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Aspects of predication and their influence on reasoning about logic in discrete mathematics

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Abstract

This theoretical paper sets forth two *aspects of predication*, which describe how students perceive the relationship between a property and an object. We argue these are consequential for how students make sense of discrete mathematics proofs related to the properties and how they construct a logical structure. These aspects of predication are (1) *populating* the way students generate sets of examples of the property, and (2) *testing membership* how one tests whether or not a given object has a specific property. Using data from two teaching experiments in which undergraduate students read proofs of theorems about the discrete concept of multiple relations, we illustrate the nature of these aspects of predication and demonstrate how they help explain student interpretations of the proofs. We argue that these particular properties from number theory likely have correlates in many other discrete mathematics topics because of the role of computation/algorithms for defining and testing properties as well as the role of iteration and recursion in populating examples. We anticipate that these constructs will be useful to teachers and researchers of discrete mathematics to foster and assess student understanding of various mathematical properties. They provide tools for thinking about what it means to understand properties in a rich and coherent way that supports understanding complex lines of inference and generalizations.

Keywords Aspects of predication · Logic · Multiple relations · Proof · Discrete mathematics

1 An illustrative example proof task in graph theory

In this theoretical paper, we are concerned with the cognitive processes involved in students making sense of mathematical proofs and discerning logical structure in those proofs and their associated theorems. Consider the following statement of a graph theory theorem (along with the surrounding text) from an inquiry-based learning textbook meant to introduce students to mathematical proving by having them discover such proofs on their own:

> “We are now ready to characterize those graphs that have an Euler circuit. Determining whether a graph has an Euler circuit turns out to be easy to check. It is always a pleasure when a property that appears to be difficult to determine actually is rather simple.
>
> **Euler Circuit Theorem 2.31.** A graph G has an Euler circuit if and only if it is connected and every vertex in G has an even, positive degree.
>
> If you truly understand the proof of this theorem, you should be able to take a graph and produce an Euler circuit, if it has one, using the technique implicit in your proof.” (Katz & Starbird, 2013, p. 21)

In analyzing what is cognitively involved in proving this theorem, we note that the existence of an Euler circuit is treated as a property that any graph either has or lacks. This property is attributed to the graph, not just the circuit. Also, the text describes the existence of an Euler circuit as “difficult to determine,” which motivates the theorem’s value by providing an alternative means of identifying the property. The theorem’s statement alone does not alleviate the challenge since students must generate a method of producing such circuits to prove the theorem, as pointed out in the latter part of the quote. While not unique to discrete mathematics, discrete mathematics frequently defines objects via properties that involve some computational process (as well as properties of computation processes themselves, Modeste

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& Ouvrier-Buffet, 2011) or introduced objects (such as an Euler circuit). In this paper, we consider the particular challenges posed to students learning to read and write proofs involving such properties.

To prove the theorem above, students must engage in at least two critical ways of reasoning about Euler circuits as characterizations of the structure of graphs. To prove the "if" direction of the theorem, students must stipulate that a connected graph has all vertices of even, positive degree. They must then generate a procedure for constructing an Euler circuit on this imagined object. This procedure must be general enough to anticipate how it would be produced on any such graph. To prove the "only if" direction, students must (by the power of mathematical imagination or authority) stipulate that a graph has an Euler circuit. Note this is a bit strange since it bypasses the work of construction by the mere power of assumption. Assuming a circuit exists entails anticipating the completion of the construction process without carrying it out and anticipating that one may simply call forth any member of the set of graphs upon which such circuits can be constructed.

In the data featured in this paper, students reason about proofs about multiple relations (i.e., the relationship of one number being a multiple of/divisible by another). Though much more familiar, the definition of multiple shares some features with having an Euler circuit. First, x being a multiple of/ divisible by d is defined in terms of an introduced object k, such that $x = k * d$. We introduce this object differently in various proofs depending upon whether the multiple of claim is a hypothesis or conclusion. In the former case, k is merely stipulated to exist; in the latter, it must somehow be constructed from other assumptions. Second, just as the equivalent property in the theorem—connected graph with all vertices of even, positive degree—does not make any reference to the circuit, students often reason about multiple relations without attending to k. In particular, when thinking about "multiples," we often find students skip-count to enumerate examples. When thinking about "divisible," they may engage in dividing by d to determine whether the quotient k is an integer. Enumerating examples (by an iterative process) is a form of what we call *populating* and dividing by d is a *membership test*, which are two features of these properties that we shall consider in this paper.

2 The goals of this investigation

In our ongoing efforts to teach mathematical logic to university students, we have become particularly sensitive to the challenges posed by the ways of reasoning required by the proof task above—constructing an introduced object to verify a property and stipulating that such an object exists by hypothesis. In a series of teaching experiments, we have students read theorem/proof pairs, which vary by context and logical form, to determine whether the proof proves the given theorem or proves something else. Our goal is for students to construct a sense of sameness across the theorem/proof pairs tantamount to logical structure. By attending to the different contours of student reasoning in number theory versus geometry, we have learned both about the opportunities for abstraction across the settings (see Dawkins et al., 2021) and the more local features that might impede it. In this paper, we focus on the local features we have observed in students' reasoning about multiple relations that we predict will be indicative of broader phenomena in discrete settings, as portrayed in our analysis of Euler circuits above.

Consistent with teaching experiment methodology (Steffe & Thompson, 2000), we operated on a learning hypothesis about how students can abstract the logic of theorems and proofs using set theory. We guided students to associate properties with the entire set of objects that have the property and to attend to how chains of argument in proof apply to a range of objects. If students can do this, they can construct analogies between quite different theorems based on a shared set structure (see Dawkins, 2017). However, pursuing such generalization has helped us appreciate how difficult it is for students to reason about these sets when the properties in question involve computation and/or introduced objects. This theoretical paper identifies some constructs that describe these aspects of the relationship between a property and referent objects that influence how students reason about proofs. We refer to them as *aspects of predication*. The two aspects of predication we highlight are:

1. *Populating*—how one imagines generating examples, and
2. *Membership testing*—how one tests whether a particular object has a given property.

We can illustrate these two constructs in relation to the Euler circuit theorem above. Constructing an Euler circuit constitutes a membership test for an object. When comparing the two parts of the proof ("if" and "only if"), students must engage in populating in quite different ways. In one case, they must generate the circuit to show that the test is met while in the other they must simply anticipate the circuit being given without any act of construction.

