



A simple model of radiating solitary waves

J. Douglas Wright

Department of Mathematics, Drexel University, Philadelphia, PA, United States of America



ARTICLE INFO

Article history:

Received 7 February 2022

Received in revised form 31 May 2022

Accepted 1 June 2022

Available online 22 June 2022

Keywords:

Radiating Solitary Waves

Metastability

Beyond all orders asymptotics

Singular perturbation

ABSTRACT

To understand an oft-observed but poorly understood phenomenon in which a solitary wave in a dispersive equation slowly deteriorates due to a persistent emission of radiation (i.e. a “radiating solitary wave”), we propose a bare-bones model which captures many essential features and which we are capable of analyzing completely by way of the Laplace transform. We find that wave amplitude decreases at an exponential rate but with a decay constant that is (in many cases) small beyond all orders of the frequency.

© 2022 Elsevier B.V. All rights reserved.

1. Introduction

In the articles [1–3], simulations of solitary waves in spatially heterogeneous variants of the Fermi–Pasta–Ulam–Tsingou (FPUT) and Toda lattices demonstrated that such waves do not propagate without change of form but instead continuously emit a small trailing ripple. The systems conserve energy and consequently the solitary waves experience a commensurate and extremely slow attenuation in amplitude. See Fig. 1 for a representative depiction. A similar phenomena occurs in simulations of mass-in-mass lattices [4,5], various coupled Boussinesq systems [6,7] as well as in certain fifth-order Korteweg–de Vries equations [8,9]. In some of these cases, especially those which model capillary-gravity waves [7–9], the emission runs ahead of the solitary wave. Simulations of water waves over variable bottom topography evoke a similar dynamic [10–13]. These sorts of waves, often referred to as *radiating solitary waves*, are examples of metastable structures in nonlinear dispersive systems.

The observed attenuation is so slow that one of the major tools for analyzing the dynamics of solitary wave-like solutions, namely approximations of the problem with the KdV (or similar) equations [14–18], is incapable of capturing the phenomena. Such approximations are valid over very long but nevertheless finite time intervals; the erosion is so subtle during the period of good approximation that it falls within the natural error bounds. Moreover, radiating solitary waves very often occur in problems where the construction of genuinely localized solitary waves fails and what is found instead are *generalized solitary waves* (also known as a *nanopterons*) [19–29]. These traveling wave solutions are asymptotic at spatial infinity to very small amplitude co-propagating periodic waves and are consequently of infinite energy, further evidence that the dynamics of finite energy solitary wave-like solutions is subtle.

Unlike their steady counterparts the nanopterons, there is at this time no fully rigorous mathematical explanation or description of these radiating solitary waves in any of the many problems in which they arise. There are several careful non-rigorous treatments (especially [6,8]), but in the main investigations are heuristic or numerical. The phenomenon is usually attributed to an excitation of some sort of high-frequency oscillation by the solitary wave, due, for instance, to band structure considerations [3,6], Bragg scattering [12], internal resonances [5] or non-monotone dispersion relations [8]. Notably, the amplitude decays so slowly that in most cases it is very hard to determine the rate with any level of precision.

E-mail address: jdw66@drexel.edu.

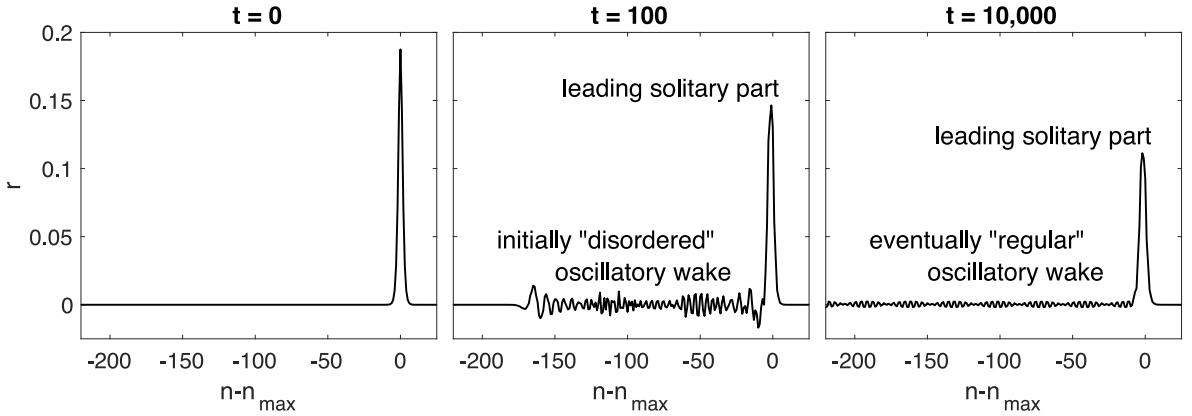


Fig. 1. An example of a “radiating solitary wave” in a diatomic FPUT lattice. The wave is propagating to the right and the horizontal axis is in a moving reference frame.

Source: Figure taken from [3].

For instance, the investigation in [6] finds that the amplitude of the radiating tail is exponentially small with respect to their parameter but, since their analysis proceeds from a quasi-steady state assumption, no prediction of decay rate for the solitary wave is made. And in those articles which do hazard a guess, they do not always agree; the formal asymptotics of [8] predict that the decay is exponential with a very small decay rate whereas [3] conjectures an algebraic rate of decay based on numerical evidence.

To better understand radiating solitary waves, in this article we formulate a rather bare-bones model which captures many essential features at play and which we can rigorously analyze. For our system we are able to exactly pin down the rate at which the solitary wave decays as well as a rather complete description of the radiating tail. Note that our model is decidedly not meant to quantitatively portray any of the specific systems mentioned but instead provide a partial skeleton for the rigorous analysis of such systems down the line.

Here are the main ingredients of our model:

- A spatial variable $x \in \mathbf{R}$ and time variable $t \in \mathbf{R}$.
- A solitary wave of fixed speed and profile but variable amplitude: $a(t)q(x - t)$. Here $q(x)$ is the profile and $a(t) \in \mathbf{R}$ is the amplitude. We assume that $q(x)$ is real-valued and satisfies some decay/regularity conditions we make precise below.
- A high-frequency simple harmonic oscillator located at each point $x \in \mathbf{R}$. We represent each oscillator by its complex amplitude $\psi(x, t) \in \mathbf{C}$. This field of oscillators is driven by the solitary wave in a naive way:

$$\psi_t = i\omega\psi + \omega a(t)q(x - t). \quad (1.1)$$

In the above, the oscillators’ natural frequency is $\omega \gg 1$.

- A conserved energy

$$E := \frac{1}{2}a(t)^2 + \frac{1}{2\omega^2} \int_{\mathbf{R}} |\psi(x, t)|^2 dx. \quad (1.2)$$

The first term is proportional to the square of the L^2 -norm (i.e. energy) of the solitary wave and the second is the energy of the oscillator field.

Together (1.1) and (1.2) form a closed system for the variables $a(t)$ and $\psi(x, t)$. Our main result states that $a(t) \approx a_0 e^{-\theta_\omega t}$ where θ_ω is all but equal to $2\pi^2 |\widehat{q}(\omega)|^2$ and \widehat{q} is the Fourier transform¹ of q . Weak assumptions on $q(x)$ tell us that $|\widehat{q}(\omega)| \rightarrow 0$ as $\omega \rightarrow \infty$ and as such the rate of decay is very slow when ω is big. In particular, if $q(x)$ is analytic then one knows that $|\widehat{q}(\omega)| \leq c_1 e^{-c_2 |\omega|}$ for some positive constants c_1 and c_2 . As such the decay rate of $a(t)$ is incredibly small, beyond all algebraic orders of ω . For instance, we find that if $\omega = 10$ and $q(x) = \text{sech}(x)$ then the time it will take for the amplitude to deteriorate to 95% of its original amplitude is on the order of 10^{11} .

We also prove several results about the asymptotics of the oscillator field, the most salient of which is that $e^{\theta_\omega t} \psi(x, t)$ converges to a scalar multiple of a specific profile function $\sigma_\omega(x - t)$. This convergence is pointwise in the moving reference frame $x - t$. We have an explicit formula for this asymptotic profile which ultimately leads to the description

$$\sigma_\omega(x - t) \approx i q(x - t) / \omega + \text{trailing periodic wave of extremely small amplitude.}$$

¹ The normalization/notation for the Fourier transform of a function $f : \mathbf{R} \rightarrow \mathbf{C}$ we use is: $\mathfrak{F}[f](k) := \widehat{f}(k) := \frac{1}{2\pi} \int_{\mathbf{R}} e^{-ikx} f(x) dx$.

That is to say, when viewed in a frame moving along with the solitary wave the oscillator field looks like roughly like a copy of the solitary wave plus a tiny periodic tail, exactly the qualitative behavior seen in the systems we are hoping to understand.

In Section 2 we carry out a number of reformulations of (1.1) and (1.2) which put the analysis and simulation of solutions within grasp. In Section 3 we use the Laplace transform to describe the dynamics of $a(t)$ for large times, the specifics of which are contained in Theorem 3.2. In Section 4 we similarly analyze the oscillatory field $\psi(x, t)$; see Theorem 4.2. Lastly, in Section 5, we describe the results of simulations of solutions of our model and compare them with our analytical results.

