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Identifying mental actions for abstracting the logic of conditional statements

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ABSTRACT

In this paper, we provide a theoretical development of the mental actions that underlie reasoning about logic. Building explicitly on Piaget's epistemology, we propose populating, inferring, expanding, and negating as four mental actions that, upon becoming reversible and composable, can give rise to the logic of universally quantified conditional statements. We adopt the view that logic is a metacognitive activity in which people abstract content-general relationships by reflecting across their content-specific reasoning activity. We explore how these four actions become reversible and composable in mathematics, suggesting that logic can be built psychologically upon the foundation of mathematical reasoning. Further, by exploring what it means for these actions to be reversible and composable, we propose how students may need to engage in these actions to refine and reflect on them so as to afford logical abstraction (in the manner we envision).

1. Introduction

The keystone in Jean Piaget's characterization of logico-mathematical reasoning is the centrality of action (physical or mental) to such reasoning (Piaget, 1970). Applied to studies of students' mathematical thinking and learning, this tenet has yielded key insights in mathematics education, especially in number concepts (Clements, 1984; Kamii, 1999), multiplicative reasoning (Hackenberg, 2010; Ramful & Olive, 2008; Steffe, 1994), and fractions (Norton & Wilkins, 2012; Simon, Kara, Placa, & Sandir, 2016; Tzur, 2004). The principle has often been applied superficially to imply that students must be "active" in learning, which does not maintain the deeper insight that learners' actions must become reversible and composable to yield the power of mathematical reasoning. These qualities of mathematical activity imbue mathematics with its sense of necessity, and reliability, and its potential for extension (Piaget, 1970). Specifically, every action can be reversed by an inverse action (e.g., partitioning a whole into five parts and iterating any one of them to reproduce the whole), returning to a starting point from which to begin again with perfect reliability; and actions can be composed with one another to extend those actions into a single new action (e.g., composing two reflections to form a rotation).

In contrast to this Piagetian principle, much modern instruction of logic seems to divorce itself from students' mathematical mental actions (c.f., Dawkins, Zazkis, & Cook, 2022; Durand-Guerrier & Dawkins, 2018). This is because meaningful concepts and claims are replaced by linguistic variables, such as p or q. The mathematical objects about which students might make inferences are replaced by mere "truth-values," true or false (Dawkins, 2019). Often such instruction gets bogged down in counterintuitive issues such as why a

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conditional is true when the antecedent is false or how a false claim can imply anything at all. Such approaches will thereby fail to address what should be more important, such as how we draw universally true and reliable inferences. How can we infer that any multiple of an abundant number is abundant, that the midpoints of the sides of any quadrilateral form a parallelogram, or that any differentiable function must be continuous? How can we be certain about our lines of inference in terms of their quality and scope? In what sense are these three inferences all instances of the same pattern of activity (i.e., logically analogous)? Certainly, students can learn some facts about logic – such as contrapositive equivalence or how to write negations – without reflecting on their own mental activity. However, in this paper we consider pursuing a richer form of logical understanding, which we deem both achievable and desirable. The claims we make throughout this paper about logic learning should all be qualified to within the scope of this particular commitment.

Identifying reversible and composable mental actions that might undergird the learning of mathematical logic strikes us as particularly valuable in the context of proof-based mathematical activity, where the sense of logical necessity is strongest for mathematicians and unfortunately elusive for many students. In this paper, we build on our recent investigations in undergraduate students' reasoning about logic (Dawkins, 2017; Dawkins & Cook, 2017) and teaching mathematical induction (Norton, Arnold, Kokushkin, & Tiraphatna, 2022) to posit four types of mental actions that students can abstract to construct the logic of conditional mathematical statements ("for all $x \in S$, if $P(x)$, then $Q(x)$").

The four actions we propose (along with their inverse actions) seek to correct what we perceive are the errors of current logic instruction mentioned above. The *populating* action (whose inverse is *describing*) captures how we use properties to characterize whole classes of mathematical objects (e.g., rhombi are quadrilaterals with four equal sides). This activity is the foundation of reference or semantics, which is key to how students interpret mathematical language (Dawkins, 2019). The *inferring* action (whose inverse is *assuming*) captures how we draw inferences from given properties (e.g., whenever a quadrilateral has four equal sides, it will also have parallel opposite sides). As mentioned above, considering the quality of such inferences is at the heart of mathematical arguing and proving, which we claim logic should formalize and refine. Inviting students to abstract logic from such activity would link the logical notion of implication with the psychological activity of drawing inferences. Once students compose populating with inferring, they might see why a finite sequence of inferences can justify a claim with infinite scope (e.g., all rhombi are parallelograms).

The other two actions are necessary for making the system complete (in the spirit of Piaget's grouping structures, Beth & Piaget, 1966). The action of *expanding* (whose inverse is *nesting*) captures when we notice a pattern between two classes of objects without a property-based inference (e.g., all perfect numbers are even). The action of *negation* (its own inverse action) allows us to treat negative claims as having the same structure as positive ones. This is not only common in mathematics, but also key to logical relationships such as contraposition (e.g., if a quadrilateral is not a parallelogram, then it is not a rhombus). We seek to maintain negation as a psychological activity of constructing an opposite property that refers to the complement set of objects (Dawkins, 2017) rather than merely switching truth-values.

These four activities are not particularly novel. Our contribution is to argue how and why they should be central to the teaching and learning of logic, namely as the raw material for reflection and abstraction. Students engage in them often, but that does not mean they see all such actions as instances of the same activity or that they observe a shared and stable structure across such activities. Also, we acknowledge that students engage in these actions outside of mathematics. We posit that these actions are uniquely reversible and composable within mathematics. Only when they become reversible and composable do they render mathematical proof with the power it is taken to possess by the mathematical community. This connection between mental actions and proving has explanatory power. Proofs may fail to prove in students' understanding because these actions are not reversible or composable in their reasoning (as we discuss in Section 3.3). More positively, our paper offers a picture of logic learning as a meta-cognitive, reflective activity in which students may engage. They may abstract logic as the study of the scope, quality, and nature of mathematical language and inferences. While we draw on our empirical insights about how students engage in these actions and the conditions under which they abstract those actions, we dedicate this report to providing a theoretical development of the actions themselves.

The rest of the paper is organized in the following way. Section 2 further elaborates some of the background considerations and assumptions that guide our particular approach to the teaching and learning of logic. Section 3 describes the four actions and their inverse actions, connecting them to the work of proving and to student activity. Section 4 discusses some implications from this activity-based approach to logic, outlining the contributions of the paper to teaching and learning.

