity of the functional

$$J(t) = \frac{(\alpha+1)^2}{\alpha^2} \int (\kappa^\alpha)_\theta^2 - (\kappa^\alpha)^2 d\theta.$$

Such a functional was used in [11] for the classification of closed convex ancient solutions to the CSF. Note that on a closed convex solution the function κ is 2π-periodic and one can simply obtain $\partial_t J \leq 0$ by integration by parts. However, in our non-compact case boundary terms appear after we integrate by parts (see Proposition 2.1). Heuristically, we have

$$\partial_t J = \frac{2(\alpha+1)^2}{\alpha^2} \Big(-\alpha \int_0^\pi \kappa^{\alpha+1} [(\kappa^\alpha)_{\theta\theta} + \kappa^\alpha]^2 d\theta + \Big((\kappa^\alpha)_\theta (\kappa^\alpha)_t \Big)_{\theta=0}^{\theta=\pi} \Big). \tag{1.2}$$

The most challenging part of our proof is to show that the boundary terms vanish. For that it is crucial to derive local derivative estimates on the speed κ^α in (see Section 3). We then combine these estimates which we then combined with our Hölder estimate for κ^α (see in Section 2). Notice that even if the curvature $\kappa(\cdot, 0)$ of the initial data does not converge to zero at infinity (i.e. as $\theta \to 0$ or $\theta \to \pi$), Theorem 2.9 shows that κ^α decays in a sufficient Hölder norm at the two boundary points after some finite time.

The derivative decay estimates in Section 3 are conducted in Euclidean space by using an extrinsic cut-off function up to the critical exponent $\alpha > \frac{1}{3}$. Note that the local estimate does not depend on the global structure, asymptotic lines. Hence, the local estimates are naturally obtained up to $\alpha > \frac{1}{3}$. In the critical case $\alpha = \frac{1}{3}$, one would need to introduce an affine-invariant cut-off function.

To apply the derivative estimate with the arclength parameter s, we have to use the change of variable $\partial_s = \kappa\, \partial_\theta$. Therefore, we need to derive a lower bound for κ. We do so by considering the flow as a fast diffusion equation. Then, for $\alpha > \frac{1}{2}$ we obtain the required lower bound in Theorem 3.5.

In the last section, we show (by utilizing our estimates in previous sections) that $J(t)$ converges to zero as time tends to infinity on each compact interval in $(0, \pi)$. Thus, $\kappa^\alpha_{\theta\theta} - \kappa^\alpha$ converges to zero in L^2-sense (see in Lemma 4.4). We then conclude the convergence of $\kappa^\alpha(\theta, t)$ to $c \sin\theta$ in the C^∞_{loc}-topology, for some $c > 0$ depending on the width of the smallest slab region which encloses our solution. This yields Theorem 2.3. Finally, Theorem 2.3 combined with Proposition 2.1 implies our main result Theorem 1.1.

2. Preliminaries and Curvature Estimates

We begin by defining the following notation. We denote by $\mathcal{N}_t$ the *normal image* of M_t at a given instant t, namely:

$$\mathcal{N}_t := \{n \in S^1 : n \text{ is an inward unit normal vector to } M_t\} \tag{2.1}$$

and denote by $S(n, t) : \mathcal{N}_t \to \mathbb{R}$ the *support function*

$$S(n, t) := \sup_{X \in M_t} \langle -n, X\rangle. \tag{2.2}$$

In the next Proposition we gather some basic properties of any solution M_t to the α-CSF which satisfies the assumptions of Theorem 1.1 and sketch its proof for the reader's convenience.

Proposition 2.1. *Assume that M_0 is a strictly convex smooth non-compact complete curve embedded in $\mathbb{R}^2$ such that its two ends are asymptotic to the two lines $\{x_1 = \pm 1\}$, as $x_2 \to +\infty$. Then, the α-CSF ($\alpha > 0$) has a unique convex complete solution M_t existing for all time $t \in [0, +\infty)$. Moreover, each M_t is a graph over $(-1, 1)$ with $\mathcal{N}_t = \{\langle n, e_2\rangle > 0\}$.*

Proof. First, by the strict convexity and the completeness of M_0, N_0 is open in S^1 and we easily see that M_0 is a convex graph over $(-1, 1)$.

Next, we claim that a complete convex solution M_t (if it exists) remains as a graph for a short time $t \in [0, T]$. We consider closed circular solutions

$$\Gamma^h_t = \{x \in \mathbb{R}^2 : (1/2)^{\alpha+1} = |x - (0, h)|^{\alpha+1} + (\alpha + 1)t\} \text{ for } h \in \mathbb{R}.$$

Since the convex hull of M_0 contains Γ^h_0 for $h \gg 1$, the convex hull of M_t contains Γ^h_t for $h \gg 1$. Let T be the singular time of Γ^h_t. Then, the convex hull of M_t contains Γ^h_t for $t \in [0, T]$ for $h \gg 1$. This implies $N_t \cap \{\langle n, e_2\rangle < 0\} = \phi$ for $t \in [0, T]$. M_t is strictly convex by the strong maximum principle and hence again N_t is open in S^1. Therefore, $N_t \cap \{\langle n, e_2\rangle \le 0\} = \phi$. i.e. it is a graph.

The all time existence of complete convex graph solutions is given in [5]. Moreover, it was also shown in [5] that the domain of every graphical solution is fixed over time. Therefore, each M_t is a convex complete graph over $(-1, 1)$. Since M_t is a complete convex graph over $(-1, 1)$, it follows that $N_t = \{\langle n, e_2\rangle > 0\}$.

Finally, let us sketch the proof of the uniqueness assertion of the proposition. Let M_t and $\bar{M}_t$ be two solutions with the same initial data $M_0 = \bar{M}_0$. We may assume that the convex hull of M_0 contains the origin. Consider, for $\epsilon \in (0, 1)$ the rescaled solution $\hat{M}_t := (1 - \epsilon)\bar{M}_{(1-\epsilon)^{-(1+\alpha)}t}$. Then, each $\hat{M}_t$ is a graph over $(-1 + \epsilon, 1 - \epsilon)$ and the convex hull of M_0 contains $\hat{M}_0$. Thus, the convex hull of M_t contains $\hat{M}_t$ by the comparison principle. Passing $\epsilon \downarrow 0$, we conclude that the convex hull of M_t contains $\bar{M}_t$. Similarly, the convex hull of $\bar{M}_t$ contains M_t, yielding that the solution is unique. □

Lemma 2.2. *Assume that M is a complete graph of a smooth strictly convex function defined on $(-1,1)$ which implies that for any unit vector n satisfying $\langle n, e_2\rangle > 0$, there exists a point $X(n) \in M$ such that n is the inner unit normal at $X(n) \in M$. Then, we have $S(n) := \sup_{X\in M}\langle -n, X\rangle = \langle -n, X(n)\rangle$ and*

$$\lim_{n\to\pm e_1} S(n) = 1.$$

Proof. We will only show that $\lim_{n\to e_1} S(n) = 1$, as the other limit follows similarly. Let $X(n) := (x_1(n), x_2(n))$. If $x_1(n)$ is sufficiently close to -1, we have $x_2(n) > 0$, $0 < \langle n, e_1\rangle < 1$. Thus, since $\langle n, e_2\rangle > 0$ we have

$$\limsup_{n\to e_1} S(n) = \limsup_{n\to e_1}\langle -n, X(n)\rangle = \limsup_{n\to e_1} -x_1(n)\langle n, e_1\rangle - x_2(n)\langle n, e_2\rangle \le 1.$$

Now, we assume that there exists a sequence of unit vectors n_i such that $\langle n_i, e_2\rangle > 0$, $\lim_{i\to\infty} n_i = e_1$, and $S(n_i) \le 1-\epsilon$ for some $\epsilon > 0$. We denote by L_i the tangent line to M at $X(n_i)$. We observe that there exists the closed half plane $E_i \subset \mathbb{R}^2$ such that $\partial E_i \parallel L_i$, $L_i \subset E_i$, and $-(1-\epsilon)\, n_i \in \partial E_i$. Then, we have $M \subset E_i$ and $\overline{\lim}_{i\to\infty} E_i = \{x_1 \ge -1+\epsilon\}$. To be more precise, for every $X' = (x_1', x_2') \in \mathbb{R}^2$ with $x_1' < -1+\epsilon$, $X' \notin E_i$ for large i. This contradicts the condition that M is a graph over $(-1,1)$. □

After scaling and rotating our initial data M_0, Proposition 2.1 implies that we only need to prove the following result instead of Theorem 1.1.

Theorem 2.3 (Local convergence to solitons)**.** *Let M_0 be a strictly convex smooth non-compact complete curve embedded in $\mathbb{R}^2$ such that its two ends are asymptotic to the two lines $\{x_1 = \pm 1\}$, as $x_2 \to +\infty$. For any $\alpha > 1/2$, let M_t, $t \in (0,+\infty)$ be the unique solution of the α-CSF with the initial data M_0 and denote by $f : (-1,1)\times[0,+\infty) \to \mathbb{R}$ the graphical parametrization of M_t.*

Then, the gradient $f_x(x,t)$ converges to $f_\alpha'(x)$ in $C^\infty_{loc}[(-1,1)]$ as $t \to +\infty$, where the graph of the function $f_\alpha(x) = \int_0^x f_\alpha'(s)\,ds$ is the translating soliton to the α-CSF moving in e_2 direction whose two ends are asymptotic to $\{x_1 = \pm 1\}$.

2.1. **Parametrization of a convex curve by its normal vector.** Let $M \subset \mathbb{R}^2$ be a strictly convex C^2 curve which is the boundary of a convex body $\hat{M} \subset \mathbb{R}^2$. We denote by n the normal vector at $X = (x_1, x_2) \in M$ and $\theta \in [0, 2\pi)$ the angle between n and e_1. This parametrization was used in Gage-Hamilton [16]. Note that a *convex* curve is completely determined by the curvature function parametrized by θ, namely $\kappa(\theta)$, up to a translation.

Recall the well known facts that the arc-length parameter s satisfies $\kappa = \dfrac{\partial\theta}{\partial s}$, thus

$$\frac{\partial X}{\partial\theta} = \frac{\partial X}{\partial s}\frac{\partial s}{\partial\theta} = \frac{1}{\kappa}\frac{\partial X}{\partial s}$$

and

$$\frac{\partial X}{\partial s} = \Big(\cos(\theta - \frac{\pi}{2}), \sin(\theta - \frac{\pi}{2})\Big) = (\sin\theta, -\cos\theta)$$

yielding

$$X(\theta_1) - X(\theta_0) = \left(\int_{\theta_0}^{\theta_1} \frac{\sin\theta}{\kappa(\theta)} d\theta, \int_{\theta_0}^{\theta_1} -\frac{\cos\theta}{\kappa(\theta)} d\theta \right). \tag{2.3}$$

As mentioned earlier, J. Urbas [21] showed that for exponents $\alpha > 1/2$, all the translators of the α-GCF (which includes the $n = 1$ case of the α-CSF) are enclosed inside a cylinder. Moreover, M_0 is a translating soliton of the α-CSF moving in e_2 direction with the speed $c > 0$ if and only if $\kappa^\alpha = \langle n, c\, e_2 \rangle = c \sin\theta$. Let us observe next that this fact and (2.3) give a short proof of Urbas's result when $n = 1$.

Proposition 2.4. *For $\alpha > 1/2$, there exists a strictly convex function $f_\alpha : (-1, 1) \to \mathbb{R}$ such that $\lim_{|x|\to 1} |f'_\alpha(x)| = +\infty$ and the graph of f_α is a translating soliton to the α-CSF. f_α is unique up to addition by a constant. Moreover,*

$$\lim_{|x|\to 1} |f_\alpha(x)| = +\infty \quad \text{if } \alpha \in (\frac{1}{2}, 1], \qquad \text{and} \quad \lim_{|x|\to 1} |f_\alpha(x)| = C < +\infty \quad \text{if } \alpha > 1.$$

For $\alpha \in (0, 1/2]$, translating solitons are entire graphs on $\mathbb{R}$.

Proof. Given $\alpha > \frac{1}{2}$, we define the positive finite constant $m(\alpha)$ by

$$m(\alpha) := \Big(\int_0^\pi \frac{\sin y}{\sin^{1/\alpha} y} dy \Big)^\alpha. \tag{2.4}$$

If we fix a point $X_\alpha(\frac{\pi}{2}) = (x^1_\alpha, x^2_\alpha)(\frac{\pi}{2}) = (0, 0)$, the equation $\kappa^\alpha(\theta) = m(\alpha)\sin\theta$ defines a translating soliton of the α-CSF by (2.3). Namely, $x^i_\alpha : (0, \pi) \to \mathbb{R}$ for $i = 1, 2$ by

$$x^1_\alpha(\theta) = m(\alpha)^{-\frac{1}{\alpha}} \int_{\pi/2}^\theta (\sin y)^{1-\frac{1}{\alpha}} dy, \qquad x^2_\alpha(\theta) = -m(\alpha)^{-\frac{1}{\alpha}} \int_{\pi/2}^\theta (\sin y)^{-\frac{1}{\alpha}} \cos y\, dy.$$

Note that we have $x^2_\alpha \geq 0$, $x^1_\alpha \in (-1, 1)$, $\lim_{\theta\to 0} x^1_\alpha(\theta) = -1$, $\lim_{\theta\to\pi} x^1_\alpha(\theta) = 1$. The graph of (x^1_α, x^2_α) could be written as a graph of a function f_α on $(-1, 1)$. All the other properties of f_α can be checked directly from x^i_α. Note the the speed $m(\alpha)$ is fixed, as we have fixed the size of the interval $I := (-1, 1)$ over which our translator f_α is defined. For $\alpha \in (0, 1/2]$, $m(\alpha) = \infty$ implies every soliton has to be an entire graph. □

2.2. **Evolution equations.** We first recall well-known equations for the normal vector $n(p, t)$, the speed $\bar\kappa^\alpha(p, t)$ and the extrinsic distance $|X(p, t)|$, where all are considered with respect to the geometric parametrization which defines the flow in (1.1), in particular ∂_s and ∂_{ss} denote as usual the first and second order derivatives with respect to arc-length parameter s. The base point of this arc-length could be any point, but we choose an orientation of this parameter s in such a way that $\frac{\partial\theta}{\partial s} = \kappa$.