2. Reformulations and reductions

2.1. An ODE on a Banach space

Differentiation of (1.2) with respect to time and the use of (1.1) to eliminate ψ_t get us:

$$0 = a(t)\dot{a}(t) + \frac{1}{\omega} \int_{\mathbf{R}} \Re [\psi^*(x, t)(i\psi(x, t) + a(t)q(x - t))] dx.$$

The quantity $\psi^*i\psi$ is purely imaginary whereas a and q are purely real so we can simplify this to:

$$\dot{a}(t) = -\frac{1}{\omega} \int_{\mathbf{R}} \Re [\psi(x, t)] q(x - t) dx. \quad (2.1)$$

Note that now (1.1) and (2.1) are an ODE for the variables (ψ, a) on the Banach space $L^2(\mathbf{R}; \mathbf{C}) \times \mathbf{R}$. Under the assumption that $q(x) \in L^2(\mathbf{R}; \mathbf{R})$ the right-hand sides of (1.1) and (2.1) satisfy the following estimates:

$$\|i\omega\psi + \omega a(t)q(\cdot - t)\|_{L^2(\mathbf{R}; \mathbf{C})} \leq \omega \|\psi\|_{L^2(\mathbf{R}; \mathbf{C})} + \omega |a(t)| \|q\|_{L^2(\mathbf{R}; \mathbf{R})}$$

and

$$\left| \frac{1}{\omega} \int_{\mathbf{R}} \Re [\psi(x, t)] q(x - t) dx \right| \leq \frac{1}{\omega} \|\psi\|_{L^2(\mathbf{R}; \mathbf{C})} \|q\|_{L^2(\mathbf{R}; \mathbf{R})}.$$

We have used the Cauchy–Schwarz inequality in the latter estimate. These estimates indicate that the right-hand sides are bounded when viewed as a map from $L^2(\mathbf{R}; \mathbf{C}) \times \mathbf{R}$ into itself. Since they are also linear in (ψ, a) they are consequently continuous (see, for instance, Section 3.1 of [30]). Thus Picard’s Theorem (see Theorem 1.8.1 in [31]) gives the existence of solutions to the initial value problem for short times and this solution depends continuously on the data (Proposition 1.10.1 of [31]). By the Continuation Theorem (Theorem 1.8.3 of [31]), the solution will exist so long as the norm of the solution in $L^2(\mathbf{R}; \mathbf{C}) \times \mathbf{R}$ remains finite. The conservation law (1.2) implies that the norm is constant and therefore cannot become infinite. Thus the solution exists for all times. Summing up, we have shown:

Theorem 2.1. *Fix $q(x) \in L^2(\mathbf{R}; \mathbf{R})$ and $\omega > 0$. Then for any $\psi_0(x) \in L^2(\mathbf{R}; \mathbf{C})$ and $a_0 \in \mathbf{R}$ there exists unique $(\psi, a) \in C^1(\mathbf{R}; L^2(\mathbf{R}; \mathbf{C}) \times \mathbf{R})$ which satisfy (1.1), (2.1), $\psi(x, 0) = \psi_0(x)$ and $a(0) = a_0$. The map carrying (ψ_0, a_0) to (ψ, a) is continuous (for t in compact sets). Additionally for all t*

$$\frac{1}{2}a(t)^2 + \frac{1}{2\omega^2} \int_{\mathbf{R}} |\psi(x, t)|^2 dx = \frac{1}{2}a_0^2 + \frac{1}{2\omega^2} \int_{\mathbf{R}} |\psi_0(x)|^2 dx.$$

Before moving on, note that formulating our system as in (1.1) and (2.1) is a convenient starting point for performing simulations, which we do in Section 5.

2.2. A scalar delay differential equation

We can eliminate the oscillatory field from the governing equations entirely. We solve (1.1) by Duhamel’s formula:

$$\psi(x, t) = e^{i\omega t} \psi_0(x) + \omega \int_0^t e^{i\omega(t-s)} a(s) q(x - s) ds. \quad (2.2)$$

Putting this into (2.1):

$$\dot{a}(t) = -\frac{1}{\omega} \underbrace{\int_{\mathbf{R}} \Re [e^{i\omega t} \psi_0(x)] q(x - t) dx}_{j_{\omega}(t)} - \int_{\mathbf{R}} \Re \left[\int_0^t e^{i\omega(t-s)} a(s) q(x - s) ds \right] q(x - t) dx.$$

Rearranging the order of integration in the second term plus some algebra yields

$$\dot{a}(t) = j_{\omega}(t) - \int_0^t \cos(\omega(t - s)) q \star q(t - s) a(s) ds \quad (2.3)$$

where

$$q \star q(t) := \int_{\mathbf{R}} q(x+t)q(x)dx$$

is the autocorrelation of q . It is worth pointing out here that $q \star q(t)$ is an even function of t .

The scalar continuous delay differential Eq. (2.3) is equivalent to our original system and will be the formulation on which we do our analysis. Nevertheless there is one more change we make that permits a short formal analysis.

2.3. A scalar renewal equation

Integrating (2.3) from 0 to t gives

$$a(t) = a_0 + \underbrace{\int_0^t j_\omega(\tau)d\tau}_{f_\omega(t)} - \int_0^t \int_0^\tau \cos(\omega(\tau-s))q \star q(\tau-s)a(s)dsd\tau.$$

Exchanging the order of integration in the second term leads to

$$a(t) = a_0 + f_\omega(t) + \int_0^t \phi_\omega(t-s)a(s)ds \quad (2.4)$$

where

$$\phi_\omega(t) := - \int_0^t \cos(\omega s)q \star q(s)ds.$$

Eq. (2.4) is a renewal equation of the sort studied in [32], though in most applications of renewal equations the function $\phi_\omega(t)$ is positive with finite first moment (neither of which is the case for us).

We do a quick non-rigorous analysis of (2.4) now, in the case where $f_\omega(t) = 0$. Since $q \star q(t)$ is even, the Fourier Convolution Theorem implies $\phi_\omega(t) \rightarrow -2\pi^2|\hat{q}(\omega)|^2$ as $t \rightarrow \infty$. If we simply replace ϕ_ω with this asymptotic state in (2.4) we have $a(t) = a_0 - 2\pi^2|\hat{q}(\omega)|^2 \int_0^t a(s)ds$. Differentiation of this gives $\dot{a} = -2\pi^2\pi^2|\hat{q}(\omega)|^2a$ and so the solution of this simplified equation is $a_0 e^{-2\pi^2|\hat{q}(\omega)|^2 t}$. This gives us our first glimpse as to what the rate of decay for $a(t)$ is going to be. Of course this replacement of ϕ_ω is completely unjustified which is why we now move on to rigorous analysis.

3. The fate of $a(t)$

In this section we determine the long time asymptotics of $a(t)$ using (2.3) as the starting point. The integral on the right hand side of that equation is a time-domain convolution of $a(t)$ and

$$k_\omega(t) := \cos(\omega t)q \star q(t).$$

So if we apply the Laplace transform² to (2.3) we get

$$zA(z) - a_0 = J_\omega(z) - K_\omega(z)A(z). \quad (3.1)$$

We have made use of the well-known formulas for the Laplace transform of a derivative and of a time-domain convolution (see, for instance, Sections 6.2 and 6.6 in [33]). We adhere to the usual convention that the Laplace transform of a variable whose name is in lower case is represented by the corresponding capital.

We isolate $A(z)$ in (3.1):

$$A(z) = \frac{a_0}{z + K_\omega(z)} + \frac{J_\omega(z)}{z + K_\omega(z)}. \quad (3.2)$$

From this we ascertain that the key to understanding the evolution of $a(t)$ is the function $v_\omega(t)$ whose Laplace transform is

$$V_\omega(z) := \frac{1}{z + K_\omega(z)}. \quad (3.3)$$

The following result tells us everything we need to know about $V_\omega(z)$:

Lemma 3.1. *Suppose that, for some $\rho > 0$, $e^{\rho t}q \star q(t) \in W^{1,\infty}(\mathbf{R}^+; \mathbf{R})$. Then, for all ω sufficiently large, $V_\omega(z)$ has exactly one pole in the set $\Re(z) \geq -\rho/2$. This pole is simple and located at $z = -\theta_\omega \in \mathbf{R}$ where*

$$\theta_\omega = 2\pi^2|\hat{q}(\omega)|^2(1 + \mathcal{O}(1/\omega)). \quad (3.4)$$

² Here, we use the notation that the Laplace transform of a function $f : \mathbf{R}^+ \rightarrow \mathbf{C}$ is $\mathcal{L}[f](z) := F(z) := \int_0^\infty e^{-zt}f(t)dt$.