3 Our experiments and their connections to discrete mathematics

We have conducted five complete teaching experiments in this sequence, two of which are featured in this paper. The subjects of the experiment were pairs of undergraduate mathematics students recruited from multi-variable calculus

Table 1 Forms of proof presented for comparison (only columns 1 and 4 prove)

Direct proof	Converse proof	Converse disproof	Contrapositive proof	Inverse disproof
Original theorem	"For any $x \in S$, if $Q(x)$, then $P(x)$."	"For any $x \in S$, if $Q(x)$, then P (x)."	"For any $x \in S$, if not $Q(x)$, then not P (x)."	"For any $x \in S$, if not $P(x)$, then not $Q(x)$."
Proof: Let x have property P … Thus, x has property Q	Proof: Let x have property Q … Thus, x has property P	Proof: Let x have property Q x could be a a does not have property P	Proof: Let x not have property Q … Thus, x does not have property P	Proof: Let x not have property P x could be a a does have property Q

courses in the United States. We provided a pre-screening survey to confirm volunteers did not already provide normative interpretations of whether proofs prove theorems. The tasks in the experiment primarily consisted of universally quantified conditional theorems (of the form "for all $x \in S$, if $P(x)$, then $Q(x)$"), paired with various forms of proof. The theorems were all true, and the proofs contained no errors. The proofs presented were of a few basic forms, as presented in Table 1. Our goals were for students to link the various proofs with the same logical form and to develop self-consistent and normative rules about whether each proof type proves the given theorem. For instance, the *principle of universal generalization* claims that direct proofs prove the theorem, *contrapositive equivalence* justifies why contrapositive proofs do prove the theorem, and *converse independence* asserts that converse proofs do not prove the theorem.

One of the two authors served as the teacher/researcher (Cobb & Steffe, 1983) for each teaching experiment, with the other acting as an outside observer. The researcher met with students in a classroom or over zoom (during the pandemic) for 1–1.5 h/week for 6–12 weeks, with all sessions video recorded and all student work maintained. The authors attempted to model student thinking and student learning iteratively, both during and between sessions to form hypotheses about student reasoning that could be tested in future interactions. Relevant to the current paper, we attended to (1) the students' meanings for the properties in the theorems and proofs, (2) their arguments for why the theorem was true or false, (3) their interpretations of the argument in the proof, and (4) their ways of determining whether the proofs did or did not justify the theorem. Central to the teaching hypothesis of the experiment was the idea that students construct logic by reflecting on their own reasoning across parallel tasks. To foster this activity, the teacher/researcher invited students to compare their reasoning with their partner's and with their own reasoning on previous tasks. To invite abstraction, the teacher/researcher often selected student-produced diagrams or student-generated lines of argument and asked students to apply them again on subsequent theorem/proof pairs. As noted above, such analysis entailed attending both to the specific features of student reasoning within each semantic context (number theory or geometry) and efforts to identify what might generalize across those contexts.

The teaching sessions were conducted to initially elicit students' ways of reasoning about the theorems and proofs. We formed models of their ways of thinking about the predicates in each theorem, the negations of those predicates, whether they could reason about the truth sets of the predicates, and how they interpreted the relationships between the predicates and truth sets. As noted above, this provided a natural occasion for us to consider students' links between properties and objects and to trace the consequences of these ways of reasoning. Following Dawkins (2017), we refer to the propensity to associate a property to the entire set of objects that have the property *reasoning with predicates*. We worked over the course of the experiments to foster this way of reasoning as we perceive it to be propitious to students' abstraction of logical structure. We have observed that this activity operates differently for students in discrete settings such as number theory because students can iteratively produce examples, in contrast with geometry settings in which no such iterative structure is typically available. We also see connections between reasoning with predicates and Lockwood's (2013) focus on enumerating sets of outcomes in combinatorics settings. It is for these reasons that we highlight the unique modes of populating available among particular properties in discrete settings.

The data in this paper features students' reasoning about multiple relations in number theory. Tests for divisibility relations may constitute one of the simplest and most familiar membership tests in discrete mathematics. However, the complexity of student reasoning we illustrate in this report suggests that these phenomena might be all the more challenging regarding complex properties like having an Euler circuit. As described above, we specifically investigate student reasoning about proofs using *arbitrary particular* objects.[1] Discrete settings often also feature iteration and recursion (Ouvrier-Buffet et al., 2018), which are other modes of generalization (Ellis et al., 2021) worthy of study. As alluded to above, we shall observe iterative thinking in our data, though not recursion.

3.1 Situating our work and our choice of terms

The association between a property and (sets of) objects has most frequently been studied under the lens of defining (Alcock & Simpson, 2002; Edwards & Ward, 2008; Zaslavsky & Shir, 2005) or concept formation (Ourvrier-Buffet 2006, 2011; Vergnaud, 1996). Freudenthal (1973) distinguished between classical *descriptive defining* that seeks to capture a familiar category or to express an existing meaning and *constructive defining* that produces new knowledge. Over the centuries, mathematicians have shifted from desiring the former to focusing on the latter. Specifically, modern mathematical definitions need not tie to any intuitive meaning. Lakatos' (1976) exploration of the Euler conjecture features a stark example of this when modern mathematicians define polyhedra using matrices, dispensing with any apparent ties between that concept and 3-dimensional geometry. Students' understandings of a concept often exist as a constellation of meanings and ways of reasoning that need not be internally consistent and may lack internal structure (such as what properties define and whether other properties are equivalent, necessary, or sufficient for the defining property; c.f., Hershkowitz, 1987). As Vergnaud (1996) argues, students' understandings of concepts and definitions are integrally rooted in the problems solved with and about them over years of experience. For these reasons, we draw upon research on defining. Still, we find it helpful to avoid terms such as *definition* or *concept* as those terms may assume too much about the internal organization of student meanings. We view reasoning with predicates as a way of reasoning that students must construct in practice and continually reconstitute in new mathematical contexts.

3.2 On the link between a property and an object

Psychological research on how people reason about categories has tended to emphasize the influence of exemplars or characteristic attributes (e.g., Alcock & Simpson, 2002; Murphy & Hoffman, 2012). In our studies, we have generally found it useful to categorize students' strategies for thinking about a concept as focusing on (1) examples, (2) the set of all examples, or (3) their meanings for a property. Ouvrier-Buffet's (2006, 2011) work on student defining of discrete mathematics concepts first sensitized us to the additional notion that students' reasoning about a category may take the form of (4) a test for membership. When students in her study were tasked with determining which discrete sets of squares in a grid constituted [a line],[2] students adopted various in-action criteria that could serve to help link possible exemplars and exclude others.