2. Our perspective on logic as it pertains to reasoning

A number of challenges arise in conceptualizing research on the learning of logic. One must avoid conflating various meanings of logic itself and its relationship to human reasoning. The modern formalizations of logic taught in undergraduate mathematics courses developed relatively late even as compared to other parts of the undergraduate proof-based curriculum. These formalizations seek to realize the ambitions outlined by earlier mathematicians to render human language and reasoning into a form of calculation (or "calculus") that was completely rule-based and reliable (Leibniz, 1969; Thomson, 2001). A key problem is that neither natural language nor everyday concepts are sufficiently precise and unambiguous to afford this (relevant to this paper, Johnson-Laird & Byrne, 2002, provide an account of diverse everyday uses of *if*). Accordingly, the emergence of modern logic required and utilized the development of formal languages (Azzouni, 2004; Wittgenstein, 1961). Creating languages that are rule-based and unambiguous helped realize the ideal of logic as a calculus. This came at the cost of divorcing the logical systems from the kind of everyday reasoning some hoped to improve. Some early modern logicians like Frege and Husserl recognized the incompatibility between these formal systems of logic they created and actual human reasoning. They decried any connection between logic and psychology, which they called *psychologism*. Many modern psychologists who study human reasoning and decision-making have come to agree quite strongly that propositional logic is an inadequate tool for modeling how people think (e.g., Schroyens, 2010). Logic is not innate, but learned (see Dutihl Novaes, 2021, for a parallel argument concerning the related notion of "deductive reasoning" as a learned skill that is fostered in particular fields of study, namely mathematics, law, and philosophy).

2.1. Mathematics instruction requires an interface between logic and reasoning

As instructors of proof-based courses, we cannot ignore the interface of logic and reasoning. For undergraduate mathematics students to engage in proving, their mathematical activity needs to conform to the norms of logic. This means that they somehow must learn to guide or curtail their mathematical reasoning in light of logical rules.[1] Piaget (Beth & Piaget, 1966) agreed that logic and psychology must be kept distinct in their scope and methods, but also pointed out that logicians are people and so there are psychological processes that underlie their construction of logic. What then is the underlying mental activity of reasoning about formal logic? What mental activity should students engage in to abstract the relationships studied by mathematical logicians?

To answer these questions, we propose three points about the nature of logic as a mental construction: logic is a metacognitive activity, logic is normative, and logic is primarily viable in mathematical contexts. (1) We say logic is a metacognitive activity, because it must be abstracted from the learner's reflection upon (Wheatley, 1992) their own (or someone else's) language use and reasoning. (2) We say logic is normative because it entails judgments about the quality of arguments or inferences, such as whether they are valid. This is why we stated above that students must "conform to the norms of logic." Logic does not describe their process of reasoning, but it may critique the arguments they produce. More positively, we hope that logic's assertions of validity would correspond to students' experiencing a sense of necessity and reliability in their mathematical reasoning. While children may be fully convinced of their reasoning without metacognizing about logic, we hope that the metacognitive activity of logic would support a more conscious and certain understanding of mathematical justifications. (3) Only in mathematical contexts can the actions that underlie logic become adequately reversible and composable. One consequence of these points is that logic then should be built upon the foundation of mathematical reasoning (abstracted from that activity). This is ironic inasmuch as many mathematicians think of logic as the foundation of mathematics, but psychologically it is unsurprising since most logicians have also been mathematicians (they had much mathematical reasoning to reflect on).

In much prior research on student reasoning about key mathematical relationships expressed as theorems, there emerges almost a dichotomy between students apprehending the relationship with an internal sense of necessity rooted in their own transformational reasoning (e.g., Simon, 1996; Harel & Sowder, 1998) and students understanding the theorem as a syntactic object that warrants moves in a proof (e.g., Duval, 2007). This dichotomy points to a fundamental challenge in teaching logic. How can students simultaneously develop insights that stem from semantic connections while appreciating the role of structuring inferences, warrants, and language in a precise syntax (Durand-Guerrier, et al., 2012)?

On the one hand, being convinced of a theorem such as the Mean Value Theorem (MVT: "Given any $f : [a, b] \to \mathbb{R}$, if f is continuous on $[a, b]$ and differentiable on (a,b), then $\exists c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.") will largely depend upon meanings very specific to that theorem (continuity, rate of change, derivative, etc.). The particular reasoning by which students may convince themselves is content specific. In contrast, one defining aspect of logic is that it is in some sense content-general. While transformational reasoning may provide a means of fostering logical necessity regarding the MVT, it does not directly address how students may use their reasoning about that theorem to understand logic or how they would use logic to help understand that statement.

On the other hand, logic is frequently taught as syntax, which is to say rule-based manipulation of symbols that have no or unspecified referents. Rendering the MVT as "if *P*, then *Q*" offloads the complex meanings in the theorem onto the hypotheses and conclusions being "true." This accomplishes the goal of creating a way to connect this theorem to others like it: "if the hypotheses are true, the conclusions must be true." Unfortunately, the notion of asserting truth dissociates logic from the rich meaning of the MVT in calculus and analysis. It is helpful to compare logic in this sense to algebra. Algebra can be taught as meaningless, rule-based manipulation of symbols with no connection to underlying arithmetic, but mathematics educators have long found this possibility dissatisfying (Boyce & Moss, 2019; Hackenberg & Lee, 2015; Kaput, 2008). We find logic as mere syntax similarly unhelpful and unproductive for instruction. There is evidence that learning mathematics improves performance on abstract reasoning tasks (Attridge & Inglis, 2013; Inglis & Simpson, 2008, 2009), but the reverse relationship is not well-established.

By framing logic as metacognitive (reasoning about reasoning), normative (entailing rules for acceptable inference and justification), and rooted in mathematics, we propose a way to bridge this dichotomy to foster content-general insights rooted in the necessity that arises in content-specific lines of reasoning. The logical actions defined in this paper specify what is repeated, and thus what might be abstracted, in student reasoning across a broad range of mathematical theorems.

2.2. Reversible and composable mental actions

Piaget argued from his extensive research program that children construct their understanding of the world through interactions

[1] We infer that mathematicians to some extent guide their reasoning in anticipation of rendering their final arguments in logically valid form, though we do not infer that all intermediate stages of reasoning are logically valid. Inglis (2006) and Inglis, Mejía-Ramos, and Simpson (2007) provide some helpful insights into the structure of mathematician argumentation, but much more could be done to study the integration of logical structure in the course of mathematicians' reasoning and how mathematicians render their reasoning into a more logically structured argument in the course of writing proofs for publication.

within it. The revolutionary idea was that, by repeated action and abstraction of experience, children actively construct many seemingly obvious aspects of the adult experiential world, such as space (Piaget & Inhelder, 1967), time (Piaget, 1969), number (Piaget & Szeminska, 1952), and causality (Piaget, 1974). Conversely, once constructed, children use those constructs to organize and make sense of the worlds they experience. Thus, over time, we take time, space, number, and causality for granted as if they are properties of a world apart from ourselves.