Proof. Preliminary estimates: For functions $f : \mathbf{R}^+ \rightarrow \mathbf{C}$ we put:

$$\|f\|_\rho := \sup_{t \geq 0} e^{\rho t} |f(t)|.$$

For functions with $\|f\|_\rho < \infty$ we have the following elementary estimate, which holds for $\Re(z) > -\rho$:

$$|\mathcal{L}[f](z)| = \left| \int_0^\infty e^{-zt} f(t) dt \right| \leq \int_0^\infty e^{-\Re(z)t} \|f\|_\rho e^{-\rho t} dt \leq \frac{1}{\Re(z) + \rho} \|f\|_\rho. \quad (3.5)$$

If we have $\|\dot{f}(t)\|_\rho < \infty$ then an integration by parts in the above gets:

$$|\mathcal{L}[f](z)| = \left| \frac{1}{z} f(0) + \frac{1}{z} \int_0^\infty e^{-zt} \dot{f}(t) dt \right| \leq \frac{1}{|z|} \left(|f(0)| + \frac{1}{\Re(z) + \rho} \|\dot{f}\|_\rho \right). \quad (3.6)$$

This again holds for $\Re(z) > -\rho$. Lastly, the famous identity $\mathcal{L}[tf](z) = -\frac{d}{dz} \mathcal{L}[f](z)$, combined with (3.6), implies

$$\left| \frac{d}{dz} \mathcal{L}[f](z) \right| \leq \frac{1}{|z|} \frac{1}{\Re(z) + \rho} \left\| \frac{d}{dt} (tf) \right\|_\rho. \quad (3.7)$$

There are no poles far away from the origin: Note that the poles of $V_\omega(z)$ are precisely the zeros of $z + K_\omega(z)$ and consequently our analysis will focus on $K_\omega(z)$. The assumptions on $q \star q$ and the definition of k_ω imply that $K_\omega(z)$ is analytic in the set $\Re(z) > -\rho$. Moreover we have $\|k_\omega\|_\rho \leq \|q \star q\|_\rho < \infty$. Importantly, $\|k_\omega\|_\rho$ can be controlled independently of ω . Using (3.5) we have $|K_\omega(z)| \leq \|q \star q\|_\rho / (\Re(z) + \rho)$. The restriction that $\Re(z) \geq -\rho/2$ tells us that $|K_\omega(z)| \leq 2\|q \star q\|_\rho / \rho$. Therefore $z + K_\omega(z) \neq 0$ when $|z| > 2\|q \star q\|_\rho / \rho$ and the only place to look for the zeros is in inside the set

$$U_\rho := \{ |z| \leq 2\|q \star q\|_\rho / \rho \} \cap \{ \Re(z) \geq -\rho/2 \}.$$

There is just one pole near the origin: We use the fact that $\cos(\omega t)$ is of high-frequency; the frequency shifting formula for the Laplace transform tells us

$$K_\omega(z) = \frac{1}{2} \mathcal{L}[q \star q](z + i\omega) + \frac{1}{2} \mathcal{L}[q \star q](z - i\omega). \quad (3.8)$$

Using (3.6) on the right hand side we have:

$$|K_\omega(z)| \leq \left(\frac{1}{2|z + i\omega|} + \frac{1}{2|z - i\omega|} \right) \left(q \star q(0) + \frac{1}{\Re(z) + \rho} \left\| \frac{d}{dt} q \star q \right\|_\rho \right). \quad (3.9)$$

We have assumed that $\|(d/dt)q \star q\|_\rho < \infty$ and this quantity is clearly independent of ω , as is $q \star q(0)$. In U_ρ we have $\Re(z) + \rho \geq \rho/2$ and $|z \pm i\omega| \geq \omega - |z| \geq \omega - 2\|q \star q\|_\rho / \rho$. All these together tell us that there are constants $\omega_0 > 0$ and $C_0 > 0$ for which

$$\omega \geq \omega_0 \text{ and } z \in U_\rho \implies |K_\omega(z)| \leq C_0 / \omega. \quad (3.10)$$

Now let $r := \min \{ |z| : z \in \delta U_\rho \} > 0$, which is independent of ω . Thus the preceding estimate allows us to find $\omega_1 > 0$ so that

$$\omega > \omega_1 \text{ and } z \in U_\rho \implies |K_\omega(z)| \leq r/2.$$

Thus we have $|K_\omega(z)| < |z|$ on the boundary of U_ρ . Since z and $K_\omega(z)$ are analytic in U_ρ we conclude, by Rouche's Theorem, that z and $z + K_\omega(z)$ have the same number of zeros (counted with multiplicity) inside U_ρ . Which is to say $z + K_\omega(z)$ has one simple zero in that set.

And so, as foretold in Lemma 3.1, we have shown that $V_\omega(z) = 1/(z + K_\omega(z))$ has exactly one simple pole (whose location is denoted by $-\theta_\mu$) in the set $\Re(z) \geq -\rho/2$. What remains is to further pin down this pole as described in (3.4).

Characterizing the pole: Since $q \star q(t)$ is even in t we have

$$K_\omega(0) = \int_0^\infty \cos(\omega t) q \star q(t) dt = \pi \Im[q \star q](\omega) = 2\pi^2 |\hat{q}(\omega)|^2. \quad (3.11)$$

The final equality is due to the Convolution Theorem. The regularity/decay condition on $q \star q(t)$ implies $\Im[q \star q](\omega) \rightarrow 0$ as $\omega \rightarrow \infty$, and thus so does $K_\omega(0)$.

Differentiating (3.8) we have:

$$K'_\omega(z) = \frac{1}{2} \frac{d}{dz} \mathcal{L}[q \star q](z + i\omega) + \frac{1}{2} \frac{d}{dz} \mathcal{L}[q \star q](z - i\omega).$$

Then we use (3.7) to get

$$|K'_\omega(z)| \leq \frac{1}{2} \left(\frac{1}{|z + i\omega|} + \frac{1}{|z - i\omega|} \right) \frac{1}{\Re(z) + .9\rho} \left\| \frac{d}{dt} (tq \star q) \right\|_{.9\rho}.$$

Note that we have used the fact that $\|f\|_\rho < \infty$ implies $\|tf\|_{\rho'} < \infty$ for $\rho' < \rho$. Much as in the run up to (3.10), the above estimate implies the existence of $\omega_2 > 0$ and $C_2 > 0$ such that

$$\omega \geq \omega_2 \text{ and } z \in U_\rho \implies |K'_\omega(z)| \leq C_2/\omega. \quad (3.12)$$

To complete the proof, we next note that $K(z)$ is real-valued if $z = s \in \mathbf{R}$. The fundamental theorem of calculus implies that $|K_\omega(s) - K_\omega(0)| = \left| \int_0^s K'(s')ds' \right|$ and so (3.12) leads to:

$$K_\omega(0) + (1 - C_2/\omega)s \leq s + K_\omega(s) \leq K_\omega(0) + (1 + C_2/\omega)s.$$

The function on the right is zero at $s = -K_\omega(0)/(1 + C_2/\omega)$ and the function on the left at $s = -K_\omega(0)/(1 - C_2/\omega)$. The intermediate value theorem then tells us that $s + K_\omega(s)$ has a zero somewhere between these two points, which, because $K_\omega(0) \rightarrow 0$ as $\omega \rightarrow \infty$, are both in U_ρ . Consequently that zero is $-\theta_\omega$, the unique zero of $z + K_\omega(z)$ in U_ρ we found earlier. All this, and (3.11), give:

$$\frac{2\pi^2 |\widehat{q}(\omega)|^2}{1 + C_2/\omega} \leq \theta_\omega \leq \frac{2\pi^2 |\widehat{q}(\omega)|^2}{1 - C_2/\omega}.$$

This is equivalent to (3.4) and the proof is complete. \square

Lemma 3.1 tell us that the rightmost pole of $V_\omega(z)$ is located at $-\theta_\omega$. The common wisdom holds that the placement of the poles of the transform dictate the asymptotic decay rate of the function and this thinking leads us to expect $v_\omega(t) = \mathcal{L}^{-1}[V_\omega](t)$ will behave like $e^{-\theta_\omega t}$ as $t \rightarrow \infty$. Nevertheless inverting the Laplace transform is a sometimes subtle business and so we make a precise statement and proof.

Theorem 3.2. Suppose that, for some $\rho > 0$, $e^{\rho|x|}q(x) \in W^{1,\infty}(\mathbf{R}; \mathbf{R})$. Then the solution of (1.1) and (2.1) with initial data $\psi(x, 0) = 0$ and $a(0) = a_0$ satisfies (for ω sufficiently large)

$$a(t) = a_0 r_\omega e^{-\theta_\omega t} + b_\omega(t) \quad (3.13)$$

where $\theta_\omega = 2\pi^2 |\widehat{q}(\omega)|^2 (1 + \mathcal{O}(1/\omega))$, $r_\omega = 1 + \mathcal{O}(1/\omega)$ and $\|b_\omega\|_{\rho'} < \infty$ for any $\rho' < \rho/2$.

Additionally if $e^{\rho|x|}\psi_0(x) \in L^\infty(\mathbf{R}; \mathbf{C})$ then the solution of (1.1) and (2.1) with initial data $\psi(x, 0) = \psi_0(x)$ and $a(0) = a_0$ satisfies (for ω sufficiently large)

$$a(t) = \alpha e^{-\theta_\omega t} + b_\omega(t) \quad (3.14)$$

for some constant $\alpha = \alpha(\psi_0, a_0) \in \mathbf{R}$; $b_\omega(t)$ satisfies $\|b_\omega\|_{\rho'} < \infty$ for any $\rho' < \rho/2$.