3.3 Some emergent distinctions in how students reason about various properties

Our experiments involved presenting proof texts to undergraduates in the United States who have not yet received any instruction in mathematical proofs. We used topics that are familiar to students and do not provide any instruction on these topics (though we provided definitions and refreshing explanations). We thus relied on whatever meanings students evoke. The data we present in this paper regards proofs about multiple relations, which both connect these experiments to this special issue and are the very first tasks our study participants see. Before exploring student data, we will outline some subtle distinctions between properties based on the aspects of predication.

Example 1 [Even] and [odd] are good examples of divisibility properties that students tend to easily associate with their truth sets. Students are highly familiar with [even] and [odd], can easily enumerate examples of such numbers, and have an easy way to classify any given example without much effort (using the value in the one's place). As a result, students have little trouble populating or reasoning about the tests for membership.

However, students' familiarity with [even] and [odd] is such that defining attributes may not play a prominent role in student reasoning about them. The disconnect between the one's digit test and the defining attributes likely explains this effect. We are fascinated by Morris' (2002) evidence of college students who question whether even and odd numbers alternate throughout the set of counting numbers. One explained their rejection of both deductive and inductive justifications of that claim, saying:

> Who's to say number one zillion may not disprove the theory [that even and odd numbers alternate]?

[1] We shall call these *carriers* of properties rather than *examples*, since they operate to identify the property without a particular object rather than to identify an object that has the property.

[2] For the remainder of the paper, we shall notate predicates in square brackets for clarity.

Definition 1: We say the integer n is a **multiple of d** whenever there exists some integer k such that $n = k * d$.
This can also be stated as "n is divisible by d" or that "d divides n."
Notice that in the case that n is a multiple of d meaning $\frac{n}{d} = k$, k is called the **quotient** of dividing n by d.

Theorem to be proven 1: For every integer x, if x is a multiple of 6, then x is a multiple of 3.

Proof 1.1: Let x be any integer that is a multiple of 6.
Then by Definition 1, there exists some k such that $x = k * 6$.
Since $6 = 2 * 3$, $x = k * 6 = (k * 2) * 3$.
Since $k * 2$ is also an integer, according to Definition 1 x is a multiple of 3.

Theorem to be proven 2: For any integer x, if x is a multiple of 2 and a multiple of 7, then x is a multiple of 14.

Proof 2.1: Let x be an integer that is a multiple of 2 and a multiple of 7.
Then by definition there is some integer k such that $x = k * 7$.
Since x is a multiple of 2, either k or 7 must be even.
Thus k is even, and for some integer j, $k = j * 2$.
Substituting into the equation for x, we see $x = (j * 2) * 7 = j * (2 * 7) = j * 14$.
Thus, x is a multiple of 14.

Proof 2.2: Let x be an integer that is a multiple of 14.
Then by definition there is some integer k such that $x = k * 14$.
This means $x = (k * 7) * 2$, so x is a multiple of 2.
This also means $x = (k * 2) * 7$, so x is a multiple of 7.
Thus, x is a multiple of 7 and a multiple of 2.

Fig. 1 Proofs presented to students on the first day of the teaching experiments

> Who's to say that that number wouldn't just happen to have a unique quality of some sort? Maybe there's gonna be something nobody ever realized before… We think we know all the numbers but maybe we really don't. (Morris, 2002, p. 103)

We interpret this student as conceiving of even and odd as species of numbers that exist in the wilds of the number sequence. If we cannot rule out the existence of an animal in the practically inexhaustible wilds of the earth, how much less can we rule out the existence of a number in the literally inexhaustible wilds of the number sequence? Such stories inform and challenge us regarding the local features of how students relate defining attributes, sets of objects, and tests for properties.

Example 2 [Multiple of 14] is mathematically similar to [even], but students are less likely to reason about it in quite the same way. They have had much less opportunity to make use of this property for any other mathematical work. Students could easily work with a number without ever knowing or checking whether it had [multiple of 14] without being prompted. Students could probably generate some examples of [multiple of 14] by skip counting, but testing whether something has [multiple of 14] requires performing division that most students would be loath to complete mentally.

Example 3 [Divisible by 14] is mathematically equivalent to [multiple of 14], but we have often found it produces slightly different reasoning for students. First, divisibility focuses students on the test of dividing by, which is the most labor-ridden aspect of their understanding of [multiple of 14]. Furthermore, many undergraduates only infrequently state the requirement that the outcome of division is an integer and, at times, question whether all numbers are [divisible by 14] because any number can be so divided. Students in such cases conflate the property [divisible by 14] with the instructions "divide by 14." Even when this computation is encapsulated algebraically by an expression such as "$\frac{x}{14} = k$," our study participants interpret either side of the equation as being "divisible" rather than x as bearing the property.

We hope this brief preview gives the reader a sense of how concepts that are, on the one hand, formally isomorphic can, on the other, operate in distinct ways for students. Furthermore, we hope this clarifies our interest in the aspects of predication that help us account for the distinctions in meaning that seem salient to students. We now turn to present data from two teaching experiments to illustrate these distinctions in student activity and to demonstrate how the aspects of predication seem to clarify some phenomena we observe therein.

4 Data from teaching experiment 1 with Jess and Zandra

In this section, we shall present some data from the first of two teaching experiments featured in this paper. On the first day of both experiments, we invited the pair of undergraduates to reason about the proofs shown in Fig. 1. The participants in the first experiment, whom we call Jess and Zandra, were engineering majors.

4.1 Jess and Zandra's reasoning: conflating operations on *x* and properties of *x*

Early in the reading of Proof 1.1 (direct proof), Jess explained that the relationship between k and x was confusing to her. After providing some clarifying explanation about what the theorem itself said, the interviewer returned to the question, "what is k, or when we put k into the picture there on the second line, what's going on?" Zandra answered, "We need to figure that out I think." From our perspective, k is being stipulated to exist by hypothesis. We interpret Zandra's response to mean that she wants to engage in the familiar activity of solving for k to find the value of an unknown (c.f., Dawkins & Zazkis, 2021). Based on line 3, they inferred that k could have the values 2 or 3.

The interviewer decided to elicit their meanings for multiple given their difficulties in thinking about the given text. He proceeded:

Int: Before I gave you this, how did you think about, if I say, one number is a multiple of a number, how do you think about that?... Let's talk about multiples of 5. What do you think it means to say some number's a multiple of 5?
Zan: 5, 10, 20, 25.
Int: So you think about listing them all?
Zan: Yeah, listing them all.
Int: Okay, so what about 20 makes it a multiple of 5?
Zan: 4? 5 times 4 is 20.
Jes: Yeah. How many times by that number, I guess makes it a multiple.