Piaget characterized advances in children's understanding using group structures (as studied in abstract algebra) because after a period of organization, children render their activity reversible (for every action, there was an inverse action) and composable (they could link actions in chains and anticipate the outcome of such compositions). For example, infants begin to construct space by learning to manipulate their perceptual fields through self-locomotion (e.g., crawling). Ultimately, they learn to compose these movements by continuing or otherwise combining them; they also learn to reverse these movements by returning to a starting point. They become able to coordinate "where they are" with "where they were" by imagining reversing their movement in space, such that the reversibility of action imbues their environment with sense of permanence and invariance. Thus, they construct space through a coordination of their own actions, which Piaget and Inhelder (1967) modeled as a "group of displacements."

In this same spirit, we intend to problematize the learning of seemingly obvious aspects of proof-based mathematics. There must be an intellectual history to ubiquitous mathematical ideas such as *all*, *any*, *implies*, *not*, *true*, and *false*. We claim that by a series of abstractions—from the way students use language to refer to objects and use actions on those objects to discern their properties—students can render mathematical language as having a group-like structure. This is tantamount to rendering that language formal for the purposes of reasoning about logic.

3. Defining four mental actions underlying the logic of mathematical conditionals

We introduce four mental actions related to the way students reason about universally quantified conditional statements: statements of the form "For all $x \in S$, if $P(x)$, then $Q(x)$." We choose such statements for a few strategic reasons. First, a huge number of mathematical theorems are of this form, even if not stated in this exact language. Second, such statements form an interface between logical concepts such as truth-conditions, reference, inference, and proof techniques.

For any universally quantified conditional, we need to identify the various entities at play. In mathematics, one generally specifies a *universal set* over which a given statement is quantified (S). Also, the predicates P and Q are made up of well-defined *properties* that members of S either have or do not have. These predicates are notated as functions because they can be thought of as truth functions that input each $x \in S$ and return T if x has the defining property and F if x does not. As a result, the members of S can be partitioned into the set of things that have property P – called the *truth set of P* that can be written $\{x \in S : P(x) = T\}$ – and the set of things without the property P – called the *falsehood set of P*. By the law of excluded middle, these sets should be complements of each other and either of these sets may be empty. We use the term *predicate* to refer to the whole complex of property, truth-function, truth-set, and examples.

In general, we may not always be able to talk about the property P without attributing it to some object, in which case we call that object a *carrier*. This is distinct from an *example* with the property since a carrier may be intended to represent any element of the truth set (a carrier is to logic as a variable is to algebra). This has sometimes been represented in logical notation by using letters early in the alphabet to represent examples and letters late in the alphabet to represent carriers. A carrier is often described as an "arbitrary, but fixed" element of the truth-set. This distinction between carriers and examples is subtle but essential for making sense of the mental actions that underlie logic.

The next four Sections (3.1–3.4) are organized to describe direct proof, which is to say how a proof that property Q follows from property P proves that the truth set of P is contained in the truth set of Q. Stated more briefly, "P implies Q" means that "all objects with property P must also have property Q." To justify this, one must pass from the truth set of P, through the properties, back to the truth set of Q. Section 3.1 introduces and defines the four actions and their inverse actions. Section 3.2 relates the truth sets to their properties. This establishes our approach to matters of reference (how statements and/or properties refer to objects). Section 3.3 addresses the inferences that connect properties and how composition of actions links the truth sets, which is the heart of proving. We begin with populating and inferring because together they form the basis for how finite proofs can reliably prove claims with infinite scope. Section 3.4 considers reflection on and abstraction of this recurrent structure, consistent with our view of logic as metacognitive activity. This section addresses how the actions we describe foster the emergence of logic as a content-general theory of mathematical conditional statements and proofs. The final two subsections then address other actions less central to direct proofs. We add them to complete the structure and maintain closure. Section 3.5 addresses *assuming*, which is the inverse action of inferring. Finally, Section 3.6 addresses the role of negation in the preceding structure.

3.1. Four actions and their inverse actions broadly defined

We may now define the four actions and their inverse actions. The four actions relate to conditionals in the following way.

1) *Populating* captures how such statements refer to examples.
2) *Inferring* links the properties P and Q.
3) *Expanding* describes observing an apparent overlap between examples with P and examples with Q.
4) *Negating* captures the notion of opposing properties and how lacking P can be understood as a new property $\sim P$.

At this point, we shall define these actions broadly to entail the ways that students may engage in them regardless of whether they have been rendered reversible and composable. In subsequent sections, we discuss them in more detail to describe what it means for someone to refine these mental actions to afford mathematical logic, namely by organizing them within reversible and composable structures (Piaget, 1970). For convenience, we shall refer to the statement "Given any integer x, if x is a multiple of 6, then x is a multiple of 3."

Populating is the action by which someone associates an example, set of examples, or carrier to a given property. The inverse action is *describing* by which someone associates a property to an example or set of examples. These actions move between a property and some aspect of its associated truth-set (see Fig. 1). Populating the predicate "is a multiple of 6″ could mean selecting one or a few specific multiples of 6 or it could mean expressing $x = 6k$ where k is an integer.

Inferring is the action of identifying some new property Q based on one's meaning for a given property P. This suggests that property Q is closely associated with property P or possibly entailed in the meaning of P. This action moves between properties (Fig. 1). Using the same example statement from the previous paragraph, we can understand *inferring* as moving from "is a multiple of 6" to "is a multiple of 3." This action depends on the meaning someone evokes for "multiple of 6." For instance, if one thinks of numbers made up of groups of 6, one may imagine splitting those groups in half to infer that this number is also made up of groups of 3. If one instead thinks of $x = k * 6$, then one can infer that $x = (k * 2) * 3$ to see why x is also a multiple of 3. The inverse action is *assuming* or *specifying* in which case one moves from a property Q to some other property P that has (possibly) more specified qualities. If one begins with a multiple of 3 and then assumes further that the number is also even, then the number will be a multiple of 6.