Proof. From Lemma 3.1 we know that the pole of $V_\omega(z)$ at $z = -\theta_\mu$ is simple. We now compute the residue at the pole in the usual way:

$$r_\omega := \text{Res}(V_\omega, -\theta_\omega) = \lim_{z \rightarrow -\theta_\omega} (z + \theta_\omega)V_\omega = \lim_{z \rightarrow -\theta_\omega} \frac{z + \theta_\omega}{z + K_\omega(z)} = \frac{1}{1 + K'(-\theta_\omega)}.$$

Estimate (3.12) tells us that $r_\omega = 1 + \mathcal{O}(1/\omega)$.

With this we now know that

$$W_\omega(z) := V_\omega(z) - \frac{r_\omega}{z + \theta_\omega}$$

is analytic for $\Re(z) > -\rho/2$. Because of (3.9) we have

$$\lim_{z \rightarrow \infty} zW_\omega(z) = \frac{z}{z + K_\omega(z)} - \frac{r_\omega z}{z + \theta_\omega} = 1 - r_\omega$$

provided $\Re(z) > -\rho/2$. Thus we have $|W_\omega(z)| \leq C/|z|$ for $|z|$ big enough. This is not a rapid enough decay to for us to deploy standard inversion results from (for instance) [34].

To get around this we let

$$B_\omega(z) := V_\omega(z) - \frac{r_\omega}{z + \theta_\omega} - \frac{1 - r_\omega}{z + 2\rho}.$$

The final term there is a sort of “fudge factor”. Note that $B_\omega(z)$ is analytic for $\Re(z) > -\rho/2$. It is easy to see that $\lim_{z \rightarrow \infty} zB_\omega(z) = 0$ provided $\Re(z) > -\rho/2$. Moreover a routine computation gives:

$$\lim_{z \rightarrow \infty} z^2 B_\omega(z) = (-2r_\omega + 2)\rho + r_\omega \theta_\omega. \quad (3.15)$$

Thus $|B_\omega(z)| < C/|z|^2$ for $|z|$ big enough. This rate is fast enough to use Lemma 76.4 in [34] and conclude that $e^{\rho't} \mathcal{L}^{-1}[B_\omega](t) \rightarrow 0$ as $t \rightarrow \infty$ for any $\rho' < \rho/2$.

And so all together we find that

$$v_\omega(t) = \mathcal{L}^{-1} \left[\frac{r_\omega}{z + \theta_\omega} + \frac{1 - r_\omega}{z + 2\rho} + B_\omega \right] (t) = r_\omega e^{-\theta_\omega t} + (1 - r_\omega)e^{-2\rho t} + \mathcal{L}^{-1}[B_\omega](t).$$

If $\psi_0(x, 0) = 0$ then from (3.2) we have $a(t) = a_0 v_\omega(t)$ and the above leads directly to (3.13).

Now for (3.14). If $\psi_0(x) \neq 0$ the decay conditions placed upon it imply that $\|j_\omega\|_\rho \leq C/\omega$; here is the calculation:

$$|e^{\rho t} j_\omega(t)| \leq \frac{1}{\omega} e^{\rho t} \int_{\mathbf{R}} |\psi_0(x)| |q(x-t)| dx \leq \frac{C}{\omega} e^{\rho t} \int_{\mathbf{R}} e^{-\rho|x|} e^{-\rho|x-t|} dx \leq \frac{C}{\omega}.$$

This tells us that $J_\omega(z)$ is analytic for $\Re(z) > -\rho$. Thus the term $J_\omega(z)V_\omega(z)$ in (3.2) is analytic in $\Re(z) > -\rho/2$ except for the pole of $V_\omega(z)$ at $-\theta_\omega$. The same sorts of steps as above can be repeated to show that $\lim_{t \rightarrow \infty} e^{\theta_\omega t} \mathcal{L}^{-1}[J_\omega V_\omega](t)$ exists, from which (3.14) follows. \square

4. The fate of $\psi(x, t)$

Now that we have determined the dynamics of $a(t)$ for large values of t , we do the same for $\psi(x, t)$. We begin by observing that because $a(t) \rightarrow 0$ and we have the conservation of the energy (1.2) we know that $\psi(x, t)$ does not converge to zero in the $L^2(\mathbf{R}; \mathbf{C})$ norm. This is not much of a statement, but it does indicate that the ultimate behavior of $\psi(x, t)$ is not disintegration.

For a more refined analysis, our starting point is (2.2) which expresses $\psi(x, t)$ explicitly in terms of $a(t)$. Here is our first result:

Corollary 4.1. *If $e^{\rho|x|}q(x) \in W^{1,\infty}(\mathbf{R}; \mathbf{R})$ and $e^{\rho|x|}\psi_0(x) \in L^\infty(\mathbf{R}; \mathbf{C})$ then the solution of (1.1) and (2.1) with initial data $\psi(x, 0) = \psi_0(x)$ and $a(0) = a_0$ satisfies (for ω sufficiently large)*

$$\lim_{t \rightarrow \infty} e^{-i\omega t} \psi(x, t) = \psi_0(x) + \omega \int_0^\infty e^{-i\omega s} a(s) q(x-s) ds$$

pointwise in x .

Proof. Divide both sides of (2.2) by $e^{i\omega t}$ and take the limit. The integral converges due to the restrictions placed upon ψ_0 and q . \square

The most important takeaway from this result is that it indicates that $\psi(x, t)$ does not decay to zero (in the supremum norm) in the large time limit. This can be made rigorous by drilling down into the integral term above to get a more refined picture, but it turns out it is more interesting to view $\psi(x, t)$ in a frame that moves along with the solitary wave $q(x-t)$. And so we put $x-t = l$ and $\varphi(l, t) = \psi(x, t)$. This converts (2.2) to

$$\varphi(l, t) = e^{i\omega t} \psi_0(t+l) + \omega \int_0^t e^{i(t-s)} a(s) q(t+l-s) ds. \quad (4.1)$$

We begin by considering the situation where $\psi_0(x) = 0$ and $a_0 = 1$, in which case $\varphi(l, t) = \gamma_\omega(l, t)$ with

$$\gamma_\omega(l, t) := \omega \int_0^t e^{i\omega(t-s)} v_\omega(s) q(t+l-s) ds.$$

If we fix l and take the Laplace transform of the above with respect to t , the convolution and frequency shifting identities get us

$$\Gamma_\omega(l, z) = \omega Q_l(z - i\omega) V_\omega(z)$$

where

$$Q_l(z) := \mathcal{L}[q(t+l)](z).$$

To be clear here, $Q_l(z)$ is the Laplace transform of $q(t+l)$ with respect to t .

With our standard assumption that $e^{\rho|x|}q(x) \in W^{1,\infty}(\mathbf{R}; \mathbf{R})$ we have $\|q(\cdot+l)\|_\rho < \infty$ for any $l \in \mathbf{R}$. Thus $Q_l(z)$ will be analytic when $\Re(z) > -\rho$. In turn this implies that $\Gamma_\omega(l, z)$ will inherit the simple pole at $z = -\theta_\omega$ from $V_\omega(z)$ and that this is the only singularity when $\Re(z) \geq -\rho/2$. It is easy to compute that

$$\text{Res}(\Gamma_\omega(l, z), -\theta_\omega) = \omega r_\omega \underbrace{Q_l(-\theta_\omega - i\omega)}_{\sigma_\omega(l)}. \quad (4.2)$$

And so we can conclude that

$$\Gamma_\omega(l, z) = \frac{\omega r_\omega \sigma_\omega(l)}{z + \theta_\omega} + Y_\omega(l, z)$$

where $Y_\omega(l, z)$ is analytic with respect to z when $\Re(z) \geq -\rho/2$ and for any l . Thus we have

$$\gamma_\omega(l, t) = \mathcal{L}^{-1} \left[\frac{\omega r_\omega \sigma_\omega(l)}{z + \theta_\omega} + Y_\omega(l, z) \right] = \omega r_\omega \sigma_\omega(l) e^{-\theta_\omega t} + \mathcal{L}^{-1}[Y_\omega(l, z)](t).$$

For $\mathcal{L}^{-1}[Y_\omega(l, z)](t)$ we can show, using exactly the same sort of techniques that we used in the proof of [Theorem 3.2](#), that it decays faster than $e^{-\rho't}$ for any $\rho' < \rho/2$; we omit the details.

The above line of reasoning leads us to:

Theorem 4.2. Suppose that, for some $\rho > 0$, $e^{\rho|x|}q(x) \in W^{1,\infty}(\mathbf{R}; \mathbf{R})$. Then the solution of [\(1.1\)](#) and [\(2.1\)](#) with initial data $\psi(x, 0) = 0$ and $a(0) = a_0$ satisfies (for ω sufficiently large)

$$\psi(t + l, t) = a_0 \omega r_\omega \sigma_\omega(l) e^{-\theta_\omega t} + \eta(l, t). \quad (4.3)$$

Here $\sigma_\omega(l)$ is given in [\(4.2\)](#), $r_\omega = 1 + \mathcal{O}(1/\omega)$ and $\|\eta(l, \cdot)\|_{\rho'} < \infty$ for all $\rho' < \rho/2$ and every $l \in \mathbf{R}$.