Notice that Zandra's first evoked meaning for [multiple of 5] was to list examples by skip-counting (an iterative form of populating). This meaning gave no role to k, which Zandra addressed by shifting to a multiplication meaning, more consistent with the definition provided. When the interviewer asked, "What's the equation say?" Zandra answered, "n gets divided by 5." We note that her wording focuses on a process of computation, not a property.

The interviewer anticipated that Zandra and Jess would think of k as the number of copies of 5 in skip-counting or as the quotient when dividing. But when the interviewer asked again "what is the k?," Zandra instead answered in terms of k's value. "If you're assuming that n is some integer, then it could be anything. If you're dividing, assuming n as 5, then k would be 1. If you're assuming n as 10, then k could be 2." We note that there is an ambiguity here identifying k as a number (i.e., a value) or as a carrier of a property in the definition of [multiple of d]. The interviewer intended the students to interpret k in terms of its role, but Zandra at this point focused on its value.

The interviewer then asked, "What values do you think k will take on? How should we think about k under there in the second line?" Zandra replied, "We could do x divides by 6, so x could be, if x is 2, then it would be 1 divided by 3, k could be… one third… If x is 3, then it would be one half, k would be half." To interpret k, Zandra and Jess seemed drawn to division since k was the outcome of dividing x by 6. However, by adopting a division meaning they conflated the property [multiple of 6] with the operation dividing by 6, without the constraint of yielding an integer. When the interviewer claimed that 2 and 3 were not multiples of 6, Zandra asked, "x can be any integer, right?" Zandra's interpretation of [multiple of 6] at this point did not function to distinguish objects that have and do not have the property.

The interviewer then returned to Zandra's original meaning by having her list the multiples of 6, which she did by skip-counting. She affirmed that 2 and 3 were not multiples of 6. However, both students continued in preferring the division meaning, showing some sense of satisfaction by rewriting the equation in line 2 as $\frac{x}{6} = k$. The interviewer invited them to think of [multiple of 6] in terms of division. Zandra elaborated her reasoning as follows:

k would be a multiple of 6, right? If you're dividing x by 6… Because any multiple, anything that divides by k could be, would be a multiple of k, right? Multiple of 6, right? So k would be the answer if you're assuming x as 12, k would be 2, that would be a multiple of 6, yeah.

We interpret that Zandra recognized that the division meaning gave a clearer role to k in the definition and that it was consequential in determining the property of x. However, this led her to associate the property to the introduced object k, rather than or in addition to x. This further reflects a form of conflating the property (of x) with the operation (on x) involved in a membership test.

4.2 Jess and Zandra's reasoning: the need to coordinate [multiple of 6] and [multiple of 3]

The interviewer then described how k could also be interpreted as the number of groups of 6 in the number, trying to give k a meaning in Zandra's skip-counting notion of [multiple of 6]. He then invited the students to revisit interpreting the proof, specifically the third and fourth lines. At times, Jess and Zandra discussed choosing values for k to generate values of x, consistent with their skip-counting meaning. Later on, Zandra began rather consistently applying her division meaning to each instance of [multiple of d] in the text. This led her to claim that when $x = 12$, k was 2 in line 2 (when dividing by 6) and k was 4 in line 4 (when dividing by 3). In this way, k took on the role of quotient in the division process entailed in the definition; she did not interpret the proof such that k retained its value throughout the text. She showed no evidence of coordinating the outcome of dividing by 6 and dividing by 3 since they constituted separate tests for distinct properties. By skip-counting the multiples of 6 and finding the k-values when dividing by 3, she noticed that the k-values in line 4 were all even, but she could not explain this pattern. She explained the equation in line 3 as "splitting" the 6 into 2*3. However, she did not interpret the proof text as linking the quotient when dividing by 6 and the quotient when dividing by 3.

At this point, the interviewer noticed that Jess and Zandra's reasoning about division and multiplication seemed to operate distinctly and neither student interpreted the equations as coordinating [multiple of 6] with [multiple of 3]. To test this conjecture, the interviewer asked Jess and Zandra to interpret the proof when $x = -54$. The students seemed hesitant when needing to divide −54 by 6, so the interviewer helped them identify that $k = -9$. Zandra noted that substituting −9 into the equation $x = k * 6 = (k * 2) * 3$ would yield $-18 * 3$. The interviewer asked, "what is $-18 * 3$" to which Zandra replied, "I used too much of my calculator." We take this as evidence that she indeed has not connected x's multiplicative relationship with 6 and 3 through the equation, since she did not anticipate that $-18 * 3$ preserved the value of x. When Jess later affirmed that −54 was a [multiple of 3], the interviewer asked how she knew this. She said, "We can divide 3, and get an integer." When the interviewer asked her what integer they would get, Jess again anticipated needing to divide −54 by 3. She did not anticipate from the previous work that the answer was −18.

Why did Jess and Zandra have trouble using the equations to relate [multiple of 6] and [multiple of 3]? We explain this in terms of two challenges Zandra and Jess faced:

1. Jess and Zandra did not perceive the multiplicative equation $x = k * 6$ or $x = (k * 2) * 3$ as expressing the quotative fact that $\frac{x}{6} = k$ or $\frac{x}{3} = k * 2$. We have further evidence of this inasmuch as both students expressed some satisfaction when the interviewer wrote down the quotative equations, since they found them more insightful.
2. They did not have a way to coordinate the quotients from [multiple of 6] with the quotients from [multiple of 3].

To further explore Jess and Zandra's understanding of Proof 1.1, the interviewer asked, "which values of x are we really concerned about in this whole argument?… All multiples of 6, all the multiples of 3?" This was both to determine whether the students believed this inference would hold for any possible multiple of 6 and whether they understood the continuity of x throughout the argument such that the first line set the scope for x. Both students agreed that x could be a [multiple of 6] or a [multiple of 3]. The interviewer asked them to reread the proof considering when $x = 9$. Jess then said this was a problem because k was no longer an integer. Asked to interpret this, Zandra revised her scope for x: "multiple of 6, since anything that is a multiple of 6 will be, can be divided by… 2 and 3, anything that is divisible by 6 automatically gets divisible by 2, and 3, so anything, any multiple of 6 can be x." At this point, not only did Zandra show greater certainty of the generality of the theorem, but she introduced the new claim that [multiple of 6] entails being a [multiple of 2] and she revised her conclusion from "can be divided by" to "gets divisible by." We see this as possibly reflecting a shift from focusing on the operation to be performed to a property that x either has or does not have.

