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Well-Posedness and Asymptotics of a Coordinate-Free Model of Flame Fronts*

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Abstract. We investigate a coordinate-free model of flame fronts introduced by Frankel and Sivashinsky; this model has a parameter α which relates to how unstable the front might be. We first prove short-time well-posedness of the coordinate-free model for any value of $\alpha > 0$. We then argue that near the threshold $\alpha \approx 1$, the solution stays arbitrarily close to the solution of the weakly nonlinear Kuramoto–Sivashinsky equation, as long as the initial values are close.

Key words. Kuramoto–Sivashinsky, coordinate-free model of flame front, well-posedness, asymptotic behavior

AMS subject classifications. 35B40, 35B65

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1. Introduction.

The Kuramoto–Sivashinsky equation,

$$f_t + \frac{1}{2}f_x^2 + (\alpha - 1)f_{xx} + 4f_{xxxx} = 0, \tag{1.1}$$

is a weakly nonlinear model for flame fronts [23], [32]. Frankel and Sivashinsky have shown that it can be formally derived from coordinate-free models [15] of flame propagation. In such a coordinate-free model, the normal velocity of the front is specified in terms of intrinsic geometric information such as curvature and arclength. One such model put forward by Frankel and Sivashinsky is

$$V_n = 1 + (\alpha - 1)\kappa + \left(1 + \frac{1}{2}\alpha^2\right)\kappa^2 + \left(2\alpha + 5\alpha^2 - \frac{1}{3}\alpha^3\right)\kappa^3 + \alpha^2(\alpha + 3)\kappa_{ss}, \tag{1.2}$$

where V_n is the normal velocity of the front, κ is the curvature of the front, s is arclength, and α is a parameter measuring instability of the interface. Note that V_n is the normal velocity of a curve in the plane and therefore is (related to) the time derivative of the position of the curve. To make the relationship precise, a parameterization must be chosen. Setting this parameterization is equivalent to specifying the tangential velocity of the front. In the next section we specify a parameterization (choosing a graph parameterization), and we thus arrive at a more traditional evolution equation for the flame front. Frankel and Sivashinsky perform asymptotic analysis of (1.2) in the case $\alpha \approx 1$, finding the simplified coordinate-free model

$$V_n = 1 + (\alpha - 1)\kappa + 4\kappa_{ss}. \tag{1.3}$$

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As discussed by Brauner et al. [8], there are two primary destabilization mechanisms for premixed gas combustion: hydrodynamic instability (stemming from thermal expansion of the gas) and thermal-diffusive instability. The derivation of the models (1.2) and (1.3) in [15] starts from a constant density flame model, neglecting thermal expansion of the gas. Thus these are models exploring thermal-diffusive instability. This instability generates cellular structures which may be modeled with free interface problems [11], [12], and models such as (1.2) and (1.3) give the velocity of this interface. In addition to [15], coordinate-free models for flame front propagation have been developed in [14] and [16]. Some analytical studies have been made of these models, such as studying a quasi-steady problem [9], [10].

The Kuramoto–Sivashinsky equation as given in (1.1) is a form of the more general Kuramoto–Sivashinsky equation

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 = -c_1^2\Delta^2 u - c_2^2\Delta u, \tag{1.4}$$

in the case of one spatial dimension. The two linear terms on the right-hand side play different roles, as the fourth-order term is stabilizing and makes the problem well-posed, while the second-order term is destabilizing and can lead to growth of solutions. The interaction of the nonlinear term on the left-hand side with the linear terms leads to rich and highly nontrivial dynamics, especially given the lack of a maximum principle for the equation owing to its fourth-order nature. (We mention that there are versions of the coordinate-free models such as (1.2) available in higher dimension as well [16].)

The Kuramoto–Sivashinsky equation has been widely studied over the years, with global existence of solutions and stability of the zero solution both established in one spatial dimension [19], [29], [33]. Detailed estimates have been developed in one spatial dimension for the dependence of the solutions on the size of the periodic domain [17], [18]. Many results for the Kuramoto–Sivashinsky equation in one spatial dimension rely on structure not present in higher-dimensional problems, especially that an estimate for the L^2 norm of the first spatial derivative of the unknown is available. In higher dimensions this estimate is not available, and there are fewer results. If the right-hand side of (1.4) is modified to instead be $c_1^2\Delta u + c_2^2 u$, then a maximum principle is available and this structure may be used to find some global existence results [19], [27]; the equation is then known instead as the Burgers–Sivashinsky equation. Larios and Yamazaki have also leveraged this structure for a system which blends features of the Kuramoto–Sivashinsky and Burgers–Sivashinsky models [25]. For the full Kuramoto–Sivashinsky equation in two spatial dimensions, Sell and Taboada have proven global existence of solutions in thin domains [31], and Ambrose and Mazzucato have shown global existence in the absence of linearly growing modes (which happens when the domain is a sufficiently small torus) [3] and in the case of a single linearly growing mode in each spatial dimension [4]. Additional results for the Kuramoto–Sivashinsky equation on thin domains may be found in [6], [22], [28].

The distinction between known behavior in one spatial dimension and in two spatial dimensions indicates that the structures present in (1.1) used to demonstrate, for example, global existence of solutions are perhaps a bit delicate and may not be present in closely related systems. Indeed, while Frankel and Sivashinsky have formally derived (1.1) from the coordinate-free models (1.2) and (1.3), the authors are unaware of any analytical theory for these relationships. While the question of global existence of solutions for the coordinate-free models remains open, we demonstrate short-time well-posedness here, focusing on (1.3) for simplicity, and show rigorously the connection between solutions of (1.3) and (1.1).

There is a long history of demonstrating that weakly nonlinear models serve as valid approximations for more fully nonlinear models; a key example of such work is the proof that the Korteweg–de Vries equation is a good approximation of the irrotational Euler equations with a free surface [7], [30], [34]. For more such works in the theory of water waves, the interested reader might consult the book of Lannes and the references therein [24]. While the Kuramoto–Sivashinsky equation is a widely studied weakly nonlinear model for the propagation of flame fronts, the authors are unaware of any prior proofs of its validity in approximating more highly nonlinear models. The result in the literature most similar to the present work appears to be the main result of [8], in which solutions of the Kuramoto–Sivashinsky equation are shown to remain close to solutions of another weakly nonlinear model; this weakly nonlinear model is derived from coordinate-free models similar to (1.2), but also incorporating temperature effects.

As we will first prove well-posedness of the initial value problem for the coordinate-free model given by (1.3), we first convert it into an evolutionary problem, which requires setting coordinates. We do so with an eye toward our approximation theorem, and so not making the most general possible choice. As the approximation theorem we prove is for the Kuramoto–Sivashinsky equation, and the flame front in the Kuramoto–Sivashinsky equation is parameterized as a graph over the horizontal coordinate, x, we thus make this choice of frame for the coordinate-free model. We make the relevant calculations in section 1.1. We prove well-posedness of the initial value problem when the initial data is relatively smooth, namely we take the data in the Sobolev space H^5. We do this so as to deal only with classical solutions, and since regularity theory is not the focus of the present work. If we were to take a mild solutions viewpoint instead, then the parabolic nature of the evolution would certainly allow for rough data. In [3], Ambrose and Mazzucato constructed mild solutions of the two-dimensional Kuramoto–Sivashinsky equation with initial data in L^2. We expect the same would be possible for the coordinate-free model studied here.

This choice of restricting (1.3) to the case of a graph over the horizontal coordinate is not a limitation on our well-posedness theory; indeed it would be no more difficult to treat (1.3) for flame fronts which could have multivalued height or which might be closed curves. To treat such scenarios, the parameterization of the curve could be set using tangent angle and arclength, as was done for interfaces between fluids in the numerical work of Hou, Lowengrub, and Shelley [20], [21]. The formulation of Hou, Lowengrub, and Shelley was subsequently used by Ambrose and collaborators a number of times to prove well-posedness of initial value problems in interfacial fluid mechanics, for example, in the works [2], [5], [26]. The advantage of the tangent angle and arclength formulation is that these are naturally related to the curvature, and the curvature of the front is what appears on the right-hand sides of (1.2) and (1.3). Ambrose and Akers have implemented numerical methods to compute the propagation of fronts using the angle-arclength formulation for the models (1.2) and (1.3) using further ideas from [20] in [1].

1.1. Reformulation: Setting coordinates. In order to compare solutions of (1.1) with those of (1.2), we need to have a more convenient form of (1.2). This convenient form of (1.2) is found by specifying a parameterization of the curve which evolves according to (1.2); since the Kuramoto–Sivashinsky equation (1.1) assumes the front is a graph, we choose a graph parameterization for (1.2) as well. We take the horizontal spatial variable x to be in $\mathbb{T}^1$, and we then take the front height to be a function of x. Clearly we need to rewrite V_n and κ_{ss} in the new variables.

Function V_n. Consider artificial parameters (β, τ); for any curve $(x(\beta,\tau), y(\beta,\tau))$ we can write the motion as a combination of the normal vector $n = \frac{(y_\beta, -x_\beta)}{|(y_\beta,-x_\beta)|}$ and the tangent vector $T = \frac{(x_\beta, y_\beta)}{|(x_\beta,y_\beta)|}$. Furthermore, we have the following decomposition of the time derivative of the curve $(x,y)_t$:

$$(x,y)_t = V_n \cdot n + V_\tau \cdot T, \tag{1.5}$$

where V_n is the normal velocity of the interface, and the tangential velocity V_τ is related to the choice of the parameters. As mentioned above, our model covers the case of $(x,y) = (x, f(x))$ and $x_t = 0$ (i.e., $x = \beta$), therefore

$$x_t = \frac{y_x V_n}{\sqrt{1+y_x^2}} + \frac{V_\tau}{\sqrt{1+y_x^2}} = 0 \Rightarrow V_\tau = -y_x V_n.$$

We can use the above to find y_t. Indeed,

$$y_t = \frac{-V_n}{\sqrt{1+y_x^2}} + \frac{y_x \cdot V_\tau}{\sqrt{1+y_x^2}} = \frac{-(1+y_x^2)\cdot V_n}{\sqrt{1+y_x^2}} = -\sqrt{1+y_x^2}\cdot V_n.$$

This clearly yields

$$V_n = \frac{-y_t}{\sqrt{1+y_x^2}}. \tag{1.6}$$

Function κ_{ss}. Note that $\frac{ds}{dx} = \sqrt{1+y_x^2}$, therefore

$$\frac{d\kappa}{dx} = \frac{d\kappa}{ds}\cdot\frac{ds}{dx} = \frac{d\kappa}{ds}\cdot\sqrt{1+y_x^2},$$

and consequently,

$$\begin{aligned}\frac{d^2\kappa}{dx^2} &= \frac{d}{dx}\left(\frac{d\kappa}{ds}\cdot\sqrt{1+y_x^2}\right) = \frac{d^2\kappa}{ds^2}\cdot\frac{ds}{dx}\cdot\sqrt{1+y_x^2} + \frac{d\kappa}{ds}\cdot\frac{y_x y_{xx}}{\sqrt{1+y_x^2}}\\ &= \frac{d^2\kappa}{ds^2}\cdot(1+y_x^2) + \frac{d\kappa}{dx}\cdot\frac{y_x y_{xx}}{1+y_x^2}.\end{aligned}$$

In other words,

$$\frac{d^2\kappa}{ds^2} = \frac{1}{1+y_x^2}\cdot\frac{d^2\kappa}{dx^2} - \frac{y_x y_{xx}}{(1+y_x^2)^2}\cdot\frac{d\kappa}{dx}. \tag{1.7}$$

Now we insert (1.6) and (1.7) into (1.2) and get the following equation:

$$(1.8)\quad \begin{cases} y_t + \frac{(\alpha-1)\cdot y_{xx}}{1+y_x^2} + \left(1+\frac{1}{2}\alpha^2\right)\frac{y_{xx}^2}{(1+y_x^2)^{\frac{5}{2}}} + \left(2\alpha+5\alpha^2-\frac{1}{3}\alpha^3\right)\frac{y_{xx}^3}{(1+y_x^2)^4} + \frac{\alpha^2(\alpha+3)}{\sqrt{1+y_x^2}}\cdot\frac{d^2\kappa}{dx^2} + \\ \quad +\sqrt{1+y_x^2} = \alpha^2(\alpha+3)y_x\cdot\kappa\cdot\frac{d\kappa}{dx}, \\ y(x,0) = y_0(x). \end{cases}$$

where,

$$\frac{d\kappa}{dx} = \frac{y_{xxx}}{(1+y_x^2)^{\frac{3}{2}}} - \frac{3y_x\cdot(y_{xx})^2}{(1+y_x^2)^{\frac{5}{2}}},$$

$$\frac{d^2\kappa}{dx^2} = \frac{y_{xxxx}}{(1+y_x^2)^{\frac{3}{2}}} - \frac{3(y_{xx})^3+9y_xy_{xx}y_{xxx}}{(1+y_x^2)^{\frac{5}{2}}} + \frac{15(y_x)^2(y_{xx})^3}{(1+y_x^2)^{\frac{7}{2}}}.$$

In section 2.1 we recall some definitions, standard estimates from Harmonic analysis, as well as a form of Grönwall's inequality which fits our Grönwall's type inequalities. In section 3 we present the existence of the solution of (1.8) in H^4. In other words, section 3 covers the proof of Theorem 2.3. This is done via an approximate equation. Finally, in section 4 we present a proof of Theorem 2.4. This is done via a coordinate scaling, where the scaling has been chosen carefully.

2. Preliminaries.

2.1. Fourier series, function spaces, and mulitpliers.

We will consider periodic function spaces, although this is not essential. A sufficiently regular function f on a periodic interval may be written with its Fourier series,

$$f(x) = \sum_{p\in\mathbb{Z}} \hat{f}(p)e^{ipx}.$$

Consequently, since $\widehat{-\Delta f}(p) = |p|^2\hat{f}(p)$, we define the operators $|\nabla|^a := (-\Delta)^{a/2}, a > 0$, via its action on the Fourier side $\widehat{|\nabla|^a f}(p) = |p|^a\hat{f}(p)$.

The $L^p(\mathbb{R}^n)$, $n \geq 1$, spaces are defined by the norm $\|f\|_{L^p(\mathbb{R}^n)} = (\int_{\mathbb{R}^n} |f(x)|^p\, dx)^{\frac{1}{p}}$. For $p \in (1,\infty)$, the Sobolev spaces are the closure of the Schwartz functions in the norm

$$\|f\|_{W^{k,p}(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)} + \sum_{|\alpha|\leq k} \|\partial^\alpha f\|_{L^p(\mathbb{R}^n)},$$

while for a noninteger s one takes

$$\|f\|_{W^{s,p}(\mathbb{R}^n)} = \|(1-\Delta)^{s/2}f\|_{L^p(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n)} + \||\nabla|^s f\|_{L^p(\mathbb{R}^n)}.$$

The Sobolev embedding theorem states $\|f\|_{L^p(\mathbb{T}^1)} \leq C\||\nabla|^s f\|_{L^q(\mathbb{T}^1)}$, where $1 < p < q < \infty$ and $\frac{1}{p} - \frac{1}{q} = s$, with the usual modification for $p = \infty$, namely $\|f\|_{L^\infty(\mathbb{T}^1)} \leq C_s\|f\|_{W^{s,q}(\mathbb{T}^1)}$, $s > \frac{1}{p}$. Another useful ingredient will be the Gagliardo–Nirenberg interpolation inequality,

$$(2.1)\quad \||\nabla|^s f\|_{L^p(\mathbb{R}^n)} \leq \||\nabla|^{s_1} f\|_{L^q(\mathbb{R}^n)}^{\theta}\||\nabla|^{s_2} f\|_{L^r(\mathbb{R}^n)}^{1-\theta},$$

where $s = \theta s_1 + (1-\theta)s_2$ and $\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{r}$.

Throughout this work we make use of a particular version of mollifier operators $\mathcal{J}^\delta$, $0 < \delta << 1$, which represent the truncation of the Fourier series, zeroing out modes with wave number larger than $\frac{1}{\delta}$. We frequently use the following two essential properties of the mollifiers, which can be easily proved in a straightforward way using the Hausdorff–Young inequality, or alternatively the Plancherel theorem:

$$\|\mathcal{J}^\delta f\|_{H^s(\mathbb{R}^n)} \leq \|f\|_{H^s}, \tag{2.2}$$

$$\|\mathcal{J}^\delta \partial^s f\|_{L^2(\mathbb{R}^n)} \leq \frac{C}{\delta^s}\|f\|_{L^2(\mathbb{R}^n)}. \tag{2.3}$$

Note that the operator $\mathcal{J}^\delta$ is both a self-adjoint operator and a projection, i.e., $\mathcal{J}^\delta(\mathcal{J}^\delta f) = \mathcal{J}^\delta f$. Moreover, it commutes with the derivative operator, $\mathcal{J}^\delta \partial f = \partial \mathcal{J}^\delta f$.

2.2. Grönwall's inequality. We need the following two versions of the Grönwall's inequality.

Lemma 2.1. *Let the functions x, a, b, and k be continuous and nonnegative on the interval $J = [\alpha, \beta]$, and let n be a positive integer ($n \geq 2$). Assume $\frac{a}{b}$ is a nondecreasing function. If*

$$x(t) \leq a(t) + b(t)\int_\alpha^t k(s)(x(s))^n ds, \quad t \in J, \tag{2.4}$$

then

$$x(t) \leq a(t)\left\{1 - (n-1)\int_\alpha^t k(s)b(s)a^{n-1}(s)ds\right\}^{\frac{1}{n-1}}, \quad \alpha \leq t \leq \beta_n, \tag{2.5}$$

where β_n is given by

$$\beta_n = \sup\left\{t \in J : (n-1)\int_\alpha^t k(s)b(s)a^{n-1}(s)ds < 1\right\}. \tag{2.6}$$

Lemma 2.2. *Fix τ_* and $\Gamma_* > 0$. Assume the function $E(t) > 0$ satisfies the relation*

$$\frac{d}{dt}E(t) \leq \alpha E(t) + \beta E^2(t) + \epsilon^n (E(t))^m, \tag{2.7}$$

where $0 < \epsilon << 1$, $n \geq 0$, and $m \geq 1$. Then there exists E_* and ϵ_* so that for any $E(0) = E_0 \leq E_*$ and $0 < \epsilon \leq \epsilon_*$

$$\sup_{0<\tau<\tau_*} E(t) \leq \Gamma_*. \tag{2.8}$$

We provide the proof of Lemma 2.2. One can find the proof of Lemma 2.1 in [13, Theorem 25].