Additionally if $e^{\rho|x|}\psi_0(x) \in L^\infty(\mathbf{R}; \mathbf{C})$ then the solution of [\(1.1\)](#) and [\(2.1\)](#) with initial data $\psi(x, 0) = \psi_0(x)$ and $a(0) = a_0$ satisfies (for ω sufficiently large)

$$\psi(t + l, t) = \beta \omega r_\omega \sigma_\omega(l) e^{-\theta_\omega t} + \eta(l, t) \quad (4.4)$$

for some finite constant $\beta = \beta(\psi_0, a_0) \in \mathbf{R}$. As above, $r_\omega = 1 + \mathcal{O}(1/\omega)$ and $\|\eta(l, \cdot)\|_{\rho'} < \infty$ for all $\rho' < \rho/2$ and every $l \in \mathbf{R}$.

Remark 1. The result above is pointwise in l . It is quite possible that a stronger mode of convergence holds here, though the technical difficulty in establishing this is motivation enough to leave that for another article.

To close out this section, we now describe the “asymptotic profile” $\sigma_\omega(l)$ in greater detail. The analysis here is formal though it could be made rigorous if we stack enough hypotheses on $q(x)$. Recall that $\sigma_\omega(l) = Q_l(-\theta_\omega - i\omega)$. Since $\theta_\omega \rightarrow 0$ as $\omega \rightarrow \infty$ and $Q_l(z)$ is analytic in z we have $Q_l(\theta_\omega - i\omega) = Q_l(-i\omega) + \mathcal{O}(\theta_\omega)$. Then by definition we have

$$Q_l(-i\omega) = \int_0^\infty e^{i\omega t} q(t + l) dt = e^{-i\omega l} \int_l^\infty e^{i\omega x} q(x) dx.$$

And so we see that $e^{i\omega l} Q_l(-i\omega) \rightarrow 2\pi \widehat{q}(-\omega)$ as $l \rightarrow -\infty$ and $e^{i\omega l} Q_l(-i\omega) \rightarrow 0$ as $l \rightarrow \infty$. This latter convergence will be exponentially fast. Lastly the same sort of calculation that led to [\(3.6\)](#) gives $Q_l(-i\omega) = iq(l)/\omega + \mathcal{O}(1/\omega^2)$.

Putting everything together we have

$$\sigma_\omega(l) \sim \begin{cases} 2\pi \widehat{q}(-\omega) e^{-i\omega l} & l \ll 0 \\ iq(l)/\omega & l \sim 0 \\ \text{exponential decay} & l \gg 0. \end{cases}$$

In short, $\sigma_\omega(l)$ looks like a scalar multiple of the solitary wave profile plus a trailing periodic wave of small amplitude and frequency ω .

5. Simulations

We have simulated solutions of our model with a variety choices of the profile $q(x)$ and frequency ω . We always take $\psi_0(x) = 0$ since our analytic results indicate the effects of this part of the initial data are transient and do not alter the long time behavior (or at least the rate of decay of $a(t)$). Likewise $a_0 = 1$ in all cases.

Our method is straightforward: we treat the system as an ODE for (ψ, a) as in [\(1.1\)](#) and [\(2.1\)](#) and simulate using an RK4 algorithm. We implement the integrals in [\(2.1\)](#) via Simpson’s rule. We also compute $\sigma_\omega(l)$ numerically, as for most functions closed expressions are hard to obtain. We again use Simpson’s rule for the computation. All simulations were done in MATLAB.

Before we get into the results we note that with our method high accuracy/long time simulations of the problem are challenging to obtain even for modestly large values of ω . Roughly speaking, to accurately resolve a decay rate like $e^{-\theta_\omega t}$ we would need to simulate out to times of $\mathcal{O}(1/\theta_\omega)$ and (for the RK4 method we use) have a temporal step size which is $\mathcal{O}(\theta_\omega)$. The long time of integration in turn implies a large spatial domain (also $\mathcal{O}(1/\theta_\omega)$) is needed, since $q(x-t)$ propagates in space. And we need to resolve that spatial domain at the same step size as the temporal one. All these considerations tell us that we need $\mathcal{O}(1/\theta_\omega^4)$ operations, at a bare minimum, to experimentally determine θ_ω in a quantitatively reliable fashion. And in the most interesting cases θ_ω is exponentially small in ω , meaning that we quickly reach a computational bottleneck.

In light of these inherent difficulties we simply integrate out to $t = 1000$ and choose our step size to be $\pi(10\omega)^{-1}$. For the largest values of ω , our simulations have been pushed past the point at which we can be convinced of their quantitative reliability and instead we view them as a qualitative illustration of our results (which are, after all, fully justified) and the phenomena they describe.

We show our results in the figures which follow. Each figure contains the same four sorts of graphs.

- *Upper left panel:* A semilog plot of the numerical solution $a(t)$ vs t for the entirety of the run. In the same panel is shown the graph of $\alpha e^{-2\pi|\widehat{q}(\omega)|^2 t}$ where α is just a scaling factor used to make the graph readable. [Theorem 3.2](#) tells us that these should be nearly parallel when ω is large and in fact we do see this.

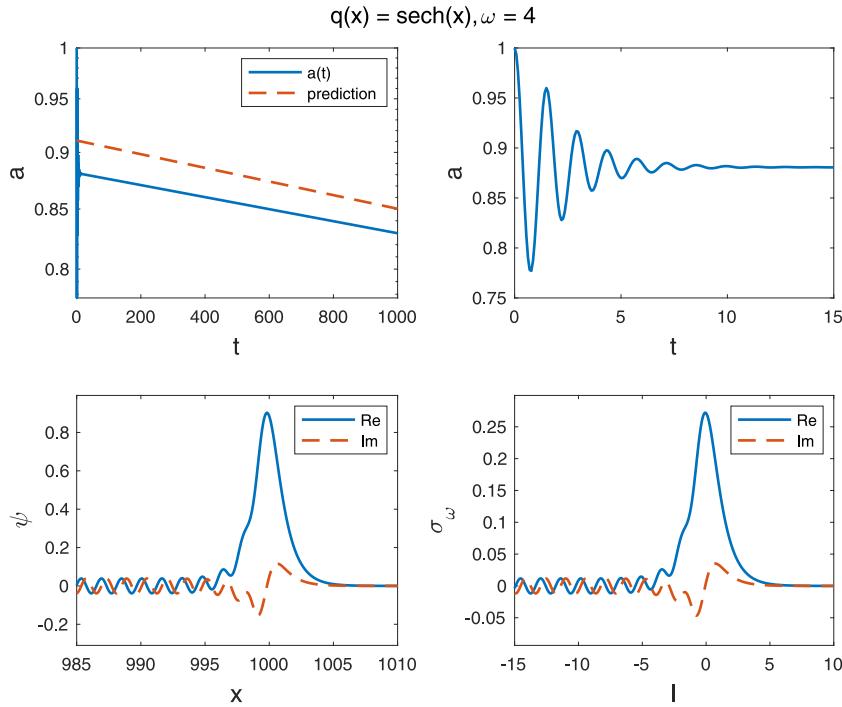


Fig. 2. Simulation results for sech profile, low frequency oscillators.

- *Upper right panel:* A plot of $a(t)$ vs t during the very beginning of the simulation. In each case we see that the $a(t)$ oscillates a few times and quickly “settles down” into the slow decay. The time it takes to settle down does not seem to depend on ω , as predicted by [Theorem 3.2](#).
- *Lower left panel:* The graphs of the real and imaginary parts of ψ vs x , at $t = 1000$, the end of the run. The figure is zoomed in on the leading edge of the solution, located near $x = 1000$.
- *Lower right panel:* The numerically computed asymptotic profile function $\sigma_\omega(l)$ vs l . The horizontal scale is arranged to match that of the previous panel. [Theorem 4.2](#) tells us that $\psi(x, t)$ should look much like (a scalar multiple of) $\sigma_\omega(l)$ as $t \rightarrow \infty$ and indeed we see exactly this.

5.1. $q(x) = \operatorname{sech}(x)$

We lead off with this choice for $q(x)$ because of the ubiquity of hyperbolic secant in profiles for solitary waves. One has $\hat{q}(\omega) = \operatorname{sech}(\pi\omega/2)/2$ and consequently we have

$$\theta_\omega = \frac{\pi^2}{2} \operatorname{sech}^2\left(\frac{\pi\omega}{2}\right) (1 + \mathcal{O}(1/\omega)).$$

We take $\omega = 4$ and 8 . Results are shown in [Figs. 2](#) and [3](#). When $\omega = 8$ both the decay of $a(t)$ and the trailing oscillatory tails of $\psi(x, 1000)$ and $\sigma_\omega(l)$ have vanished to the naked eye.