4.3 Jess and Zandra's reasoning: the productive influence of adopting a measurement meaning for [multiple of *d*]

Later in that same interview, Jess and Zandra read Proof 2.2 (converse proof), which strongly mirrors Proof 1.1's argument, though it is applied to [multiple of 14]. Jess and Zandra affirmed that the proof worked.[3] Zandra summarized, "like I said in the previous one we just plug it in, and we're just thinking that x can be divided by 2, and then over here we're presenting that x can be divided by 7, so it's like x is a multiple of 7, right, x is a multiple of 2." The interviewer, to connect to the students' quotative preference, asked them what $\frac{x}{7}$ and $\frac{x}{2}$ would be, which they correctly identified as $k * 2$ and $k * 7$ respectively.

As a final exploration of Jess and Zandra's meanings for these multiple properties and the proofs they read, the

[3] Jess and Zandra judged that Proof 2.2 proved Theorem 2 and preferred it to Proof 2.1. This is contrary to standard mathematics and common among our study participants. The matter of converses is not relevant to this analysis.

interviewer asked the pair to imagine that a number was made up of 250 groups of 14 and to determine how many 2's and how many 7's would be in the number. It took the students some time to imagine [multiple of 14] in this measurement meaning (for instance Zandra asked whether he meant the number 250 was divisible by 7). This suggests the measurement meaning "made up of groups of" was not compatible with the various meanings they had been drawing on to this point. Their meaning for k as quotient when dividing by 14 did not entail x being made of groups of 14. As a result, they still struggled to coordinate the value of k in [multiple of 14] and in [multiple of 7]. After some time, Jess proposed, "would it be twice as many, would it be 500?" She explained "14 is a multiple of 7, so times 2, if I am thinking of it right." It is unclear whether her explanation reflects her partitioning groups of 14 into groups of 7 or whether she was imagining the symbolic move in the proof $k * 14 = (k * 2) * 7$. When asked how many groups of 2, Jess was much faster to reply "multiply by 7."

We present this extended episode with Jess and Zandra to exemplify some of the points we raised in the previous section about the aspects of predication in student thinking. Clearly, their difficulties were affected by weak procedural fluency, which may be expected since their experiences as college students likely do not involve much arithmetic (especially without a calculator). We care much more about the conceptual roots of their difficulties, specifically how the aspects of predication influence their ability to coherently interpret equations and proofs.

First, we wish to highlight the challenge Zandra faced in coordinating the test for the property and the property itself. Interpreting [multiple of 6] in terms of dividing by 6 led Zandra to conflate the property with both the process of dividing and the outcome of that process. She acted as though 2 and 3 were [multiples of 6] (contrary to her skip-counting reasoning) since the operation of dividing by 6 could be performed on them. At other points, she wondered if k had the property [multiple of 6]. We gain further insight from the later discussion in which it was clear that Zandra and Jess had trouble interpreting [multiple of 14] as saying something about the structure of the number (being made up of equal groups). Neither their skip-counting meaning nor their quotative meaning seemed to directly provide information about x in this way; [multiple of 6] did not provide information about the structure of the number.

We conjecture that the lack of operationalizable meaning for k combined with the lack of a way to interpret [multiple of 6] as saying something about the structure of x may explain why Jess and Zandra could not follow Proof 1.1. It seemed as though Jess had to reconstitute the algebraic transformation in the third line of the proof as though for the first time. The task of relating groups of 14 and groups of 7 seemed pivotal for her thinking. The interviewer providing the measurement meaning of [multiple of 14] seemed sufficient to help Jess produce a novel insight into the key inference in Proofs 1.1 and 2.2. We conjecture this occurred because k was provided an operationalizable role by the measurement meaning of [multiple of 14].

5 Data from teaching experiment 2 with Moria and April

In this section, we present episodes from a second teaching experiment we conducted the semester after the previous. Moria and April were both computer science majors. They completed minorly revised versions of the same tasks. Moria and April showed quite different ways of interpreting the proofs, which were rooted in a much stronger conceptual understanding of [multiple of d] (for instance they better coordinated multiplicative and quotative meanings). Nevertheless, Moria faced related challenges in reasoning about the proofs related to our aspects of predication. Moria and April's greater fluency with multiple relations seemed influenced by their coursework as computer science majors. For instance, they were familiar with modular arithmetic.

5.1 Moria and April's reasoning: conceptual understanding and tests for divisibility

Upon reading Proof 1.1, Moria showed a quick disposition to reason with predicates. For example, she asked whether Proof 1.1 was "bulky enough" to prove Theorem 1, given that it only discussed [multiples of 6]. This means she interpreted the first line of Proof 1.1 as setting a scope of variation. Regarding the string of equations "$\frac{x}{3} = \frac{x}{6} * 2 = k * 2$" in line 3,[4] Moria wondered, "If you are just taking something by x over 3, it's not always going to be divisible by 6. So that makes me little bit uncomfortable." This revealed a few things about her reasoning. First, she knew that all [multiples of 6] are [multiples of 3], but that some in the latter category are not in the former. Second, she associated the expression "$\frac{x}{3}$" with asserting the property [multiple of 3], similar to how Zandra conflated the operation with the property. Third, she inferred the property [multiple of 3] referred to all such numbers (reasoning with predicates), and thus wanted to ward against making a claim she knew was false. This is similar to the way Zandra did not maintain the scope of a variable across the text. April tried to alleviate this worry, saying, "We already know here that x is a multiple of 6" (referring to line 1).

[4] In response to Jess and Zandra's preferences, we had revised the text of Proof 1.1 presented to Moria and April to use quotative equations.

Definition 1: We say the integer n is a **multiple of d** whenever there exists some integer k such that $n = k * d$.
This can also be stated as "n is divisible by d" or that "d divides n."
Notice that in the case that n is a multiple of d meaning $\frac{n}{d} = k$, k is called the **quotient** of dividing n by d.

Theorem to be proven 1: For every integer x, if x is a multiple of 6, then x is a multiple of 3.

Proof 1.2: Let x be an integer that is a multiple of 3.
Then x could be 15, which is not a multiple of 6.
Thus it is not necessarily the case that x is a multiple of 6.