Evolution of the normal:

$$\partial_t n = -\nabla \bar\kappa^\alpha = -\alpha\, \kappa^{\alpha-1} \bar\kappa_s\, \partial_s \quad \text{or equivalently} \quad \partial_t \theta = \alpha\, \kappa^{\alpha-1} \bar\kappa_s. \tag{2.5}$$

Evolution of the speed $\bar\kappa^\alpha$:

$$(\partial_t - \alpha \bar\kappa^{\alpha-1} \partial_{ss}) \bar\kappa^\alpha = \alpha\, \bar\kappa^{2\alpha+1}. \tag{2.6}$$

Evolution of the curvature $\bar\kappa$:

$$\bar\kappa_t = \partial_{ss}\bar\kappa^\alpha + \bar\kappa^{\alpha+2}. \tag{2.7}$$

Evolution of the extrinsic distance:

$$(\partial_t - \alpha\bar\kappa^{\alpha-1}\partial_{ss})|X|^2 = 2\,\alpha\bar\kappa^{\alpha-1}(-1 + (\alpha^{-1}-1)\langle X, n\rangle\,\bar\kappa). \tag{2.8}$$

Next, we will compute the evolution of the derivatives of the speed κ^α by differentiating equation (2.6). Before this, let us note that the parameter s is not a fixed coordinate and changes with respect to time. In fact,

$$\frac{\partial}{\partial s} = \frac{1}{\sqrt{g_{11}}}\frac{\partial}{\partial x_1}, \qquad \frac{\partial^2}{\partial t\partial s} = \frac{\partial^2}{\partial s\partial t} - \frac{\partial_t\, g_{11}}{2g_{11}\sqrt{g_{11}}}\frac{\partial}{\partial x_1}$$

and hence the commutator satisfies

$$\frac{\partial^2}{\partial t\partial s} = \frac{\partial^2}{\partial s\partial t} + \bar\kappa^{\alpha+1}\frac{\partial}{\partial s}. \tag{2.9}$$

To simplify the notation we set $u := \bar\kappa^\alpha$, and express equation equation (2.6) as

$$u_t = \alpha\, u^{1-\frac{1}{\alpha}} u_{ss} + \alpha u^{2+\frac{1}{\alpha}} = \alpha u^{1-\frac{1}{\alpha}}\left(u_{ss} + u^{1+\frac{2}{\alpha}}\right) \tag{2.10}$$

Differentiating (2.10) while using the commutator identity (2.9), we obtain the following evolution equations for the higher order derivatives of u:

$$\partial_t u_s = \partial_s u_t + u^{1+\frac{1}{\alpha}} u_s = \alpha u^{1-\frac{1}{\alpha}}\partial^2_{ss} u_s + (\alpha-1)u^{-\frac{1}{\alpha}} u_s u_{ss} + 2(\alpha+1)\, u_s u^{1+\frac{1}{\alpha}} \tag{2.11}$$

and

$$\begin{aligned} \partial_t u_{ss} &= \partial_t\partial_s u_s = \partial_s\partial_t u_s + u^{1+\frac{1}{\alpha}} u_{ss} \\ &= \alpha u^{1-\frac{1}{\alpha}}(u_{ss})_{ss} + 2(\alpha-1)u^{-\frac{1}{\alpha}} u_s u_{sss} + (\alpha-1)u^{-\frac{1}{\alpha}} u_{ss}^2 + (\frac{1}{\alpha}-1)u^{-1-\frac{1}{\alpha}} u_s^2 u_{ss} \\ &\quad + 2(\alpha+1)(1+\frac{1}{\alpha})u^{\frac{1}{\alpha}} u_s^2 + (2\alpha+3)u^{1+\frac{1}{\alpha}} u_{ss}. \end{aligned} \tag{2.12}$$

For a smooth strictly convex solution, $\theta(p,t)$ is a smooth invertible function. Thus, for a fixed θ' in the image of $\theta(p,t)$ for a time interval $t \in I$, we may define a curve $\gamma_{\theta'}(t)$ for $t \in I$ so that $\theta(\gamma_{\theta'}(t),t) = \theta'$. Let us parametrize the curvature $\bar\kappa$ by (θ,t) as follows

$$\kappa(\theta,t) = \bar\kappa(\gamma_\theta(t),t).$$

We will often abuse the notation and continue to use $\kappa(n,t) = \kappa(\theta,t)$, for $n = (\cos\theta, \sin\theta)$. Let us next derive the evolution equation of $\kappa(\theta,t)$. Note that

$$\partial_t\kappa = \partial_t\bar\kappa + \partial_s\bar\kappa\,\dot\gamma_\theta, \qquad \text{where } \dot\gamma_\theta = \frac{\partial}{\partial t}(s(\gamma_\theta(t))).$$

On the other hand, since $\theta(\gamma_{\theta'}(t),t)$ is constant in t we have

$$0 = \frac{d}{dt}\theta(\gamma_{\theta'}(t),t) = \bar\kappa(\gamma_{\theta'}(t),t)\,\dot\gamma_{\theta'} + \partial_s\bar\kappa^\alpha(\gamma_{\theta'}(t),t) \quad \text{thus} \quad \dot\gamma_\theta = -\alpha\bar\kappa_s\bar\kappa^{\alpha-2}.$$

Hence

$$\partial_t\kappa = \partial_t\bar\kappa - \alpha\bar\kappa^{\alpha-2}\bar\kappa_s^2 \tag{2.13}$$

and use $\partial_s = \kappa\,\partial_\theta$ to conclude that

$$\begin{aligned}\partial_t \kappa &= (\kappa^\alpha)_{ss} + \kappa^{\alpha+2} - \alpha\kappa^{\alpha-2}\kappa_s^2 \\ &= \alpha\kappa^{\alpha+1}\kappa_{\theta\theta} + \alpha(\alpha-1)\kappa^\alpha\kappa_\theta^2 + \kappa^{\alpha+2} \\ &= \kappa^2\big((\kappa^\alpha)_{\theta\theta} + \kappa^\alpha\big)\end{aligned}$$

which also implies the equation

$$\partial_t(\kappa^\alpha) = \alpha(\kappa^\alpha)^{1+\frac{1}{\alpha}}\big((\kappa^\alpha)_{\theta\theta} + \kappa^\alpha\big). \tag{2.14}$$

The derivation of equation (2.14) is well known, however we included it here for the reader's convenience. Sometimes, it is useful to define $p := \kappa^{\alpha+1}$ which we call the *pressure function* following the terminology of the porous medium and fast-diffusion equations. The evolution of $p(\theta,t)$ is given by

$$\partial_t p = \alpha\, p p_{\theta\theta} - \frac{\alpha}{\alpha+1} p_\theta^2 + (\alpha+1)\, p^2. \tag{2.15}$$

2.3. **Harnack Estimates.** We need a following pointwise Harnack estimate in (θ,t) variables derived from Li-Yau-Hamilton differential Harnack estimate which appears in [18] and [7] for the mean curvature flow and the α-Gauss curvature flow, respectively.

Proposition 2.5 (Harnack Estimate)**.** *Let M_t be a smoothly strictly convex solution of the α-CSF. Then, the curvature $\kappa(\theta,t)$ satisfies*

$$\kappa_t \geq -\frac{1}{\alpha+1}\frac{\kappa}{t}$$

implying for $0 < t_1 < t_2$ the inequality

$$\kappa(\theta,t_2) \geq \Big(\frac{t_1}{t_2}\Big)^{\frac{1}{\alpha+1}}\kappa(\theta,t_1).$$

Proof. From 478 page in [7], for $\bar\kappa(p,t)$,

$$\alpha\frac{\bar\kappa_t}{\bar\kappa} - \alpha^2\frac{\bar\kappa_s^2}{\bar\kappa^2}\frac{\bar\kappa^\alpha}{\bar\kappa} \geq -\frac{1}{1+\alpha^{-1}}\frac{1}{t}.$$

Since

$$\partial_t\kappa = \partial_t\bar\kappa - \alpha\bar\kappa^{\alpha-2}\bar\kappa_s^2$$

this directly implies the proposition. □

2.4. **Curvature Upper and Lower bounds.** The goal of this section is to prove Proposition 2.7, which gives global upper bounds on the speed κ^α for $t>0$ and local (in θ) lower bounds on the speed κ^α for large times. We first show a simple lemma which says that the support functions of convex surfaces are ordered if one surface contains the other.

Lemma 2.6. *Suppose M_1 and M_2 are convex hypersurfaces in $\mathbb{R}^{n+1}$ and the convex hull of M_1 contains M_2. Then, their support functions $S_i(n) = \sup_{x\in M_i}\langle -n, x\rangle$ satisfy $S_1(n) \geq S_2(n)$.*

Proof. We denote by $E_i \subset \mathbb{R}^3$ the convex hull of M_i. Then, we have $S_0 = \sup\limits_{x\in M_i}\langle -n,x\rangle = \sup\limits_{x\in E_i}\langle -n,x\rangle$ by the convexity. Hence, $E_2 \subset E_1$ implies the desired result. □

Proposition 2.7. *Let M_t be a solution of the α-CSF as in Proposition 2.1. Then, given $t_0 > 0$, we have*

$$\lim_{n\to\pm e_1}\sup_{s\in[t,t+3]}\kappa^\alpha(n,s)=0 \qquad \text{and} \qquad \kappa^\alpha(n,t)\le C$$

for all $n\in\mathcal{N}=\{\langle n,e_2\rangle>0\}$ and $t\ge t_0$, where the constant C depends on t_0 and M_0. In addition, for each $\delta\in(0,\frac{1}{10})$, there is a large $T>0$ and $c(\delta)>0$ such that

$$\kappa^\alpha(n,t)\ge c(\delta)$$

whenever $\langle n,e_2\rangle\ge\delta$ and $t\ge T$. The constants T and $c(\delta)$ may depend on M_0 and δ.

Proof. We begin by observing that the support function $S(n,t)$ of a solution M_t of the α-CSF (defined by (2.2)) satisfies $\partial_t S(n,t)=-\kappa^\alpha(n,t)$.

Therefore, the Harnack inequality 2.5 and the above observation yield

$$c(t_0)\,\kappa^\alpha(n,\tau)\le\int_\tau^{\tau+1}\kappa^\alpha(n,s)\,ds=S(n,\tau)-S(n,\tau+1)\le S(n,t)-S(n,t+4),$$

for $\tau\in[t,t+3]$ and $t\ge t_0>0$. Since $\lim\limits_{n\to\pm e_1}S(n,t)=\lim\limits_{n\to\pm e_1}S(n,t+4)=1$ by Lemma 2.2, we have the first desired result

$$\lim_{n\to\pm 1}\sup_{\tau\in[t,t+3]}\kappa^\alpha(n,\tau)=0.$$

Given $\alpha\in(\frac{1}{2},1]$, we denote by M_α the translator $M_\alpha=\{(x,f_\alpha(x)):|x|<1\}$, where f_α is given in Proposition 2.4. For $\alpha\in(1,\infty]$, we define M_α by

$$M_\alpha=\{(x_1,f_\alpha(x_1)):|x_1|<1\}\cup\{(\pm1,x_2):x_2\ge\lim_{|y|\to1}f_\alpha(y)\}.$$

Let us fix a small $\epsilon_0\in(0,1/10)$. Then depending on M_0, there is $L>0$ so that the convex hull of

$$\hat{M}_\alpha:=\frac{1}{(1-\epsilon_0)^{1/\alpha}}\,M_\alpha-Le_2$$

contains initial surface M_0 and

$$\bar{M}_\alpha:=\frac{1}{(1+\epsilon_0)^{1/\alpha}}\,M_\alpha+Le_2$$

is contained in the convex hull of M_0. Then, $\hat{M}^\alpha_t:=\hat{M}_\alpha+(1-\epsilon_0)\,mt\,e_2$ and $\bar{M}^\alpha_t:=\bar{M}^\alpha+(1+\epsilon_0)\,mt\,e_2$ are solutions of the α-CSF, where $m=m(\alpha)$ is the positive constant given in (2.4).

Let us denote $\hat{S}_\alpha$ and $\bar{S}_\alpha$ by the support functions of the outer barrier $\hat{M}^\alpha_t$ and the inner barrier $\bar{M}^\alpha_t$, respectively. Thus $\partial_t\bar{S}^\alpha=-(1+\epsilon_0)m\,\langle n,e_2\rangle$, and $\partial_t\hat{S}^\alpha=-(1-\epsilon_0)m\,\langle n,e_2\rangle$. Moreover, if $K=\sup\{S(n,0):\langle n,e_2\rangle>0\}$, we have

$$0<\hat{S}^\alpha(n,t)-\bar{S}^\alpha(n,t)\le 2(L+\epsilon_0\,mt)\langle n,e_2\rangle+\Big(\frac{1}{(1-\epsilon_0)^{1/\alpha}}-\frac{1}{(1+\epsilon_0)^{1/\alpha}}\Big)K \tag{2.16}$$

for all $(n,t)\in\mathcal{N}\times(0,\infty)$. Let us set $M:=\Big(\frac{1}{(1-\epsilon_0)^{1/\alpha}}-\frac{1}{(1+\epsilon_0)^{1/\alpha}}\Big)$ from now on.