5.2. $q(x) = e^{-x^2}$

We selected this because the ultra-rapid decay of its Fourier transform renders the decay rate incredibly small even at modest values of ω . We have

$$\theta_\omega = \frac{\pi}{2} e^{-\omega^2/2} (1 + \mathcal{O}(1/\omega)).$$

We show results for $\omega = 4$ and $\omega = 6$, in [Figs. 4](#) and [5](#). Note how there is essentially no decay even at $\omega = 6$. Likewise at this value the trailing oscillatory waves in $\psi(x, 1000)$ and $\sigma_\omega(l)$ are invisible.

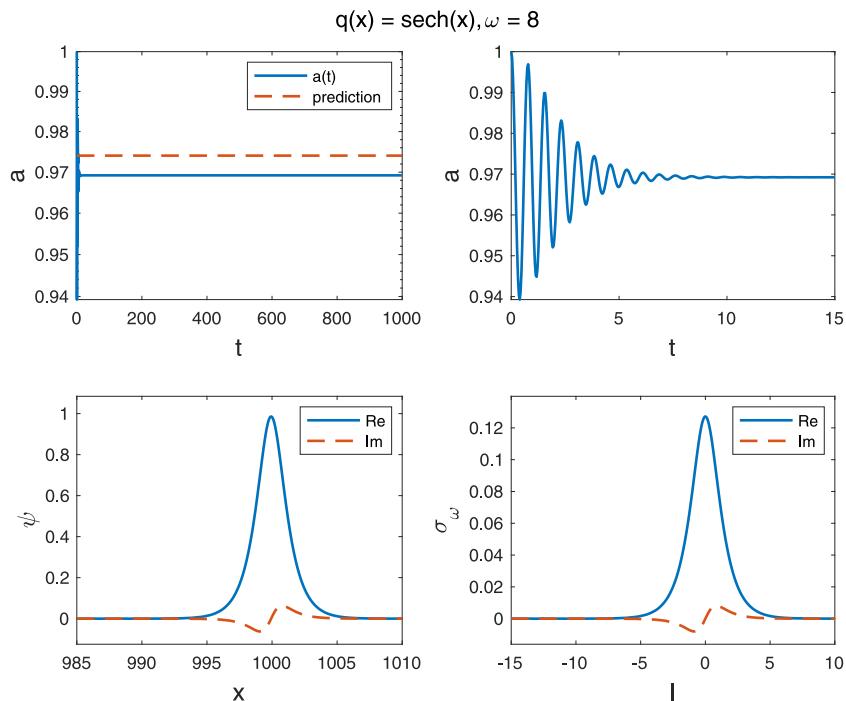


Fig. 3. Simulation results for sech profile, high frequency oscillators.

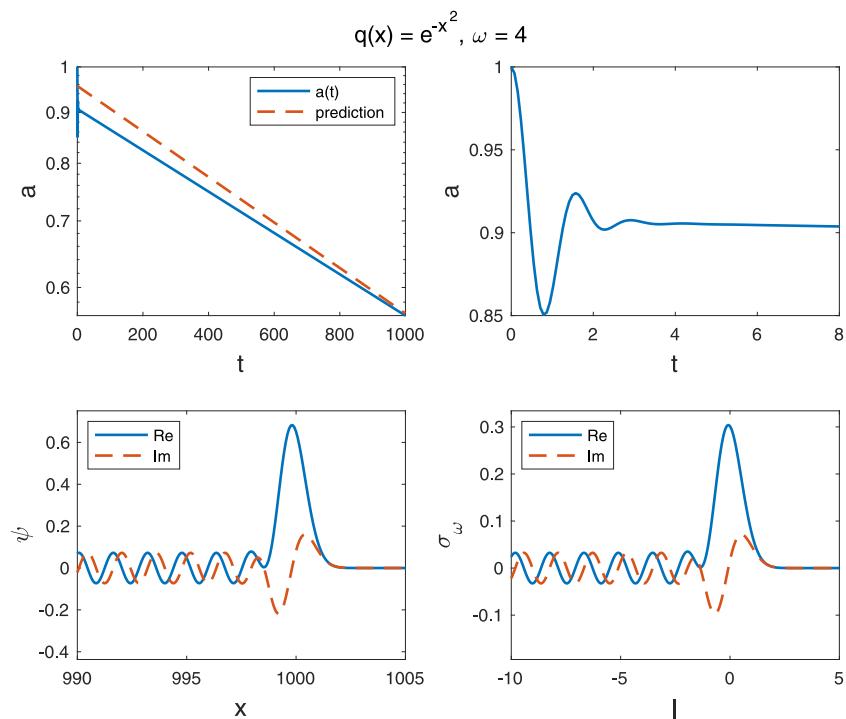


Fig. 4. Simulation results for gaussian profile, low frequency oscillators.

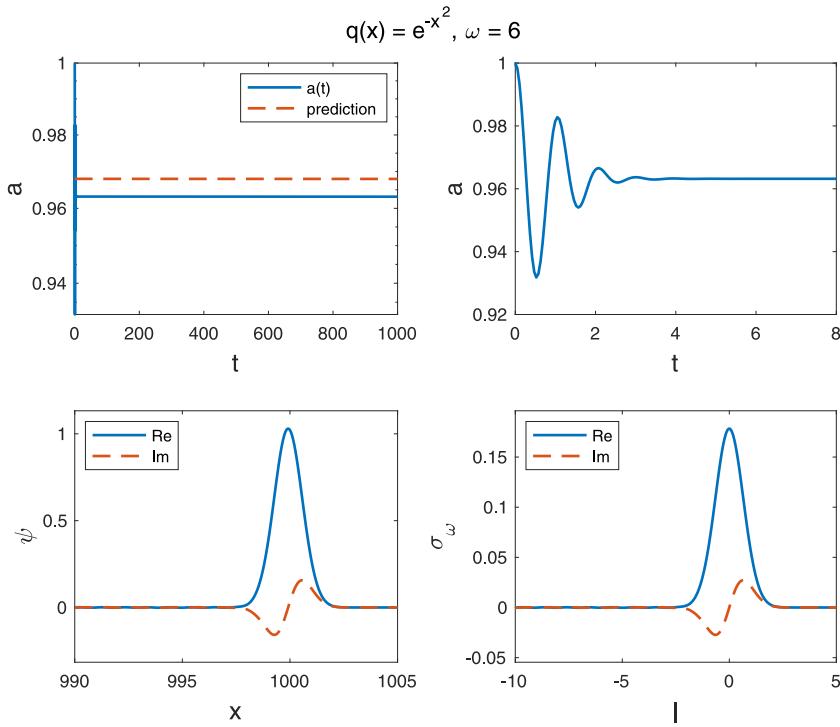


Fig. 5. Simulation results for gaussian profile, high frequency oscillators.

5.3. $q(x) = e^{-|x|}$

This “peakon” profile is not analytic and consequently the decay rate merely goes to zero algebraically fast and as such the decay is more obvious at larger values of ω . To wit

$$\theta_\omega = \frac{2}{(1 + \omega^2)^2} (1 + \mathcal{O}(1/\omega)).$$

We show results for $\omega = 8$ and 16 in [Figs. 6](#) and [7](#). Both the decay and the tails are visible at $\omega = 16$.

5.4. $q(x) = (1 - |x|)_+$

The subscript “+” mean to take the positive part, which is to say that $q(x)$ is the “tent” map. This is also non-analytic and its Fourier transform is sometimes zero (unlike the others) and as such if we fine tune ω we can get solutions which do not decay at all. Specifically we have

$$\theta_\omega = \frac{1}{2} \text{sinc}(\omega/2)^4 (1 + \mathcal{O}(1/\omega)).$$

so if we take ω to be an even multiple of π we should see no decay/no tails. We show results here ([Figs. 8](#) and [9](#)) for $\omega = 4\pi$ and 5π and we see exactly this behavior.

6. Conclusions, remarks and future directions

A key takeaway of this article is that in this simple model of radiating solitary waves the rate of attenuation in the amplitude is very slow, but nevertheless exponential. This is in line with the results of [\[8\]](#) and not the algebraic decay rate the author predicted in [\[3\]](#). The extremely slow rate of decay predicted by the main results provides further evidence that radiating solitary waves present challenging complications both on the numerical and analytic sides. Indeed, even in this simple model, really capturing the rate of decay numerically for large values of ω would require a much more sophisticated approach than is used here.