Expanding is the action of recognizing significant overlap between sets or that one set is contained in another. This action thus moves between two sets of objects that have at least some common elements (Fig. 1). This action occurs in pattern noticing; for instance, when children sum groups of three consecutive integers and notice that each sum is a multiple of 3, they note an *expanding* relationship in which the set of such sums is contained in the multiples of 3 (though many students do not think of this action in terms of sets). Similarly, students may note that the quadrilateral formed by the midpoints of the sides of any quadrilateral is always a parallelogram. This connection, being initially empirical, is not a movement between properties but rather between the sets of objects themselves. The inverse action of *nesting* occurs when one moves from a possibly broader set of examples to a narrower contained set. This occurs when students are looking for patterns among examples and choose to focus on narrower groups within the set of interest (which occurs in any argument by cases). For instance, they may choose to focus on even or odd numbers, concave or convex polygons, monotone functions, etc.

Negating is the action of constructing an opposite property or predicate to a given one. This action is its own inverse action, though students often reason about the inverse action as removing a "not" rather than applying the same action of negating again. This action is non-trivial inasmuch as students frequently interpret negative claims in ways that render different conditional statements as having distinct forms, as we shall discuss in Section 3.6. For instance, students may interpret the statement above as saying "all multiples of 6 are multiples of 3," but its contrapositive as saying "no non-multiples of 3 are multiples of 6" (Hub & Dawkins, 2018). In general, negation can be carried out on many logical objects (a quantifier, a conditional, or a predicate). For the sake of clarity, we shall only address negation of predicates in this paper in which case this action goes from one predicate P to another predicate $\sim P$.

We hope it is clear that people engage in these mental actions in many contexts beyond mathematics. People exemplify properties with examples and they generate unifying properties to describe collections of examples (Murphy & Hoffman, 2012). People make inferences based on their understanding of certain ideas and they make assumptions to explore conceptual relationships. People note when there is overlap between sets of objects[2] and consider whether "All A are B" as Aristotle discussed in his syllogisms. What we will argue in the following discussion is that these actions can become uniquely reversible and composable in mathematics. Further, this reversibility and composability is essential to mathematical logic and valid proof. Finally, we posit that students' reflections on their engagement in these four actions forms the underlying basis for abstracting logic.

3.2. Populating becoming reversible

Much research has considered how students relate properties and examples of those properties, most often under the domain of student use of *definitions*. These studies note that students need experiences with mathematical defining because such definitions do not operate like everyday definitions (Alcock & Simpson, 2002; Edwards & Ward, 2008). Everyday definitions for category terms may be good or bad to the extent that they properly capture the intended class of objects (Edwards & Ward, 2008), which means definition writers may engage in different acts of *describing* so as to match the property and the intended truth-set. This is not easily done in many cases because everyday categories are often not truly property-based. For instance, can we give precise properties that unify all forks, chairs, or lawyers? All of these categories have marginal cases (sporks, flat rocks being sat upon, those arguing in self-defense, or those who have been disbarred) that make the match between the defining properties and the set of exemplars imprecise. This means that efforts to *describe* the set of exemplars and to *populate* truth-sets for the properties will not preserve perfect fit. Thus, populating and describing actions will not be perfectly reversible (composition does not yield identity). This is not only a problem for everyday categories. Even scientific terms such as "species" or "electron" have undergone shifts in definition over time as our understanding of

[2] One psychological model for how people reason about conditional claims in everyday settings is the suppositional account, which claims people test the truth of "if P, then Q" in terms of an implicit judgment of the conditional probability of Q given P. This model, which has widespread empirical support (Evans & Over, 2004; Oaksford & Chater, 2020), suggests that people affirm conditionals when there is broad overlap between the occurrence of P and Q, at times even in the presence of known counterexamples (cases where P is true and Q false). This suggests that people attend to the co-occurrence of conditions as *expanding* suggests, though how this is carried out in various contexts is a matter for further investigation.

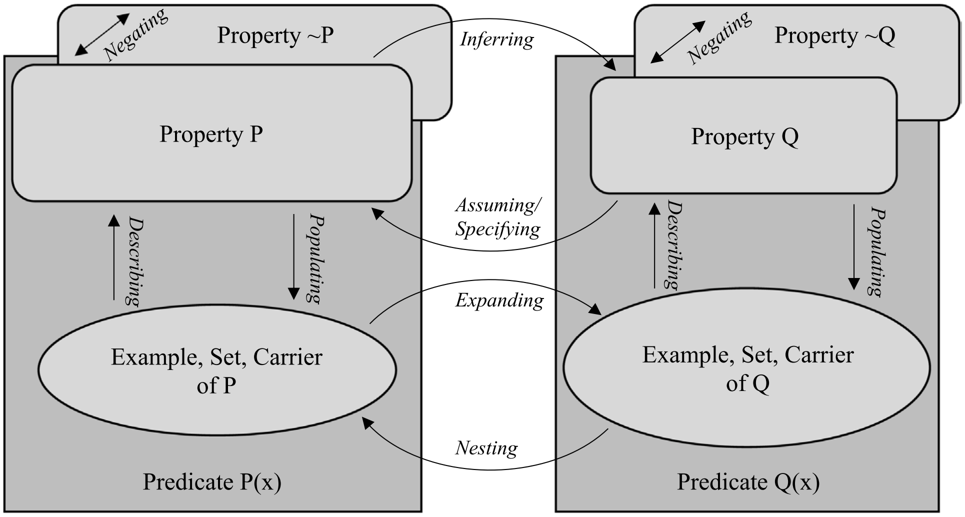

Fig. 1. How the four actions operate between elements of the two predicates.

the defining properties and set of referent objects have developed.

Mathematical definitions are property-based in the sense that there should be perfect fit between the defining property and the truth-set such that the populating and describing actions are truly reversible. Note this has not always been the case, as demonstrated by Euclid's definitions of point and line about which Russell famously noted that the "definitions do not always define" (Russell, 1951) as well as Lakatos (1976) now famous account of the definition of polyhedron. Lakatos' dialogue traces a complex interplay between various definitions for polyhedron (describing actions) and the introduction of strange examples such as large cubes with a small cube glued on the side (populating actions). The new examples prompted revisions of the definition until the defining properties and class of exemplars could be brought into a stable, reversible relationship. Part of the emergence of modern logic is the development of more precise mathematical definitions that afford tighter reversibility between *populating* and *describing*. Durand-Guerrier and Arsac (2005) share an interesting case in which Abel proved claims that were false because the theorems assumed pointwise convergence and yet he drew inferences from uniform convergence. This mismatch between stated hypothesis and inference led to the apparent co-existence of proof and counterexample to the theorems, until the definitions were properly distinguished.