Next, by the comparison principle and Lemma 2.6, $\bar{S}^\alpha \le S \le \hat{S}^\alpha$ for all $(n,t) \in \mathcal{N} \times (0,\infty)$. Now, if $\kappa^\alpha(n_0,t_0) = C$ then by the Harnack estimate, $\kappa^\alpha(n_0,t) \ge \eta C$ for $t \in [t_0, 4t_0]$ and some $\eta = \eta(\alpha) \in (0,1)$. By (2.16),

$$0 \le S - \bar{S}^\alpha \le \hat{S}^\alpha - \bar{S}^\alpha \le 2(L + \epsilon_0\, mt) + MK \tag{2.17}$$

on $(n,t) \in \mathcal{N} \times (0,\infty)$ and hence using that $\partial_t S = -\kappa^\alpha$, we obtain

$$\begin{aligned} 0 \le S(n_0,4t_0) - \bar{S}^\alpha(n_0,4t_0) &= S(n_0,t_0) - \bar{S}^\alpha(n_0,t_0) + \int_{t_0}^{4t_0} \partial_t (S - \bar{S}^\alpha)(n_0,t)dt \\ &\le 2(L + \epsilon_0\, mt_0) + MK + \big(-\eta C + (1+\epsilon_0)\, m\big)\, 3t_0. \end{aligned} \tag{2.18}$$

Now we observe that there is a constant $C(t_0)$ such that $C \ge C_0(t_0)$ makes last line negative. i.e. contradiction. It is also clear that such a $C(t_0)$ can be made uniformly bounded as $t_0 \to \infty$. This proves the uniform curvature upper bound

$$\kappa^\alpha(n,t) \le C(t_0, M_0).$$

We suppose next that $\kappa^\alpha(n_0,t_0) = c$, for some $\langle n_0, e_2\rangle \ge \delta$ with $\delta \in (0, \frac{1}{10})$. By the Harnack estimate, $\kappa^\alpha(n_0,t) \le c\,\delta^{-\frac{\alpha}{1+\alpha}}$ for $t \in [\delta t_0, t_0]$. Similar computation yields

$$\begin{aligned} 0 \le \hat{S}^\alpha(n_0,t_0) - S(n_0,t_0) &= \hat{S}^\alpha(n_0,\delta t_0) - S(n_0,\delta t_0) + \int_{\delta t_0}^{t_0} \partial_t(\hat{S}^\alpha - S)(n_0,t)dt \\ &\le 2(L + \epsilon_0 m\delta t_0) + MK + \Big(c\,\delta^{-\frac{\alpha}{1+\alpha}} t_0 - (1-\epsilon_0)m\,\langle n_0, e_2\rangle (1-\delta)t_0\Big) \\ \text{(since } 0<\epsilon_0,\ \delta < 1/10) \quad &\le 2L + \frac{1}{5}\delta m t_0 + MK + \Big(c\,\delta^{-\frac{\alpha}{1+\alpha}} - \frac{81}{100} m\delta\Big)t_0 \\ &\le 2L + MK + \Big(c\,\delta^{-\frac{\alpha}{1+\alpha}} - \frac{1}{2} m\delta\Big)t_0. \end{aligned}$$

Now, we set $T = 4(m\delta)^{-1}(2L + MK + 1)$. Then, for $t_0 \ge T$ we must have

$$c\,\delta^{-\frac{\alpha}{1+\alpha}} \ge \frac{1}{4} m\,\delta$$

in order to satisfy the inequality above. We conclude that

$$\kappa^\alpha(n_0,t_0) = c \ge \frac{1}{4}\, m\, \delta^{1+\frac{\alpha}{1+\alpha}} := c(\delta)$$

completing the proof of the last claim of our proposition. □

2.5. **Barrier Construction.** Based on our uniform curvature bound given in Proposition 2.7 and the fact that $\lim_{\theta \to 0} \kappa = 0$, the following barrier shows that the modulus of continuity of $k(\theta,t)$ at $\theta = 0$ is $\kappa(\theta,t) = O(\kappa^{1-\epsilon})$, for every $\epsilon > 0$ and $t \ge t_1 \gg 1$.

Lemma 2.8. *For every $t_0 > 0$ with $t_0 < \min(3, \frac{6}{1+\frac{1}{\alpha}})$, there is $A_0 > 0$ such that for every $A > A_0$, the function defined by*

$$h_\delta := A \sin^{t/3}\left(\frac{3\theta}{t}\right) + \delta \qquad \text{on } \theta \in \Big(0, \frac{t\pi}{6}\Big),\ t \in (0, t_0]$$

is a viscosity supersolution of (2.14) for all $0 < \delta \le \delta_0(t_0, A)$.

Proof. Set

$$w(\theta,t) = A\,\varphi(\theta,t)^{t/3} \qquad \text{where} \quad \varphi(\theta,t) := \sin(\frac{3\theta}{t}).$$

Then, for $\theta \in (0, \frac{\pi}{6}t)$ and $0 < t \le t_0$, we compute

$$w_\theta = A\,\varphi^{t/3-1}\cos(3\theta t^{-1})$$

and

$$\begin{aligned} w_{\theta\theta} &= A\,(1-3/t)\,\varphi^{t/3-2}\,\cos^2(3\theta t^{-1}) - A\,(3/t)\,\varphi^{t/3} \\ &= A\,(1-3/t)\,\varphi^{t/3-2} - A\,\varphi^{t/3} = A\,(1-3/t)\,\varphi^{t/3-2} - w. \end{aligned}$$

Therefore,

$$\alpha\, w^{1+\frac{1}{\alpha}}\big(w_{\theta\theta}+w\big) = \alpha\, A^{2+\frac{1}{\alpha}}\,(1-3/t)\varphi^{\frac{2\alpha+1}{3\alpha}t-2}.$$

On the other hand, expressing $w/A = \exp\big((t/3)\log\varphi\big)$, we have

$$\begin{aligned} \frac{w_t}{A} &= \frac{1}{3}\varphi^{t/3}\Big(-(3\theta t^{-1})\cot(3\theta t^{-1}) + \log\varphi\Big) \\ &\ge -\frac{1}{3}\varphi^{t/3} + \frac{1}{3}\varphi^{\frac{2\alpha+1}{3\alpha}t-2}\varphi^{2-\frac{1+\alpha}{3\alpha}t}\log\varphi \\ &\ge -\frac{1}{3}\varphi^{\frac{2\alpha+1}{3\alpha}t-2} - \frac{1}{3}\varphi^{\frac{2\alpha+1}{3\alpha}t-2}\frac{3\alpha}{(6\alpha-(1+\alpha)t)e} \\ &\ge -\frac{1}{3}\Big(1+\frac{3}{(6-(1+\frac{1}{\alpha})t)e}\Big)\,\varphi^{\frac{2\alpha+1}{3\alpha}t-2}. \end{aligned}$$

Combining the two inequalities, yields

$$w_t - \alpha\, w^{1+\frac{1}{\alpha}}(w_{\theta\theta}+w) \ge \frac{A}{3}\Big[\Big(\frac{3}{t}-1\Big)3\alpha A^{1+\frac{1}{\alpha}} - 1 - \frac{3}{(6-(1+\frac{1}{\alpha})t)e}\Big)\varphi^{\frac{2\alpha+1}{3\alpha}t-2}.$$

Hence, for $t_0 < \min(3, \dfrac{6}{1+\frac{1}{\alpha}})$, there is $\epsilon = \epsilon(t_0,\alpha) > 0$ such that

$$w_t - \alpha w^{1+\frac{1}{\alpha}}(w_{\theta\theta}+w) \ge \frac{A}{3}\Big[\epsilon A^{1+\frac{1}{\alpha}} - \frac{1}{\epsilon}\Big)\varphi^{\frac{2\alpha+1}{3\alpha}t-2}.$$

This proves there exists $A_0(t_0,\alpha) > 0$ and $\epsilon_0(t_0,\alpha) > 0$ such that if $A \ge A_0$,

$$w_t - \alpha w^{1+\frac{1}{\alpha}}(w_{\theta\theta}+w) \ge \epsilon_0\varphi^{\frac{2\alpha+1}{3\alpha}t-2} \ge 0 \tag{2.19}$$

holds on $\theta \in (0, \frac{\pi t}{6})$ and $t \in (0, t_0)$.

For the next step, we set $h_\delta := w + \delta$ for a small constant $\delta > 0$ and to simplify the notation we drop the index δ from h for the rest of the proof, denoting $h := h_\delta$. Then for $\theta \in (0, \frac{\pi}{6}t)$ and $0 < t \le t_0$, we compute

$$h_t - \alpha h^{1+\frac{1}{\alpha}}(h_{\theta\theta}+h) = (w_t - \alpha w^{1+\frac{1}{\alpha}}(w_{\theta\theta}+w)) - \alpha(h^{1+\frac{1}{\alpha}} - w^{1+\frac{1}{\alpha}})\,w_{\theta\theta} - \alpha(h^{2+\frac{1}{\alpha}} - w^{2+\frac{1}{\alpha}}).$$

Observe that, by Taylor's Theorem, we have

$$\begin{aligned} -\alpha(h^{1+\frac{1}{\alpha}} - w^{1+\frac{1}{\alpha}})\,w_{\theta\theta} - \alpha(h^{2+\frac{1}{\alpha}} - w^{2+\frac{1}{\alpha}}) &= -\alpha\Big(\delta(1+\frac{1}{\alpha})\bar{w}^{\frac{1}{\alpha}}w_{\theta\theta} + \delta(2+\frac{1}{\alpha})\hat{w}^{1+\frac{1}{\alpha}}\Big) \\ &\ge -\delta(\alpha+1)w^{\frac{1}{\alpha}}w_{\theta\theta} - \delta(2\alpha+1)(w+\delta)^{1+\frac{1}{\alpha}} \end{aligned}$$

where we used that $w \le \bar w, \hat w \le w+\delta$ and $w_{\theta\theta} \le 0$, $w \ge 0$. Hence, using (2.19) we obtain

$$\begin{aligned} h_t - \alpha h^{1+\frac1\alpha}(h_{\theta\theta}+h) &= \big(w_t - \alpha w^{1+\frac1\alpha}(w_{\theta\theta}+w)\big) - \alpha(h^{1+\frac1\alpha} - w^{1+\frac1\alpha})w_{\theta\theta} - \alpha(h^{2+\frac1\alpha} - w^{2+\frac1\alpha}) \\ &\ge \epsilon_0 \varphi^{\frac{2\alpha+1}{3\alpha}t-2} - \delta(\alpha+1)w^{\frac1\alpha}w_{\theta\theta} - \delta(2\alpha+1)(w+\delta)^{1+\frac1\alpha} \\ &\ge \big(\epsilon_0\varphi^{\frac{2\alpha+1}{3\alpha}t-2} - \delta(2\alpha+1)w^{1+\frac1\alpha}\big) \\ &\qquad + \delta\big(-(\alpha+1)w^{\frac1\alpha}w_{\theta\theta} + (2\alpha+1)(w^{1+\frac1\alpha} - (w+\delta)^{1+\frac1\alpha})\big) \\ &\ge \big(\epsilon_0\varphi^{\frac{2\alpha+1}{3\alpha}t-2} - \delta(2\alpha+1)A^{1+\frac1\alpha}\varphi^{\frac{\alpha+1}{3\alpha}t}\big) \\ &\qquad + \delta\,\big(-(\alpha+1)w^{\frac1\alpha}w_{\theta\theta} - (2\alpha+1)(1+\frac{1}{\alpha})\delta(w+\delta)^{\frac1\alpha}\big) \\ &\ge \big(\epsilon_0 - \delta(2\alpha+1)A^{1+\frac1\alpha}\big)\varphi^{\frac{2\alpha+1}{3\alpha}t-2} \\ &\qquad + \delta[-(\alpha+1)w^{\frac1\alpha}w_{\theta\theta} - (2\alpha+1)(1+\frac{1}{\alpha})\delta(w+\delta)^{\frac1\alpha}\big). \end{aligned}$$

Moreover, using the earlier calculation of $w_{\theta\theta}$ and $t \ge t_0$, we have

$$-w_{\theta\theta}w^{\frac1\alpha} \ge \Big(A(\frac{3}{t_0}-1)\varphi^{\frac t3 - 2}\Big)w^{\frac1\alpha} \ge A^{1+\frac1\alpha}(\frac{3}{t_0}-1)\varphi^{\frac{1+\alpha}{3\alpha}t-2} \ge A^{1+\frac1\alpha}(\frac{3}{t_0}-1).$$

The last two inequalities imply that there is small a $\delta_0 = \delta_0(\epsilon_0, A, t_0) = \delta_0(A,t_0)$ such that, for $0<\delta\le\delta_0$,

$$h_t - \alpha h^{1+\frac1\alpha}(h_{\theta\theta}+h) \ge 0 \quad \text{holds on } \ \theta \in (0, \frac{\pi}{6}t), \ \ t\in(0,t_0).$$

This completes the proof. □

This barrier gives the following, important for our purposes, curvature decay estimate at the two boundary points $\theta = 0, \pi$:

Theorem 2.9 (Curvature decay)**.** *For $\alpha > 1/2$ and $t>3$, we have*

$$\kappa^\alpha(\theta,t),\, \kappa^\alpha(\pi-\theta,t) \le C(M_0,\alpha)\,\theta^{\frac23} \qquad \text{on } \ \theta\in(0,\pi).$$

Proof. For $\alpha > \frac12$, note we have $2 < \min(3, \frac{6}{1+\alpha^{-1}})$. It suffices to show for any fixed $t_1 > 1$ the statement holds at $t = t_1+2$.

Setting $t_0 = 2$, let $A_0 = A_0(t_0)$ be the constant given in Lemma 2.8. By Proposition 2.7 we can choose a constant $A > \max\{A_0, \sup_{t\ge1}\sup_{\theta\in(0,\pi)}\kappa^\alpha(\theta,t)\}$, so that

$$\tilde u_{t_1}(\theta,t) := \kappa^\alpha(\theta, t_1+t) < A+\delta = w_\delta(t\pi/6, t)$$

for $t\in(0,2]$ and $\theta\in(0,\pi)$, where $w_\delta = A\sin^{t/3}\left(\frac{3\theta}{t}\right)+\delta$ as given in Lemma 2.8. Moreover, Proposition 2.7 implies that there exists a small constant $c(t_1,\delta)$ such that

$$\kappa^\alpha(\theta,t_1+t) < \delta \le w_\delta(\theta,t)$$

holds for $0<\theta\le c(t_1,\delta)$ and $t\in(0,2]$. Let us denote $t_2 = \frac{6c}{\pi}$. Then, $t\in(0,t_2]$ satisfies $\theta\in(0,\frac{t\pi}{6}) \subset (0,\frac{t_2\pi}{6}) = (0,c)$. Thus,

$$\kappa^\alpha(\theta,t_1+t) < \delta \le w_\delta(\theta,t_1+t),$$

holds for $t \in (0, t_2]$ and $\theta \in (0, \frac{t\pi}{6})$. Hence, by the comparison principle $\kappa^\alpha(\theta, t_1 + t) \leqq w_\delta(\theta, t_1 + t)$ holds for $\theta \in (0, \frac{t\pi}{6})$ and $t \in (0, 2]$, which implies $\kappa^\alpha(\theta, t_1 + 2) \leq w_\delta(\theta, t_1 + 2)$ for $\theta \in (0, \pi/3)$. Passing δ to zero in the last inequality, the following holds for $\theta \in (0, \pi/3)$

$$\kappa^\alpha(\theta, t_1 + 2) \leq A \sin^{\frac{t_1+2}{3}} \big(\frac{3\theta}{t_1 + 2}\big) \leq A \sin^{\frac{2}{3}} \big(\frac{3\theta}{2}\big) \leq C\, \theta^{\frac{2}{3}}$$

where the constant C depends on M_0 and α. This concludes the proof of our theorem. □

3. Decay Estimates (Pointwise curvature derivative estimates)

In this section we will use the curvature decay estimate at the boundary points $\theta = 0, \pi$ proven in Theorem 2.9 to obtain decay estimates for the first and the second order derivative at the boundary points $\theta = 0, \pi$ for $u := \bar{\kappa}^\alpha$. As a consequence we will obtain the estimate in Theorem 3.5 which will allow us to control the boundary terms when we prove our convergence result in the next section. We begin with a first order derivative decay estimate. Throughout this section we will assume that M_t is a solution of the α-CSF as in Proposition 2.1. We will only use the geometric parametrization in terms of arclength, i.e. we will assume that $u = u(s, t) = \bar{\kappa}^\alpha(s, t)$. Here and in what follows $B_r := \{x \in \mathbb{R}^2 \,|\, |x| < r\}$ denotes the Euclidean extrinsic ball of radius r.