Proof. In order to prove Lemma 2.2 fix Γ_*, and let $E(t_0)$ be the first time at which $E(t_0) = \Gamma_*$ (if for all $t > 0$, $E(t_0) < \Gamma_*$ then let $t_0 = \infty$, in which case the proof is completed). Hence, for any $t \in [0, t_0]$ we have $E^m \leq \Gamma_*^{m-1}E$. Therefore,

$$\frac{d}{dt}E(t) \leq \left(\alpha + \beta\Gamma_* + \epsilon^n\Gamma_*^{m-1}\right)E(t). \tag{2.9}$$

Now we apply the routine Grönwall's inequality to this relation, and we get, for any $t \in [0, t_0]$,

$$E(t) \leq \exp\left((\alpha + \beta\Gamma_* + \epsilon^n \Gamma_*^{m-1})t\right) E_0. \tag{2.10}$$

At $t = t_0$, we have $E(t_0) = \Gamma_*$, hence

$$\Gamma_* \leq \exp\left((\alpha + \beta\Gamma_* + \epsilon^n \Gamma_*^{m-1})t_0\right) E_0,$$

which implies

$$t_0 \geq \frac{\ln\left(\frac{\Gamma_*}{E_0}\right)}{\alpha + \beta\Gamma_* + \epsilon^n \Gamma_*^{m-1}} =: \tau_0(\Gamma_*, E_0, \epsilon).$$

Note that $\tau_0(\Gamma_*, E_0, \epsilon)$ is decreasing with ϵ and with E_0. What we have shown so far asserts that if $0 \leq t \leq \tau_0(\Gamma_*, E_0, \epsilon)$, then

$$E(t) \leq \Gamma_*. \tag{2.11}$$

Now fix a time t_*, and Γ_* as well as $\epsilon \leq 1 := \epsilon_*$, and solve $\tau_0(\Gamma_*, E_0, \epsilon) = t_*$ for E_*, namely

$$E_* = \Gamma_* \exp\left((\alpha + \beta\Gamma_* + \epsilon^n \Gamma_*^{m-1})t_*\right). \tag{2.12}$$

Now we claim that with t_*, Γ_*, and E_* as above, then if $E_0 \leq E_*$ and $\epsilon < 1$ we have

$$\sup_{0<\tau<\tau_*} E(t) \leq \Gamma_*. \tag{2.13}$$

Indeed, by (2.11) we have $E(t) \leq \Gamma_*$ for $0 \leq t \leq \tau_0(\Gamma_*, E_0, \epsilon)$. Since $\tau_0(\Gamma_*, E_0, \epsilon)$ is decreasing with respect to E_0 and ϵ, we know

$$t_* = \tau_0(\Gamma_*, E_*, 1) \leq \tau_0(\Gamma_*, E_0, \epsilon).$$

Thus

$$\{t : 0 \leq t \leq t_*\} \subset \{t : 0 \leq t \leq \tau_0(\Gamma_*, E_0, \epsilon)\},$$

and we get

$$\sup_{0<\tau<\tau_*} |E(t)| \leq \Gamma_*. \tag{2.14}$$ ■

2.3. Main result. As mentioned, we pursue two main goals in this article. First we aim to prove the well-posedness of the initial value problem associated to (1.8). This is the content of Theorem 2.3. Our second goal is to show that the solution to (1.8) stays close enough to the solution of (1.1), in a sense to be made precise. In Theorem 2.4 we present the related result.

Theorem 2.3. *Let $y_0 \in H^5$ be given. Then there exists a time $T = T(\|y_0\|_{H^5})$ and a function $y \in C([0, T], H^5)$ which satisfies (1.8), and the initial condition $y(\cdot, 0) = y_0$.*

Theorem 2.4. *Fix $\tau_* > 0$ and $\Gamma_* > 0$. Then there exists ϵ_* and E_* so that whenever $0 < \epsilon < \epsilon_*$ and $\|U_0(\cdot)\|_{H^5} \le E_*$, the following hold.*

Let $y(x,t)$ be the solution of (1.8) with $\alpha - 1 = \epsilon$, and

$$y(x,0) = \epsilon U_0(\sqrt{\epsilon}x) + y_{\epsilon,0}(x) \tag{2.15}$$

with $\|y_{\epsilon,0}\|_{H^5} \le \epsilon^{7/4}$. Let $U(\xi,\tau)$ be the solution of the Kuramoto–Sivashinsky equation

$$\partial_\tau U + \frac{1}{2}(\partial_\xi U)^2 + \partial_\xi^2 U + 4\partial_\xi^4 U = 0 \tag{2.16}$$

with $U(\xi,0) = U_0(\xi)$. Then

$$\sup_{0<t<\frac{\tau_*}{\epsilon^2}} \|y(\cdot,t) + t - \epsilon U(\sqrt{\epsilon}\ \cdot, \epsilon^2 t)\|_{L^2} \le \Gamma_* \epsilon^{\frac{7}{4}}. \tag{2.17}$$

The proofs of Theorems 2.3 and 2.4 are presented in Lemma 3.4 and Remark 4.4, respectively.

Remark 2.5. The time interval presented in Theorem 2.3 increases for a smaller $\|y_0\|_{H^5}$. In fact the time interval $[0,T]$ increases as the upper bound of T, i.e., $C\ln(1 + \frac{C}{\|y_0\|_{H^5}^{m-2}})$, increases with smaller $\|y_0\|_{H^5}$.

3. Existence of the solution.

The first step toward the completion of the argument is to show that (1.8) has a unique solution in some Sobolev spaces, over a time interval $[0,T]$, with T to be determined. The proof follows the energy method. To that end, we first introduce approximate equations, where the approximation is introduced via a multiplier operator (mollifier) $\mathcal{J}^\delta$. We next use the Picard theorem to find that the approximate equations admit unique solutions in some Sobolev spaces over a time interval $[0,T_\delta]$. This T_δ might be small (i.e., this time depends badly on the approximation parameter δ). Therefore, in an attempt to increase T_δ, we prove bounds on the solution which are uniform with respect to δ. Once the uniform bounds are in hand, since norms of the solutions of the approximate equations are not increasing fast, the solutions may be continued to a time interval $[0,T]$, where T can be taken to be independent of δ. Finally, with solutions existing on a uniform time interval, the limit may be taken as δ vanishes, and this limit can be seen to satisfy the correct initial value problem.

We define y^δ to be the solution of the following initial value problem:

$$\begin{cases} y_t^\delta + (\alpha-1)\mathcal{J}^\delta\left[\frac{\mathcal{J}^\delta y_{xx}^\delta}{1+(\mathcal{J}^\delta y_x^\delta)^2}\right] + \left(1+\frac{1}{2}\alpha^2\right)\mathcal{J}^\delta\left[\frac{(\mathcal{J}^\delta y_{xx}^\delta)^2}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^{\frac{5}{2}}}\right] + \left(2\alpha+5\alpha^2-\frac{1}{3}\alpha^3\right)\mathcal{J}^\delta\left[\frac{(\mathcal{J}^\delta y_{xx}^\delta)^3}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^4}\right] \\ \qquad + \alpha^2(\alpha+3)\mathcal{J}^\delta\left[\frac{1}{\sqrt{1+(\mathcal{J}^\delta y_x^\delta)^2}}\cdot\frac{d^2\kappa^\delta}{dx^2}\right] + \mathcal{J}^\delta\left[\sqrt{1+(\mathcal{J}^\delta y_x^\delta)^2}\right] = \alpha^2(\alpha+3)\mathcal{J}^\delta\left[(\mathcal{J}^\delta y_x^\delta)\cdot\kappa^\delta\cdot\frac{d\kappa^\delta}{dx}\right], \\ y^\delta(x,0) = \mathcal{J}^\delta y_0(x), \end{cases} \tag{3.1}$$

where

$$\kappa^\delta = \frac{\mathcal{J}^\delta y^\delta_{xx}}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^{\frac{3}{2}}}, \tag{3.2}$$

$$\frac{d\kappa^\delta}{dx} = \frac{\mathcal{J}^\delta y^\delta_{xxx}}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^{\frac{3}{2}}} - 3\frac{(\mathcal{J}^\delta y^\delta_x)\cdot(\mathcal{J}^\delta y^\delta_{xx})^2}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^{\frac{5}{2}}}, \tag{3.3}$$

$$\frac{d^2\kappa^\delta}{dx^2} = \frac{\mathcal{J}^\delta y^\delta_{xxxx}}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^{\frac{3}{2}}} - \frac{3(\mathcal{J}^\delta y^\delta_{xx})^3+9(\mathcal{J}^\delta y^\delta_x)(\mathcal{J}^\delta y^\delta_{xx})(\mathcal{J}^\delta y^\delta_{xxx})}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^{\frac{5}{2}}} + \frac{15(\mathcal{J}^\delta y^\delta_x)^2(\mathcal{J}^\delta y^\delta_{xx})^3}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^{\frac{7}{2}}}. \tag{3.4}$$

We now present the first step toward the existence argument. We show that (3.1) admits a solution up to a small time T_δ.

Lemma 3.1. *Let $y(0)\in H^5$ be given. For any $\delta>0$, for any $s\ge 0$, there is a time T_δ and a function $y^\delta\in C^1([0,T_\delta],H^s)$ that satisfies (3.1), as well as $y^\delta(\cdot,0)=\mathcal{J}^\delta y(0)$.*

Proof. Since the initial data is mollified, it is in any Sobolev space. With the abundance of mollifiers present on the right-hand side of the evolution equation, it is not difficult to demonstrate that the relevant operator is a Lipschitz map. The Picard theorem applies, leading to the conclusion of the theorem. We omit further details. ∎

The next two lemmas concern some uniform bounds on the solution of (3.1). In the first lemma we prove an H^4 bound, and we then use it in the subsequent lemma for an H^5 bound.

Lemma 3.2. *Assume y^δ is the solution of (1.8). Then there exists $T=T(\alpha)$ and $C=C(y_0,\alpha)$, independent of δ, so that for any $0<t<\frac{\ln(1+\frac{\gamma}{\|y_0\|^{m-2}_{H^4}})}{\gamma}$ (m and γ to be defined later),*

$$\sup_{0<t<T}\|y^\delta\|^2_{H^4} + \int\frac{(\mathcal{J}^\delta y^\delta_{xx})^2+(\partial_x^6\mathcal{J}^\delta y^\delta)^2}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^2}dx \le C. \tag{3.5}$$

Proof. During the proof, we assume that $\|y^\delta\|^2_{L^2}+\|\partial_x^4 y^\delta\|^2_{L^2}>1$; otherwise there is nothing to prove.

In order to prove this lemma, we combine two energy estimates, one on $\|y^\delta\|_{L^2}$, and the other one on $\|\partial_x^4 y^\delta\|_{L^2}$. Indeed,

(3.6)

$$\begin{aligned}
&\frac{1}{2}\partial_t\|y^\delta\|^2_{L^2} + (\alpha-1)\int(\mathcal{J}^\delta y^\delta)\cdot\left[\frac{(\mathcal{J}^\delta y^\delta_{xx})}{1+(\mathcal{J}^\delta y^\delta_x)^2}\right]dx\\
&\quad+\left(1+\frac{1}{2}\alpha^2\right)\int(\mathcal{J}^\delta y^\delta)\cdot\mathcal{J}^\delta\left[\frac{(\mathcal{J}^\delta y^\delta_{xx})^2}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^{\frac{5}{2}}}\right]dx+\alpha^2(\alpha+3)\int(\mathcal{J}^\delta y^\delta)\\
&\quad\cdot\left[\frac{1}{\sqrt{1+(\mathcal{J}^\delta y^\delta_x)^2}}\cdot\frac{d^2\kappa^\delta}{dx^2}\right]dx\\
&\quad+\int(\mathcal{J}^\delta y^\delta)\cdot\left[\sqrt{1+(\mathcal{J}^\delta\mathcal{J}^\delta y^\delta_x)^2}\right]dx+\left(2\alpha+5\alpha^2-\frac{1}{3}\alpha^3\right)\int(\mathcal{J}^\delta y^\delta)\cdot\left[\frac{(\mathcal{J}^\delta y^\delta_{xx})^3}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^4}\right]dx\\
&=\alpha^2(\alpha+3)\int(\mathcal{J}^\delta y^\delta)\cdot\left[(\mathcal{J}^\delta y^\delta_x)\cdot\kappa^\delta\cdot\frac{d\kappa^\delta}{dx}\right]dx.
\end{aligned}$$

We use integration by parts to arrive at a more convenient form for this expression.

The first term we simplify produces a useful term in the left-hand side of (3.6), namely $\int \frac{(\mathcal{J}^\delta y^\delta_{xx})^2}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^2}dx$. Indeed, when we substitute from (3.4) into the fourth term on the left-hand side of (3.6), we find

$$\begin{aligned}
&\alpha^2(\alpha+3)\int(\mathcal{J}^\delta y^\delta)\cdot\left[\frac{1}{\sqrt{1+(\mathcal{J}^\delta y^\delta_x)^2}}\cdot\frac{d^2\kappa^\delta}{dx^2}\right]dx\\
&=\alpha^2(\alpha+3)\int\frac{(\mathcal{J}^\delta y^\delta)\cdot(\mathcal{J}^\delta y^\delta_{xxxx})}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^2}\,dx\\
&\quad-3\alpha^2(\alpha+3)\int\frac{(\mathcal{J}^\delta y^\delta)\cdot(\mathcal{J}^\delta y^\delta_{xx})^3}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^3}\,dx-9\alpha^2(\alpha+3)\int\frac{(\mathcal{J}^\delta y^\delta)(\mathcal{J}^\delta y^\delta_x)(\mathcal{J}^\delta y^\delta_{xx})(\mathcal{J}^\delta y^\delta_{xxx})}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^3}\,dx\\
&\quad+15\alpha^2(\alpha+3)\int\frac{(\mathcal{J}^\delta y^\delta)\cdot(\mathcal{J}^\delta y^\delta_x)^2(\mathcal{J}^\delta y^\delta_{xx})^3}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^4}\,dx.
\end{aligned}$$

The term we wish to draw out can now be found after integrating by parts twice:

$$\begin{aligned}
\int\frac{(\mathcal{J}^\delta y^\delta)\cdot(\mathcal{J}^\delta y^\delta_{xxxx})}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^2}&=-\int\frac{(\mathcal{J}^\delta y^\delta_{xxx})\cdot(\mathcal{J}^\delta y^\delta_x)}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^2}dx+4\int\frac{(\mathcal{J}^\delta y^\delta)\cdot(\mathcal{J}^\delta y^\delta_x)\cdot(\mathcal{J}^\delta y^\delta_{xx})(\mathcal{J}^\delta y^\delta_{xxx})}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^3}dx\\
&=\int\frac{(\mathcal{J}^\delta y^\delta_{xx})^2}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^2}dx-4\int\frac{(\mathcal{J}^\delta y^\delta_x)^2\cdot(\mathcal{J}^\delta y^\delta_{xx})^2}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^3}dx\\
&\quad+4\int\frac{(\mathcal{J}^\delta y^\delta)\cdot(\mathcal{J}^\delta y^\delta_x)\cdot(\mathcal{J}^\delta y^\delta_{xx})(\mathcal{J}^\delta y^\delta_{xxx})}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^3}dx.
\end{aligned}$$

Our conclusion is

$$\begin{aligned}
\alpha^2(\alpha+3)\int(\mathcal{J}^\delta y^\delta)\cdot\left[\frac{1}{\sqrt{1+(\mathcal{J}^\delta y^\delta_x)^2}}\cdot\frac{d^2\kappa^\delta}{dx^2}\right]dx&=\alpha^2(\alpha+3)\int\frac{(\mathcal{J}^\delta y^\delta_{xx})^2}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^2}dx\\
-4\alpha^2(\alpha+3)\int\frac{(\mathcal{J}^\delta y^\delta_x)^2\cdot(\mathcal{J}^\delta y^\delta_{xx})^2}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^3}dx&-3\alpha^2(\alpha+3)\int\frac{(\mathcal{J}^\delta y^\delta)\cdot(\mathcal{J}^\delta y^\delta_{xx})^3}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^3}dx\\
-5\alpha^2(\alpha+3)\int\frac{(\mathcal{J}^\delta y^\delta)\cdot(\mathcal{J}^\delta y^\delta_x)\cdot(\mathcal{J}^\delta y^\delta_{xx})(\mathcal{J}^\delta y^\delta_{xxx})}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^3}dx&\\
+15\alpha^2(\alpha+3)\int\frac{(\mathcal{J}^\delta y^\delta)\cdot(\mathcal{J}^\delta y^\delta_x)^2\cdot(\mathcal{J}^\delta y^\delta_{xx})^3}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^4}dx.&
\end{aligned}$$

For the right-hand side of (3.6), we substitute from (3.3), finding

$$\begin{aligned}
\alpha^2(\alpha+3)\int(\mathcal{J}^\delta y^\delta)\cdot\left[(\mathcal{J}^\delta y^\delta_x)\cdot\kappa^\delta\cdot\frac{d\kappa^\delta}{dx}\right]dx&=\alpha^2(\alpha+3)\int\frac{(\mathcal{J}^\delta y^\delta)\cdot(\mathcal{J}^\delta y^\delta_x)\cdot(\mathcal{J}^\delta y^\delta_{xx})(\mathcal{J}^\delta y^\delta_{xxx})}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^3}dx\\
&\quad-3\alpha^2(\alpha+3)\int\frac{(\mathcal{J}^\delta y^\delta)\cdot(\mathcal{J}^\delta y^\delta_x)^2\cdot(\mathcal{J}^\delta y^\delta_{xx})^3}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^4}dx.
\end{aligned}$$

We also rewrite the fifth term on the left-hand side of (3.6) as

$$\int(\mathcal{J}^\delta y^\delta)\sqrt{1+(\mathcal{J}^\delta y^\delta_x)^2}\,dx=\int\frac{(\mathcal{J}^\delta y^\delta)\Big(1+(\mathcal{J}^\delta y^\delta_x)^2\Big)}{\sqrt{1+(\mathcal{J}^\delta y^\delta_x)^2}}\,dx.$$