One can reasonably ask, however, whether or not our model here will truly be reflective of the systems which possess radiating solitary waves. After all, the derivation of [\(1.1\)](#)–[\(1.2\)](#) is *ad hoc* and (as is always the case in such models) important features/considerations have been omitted. Indeed, in our model only the amplitude of the solitary wave is

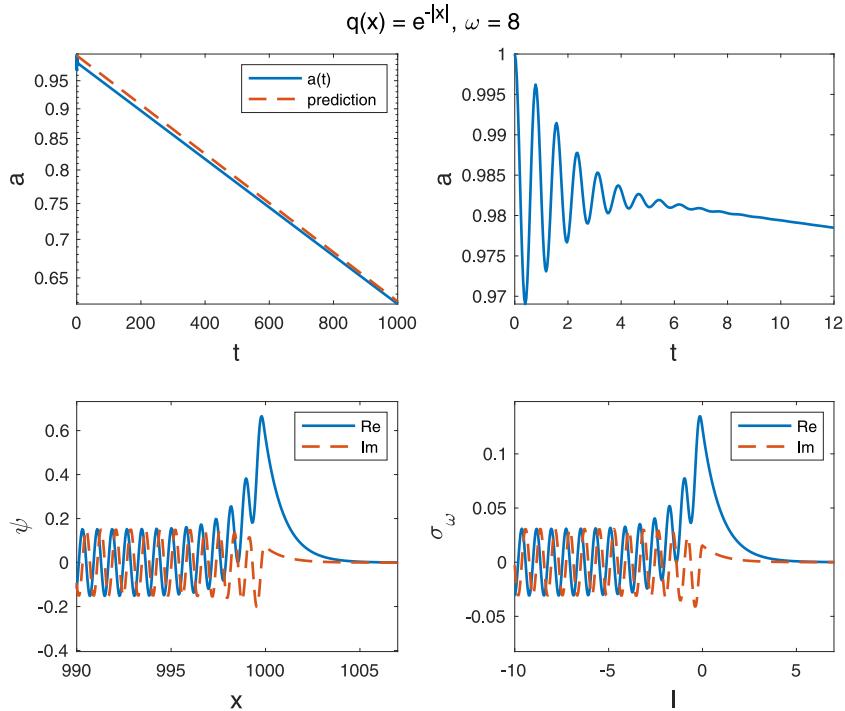


Fig. 6. Simulation results for peakon profile, middling frequency oscillators.

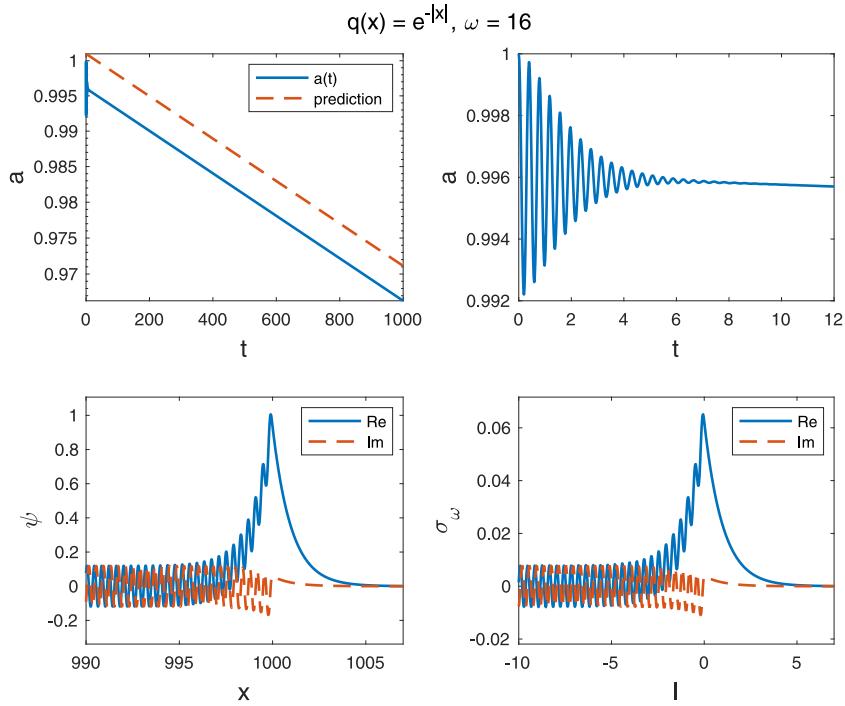


Fig. 7. Simulation results for peakon profile, high frequency oscillators.

variable and it has fixed speed. In nearly all systems with solitary waves, the speed, amplitude and wavelength of the solitary wave are linked. This is particularly important in the proofs of stability of solitary waves in KdV [35] and FPUT [36] and in the derivation of effective equations for solitary waves in potentials [37]. By restricting to a fixed speed and

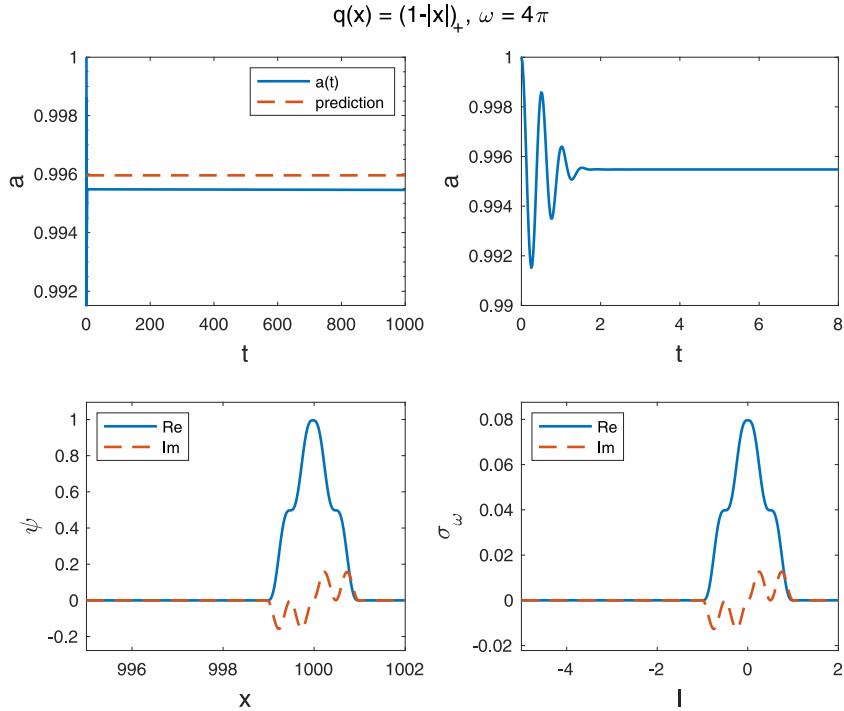


Fig. 8. Simulation results for tent profile, low frequency oscillators, tuned for no radiation.

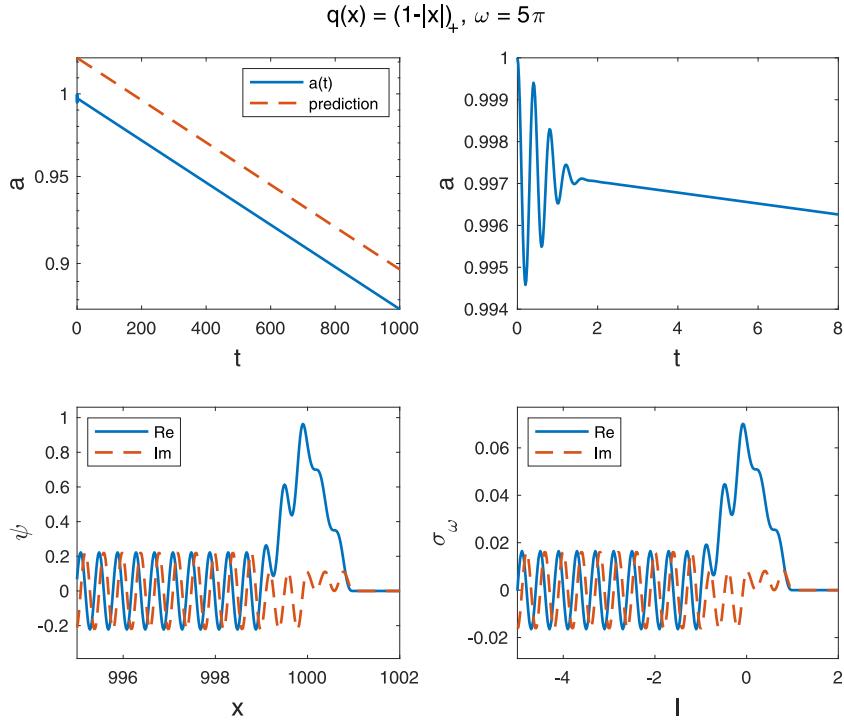


Fig. 9. Simulation results for tent profile, middling frequency oscillators, tuned to emit radiation.

width, our model is ultimately linear in its unknowns and this in turn makes the analysis by Laplace transform possible. Preliminary attempts at incorporating variable speed/wavelength result in nonlinear systems and, consequently, there are substantial technical challenges that make the application of the methods used here non-obvious. One avenue is to adapt

the refined asymptotics for stationary problems in [20] to this time-dependent setting. Success in that venture will lead to the next, and most exciting challenge: connecting such results rigorously to a full system with radiating solitary waves. Work is underway.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The author is grateful to the NSF, USA who funded this work under grant DMS-2006172.