Proposition 3.1. *Suppose $0 < u := \bar{\kappa}^\alpha \leq L$ on B_1 for $t \geq 0$. Then for every $\beta > 0$ with $\beta < \min(1, \alpha^{-1})$, we have*

$$|u_s| \leq C(1 + t^{-1/2})\, u^\beta \quad \text{on } B_{1/2}$$

for some $C = C(\alpha, \beta, L)$.

Proof. Let η be a cut-off function with compact support, and denote by $\Box\eta$ the term

$$\Box\eta := \partial_t \eta - \alpha u^{1-\frac{1}{\alpha}} \partial_{ss}\eta$$

which is defined on the support of η. Consider the continuous function $w := t^2 \eta^2 u_s^2 u^{-2\beta}$ with a fixed constant $\beta > 0$ satisfying the condition of the theorem. Then, on the set $\{w > 0\}$

$$\frac{\partial_t w}{2w} = \frac{1}{t} + \frac{\partial_t \eta}{\eta} + \frac{\partial_t u_s}{u_s} - \beta \frac{\partial_t u}{u}, \qquad \frac{\partial_s w}{2w} = \frac{\partial_s \eta}{\eta} + \frac{\partial_s u_s}{u_s} - \beta \frac{\partial_s u}{u}.$$

We differentiate the second equation above again

$$\frac{\partial_{ss} w}{2w} - \frac{w_s^2}{2w^2} = \frac{\eta_{ss}}{\eta} - \frac{\eta_s^2}{\eta^2} + \frac{\partial_{ss} u_s}{u_s} - \frac{u_{ss}^2}{u_s^2} - \beta \frac{\partial_{ss} u}{u} + \beta \frac{u_s^2}{u^2}.$$

Combining (2.10), (2.11), and the equations above yields

$$\begin{aligned}\frac{\partial_t w}{2w} - \alpha u^{1-\frac{1}{\alpha}} \frac{\partial_{ss} w}{2w} + \alpha u^{1-\frac{1}{\alpha}} \frac{w_s^2}{2w^2} =& \frac{1}{t} + \frac{\Box\eta}{\eta} + \alpha u^{1-\frac{1}{\alpha}} \frac{\eta_s^2}{\eta^2} + \frac{(\alpha - 1) u^{-\frac{1}{\alpha}} u_s u_{ss} + (2\alpha + 2) u_s u^{1+\frac{1}{\alpha}}}{u_s} \\ &+ \alpha u^{1-\frac{1}{\alpha}} \frac{u_{ss}^2}{u_s^2} - \beta\alpha u^{1-\frac{1}{\alpha}} \frac{u^{1+\frac{2}{\alpha}}}{u} - \beta\alpha u^{1-\frac{1}{\alpha}} \frac{u_s^2}{u^2}.\end{aligned}$$

Given $T > 0$, we assume that w attains its nonzero maximum at (p_0, t_0) for $t \in [0, T]$. Then, at the maximum point we have $w(p_0, t_0) > 0$, and thus $t_0 > 0$ and the following holds

$$0 \leq \frac{1}{t_0} + \frac{\Box\eta}{\eta} + (2\alpha + 2 - \beta\alpha)u^{1+\frac{1}{\alpha}} + \alpha u^{1-\frac{1}{\alpha}} I,$$

where

$$I = \Big(1 - \frac{1}{\alpha}\Big)\frac{u_{ss}}{u} + \frac{\eta_s^2}{\eta^2} + \frac{u_{ss}^2}{u_s^2} - \beta\frac{u_s^2}{u^2}.$$

Moreover, since $w_s = 0$ at the maximum point (p_0, t_0), the following hold

$$\frac{u_{ss}}{u} = \frac{u_s}{u}\frac{u_{ss}}{u_s} = \frac{u_s}{u}\Big(\beta\frac{u_s}{u} - \frac{\eta_s}{\eta}\Big) = \beta\frac{u_s^2}{u^2} - \frac{u_s}{u}\frac{\eta_s}{\eta}, \qquad \frac{u_{ss}^2}{u_s^2} = \beta^2\frac{u_s^2}{u^2} + \frac{\eta_s^2}{\eta^2} - 2\beta\frac{u_s}{u}\frac{\eta_s}{\eta}.$$

Substituting these derivatives in I and using the condition $\beta < \frac{1}{\alpha}$ and Young's inequality, we obtain by direct calculation that at the maximum point (p_0, t_0)

$$I = -\beta\Big(\frac{1}{\alpha} - \beta\Big)\frac{u_s^2}{u^2} + 2\frac{\eta_s^2}{\eta^2} - \Big(1 - \frac{1}{\alpha} + 2\beta\Big)\frac{u_s}{u}\frac{\eta_s}{\eta} \leq -\frac{\beta}{2}\Big(\frac{1}{\alpha} - \beta\Big)\frac{u_s^2}{u^2} + C\frac{\eta_s^2}{\eta^2},$$

for some $C = C(\alpha, \beta) > 0$. This implies that there exist $C > 0$ and $\delta > 0$ which depend on α and β such that, at (p_0, t_0),

$$\delta u^{1-\frac{1}{\alpha}}\frac{u_s^2}{u^2} = \delta\frac{u_s^2}{u^{2\beta}}u^{2\beta-1-\frac{1}{\alpha}} \leq \frac{1}{t_0} + \frac{\Box\eta}{\eta} + (2\alpha + 2 - \beta\alpha)u^{1+\frac{1}{\alpha}} + Cu^{1-\frac{1}{\alpha}}\frac{\eta_s^2}{\eta^2}. \tag{3.1}$$

Next, we define the cut-off function η by

$$\eta = \big(1 - |X(p,t)|^2\big)_+,$$

and observe that we have $|\eta_s| \leq 2$, $|\langle X, n\rangle| \leq |X| \leq 1$ and

$$|\Box\eta| = \Big|\partial_t\eta - \alpha u^{1-\frac{1}{\alpha}}\partial_{ss}\eta\Big| = \Big|\alpha u^{1-\frac{1}{\alpha}}(2 + 2(1-\alpha^{-1})u^{\frac{1}{\alpha}}\langle X, n\rangle)\Big| \leq Cu^{1-\frac{1}{\alpha}}, \tag{3.2}$$

for some $C = C(\alpha, L)$. Since $t_0 \leq T$, $0 \leq \eta \leq 1$ and $|\eta_s| \leq 2$, at the point (p_0, t_0) we have

$$\frac{u_s^2}{u^{2\beta}}t_0^2\eta^2 \leq C'\Big(Tu^{1+\frac{1}{\alpha}-2\beta} + u^{2+\frac{2}{\alpha}-2\beta}T^2 + u^{2-2\beta}T^2\Big) \leq C'T^2(1 + T^{-1}).$$

Here $C' = C'(\alpha, \beta, L) > 0$ and this is possible because $\beta \leq \min(\alpha^{-1}, 1)$. At any point (p, T),

$$\frac{u_s^2}{u^{2\beta}}\eta^2 \leq \frac{w(p_0, t_0)}{T^2} \leq C'(1 + T^{-1}).$$

Therefore, replacing T by t yields the desired result. □

Proposition 3.2. *For $\alpha \in (0,1)$, for each $t > 0$,*

$$\liminf_{|X|\to\infty} |X|^2\,\bar{\kappa}^{1-\alpha} = \liminf_{|X|\to\infty} |X|^2\,u^{\frac{1}{\alpha}-1} \geq \frac{2\alpha(1+\alpha)}{1-\alpha}t.$$

Here, $|X| = |X(p,t)|$ is the extrinsic distance from the origin. More generally, this is uniform in t for all compact time interval which is away from $t = 0$. In other words, for $0 < \tau_1 < \tau_2 < \infty$ and $\epsilon > 0$, there is $R > 0$ such that

$$|X(p,t)|^2\,\bar{\kappa}^{1-\alpha}(p,t) \geq \frac{2\alpha(1+\alpha)}{1-\alpha}t - \epsilon$$

for all (p,t) *with* $t \in [\tau_1, \tau_2]$ *and* $|X(p,t)| \geq R$.

Proof. We will follow the idea and the proof of [19] Theorem 2.4, where the same inequality is shown for a solution of the Euclidean fast diffusion equation $w_t = \Delta w^\alpha$. Recall that the curvature $\bar{\kappa}$ satisfies the equation $\bar{\kappa}_t = (\bar{\kappa}^\alpha)_{ss} + \bar{\kappa}^{\alpha+2}$. Since here we are on a Riemannian manifold (though it's 1D) and the metric is changing with respect to time, we need to modify the proof.

Let us define the constant b by $b^{-1} := \frac{2\alpha(1+\alpha)}{1-\alpha}$ and consider then function $U : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ defined by

$$U(x,t) = t^{-\frac{1}{\alpha+1}}(1 + b|x|^2 t^{-\frac{2}{\alpha+1}})^{-\frac{1}{1-\alpha}}.$$

Then, it can be directly checked that

$$U_\mu(x,t) := \mu^{\frac{2}{1-\alpha}} U(t, \mu x) = t^{-\frac{1}{\alpha+1}}(\mu^{-2} + b|x|^2 t^{-\frac{2}{\alpha+1}})^{-\frac{1}{1-\alpha}}$$

are solutions of the 1D fast diffusion equation $f_t = (f^\alpha)_{xx}$ for all parameters $\mu > 0$.

Case 1: Assume first that our solution of the α-CSF is smooth for $t \geq 0$ and has the positive and bounded curvature $0 < \kappa^\alpha \leq L$. Pick a point $p \in N$. Then the intrinsic distance function $s_p(q,t) = \mathrm{dist}_{g(t)}(p,q) \geq 0$ is smooth away from p for $t \geq 0$. Moreover, $ds = \sqrt{g_{11}}dx$ implies that $\frac{\partial}{\partial t} ds = \frac{-2\bar{\kappa}^{\alpha+1} g_{11}}{2\sqrt{g_{11}}} dx = -\bar{\kappa}^{\alpha+1} ds$. Thus, for a curve $\gamma : [0,1] \to M$ which joins p to q, we have

$$\frac{\partial}{\partial t} s_p(q,t) = \int_0^1 -\bar{\kappa}^{\alpha+1} |\dot{\gamma}(\tau)| d\tau \geq \int -\bar{\kappa}^\alpha |d\theta| \geq -L\pi.$$

Define $\bar{U}_\mu : M \times (0,\infty) \to \mathbb{R}$ by

$$\bar{U}_\mu(q,t) = U_\mu(s_p(q,t) + L\pi t, t) = U_\mu(\bar{s}_p(q,t), t), \qquad \bar{s}_p(q,t) := s_p(q,t) + L\pi t.$$

One can easily check using the chain rule and the fact that $\partial_t \bar{s}_p = \partial_t s_p + L\pi \geq 0$ and $\partial_x U_\mu \leq 0$, that

$$\partial_t \bar{U}_\mu = \partial_t \bar{s}_p \partial_x U_\mu + \partial_t U_\mu \leq \partial_t U_\mu = (U_\mu^\alpha)_{xx} = (\bar{U}_\mu^\alpha)_{ss}.$$

i.e. $\partial_t \bar{U}_\mu - (\bar{U}_\mu^\alpha)_{ss} \leq 0$ away from the non-smooth point p. From this point, we can follow the proof of Theorem 2.4 [19] using these barriers $\bar{U}_\mu$. Let us choose two different points p_1, $p_2 \in N$ such that $s_p(p_i, 0) = 1$. For a fixed $T > 0$, let's denote $\delta = \delta(T) > 0$ by

$$\delta = \frac{1}{2} \min_{i=1,2,\, t\in[0,T]} \bar{\kappa}(p_i, t).$$

We can find small $\mu > 0$ such that

$$\bar{U}_\mu(q,t) \leq \delta \quad \text{when } 0 \leq t \leq T,\ q \in M \setminus B_{g(0)}(1,p).$$

This is possible because

$$\bar{U}_\mu(q,t) = U_\mu(\bar{s}_p(q,t), t) \leq U_\mu(s_p(q,0), t) \leq U_\mu(1,t).$$

Recall that $\bar{\kappa}_t = (\bar{\kappa}^\alpha)_{ss} + \bar{\kappa}^{\alpha+2} \geq (\bar{\kappa}^\alpha)_{ss}$. Since $\bar{U}_\mu(q,0) = 0$, by the comparison principle (c.f. Lemma 3.4 [19]),

$$\bar{U}_\mu(q,t) \leq \bar{\kappa}(q,t) \quad \text{for } 0 \leq t \leq T,\ q \in M \setminus B_{g(0)}(1,p).$$

The proof of Lemma 3.4 [19] uses the Kato's inequality

$$\Delta(f^+) \geq (\Delta f)^+ \quad \text{in distribution sense.}$$

At each fixed time slice and thus fixed metric, this is again true in our (1D) Riemannian case. Thus the proof actually works in our setting, thus the comparison principle holds. Therefore, comparing with our barrier $\bar U_\mu$ yields that for each $0<\tau_1<\tau_2\le T$ and $\epsilon>0$, there is $R>0$ such that

$$\bar s_p^2(q,t)\,\bar\kappa^{1-\alpha}(q,t)\ge \bar s_p^2(q,t)\,\bar U_\mu^{1-\alpha}(q,t)\ge \Big(tb^{-1}\Big)-\epsilon$$

holds, for all (q,t) with $t\in[\tau_1,\tau_2]$ and $\bar s_p(q,t)\ge R$. Also, observe

$$\frac{\bar s_p(q,t)}{|X(q,t)|}=\frac{s_p(q,t)+L\pi t}{|X(q,t)|}\to 1$$

uniformly for $t\in[0,T]$ as $|X(q,t)|\to\infty$. This follows from the fact that M_t is convex, it is located between to two parallel lines, it is asymptotic to these parallel lines and $|X(p,t)-X(p,0)|\le Lt$. We can choose $T>0$ arbitrary large and repeat the same argument to conclude that the proposition holds, under the extra assumption that $0<\bar\kappa^\alpha\le L$.