With all of these considerations, (3.6) now may be written as

(3.7)

$$\frac{1}{2}\partial_t\|y^\delta\|_{L^2}^2+\alpha^2(\alpha+3)\int\frac{(\mathcal{J}^\delta y^\delta_{xx})^2}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^2}dx=-(\alpha-1)\int\frac{(\mathcal{J}^\delta y^\delta)\cdot(\mathcal{J}^\delta y^\delta_{xx})}{(1+(\mathcal{J}^\delta y^\delta_x)^2)}dx+$$
$$-\int\frac{(\mathcal{J}^\delta y^\delta)\Big(1+(\mathcal{J}^\delta y^\delta_x)^2\Big)}{\sqrt{1+(\mathcal{J}^\delta y^\delta_x)^2}}dx+3\alpha^2(\alpha+3)\int\frac{(\mathcal{J}^\delta y^\delta)\cdot(\mathcal{J}^\delta y^\delta_{xx})^3}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^3}dx$$
$$+4\alpha^2(\alpha+3)\int\frac{(\mathcal{J}^\delta y^\delta_x)^2\cdot(\mathcal{J}^\delta y^\delta_{xx})^2}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^3}dx+6\alpha^2(\alpha+3)\int\frac{(\mathcal{J}^\delta y^\delta)\cdot(\mathcal{J}^\delta y^\delta_x)\cdot(\mathcal{J}^\delta y^\delta_{xx})(\mathcal{J}^\delta y^\delta_{xxx})}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^3}dx$$
$$-15\alpha^2(\alpha+3)\int\frac{(\mathcal{J}^\delta y^\delta)\cdot(\mathcal{J}^\delta y^\delta_x)^2\cdot(\mathcal{J}^\delta y^\delta_{xx})^3}{(1+(\mathcal{J}^\delta y^\delta_{xx})^2)^4}dx-\left(1+\frac{1}{2}\alpha^2\right)\int\left[\frac{(\mathcal{J}^\delta y^\delta)\cdot(\mathcal{J}^\delta y^\delta_{xx})^2}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^{\frac{5}{2}}}\right]dx$$
$$+\left(2\alpha+5\alpha^2-\frac{1}{3}\alpha^3\right)\int\left[\frac{(\mathcal{J}^\delta y^\delta)\cdot(\mathcal{J}^\delta y^\delta_{xx})^3}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^4}\right]dx.$$

All the terms on the right-hand side are controlled by terms of the form of $C(\|y^\delta\|_{L^2}^a+\|\partial_x^4 y^\delta\|_{L^2}^a)$, where $2\le a\le 4$. Overall, we have the following simplified inequality:

(3.8)
$$\frac{1}{2}\partial_t\|y^\delta\|_{L^2}^2+\alpha^2(\alpha+3)\int\frac{(\mathcal{J}^\delta y^\delta_{xx})^2}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^2}dx\le C\Big(\|y^\delta\|_{L^2}^2+\|\partial_x^4 y^\delta\|_{L^2}^2\Big)+\Big(\|y^\delta\|_{L^2}^4+\|\partial_x^4 y^\delta\|_{L^2}^4\Big).$$

This is straightforward to see (it mainly consists of counting derivatives) and we omit further details of the proof of (3.8).

We now turn our attention to the rest of the energy estimate. We take four spatial derivatives of (3.1), and then find its inner product with $\partial_x^4 y^\delta$:

(3.9)
$$\frac{1}{2}\partial_t\|\partial_x^4 y^\delta\|_{L^2}^2+(\alpha-1)\int(\mathcal{J}^\delta\partial_x^4 y^\delta)\cdot\partial_x^4\left[\frac{(\mathcal{J}^\delta y^\delta_{xx})}{1+(\mathcal{J}^\delta y^\delta_x)^2}\right]dx+$$
$$+\left(1+\frac{1}{2}\alpha^2\right)\int(\mathcal{J}^\delta\partial_x^4 y^\delta)\cdot\partial_x^4\left[\frac{(\mathcal{J}^\delta y^\delta_{xx})^2}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^{\frac{5}{2}}}\right]dx$$

(3.10)
$$+\alpha^2(\alpha+3)\int(\mathcal{J}^\delta\partial_x^4 y^\delta)\cdot\partial_x^4\left[\frac{1}{\sqrt{1+(\mathcal{J}^\delta y^\delta_x)^2}}\cdot\frac{d^2\kappa^\delta}{dx^2}\right]dx$$
$$+\int(\mathcal{J}^\delta\partial_x^4 y^\delta)\cdot\partial_x^4\left[\sqrt{1+(\mathcal{J}^\delta y^\delta_x)^2}\right]dx+\left(2\alpha+5\alpha^2-\frac{1}{3}\alpha^3\right)\int(\mathcal{J}^\delta\partial_x^4 y^\delta)\cdot\partial_x^4\left[\frac{(\mathcal{J}^\delta y^\delta_{xx})^3}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^4}\right]dx$$
$$=\alpha^2(\alpha+3)\int(\mathcal{J}^\delta\partial_x^4 y^\delta)\cdot\partial_x^4\left[(\mathcal{J}^\delta y^\delta_x)\cdot\kappa^\delta\cdot\frac{d\kappa^\delta}{dx}\right]dx.$$

As before, for the fourth term on the left-hand side of (3.9), we substitute from (3.4):

$$\alpha^2(\alpha+3)\int(\mathcal{J}^\delta\partial_x^4 y^\delta)\cdot\partial_x^4\left[\frac{1}{\sqrt{1+(\mathcal{J}^\delta y^\delta_x)^2}}\cdot\frac{d^2\kappa^\delta}{dx^2}\right]dx$$
$$=\alpha^2(\alpha+3)\int(\mathcal{J}^\delta\partial_x^6 y^\delta)\cdot\partial_x^2\left[\frac{\mathcal{J}^\delta y^\delta_{xxxx}}{(1+(\mathcal{J}^\delta y^\delta_x)^2)^2}\right]dx$$

$$- 3\alpha^2(\alpha+3)\int(\mathcal{J}^\delta\partial_x^6 y^\delta)\cdot\partial_x^2\left[\frac{(\mathcal{J}^\delta y_{xx}^\delta)^3}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^3}\right]dx$$
$$- 9\alpha^2(\alpha+3)\int(\mathcal{J}^\delta\partial_x^6 y^\delta)\cdot\partial_x^2\left[\frac{(\mathcal{J}^\delta y_x^\delta)(\mathcal{J}^\delta y_{xx}^\delta)(\mathcal{J}^\delta y_{xxx}^\delta)}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^3}\right]dx$$
$$+ 15\alpha^2(\alpha+3)\int(\mathcal{J}^\delta\partial_x^6 y^\delta)\cdot\partial_x^2\left[\frac{(\mathcal{J}^\delta y_x^\delta)^2(\mathcal{J}^\delta y_{xx}^\delta)^3}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^4}\right]dx.$$

We expand the first term on the right-hand side of the above equality as follows:

$$\alpha^2(\alpha+3)\int(\mathcal{J}^\delta\partial_x^4 y^\delta)\cdot\partial_x^4\left[\frac{\mathcal{J}^\delta y_{xxxx}^\delta}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^2}\right]dx$$
$$= \alpha^2(\alpha+3)\int\frac{(\mathcal{J}^\delta\partial_x^6 y^\delta)^2}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^2}dx$$
$$- 4\alpha^2(\alpha+3)\int(\mathcal{J}^\delta\partial_x^6 y^\delta)\left[\frac{(\mathcal{J}^\delta y_x^\delta)(\mathcal{J}^\delta y_{xx}^\delta)(\mathcal{J}^\delta\partial^5 y^\delta)}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^3}\right]dx$$
$$- 4\alpha^2(\alpha+3)\int(\mathcal{J}^\delta\partial_x^6 y^\delta)\partial\left[\frac{(\mathcal{J}^\delta y_x^\delta)(\mathcal{J}^\delta y_{xx}^\delta)(\mathcal{J}^\delta\partial^4 y^\delta)}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^3}\right]dx.$$

Integrating by parts twice, and using (3.3), we also have the formula

$$\alpha^2(\alpha+3)\int(\mathcal{J}^\delta\partial_x^4 y^\delta)\cdot\partial_x^4\left[(\mathcal{J}^\delta y_x^\delta)\cdot\kappa^\delta\cdot\frac{d\kappa^\delta}{dx}\right]dx$$
$$= \alpha^2(\alpha+3)\int(\mathcal{J}^\delta\partial_x^6 y^\delta)\partial^2\left[\frac{(\mathcal{J}^\delta y_x^\delta)(\mathcal{J}^\delta y_{xx}^\delta)(\mathcal{J}^\delta y_{xxx}^\delta)}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^3}\right]dx$$
$$-3\alpha^2(\alpha+3)\int(\mathcal{J}^\delta\partial_x^6 y^\delta)\partial^2\left[\frac{(\mathcal{J}^\delta y_x^\delta)^2\cdot(\mathcal{J}^\delta y_{xx}^\delta)^3}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^4}\right]dx.$$

Therefore the identity (3.9) becomes the following:

$$\frac{1}{2}\partial_t\|\partial_x^4 y^\delta\|_{L^2}^2 + \alpha^2(\alpha+3)\int\frac{(\mathcal{J}^\delta\partial_x^6 y^\delta)^2}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^2}dx \tag{3.11}$$
$$= -(\alpha-1)\int(\mathcal{J}^\delta\partial_x^6 y^\delta)\cdot\partial_x^2\left[\frac{(\mathcal{J}^\delta y_{xx}^\delta)}{1+(\mathcal{J}^\delta y_x^\delta)^2}\right]dx$$
$$-\left(1+\frac{1}{2}\alpha^2\right)\int(\mathcal{J}^\delta\partial_x^6 y^\delta)\cdot\partial_x^2\left[\frac{(\mathcal{J}^\delta y_{xx}^\delta)^2}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^{\frac{5}{2}}}\right]dx$$
$$+ 4\alpha^2(\alpha+3)\int(\mathcal{J}^\delta\partial_x^6 y^\delta)\left[\frac{(\mathcal{J}^\delta y_x^\delta)(\mathcal{J}^\delta y_{xx}^\delta)(\mathcal{J}^\delta\partial^5 y^\delta)}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^3}\right]dx$$
$$+ 4\alpha^2(\alpha+3)\int(\mathcal{J}^\delta\partial_x^6 y^\delta)\partial\left[\frac{(\mathcal{J}^\delta y_x^\delta)(\mathcal{J}^\delta y_{xx}^\delta)(\mathcal{J}^\delta\partial^4 y^\delta)}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^3}\right]dx$$
$$- \int(\mathcal{J}^\delta\partial_x^6 y^\delta)\cdot\partial_x^2\left[\sqrt{1+(\mathcal{J}^\delta y_x^\delta)^2}\right]dx$$

$$- \left(2\alpha + 5\alpha^2 - \frac{1}{3}\alpha^3\right) \int (\mathcal{J}^\delta \partial_x^6 y^\delta) \cdot \partial_x^2 \left[\frac{(\mathcal{J}^\delta y_{xx}^\delta)^3}{(1 + (\mathcal{J}^\delta y_x^\delta)^2)^4} \right] dx$$
$$3\alpha^2(\alpha + 3) \int (\mathcal{J}^\delta \partial_x^6 y^\delta) \cdot \partial_x^2 \left[\frac{(\mathcal{J}^\delta y_{xx}^\delta)^3}{(1 + (\mathcal{J}^\delta y_x^\delta)^2)^3} \right] dx$$
$$+ 10\alpha^2(\alpha + 3) \int (\mathcal{J}^\delta \partial_x^6 y^\delta) \cdot \partial_x^2 \left[\frac{(\mathcal{J}^\delta y_x^\delta)(\mathcal{J}^\delta y_{xx}^\delta)(\mathcal{J}^\delta y_{xxx}^\delta)}{(1 + (\mathcal{J}^\delta y_x^\delta)^2)^3} \right] dx$$
$$- 18\alpha^2(\alpha + 3) \int (\mathcal{J}^\delta \partial_x^6 y^\delta) \cdot \partial_x^2 \left[\frac{(\mathcal{J}^\delta y_x^\delta)^2(\mathcal{J}^\delta y_{xx}^\delta)^3}{(1 + (\mathcal{J}^\delta y_x^\delta)^2)^4} \right] dx = J_1 + J_2 + \cdots + J_9.$$

We claim that we can reduce the right-hand side of this equality into a manageable form. In fact we will show that, for some $m > 2$ and C_1 small enough,

$$\left| J_1 + \cdots + I_9 \right| \le C_1 \int \frac{(\mathcal{J}^\delta \partial_x^6 y^\delta)^2}{(1 + (\mathcal{J}^\delta y_x^\delta)^2)^2} dx + C_2 \left(\|y^\delta\|_{L^2}^2 + \|\partial_x^4 y^\delta\|_{L^2}^2 \right) + C_3 \left(\|y^\delta\|_{L^2}^m + \|\partial_x^4 y^\delta\|_{L^2}^m \right). \tag{3.12}$$

To prove this, we find bounds for each of the terms $J_1, \ldots, J_9$. Instead of demonstrating the full bound for every single integral, we focus on the most singular part of each of $J_1, \ldots, J_9$, with these most singular parts being the terms with the highest derivatives when distributing spatial derivatives according to the product rule. We will label collections of the less singular terms as $G(t)$, which stands for good terms.

We begin with J_1, estimating its most singular term by means of Young's inequality:

$$|J_1| = \left| (\alpha - 1) \int (\mathcal{J}^\delta \partial_x^6 y^\delta) \cdot \partial_x^2 \left[\frac{(\mathcal{J}^\delta y_{xx}^\delta)}{1 + (\mathcal{J}^\delta y_x^\delta)^2} \right] dx \right| \le \left| (\alpha - 1) \int (\mathcal{J}^\delta \partial_x^6 y^\delta) \cdot \left[\frac{(\mathcal{J}^\delta \partial_x^4 y^\delta)}{1 + (\mathcal{J}^\delta y_x^\delta)^2} \right] dx \right| + G(t)$$
$$\le \frac{1}{100} \left\| \frac{\mathcal{J}^\delta \partial_x^6 y^\delta}{1 + (\mathcal{J}^\delta y_x^\delta)^2} \right\|_{L^2}^2 + C \|\mathcal{J}^\delta \partial_x^4 y^\delta\|_{L^2}^2 + G(t).$$

We proceed similarly for the most singular term in J_2 :

$$|J_2| = \left| \left(1 + \frac{1}{2}\alpha^2\right) \int (\mathcal{J}^\delta \partial_x^6 y^\delta) \cdot \partial_x^2 \left[\frac{(\mathcal{J}^\delta y_{xx}^\delta)^2}{(1 + (\mathcal{J}^\delta y_x^\delta)^2)^{\frac{5}{2}}} \right] dx \right|$$
$$\le C \left| \int \frac{(\mathcal{J}^\delta \partial_x^6 y^\delta)}{1 + (\mathcal{J}^\delta y_x^\delta)} \cdot \left[\frac{(\mathcal{J}^\delta \partial_x^2 y^\delta)(\mathcal{J}^\delta \partial_x^4 y^\delta)}{1 + (\mathcal{J}^\delta y_x^\delta)^{\frac{3}{2}}} \right] dx \right| + G(t)$$
$$\le \frac{1}{100} \left\| \frac{\mathcal{J}^\delta \partial_x^6 y^\delta}{1 + (\mathcal{J}^\delta y_x^\delta)^2} \right\|_{L^2}^2 + C \left\| \mathcal{J}^\delta \partial_x^2 y^\delta \right\|_{L^\infty}^2 \left\| \mathcal{J}^\delta \partial_x^4 y^\delta \right\|_{L^2}^2 + G(t).$$

We now use the Sobelev and Gagliardo–Nirenberg inequalities to control $\|\mathcal{J}^\delta \partial_x^2 y^\delta\|_{L^\infty}$:

$$\|\partial_x^2 y^\delta\|_{L^\infty} \le \||\nabla|^{\frac{1}{2}} \partial_x^2 y^\delta\|_{L^2} \le \|y^\delta\|_{L^2}^{\frac{3}{8}} \|\partial_x^4 y^\delta\|_{L^2}^{\frac{5}{8}}.$$

This then implies

$$|J_2| \leq \frac{1}{100}\left\|\frac{\mathcal{J}^\delta \partial_x^6 y^\delta}{1+(\mathcal{J}^\delta y_x^\delta)^2}\right\|_{L^2}^2 + C\left(\|y^\delta\|_{L^2}^{\frac{3}{8}}\|\partial_x^4 y^\delta\|_{L^2}^{\frac{5}{8}}\right)^2\|\partial_x^4 y^\delta\|_{L^2}^2 + G(t)$$
$$\leq \frac{1}{100}\left\|\frac{\mathcal{J}^\delta \partial_x^6 y^\delta}{1+(\mathcal{J}^\delta y_x^\delta)^2}\right\|_{L^2}^2 + C\left(\left\|y^\delta\right\|_{L^2}^4 + \left\|\partial_x^4 y^\delta\right\|_{L^2}^4\right) + G(t).$$

We turn our attention to estimating J_3; to begin, we have

$$|J_3| = \left|4\alpha^2(\alpha+3)\int(\mathcal{J}^\delta\partial_x^6 y^\delta)\left[\frac{(\mathcal{J}^\delta y_x^\delta)(\mathcal{J}^\delta y_{xx}^\delta)(\mathcal{J}^\delta\partial^5 y^\delta)}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^3}\right]\,dx\right|$$
$$\leq C\left|\int\frac{(\mathcal{J}^\delta\partial_x^6 y^\delta)}{1+(\mathcal{J}^\delta y_x^\delta)^2}\cdot\left[\frac{(\mathcal{J}^\delta y_x^\delta)(\mathcal{J}^\delta y_{xx}^\delta)(\mathcal{J}^\delta\partial_x^5 y^\delta)}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^2}\right]\,dx\right| + G(t)$$
$$\leq \frac{1}{100}\left\|\frac{\mathcal{J}^\delta \partial_x^6 y^\delta}{1+(\mathcal{J}^\delta y_x^\delta)^2}\right\|_{L^2}^2 + C\left\|\mathcal{J}^\delta\partial_x^2 y^\delta\right\|_{L^\infty}^2\left\|\mathcal{J}^\delta\partial_x^5 y^\delta\right\|_{L^2}^2 + G(t).$$

Here we have used the fact that $|\frac{(\mathcal{J}^\delta y_x^\delta)}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^2}| \leq 1$.