Mathematics education studies note multiple ways that students' use of mathematical definitions deviate from this logical ideal (e.g., Alcock & Simpson, 2002; Edwards & Ward, 2008). From the standpoint of the mental action of populating, it is quite natural that students will reason about a property by selecting some examples that exhibit the property. The challenge in the context of universally quantified conditionals is how one can reason about those examples in a manner that nevertheless generalizes to the whole category. Once a student populates with particular examples, they may notice features of the example(s) that are not entailments of the defining property. This kind of inference may not be challenged if they reject unfamiliar examples that have the defining property (monster-barring, Lakatos, 1976; Larsen & Zandieh, 2008). For instance, students are mostly familiar with differentiable functions and may thus draw false inferences about the absolute value function, to which the MVT does not apply on some intervals. Instances like this point to the fact that a reversible relationship between a property and its truth-set becomes highly important when composed with the act of inferring, as we shall discuss in the next section.

We do not deny the value of reasoning with examples, but we emphasize the limitations for abstracting logical relationships. In Dawkins (2017) we argued instead that students need to *reason about predicates*, which means reasoning about a property by anticipating the formation of its entire truth set. This may not be necessary for mathematical reasoning in all settings, but we argue that it is key for students to abstract the logic of mathematical conditionals. For students to abstract the logic of conditional statements they must be able to populate and describe quite complex predicates. The hypothesis of the MVT is a conjunction of multiple conditions, and its conclusion is existentially quantified. It is beyond the scope of this paper to explore how students expand their ability to populate and describe for these more complex predicates. What we observe is that for them to assimilate such theorems to their understanding of the logic of universally quantified conditionals, they need to construe these various conditions as predicates with truth sets and falsehood sets determined by the stipulated properties. Only when all kinds of mathematical predicates can be understood as forming categories does the categorical structure of logic generalize across mathematical statements (which we shall discuss further with respect to negations in Section 3.6).

3.3. *Composing inferences with populating/describing*

People consistently draw new information from their understanding of given information. Philosophers and logicians have long considered how to evaluate the quality of various sorts of inferences, with Leibniz (1969) dream of a reasoning calculus as one expression. Classically, the conditional form of statement is used to express an inference ($P \Rightarrow Q$ is expressed "if P, then Q"). One way to question the quality of such an inference is to ask whether it is "valid." This question of the quality of inferences is central to mathematical proving since the mathematical standard of truth is closely tied to logical validity (Duval, 2007; Weber & Alcock, 2005). For the purpose of our explication of how these ideas operate we shall consider one common definition provided for validity: there is no instance where the hypothesis is true and the conclusion false. In other words, the inference "$P \Rightarrow Q$" is valid if there is no example where P is true and Q is not. In the case of specific conditionals, this is the same as claiming there are no counterexamples. Note that this reference to "instances" or "examples" demonstrates how inferences between properties are almost always implicitly composed with populating actions on the truth sets of the predicates. For instance, the MVT can be explained as saying "the property that the instantaneous rate of change somewhere equals the average rate of change can be inferred from the properties continuity and differentiability" or as saying "every function that is continuous and differentiable on a closed interval will always have an input where $f'(c) = \frac{f(b)-f(a)}{b-a}$." We question whether this definition of validity really adds any information about the quality of the inference itself. Consistent with our claim that logic is normative, this definition helps judge when an inference fails (by identifying a counterexample) more than it allows us to understand what makes an inference valid or how to draw such inferences.

This composition of inferring with populating shows why it is so important for the sets associated with a category to be precisely the truth-sets for the given properties (as discussed in the last section). A valid inference on the property P will only carry to the whole set of objects if the set is populated with precisely the set of objects with property P. For instance, we may infer that all exponential functions are solutions to a differential equation of the form $f' = Kf$ (one meaning for an exponential function is that its rate of change is proportional to the function value). Many students may populate "exponential function" with higher order polynomial functions because they have exponents in their equations, but this inference does not carry to those examples (it admits counterexamples within the set so populated).

Another implication of this composition of populating with inferencing is that students may only perceive that the inference holds on the whole truth set when the inferences are rooted in their meanings for the relevant properties. In our previous studies inviting university students to read mathematical proofs, we noted the stark difference between their attempts to make sense of the argument written in the proof and their own argumentation rooted in their personal meanings for the relevant concepts (Dawkins & Roh, 2022). In an example we share toward the end of this section, we noticed differences among the inferences students drew regarding the same proof depending upon whether they reasoned about listing "multiples" by skip counting or testing which numbers were "divisible" by dividing. From an epistemological standpoint, we propose to distinguish inferences rooted in reversible and composable mental actions on the underlying mathematical objects from inferences that students cannot so construe, either because their sense of reference is not reversible or the implications in the proof do not correspond to inferences the learner is ready to draw. Previous studies of student understanding of proof lament when students believe every step in a proof without being convinced that the corresponding mathematical relationship admits no counterexamples (Fischbein & Kedem, 1982; Fischbein, 1982). We propose that this is unsurprising if students have not constructed a reversible relationship between the properties and sets of objects or if students reason about properties that are not the basis of the inferences in the proof.

If a student is able to compose the inference "P implies Q" with populating the two predicates with their truth sets, one ultimately affirms a subset relationship: the truth set of P is a subset of the truth set of Q. This is still distinct from the mental act of *expanding* since one went from one set to the other through the properties. The distinction between recognizing a subset relationship through *expanding* and by composing *populating* with *inferring* is the same as the distinction between an empirical generalization (between the sets of objects) and a property-based inference or proof (between the defining properties), as portrayed in Fig. 2.

As an example of the distinctions described above, we will briefly share observations about two students, whom we call Jess and Zandra, in a recent teaching experiment reasoning about the proof shown in Fig. 3 (reported in Dawkins & Roh, 2022). Though Jess and Zandra populated the properties in the proof and worked through the computations with those examples, they did not think of the algebraic expressions (k and $k * 2$) in terms of their role in the computation (quotient when dividing). The students alternated between thinking about "multiple of 6" in terms of dividing to test for "divisibility" and skip-counting to list "multiples." Their own reasoning shifted based on which meaning they adopted in the moment. However, their different personal meanings did not afford the same inferences and they did not populate to the same set. Furthermore, they did not use the factor-out argument in the proof ($x = k * 6 = (k * 2) * 3$) as a means to relate the division operation for 6 and 3 or the lists of multiples. For instance, Zandra noticed that the quotients when dividing by 3 were even, but did not know why (an instance of expanding). Because the algebraic equations in the proof did not serve as expressions of their computational activity, they did not perceive the proof as justifying that all multiples of 6 are multiples of 3. Specifically, they did not perceive that the quotient when dividing by 3 is twice the quotient when dividing by 6, as could be inferred in line 3. Thus, they populated and described in distinct ways that were not always compatible one to another (not reversible) and they did not see necessity for the inferences in the proof since they were reasoning with different properties (skip counting and dividing instead of factoring). They could only follow the inferences in the proof after the interviewer engaged them in a new sequence of activity, interpreting "multiple" as being made up of equal groups. For instance, he asked them if we divided by 14 and found a number was made up of 250 groups of 14, how many groups of 7 would be in the number? By engaging in the activity of partitioning groups of 14 into groups of 7, Jess and Zahra were able to draw the inference that the quotient when dividing by 7 would be twice the quotient when dividing by 14. This is because the inference was entailed by their partitioning activity. Furthermore, partitioning into equal groups matched their skip-counting meaning for being a multiple. This brief example illustrates our claim that a proof's ability to justify a general claim will interact with students' meanings for the properties and the inferences available from those meanings.