Proof 1.3: Let x be any integer that is not a multiple of 3.
That means when we divide x by 3, we get a remainder of 1 or 2.
Then there exists some integer k such that $x = k * 3 + 1$ or $x = k * 3 + 2$.
If k is even, then there exists some integer s such that $k = s * 2$.
Substituting into the equations for x, we see:

$$x = (s * 2) * 3 + 1$$
$$= s * 6 + 1$$
or
$$x = (s * 2) * 3 + 2$$
$$= s * 6 + 2.$$

This means x is not a multiple of 6, because x is 1 or 2 greater than a multiple of 6.
If k is odd, then there exists some integer t such that $k = t * 2 + 1$.
Substituting into the equations for x, we see

$$x = (t * 2 + 1) * 3 + 1$$
$$= t * 6 + 4$$
or
$$x = (t * 2 + 1) * 3 + 2$$
$$= t * 6 + 5$$

This means x is not a multiple of 6, because it is 4 or 5 greater than a multiple of 6.

Fig. 2 The latter proofs regarding Theorem 1

Figure 2 presents the latter two proofs that the students read in tandem with Theorem 1. Their task was to determine if each proof proved the theorem, and if not what statement it proved. April introduced an important metaphor as she explained her reasoning about Proof 1.2, saying, "I don't think this proof is trying to prove this theorem because this is iterating through everything that is divisible by 3. The theorem could still apply because it's 'If it's a multiple of 6, then it's divisible by 3,' but it's not looking to prove that theorem." The metaphor we refer to is "iterating through" a range of values. On the surface, "iterating through" is similar to Zandra's skip-counting, but for April and Moria it referred to a process that could be embedded in computer code. The proof text itself entails no such language, but this became a powerful tool for April and Moria to interpret the number theory proofs they read, building productively on the structure of integers in terms of successor relations. Furthermore, April used the scope of this iterating process to distinguish Proof 1.2 from Theorem 1. Proof 1.2 iterated through [multiples of 3] while Theorem 1 refers to [multiples of 6], meaning the proof is not trying to prove (or disprove) that theorem.

5.2 Moria and April's reasoning: remainders and reasoning with predicates

Proof 1.3 (which proves Theorem 1 by contrapositive) is an example of the kind of text in which we see reasoning with predicates as highly productive. The reader must recognize how the various remainder equations justify the claim for different groups of integers that together cover all [non-multiples of 3]. April and Moria's reading demonstrated this well. Moria began interpreting the text substituting values of k in the equations (treating x as a function of k) and double-checking the algebraic steps of substitution and manipulation of the expressions in each line. She affirmed that the x-values that resulted in the equations were neither [multiples of 3] nor [multiples of 6]. Her initial work seemed to focus on examples and algebraic steps.

April provided a more conceptual interpretation of the equations and the argument of Proof 1.3 when she said "Everything that you throw into it is going to give a remainder, how it's set up… you've already eliminated the fact that there's ever going to be a 3 in this, so it just doesn't formulate. It's like making a cake without the flour or the sugar."

Theorem to be proven 2: For any integer x, if x is a multiple of 2 and a multiple of 7, then x is a multiple of 14.

Proof 2.1: Let x be an integer that is a multiple of 2 and a multiple of 7.
Then by definition there is some integer k such that $\frac{x}{7} = k$.
Since x is a multiple of 2, either k or 7 must be even.
Thus k is even, and for some integer j, $\frac{k}{2} = j$.
Then, $\frac{x}{14} = \frac{x}{7} * \frac{1}{2}$
$= \frac{k}{2}$
$= j$.
Thus, x is a multiple of 14.

Theorem to be proven 2': For any integer x, if x is a multiple of 4 and a multiple of 6, then x is a multiple of 24.

Proof 2.1': Let x be an integer that is a multiple of 4 and a multiple of 6.
Then by definition there is some integer k such that $\frac{x}{6} = k$.
Since x is a multiple of 4, either k or 6 must be a multiple of 4.
Thus k is a multiple of 4, and for some integer j, $\frac{k}{4} = j$.
Then, $\frac{x}{24} = \frac{x}{6} * \frac{1}{4}$
$= \frac{k}{4}$
$= j$.
Thus, x is a multiple of 24.

Fig. 3 The original and modified versions of Theorem 2 and Proof 2.1

First, April interprets [not a multiple of 3] as "give[s] a remainder," which allows her to interpret the equations in Proof 1.3 as expressing the property. She then connects this to eliminate the possibility of having "a 3 in this," which we interpret as the ability to factor a 3 out of the number. We note that this explanation shows how April saw [multiple of 3] as key features that numbers either had or lacked, which she expressed in the metaphor of ingredients in baking (she saw 2 and 3 as necessary "ingredients" to make 6). She perceived strict incompatibility between having a remainder in a division process (outcome of a property test) and quality of the number as expressed algebraically (a property attribution).

Moria then introduced the idea of interpreting the proof in terms of "for loops" in computer programming. Building on this idea of running tests, she summarized the proof as saying, "If it's not a multiple of 3, then it's going to be a multiple of 3 plus 1, or 3 plus 2, or 3 plus 4, or 3 plus 5." There are a few powerful ideas embedded in this new construal of the text. First, loops are a way of dealing with cases and alternatives in coding, which is a close analog to the way the equations in this proof express possible cases for the remainder when dividing by 3 or 6. We have observed that many novices reading proofs with equations tend to treat those equations as opportunities for manipulations such as substituting or solving (as Moria initially did) rather than ways of attributing and expressing properties (Dawkins & Zazkis, 2021). Moria used the for- loop-idea to develop a more productive construal of this text as expressing alternative cases, categorizing numbers by their remainders. April and Moria recognized that all the cases represented [not a multiple of 3] and [not a multiple of 6], that the cases were mutually distinct and together exhaustive, and thus Proof 1.3 proved "if an integer is not a multiple of 3, then it is not a multiple of 6."

April and Moria's interpretations of the text implicitly rely on reasoning with predicates to recognize how the various remainder values refer to whole classes of numbers. April and Moria had recognized how [multiple of 3] and [multiple of 6] refer to whole sets through their iterating metaphor (much like skip-counting for Jess and Zandra). Recognizing how the remainder equations also entailed sets of [non-multiples of 3] was more sophisticated and powerful. Moria was later able to explain the internal relations between dividing by 6 and 3 when she noted that if the quotient of dividing by 3 is odd, you get a larger remainder when dividing by 6 (though she used pronouns instead of these formal terms).