We turn our attention to bounding $\|\mathcal{J}^\delta\partial_x^2 y^\delta\|_{L^\infty}^2$ and $\|\mathcal{J}^\delta\partial_x^5 y^\delta\|_{L^2}^2$ as follows. We use the Sobolev inequality as well as the Gagliardo–Nirenberg inequality, finding

$$\|\mathcal{J}^\delta\partial_x^2 y^\delta\|_{L^\infty} \leq \|\mathcal{J}^\delta|\nabla|^{\frac{1}{2}}\partial_x^2 y^\delta\|_{L^2} \leq \|y^\delta\|_{L^2}^{\frac{3}{8}}\|\partial_x^4 y^\delta\|_{L^2}^{\frac{5}{8}}. \tag{3.13}$$

Moreover,

$$\|\mathcal{J}^\delta\partial_x^5 y^\delta\|_{L^2} \leq \|\mathcal{J}^\delta\partial_x^6 y^\delta\|_{L^2}^{\frac{1}{2}}\|\partial_x^4 y^\delta\|_{L^2}^{\frac{1}{2}} \leq C\|\frac{\mathcal{J}^\delta\partial_x^6 y^\delta}{1+(\mathcal{J}^\delta y_x^\delta)^2}\|_{L^2}^{\frac{1}{2}}\|\partial_x^4 y^\delta\|_{L^2}^{\frac{1}{2}}\|1+(\mathcal{J}^\delta y_x^\delta)^2\|_{L^\infty}^{\frac{1}{2}},$$

and also,

$$\|1+(\mathcal{J}^\delta y_x^\delta)^2\|_{L^\infty} \leq 1+\|\mathcal{J}^\delta y_x^\delta\|_{L^\infty}^2 \leq 1+\|\mathcal{J}^\delta|\nabla|^{\frac{1}{2}}y_x^\delta\|_{L^2}^2 \leq 1+\left(\|\mathcal{J}^\delta y^\delta\|_{L^2}^{\frac{5}{8}}\|\mathcal{J}^\delta\partial_x^4 y^\delta\|_{L^2}^{\frac{3}{8}}\right)^2.$$

We may thus conclude our bound for J_3:

$$|J_3| \leq \frac{1}{100}\left\|\frac{\mathcal{J}^\delta \partial_x^6 y^\delta}{1+(\mathcal{J}^\delta y_x^\delta)^2}\right\|_{L^2}^2 + C\|\frac{\mathcal{J}^\delta \partial_x^6 y^\delta}{1+(\mathcal{J}^\delta y_x^\delta)^2}\|_{L^2}\|y^\delta\|_{L^2}^2\|\partial_x^4 y^\delta\|_{L^2}^3 + C + G(t)$$
$$\leq \frac{2}{100}\left\|\frac{\mathcal{J}^\delta \partial_x^6 y^\delta}{1+(\mathcal{J}^\delta y_x^\delta)^2}\right\|_{L^2}^2 + C\|y^\delta\|_{L^2}^4\|\partial_x^4 y^\delta\|_{L^2}^6 + C + G(t)$$
$$\leq \frac{2}{100}\left\|\frac{\mathcal{J}^\delta \partial_x^6 y^\delta}{1+(\mathcal{J}^\delta y_x^\delta)^2}\right\|_{L^2}^2 + C\left(\|y^\delta\|_{L^2}^{10} + \|\partial_x^4 y^\delta\|_{L^2}^{10}\right) + G(t).$$

Note that above we used the assumption that $\|y^\delta\|_{L^2}^2 + \|\partial_x^4 y^\delta\|_{L^2}^2 \geq 1$ (otherwise there would be nothing to prove), and consequently $C < (\|y^\delta\|_{L^2}^{10} + \|\partial_x^4 y^\delta\|_{L^2}^{10})$.

We estimate J_4 similarly to how we estimated J_3:

$$|J_4| = \left|4\alpha^2(\alpha+3)\int(\mathcal{J}^\delta\partial_x^6 y^\delta)\partial\left[\frac{(\mathcal{J}^\delta y_x^\delta)(\mathcal{J}^\delta y_{xx}^\delta)(\mathcal{J}^\delta\partial^4 y^\delta)}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^3}\right]dx\right|$$
$$\leq C\left|\int\frac{(\mathcal{J}^\delta\partial_x^6 y^\delta)}{1+(\mathcal{J}^\delta y_x^\delta)^2}\cdot\left[\frac{(\mathcal{J}^\delta y_x^\delta)(\mathcal{J}^\delta y_{xx}^\delta)(\mathcal{J}^\delta\partial_x^5 y^\delta)}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^2}\right]dx\right|+G(t)$$
$$\leq \frac{2}{100}\left\|\frac{\mathcal{J}^\delta\partial_x^6 y^\delta}{1+(\mathcal{J}^\delta y_x^\delta)^2}\right\|_{L^2}^2+C\left(\|y^\delta\|_{L^2}^{10}+\|\partial_x^4 y^\delta\|_{L^2}^{10}\right)+G(t).$$

We next consider J_5, beginning as follows:

$$|J_5| = \left|\int(\mathcal{J}^\delta\partial_x^6 y^\delta)\cdot\partial_x^2\left[\frac{1+(\mathcal{J}^\delta y_x^\delta)^2}{\sqrt{1+(\mathcal{J}^\delta y_x^\delta)^2}}\right]dx\right|$$
$$\leq C\left|\int(\mathcal{J}^\delta\partial_x^6 y^\delta)\cdot\left[\frac{(\mathcal{J}^\delta y_x^\delta)(\mathcal{J}^\delta\partial_x^3 y^\delta)}{\sqrt{1+(\mathcal{J}^\delta y_x^\delta)^2}}\right]dx\right|+G(t)$$
$$\leq \frac{1}{100}\left\|\frac{\mathcal{J}^\delta\partial_x^6 y^\delta}{1+(\mathcal{J}^\delta y_x^\delta)^2}\right\|_{L^2}^2+C\|(\mathcal{J}^\delta y_x^\delta)(\mathcal{J}^\delta\partial_x^3 y^\delta)\sqrt{1+(\mathcal{J}^\delta y_x^\delta)^2}\|_{L^2}^2+G(t)$$
$$\leq \frac{1}{100}\left\|\frac{\mathcal{J}^\delta\partial_x^6 y^\delta}{1+(\mathcal{J}^\delta y_x^\delta)^2}\right\|_{L^2}^2+C\|(\mathcal{J}^\delta y_x^\delta)\|_{L^\infty}^2\|(\mathcal{J}^\delta\partial_x^3 y^\delta)\|_{L^2}^2\|\sqrt{1+(\mathcal{J}^\delta y_x^\delta)^2}\|_{L^\infty}^2+G(t).$$

By the Sobelev and Gagliardo–Nirenberg inequalities, we have

$$\left\|(\mathcal{J}^\delta y_x^\delta)\right\|_{L^\infty}\leq\left\||\nabla|^{\frac{1}{2}}y_x^\delta\right\|_{L^2}\leq\left\|y^\delta\right\|_{L^2}^{\frac{5}{8}}\left\|\partial_x^4 y^\delta\right\|_{L^2}^{\frac{3}{8}},$$

as well as

$$\left\|\sqrt{1+(\mathcal{J}^\delta y_x^\delta)^2}\right\|_{L^\infty}^2\leq C\left(1+\|\mathcal{J}^\delta y_x^\delta\|_{L^\infty}\right)^2\leq C\left(1+\||\nabla|^{\frac{1}{2}}y_x^\delta\|_{L^2}\right)^2\leq C\left(1+\|y^\delta\|_{L^2}^{\frac{5}{8}}\|\partial_x^4 y^\delta\|_{L^2}^{\frac{3}{8}}\right)^2.$$

Moreover,

$$\|(\mathcal{J}^\delta\partial_x^3 y^\delta)\|_{L^2}\leq\|y^\delta\|_{L^2}^{\frac{1}{4}}\|\partial_x^4 y^\delta\|_{L^2}^{\frac{3}{4}}.$$

We may then conclude our bound for J_5 as

$$|J_5|\leq\frac{1}{100}\left\|\frac{\mathcal{J}^\delta\partial_x^6 y^\delta}{1+(\mathcal{J}^\delta y_x^\delta)^2}\right\|_{L^2}^2+C\|y^\delta\|_{L^2}^3\|\partial_x^4 y^\delta\|_{L^2}^3+1+G(t)$$
$$\leq\frac{1}{100}\left\|\frac{\mathcal{J}^\delta\partial_x^6 y^\delta}{1+(\mathcal{J}^\delta y_x^\delta)^2}\right\|_{L^2}^2+C\|y^\delta\|_{L^2}^4+\|\partial_x^4 y^\delta\|_{L^2}^4+C\|y^\delta\|_{L^2}^6+\|\partial_x^4 y^\delta\|_{L^2}^6+G(t).$$

We begin the estimate for J_6 similarly to how we estimated J_3 above:

$$|J_6| = \left|\left(2\alpha+5\alpha^2-\frac{1}{3}\alpha^3\right)\int(\mathcal{J}^\delta\partial_x^6 y^\delta)\cdot\partial_x^2\left[\frac{(\mathcal{J}^\delta y_{xx}^\delta)^3}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^4}\right]dx\right|$$
$$\leq C\left|\int\frac{(\mathcal{J}^\delta\partial_x^6 y^\delta)}{(1+(\mathcal{J}^\delta y_x^\delta)^2)}\cdot\left[\frac{(\mathcal{J}^\delta y_{xx}^\delta)^2(\mathcal{J}^\delta\partial^4 y^\delta)}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^3}\right]dx\right|+G(t)$$
$$\leq\frac{2}{100}\left\|\frac{\mathcal{J}^\delta\partial_x^6 y^\delta}{1+(\mathcal{J}^\delta y_x^\delta)^2}\right\|_{L^2}^2+C\left\|(\mathcal{J}^\delta y_{xx}^\delta)^2(\mathcal{J}^\delta\partial^4 y^\delta)\right\|_{L^2}+G(t).$$

We then make use of relation (3.13), finding

$$\left\|(\mathcal{J}^\delta y^\delta_{xx})^2(\mathcal{J}^\delta\partial^4 y^\delta)\right\|_{L^2} \le \|y^\delta\|_{L^2}^{\frac{3}{2}}\|\mathcal{J}^\delta\partial^4 y^\delta\|_{L^2}^{\frac{9}{2}} \le C\Big(\|y^\delta\|_{L^2}^6 + \|\mathcal{J}^\delta\partial^4 y^\delta\|_{L^2}^6\Big).$$

Therefore, we have the conclusion

$$|J_6| \le \frac{2}{100}\left\|\frac{\mathcal{J}^\delta\partial_x^6 y^\delta}{1+(\mathcal{J}^\delta y_x^\delta)^2}\right\|_{L^2}^2 + C\Big(\|y^\delta\|_{L^2}^6 + \|\mathcal{J}^\delta\partial^4 y^\delta\|_{L^2}^6\Big) + G(t).$$

We estimate J_7 as follows:

$$\begin{aligned}
|J_7| &= \left|3\alpha^2(\alpha+3)\int \frac{(\mathcal{J}^\delta\partial_x^6 y^\delta)}{1+(\mathcal{J}^\delta y_x^\delta)^2}\cdot\partial_x^2\left[\frac{(\mathcal{J}^\delta y^\delta_{xx})^3}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^2}\right]\,dx\right| \\
&\le C\left|\int \frac{(\mathcal{J}^\delta\partial_x^6 y^\delta)}{1+(\mathcal{J}^\delta y_x^\delta)^2}\cdot\left[\frac{(\mathcal{J}^\delta y^\delta_{xx})^2(\mathcal{J}^\delta\partial_x^4 y^\delta)}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^2}\right]\,dx\right| + G(t) \\
&\le \frac{1}{100}\left\|\frac{\mathcal{J}^\delta\partial_x^6 y^\delta}{1+(\mathcal{J}^\delta y_x^\delta)^2}\right\|_{L^2}^2 + C\left\|\frac{(\mathcal{J}^\delta y^\delta_{xx})^2(\mathcal{J}^\delta\partial_x^4 y^\delta)}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^2}\right\|_{L^2}^2 + G(t) \qquad (3.14)\\
&\le \frac{1}{100}\left\|\frac{\mathcal{J}^\delta\partial_x^6 y^\delta}{1+(\mathcal{J}^\delta y_x^\delta)^2}\right\|_{L^2}^2 + C\left(\|\mathcal{J}^\delta y^\delta_{xx}\|_{L^\infty}^2\|\mathcal{J}^\delta\partial_x^4 y^\delta\|_{L^2}\right)^2 + G(t) \\
&\le \frac{1}{100}\left\|\frac{\mathcal{J}^\delta\partial_x^6 y^\delta}{1+(\mathcal{J}^\delta y_x^\delta)^2}\right\|_{L^2}^2 + C\left(\|y^\delta\|_{L^2}^{\frac{3}{8}}\|\partial_x^4 y^\delta\|_{L^2}^{\frac{5}{8}}\right)^4\|\mathcal{J}^\delta\partial_x^4 y^\delta\|_{L^2}^2 + G(t).
\end{aligned}$$

Our conclusion for J_7 is then

$$|J_7| \le \frac{1}{100}\left\|\frac{\mathcal{J}^\delta\partial_x^6 y^\delta}{1+(\mathcal{J}^\delta y_x^\delta)^2}\right\|_{L^2}^2 + C\left(\|y^\delta\|_{L^2}^6 + \|\mathcal{J}^\delta\partial_x^4 y^\delta\|_{L^2}^6\right) + G(t).$$

For J_8, we begin with the following estimate:

$$\begin{aligned}
|J_8| &= 10\left|\alpha^2(\alpha+3)\int(\mathcal{J}^\delta\partial_x^6 y^\delta)\cdot\partial_x^2\left[\frac{(\mathcal{J}^\delta y_x^\delta)(\mathcal{J}^\delta y^\delta_{xx})(\mathcal{J}^\delta y^\delta_{xxx})}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^3}\right]\,dx\right| \\
&\le C\left|\int(\mathcal{J}^\delta\partial_x^6 y^\delta)\cdot\left[\frac{(\mathcal{J}^\delta y_x^\delta)(\mathcal{J}^\delta y^\delta_{xx})(\mathcal{J}^\delta\partial^5 y^\delta)}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^3}\right]\,dx\right| + G(t).
\end{aligned}$$

The integral on the right-hand side is similar to J_3, and we handle it in the same way.

This brings us to the final term to estimate, J_9; for this, we have the following:

$$\begin{aligned}
|J_9| &= \left|18\alpha^2(\alpha+3)\int(\mathcal{J}^\delta\partial_x^6 y^\delta)\cdot\partial_x^2\left[\frac{(\mathcal{J}^\delta y_x^\delta)^2(\mathcal{J}^\delta y^\delta_{xx})^3}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^4}\right]dx\right| \\
&\le C\left|\int\frac{(\mathcal{J}^\delta\partial_x^6 y^\delta)}{1+(\mathcal{J}^\delta y_x^\delta)^2}\cdot\left[\frac{(\mathcal{J}^\delta y_x^\delta)^2(\mathcal{J}^\delta y^\delta_{xx})^2(\mathcal{J}^\delta\partial^4 y^\delta)}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^3}\right]dx\right| + G(t) \\
&\le \frac{1}{100}\left\|\frac{\mathcal{J}^\delta\partial_x^6 y^\delta}{1+(\mathcal{J}^\delta y_x^\delta)^2}\right\|_{L^2}^2 + C\left\|(\mathcal{J}^\delta y^\delta_{xx})^2(\mathcal{J}^\delta\partial_x^4 y^\delta)\right\|_{L^2}^2 + G(t).
\end{aligned}$$

Here we have used the fact that $|\frac{(\mathcal{J}^\delta y_x^\delta)^2}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^3}| \leq 1$. The last inequality is similar to (3.14), and we proceed in the same way to get

$$|J_9| \leq \frac{1}{100}\left\|\frac{\mathcal{J}^\delta \partial_x^6 y^\delta}{1+(\mathcal{J}^\delta y_x^\delta)^2}\right\|_{L^2}^2 + C\Big(\|y^\delta\|_{L^2}^6 + \|\mathcal{J}^\delta \partial_x^4 y^\delta\|_{L^2}^6\Big) + G(t).$$

Putting the above together leads to the relation (3.12), which we had been aiming to prove. We now may add (3.8) and (3.12), finding

$$\begin{aligned}\frac{1}{2}\partial_t\left(\|y^\delta\|_{L^2}^2 + \|\partial_x^4 y^\delta\|_{L^2}^2\right) + C\int \frac{(\mathcal{J}^\delta \partial_x^2 y^\delta)^2 + (\mathcal{J}^\delta \partial_x^6 y^\delta)^2}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^2}dx \\ \leq C_0\left(\|y^\delta\|_{L^2}^2 + \|\partial_x^4 y^\delta\|_{L^2}^2\right) + C_1\left(\|y^\delta\|_{L^2}^m + \|\partial_x^4 y^\delta\|_{L^2}^m\right).\end{aligned}$$

This inequality clearly implies a uniform bound (uniform with respect to δ) for the function $y^\delta(x,t)$ in the space H^4 until a time $T = T(\alpha, \|y_0\|_{H^4})$. We will study the size of this time interval $[0,T]$ in a bit more detail, and to this end we define $I(t) = \|y^\delta\|_{L^2}^2 + \|\partial_x^4 y^\delta\|_{L^2}^2$. We also fix C_0 with $\gamma = C_0$.

We may then say

$$I(t) \leq e^{\gamma t} I(0) + \int_0^t e^{\gamma(t-s)} I^{\frac{m}{2}}(s)ds.$$

We use Lemma 2.1 to see that $I(t)$ remains bounded as long as $t \in [0, \beta_{\frac{m}{2}}]$, where

$$\beta_{\frac{m}{2}} = \sup\left\{t : \left(\frac{m}{2}-1\right)\int_0^t e^{-C\gamma s} e^{\gamma s}\left[(I(0))^{(\frac{m}{2}-1)} e^{(\frac{m}{2}-1)\gamma s}\right] ds < 1\right\}. \tag{3.15}$$

A simple calculation then shows that we have guaranteed existence of our solutions over the interval

$$0 < t < \frac{\ln\left(1 + \frac{\gamma}{\|y_0\|_{H^4}^{m-2}}\right)}{\gamma}. \tag{3.16}$$

Clearly, this bound for the time of existence depends on the initial values; that is, if the value $I(0) = \|y_0\|_{L^2}^2 + \|\partial_x^4 y_0\|_{L^2}^2$ stays small, the time interval is large. Note that we will take this $m > 2$ and γ to be fixed throughout the following. ■

Lemma 3.2 provides a uniform bound in H^4 for the solutions of the approximate equations (3.1). Although this is a good and useful estimate, as we are aiming to show the existence of classical solutions, we need a little more. In what follows, we will pass to the limit of solutions of the approximate equations (3.1) to find solutions of the original equation (1.8). In order to do this, we need to have at least the continuity of the function $F(y^\delta)$, where $F(y^\delta)$ denotes

$$y_t^\delta = F(y^\delta), \tag{3.17}$$

with y^δ determined from (3.1). To guarantee continuity of this function, one approach is to prove an H^5 uniform bound. This clearly means $\partial_x^4 y$ is continuous and hence the function $F(y^\delta)$ is continuous as well. The following lemma concerns the appropriate bound.