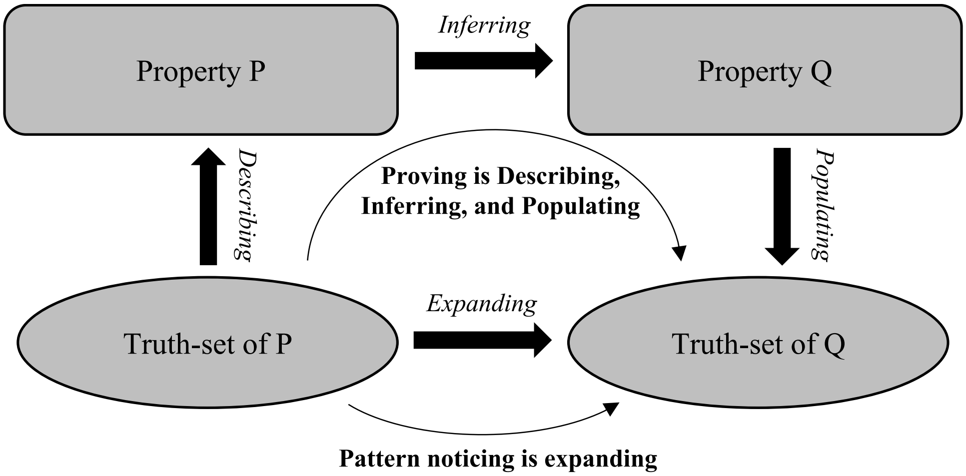

Fig. 2. Contrasting pattern noticing and proving that "All x with P are also x with Q.".

Definition 1: We say the integer n is a **multiple of d** whenever there exists some integer k such that $n = k * d$. This can also be stated as "n is divisible by d" or that "d divides n."
Notice that in the case that n is a multiple of d meaning $\frac{n}{d} = k$, k is called the **quotient** of dividing n by d.

Theorem to be proven 1: For every integer x, if x is a multiple of 6, then x is a multiple of 3.

Proof 1.1: Let x be any integer that is a multiple of 6.
Then by Definition 1, there exists some k such that $x = k * 6$.
Since $6 = 2 * 3$, $x = k * 6 = (k * 2) * 3$.
Since $k * 2$ is also an integer, according to Definition 1 x is a multiple of 3.

Fig. 3. A proof that Jess and Zandra struggled to coordinate with their computational activity.

3.4. *Abstracting the logic of inference*

In this section, we shall argue how the kinds of proofs discussed in the previous sections help satisfy two central goals of the paper. First, they link the logical notion of implication to the psychological activity of inferring. Second, they form the raw material for the abstraction of logical structure that might unify all conditional claims in mathematics. To say that inferring is an important mental act relevant to the learning of logic bypasses a more surprising claim from a psychological standpoint. The logical notion of "validity" suggests that all acts of *inferring* can be seen as the same sort of mental act, among which we may distinguish certain qualities.[3] Combining all mental actions on a property in this way constitutes a very broad and general abstraction. The previous paragraphs consider the relevance of composing these mental actions for a student's sense of confidence in a particular mathematical relationship. While we as researchers can portray the parallel structure between various inferences of this sort, this is quite distinct from a student consciously recognizing some sameness across various inferences they draw. Learning logic further entails abstraction of such categorical relationships to construct a content-general understanding of implication itself.

From a psychological standpoint, inferences vary greatly in scope and significance. Some inferences may be considered rewordings ("a square is equilateral and equiangular"), others are modest in scope ("a quadrilateral has interior angles summing to four right angles"), and still others represent notable shifts in meaning (MVT). Outside the mathematical realm, we may associate properties in a range of ways (what comes to mind from your meaning for "politician"?). How does it occur that a mathematics student reflects on their own reasoning so as to consider these varied "inferences" as instances of the same activity and further consider the distinction between "valid" or "invalid" inferences?

[3] We acknowledge that logicians have tried to define validity as a property of the form of an argument so as to remove any links to mental acts or psychology. This divorces logic from the human activity of constructing proofs and does not fit with our commitment to rooting logic in student activity. We thus seek to maintain the link between logical implications and psychological inferences. Some authors manage this distinction by using "validity" to refer the logical property and "deductive reasoning" to refer to the cognitive activity, but in either case our standpoint on logic commits us to avoiding that gloss.

We do not propose in this paper to answer this question, but we argue that the mental acts of inferring (which entails a range of content-specific mental actions) and populating constitute the mental activity that would be reflected on to reason about logic. We posit that for students to abstract inferencing as an action upon which logical concepts can be constructed, they must be able to reflect on their sequence of activity leading from P to Q and anticipate that sequence of activity being repeated for any other member of the truth-set of P. From a teaching standpoint, this has been carried out in the context of proof by induction (Norton et al., 2022). Specifically, mathematical induction requires students to abstract a general inference, $P(k) \to P(k+1)$, that they can apply to an arbitrary value of k; though pedagogically, it may be useful for them to begin by drawing inferences from particular cases, as in $P(1) \to P(2)$ (termed "naive induction" by Avital & Libeskind, 1978).

By allowing students to follow a proof of an implication with various particular instances, they may come to anticipate the inference applying to new cases without completing the sequence of mental actions. The composition of the sequence of mental actions into a single entity that can be repeated, unitized, and inverted is what we mean by inferencing becoming reversible and composable. It may be that it is actually in the refinement of these inferring actions that an example is able to become a carrier, since the student anticipates the generality of the object exhibiting the properties. What is still little understood is how a student would curtail their mental actions to only those available on any member of the truth set or how students would then construct analogies between sequences of mental action in geometry, number theory, and beyond to see a general logical notion of "implies."

Once students have some experience seeing proofs as inferences or chains of inference from hypotheses to conclusions, we expect that they can abstract this structure to all conditional claims. This is the basis on which we expect students to see all universally quantified conditional statements as being logically the same. This approximates Weber and Alcock (2005) notion of the warranted conditional, which referred to how a mathematician only affirmed a conditional claim if the conclusion could be inferred from the hypothesis using acceptable warrants. Notice this is quite different from most everyday instances of conditional statements. Even a quite general conditional like "if something is made of metal, then it conducts electricity" is not an expression of an inference from the defining property of metals to the defining property for conducting electricity for most non-scientists.