The interviewer later asked Moria to interpret Proof 1.1 when $x = 54$. Interestingly, Moria asked for a calculator much as Jess and Zandra had. It seems her procedural fluency in the moment was also somewhat weak since such arithmetic was unlikely part of her regular activity as a college student. April similarly commented, "That took way too long for me to divide in my head. I'm ashamed." However, once they knew that $k = 9$ when $x = 54$, they substituted into the equations in line 3 of Proof 1.1 and anticipated that $3 * 18 = 54$ without performing arithmetic. In other words, they saw the algebraic equation as expressing a sameness that carried to particular values when substituted. We infer that this is related to their sense that [multiple of 6] and [multiple of 3] were structural qualities of numbers and that [multiple of 6] entailed [multiple of 3] (since being able to factor out a 6 entailed being able to factor out a 3).

5.3 Moria and April's reasoning: subtle challenges in coordinating tests for membership and property attribution

In Fig. 3 below, we present the version of Proof 2.1 that Moria and April read alongside a reproduction of the way we asked them to revise the theorem and proof by replacing (2,7,14) with (4,6,24). Notice that Theorem 2' is false and Proof 2.1' is invalid. Some students had trouble with revising the proof in this way because they did not recognize to change [even] to [multiple of 4], since they do not interpret [even] as an instance of being a multiple in the same way as the others.

When Moria read Proof 2.1, she continued interpreting the claims of divisibility by imagining writing computer code that would iterate through the values of the variables and then test for divisibility. This way of construing the text had some interesting consequences as she tried to understand how the proof related to the theorem. First, she interpreted the proof as running two nested loops, one testing for [multiple of 7] and another testing for [multiple of 2]. This allowed her to distinguish this property from [multiple of 14], since it only involved a single loop. This struck us as significant because both Moria and April recognized that the two were equivalent in some sense ([multiple of 2 and multiple of 7] and [multiple of 14] have the same truth-sets). However, April's ingredients metaphor led her to see the two as simply the same properties, inasmuch as factoring out a 2 and a 7 meant factoring out a 14 and vise versa. Moria saw the two properties as distinct and equivalent, since her membership tests were distinct even if the exact same set of numbers passed both tests.

When Moria first read Theorem 2' during the first teaching session, she anticipated that it should conclude that "x is a multiple of 12." This showed her relative fluency with multiple relations. When the group revisited Proof 2.1' during the second teaching session, Moria used their iterating test idea to argue why Theorem 2' was false. She said, [multiple of 24] would "skip over some of the potential multiples, 'cause it would go 24 and 48 and it would skip over 12 and 36, which are also multiples of 6 and 4." In contrast, when Moria was interpreting Proof 2.1' she imagined running the test for [multiple of 4] on the quotient from the test for [multiple of 6]. As a result, she inferred that any number that was a [multiple of 4 and a multiple of 6] would be a [multiple of 24]. While this claim is not normatively true and conflicts with her reasoning in the previous interview, she was correct in claiming that if $\frac{x}{6}$ is a [multiple of 4], then x is a [multiple of 24]. This was a more sophisticated and subtle form of conflating the property [multiple of 6] with the outcome of the test for the property.

In some sense, this seemed a persistent issue in Moria's reasoning about the conjunctive property [multiple of 4 and multiple of 6]. In discussing this difficulty in the first session, the interviewer asked, "Does 12 have the property of being a multiple of four and multiple of 6?" Moria replied, "12 certainly does because. No, because you could divide 12 by 4 and it would come out to be 3. But you could also divide it by 6 and it would come out to be 2. So it's a multiple of both of them, but not at the same time." She used similar language during the second session when she asked, "Is it a multiple of 4 and 6 simultaneously, or is it individually a multiple of 6 and a multiple of 4?… It's an 'and' statement, not an 'or' statement." We interpret Moria's final point in light of her computations on 12 from the previous session. She could divide 12 by 4 or by 6 and get an integer, but you cannot divide by both in sequence and yield an integer. Moria's notion of "at the same time" or "simultaneously" seemingly moves away from her previous sense that the numbers either have or do not have these properties (and to have both means to have each). Rather, when she focused on the tests for the properties, as expressed in her nested for loops, she found multiple ways to interpret the conjunction "and." She noted at some points how the division tests could be carried out separately, but at other points interpreted "and" as sequencing the tests for divisibility (the input to one was the output of the other). What we note here is that even for a student with relatively coherent and sophisticated meanings for properties, the work of coordinating tests for properties, sets of objects, and proofs is complex and challenging.

6 Summary

We set forth in this paper to describe some subtle distinctions among how students link properties to objects, which we call *aspects of predication*, in the context of reading number theory proofs. These distinctions hold consequences for these students' ability to reason with predicates, which in turn has consequences for their understanding of the logic in proofs. We provided examples from two teaching experiments of how these aspects of predication influenced how students interpreted number theory proofs. While almost every property discussed in our examples was either of the form [multiple of d] or [not a multiple of d], we noted quite a bit of variation in the aspects of predication based on various meanings students evoked. Indeed, this provides the reason for our investigation of how features of discrete concepts interact with student understandings to constrain or afford the emergence of a unifying logical structure across various semantic contexts.

We can now more easily summarize how these aspects of predication influence reasoning with predicates. The formal notion of interpreting a property as a predicate is to transform the property into a "truth function" that maps each object in a set to "T" if the object has the property or to "F" if it does not have the property. From this, truth-sets and false-sets can be formed.

Populating sets of examples is an aspect of forming truth sets. Populating is a particular affordance of discrete concepts since integers, graphs, and combinatorial patterns all allow many forms of iteration and recursion (e.g., Lockwood & De Chenne, 2020; Ouvrier-Buffet et al., 2018). However, we observed the need to coordinate such iteration with meanings for properties and tests for properties. This is particularly important in the context of proving since proofs operate on properties or carriers of properties, not on exemplars. It is also crucially relevant to students' ability to make sense of equivalence between properties since as they need to treat properties as distinct even when the set of exemplars is the same. Whenever two properties P and Q have the same truth set, then they are in a bi-conditional relationship ("Given any $x \in S$, $P(x)$ if and only if $Q(x)$"). Students in our studies often find it hard to interpret such bi-conditional claims as two one-way relationships when they see the two properties as simply interchangeable or "the same."