References

- [1] Y. Okada, S. Watanabe, H. Tanaka, Solitary wave in periodic nonlinear lattice, *J. Phys. Soc. Japan* 59 (8) (1990) 2647–2658, <http://dx.doi.org/10.1143/JPSJ.59.2647>.
- [2] Y. Tabata, Stable solitary wave in diatomic Toda lattice, *J. Phys. Soc. Japan* 65 (12) (1996) 3689–3691, <http://dx.doi.org/10.1143/JPSJ.65.3689>.
- [3] N. Giardetti, A. Shapiro, S. Windle, J.D. Wright, Metastability of solitary waves in diatomic FPUT lattices, *Math. Eng.* 1 (3) (2019) 419–433, <http://dx.doi.org/10.3934/mine.2019.3.419>.
- [4] S. Hauver, X. He, D. Mei, E.G. Charalampidis, P.G. Kevrekidis, E. Kim, J. Yang, A. Vainchtein, Lattices with internal resonator defects, *Phys. Rev. E* 98 (2018) 032902, <http://dx.doi.org/10.1103/PhysRevE.98.032902>.
- [5] T.E. Faver, R.H. Goodman, J.D. Wright, Solitary waves in mass-in-mass lattices, *Z. Angew. Math. Phys.* 71 (6) (2020) <http://dx.doi.org/10.1007/s0033-020-01384-8>, Paper No. 197, 20.
- [6] R.H.J. Grimshaw, K.R. Khusnutdinova, K.R. Moore, Radiating solitary waves in coupled Boussinesq equations, *IMA J. Appl. Math.* 82 (4) (2017) 802–820, <http://dx.doi.org/10.1093/imamat/hwx014>.
- [7] J.L. Bona, V.A. Dougalis, D.E. Mitsotakis, Numerical solution of Boussinesq systems of KdV-KdV type. II. Evolution of radiating solitary waves, *Nonlinearity* 21 (12) (2008) 2825–2848, <http://dx.doi.org/10.1088/0951-7715/21/12/006>.
- [8] E.S. Benilov, R. Grimshaw, E.P. Kuznetsova, The generation of radiating waves in a singularly-perturbed Korteweg-de Vries equation, *Physica D* 69 (3–4) (1993) 270–278, [http://dx.doi.org/10.1016/0167-2789\(93\)90091-E](http://dx.doi.org/10.1016/0167-2789(93)90091-E).
- [9] Y. Tan, J. Yang, D.E. Pelinovsky, Semi-stability of embedded solitons in the general fifth-order KdV equation, *Wave Motion* 36 (3) (2002) 241–255, [http://dx.doi.org/10.1016/S0165-2125\(02\)00016-1](http://dx.doi.org/10.1016/S0165-2125(02)00016-1).
- [10] J. Garnier, J.C. Muñoz Grajales, A. Nachbin, Effective behavior of solitary waves over random topography, *Multiscale Model. Simul.* 6 (3) (2007) 995–1025, <http://dx.doi.org/10.1137/060676064>.
- [11] A. Nachbin, G.C. Papanicolaou, Water waves in shallow channels of rapidly varying depth, *J. Fluid Mech.* 241 (1992) 311–332, <http://dx.doi.org/10.1017/S0022112092002052>.
- [12] Y. Liu, D.K.P. Yue, On generalized Bragg scattering of surface waves by bottom ripples, *J. Fluid Mech.* 356 (1998) 297–326, <http://dx.doi.org/10.1017/S0022112097007969>.
- [13] O. Nakoulima, N. Zahibo, E. Pelinovsky, T. Talipova, A. Kurkin, Solitary wave dynamics in shallow water over periodic topography, *Chaos* 15 (3) (2005) 037107, <http://dx.doi.org/10.1063/1.1984492>, 8.
- [14] J. Gaison, S. Moskow, J.D. Wright, Q. Zhang, Approximation of polyatomic FPU lattices by KdV equations, *Multiscale Model. Simul.* 12 (3) (2014) 953–995, <http://dx.doi.org/10.1137/130941638>.
- [15] G. Schneider, C.E. Wayne, The rigorous approximation of long-wavelength capillary-gravity waves, *Arch. Ration. Mech. Anal.* 162 (3) (2002) 247–285, <http://dx.doi.org/10.1007/s002050200190>.
- [16] F. Hadadifard, J.D. Wright, Mass-in-mass lattices with small internal resonators, *Stud. Appl. Math.* 146 (1) (2021) 81–98, <http://dx.doi.org/10.1111/sapm.12340>.
- [17] M. Chirilus-Bruckner, C. Chong, O. Prill, G. Schneider, Rigorous description of macroscopic wave packets in infinite periodic chains of coupled oscillators by modulation equations, *Discrete Contin. Dyn. Syst. Ser. S* 5 (5) (2012) 879–901, <http://dx.doi.org/10.3934/dcdss.2012.5.879>.
- [18] D.E. Pelinovsky, G. Schneider, The monoatomic FPU system as a limit of a diatomic FPU system, *Appl. Math. Lett.* 107 (2020) 106387, <http://dx.doi.org/10.1016/j.aml.2020.106387>, 8.
- [19] J.T. Beale, Exact solitary water waves with capillary ripples at infinity, *Comm. Pure Appl. Math.* 44 (2) (1991) 211–257, <http://dx.doi.org/10.1002/cpa.3160440204>.
- [20] T.R. Akylas, T.-S. Yang, On short-scale oscillatory tails of long-wave disturbances, *Stud. Appl. Math.* 94 (1) (1995) 1–20, <http://dx.doi.org/10.1002/sapm19959411>.
- [21] S.M. Sun, Existence of a generalized solitary wave solution for water with positive Bond number less than 1/3, *J. Math. Anal. Appl.* 156 (2) (1991) 471–504, [http://dx.doi.org/10.1016/0022-247X\(91\)90410-2](http://dx.doi.org/10.1016/0022-247X(91)90410-2).
- [22] T.E. Faver, J.D. Wright, Exact diatomic Fermi-Pasta-Ulam-Tsingou solitary waves with optical band ripples at infinity, *SIAM J. Math. Anal.* 50 (1) (2018) 182–250, <http://dx.doi.org/10.1137/15M1046836>.
- [23] T.E. Faver, Nanopteron-stegoton traveling waves in spring dimer Fermi-Pasta-Ulam-Tsingou lattices, *Quart. Appl. Math.* 78 (3) (2020) 363–429, <http://dx.doi.org/10.1090/qam/1548>.
- [24] A. Hoffman, J.D. Wright, Nanopteron solutions of diatomic Fermi-Pasta-Ulam-Tsingou lattices with small mass-ratio, *Physica D* 358 (2017) 33–59, <http://dx.doi.org/10.1016/j.physd.2017.07.004>.
- [25] T.E. Faver, Small mass nanopteron traveling waves in mass-in-mass lattices with cubic FPUT potential, *J. Dynam. Differential Equations* 33 (4) (2021) 1711–1752, <http://dx.doi.org/10.1007/s10884-020-09865-0>.
- [26] E. Lombardi, Oscillatory Integrals and Phenomena beyond All Algebraic Orders, in: *Lecture Notes in Mathematics*, vol. 1741, Springer-Verlag, Berlin, 2000, p. xvi+413, With applications to homoclinic orbits in reversible systems, <http://dx.doi.org/10.1007/BFb0104102>.
- [27] M.A. Johnson, J.D. Wright, Generalized solitary waves in the gravity-capillary Whitham equation, *Stud. Appl. Math.* 144 (1) (2020) 102–130, <http://dx.doi.org/10.1111/sapm.12288>.
- [28] N. Joshi, C.J. Lustri, Generalized solitary waves in a finite-difference Korteweg-de Vries equation, *Stud. Appl. Math.* 142 (3) (2019) 359–384, <http://dx.doi.org/10.1111/sapm.12252>.
- [29] C.J. Lustri, M.A. Porter, Nanoptera in a period-2 Toda chain, *SIAM J. Appl. Dyn. Syst.* 17 (2) (2018) 1182–1212, <http://dx.doi.org/10.1137/16M108639X>.

- [30] M. Schechter, Principles of Functional Analysis, second ed., in: Graduate Studies in Mathematics, vol. 36, American Mathematical Society, Providence, RI, 2002, p. xxii+425, <http://dx.doi.org/10.1090/gsm/036>.
- [31] H. Cartan, Differential Calculus, Hermann, Paris; Houghton Mifflin Co., Boston, Mass., 1971, p. 160, Exercises by C. Buttin, F. Rideau and J. L. Verley, Translated from the French.
- [32] R. Bellman, K.L. Cooke, Differential-Difference Equations, Academic Press, New York-London, 1963, p. xvi+462.
- [33] E. Kreyszig, Advanced Engineering Mathematics, seventh ed., Wiley, New York, NY, 1992.
- [34] T.W. Körner, Fourier Analysis, Cambridge University Press, Cambridge, 1988, p. xii+591, <http://dx.doi.org/10.1017/CBO9781107049949>.
- [35] R.L. Pego, M.I. Weinstein, Asymptotic stability of solitary waves, *Comm. Math. Phys.* 164 (2) (1994) 305–349.
- [36] G. Friesecke, R.L. Pego, Solitary waves on Fermi-Pasta-Ulam lattices. IV. Proof of stability at low energy, *Nonlinearity* 17 (1) (2004) 229–251, <http://dx.doi.org/10.1088/0951-7715/17/1/014>.
- [37] J. Holmer, Dynamics of KdV solitons in the presence of a slowly varying potential, *Int. Math. Res. Not. IMRN* (23) (2011) 5367–5397, <http://dx.doi.org/10.1093/imrn/rnq284>.