Case 2: For a general solution of the α-CSF which is not smooth up to $t=0$ or does not satisfy the curvature bound $0<\bar\kappa^\alpha\le L$, we may apply the previous proof on $t\in[\tau,\infty)$, for small fixed $\tau>0$ and conclude that for $0<\tau<\tau_1<\tau_2<\infty$ and $\epsilon>0$, there is $R>0$ such that

$$|X(p,t)|^2\,\bar\kappa^{1-\alpha}(p,t)\ge b^{-1}(t-\tau)-\epsilon$$

for all (p,t) with $t\in[\tau_1,\tau_2]$ and $|X(p,t)|\ge R$. We may chose τ small enough so that $b^{-1}\tau\le\epsilon/2$, finishing the proof. □

In the range of exponents $\alpha\in(0,1)$ we have the following global and somewhat improved estimate than Proposition 3.1.

Proposition 3.3. *For a fixed $\alpha\in(0,1)$, suppose $0<u:=\bar\kappa^\alpha\le L$, for $t\ge0$. Then, there exists some $C=C(\alpha,L,M_0)$ such that*

$$|u_s|\le C(1+\frac{1}{\sqrt t})\,u^{\frac12\left(1+\frac1\alpha\right)}$$

holds for $t>0$.

Proof. Given T and τ with $0<\tau<T$, we set $w:=(t-\tau)^2\eta^2u_s^2u^{-2\beta}$ for $t\in(\tau,T]$ with the fixed exponent $\beta:=\frac12\left(1+\frac1\alpha\right)$ and a smooth cut off function η. We are going to choose η in a different way to use the asymptotic bound in Proposition 3.2. Let us fix a usual cut off function $\xi:[0,\infty)\to\mathbb R$ such that

$$0\le\xi\le1,\quad \xi=1 \text{ on } [0,1/2],\quad \operatorname{supp}\xi\subset[0,1],\quad 0\le-\xi',\ |\xi''|\le C.$$

Define $\eta(p,t):=\xi\big(\frac{|X(p,t)|}{R}\big)$ for $R\gg1$. Then the following holds by direct computation.

Claim 3.1.

$$\eta_s^2\le\frac{C}{R^2},\qquad \Box\eta:=\partial_t\eta-\alpha u^{1-\frac1\alpha}\partial_{ss}\eta\le C\left(\frac{u^{1-\frac1\alpha}}{R^2}+1\right)$$

for some $C=C(\alpha,L,M_0)$.

Proof of Claim 3.1. We begin by observing that $\partial_s |X| = \frac{|X|^2_s}{2|X|} = \langle \frac{X}{|X|}, \partial_s X\rangle$ implying $|\partial_s |X|| \leq 1$, thus $\eta_s^2 = \left(\frac{|X|_s}{R}\xi'\right)^2 \leq R^{-2}C$ yields the first estimate. Next, by the chain rule and (2.8), we compute

$$\begin{aligned}\Box\eta = (\partial_t - \alpha u^{1-\frac{1}{\alpha}}\partial_{ss})\eta &= \frac{\xi'}{R}(\partial_t - \alpha u^{1-\frac{1}{\alpha}}\partial_{ss})|X| - \alpha u^{1-\frac{1}{\alpha}}\frac{\xi''}{R^2}(\partial_s|X|)^2\\ &= \alpha u^{1-\frac{1}{\alpha}}\frac{\xi'}{R}\Big(\frac{(\partial_t - \alpha u^{1-\frac{1}{\alpha}}\partial_{ss})|X|^2}{\alpha u^{1-\frac{1}{\alpha}}\, 2|X|} + \frac{(\partial_s(|X|^2))^2}{4(|X|^2)^{3/2}}\Big) - \alpha u^{1-\frac{1}{\alpha}}\frac{\xi''}{R^2}(\partial_s|X|)^2\\ &= \alpha u^{1-\frac{1}{\alpha}}\Big(\frac{\xi'}{R}\frac{(\partial_s|X|)^2 - 1 + (\alpha^{-1}-1)\bar\kappa\langle X, n\rangle}{|X|} - \frac{\xi''}{R^2}(\partial_s|X|)^2\Big).\end{aligned}$$

Note that since since $\xi'(|X|/R) = 0$ for $|X|/R < \frac{1}{2}$, we have $\dfrac{|\xi'|}{R}\dfrac{1}{|X|} \leq \dfrac{2|\xi'|(|X|/R)}{R}\dfrac{1}{|X|} \leq CR^{-2}$. Therefore, $\bar\kappa = u^{\frac{1}{\alpha}}$, $|\xi''| \leq C$ and $\langle X, n\rangle \leq |X|$ yield

$$\Box\eta \leq C\frac{u^{1-\frac{1}{\alpha}}}{R^2} + (1-\alpha)u^{1-\frac{1}{\alpha}}\,\bar\kappa\,\frac{\xi'}{R}\frac{\langle X,n\rangle}{|X|} \leq C\,\frac{u^{1-\frac{1}{\alpha}}}{R^2} + (1-\alpha)\,u\frac{|\xi'|}{R}.$$

By using $u \leq L$, $R \geq 1$, $|\xi'| \leq C$, we obtain the second estimate in the claim. □

We will now continue with the proof of the proposition. Assume that a nonzero maximum of $w(p,t)$ on $t \in [\tau, T]$ is obtained at (p_0, t_0) with $t_0 \in (\tau, T]$. Since the proof of Proposition 3.1 does not make use of the specific η until (3.1), except that it has a compact support, we may use the calculation in (3.1) and combine it with the above claim to conclude that at the point (p_0, t_0) we have

$$\delta u^{1-\frac{1}{\alpha}}\frac{u_s^2}{u^2} = \delta\frac{u_s^2}{u^{2\beta}} \leq \frac{1}{t_0-\tau} + \frac{C}{\eta}\Big(\frac{u^{1-\frac{1}{\alpha}}}{R^2} + 1\Big) + Cu^{1+\frac{1}{\alpha}} + \frac{C}{\eta^2}\frac{u^{1-\frac{1}{\alpha}}}{R^2},$$

for some $C = C(\alpha, L, M_0)$. Therefore, multiplying the last inequality by $(t_0-\tau)^2\eta(p_0,t_0)^2$ and using $0 < t_0 - \tau \leq t_0 \leq T$, $\eta \leq 1$, $\beta := \frac{1}{2}\left(1 + \frac{1}{\alpha}\right)$ yield that for $t \in [\tau, T]$ and $|x| \leq R/2$ the following holds

$$\frac{u_s^2}{u^{2\beta}}(t-\tau)^2 =: w(p,t) \leq w(p_0,t_0) := \frac{u_s^2}{u^{2\beta}}\eta^2(t_0-\tau)^2 \leq C\,T\Big(1 + T + \sup_{\operatorname{supp}\eta,\, t=t_0}\frac{u^{1-\frac{1}{\alpha}}}{R^2}t_0\Big),$$

where $C = C(\alpha, L, M_0)$ but independent of $R > 0$ and $\tau > 0$.

Now, we apply Proposition 3.2 with $\tau_1 = \tau$, $\tau_2 = T$, $\epsilon = \frac{\alpha(1+\alpha)}{1-\alpha}t_0$, which implies that there exists some $R_0 > 0$ such that $|X|^2u^{\frac{1}{\alpha}-1} \geq \frac{\alpha(1+\alpha)}{1-\alpha}t_0$ for $|X|(p,t_0) \geq R_0$. Combining this with the above estimate yields that for $R \geq R_0$

$$\sup_{R_0 \leq |X| < R}\frac{u^{1-\frac{1}{\alpha}}(p,t_0)}{R^2}t_0 \leq \sup_{R_0\leq|X|<R}\frac{t_0}{|X|^2u^{\frac{1}{\alpha}-1}(p,t_0)} \leq \frac{1-\alpha}{\alpha(1+\alpha)}.$$

We conclude that if $t \in [\tau, T]$ and $|x| \leq R/2$ with $R \geq R_0$ then we have

$$\frac{u_s^2}{u^{2\beta}}(t-\tau)^2 \leq CT\Big(1 + T + KTR^{-2}\Big), \quad \text{where } K = \sup_{|x|(p,t_0)\leq R_0} u^{1-\frac{1}{\alpha}}(p,t_0).$$

Passing $\tau \to 0$ and $R \to +\infty$ and then setting $t = T$, we finally obtain the bound

$$\frac{u_s^2}{u^{2\beta}} T^2 \le C\, T^2(1 + T^{-1}),$$

which holds for all $T > 0$. By replacing T by t, we have the desired result. □

Proposition 3.4. *Let $\alpha \in [\frac{1}{3}, \infty)$ be fixed. If $|u_s| \le K u^\beta$ for some $\beta \in (0, \frac{1}{\alpha})$ and $L \ge \sup_{B_1} u$ for $t \ge 0$, then for every $\epsilon \in (0, \min(1, \beta))$ we have*

$$|u_{ss}| \le C(1 + t^{-1/2})\, u^{\min(\beta, 2\beta - 1) - \epsilon} \qquad \text{on } B_{1/2}$$

with some $C = C(\alpha, \beta, \epsilon, K, L)$.

Proof. Let us define $v := u^{1-\beta}(u_s + 2Ku^\beta) = u^{1-\beta}u_s + 2Ku$. Note that

$$Ku \le v \le 3Ku.$$

Claim 3.2. *There is $C' = C'(\alpha, \beta, K) > 0$ such that*

$$\frac{|(\partial_t - \alpha u^{1-\frac{1}{\alpha}}\partial_{ss})v|}{\alpha u^{1-\frac{1}{\alpha}}} \le C'\Big(|v_s| u^{\beta - 1} + v u^{2\beta - 2} + v u^{\frac{2}{\alpha}}\Big)$$

and

$$\frac{|(\partial_t - \alpha u^{1-\frac{1}{\alpha}}\partial_{ss})v_s|}{\alpha u^{1-\frac{1}{\alpha}}} \le C'\Big(|v_s|\big(\frac{|v_{ss}|}{|v_s|} + \frac{|v_s|}{v}\big)u^{\beta-1} + |v_s|(u^{2\beta-2} + u^{\frac{2}{\alpha}}) + v(u^{2\beta} + u^{\beta - 1 + \frac{2}{\alpha}} + u^{3\beta - 3})\Big).$$

Since a proof of this claim is long, let us postpone it for the end of this proposition and assume it is true. Define $w := v_s^2 v^{-2\gamma} t^2 \eta^2$ for some $0 < \gamma < 1$ to be determined later, where $\eta = (1 - |X(p,t)|)_+^2$. Then, on the support $\{w > 0\}$ we have

$$\frac{\partial_t w}{2w} = \frac{1}{t} + \frac{\partial_t \eta}{\eta} + \frac{\partial_t v_s}{v_s} - \gamma\frac{\partial_t v}{v} \quad \text{ and } \quad \frac{\partial_s w}{2w} = \frac{\partial_s \eta}{\eta} + \frac{\partial_s v_s}{v_s} - \gamma \frac{\partial_s v}{v}$$

and

$$\frac{\partial_{ss} w}{2w} - \frac{w_s^2}{2w^2} = \frac{\eta_{ss}}{\eta} - \frac{\eta_s^2}{\eta^2} + \frac{\partial_{ss} v_s}{v_s} - \frac{v_{ss}^2}{v_s^2} - \gamma\frac{v_{ss}}{v} + \gamma\frac{v_s^2}{v^2}.$$

Suppose that a nonzero maximum of w on $t \in [0, T]$ is attained at (p_0, t_0). At this point,

$$\begin{aligned}
0 &\le \frac{\partial_t w}{2w} - \alpha u^{1-\frac{1}{\alpha}}\frac{\partial_{ss} w}{2w} + \alpha u^{1-\frac{1}{\alpha}}\frac{w_s^2}{2w^2} \\
&= \frac{1}{t_0} + \alpha u^{1-\frac{1}{\alpha}}\Big(\frac{(\partial_t - \alpha u^{1-\frac{1}{\alpha}}\partial_{ss})\eta_{ss}}{\alpha u^{1-\frac{1}{\alpha}}\eta} + \frac{(\partial_t - \alpha u^{1-\frac{1}{\alpha}}\partial_{ss})v_s}{\alpha u^{1-\frac{1}{\alpha}} v_s} - \gamma\frac{(\partial_t - \alpha u^{1-\frac{1}{\alpha}}\partial_{ss})v}{\alpha u^{1-\frac{1}{\alpha}} v}\Big) \\
&\quad + \alpha u^{1-\frac{1}{\alpha}}\Big(\frac{v_{ss}^2}{v_s^2} - \gamma\frac{v_s^2}{v^2} + \frac{\eta_s^2}{\eta^2}\Big).
\end{aligned} \tag{3.3}$$

Since $w_s = 0$ at the maximum (p_0, t_0), we have

$$\frac{v_{ss}^2}{v_s^2} - \gamma\frac{v_s^2}{v^2} + \frac{\eta_s^2}{\eta^2} = (-\gamma + \gamma^2)\frac{v_s^2}{v^2} + 2\frac{\eta_s^2}{\eta^2} - 2\gamma\frac{\eta_s}{\eta}\frac{v_s}{v} \le -\frac{\gamma(1-\gamma)}{2}\frac{v_s^2}{v^2} + C\frac{\eta_s^2}{\eta^2}$$

and by using $\gamma < 1$

$$\Big|\frac{v_{ss}}{v_s}\Big| = \Big|\gamma\frac{v_s}{v} - \frac{\eta_s}{\eta}\Big| \le \frac{|v_s|}{v} + \frac{|\eta_s|}{\eta}.$$