Student interpretations of Membership Testing revealed themselves as highly complex and challenging. In some cases, the mental work of carrying out a test simply inhibited students' ability to map particular objects to "T" or "F." By making this truth-function mentally laborious, students faced challenges in anticipating the formation of truth-sets. In other cases, students conflated the property with the test itself or with the outcome of the test. We found this particularly interesting in Moria's case where she had trouble making sense of the conjunction of two properties since her coding way of reasoning led her to sequence the two tests rather than simply to affirm that both tests returned "T." We had previously noted in our experiments that students often use "and" to express a union operation when they were thinking of sets of objects (e.g., [multiple of 3 or multiple of 5] is true of all [multiples of 3] and all [multiples of 5], combining the sets). We now see that conjunctions can be even more complex as students try to conjoin: sets of objects (union), properties (intersection), or tests for properties (stronger than intersection).

7 Contributions

Mathematicians seem to become very fluent in anticipating, regardless of how easily they can populate a truth set or construct the predicate function, that every object either has or does not have a given property (that is well-defined). Consequently, we offer reasoning with predicates and the aspects of predication as a characterization of what it means for students to understand mathematical properties in sophisticated ways that might afford proving and reasoning about logical structure. For any given property that is being taught, we think the following questions and sub-questions about the aspects of predication might help generate tasks to help stimulate student thinking and assess student understanding:

1. Do students have productive ways to enumerate examples through some iterative or case-based structure?
   (a) Is that generation process coordinated with the meaning of the property itself?
   (b) Can they harness that generation process to prove?
   (c) Do students have ways to argue that case-based structures exhaust all possibilities, such that by reasoning with predicates they can recognize the universality of an argument by cases?
2. If there is a process by which one tests whether a given object has the property, can students anticipate this process as yielding an unknown result without carrying it out?
   (a) Can they coordinate the various roles in the test without conflating the objects and their respective properties?
   (b) Can students reason about how these tests relate among closely-related objects (e.g., properties of factors or multiples, properties of subgraphs, properties of substrings in counting)?
3. If a property is defined in terms of an introduced object, can students alternate productively between processes for constructing the introduced object and stipulating that such objects exist, depending upon the status of the definition in a proof task (hypothesis or conclusion)?

As these questions suggest, we anticipate that reasoning with predicates and these aspects of predication hold important application across discrete settings. This is because of the shared iterative structure (Ouvrier-Buffet, 2020), the prevalence of classifying objects using introduced objects, the common use of algorithms to determine properties, and the analysis of properties of algorithms themselves (Modeste & Ouvrier-Buffet, 2011). Future work should explore whether and how supporting students in reasoning productively about membership testing and forming truth sets can facilitiate student comprehension and production of proofs in discrete contexts. We hope that these aspects of predication will help future researchers and teachers make sense of students' justification and generalization activities.

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References

Alcock, L., & Simpson, A. (2002). Definitions: Dealing with categories mathematically. *For the Learning of Mathematics, 22*(2), 28–34.

Cobb, P., & Steffe, L. P. (1983). The constructivist researcher as teacher and model builder. *Journal for Research in Mathematics Education, 14*(2), 83–94.

Dawkins, P. C. (2017). On the importance of set-based meanings for categories and connectives in mathematical logic. *International Journal for Research in Undergraduate Mathematics Education, 3*(3), 496–522.

Dawkins, P. C., Roh, K. H., Eckman, D., & Cho, Y. K. (2021). Theo's reinvention of the logic of conditional statements' proofs rooted in set-based reasoning. *Proceedings of the 43rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, USA.

Dawkins, P. C., & Zazkis, D. (2021). Using moment-by-moment reading protocols to understand students' processes of reading mathematical proof. *Journal for Research in Mathematics Education, 52*(5), 510–538.

Edwards, B., & Ward, M. (2008). The role of mathematical definitions in mathematics and in undergraduate mathematics courses. In M. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education MAA notes #73* (pp. 223–232). Mathematics Association of America.

Ellis, A., Lockwood, E., Tillema, E., & Moore, K. C. (2021). Generalization across multiple mathematical areas: Relating, forming, and extending. *Cognition and Instruction*. https://doi.org/10.1080/07370008.2021.2000989

Freudenthal, H. (1973). *Mathematics as an educational task*. D. Reidel Publishing.

Hershkowitz, R. (1987). The acquisition of concepts and misconceptions in basic geometry: Or when "a little learning is a dangerous thing. In J. D. Novak (Ed.), *Proceedings of the second international seminar on misconceptions and educational strategies in science and mathematics, Vol. 3* (pp. 236–251). Cornell University.

Katz, B., & Starbird, M. (2013). *Distilling ideas: An introduction to mathematical thinking*. Mathematical Association of America.

Lakatos, I. (1976). *Proofs and refutations*. Cambridge University Press.

Lockwood, E. (2013). A model of students' combinatorial thinking. *The Journal of Mathematical Behavior, 32*(2), 251–265.

Lockwood, E., & De Chenne, A. (2020). Using conditional statements in Python to reason about sets of outcomes in combinatorial problems. *International Journal of Research in Undergraduate Mathematics Education, 6*, 303–346. https://doi.org/10.1007/s40753-019-00108-2

Modeste, S., & Ouvrier-Buffet, C. (2011). The appearance of algorithms in curricula, a new opportunity to deal with proof? In *Proceedings of CERME 7*. University of Rzeszów. http://www.cerme7.univ.rzeszow.pl/index.php?id=wg1. Retrieved 9 June 2021.

Morris, A. K. (2002). Mathematical reasoning: Adults' ability to make the inductive-deductive distinction. *Cognition and Instruction, 20*(1), 79–118.

Murphy, G., & Hoffman, A. (2012). Concepts. In K. Frankish & W. Ramsey (Eds.), *The Cambridge handbook of cognitive science* (pp. 151–170). Cambridge University Press.

Ouvrier-Buffet, C. (2006). Exploring mathematical definition construction processes. *Educational Studies in Mathematics, 63*(3), 259–282.

Ouvrier-Buffet, C. (2011). A mathematical experience involving defining processes: In-action definitions and zero-definitions. *Educational Studies in Mathematics, 76*(2), 165–182.

Ouvrier-Buffet, C. (2020). Discrete mathematics teaching and learning. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 227–233). Springer.

Ouvrier-Buffet, C., Meyer, A., & Modeste, S. (2018). Discrete mathematics at university level. Interfacing mathematics, computer science and arithmetic. *INDRUM* (pp. 255–264). INDRUM Network, University of Agder, Kristiansand, Norway.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267–307). Erlbaum.

Vergnaud, G. (1996). The theory of conceptual fields. In L. Steffe, P. Nesher, P. Cobb, G. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 219–239). Erlbaum.

Zaslavsky, O., & Shir, K. (2005). Students' conceptions of a mathematical definition. *Journal for Research in Mathematics Education, 36*, 317–346.

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