Lemma 3.3. *Let y^δ be the solution of (1.8). Then there exists $T = T(\alpha)$ and $C = C(y_0, \alpha)$, independent of δ, so that for any $0 < t < \frac{\ln(1+\frac{\gamma}{\|y_0\|_{H^4}^{m-2}})}{\gamma}$,*

$$\sup_{0<t<T} \|y^\delta\|_{H^5}^2 + \int \frac{(\partial_x^7 \mathcal{J}^\delta y^\delta)^2}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^2} dx \le C. \tag{3.18}$$

Proof. We take five spatial derivatives of (3.1), and its inner product with the function $\partial_x^5 y^\delta$

$$\begin{aligned} \frac{1}{2}\partial_t \|\partial_x^5 y^\delta\|_{L^2}^2 &+ (\alpha-1)\int (\mathcal{J}^\delta \partial_x^5 y^\delta)\cdot \partial_x^5\left[\frac{(\mathcal{J}^\delta y_{xx}^\delta)}{1+(\mathcal{J}^\delta y_x^\delta)^2}\right]dx \\ &+ \alpha^2(\alpha+3)\int (\mathcal{J}^\delta \partial_x^5 y^\delta)\cdot \partial_x^5\left[\frac{1}{\sqrt{1+(\mathcal{J}^\delta y_x^\delta)^2}}\cdot\frac{d^2\kappa^\delta}{dx^2}\right]dx \\ &+ \int (\mathcal{J}^\delta \partial_x^5 y^\delta)\cdot \partial_x^5\left[\sqrt{1+(\mathcal{J}^\delta \mathcal{J}^\delta y_x^\delta)^2}\right]dx \\ &+ \left(1+\frac{1}{2}\alpha^2\right)\int (\mathcal{J}^\delta \partial_x^5 y^\delta)\cdot \partial_x^5\left[\frac{(\mathcal{J}^\delta y_{xx}^\delta)^2}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^{\frac{5}{2}}}\right]dx \\ &+ \left(2\alpha+5\alpha^2-\frac{1}{3}\alpha^3\right)\int (\mathcal{J}^\delta \partial_x^5 y^\delta)\cdot \partial_x^5\left[\frac{(\mathcal{J}^\delta y_{xx}^\delta)^3}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^4}\right]dx \\ &= \alpha^2(\alpha+3)\int (\mathcal{J}^\delta \partial_x^5 y^\delta)\cdot \partial_x^5\left[(\mathcal{J}^\delta \mathcal{J}^\delta y_x^\delta)\cdot\kappa^\delta\cdot\frac{d\kappa^\delta}{dx}\right]dx. \end{aligned} \tag{3.19}$$

For a more convenient form of this identity, we simplify some of these terms. To begin, we have

$$\begin{aligned} \alpha^2(\alpha+3)&\int (\mathcal{J}^\delta \partial_x^5 y^\delta)\cdot \partial_x^5\left[\frac{1}{\sqrt{1+(\mathcal{J}^\delta y_x^\delta)^2}}\cdot\frac{d^2\kappa^\delta}{dx^2}\right]dx \\ &= \alpha^2(\alpha+3)\int (\mathcal{J}^\delta \partial_x^5 y^\delta)\cdot \partial_x^5\left[\frac{\mathcal{J}^\delta y_{xxxx}^\delta}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^2}\right]dx \\ &\quad -3\alpha^2(\alpha+3)\int (\mathcal{J}^\delta \partial_x^5 y^\delta)\cdot \partial_x^5\left[\frac{(\mathcal{J}^\delta y_{xx}^\delta)^3}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^3}\right]dx \\ &\quad +9\alpha^2(\alpha+3)\int (\mathcal{J}^\delta \partial_x^5 y^\delta)\cdot \partial_x^5\left[\frac{(\mathcal{J}^\delta y_x^\delta)(\mathcal{J}^\delta y_{xx}^\delta)(\mathcal{J}^\delta y_{xxx}^\delta)}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^3}\right]dx \\ &\quad +15\alpha^2(\alpha+3)\int (\mathcal{J}^\delta \partial_x^5 y^\delta)\cdot \partial_x^5\left[\frac{(\mathcal{J}^\delta y_x^\delta)^2(\mathcal{J}^\delta y_{xx}^\delta)^3}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^4}\right]dx. \end{aligned}$$

The first term in the right-hand side of this relation is

$$\begin{aligned} \alpha^2(\alpha+3)&\int (\mathcal{J}^\delta \partial_x^5 y^\delta)\cdot \partial_x^5\left[\frac{\mathcal{J}^\delta y_{xxxx}^\delta}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^2}\right]dx \\ &= \alpha^2(\alpha+3)\int \frac{(\mathcal{J}^\delta \partial_x^7 y^\delta)^2}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^2}dx \end{aligned}$$

$$- 4\alpha^2(\alpha+3)\int (\mathcal{J}^\delta \partial_x^7 y^\delta)\cdot \frac{(\mathcal{J}^\delta \partial_x^6 y^\delta)(\mathcal{J}^\delta y_x^\delta)(y_{xx}^\delta)}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^3}dx$$
$$- 4\alpha^2(\alpha+3)\int (\mathcal{J}^\delta \partial_x^7 y^\delta)\cdot \partial_x^2\left[\frac{(\mathcal{J}^\delta \partial_x^4 y^\delta)(\mathcal{J}^\delta y_x^\delta)(y_{xx}^\delta)}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^3}\right]dx.$$

We next rewrite another term appearing in (3.19):

$$\int (\mathcal{J}^\delta \partial_x^5 y^\delta)\cdot \partial_x^5\left[\sqrt{1+(\mathcal{J}^\delta\mathcal{J}^\delta y_x^\delta)^2}\right]dx = \int (\mathcal{J}^\delta \partial_x^7 y^\delta)\cdot \partial_x^3\left[\frac{1+(\mathcal{J}^\delta\mathcal{J}^\delta y_x^\delta)^2}{\sqrt{1+(\mathcal{J}^\delta\mathcal{J}^\delta y_x^\delta)^2}}\right]dx.$$

Finally, another term in (3.19) can be written as follows:

$$\alpha^2(\alpha+3)\int (\mathcal{J}^\delta \partial_x^5 y^\delta)\cdot \partial_x^5\left[(\mathcal{J}^\delta\mathcal{J}^\delta y_x^\delta)\cdot \kappa^\delta \cdot \frac{d\kappa^\delta}{dx}\right]dx$$
$$= 4\int (\mathcal{J}^\delta \partial_x^7 y^\delta)\cdot \partial_x^3\left[\frac{(\mathcal{J}^\delta\mathcal{J}^\delta y_x^\delta)(\mathcal{J}^\delta\mathcal{J}^\delta y_{xx}^\delta)(\mathcal{J}^\delta y_{xxx}^\delta)}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^3}\right]dx$$
$$-4\alpha^2(\alpha+3)\int (\mathcal{J}^\delta \partial_x^7 y^\delta)\cdot \partial_x^3\left[\frac{(\mathcal{J}^\delta y_x^\delta)^2\cdot(\mathcal{J}^\delta y_{xx}^\delta)^3}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^4}\right]dx.$$

One can put in more effort and simplify other terms for a more convenient form, but we avoid long calculations and work with the following simplified version, as it is enough for our purpose:

$$\begin{aligned}
&\frac{1}{2}\partial_t\|\partial_x^5 y^\delta\|_{L^2}^2 + \alpha^2(\alpha+3)\int \frac{(\mathcal{J}^\delta \partial_x^7 y^\delta)^2}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^2}dx \\
&\le |\alpha-1|\cdot\left|\int (\mathcal{J}^\delta \partial_x^7 y^\delta)\cdot \partial_x^3\left[\frac{(\mathcal{J}^\delta y_{xx}^\delta)}{1+(\mathcal{J}^\delta y_x^\delta)^2}\right]dx\right| \\
&\quad + 4\alpha^2(\alpha+3)\left|\int (\mathcal{J}^\delta \partial_x^7 y^\delta)\cdot \frac{(\mathcal{J}^\delta \partial_x^6 y^\delta)(\mathcal{J}^\delta y_x^\delta)(y_{xx}^\delta)}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^3}dx\right| \\
&\quad + 4\alpha^2(\alpha+3)\left|\int (\mathcal{J}^\delta \partial_x^7 y^\delta)\cdot \partial_x^2\left[\frac{(\mathcal{J}^\delta \partial_x^4 y^\delta)(\mathcal{J}^\delta y_x^\delta)(y_{xx}^\delta)}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^3}\right]dx\right| \\
&\quad + 3\alpha^2(\alpha+3)\left|\int (\mathcal{J}^\delta \partial_x^7 y^\delta)\cdot \partial_x^3\left[\frac{(\mathcal{J}^\delta y_{xx}^\delta)^3}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^3}\right]dx\right| \\
&\quad + 10\alpha^2(\alpha+3)\left|\int (\mathcal{J}^\delta \partial_x^7 y^\delta)\cdot \partial_x^3\left[\frac{(\mathcal{J}^\delta y_x^\delta)(\mathcal{J}^\delta y_{xx}^\delta)(\mathcal{J}^\delta y_{xxx}^\delta)}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^3}\right]dx\right| \\
&\quad + 18\alpha^2(\alpha+3)\left|\int (\mathcal{J}^\delta \partial_x^7 y^\delta)\cdot \partial_x^3\left[\frac{(\mathcal{J}^\delta y_x^\delta)^2(\mathcal{J}^\delta y_{xx}^\delta)^3}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^4}\right]dx\right| \\
&\quad + \left|\int (\mathcal{J}^\delta \partial_x^7 y^\delta)\cdot \partial_x^3\left[\frac{1+(\mathcal{J}^\delta y_x^\delta)^2}{\sqrt{1+(\mathcal{J}^\delta y_x^\delta)^2}}\right]dx\right| \\
&\quad + \left(1+\frac{1}{2}\alpha^2\right)\left|\int (\mathcal{J}^\delta \partial_x^7 y^\delta)\cdot \partial_x^2\left[\frac{(\mathcal{J}^\delta y_{xx}^\delta)^2}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^{\frac{5}{2}}}\right]dx\right| \\
&\quad + \left|\left(2\alpha+5\alpha^2-\frac{1}{3}\alpha^3\right)\int (\mathcal{J}^\delta \partial_x^7 y^\delta)\cdot \partial_x^2\left[\frac{(\mathcal{J}^\delta y_{xx}^\delta)^3}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^4}\right]dx\right| \\
&:= I_1 + \cdots + I_7 + I_8 + I_9.
\end{aligned} \tag{3.20}$$

We will omit most details of the estimates of these terms, as the proof is similar in many respects to the previous lemma. For those details that we do show, in order to control $I_1, \ldots, I_9$ we will focus on the worst term in each of them. In fact, the worse terms are the ones for which the derivative behind the fractions hits the highest degree in the numerator. Note that as long as we restrict the time interval to the interval in (3.16), Lemma 3.2 already provides us with H^4 bounds, and hence, for $a = 0, 1, 2, 3$, there is a constant C so that

$$\|\partial_x^a y^\delta\|_{L^\infty} \leq C. \tag{3.21}$$

Therefore, any term of this kind which comes up in the estimates is easily bounded by a constant C. Moreover, an application of the Gagliardo–Nirenberg inequality and Lemma 3.2 leads to

$$\|\partial_x^4 y^\delta\|_{L^\infty} \leq \epsilon_0 \|\partial_x^7 y^\delta\|_{L^2} + C\|y^\delta\|_{L^2}, \tag{3.22}$$

where in our future calculations, the constant ϵ_0 will be chosen in a way that the seventh derivatives on the right-hand side could be absorbed in the left-hand side (as is frequently done in energy estimates for parabolic equations). This incurs the expense of a potentially large constant $C > 0$ on the term $\|y^\delta\|_{L^2}$.

We will start with the term I_1, and as mentioned above we only present the bound for the worst term in the expansion of I_1:

$$\begin{aligned}
I_1 &= |\alpha - 1| \cdot \left| \int \frac{(\mathcal{J}^\delta \partial_x^7 y^\delta)}{1 + (\mathcal{J}^\delta y_x^\delta)^2} \cdot \left(1 + (\mathcal{J}^\delta y_x^\delta)^2\right) \cdot \partial_x^3 \left[\frac{(\mathcal{J}^\delta y_{xx}^\delta)}{1 + (\mathcal{J}^\delta y_x^\delta)^2}\right] dx \right| \\
&\leq C \left\| \frac{(\mathcal{J}^\delta \partial_x^7 y^\delta)}{1 + (\mathcal{J}^\delta y_x^\delta)^2} \right\|_{L^2} \left\| 1 + (\mathcal{J}^\delta y_x^\delta)^2 \right\|_{L^\infty} \left\| \partial_x^3 \left[\frac{(\mathcal{J}^\delta y_{xx}^\delta)}{1 + (\mathcal{J}^\delta y_x^\delta)^2}\right] \right\|_{L^2} \\
&\leq C \left\| \frac{(\mathcal{J}^\delta \partial_x^7 y^\delta)}{1 + (\mathcal{J}^\delta y_x^\delta)^2} \right\|_{L^2} \cdot \left\| \partial_x^3 \left[\frac{(\mathcal{J}^\delta y_{xx}^\delta)}{1 + (\mathcal{J}^\delta y_x^\delta)^2}\right] \right\|_{L^2} \\
&\leq C \left\| \frac{(\mathcal{J}^\delta \partial_x^7 y^\delta)}{1 + (\mathcal{J}^\delta y_x^\delta)^2} \right\|_{L^2} \cdot \left\| \frac{(\mathcal{J}^\delta \partial_x^5 y^\delta)}{1 + (\mathcal{J}^\delta y_x^\delta)^2} \right\|_{L^2} + G \\
&\leq C \left\| \frac{(\mathcal{J}^\delta \partial_x^7 y^\delta)}{1 + (\mathcal{J}^\delta y_x^\delta)^2} \right\|_{L^2} \cdot \left\| \mathcal{J}^\delta \partial_x^5 y^\delta \right\|_{L^2} + G \\
&\leq C \left\| \frac{(\mathcal{J}^\delta \partial_x^7 y^\delta)}{1 + (\mathcal{J}^\delta y_x^\delta)^2} \right\|_{L^2} \cdot \left(\left\| \mathcal{J}^\delta \partial_x^4 y^\delta \right\|_{L^2}^{\frac{2}{3}} \left\| \mathcal{J}^\delta \partial_x^7 y^\delta \right\|_{L^2}^{\frac{1}{3}} \right) + G \\
&\leq \left\| \frac{(\mathcal{J}^\delta \partial_x^7 y^\delta)}{1 + (\mathcal{J}^\delta y_x^\delta)^2} \right\|_{L^2}^{\frac{4}{3}} \left\| 1 + (\mathcal{J}^\delta y_x^\delta)^2 \right\|_{L^\infty}^{\frac{1}{3}} \leq \frac{1}{10} \left\| \frac{(\mathcal{J}^\delta \partial_x^7 y^\delta)}{1 + (\mathcal{J}^\delta y_x^\delta)^2} \right\|_{L^2}^2 + C.
\end{aligned}$$

We omit the details for I_2 but reach the same conclusion as for I_1.

The estimate for I_3 has an interesting feature which we mention. To begin, we have

$$\begin{aligned}
I_3 &\leq |\alpha^2(\alpha + 3)| \left| \int (\mathcal{J}^\delta \partial_x^7 y^\delta) \cdot \partial_x^2 \left[\frac{(\mathcal{J}^\delta \partial_x^4 y^\delta)(\mathcal{J}^\delta y_x^\delta)(y_{xx}^\delta)}{(1 + (\mathcal{J}^\delta y_x^\delta)^2)^3}\right] dx \right| \\
&\leq C \left| \int (\mathcal{J}^\delta \partial_x^7 y^\delta) \cdot \left[\frac{(\mathcal{J}^\delta \partial_x^6 y^\delta)(\mathcal{J}^\delta y_x^\delta)(y_{xx}^\delta)}{(1 + (\mathcal{J}^\delta y_x^\delta)^2)^3}\right] dx \right| + G.
\end{aligned}$$

The integral on the right-hand side of this may be controlled as desired. The interesting feature mentioned above has to do with an estimate of a lower-order term from the collection G, namely

$$\begin{aligned}\left|\int (\mathcal{J}^\delta \partial_x^7 y^\delta)\cdot\left[\frac{(\mathcal{J}^\delta \partial_x^4 y^\delta)^2(\mathcal{J}^\delta y_x^\delta)}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^3}\right]dx\right| &\le \|\mathcal{J}^\delta\partial_x^7 y^\delta\|_{L^2}\|\mathcal{J}^\delta\partial_x^4 y^\delta\|_{L^2}\|\mathcal{J}^\delta\partial_x^4 y^\delta\|_{L^\infty}\\ &\le \frac{1}{10}\left\|\frac{\mathcal{J}^\delta\partial_x^7 y^\delta}{1+(\mathcal{J}^\delta y_x^\delta)^2}\right\|_{L^2}^2 + C\|\mathcal{J}^\delta y^\delta\|_{L^2}^2\\ &\le \frac{1}{10}\left\|\frac{\mathcal{J}^\delta\partial_x^7 y^\delta}{1+(\mathcal{J}^\delta y_x^\delta)^2}\right\|_{L^2}^2 + C\end{aligned}$$

in which we have used (3.22) and Lemma 3.2.

Omitting further details, we have the conclusion

$$\frac{1}{2}\partial_t\|\partial_x^5 y^\delta\|_{L^2}^2 + C_0\int\frac{(\mathcal{J}^\delta\partial_x^7 y^\delta)^2}{(1+(\mathcal{J}^\delta y_x^\delta)^2)^2}dx \le C,$$

where C_0 is a positive constant and satisfies $C_0 \le 4 - \frac{9}{10}$. This clearly finishes the proof. ■

Now we are ready to present the existence of the solution to the initial value problem for (1.8) in the Sobolev space H^5.