3.5. Reversing inference

The preceding sections covered the mental actions most important for constructing and justifying mathematical proofs. In the spirit of Piaget's use of algebraic structures, we sought to complete the system of actions between sets of objects and properties. This necessitates identifying an inverse for inferring (this section) and the formulation of negative predicates (the next section).

The inverse action of inferring, which we call *assuming* or *specifying*, is considered much less in logic. We benefitted from observing students engaging in the activity to understand its features.[4] We observed students engaging in *specifying* when they explored the relationship between a statement and its contrapositive (the contrapositive of "if x is a multiple of 6, then it is a multiple of 3" is "if x is not a multiple of 3, then it is not a multiple of 6."). Students in one teaching experiment understood that being a multiple of 6 entailed being a multiple of 3 based on the "factor out" argument expressed by $x = k * 6 = (k * 2) * 3$. To engage in *specifying*, they shifted from thinking of a fixed carrier that had the property "multiple of 6" to thinking of a modifiable carrier of which various properties could be assumed or applied. This means imagining adding factors to some number. They recognized that "multiple of 6" required more than "multiple of 3" and thus was more specific. In the language of assuming, one must assume something is a multiple of 3 and then assume further that something is a multiple of 2 to ensure that number is a multiple of 6. Composing this assumption of properties with populating to the truth-sets, the students recognized that assuming more properties is tantamount to narrowing or specifying the truth-set.

Reflecting on these students' work, we recognize that this kind of reasoning occurs quite often when we prove a claim using cases. When a proof divides up a general claim into cases, each sub-argument assumes properties that are not shared by the whole set in question. By assuming a new property (that is not entailed by the assumptions), the proof temporarily narrows the scope of inference. The new information affords new inferences on this smaller set. So long as the proof includes other sub-arguments that support the inferences for the rest of the set in question, this kind of argument can still prove a universal claim. Understanding proof by cases thus relies on composing assuming actions with populating actions to guarantee the generality of the overall proof.

3.6. Negating

While negating a property or predicate is by no means unique to mathematics, mathematics uses a very specific form of negation (see Dawkins, 2017). Specifically, any claim and its negation ought to obey the law of contradiction (they cannot both be true) and the law of excluded middle (in every case one of the two should be true). Negations are typically covered in standard logic, but the formal trick of simply reversing truth-values glosses over much more complex issues of how students reason in context. We have observed that students employ a number of ways of thinking about claims including *not* that are not directly compatible with negating a predicate. For instance, seeking to find a positive description to replace "not a rectangle," some students introduced "is a parallelogram." While a prototypical parallelogram is not a rectangle, the set of parallelograms is neither disjoint from the set of rectangles (law of contradiction) nor does it cover all non-rectangles (law of excluded middle). This is for us an instance of non-reversible population of the predicate "not a rectangle."

[4] Note inferring P from Q is not the inverse of inferring Q from P. This converse inference is only sometimes available and we want to consider all acts of inferring as reversible.

Our treatment of the logic of conditional statements must address negation or else not all universally quantified conditionals will really be of the same logical form. Psychological studies on how students interpret conditional statements show huge variance in interpretation based on the presence and placement of negations (Evans and Over, 2004). We have observed similar effects in student interpretation of mathematical disjunctions and conditionals (Dawkins, 2017, 2019; Dawkins & Cook, 2017; Hub & Dawkins, 2018). The key issue is that students often interpret predicates such as "is not a rectangle" not as a new predicate ($x \in \sim P$), but as a different relation ($x \notin P$). A consequence of this is that not all conditionals have the same logical form for students. A statement like "if a quadrilateral is a square, then it is a rectangle" is taken to mean "all squares are rectangles." Its contrapositive, "if a quadrilateral is not a rectangle, then it is not a square," is taken to mean "none of the non-rectangles are squares." While some of these other ways students reason about *not* are coherent and appropriate, they do not afford the same logical abstraction since it damages the logical analogy between various mathematical conditionals. In Dawkins (2017), we discussed how reasoning about logic requires students to associate the negation property with the complement set of objects, which we call the negation-complement relation. In the language of this paper, students must learn to think about negating a property as producing a new property (e.g., *irrational number*) to abstract the logical structure assumed in proof-based mathematics. Composing this form of negation with populating should support the *negation-complement relation* (Dawkins, 2017).

4. Discussion

In this section, we discuss some implications of rooting logic in these mental actions. Logic learning is challenging to study because it is so ubiquitous that it is unclear when and how it should be addressed in the curriculum. In Section 4.1 we comment on how this view of logic informs both instruction on logic and instruction trying to engage students in authentic proving activity. In Section 4.2, we reflect on the role these four mental acts play in the emergence of *arbitrariness*, which underlies much of mathematical proving and is ill understood from a learning standpoint. In Section 4.3, we summarize our claims and contributions.

4.1. Abstracting logical relationships by engaging in these four actions

As stated in Section 2.1, we view reasoning about logic as a metacognitive activity because it involves abstracting content-general relationships from students' ongoing activity regarding content-specific mathematical statements. It is not enough for students to engage in populating, inferring, expanding, and negating, since they may not note any similarity between these activities in real analysis, algebra, and number theory. Instruction must assist them to reflect on the categorical structure that arises in any of these settings to see how all universally quantified mathematical theorems with hypotheses and conclusions can be understood as having this same essential structure. We see two practical sides to this point.

First, common ways of teaching logic that appear in undergraduate texts focus on truth-values and not on the underlying sets of objects (Dawkins et al., 2022). This does not emphasize the categorical nature of mathematical predicates. It tries to capture logical relationships only in terms of implication without the other mental acts. We conjecture that this truncates the scope of the logic that students learn. Such instruction also frequently avoids meaningful mathematical contexts, which means that students are not able to metacognitively engage in these activities since they are not populating and inferring. Indeed, this framing of logic seems to conflict even with the way mathematicians reason about conditional statements (Inglis & Simpson, 2004; Weber & Alcock, 2005). Weber and Alcock in particular observed that mathematicians did not want to endorse a conditional simply based on the truth or falsehood of P and Q, but rather they considered whether Q could be proven from P. This "provability" meaning of validity is much closer to what we endorse in this paper than the "no instance where the hypotheses are true and the conclusion false" meaning offered in many textbooks.