Also, recall (3.2). Then (3.3) together with the Claim, (3.2) and the last two estimates above yield

$$\begin{aligned}
0 &\leq \frac{1}{t_0} + \alpha u^{1-\frac{1}{\alpha}}\Big(\frac{C}{\eta} + \frac{|(\partial_t - \alpha u^{1-\frac{1}{\alpha}}\partial_{ss})v_s|}{\alpha u^{1-\frac{1}{\alpha}}|v_s|} + \frac{|(\partial_t - \alpha u^{1-\frac{1}{\alpha}}\partial_{ss})v|}{\alpha u^{1-\frac{1}{\alpha}}v} - \frac{\gamma(1-\gamma)}{2}\frac{v_s^2}{v^2} + C\frac{\eta_s^2}{\eta^2}\Big) \\
&\leq \frac{1}{t_0} + Cu^{1-\frac{1}{\alpha}}\Big(\frac{1}{\eta} + \frac{|v_s|}{v}u^{\beta-1} + \frac{|\eta_s|}{\eta}u^{\beta-1} + u^{2\beta-2} + u^{\frac{2}{\alpha}} \\
&\qquad + \frac{v}{|v_s|}(u^{2\beta} + u^{\beta-1+\frac{2}{\alpha}} + u^{3\beta-3}) + \frac{\eta_s^2}{\eta^2}\Big) - \frac{\alpha\gamma(1-\gamma)}{2}\frac{u^{1-\frac{1}{\alpha}}v_s^2}{v^2} \\
&\leq \frac{1}{t_0} + Cu^{1-\frac{1}{\alpha}}\Big(\frac{1}{\eta} + u^{2\beta-2} + u^{\frac{2}{\alpha}} + \frac{v}{|v_s|}(u^{2\beta} + u^{\beta-1+\frac{2}{\alpha}} + u^{3\beta-3}) + \frac{\eta_s^2}{\eta^2}\Big) - \frac{\alpha\gamma(1-\gamma)}{4}\frac{u^{1-\frac{1}{\alpha}}v_s^2}{v^2} \\
&\leq \frac{1}{t_0} + Cu^{1-\frac{1}{\alpha}}\Big(\frac{1}{\eta} + u^{2\beta-2} + \frac{v}{|v_s|}u^{3\beta-3} + \frac{\eta_s^2}{\eta^2}\Big) - \frac{\alpha\gamma(1-\gamma)}{4}\frac{u^{1-\frac{1}{\alpha}}v_s^2}{v^2} \quad (\text{since } \beta < \frac{1}{\alpha} \leq 3)
\end{aligned}$$

for some $C = C(\alpha, \beta, \gamma, K, L)$, where the dependence of C on L takes place in the last inequality for the first time. We conclude that at the maximum point (p_0, t_0) the following holds

$$\frac{u^{1-\frac{1}{\alpha}}v_s^2}{v^2} \leq C\left(\frac{1}{t_0} + u^{1-\frac{1}{\alpha}}\big(\frac{1}{\eta} + u^{2\beta-2} + \frac{v}{|v_s|}u^{3\beta-3} + \frac{\eta_s^2}{\eta^2}\big)\right).$$

Using this estimate we now conclude that

$$\begin{aligned}
w(p_0, t_0) &= \left(\frac{v_s^2}{v^2}u^{1-\frac{1}{\alpha}}\right)u^{\frac{1}{\alpha}-1}v^{2(1-\gamma)}\,t_0^2\,\eta^2 \\
&\leq C\Big(t_0 u^{\frac{1}{\alpha}-1}\eta^2 + t_0^2\eta^2 v^{2(1-\gamma)}\Big(\frac{1}{\eta} + \frac{\eta_s^2}{\eta^2} + u^{2(\beta-1)} + \frac{v}{|v_s|}u^{3(\beta-1)}\Big)\Big) \\
&\leq C\Big(Tu^{1+\frac{1}{\alpha}-2\gamma} + T^2(u^{2(1-\gamma)} + u^{2(\beta-\gamma)}) + \frac{v^\gamma}{|v_s|t_0\eta}u^{3(\beta-\gamma)}T^3\Big) \quad (\text{since } v \leq 3Ku).
\end{aligned}$$

If we choose our $\gamma \in (0,1)$ by $\gamma := \min(1, \beta) - \epsilon$, then $1 + \frac{1}{\alpha} - 2\gamma$, $1-\gamma$, $\beta - \gamma \geq 0$ as $\gamma < 1$ and $\gamma < \beta < \frac{1}{\alpha}$. Thus

$$w(p_0, t_0) \leq C\Big(T + T^2 + \frac{1}{w^{1/2}(p_0, t_0)}T^3\Big)$$

with $C = C(\alpha, \beta, \epsilon, K, L)$. By considering the two cases $w(p_0, t_0) \geq T^2$ and $< T^2$, we finally obtain the bound

$$w(p_0, t_0) \leq CT^2(1 + T^{-1})$$

implying that at any point (p, T), we have

$$\frac{|v_s|}{u^\gamma}\,\eta \leq (3K)^\gamma\frac{|v_s|}{v^\gamma}\,\eta = (3K)^\gamma\frac{w^{1/2}(p,t)}{T} \leq C(1 + T^{-1/2}).$$

Note that $v_s = u^{1-\beta}u_{ss} + (1-\beta)u^{-\beta}u_s^2 + 2Ku_s$. Hence, $|u_s| \leq Ku^\beta$ leads to

$$\frac{u^{1-\beta}|u_{ss}|}{u^\gamma}\eta = \frac{|u_{ss}|\eta}{u^{\gamma-(1-\beta)}} \leq C(1 + T^{-1/2}) + C\frac{u^{-\beta}u_s^2 + |u_s|}{u^\gamma} \leq C(1 + T^{-1/2}).$$

We replace T by t. Then, $u^{\gamma-(1-\beta)} = u^{\min(\beta, 2\beta-1)-\epsilon}$ yields the proposition.

Proof of Claim 3.2. During the proof of the claim we will frequently use the inequalities

$$Ku \le v \le 3Ku \qquad \text{and} \qquad |u_s| \le K\, u^\beta$$

and we will denote by C various constants which depend on α, β and K.

Since

$$\begin{aligned}(\partial_t - \alpha u^{1-\alpha}\partial_{ss})v &= u_s(\partial_t - \alpha u^{1-\alpha}\partial_{ss})u^{1-\beta} + u^{1-\beta}(\partial_t - \alpha u^{1-\alpha}\partial_{ss})u_s - 2\alpha u^{1-\frac{1}{\alpha}}(u^{1-\beta})_s(u_s)_s \\ &\quad + 2K(\partial_t - \alpha u^{1-\alpha}\partial_{ss})u\end{aligned}$$

to show the first inequality in the claim, it is enough the terms on the right hand side by

$$Cu^{1-\frac{1}{\alpha}}\Big(|v_s|u^{\beta-1} + vu^{2\beta-2} + vu^{\frac{2}{\alpha}}\Big) \le Cu^{1-\frac{1}{\alpha}}\Big(|v_s|u^{\beta-1} + u^{2\beta-1} + u^{\frac{2}{\alpha}+1}\Big).$$

We begin with observing

$$v_s = u^{1-\beta}u_{ss} + (1-\beta)u^{-\beta}u_s^2 + 2Ku_s \tag{3.4}$$

and thus $|u_s| \le Ku^\beta$ and $Ku \le v$ yield

$$|u_{ss}| \le u^{\beta-1}|v_s| + Cu^{-1}u_s^2 + Cu^{\beta-1}|u_s| \le C(|v_s|u^{\beta-1} + u^{2\beta-1}). \tag{3.5}$$

Therefore,

$$\frac{|(\partial_t - \alpha u^{1-\alpha}\partial_{ss})u|}{\alpha u^{1-\frac{1}{\alpha}}} = \big|u_{ss} + u^{1+\frac{2}{\alpha}}\big| \le C\Big(|v_s|u^{\beta-1} + u^{2\beta-1} + u^{\frac{2}{\alpha}+1}\Big) \tag{3.6}$$

also implying that

$$\begin{aligned}u_s\frac{(\partial_t - \alpha u^{1-\alpha}\partial_{ss})u^{1-\beta}}{\alpha u^{1-\frac{1}{\alpha}}} &= (1-\beta)u_s u^{-\beta}\frac{(\partial_t - \alpha u^{1-\alpha}\partial_{ss})u}{\alpha u^{1-\frac{1}{\alpha}}} - \alpha(1-\beta)(-\beta)u^{-1-\beta}u_s^3 \\ &\le C\Big(|v_s|u^{\beta-1} + u^{2\beta-1} + u^{\frac{2}{\alpha}+1}\Big).\end{aligned} \tag{3.7}$$

In addition, using (3.5) we have

$$u^{1-\beta}\frac{|(\partial_t - \alpha u^{1-\alpha}\partial_{ss})u_s|}{\alpha u^{1-\frac{1}{\alpha}}} \le Cu^{1-\beta}(\frac{|u_s u_{ss}|}{u} + u^{\frac{2}{\alpha}}|u_s|) \le C\Big(|v_s|u^{\beta-1} + u^{2\beta-1} + u^{\frac{2}{\alpha}+1}\Big) \tag{3.8}$$

and

$$|(u^{1-\beta})_s(u_s)_s| = |(1-\beta)u^{-\beta}u_s u_{ss}| \le C|u_{ss}| \le C\Big(|v_s|u^{\beta-1} + u^{2\beta-1}\Big).$$

Combining the above inequalities yieds the first estimate in the claim.

Next, by using (3.4) we compute

$$\begin{aligned}(\partial_t - \alpha u^{1-\frac{1}{\alpha}}\partial_{ss})v_s &= 2K(\partial_t - \alpha u^{1-\frac{1}{\alpha}}\partial_{ss})u_s \\ &\quad + u_{ss}(\partial_t - \alpha u^{1-\frac{1}{\alpha}}\partial_{ss})u^{1-\beta} - 2\alpha u^{1-\frac{1}{\alpha}}(u^{1-\beta})_s(u_{ss})_s + u^{1-\beta}(\partial_t - \alpha u^{1-\frac{1}{\alpha}}\partial_{ss})u_{ss} \\ &\quad + (1-\beta)\Big(u_s^2(\partial_t - \alpha u^{1-\frac{1}{\alpha}}\partial_{ss})u^{-\beta} - 2\alpha u^{1-\frac{1}{\alpha}}(u^{-\beta})_s(u_s^2)_s + u^{-\beta}(\partial_t - \alpha u^{1-\frac{1}{\alpha}}\partial_{ss})u_s^2\Big).\end{aligned}$$

To show the second inequality in the claim, we bound the seven terms above by $Cu^{1-\frac{1}{\alpha}}H$ where

$$\begin{aligned} H :=& u^{\beta-1}\Big(|v_{ss}| + |v_s|^2u^{-1} + |v_s|(u^{\beta-1} + u^{1+\frac{2}{\alpha}-\beta}) + u^{\beta+2} + u^{\frac{2}{\alpha}+1} + u^{2\beta-1}\Big) \\ \geq& \Big(|v_s|\big(\frac{|v_{ss}|}{|v_s|} + \frac{|v_s|}{v}\big)u^{\beta-1} + |v_s|(u^{2\beta-2} + u^{\frac{2}{\alpha}}) + v(u^{2\beta} + u^{\beta-1+\frac{2}{\alpha}} + u^{3\beta-3})\Big). \end{aligned}$$

The inequality (3.8) implies that the first term is bounded

$$\frac{|(\partial_t - \alpha u^{1-\alpha}\partial_{ss})u_s|}{\alpha u^{1-\frac{1}{\alpha}}} \leq CH.$$

To proceed, we observe

$$u^{-\beta}\Big(|v_s|u^{\beta-1} + u^{2\beta-1}\Big)\Big(|v_s|u^{\beta-1} + u^{2\beta-1} + u^{\frac{2}{\alpha}+1}\Big) \leq CH. \tag{3.9}$$

Hence, by using (3.5) and (3.7) we estimate the second term,

$$\frac{|u_{ss}(\partial_t - \alpha u^{1-\alpha}\partial_{ss})u^{1-\beta}|}{\alpha u^{1-\frac{1}{\alpha}}} \leq CH.$$

Now, we differentiate (3.4) again so that we have

$$v_{ss} = u^{1-\beta}u_{sss} + 3(1-\beta)u^{-\beta}u_su_{ss} + (1-\beta)(-\beta)u^{-1-\beta}u_s^3 + 2Ku_{ss}.$$

Thus, by using (3.5) we have

$$\begin{aligned} |u_{sss}| =& u^{\beta-1}\Big|v_{ss} - 3(1-\beta)u^{-\beta}u_su_{ss} - (1-\beta)(-\beta)u^{-1-\beta}u_s^3 + 2Ku_{ss}\Big| \\ \leq& u^{\beta-1}|v_{ss}| + Cu^{\beta-1}|u_{ss}| + Cu^{3\beta-2} \leq CH. \end{aligned} \tag{3.10}$$

Hence, we can bound the third term, as follows

$$|(u^{1-\beta})_s(u_{ss})_s| \leq Cu^{-\beta}|u_su_{sss}| \leq C|u_{sss}| \leq CH.$$

We recall (2.12) to estimate the fourth term

$$\frac{|u^{1-\beta}(\partial_t - \alpha u^{1-\alpha}\partial_{ss})u_{ss}|}{\alpha u^{1-\frac{1}{\alpha}}} \leq Cu^{1-\beta}\Big(\frac{|u_su_{sss}|}{u} + \frac{u_{ss}^2}{u} + \frac{u_s^2|u_{ss}|}{u^2} + u^{-1+\frac{2}{\alpha}}u_s^2 + u^{\frac{2}{\alpha}}|u_{ss}|\Big).$$

This combined with (3.5), (3.9), and 3.10 yields

$$\frac{|u^{1-\beta}(\partial_t - \alpha u^{1-\alpha}\partial_{ss})u_{ss}|}{\alpha u^{1-\frac{1}{\alpha}}} \leq C|u_{sss}| + Cu^{\beta+\frac{2}{\alpha}} + Cu^{-\beta}|u_{ss}|\Big(|u_{ss}| + u^{2\beta-1} + u^{\frac{2}{\alpha}+1}\Big) \leq CH.$$