Lemma 3.4. *For all* $0 < t < \frac{1}{\gamma}\ln(1 + \frac{\gamma}{\|y_0\|_{H^4}^{m-2}})$ *there exists a function* $y \in H^5$ *that solves* (1.8) *with initial data* $y(\cdot,0) = y_0 \in H^5$. *Moreover, there is a constant* $C = C(y_0, \alpha)$ *so that*

$$\sup_{0<t<T}\|y\|_{H^5} \le C. \tag{3.23}$$

Proof. In Lemma 3.2 we have shown that $\{y^\delta\}_{\delta>0}$ is a uniformly bounded and continuous family of functions defined on $\mathbb{T}\times[0,T]$ in the Sobolev space H^5.

Thus, by Sobolev embedding, the first spatial derivatives of the solutions, y_x^δ, are uniformly bounded. Inspection of the evolution equation (3.1) also implies that y_t^δ is uniformly bounded. We conclude that the solutions y^δ form an equicontinuous family. By the Arzela–Ascoli theorem, there is a uniformly convergent subsequence (which we do not relabel). This subsequence converges uniformly to some $y \in C(\mathbb{T}\times[0,T])$. This uniform convergence implies convergence in $C([0,T];L^2)$. Combined with the interpolation inequality (2.1), convergence in $C([0,T];L^2)$ and the uniform bound in $L^\infty([0,T];H^5)$ implies convergence in $C([0,T];H^{s'})$ for any $s' \in [0,5)$.

For any $t \in [0,T]$, the sequence $y^\delta(\cdot,t)$ is bounded in the H^5, which is a Hilbert space. Since closed balls in Hilbert spaces are weakly compact, at each time $t\in[0,T]$, there is a weak limit in H^5. By uniqueness of limits, this limit must equal $y(\cdot,t)$. Thus we may also conclude that y is in H^5 at every time $t \in [0,T]$.

We claim that this $y \in H^5$ solves (1.8). Indeed, for all $\delta > 0$, the integral representation of the solution to (3.1) is in hand,

$$y^\delta(x,t) = y_0^\delta(x) + \int_0^t F(y^\delta)(x,s)ds, \tag{3.24}$$

where $F(\cdot)$ is defined in (3.17). The integrand in the right-hand side consists of continuous terms with functions $\partial_x^s y^\delta$, $0 \le s \le 4$, within. Therefore in (3.24), using the regularity we have established, there is no difficulty in passing to limit on the subsequence. Since $y^\delta \to y$ as $\delta \to 0$, that means

$$y(x,t) = y_0(x) + \int_0^t F(y)(s)ds. \tag{3.25}$$

This immediately implies that (3.25) satisfies (1.8). ■

4. Asymptotics. In this section we show that, in a special scaling limit, solutions of the system (1.8) and solutions of the Kuramoto–Sivashinsky equation (2.16) shadow one another over a time period dependent on initial values of both equations. To begin, we fix an $0 < \epsilon \ll 1$ and assume

$$\alpha = 1 + \epsilon.$$

Then we use the change of variables

$$(\xi, \tau) = \left(\epsilon^{\frac{1}{2}} x, \epsilon^2 t\right), \quad y(x,t) = \epsilon\Phi(\sqrt{\epsilon}x, \epsilon^2 t) - t. \tag{4.1}$$

A straightforward calculation transfers (1.8) into new variables as follows:

$$\begin{aligned}
&y_t = \epsilon^3\Phi_\tau - 1, \quad y_x = \epsilon^{\frac{3}{2}}\Phi_\xi, \quad \frac{(\alpha-1)y_{xx}}{1+y_x^2} = \frac{\epsilon^3\Phi_{\xi\xi}}{1+\epsilon^3(\Phi_\xi)^2},\\
&\sqrt{1+(y_x)^2} = \sqrt{1+\epsilon^3(\Phi_\xi)^2},\\
&\frac{1}{\sqrt{1+y_x^2}}\cdot\frac{d^2\kappa}{dx^2} = \frac{\epsilon^3\Phi_{\xi\xi\xi\xi}}{\left(1+(\epsilon^{\frac{3}{2}}\Phi_\xi)^2\right)^2} - \frac{3\epsilon^6(\Phi_{\xi\xi})^3 + 9\epsilon^6\Phi_\xi\Phi_{\xi\xi}\Phi_{\xi\xi\xi}}{\left(1+(\epsilon^{\frac{3}{2}}\Phi_\xi)^2\right)^3} + \frac{15\epsilon^9(\Phi_\xi)^2(\Phi_{\xi\xi})^3}{\left(1+(\epsilon^{\frac{3}{2}}\Phi_\xi)^2\right)^4},
\end{aligned}$$

and

$$y_x\cdot\kappa\cdot\frac{d\kappa}{dx} = \frac{y_x y_{xx} y_{xxx}}{(1+y_x^2)^3} - \frac{3(y_x)^2(y_{xx})^3}{(1+y_x^2)^4} = \epsilon^6\frac{\Phi_\xi\Phi_{\xi\xi}\Phi_{\xi\xi\xi}}{(1+\epsilon^3(\Phi_\xi)^2)^3} - \frac{3\epsilon^9(\Phi_\xi)^2(\Phi_{\xi\xi})^3}{(1+\epsilon^3(\Phi_\xi)^2)^4}.$$

Then

$$\begin{aligned}
&\epsilon^3\Phi_\tau + \frac{\epsilon^3\Phi_{\xi\xi}}{1+\epsilon^3(\Phi_\xi)^2} + \alpha^2(\alpha+3)\frac{\epsilon^3\Phi_{\xi\xi\xi\xi}}{(1+\epsilon^3(\Phi_\xi)^2)^2} + \sqrt{1+\epsilon^3(\Phi_\xi)^2} - 1\\
&\quad = 10\alpha^2(\alpha+3)\frac{\epsilon^6\Phi_\xi\Phi_{\xi\xi}\Phi_{\xi\xi\xi}}{(1+\epsilon^3(\Phi_\xi)^2)^3} + 3\alpha^2(\alpha+3)\frac{\epsilon^6(\Phi_{\xi\xi})^3}{(1+\epsilon^3(\Phi_\xi)^2)^3}\\
&\quad - 18\alpha^2(\alpha+3)\frac{\epsilon^9(\Phi_{\xi\xi})^3(\Phi_\xi)^2}{(1+\epsilon^3(\Phi_\xi)^2)^4} - \left(1+\frac{1}{2}\alpha^2\right)\frac{\epsilon^4(\Phi_{\xi\xi})^2}{(1+\epsilon^3(\Phi_\xi)^2)^{\frac{5}{2}}}\\
&\quad - \left(2\alpha + 5\alpha^2 - \frac{1}{3}\alpha^3\right)\frac{\epsilon^6(\Phi_{\xi\xi})^3}{(1+\epsilon^3(\Phi_\xi)^2)^4}.
\end{aligned}$$

This leads to

$$(4.2)\quad \begin{cases} \Phi_\tau + \alpha^2(\alpha+3)\frac{\Phi_{\xi\xi\xi\xi}}{(1+\epsilon^3(\Phi_\xi)^2)^2} = -\frac{\Phi_{\xi\xi}}{1+\epsilon^3(\Phi_\xi)^2} - \frac{(\Phi_\xi)^2}{1+\sqrt{1+\epsilon^3(\Phi_\xi)^2}} + 10\alpha^2(\alpha+3)\frac{\epsilon^3\Phi_\xi\Phi_{\xi\xi}\Phi_{\xi\xi\xi}}{(1+\epsilon^3(\Phi_\xi)^2)^3} \\ \quad + 3\alpha^2(\alpha+3)\frac{\epsilon^3(\Phi_{\xi\xi})^3}{(1+\epsilon^3(\Phi_\xi)^2)^3} - 18\alpha^2(\alpha+3)\frac{\epsilon^6(\Phi_{\xi\xi})^3(\Phi_\xi)^2}{(1+\epsilon^3(\Phi_\xi)^2)^4} - \left(1+\frac{1}{2}\alpha^2\right)\frac{\epsilon(\Phi_{\xi\xi})^2}{(1+\epsilon^3(\Phi_\xi)^2)^{\frac{5}{2}}} \\ \quad - \left(2\alpha+5\alpha^2-\frac{1}{3}\alpha^3\right)\frac{\epsilon^3(\Phi_{\xi\xi})^3}{(1+\epsilon^3(\Phi_\xi)^2)^4}. \end{cases}$$

In the above, if we put $\epsilon = 0$ and $\alpha = 1$ we arrive at

$$\Phi_\tau + 4\Phi_{\xi\xi\xi\xi} = -\Phi_{\xi\xi} - \frac{1}{2}\Phi_\xi^2,$$

which is just (2.16) with a new variable name. It is worth noting that putting $f(x,t) = \epsilon U(\sqrt{\epsilon}x, \epsilon^2 t)$ transfers (1.1) into (2.16) as well. The point of this section is to make rigorous the comparison of solutions of (4.2) to those of (2.16) when ϵ is small.

4.1. Some a priori estimate for the function $\Phi(\xi,\tau)$.

We now turn our attention to some bounds for the solution Φ of (4.2) in Sobolev spaces. Specifically we have the following lemma.

Lemma 4.1. *Fix τ_* and Γ_*. Then there exists constants E_* and ϵ_* so that if $\|\Phi(0)\|_{H^4} \le E_*$ and $0 < \epsilon < \epsilon_*$ and if $|\alpha - 1| = \epsilon$, then*

$$(4.3)\quad \sup_{0<\tau<\tau_*} \|\Phi(\tau)\|_{H^4} \le \Gamma_*.$$

Before the proof, note that this lemma tells us that after unraveling the scaling from (4.1) to go from Φ back to y, we find that the solution $y(x,t)$ exists on the time interval $[0, \tau_*/\epsilon^2]$, far longer than the times of existence we found in the previous section.

Proof. The proof goes by adding up two energy estimates together, one on $\|\Phi\|_{L^2}$ and the other on $\|\partial_x^4\Phi\|_{L^2}$. We first multiply (4.2) into Φ and take the integral to get the following energy estimate:

$$\begin{aligned} &\frac{1}{2}\partial_t\|\Phi\|_{L^2}^2 + \alpha^2(\alpha+3)\|\frac{\Phi_{\xi\xi}}{1+\epsilon^3(\phi_\xi)^2}\|_{L^2}^2 = -\int \frac{\Phi_{\xi\xi}\Phi}{1+\epsilon^3(\phi_\xi)^2}d\xi - \int \frac{\Phi(\Phi_\xi)^2}{1+\sqrt{1+\epsilon^3(\Phi_\xi)^2}}d\xi \\ &-\left(1+\frac{1}{2}\alpha^2\right)\epsilon\int \frac{\Phi(\Phi_{\xi\xi})^2}{(1+\epsilon^3(\Phi_\xi)^2)^{\frac{5}{2}}}d\xi + 10\alpha^2(\alpha+3)\epsilon^3\int \frac{\Phi\Phi_\xi\Phi_{\xi\xi}\Phi_{\xi\xi\xi}}{(1+\epsilon^3(\phi_\xi)^2)^3}d\xi \\ &+4\alpha^2(\alpha+3)\epsilon^3\int \frac{(\Phi_{\xi\xi})^2(\Phi_\xi)^2}{(1+\epsilon^3(\phi_\xi)^2)^3}d\xi - 4\epsilon^3\alpha^2(\alpha+3)\int \Phi_{\xi\xi}\partial_\xi\left[\frac{\Phi\Phi_\xi\Phi_{\xi\xi}}{(1+\epsilon^3(\Phi_\xi)^2)^3}\right]d\xi \\ &+3\alpha^2(\alpha+3)\epsilon^3\int \frac{(\Phi_{\xi\xi})^3\Phi}{(1+\epsilon^3(\phi_\xi)^2)^3}d\xi - \left(2\alpha+5\alpha^2-\frac{1}{3}\alpha^3\right)\epsilon^3\int \frac{\Phi(\Phi_{\xi\xi})^3}{(1+\epsilon^3(\Phi_\xi)^2)^4}d\xi \\ &-18\alpha^2(\alpha+3)\epsilon^6\int \frac{(\Phi_{\xi\xi})^3(\Phi_\xi)^2\Phi}{(1+\epsilon^3(\phi_\xi)^2)^4}d\xi := I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8+I_9. \end{aligned}$$

Above we made a simplification on one of the integrals using integration by parts,

$$\int \frac{\partial_\xi^4\Phi\cdot\Phi}{(1+\epsilon^3(\phi_\xi)^2)^2}d\xi = \int \frac{(\partial_{\xi\xi}\Phi)^2}{(1+\epsilon^3(\phi_\xi)^2)^2}d\xi - 4\epsilon^3\int \frac{(\Phi_{\xi\xi})^2(\Phi_\xi)^2}{(1+\epsilon^3(\phi_\xi)^2)^3}d\xi - 4\epsilon^3\int \Phi_{\xi\xi}\partial_\xi\left[\frac{\Phi\Phi_\xi\Phi_{\xi\xi}}{(1+\epsilon^3(\Phi_\xi)^2)^3}\right]d\xi.$$

Although this energy estimate is already in a nice form, and we can run the argument, we are still able to simplify the left-hand side, which itself reduces many calculations. In fact we make use of the bounds in Lemma 3.4 and the scaling (4.1)

$$\left\|\frac{1}{1+\epsilon^3(\Phi_\xi)^2}\right\|_{L^\infty} = \left\|\frac{1}{1+(y_x)^2}\right\|_{L^\infty} \geq C_0 \tag{4.4}$$

for some $C_0 > 0$ fixed. Therefore, the above energy estimates turns into

$$\frac{1}{2}\partial_t\|\Phi\|_{L^2}^2 + C\alpha^2(\alpha+3)\|\Phi_{\xi\xi}\|_{L^2}^2 \leq \left|I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8\right|. \tag{4.5}$$

We now try to find a proper bound for the right-hand side of this equality.

Estimate for I_1.

$$|I_1| = \left|\int \frac{\Phi_{\xi\xi}\Phi}{1+\epsilon^3(\phi_\xi)^2}d\xi\right| \leq C\|\Phi\|_{L^2}\|\Phi_{\xi\xi}\|_{L^2} \leq \|\Phi\|_{L^2}\left(\|\Phi\|_{L^2}^{\frac{1}{2}}\|\partial^4\Phi\|_{L^2}^{\frac{1}{2}}\right) \leq C\left(\|\Phi\|_{L^2}^2 + \|\partial^4\Phi\|_{L^2}^2\right)$$

Estimate for I_2.

$$\begin{aligned}|I_2| &= \left|\int \frac{\Phi(\Phi_\xi)^2}{1+\sqrt{1+\epsilon^3(\Phi_\xi)^2}}d\xi\right| \leq \left\|\frac{1}{1+\sqrt{1+\epsilon^3(\Phi_\xi)^2}}\right\|_{L^\infty}\|\Phi\|_{L^\infty}\|\Phi_\xi\|_{L^2}^2 \\ &\leq C\||\nabla|^{\frac{1}{2}}\Phi\|_{L^2}\|\Phi_\xi\|_{L^2}^2 \leq C\left(\|\Phi\|_{L^2}^{\frac{7}{8}}\|\partial^4\Phi\|_{L^2}^{\frac{1}{8}}\right)\left(\|\Phi\|_{L^2}^{\frac{3}{4}}\|\partial_\xi^4\Phi\|_{L^2}^{\frac{1}{4}}\right)^2 \\ &\leq C\left(\|\Phi\|_{L^2}^2 + \|\partial_\xi^4\Phi\|_{L^2}^2\right) + C\left(\|\Phi\|_{L^2}^4 + \|\partial_\xi^4\Phi\|_{L^2}^4\right).\end{aligned}$$

Estimate for I_3.

$$\begin{aligned}|I_3| &= \epsilon\left(1+\frac{1}{2}\alpha^2\right)\left|\int \frac{\Phi(\Phi_{\xi\xi})^2}{(1+\epsilon^3(\Phi_\xi)^2)^{\frac{5}{2}}}d\xi\right| \leq \epsilon\|\Phi\|_{L^2}\|\Phi_{\xi\xi}\|_{L^2}^2\left\|\frac{1}{(1+\epsilon^3(\Phi_\xi)^2)^{\frac{5}{2}}}\right\|_{L^\infty} \\ &\leq \epsilon\|\Phi\|_{L^2}\left(\|\Phi\|_{L^2}^{\frac{1}{2}}\|\partial^4\Phi\|_{L^2}^{\frac{1}{2}}\right)^2 \leq C\epsilon\left(\|\Phi\|_{L^2}^3 + \|\partial^4\Phi\|_{L^2}^3\right) \\ &\leq \left(\|\Phi\|_{L^2}^2 + \|\partial^4\Phi\|_{L^2}^2\right) + C\epsilon^2\left(\|\Phi\|_{L^2}^4 + \|\partial^4\Phi\|_{L^2}^4\right).\end{aligned}$$

Estimate for I_4.

$$\begin{aligned}|I_4| &= 6\alpha^2(\alpha+3)\epsilon^3\left|\int \frac{\Phi\Phi_\xi\Phi_{\xi\xi}\Phi_{\xi\xi\xi}}{(1+\epsilon^3(\phi_\xi)^2)^3}d\xi\right| \\ &\leq \epsilon^3\left\|\frac{1}{(1+\epsilon^3(\Phi_\xi)^2)^3}\right\|_{L^\infty}\|\Phi\|_{L^\infty}\|\Phi_\xi\|_{L^\infty}\|\Phi_{\xi\xi}\|_{L^2}\|\Phi_{\xi\xi\xi}\|_{L^2} \\ &\leq C\epsilon^3\||\nabla|^{\frac{1}{2}}\Phi\|_{L^2}\||\nabla|^{\frac{1}{2}}\Phi_\xi\|_{L^2}\|\Phi_{\xi\xi}\|_{L^2}\|\Phi_{\xi\xi\xi}\|_{L^2} \\ &\leq C\epsilon^3\left(\|\Phi\|_{L^2}^{\frac{7}{8}}\|\partial_\xi^4\Phi\|_{L^2}^{\frac{1}{8}}\right)\left(\|\Phi\|_{L^2}^{\frac{5}{8}}\|\partial_\xi^4\Phi\|_{L^2}^{\frac{3}{8}}\right)\left(\|\Phi\|_{L^2}^{\frac{1}{2}}\|\partial_\xi^4\Phi\|_{L^2}^{\frac{1}{2}}\right)\left(\|\Phi\|_{L^2}^{\frac{1}{4}}\|\partial_\xi^4\Phi\|_{L^2}^{\frac{3}{4}}\right) \\ &\leq C\epsilon^3\left(\|\Phi\|_{L^2}^4 + \|\partial_\xi^4\Phi\|_{L^2}^4\right).\end{aligned}$$