In contrast to the portrayal of logic in textbooks, our notion of inferring stipulates that students are enacting various mental actions on the properties/carriers that form a reversible and composable pathway from hypothetical properties to concluded properties. Only in this way is logic an extension of students ongoing mathematical construction. Logic, like proof, should constitute a reorganization of how students think about mathematical relationships and engage in mathematical reasoning. We argue this can be productively facilitated by helping students refine these four mental actions, which allows students to render parts of mathematical language as truly mathematical in nature (a reversible and composable grouping of actions; Beth & Piaget, 1966). We think this viewpoint helps explain how logicians reflect on their mathematical reasoning to abstract powerful logical understandings. It shows why we claim that, psychologically speaking, logic was built on the foundation of mathematics.

Second, many efforts to teach mathematical proving try to bypass the formalities of logic by engaging students in argumentation about problem solving activities that afford formal proof (e.g., Yee, Boyle, Ko, & Bleiler-Baxter, 2018). This is a productive approach, but we argue that students' inferring still may not afford the emergence of reasoning about logic until students reflect on and refine these essential activities. What is shared across various mathematical conditional theorems is the categorical structure. A natural approach to helping students abstract these logical relationships is to iteratively engage in the four actions across different mathematical contexts (see Dawkins & Cook, 2017; Dawkins & Roh, 2022; Hub & Dawkins, 2018). For various theorems and proofs, students need experience populating and following lines of inference to justify the implication (either arguments they construct or they read). We mentioned above that students may begin by following the inferences with a few examples until they feel confident that the argument would follow for a new example without needing to mentally carry out the full sequence. As this is repeated across contexts and with increasingly complex predicates in the theorems, students need opportunities to reflect (Wheatley, 1992) on what generalizes across the theorems and proofs. As students recognize a sameness across the reference and inference structures of these examples, they are constructing logical concepts in the sense we set forth in this paper.

4.2. The emergence of arbitrariness

Almost all mathematical proofs of universally quantified claims make use of arbitrariness (what logicians have called the principle of universal generalization, Copi, 1954). We expressed this with the notion of a *carrier* of a mathematical property, as opposed to an example. This paper sought to explicate how a reversible relationship between property and category is essential to property-based inferences holding on a (possibly infinite) set of objects. This idea is so essential to proof-based mathematics that it is taken for granted. Many theorems in undergraduate textbooks are stated not using the universal language of "for all" or "for any," but rather the language of selecting a carrier: "Let $f : [a, b] \to \mathbb{R}$ be continuous…" This replacement of universal quantification with selecting a carrier of the property hides the way that arbitrariness is essential to the proof having infinite scope (c.f., Shipman, 2016). We do not think this is a problem with the writing of textbooks. This is making productive use of the great power of mathematical proving. Rather we hope to emphasize that students need opportunities to learn how arbitrariness operates in mathematical proofs and how it guides and curtails their reasoning.

We conjecture that engaging in these four mental actions, reversing them, and reflecting on that activity can support the emergence of the concept of arbitrariness. In the classroom, this would look like activities organizing examples and non-examples in terms of what properties they share. For a proof, students may need chances to run through the inferences in the proof with a sequence of examples before they can anticipate that the inference would hold on a carrier of the hypothetical properties. To the extent that students recognize that the set of objects referred to are unified by a mathematical property and that property entails other properties, they can experience a sense of necessity that the inferences they draw carry to the whole set. Understanding this logic may help students understand what it means that a carrier is "fixed, but arbitrary" and what it means to draw "valid inferences" based only on the given assumptions. It may also be that rooting inferences in students' mental actions may implicitly resolve this issue inasmuch as the actions build on student meanings for the relevant properties. If any member of the truth-set can be assimilated to the sequence of mental actions, then the match between property and truth-set will follow from the nature of the sequence of mental activity. How these relationships co-emerge, mutually organize, and lend themselves to logical reflection is a matter of further (and ongoing) empirical investigation.

4.3. Summary

This paper characterized four types of mental actions that we argue must become reversible and composable to afford the logic of universally quantified mathematical conditionals. While many textbooks and classrooms offer an account of logic that does not build upon these actions, we find this insufficient for meaningful instruction. Defining conditionals only using truth-values hides the referential structure of actual theorems, which is a key aspect of how finite proofs can operate on infinite sets of objects. Defining in terms of truth-values also provides no underlying meaning for an inference or implication that could interface with the reasoning that students engage in while proving or reading proofs. Just as mathematics educators are dissatisfied with algebra as rule-based symbol manipulation in which variables do not refer to numbers, we should be dissatisfied with logic as rule-based syntax that cannot interface with meaningful theorems and proofs. This suggests that our frame for logic must include tools that account for reference (populating and describing), inference (inferring and assuming), and pattern noticing (expanding and nesting). We further add negating to the frame for "closure" both so statements including *not* are still of the same logical form and to afford important relationships like contrapositive equivalence. The way these actions allow us to express the interface between content-general structure and content-specific reasoning, which we deem necessary for identifying the cognitive roots of logic.

Epistemologically, we think this paper contributes to the theoretical foundations for how logic can emerge as a body of knowledge built upon the foundation of mental activity in mathematics. By organizing their engagement in these four mental actions, mathematicians have realized how to operate with language in a manner that affords the reliability of a calculus. This despite the fact that the language mathematicians use is still far from that of first-order logic (Azzouni, 2004). The surprising reliability and logical adequacy of mathematical reasoning (MacKenzie, 2001) and language depends upon the powerful way mathematicians have reorganized their engagement in these actions. Empirically studying how students construct these actions and reflect them to be consciously available as logical concepts may yield further insights into the emergence of logic from an epistemological standpoint.

How students actually curtail their reasoning to maintain validity most likely depends on the mathematical context. It may be that students (and mathematicians) must learn and relearn that skill in each new body of theory. This is because inferring depends on more particular mental actions on the specific properties and objects within the body of theory. However, we argue that mathematicians always engage in local mathematical reasoning with attention to the scope of their inferences, suggesting that they continually operate with the concepts of arbitrariness and validity at hand. We hope this paper contributes to the epistemological question of how mathematicians are able to develop such a powerful and productive metacognitive skill. We anticipate that students must reflect on these four activities to construct those same sensibilities. Toward that end, we hope these constructs will guide ongoing research on supporting students to abstract these relationships to facilitate their productive engagement in proof-based mathematics. In particular, we hope that these actions provide ways for them to construct logical understandings that operate in tandem with their mathematical activity in specific settings to refine and facilitate their mathematical growth.

CRediT authorship contribution statement

Paul Christian Dawkins: Conceptualization, Investigation, Writing – original draft **Anderson Norton:** Conceptualization, Writing – review & editing.

Declaration of interest

none.

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