The fifth term is

$$u_s^2\frac{(\partial_t - \alpha u^{1-\frac{1}{\alpha}}\partial_{ss})u^{-\beta}}{\alpha u^{1-\alpha}} = (-\beta)u_s^2u^{-\beta-1}\frac{(\partial_t - \alpha u^{1-\frac{1}{\alpha}}\partial_{ss})u}{\alpha u^{1-\alpha}} - \beta(1+\beta)u^{-\beta-2}u_s^4.$$

Therefore, by using (3.6) we have

$$u_s^2\frac{|(\partial_t - \alpha u^{1-\frac{1}{\alpha}}\partial_{ss})u^{-\beta}|}{\alpha u^{1-\alpha}} \leq Cu^{\beta-1}\frac{|(\partial_t - \alpha u^{1-\frac{1}{\alpha}}\partial_{ss})u|}{u^{1-\alpha}} + Cu^{3\beta-2} \leq CH.$$

The sixth term is bounded by (3.5)

$$|(u^{-\beta})_s(u_s^2)_s| = |-2\beta u^{-1-\beta}u_s^2u_{ss}| \leq CH.$$

The last seventh term

$$u^{-\beta}\frac{|(\partial_t-\alpha u^{1-\frac{1}{\alpha}}\partial_{ss})u_s^2|}{\alpha u^{1-\alpha}}=\Big|2u_s u^{-\beta}\frac{(\partial_t-\alpha u^{1-\frac{1}{\alpha}}\partial_{ss})u_s}{\alpha u^{1-\alpha}}-2u^{-\beta}u_{ss}^2\Big|$$
$$\leq C\frac{|(\partial_t-\alpha u^{1-\frac{1}{\alpha}}\partial_{ss})u_s|}{u^{1-\alpha}}+Cu^{-\beta}u_{ss}^2.$$

By using (3.8) and (3.9) we can estimate the first term above

$$\frac{|(\partial_t-\alpha u^{1-\frac{1}{\alpha}}\partial_{ss})u_s|}{u^{1-\alpha}}\leq Cu^{\beta-1}\Big(|v_s|u^{\beta-1}+u^{2\beta-1}+u^{\frac{2}{\alpha}+1}\Big)$$
$$=Cu^{-\beta}\Big(u^{2\beta-1}\Big)\Big(|v_s|u^{\beta-1}+u^{2\beta-1}+u^{\frac{2}{\alpha}+1}\Big)\leq CH.$$

Moreover, (3.5) and (3.9) show $u^{-\beta}u_{ss}^2\leq CH$. We conclude from the above discussion that all seven terms on $(\partial_t-\alpha u^{1-\frac{1}{\alpha}}\partial_{ss})v_s$ are bounded by $Cu^{1-\frac{1}{\alpha}}H$, finishing the proof of the claim. □

□

We are finally ready to give the proof of our main estimate which will be used in the next section to control the boundary terms. Note that while most of our previous estimates hold for $\alpha>0$ or $\alpha>1/3$, for our estimate below $\alpha>1/2$ is required.

Theorem 3.5. *Assume that M_t, $t\in[0,+\infty)$ is a solution of the α-CSF with $\alpha>1/2$ and the initial data M_0 satisfying the assumptions of Proposition 2.1. Then there exists $\epsilon=\epsilon(\alpha)>0$ so that*

$$\frac{|(\bar\kappa^\alpha)_s(\bar\kappa^\alpha)_t|}{\bar\kappa}=\frac{|u_s\,u_t|}{u^{1/\alpha}}\leq C(t_0,M_0)\,u^\epsilon,\qquad \textit{for } t>t_0.$$

Proof. By equation (2.6), $u:=\bar\kappa^\alpha$ satisfies

$$\frac{u_su_t}{u^{1/\alpha}}=u_s\,\alpha u^{1-\frac{1}{\alpha}}(u_{ss}+u^{1+2/\alpha})\,u^{-\frac{1}{\alpha}}=\alpha\,\frac{u_su_{ss}}{u^{\frac{2}{\alpha}-1}}+\alpha u_su^2.$$

By Proposition 2.7, we have a uniform upper bound on u for $t\geq t_0>0$ which combined with Proposition 3.1 yields desired bound for the second term. We will next take care the first term.

First, suppose $\alpha\in[1,\infty)$. Combining Proposition 3.1 and 3.4 together with our curvature bound (which is assumed in Proposition 3.4), implies that for $\epsilon\in(0,\frac{1}{\alpha})$,

$$|u_su_{ss}|\leq C(\alpha,\epsilon,M_0,t_0)\,u^{\frac{3}{\alpha}-1-\epsilon}\quad \text{for } \ t\geq t_0>0,$$

where we have used that for $\alpha\in[1,\infty)$, $\min(1/\alpha,2/\alpha-1)=2\alpha^{-1}-1$.

When $\alpha\in(1/3,1)$, then $\min(1/\alpha,2/\alpha-1)=1/\alpha$, thus Proposition 3.3, 3.4 and our curvature bound imply that for for $0<\epsilon<1$,

$$|u_su_{ss}|\leq C(\alpha,\epsilon,M_0,t_0)\,u^{1+\frac{1}{\alpha}-\epsilon}.$$

Since $\frac{2}{\alpha}-1<1+\frac{1}{\alpha}$ iff $\alpha>\frac{1}{2}$, we obtain the desired result for every $\alpha>\frac{1}{2}$. □

Corollary 3.6. *Under the same conditions as in Theorem 3.5 and for any $\alpha>1/2$, there is $\epsilon'(\alpha)>0$ such that*

$$|(\kappa^\alpha)_\theta(\kappa^\alpha)_t|\leq C(t_0,M_0)(\kappa^\alpha)^{\epsilon'}\quad \textit{for } t>t_0.$$

Proof. By (2.13) and Theorem 3.5,

$$|(\kappa^\alpha)_\theta(\kappa^\alpha)_t| = \frac{|(\bar\kappa^\alpha)_s|}{\bar\kappa}|(\bar\kappa^\alpha)_t - \alpha^2\bar\kappa^{2\alpha-3}\bar\kappa_s^2| \le C\Big((\bar\kappa^\alpha)^\epsilon + \frac{((\bar\kappa^\alpha)_s)^3}{\bar\kappa^2}\Big).$$

In addition, for any $\alpha > 1/3$, Propositions 3.1 and 3.3 imply that there is $\epsilon'(\alpha) > 0$ such that

$$\frac{((\bar\kappa^\alpha)_s)^3}{\bar\kappa^2} \le C(\bar\kappa^\alpha)^{\epsilon'}.$$

□

4. Convergence to Translator

In this final section, we prove our convergence result Theorem 2.3 from which Theorem 1.1 also follows. The main step in our proof is Lemma 4.4 which follows from our decay estimates in the previous section and an appropriate use of the following entropy.

Definition 4.1. For a strictly convex solution to the α-CSF, we define

$$J^\epsilon(t) := \frac{(\alpha+1)^2}{\alpha^2}\int_\epsilon^{\pi-\epsilon} (\kappa^\alpha)_\theta^2 - (\kappa^\alpha)^2\, d\theta$$

which can be also expressed in terms of the pressure function $p := \kappa^{\alpha+1}$, as

$$J^\epsilon(t) = \int_\epsilon^{\pi-\epsilon} \frac{p_\theta^2}{p^{\frac{2}{\alpha+1}}} - \frac{(\alpha+1)^2}{\alpha^2}p^{\frac{2\alpha}{\alpha+1}}\, d\theta.$$

Also, set

$$J(t) := \lim_{\epsilon\to 0} J^\epsilon(t) \in (-\infty,\infty]$$

and this is well defined due to curvature upper bound in Proposition 2.7.

Assume that M_t, $t\in[0,+\infty)$ is a solution of the α-CSF which satisfies the assumptions of Theorem 2.3. We first observe that $J(t)$ is bounded on $[t_0,+\infty)$, for all $t_0>0$.

Lemma 4.2. *For $\alpha \ge 1$, $J(t) \le C(t_0, M_0) < \infty$ for $t \ge t_0 > 0$.*

Proof. By the evolution of $p = \kappa^{\alpha+1}$ given in (2.15), we have

$$\begin{aligned}\frac{\alpha+1}{\alpha^2}\int_\epsilon^{\pi-\epsilon}\frac{p_t}{p^{\frac{2}{\alpha+1}}}\,d\theta &= \frac{\alpha+1}{\alpha}\int_\epsilon^{\pi-\epsilon} p^{\frac{\alpha-1}{\alpha+1}}p_{\theta\theta} - \frac{1}{\alpha}\frac{p_\theta^2}{p^{\frac{2}{\alpha+1}}} + \frac{(\alpha+1)^2}{\alpha^2}p^{\frac{2}{\alpha+1}}\,d\theta \\ &= -J^\epsilon(t) + \Big(\frac{\alpha+1}{\alpha}p^{\frac{\alpha-1}{\alpha+1}}p_\theta\Big)_{\theta=\epsilon}^{\theta=\pi-\epsilon}.\end{aligned} \tag{4.1}$$

Note that $p^{\frac{\alpha-1}{\alpha+1}}p_\theta = (\alpha+1)\kappa^{2\alpha-2}\kappa_s = \frac{(\alpha+1)}{\alpha}\frac{u_s}{u^{\frac{1}{\alpha}-1}}$ and this is uniformly bounded for $t \ge t_0$ when $\alpha > \frac12$ in view of Proposition 3.1. In addition, the Harnarck inequality in Proposition 2.5 implies,

$$-\frac{p_t}{p^{\frac{2}{\alpha+1}}} = -\frac{p_t}{p}p^{\frac{\alpha-1}{\alpha+1}} \le \frac{\kappa^{\alpha-1}}{t} \tag{4.2}$$

and therefore

$$\int_\epsilon^{\pi-\epsilon} -\frac{p_t}{p^{\frac{2}{\alpha+1}}} \le \int_\epsilon^{\pi-\epsilon}\frac{\kappa^{\alpha-1}}{t}d\theta \le \int_0^\pi \frac{\kappa^{\alpha-1}}{t}d\theta.$$

This integrand is uniformly bounded for $\alpha \geq 1$ and $t \geq t_0$. Combining the above shows that $J_\epsilon(t) \leq C(t_0, M_0) < \infty$, which implies the desired result. □

Proposition 4.3. *Suppose $\alpha \geq 1$. For $0 < t_1 < t_2 < \infty$, we have*

$$J(t_2) - J(t_1) = -\frac{2(\alpha+1)^2}{\alpha} \int_{t_1}^{t_2} \int_0^\pi \kappa^{\alpha+1} [(\kappa^\alpha)_{\theta\theta} + \kappa^\alpha]^2 \, d\theta dt.$$

Proof. Since everything is smooth and bounded on $[\epsilon, \pi - \epsilon] \times [t_1, t_2]$, we have

$$\begin{aligned}
\frac{d}{dt} J^\epsilon(t) &= \frac{(\alpha+1)^2}{\alpha^2} \int_\epsilon^{\pi-\epsilon} \Big(2(\kappa^\alpha)_\theta (\kappa^\alpha)_{t\theta} - 2(\kappa^\alpha)(\kappa^\alpha)_t \Big) \, d\theta \\
&= -\frac{2(\alpha+1)^2}{\alpha^2} \int_\epsilon^{\pi-\epsilon} \frac{\kappa_t (\kappa^\alpha)_t}{\kappa^2} \, d\theta + \Big(\frac{2(\alpha+1)^2}{\alpha^2} (\kappa^\alpha)_\theta (\kappa^\alpha)_t \Big)_{\theta=\epsilon}^{\theta=\pi-\epsilon} \\
&= \int_\epsilon^{\pi-\epsilon} \frac{-2(\alpha+1)^2}{\alpha} \frac{\kappa_t^2}{\kappa^{3-\alpha}} \, d\theta + \Big(\frac{2(\alpha+1)^2}{\alpha^2} (\kappa^\alpha)_\theta (\kappa^\alpha)_t \Big)_{\theta=\epsilon}^{\theta=\pi-\epsilon} \\
&= \frac{2(\alpha+1)^2}{\alpha^2} \Big(-\alpha \int_\epsilon^{\pi-\epsilon} \kappa^{\alpha+1} [(\kappa^\alpha)_{\theta\theta} + \kappa^\alpha]^2 \, d\theta + \Big((\kappa^\alpha)_\theta (\kappa^\alpha)_t \Big)_{\theta=\epsilon}^{\theta=\pi-\epsilon} \Big).
\end{aligned} \tag{4.3}$$

In view of Theorem 3.5, Theorem 2.9, and Lemma 4.2, for $\alpha \geq 1$ we can take $\epsilon \to 0$ and monotone convergence theorem implies the result. □

In the case $\alpha \in (1/2, 1)$, we cannot not show that the entropy is finite, so we avoid using the global entropy defined on $[0, \pi]$ and approach differently. Our decay estimate is sufficient to carry out this, as we see in the lemma below.

Lemma 4.4. *Assume that $\alpha > 1/2$. For fixed $\tau > 0$ and $\delta > 0$, we have*

$$\int_t^{t+\tau} \int_\delta^{\pi-\delta} \kappa^{\alpha+1} \big((\kappa^\alpha)_{\theta\theta} + \kappa^\alpha \big)^2 d\theta dt \to 0 \qquad \text{as } t \to \infty.$$

Proof. It suffices to prove that for every $\epsilon > 0$, there exist $\bar\delta \in (0, \delta)$ and $t_0 > 0$ such that

$$\int_t^{t+\tau} \int_{\bar\delta}^{\pi-\bar\delta} \kappa^{\alpha+1} \big((\kappa^\alpha)_{\theta\theta} + \kappa^\alpha \big)^2 \, d\theta dt \leq \epsilon \quad \text{for} \quad t \geq t_0.$$

In view of (4.3), for $0 < \bar\delta < \delta$ and $t \geq t_0 > 0$, we have

$$\int_t^{t+\tau} \int_{\bar\delta}^{\pi-\bar\delta} \kappa^{\alpha+1} [(\kappa^\alpha)_{\theta\theta} + \kappa^\alpha]^2 d\theta dt = \frac{\alpha}{2(\alpha+1)^2} (J^{\bar\delta}(t) - J^{\bar\delta}(t+\tau)) + \frac{1}{\alpha} \int_t^{t+\tau} \Big((\kappa^\alpha)_\theta (\kappa^\alpha)_t \Big)_{\theta=\bar\delta}^{\theta=\pi-\bar\delta} dt.$$

First, we control the boundary terms using Theorem 2.9 and Corollary 3.6

$$\begin{aligned}
\Big| \int_t^{t+\tau} \Big((\kappa^\alpha)_\theta (\kappa^\alpha)_t \Big)_{\theta=\bar\delta}^{\theta=\pi-\bar\delta} dt \Big| &\leq \tau \sup_{t \in [t_0, \tau]} [(\kappa^\alpha)_\theta (\kappa^\alpha)_t](\bar\delta, t) + \tau \sup_{t \in [t_0, \tau]} [(\kappa^\alpha)_\theta (\kappa^\alpha)_t](\pi - \bar\delta, t) \\
&\leq 2\tau \, C(t_0, \bar\delta, M_0) \quad \text{with } C(t_0, \bar\delta, M_0) \to 0 \text{ as } \bar\delta \to 0.
\end{aligned}$$

Thus, for given $\epsilon > 0$ and $t_0 > 0$, there exists δ_0 such that if $0 < \bar\delta \leq \delta_0$ and $t \geq t_0$,

$$\Big| \int_t^{t+\tau} \Big((\kappa^\alpha)_\theta (\kappa^\alpha)_t \Big)_{\theta=\bar\delta}^{\theta=\pi-\bar\delta} dt \Big| \leq \epsilon.$$

To finish the proof of the lemma it suffices to prove the following claim.