Estimate for I_5.

$$
\begin{aligned}
|I_5| &\leq 4\epsilon^3\alpha^2(\alpha+3)\left|\int \Phi_{\xi\xi}\partial_\xi\left[\frac{\Phi\Phi_\xi\Phi_{\xi\xi}}{(1+\epsilon^3(\Phi_\xi)^2)^3}\right]d\xi\right| \\
&\leq C\epsilon^3\|\Phi_{\xi\xi}\|_{L^2}\|\Phi_{\xi\xi\xi}\|_{L^2}\|\Phi\|_{L^\infty}\|\Phi_\xi\|_{L^\infty}\|\frac{1}{1+\epsilon^3(\Phi_\xi)^2}\|_{L^\infty} \\
&\leq C\epsilon^3\|\Phi_{\xi\xi}\|_{L^2}\|\Phi_{\xi\xi\xi}\|_{L^2}\||\nabla|^{\frac{1}{2}}\Phi\|_{L^2}\||\nabla|^{\frac{3}{2}}\Phi\|_{L^2} \\
&\leq C\epsilon^3\left(\|\Phi\|_{L^2}^{\frac{1}{2}}\|\partial_\xi^4\Phi\|_{L^2}^{\frac{1}{2}}\right)\left(\|\Phi\|_{L^2}^{\frac{1}{4}}\|\partial_\xi^4\Phi\|_{L^2}^{\frac{3}{4}}\right)\left(\|\Phi\|_{L^2}^{\frac{7}{8}}\|\partial_\xi^4\Phi\|_{L^2}^{\frac{1}{8}}\right)\left(\|\Phi\|_{L^2}^{\frac{5}{8}}\|\partial^4\Phi\|_{L^2}^{\frac{3}{8}}\right) \\
&\leq C\epsilon^3\left(\|\Phi\|_{L^2}^4+\|\partial_\xi^4\Phi\|_{L^2}^4\right).
\end{aligned}
$$

Estimate for I_6.

$$
\begin{aligned}
|I_6| &= 4\alpha^2(\alpha+3)\epsilon^3\left|\int\frac{(\Phi_{\xi\xi})^2(\Phi_\xi)^2}{(1+\epsilon^3(\phi_\xi)^2)^3}d\xi\right| \leq \epsilon^3\left\|\frac{1}{(1+\epsilon^3(\Phi_\xi)^2)^3}\right\|_{L^\infty}\|\Phi_{\xi\xi}\|_{L^2}^2\|\Phi_\xi\|_{L^\infty}^2 \\
&\leq C\epsilon^3\|\Phi_{\xi\xi}\|_{L^2}^2\||\nabla|^{\frac{1}{2}}\Phi_\xi\|_{L^2}^2 \leq C\epsilon^3\left(\|\Phi\|_{L^2}^{\frac{1}{2}}\|\partial^4\Phi\|_{L^2}^{\frac{1}{2}}\right)^2\left(\|\Phi\|_{L^2}^{\frac{5}{8}}\|\partial^4\Phi\|_{L^2}^{\frac{3}{8}}\right)^2 \\
&\leq \epsilon^3\left(\|\Phi\|_{L^2}^4+\|\partial^4\Phi\|_{L^2}^4\right).
\end{aligned}
$$

Estimate for I_7.

$$
\begin{aligned}
|I_6| &= 3\alpha^2(\alpha+3)\epsilon^3\left|\int\frac{(\Phi_{\xi\xi})^3\Phi}{(1+\epsilon^3(\phi_\xi)^2)^3}d\xi\right| \leq C\epsilon^3\left\|\frac{1}{(1+\epsilon^3(\Phi_\xi)^2)^3}\right\|_{L^\infty}\|\Phi_{\xi\xi}\|_{L^3}^3\|\Phi\|_{L^\infty} \\
&\leq C\epsilon^3\||\nabla|^{\frac{13}{6}}\Phi\|_{L^2}^3\||\nabla|^{\frac{1}{2}}\Phi\|_{L^2} \leq C\epsilon^3\left(\|\Phi\|_{L^2}^{\frac{11}{24}}\|\partial^4\Phi\|_{L^2}^{\frac{13}{24}}\right)^3\left(\|\Phi\|_{L^2}^{\frac{7}{8}}\|\partial^4\Phi\|_{L^2}^{\frac{1}{8}}\right) \\
&\leq C\epsilon^3\left(\|\Phi\|_{L^2}^4+\|\partial^4\Phi\|_{L^2}^4\right).
\end{aligned}
$$

Estimate for I_8. This is similar to I_7.
Estimate for I_9.

$$
\begin{aligned}
|I_9| &= 18\alpha^2(\alpha+3)\epsilon^6\left|\int\frac{(\Phi_{\xi\xi})^3(\Phi_\xi)^2\Phi}{(1+\epsilon^3(\phi_\xi)^2)^4}d\xi\right| \leq C\left\|\frac{\epsilon^3(\Phi_\xi)^2}{(1+\epsilon^3(\Phi_\xi)^2)^4}\right\|_{L^\infty}\left(\epsilon^3\|\Phi_{\xi\xi}\|_{L^3}^3\|\Phi\|_{L^\infty}\right) \\
&\leq C\epsilon^3\||\nabla|^{\frac{13}{6}}\Phi\|_{L^2}^3\||\nabla|^{\frac{1}{2}}\Phi\|_{L^2} \leq C\epsilon^3\left(\|\Phi\|_{L^2}^{\frac{11}{24}}\|\partial_\xi^4\Phi\|_{L^2}^{\frac{13}{24}}\right)^3\left(\|\Phi\|_{L^2}^{\frac{7}{8}}\|\partial_\xi^4\Phi\|_{L^2}^{\frac{1}{8}}\right) \\
&\leq C\epsilon^3\left(\|\Phi\|_{L^2}^4+\|\partial_\xi^4\Phi\|_{L^2}^4\right).
\end{aligned}
$$

Overall, the energy estimate (4.5) is transfered into

$$
\frac{1}{2}\partial_t\|\Phi\|_{L^2}^2 \leq C\left(\|\Phi\|_{L^2}^2+\|\partial_\xi^4\Phi\|_{L^2}^2\right)+C\left(\|\Phi\|_{L^2}^4+\|\partial_\xi^4\Phi\|_{L^2}^4\right). \tag{4.6}
$$

As mentioned, our argument is based on a combined energy estimate. For the other term in the energy estimate, we take 4 times derivative of (4.2), and then find the inner product of the resulting equation with $\partial_\xi^4\Phi$,

(4.7)
$$\frac{1}{2}\partial_t\|\partial_\xi^4\Phi\|_{L^2}^2+\alpha^2(\alpha+3)\int\partial_\xi^4\left[\frac{\partial_\xi^4\Phi}{(1+\epsilon^3(\Phi_\xi)^2)^2}\right]\partial_\xi^4\Phi d\xi=-\int\partial_\xi^4\left[\frac{\Phi_{\xi\xi}}{1+\epsilon^3(\Phi_\xi)^2}\right]\partial_\xi^4\Phi d\xi$$
$$-\int\partial_\xi^4\left[\frac{(\Phi_\xi)^2}{1+\sqrt{1+\epsilon^3(\Phi_\xi)^2}}\right]\partial_\xi^4\Phi d\xi+10\alpha^2(\alpha+3)\epsilon^3\int\partial_\xi^4\left[\frac{\Phi_\xi\Phi_{\xi\xi}\Phi_{\xi\xi\xi}}{(1+\epsilon^3(\phi_\xi)^2)^3}\right]\partial_\xi^4\Phi d\xi$$
$$+3\alpha^2(\alpha+3)\epsilon^3\int\partial_\xi^4\left[\frac{(\Phi_{\xi\xi})^3}{(1+\epsilon^3(\phi_\xi)^2)^3}\right]\partial_\xi^4\Phi d\xi-18\alpha^2(\alpha+3)\epsilon^6\int\partial_\xi^4\left[\frac{(\Phi_{\xi\xi})^3(\Phi_\xi)^2}{(1+\epsilon^3(\phi_\xi)^2)^3}\right]\partial_\xi^4\Phi d\xi$$
$$-\left(1+\frac{1}{2}\alpha^2\right)\epsilon\int\partial_\xi^4\left[\frac{(\Phi_{\xi\xi})^2}{(1+\epsilon^3(\Phi_\xi)^2)^{\frac{5}{2}}}\right]\partial_\xi^4\Phi d\xi-\left(2\alpha+5\alpha^2-\frac{1}{3}\alpha^3\right)\epsilon^3\int\partial_\xi^4\left[\frac{(\Phi_{\xi\xi})^3}{(1+\epsilon^3(\Phi_\xi)^2)^4}\right]\partial_\xi^4\Phi d\xi.$$

Although we can simplify most of the terms in this relation, we leave most of them in the current form, as they are easily bounded in the current form. However,

$$\alpha^2(\alpha+3)\int\partial_\xi^4[\frac{\partial_\xi^4\Phi}{(1+\epsilon^3(\Phi_\xi)^2)^2}]\partial_\xi^4\Phi d\xi=\alpha^2(\alpha+3)\int\frac{(\partial_\xi^6\Phi)^2}{(1+\epsilon^3(\Phi_\xi)^2)^2}d\xi$$
$$-4\epsilon^3\alpha^2(\alpha+3)\int\frac{(\partial_\xi^6\Phi)(\partial_\xi^5\Phi)\Phi_\xi\Phi_{\xi\xi}}{(1+\epsilon^3(\Phi_\xi)^2)^3}d\xi-4\epsilon^3\alpha^2(\alpha+3)\int(\partial_\xi^6\Phi)(\partial_\xi^4\Phi)\partial_\xi\left[\frac{\Phi_\xi\Phi_{\xi\xi}}{(1+\epsilon^3(\Phi_\xi)^2)^3}\right]d\xi.$$

Then, the energy estimates (4.7) turns into

(4.8)
$$\frac{1}{2}\partial_t\|\partial_\xi^4\Phi\|_{L^2}^2+\alpha^2(\alpha+3)\int\frac{(\partial_\xi^6\Phi)^2}{(1+\epsilon^3(\Phi_\xi)^2)^2}d\xi=-\int\partial_\xi^6\Phi\cdot\partial_\xi^2\left[\frac{\Phi_{\xi\xi}}{1+\epsilon^3(\Phi_\xi)^2}\right]d\xi$$
$$-\int\partial_\xi^6\Phi\cdot\partial_\xi^2\left[\frac{(\Phi_\xi)^2}{1+\sqrt{1+\epsilon^3(\Phi_\xi)^2}}\right]d\xi-\left(1+\frac{1}{2}\alpha^2\right)\epsilon\int\partial_\xi^6\Phi\cdot\partial_\xi^2\left[\frac{(\Phi_{\xi\xi})^2}{(1+\epsilon^3(\Phi_\xi)^2)^{\frac{5}{2}}}\right]d\xi$$
$$+6\alpha^2(\alpha+3)\epsilon^3\int\partial_\xi^6\Phi\cdot\partial_\xi^2\left[\frac{\Phi_\xi\Phi_{\xi\xi}\Phi_{\xi\xi\xi}}{(1+\epsilon^3(\phi_\xi)^2)^3}\right]\partial_\xi^4\Phi d\xi+3\alpha^2(\alpha+3)\epsilon^3\int\partial_\xi^6\Phi\cdot\partial_\xi^2\left[\frac{(\Phi_{\xi\xi})^3}{(1+\epsilon^3(\phi_\xi)^2)^3}\right]d\xi$$
$$+4\alpha^2(\alpha+3)\epsilon^3\int\frac{(\partial^6\Phi)(\partial^5\Phi)(\Phi_\xi)(\Phi_{\xi\xi})}{(1+\epsilon^3(\Phi_\xi)^2)^3}d\xi+4\alpha^2(\alpha+3)\epsilon^3\int(\partial_\xi^6\Phi)(\partial^4\Phi)\partial_\xi\left[\frac{(\Phi_\xi)(\Phi_{\xi\xi})}{(1+\epsilon^3(\Phi_\xi)^2)^3}\right]d\xi$$
$$-\left(2\alpha+5\alpha^2-\frac{1}{3}\alpha^3\right)\epsilon^3\int\partial_\xi^6\Phi\cdot\partial_\xi^2\left[\frac{(\Phi_{\xi\xi})^3}{(1+\epsilon^3(\Phi_\xi)^2)^4}\right]d\xi$$
$$-18\alpha^2(\alpha+3)\epsilon^6\int\partial_\xi^6\Phi\cdot\partial_\xi^2\left[\frac{(\Phi_{\xi\xi})^3(\Phi_\xi)^2}{(1+\epsilon^3(\phi_\xi)^2)^4}\right]d\xi:=J_1+\cdots+J_9.$$

As we argued in (4.5), we work with a simpler version of this energy estimate,

$$\frac{1}{2}\partial_t\|\partial_x^4\Phi\|_{L^2}^2+C_0\alpha^2(\alpha+3)\int\|\partial_\xi^6\Phi\|_{L^2}d\xi\le\left|J_1+\cdots+J_9\right|. \tag{4.9}$$

As stated before, in each term we present the bound for the worst part of the integral, and that happens when in the integrand, the two derivatives hit the highest degree in the numerator of the fraction. We denote the rest of the terms $G(\tau)$, where G stands for good terms. One type of such (good) terms arises when derivatives hit the denominator. Any time a derivative is applied to the denominator, which is of the form $\frac{1}{(1+\epsilon^3(\Phi_\xi)^2)^a}$, it multiplies the integrand in $\frac{\epsilon^3\Phi_\xi\Phi_{\xi\xi}}{(1+\epsilon^3(\Phi_\xi)^2)^{a+1}}$. These kind of terms are controlled in the following way:

$$\left\|\frac{\epsilon^3\Phi_\xi\Phi_{\xi\xi}}{\left(1+\epsilon^3(\Phi_\xi)^2\right)^{a+1}}\right\|_{L^\infty} \le \epsilon^{\frac{3}{2}}\left\|\frac{\epsilon^{\frac{3}{2}}\Phi_\xi}{\left(1+\epsilon^3(\Phi_\xi)^2\right)^{a+1}}\right\|_{L^\infty}\|\Phi_{\xi\xi}\|_{L^\infty}$$
$$\le C\epsilon^{\frac{3}{2}}\|\Phi_{\xi\xi}\|_{L^\infty} \le C\epsilon^{\frac{3}{2}}\||\nabla|^{\frac{5}{2}}\Phi\|_{L^2} \le C\epsilon^{\frac{3}{2}}\|\Phi\|_{L^2}^{\frac{3}{2}}\|\partial^4\Phi\|_{L^2}^{\frac{5}{8}}.$$

Although this might increase the power of $\|\Phi\|_{L^2}$ and $\|\partial_x^4\Phi\|_{L^2}$ in our final calculations, it also adds the power ϵ in front of every such term, which fits our Grönwall's inequality (2.2). For the rest of the proof, we ignore the good terms $G(\tau)$, and in each integral in (4.8), we present the proper bound for the worst term.

Estimate for J_1.

$$|J_1| = \left|\int \partial_\xi^6\Phi\cdot\partial_\xi^2\left[\frac{\Phi_{\xi\xi}}{1+\epsilon^3(\Phi_\xi)^2}\right]d\xi\right| \le C\|\partial_\xi^4\Phi\|_{L^2}\|\partial_\xi^6\Phi\|_{L^2}\left\|\frac{1}{1+\epsilon^3(\Phi_\xi)^2}\right\|_{L^\infty} + G(\tau)$$
$$\le \frac{C_0\alpha^2(\alpha+3)}{100}\|\partial_\xi^6\Phi\|_{L^2}^2 + C\left(\|\Phi\|_{L^2}^2 + \|\partial_\xi^4\Phi\|_{L^2}^2\right) + G(\tau).$$

Estimate for J_2.

$$|J_2| = \left|\int \partial_\xi^6\Phi\cdot\partial_\xi^2\left[\frac{(\Phi_\xi)^2}{1+\sqrt{1+\epsilon^3(\Phi_\xi)^2}}\right]d\xi\right| \le \|\partial_\xi^6\Phi\|_{L^2}\left\|\frac{1}{1+\sqrt{1+\epsilon^3(\Phi_\xi)^2}}\right\|_{L^\infty}\|\Phi_\xi\|_{L^\infty}\|\partial_\xi^3\Phi\|_{L^2} + G(\tau)$$
$$\le \|\partial_\xi^6\Phi\|_{L^2}\||\nabla|^{\frac{3}{2}}\Phi\|_{L^2}\|\partial_\xi^3\Phi\|_{L^2} \le \|\partial_\xi^6\Phi\|_{L^2}\left(\|\Phi\|_{L^2}^{\frac{5}{8}}\|\partial_\xi^4\Phi\|_{L^2}^{\frac{3}{8}}\right)\left(\|\Phi\|_{L^2}^{\frac{1}{4}}\|\partial_\xi^4\Phi\|_{L^2}^{\frac{3}{4}}\right) + G(\tau)$$
$$\le \frac{C_0\alpha^2(\alpha+3)}{100}\|\partial_x^6\Phi\|_{L^2}^2 + C\left(\|\partial_\xi^4\Phi\|_{L^2}^4 + \|\Phi\|_{L^2}^4\right) + G(\tau).$$

Estimate for J_3.

$$|J_3| = \left(1+\frac{1}{2}\alpha^2\right)\epsilon\left|\int \partial_\xi^6\Phi\cdot\partial_\xi^2\left[\frac{(\Phi_{\xi\xi})^2}{(1+\epsilon^3(\Phi_\xi)^2)^{\frac{5}{2}}}\right]d\xi\right| \le C\epsilon\|\partial_\xi^6\Phi\|_{L^2}\|\Phi_{\xi\xi}\|_{L^\infty}\|\partial_\xi^4\Phi\|_{L^2} + G(\tau)$$
$$\le C\epsilon\|\partial_\xi^6\Phi\|_{L^2}\||\nabla|^{\frac{5}{2}}\Phi\|_{L^2}\|\partial_\xi^4\Phi\|_{L^2} + G(\tau) \le C\epsilon\|\partial_\xi^6\Phi\|_{L^2}\left(\|\Phi\|_{L^2}^{\frac{3}{8}}\|\partial_\xi^4\Phi\|_{L^2}^{\frac{5}{8}}\right)\|\partial_\xi^4\Phi\|_{L^2} + G(\tau)$$
$$\le \frac{C_0\alpha^2(\alpha+3)}{100}\|\partial_x^6\Phi\|_{L^2}^2 + C\epsilon^2\left(\|\Phi\|_{L^2}^4 + \|\partial^4\Phi\|_{L^2}^4\right) + G(\tau).$$