Claim 4.1. *For every $\epsilon > 0$, there exists $\delta_0 > 0$ such that for each $0 < \bar{\delta} \leq \delta_0$ we can find $t_0 = t_0(\bar{\delta}) > 0$ such that*

$$|J^{\bar{\delta}}(t)| \leq \epsilon \qquad \textit{for } t \geq t_0.$$

Proof of Claim 4.1. We prove the upper and lower bound separately. The proof of the upper bound uses (4.1) i.e. we bound $J^{\bar{\delta}}(t)$ in terms of the integral term and boundary term in (4.1). To bound the integral term, we use (4.2), the curvature lower bound for $\alpha \in (1/2, 1)$, and the curvature upper bound for $\alpha \geq 1$ (both shown in Proposition 2.7) to obtain

$$\int_{\bar{\delta}}^{\pi-\bar{\delta}} -\frac{\alpha+1}{\alpha^2}\frac{p_t}{p^{\frac{2}{\alpha+1}}}d\theta \leq \int_{\bar{\delta}}^{\pi-\bar{\delta}} -\frac{\alpha+1}{\alpha^2}\frac{\kappa^{\alpha-1}}{t}d\theta \to 0 \quad \text{as} \quad t\to\infty.$$

To bound the boundary term, we note that $p^{\frac{\alpha-1}{\alpha+1}}p_\theta = (\alpha+1)\kappa^{2\alpha-21}\kappa_s = (\alpha+1)\frac{u_s}{u^{\frac{1}{\alpha}-1}}$ and $\frac{1}{\alpha}-1 < 1$ for $\alpha > \frac{1}{2}$. Therefore, Proposition 3.1 and Theorem 2.9 imply that for any given $\epsilon > 0$ and $t_0 > 0$, there exists δ_0 such that if $0 < \bar{\delta} \leq \delta_0$ and $t \geq t_0$ we have $\left|\left(\frac{\alpha+1}{\alpha}p^{\frac{\alpha-1}{\alpha+1}}p_\theta\right)_{\theta=\bar{\delta}}^{\theta=\pi-\bar{\delta}}\right| \leq \epsilon$. This completes the proof of the upper bound.

For the lower bound, we will use the 1-dim optimal Poincaré inequality, namely the bound

$$\int_{\delta}^{\pi-\delta} f'(s)^2\,ds - \Big(\frac{\pi}{\pi-2\delta}\Big)^2\int_{\delta}^{\pi-\delta} f(s)^2\,ds \geq 0$$

which holds for every smooth function f with $f(\delta) = f(\pi-\delta) = 0$. The equality holds for properly scaled sine functions. To apply it for our case, recall that

$$\frac{\alpha^2}{(\alpha+1)^2}J^{\bar{\delta}}(t) = \int_{\bar{\delta}}^{\pi-\bar{\delta}} \big((\kappa^\alpha)_\theta^2 - \kappa^\alpha\big)d\theta$$

and set $U(\theta,t) := \kappa^\alpha(\theta,t)$ and $-L(\theta,t) := \frac{U(\pi-\bar{\delta},t)-U(\bar{\delta},t)}{\pi-2\bar{\delta}}(\theta-\bar{\delta})+U(\bar{\delta},t)$ (note that we distinguish the notation of $U(\theta,t) := \kappa^\alpha(\theta,t)$ from $u(n,t) := \bar{\kappa}^\alpha(n,t)$ which uses the geometric parametrization). Since $(U+L)(\bar{\delta}) = (U+L)(\pi-\bar{\delta}) = 0$, the Poincaré inequality above combined with Young's inequality imply

$$\begin{aligned}
0 &\leq \int_{\bar{\delta}}^{\pi-\bar{\delta}} \Big((U+L)_\theta^2 - \Big(\frac{\pi}{\pi-2\bar{\delta}}\Big)^2(U+L)^2\Big)\,d\theta \\
&= \int_{\bar{\delta}}^{\pi-\bar{\delta}} \Big(U_\theta^2 - \frac{\pi^2}{(\pi-2\bar{\delta})^2}U^2 + L_\theta^2 - \frac{\pi^2}{(\pi-2\bar{\delta})^2}L^2 + 2U_\theta\,L_\theta - \frac{2\pi^2}{(\pi-2\bar{\delta})^2}UL\Big)\,d\theta \\
&\leq \int_{\bar{\delta}}^{\pi-\bar{\delta}} \Big(U_\theta^2 - \frac{\pi^2}{(\pi-2\bar{\delta})^2}U^2 + L_\theta^2 - \frac{\pi^2}{(\pi-2\bar{\delta})^2}L^2 \\
&\qquad\qquad + \frac{2\bar{\delta}}{\pi-2\bar{\delta}}U_\theta^2 + \frac{\pi-2\bar{\delta}}{2\bar{\delta}}L_\theta^2 + \frac{2\bar{\delta}\pi U^2}{(\pi-2\bar{\delta})^2} + \frac{\pi^3L^2}{2\bar{\delta}(\pi-2\bar{\delta})^2}\Big)\,d\theta \\
&= \int_{\bar{\delta}}^{\pi-\bar{\delta}} \frac{\pi}{\pi-2\bar{\delta}}\big(U_\theta^2 - U^2\big) + \frac{\pi}{2\bar{\delta}}\big(L_\theta^2 + \frac{\pi}{\pi-2\bar{\delta}}L^2\big)\,d\theta
\end{aligned}$$

We conclude that

$$J^{\bar{\delta}}(t) := \frac{(\alpha+1)^2}{\alpha^2}\int_{\bar{\delta}}^{\pi-\bar{\delta}} \big(U_\theta^2 - U\big)\,d\theta \geq \frac{(\alpha+1)^2}{\alpha^2}\frac{\pi-2\bar{\delta}}{\pi}\frac{\pi}{2\bar{\delta}}\int_{\bar{\delta}}^{\pi-\bar{\delta}} \big(L_\theta^2 + \frac{\pi}{\pi-2\bar{\delta}}L^2\big)d\theta.$$

To estimate the last integral above we observe that by Theorem 2.9, we have $|L|$ and $|L_\theta| \leq C(M_0)\bar{\delta}^{2/3}$ on $[\bar{\delta}, \pi-\bar{\delta}]$ for all $\bar{\delta} \in (0, \frac{\pi}{4})$ and $t > 3$. Hence, we have

$$J^{\bar{\delta}}(t) \geq -C(M_0,\alpha)\,\bar{\delta}^{2\frac{2}{3}-1} = -C(M_0,\alpha)\,\bar{\delta}^{\frac{1}{3}}$$

which gives the bound from below. This completes the proof of the claim. ∎

∎

We are now in position to give the proof of our main convergence result, Theorem 2.3. We have already observed in section 2 that Theorem 1.1 follows from Theorem 2.3.

Proof of Theorem 2.3. Recall $U(\theta,t) := \kappa^\alpha(\theta,t)$ solves the equation

$$U_t = \alpha U^{1+\frac{1}{\alpha}}(U_{\theta\theta} + U) \qquad \text{on } (0,\pi)\times(0,\infty). \tag{4.4}$$

For a given time sequence $t_i \to \infty$, we define the sequence of solutions $U^i(\theta,t) := U(\theta, t+t_i)$. By Proposition 2.7, the sequence $\{U^i\}$ is locally uniformly bounded from above and below in spacetime and $i \gg 1$. That is, for any compact spacetime region, there is $i_0 \gg 1$ such that $\{U^i\}_{i\geq i_0}$ is uniformly bounded from above and below by positive numbers. This implies that equation (4.4) is uniformly parabolic for $U = U^i$, $i \geq i_0$ and therefore parabolic regularity theory implies that we have locally uniform control on derivatives of the u_i of all orders. By the Arzelà-Ascoli theorem, we can find a subsequence, still denoted by U^i, such that $U^i \to \bar{U}$ uniformly on compact sets but also

$$U^i \to \bar{U} \qquad \text{in } C^\infty_{loc}((0,\infty)\times(-\infty,\infty)).$$

Then, the Lemma 4.4 implies that $\bar{U}_{\theta\theta} + \bar{U} = 0$, thus $\partial_t \bar{U} = 0$. In addition, Proposition 2.7 and Theorem 2.9 give $\bar{U} > 0$ and $\lim\limits_{\theta\to 0}\bar{U}(\theta) = \lim\limits_{\theta\to\pi}\bar{U}(\theta) = 0$. Hence, we have

$$\bar{U}(\theta) = c\sin\theta$$

for some constant $c > 0$. We will next show that $c = m(\alpha)$, where $m(\alpha)$ is given by (2.4). For this, it suffices to show that

$$U(\pi/2, t) := \kappa^\alpha(\pi/2, t) \to m(\alpha), \qquad \text{as } t\to\infty. \tag{4.5}$$

Proof of (4.5)*:* Let's suppose first that $\liminf_{t\to\infty} U(\pi/2,t) < m(\alpha)$. Then in view of the curvature lower bound in Proposition 2.7, there is a sequence $t_i \to \infty$ such that $U(\theta, t_i) \to m'\sin\theta$ locally smoothly on $(0,\pi)$ for some $m' \in (0, m(\alpha))$. Let $\big(x_1(\theta,t), x_2(\theta,t)\big)$ be the position vector of our solution M_t parametrized by θ. For small $\epsilon > 0$, this convergence and (2.3) imply that we have, for $x_1(\theta,t)$,

$$x_1(\pi-\epsilon, t) - x_1(\epsilon, t) = \int_\epsilon^{\pi-\epsilon}\frac{\sin\theta}{\kappa(\theta,t)}d\theta \to (m')^{-1/\alpha}\int_\epsilon^{\pi-\epsilon}(\sin\theta)^{1-\frac{1}{\alpha}}d\theta \qquad \text{as } t\to\infty. \tag{4.6}$$

Recall the assumptions of Theorem 2.3 and Proposition 2.1 which imply that M_{t_0} is a graph on $(-1,1)$, an interval of length 2. In view of (2.4) and $m' < m(\alpha)$, we can find a small $\epsilon(m') > 0$

depending on m' and a large $t_0(\epsilon, m') > 0$ depending on ϵ, m' such that $x_1(\pi-\epsilon, t_0) - x_1(\epsilon, t_0) > 2$. This gives a contradiction. Therefore,

$$\liminf_{t\to\infty} U(\pi/2, t) \geq m(\alpha). \tag{4.7}$$

Next, suppose $\limsup_{t\to\infty} U(\pi/2,t) > m(\alpha)$ and hence there is a sequence $t_i \to \infty$ such that $U(\pi/2, t_i) \geq (1+4\epsilon)\, m(\alpha)$, for some $\epsilon > 0$. In view of the Harnack estimate Proposition 2.5, there is $c(\epsilon) > 0$ such that $U(\pi/2,t) \geq (1+3\epsilon)\, m(\alpha)$ for $t \in [t_i, (1+c)t_i]$. Meanwhile, the inequality (4.7) implies that there is $\bar{t} > 0$ such that $U(\pi/2,t) > (1-c\epsilon)\, m(\alpha)$ for $t > \bar{t}$. Note that $\partial_t x_2(\pi/2, t) = \kappa^\alpha(\pi/2, t)$ and therefore,

$$\begin{aligned} x_2(\pi/2, (1+c)\, t_i) &= x_2(\pi/2, t_i) + \int_{t_i}^{(1+c)t_i} \kappa^\alpha(\pi/2, \tau)\, d\tau \\ &\geq [(1-c\epsilon)\, m(\alpha)\, t_i - C] + (1+3\epsilon)\, (ct_i)\, m(\alpha) \\ &= m(\alpha) \Big(1 + \frac{2c\epsilon}{1+c}\Big)(1+c)\, t_i - C. \end{aligned}$$

On the other hand, we can put a translating soliton of speed $m(\alpha)\Big(1+\frac{c\epsilon}{1+c}\Big)$ above M_0 and inside $\{|x_1| < 1-\delta\}$, for some $\delta(\epsilon, c) > 0$ depending on ϵ, c at the initial time $t = 0$. Then, by the comparison principle

$$x_2(\pi/2, (1+c)\, t_i) \leq m(\alpha) \Big(1 + \frac{c\epsilon}{1+c}\Big)(1+c)\, t_i + C$$

which contradicts the previous inequality for $t_i \gg 1$. This completes the proof of (4.5).

We have just seen that the sequence U^i smoothly converges to $\bar{U} = m(\alpha)\sin\theta$ on compact sets along arbitrary sequence. Thus, $U(\cdot, t) \to \bar{U}$ in $C^\infty_{loc}((0,\pi))$ as $t \to \infty$. From the convergence (4.6) with $m' = m(\alpha)$ and Proposition 2.1, it is easy to see $x_1(\pi/2, t)$, the x_1 coordinate of the tip, converges to 0 as $t \to \infty$. Then (2.3), Proposition 2.4 and the convergence of $\kappa(\theta, t)$ to $\big(m(\alpha)\sin\theta\big)^{1/\alpha}$ yield our desired convergence of the graphical function stated in Theorem 2.3. This completes the proof of Theorem 2.3.

□

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