Estimate for J_4.

$$|J_4| = 6\alpha^2(\alpha+3)\epsilon^3\left|\int \partial_\xi^6\Phi\cdot\partial_\xi^2\left[\frac{\Phi_\xi\Phi_{\xi\xi}\Phi_{\xi\xi\xi}}{(1+\epsilon^3(\phi_\xi)^2)^3}\right]d\xi\right|$$
$$\le \epsilon^3\|\partial_\xi^6\Phi\|_{L^2}\|\Phi_\xi\|_{L^\infty}\|\Phi_{\xi\xi}\|_{L^\infty}\|\partial_\xi^5\Phi\|_{L^2} + G(\tau)$$
$$\le \epsilon^3\|\partial_\xi^6\Phi\|_{L^2}\||\nabla|^{\frac{3}{2}}\Phi\|_{L^2}\||\nabla|^{\frac{5}{2}}\Phi\|_{L^2}\|\partial_\xi^5\Phi\|_{L^2}$$

$$\leq \epsilon^3\|\partial_\xi^6\Phi\|_{L^2}\left(\|\Phi\|_{L^2}^{\frac{5}{8}}\|\partial_\xi^4\Phi\|_{L^2}^{\frac{3}{8}}\right)\left(\|\Phi\|_{L^2}^{\frac{3}{8}}\|\partial_\xi^4\Phi\|_{L^2}^{\frac{5}{8}}\right)\left(\|\partial_\xi^4\Phi\|_{L^2}^{\frac{1}{2}}\|\partial\xi^6\Phi\|_{L^2}^{\frac{1}{2}}\right)+G(\tau)$$
$$\leq C\epsilon^3\|\partial_\xi^6\Phi\|_{L^2}^{\frac{3}{2}}\left(\|\Phi\|_{L^2}\|\partial_\xi^4\Phi\|_{L^2}^{\frac{3}{2}}\right)+G(\tau)$$
$$\leq \frac{C_0\alpha^2(\alpha+3)}{100}\|\partial_\xi^6\Phi\|_{L^2}^2+\epsilon^{12}\left(\|\Phi\|_{L^2}^4\|\partial_\xi^4\Phi\|_{L^2}^6\right)+G(\tau)$$
$$\leq \frac{C_0\alpha^2(\alpha+3)}{100}\|\partial_\xi^6\Phi\|_{L^2}^2+\epsilon^{12}\left(\|\Phi\|_{L^2}^{10}+\|\partial_\xi^4\Phi\|_{L^2}^{10}\right)+G(\tau).$$

Estimate for J_5.

$$|J_5|=3\alpha^2(\alpha+3)\epsilon^3\left|\int\partial_\xi^6\Phi\cdot\partial_\xi^2\left[\frac{(\Phi_{\xi\xi})^3}{(1+\epsilon^3(\phi_\xi)^2)^3}\right]d\xi\right|$$
$$\leq \epsilon^3\left\|\frac{1}{(1+\epsilon^3(\Phi_\xi)^2)^3}\right\|_{L^\infty}\|\partial_\xi^6\Phi\|_{L^2}\|\partial_\xi^4\Phi\|_{L^2}\|\Phi_{\xi\xi}\|_{L^\infty}^2+G(\tau)$$
$$\leq \frac{C_0\alpha^2(\alpha+3)}{100}\|\partial_\xi^6\Phi\|_{L^2}^2+C\epsilon^6\left(\|\Phi\|_{L^2}^6+\|\partial_\xi^4\Phi\|_{L^2}^6\right)+G(\tau)$$
$$\leq \frac{C_0\alpha^2(\alpha+3)}{100}\|\partial_\xi^6\Phi\|_{L^2}^2+C\left(\|\Phi\|_{L^2}^2+\|\partial_\xi^4\Phi\|_{L^2}^2\right)+C\epsilon^{12}\left(\|\Phi\|_{L^2}^{10}+\|\partial_\xi^4\Phi\|_{L^2}^{10}\right)+G(\tau).$$

Now we use the following relation.

Estimate for J_6.

$$|J_6|=4\alpha^2(\alpha+3)\epsilon^3\left|\int\frac{(\partial_\xi^6\Phi)(\partial_\xi^5\Phi)(\Phi_\xi)(\Phi_{\xi\xi})}{(1+\epsilon^3(\Phi_\xi)^2)^3}d\xi\right|$$
$$\leq C\epsilon^3\left\|\frac{1}{(1+\epsilon^3(\Phi_\xi)^2)^3}\right\|_{L^\infty}\|\partial_\xi^6\Phi\|_{L^2}\|\partial_\xi^5\Phi\|_{L^2}\|\Phi_{\xi\xi}\|_{L^\infty}\|\Phi_\xi\|_{L^\infty}+G(\tau)$$
$$\leq C\epsilon^3\|\partial_\xi^6\Phi\|_{L^2}\|\partial_\xi^5\Phi\|_{L^2}\||\nabla|^{\frac{5}{2}}\Phi\|_{L^2}\||\nabla|^{\frac{3}{2}}\Phi\|_{L^2}$$
$$\leq C\epsilon^3\|\partial_\xi^6\Phi\|_{L^2}\left(\|\partial_\xi^4\Phi\|_{L^2}^{\frac{1}{2}}\|\partial_\xi^6\Phi\|_{L^2}^{\frac{1}{2}}\right)\left(\|\Phi\|_{L^2}^{\frac{3}{8}}\|\partial_\xi^4\Phi\|_{L^2}^{\frac{5}{8}}\right)\left(\|\Phi\|_{L^2}^{\frac{5}{8}}\|\partial_\xi^4\Phi\|_{L^2}^{\frac{3}{8}}\right)+G(\tau)$$
$$\leq C\epsilon^3\|\partial_\xi^6\Phi\|_{L^2}^{\frac{3}{2}}\left(\|\Phi\|_{L^2}\|\partial_\xi^4\Phi\|_{L^2}^{\frac{3}{2}}\right)+G(\tau)$$
$$\leq \frac{C_0\alpha^2(\alpha+3)}{100}\|\partial_\xi^6\Phi\|_{L^2}^2+\epsilon^{12}\left(\|\Phi\|_{L^2}^{10}+\|\partial_\xi^4\Phi\|_{L^2}^{10}\right)+G(\tau).$$

Estimate for J_7.

$$|J_7|=4\epsilon^3\alpha^2(\alpha+3)\left|\int\partial_\xi^6\Phi\cdot\partial_\xi^4\Phi\partial_\xi\cdot\left[\frac{(\Phi_{\xi\xi})(\Phi_\xi)}{(1+\epsilon^3(\phi_\xi)^2)^3}\right]d\xi\right|$$
$$\leq C\epsilon^{\frac{3}{2}}\|\partial_\xi^6\Phi\|_{L^2}\|\partial_\xi^4\Phi\|_{L^2}\|\Phi_{\xi\xi\xi}\|_{L^\infty}\|\frac{\epsilon^{\frac{3}{2}}\Phi_\xi}{(1+\epsilon^3(\Phi_\xi))^2)^3}\|_{L^\infty}+G(\tau)$$
$$\leq C\epsilon^6\|\partial_\xi^6\Phi\|_{L^2}\|\partial_\xi^4\Phi\|_{L^2}\left(\|\Phi\|_{L^2}^{\frac{1}{8}}\|\partial_\xi^4\Phi\|_{L^2}^{\frac{7}{8}}\right)+G(\tau)$$
$$\leq \frac{C_0\alpha^2(\alpha+3)}{100}\|\partial_\xi^6\Phi\|_{L^2}^2+C\epsilon^3\left(\|\Phi\|_{L^2}^4+\|\partial_\xi^4\Phi\|_{L^2}^4\right)+G(\tau).$$

Estimate for J_8. This is similar to J_5.

Estimate for J_9.

$$
\begin{aligned}
|J_9| &= 18\alpha^2(\alpha+3)\epsilon^6 \left| \int \partial_\xi^6\Phi \cdot \partial_\xi^2 \left[\frac{(\Phi_{\xi\xi})^3(\Phi_\xi)^2}{(1+\epsilon^3(\phi_\xi)^2)^3} \right] d\xi \right| \\
&\le C\epsilon^6 \|\partial_\xi^6\Phi\|_{L^2} \|\partial_\xi^4\Phi\|_{L^2} \|\Phi_{\xi\xi}\|_{L^\infty}^2 \|\Phi_\xi\|_{L^\infty}^2 + G(\tau) \\
&\le C\epsilon^6 \|\partial_\xi^6\Phi\|_{L^2} \|\partial_\xi^4\Phi\|_{L^2} \||\nabla|^{\frac{5}{2}}\Phi\|_{L^2}^2 \||\nabla|^{\frac{3}{2}}\Phi\|_{L^2}^2 + G(\tau) \\
&\le C\epsilon^6 \|\partial_\xi^6\Phi\|_{L^2} \|\partial_\xi^4\Phi\|_{L^2} \left(\|\Phi\|_{L^2}^{\frac{3}{8}} \|\partial_\xi^4\Phi\|_{L^2}^{\frac{5}{8}} \right)^2 \left(\|\Phi\|_{L^2}^{\frac{5}{8}} \|\partial_\xi^4\Phi\|_{L^2}^{\frac{3}{8}} \right)^2 + G(\tau) \\
&\le \frac{C_0\alpha^2(\alpha+3)}{100} \|\partial_\xi^6\Phi\|_{L^2}^2 + \epsilon^{12} \left(\|\Phi\|_{L^2}^{10} + \|\partial_\xi^4\Phi\|_{L^2}^{10} \right) + G(\tau).
\end{aligned}
$$

Therefore, we can summarize the energy estimates (4.8) in the following form:

$$
\frac{1}{2}\partial_t \|\partial_\xi^4\Phi\|_{L^2}^2 \le C\left(\|\Phi\|_{L^2}^2 + \|\partial_\xi^4\Phi\|_{L^2}^2 \right) + \left(\|\Phi\|_{L^2}^4 + \|\partial_\xi^4\Phi\|_{L^2}^4 \right) + \epsilon^{12} \left(\|\Phi\|_{L^2}^{10} + \|\partial_\xi^4\Phi\|_{L^2}^{10} \right) + G(\tau). \tag{4.10}
$$

At this point we combine both energy estimates (4.6) and (4.10),

$$
\begin{aligned}
\frac{1}{2}\partial_t \left(\|\Phi\|_{L^2}^2 + \|\partial_\xi^4\Phi\|_{L^2}^2 \right) &\le C\left(\|\Phi\|_{L^2}^2 + \|\partial_\xi^4\Phi\|_{L^2}^2 \right) + C\left(\|\Phi\|_{L^2}^4 + \|\partial_\xi^4\Phi\|_{L^2}^4 \right) \\
&\quad + C\epsilon^{12} \left(\|\Phi\|_{L^2}^{10} + \|\partial_\xi^4\Phi\|_{L^2}^{10} \right) + G(\tau).
\end{aligned}
$$

Note that $G(\tau)$ is also bounded by a combination of the terms in the form of $\epsilon^a(\|\Phi\|_{L^2}^b + \|\partial_\xi^4\Phi\|_{L^2}^b)$. We define $E(t) = \|\Phi\|_{L^2}^2 + \|\partial_\xi^4\Phi\|_{L^2}^2$. Then this inequality is clearly in the form of the Grönwall's inequality in Lemma 2.2, and it finishes the proof. ∎

4.2. Asymptotics. In this section we show that the solutions of the scaled equations (2.16) and (4.2) stay close up to a time τ_*. In the previous section we established the existence of the solution of (4.2) in H^4 on a time interval $[0, \tau_*]$, under some restrictions. We also recall an important result of the global boundedness of the function $U(\xi, \tau)$ in any Sobolev spaces. This result is proved by Tadmor [33].

Lemma 4.2. *The Kuramoto–Sivashinsky equation* (2.16) *with the initial value* $U_0 \in H^4$ *admits a global smooth solution*

$$
U(\xi, \tau) \in H^4. \tag{4.11}
$$

Lemma 4.3. *Fix* $\tau_* > 0$ *and* $\Gamma_* > 0$ *and take* E_* *and* ϵ_* *as in Lemma* 4.1*. Assume that* $\|\Phi(0)\|_{H^4} \le E_*$ *and* $0 < \epsilon < \epsilon_*$*. Let* $U(\xi,\tau)$ *and* $\Phi(\xi,\tau)$ *be the solutions of* (2.16) *and* (4.2)*, respectively, where we assume* $\|U(0) - \Phi(0)\|_{L^2} \le \epsilon$*. Then*

$$
\sup_{t\in[0,\tau_*]} \|\Phi(t) - U(t)\|_{L^2} \le \Gamma_{**}\epsilon. \tag{4.12}
$$

The constant $\Gamma_{**} > 0$ *does not depend on* ϵ*.*

Remark 4.4. Lemma 4.3 leads directly to Theorem 2.4. Here is the calculation. Recalling that we define Φ from y via $y(x,t) = \epsilon\Phi(\sqrt{\epsilon}x, \epsilon^2 t) - t$, we see that the initial conditions stated in (2.4) ($y(x,0) = \epsilon U_0(\sqrt{\epsilon}x) + y_{\epsilon,0}(x)$) imply

$$\Phi(X,0) = U_0(X) + \frac{1}{\epsilon} y_{\epsilon,0}\left(\frac{X}{\sqrt{\epsilon}}\right).$$

The requirement on $y_{\epsilon,0}(x)$ in Theorem 2.4 (namely $\|y_{\epsilon,0}\|_{H^5} \le \epsilon^{7/4}$) and a routine scaling argument give us

$$\|U(0) - \Phi(0)\|_{L^2} = \frac{1}{\epsilon}\left\| y_{\epsilon,0}\left(\frac{\cdot}{\sqrt{\epsilon}}\right)\right\|_{L^2} \le \epsilon.$$

Thus the requirement on the initial condition in Lemma 4.3 is met. Then using the conclusion of Lemma 4.3 we compute

$$\begin{aligned} \sup_{0<t\le\tau_0/\epsilon^2} \|y(\cdot,t) + t - \epsilon U(\sqrt{\epsilon}\cdot, \epsilon^2 t)\|_{L^2} &= \sup_{0<t\le\tau_0/\epsilon^2} \epsilon\|\Phi(\sqrt{\epsilon}\cdot,\epsilon^2 t) - U(\sqrt{\epsilon}\cdot,\epsilon^2 t)\|_{L^2} \\ &= \sup_{0<\tau\le\tau_0} \epsilon^{3/4}\|\Phi(\cdot,\tau) - U(\cdot,\tau)\|_{L^2} \\ &\le C\epsilon^{7/4}. \end{aligned} \tag{4.13}$$

This is the concluding estimate in Theorem 2.4. In the above we used the change of variables relation $\|f(\alpha\cdot)\|_{L^2} = \alpha^{-1/2}\|f(\cdot)\|_{L^2}$ when $\alpha > 0$ several times.

Proof. From (2.16) and (4.2) we construct the equation for the quantity $v = \Phi - U$. Indeed, since for $\alpha = \epsilon + 1$, $\alpha^2(\alpha+3) = 4 + \epsilon(\epsilon+3)^2$, we have

$$\begin{aligned} &\partial_\tau(\Phi - U) + 4\left[\frac{\Phi_{\xi\xi\xi\xi}}{(1+\epsilon^3(\Phi_\xi)^2)^2} - U_{\xi\xi\xi\xi}\right] + \frac{1}{\epsilon^3}\left[\sqrt{1+\epsilon^3(\Phi_\xi)^2} - 1 - \frac{\epsilon^3}{2}(U_\xi)^2\right] \\ &\quad + (\alpha-1)\left[\frac{\Phi_{\xi\xi}}{1+\epsilon^3(\Phi_\xi)^2} - U_{\xi\xi}\right] = 10\alpha^2(\alpha+3)\frac{\epsilon^3\Phi_\xi\Phi_{\xi\xi}\Phi_{\xi\xi\xi}}{(1+\epsilon^3(\Phi_\xi)^2)^3} + 3\alpha^2(\alpha+3)\frac{\epsilon^3(\Phi_{\xi\xi})^3}{(1+\epsilon^3(\Phi_\xi)^2)^3} \\ &\quad - 18\alpha^2(\alpha+3)\frac{\epsilon^6(\Phi_{\xi\xi})^3(\Phi_\xi)^2}{(1+\epsilon^3(\Phi_\xi)^2)^4} - \left(1+\frac{1}{2}\alpha^2\right)\frac{\epsilon(\Phi_{\xi\xi})^2}{(1+\epsilon^3(\Phi_\xi)^2)^{\frac{5}{2}}} \\ &\quad - \left(2\alpha + 5\alpha^2 - \frac{1}{3}\alpha^3\right)\frac{\epsilon^3(\Phi_{\xi\xi})^3}{(1+\epsilon^3(\Phi_\xi)^2)^4} - \epsilon(\epsilon+3)^2\frac{\Phi_{\xi\xi\xi\xi}}{(1+\epsilon^3(\Phi_\xi)^2)^2}. \end{aligned}$$

Then we can simplify it in the form

$$\begin{aligned} &\partial_\tau v + 4\left[\frac{v_{\xi\xi\xi\xi}}{(1+\epsilon^3(\Phi_\xi)^2)^2}\right] + \left[\frac{v_\xi\cdot(\Phi_\xi + U_\xi)}{\left(\sqrt{1+\epsilon^3(\Phi_\xi)^2}\right) + \left(1+\frac{\epsilon^3}{2}(U_\xi)^2\right)}\right] + \left[\frac{v_{\xi\xi}}{1+\epsilon^3(\Phi_\xi)^2}\right] \\ &\quad = \alpha^2(\alpha+3)\left[\frac{2\epsilon^3(\Phi_\xi)^2 + \epsilon^6(\Phi_\xi)^4}{(1+\epsilon^3(\Phi_\xi)^2)^2}\right]U_{\xi\xi\xi\xi} + \frac{1}{4}\cdot\frac{\epsilon^3(U_\xi)^4}{\left(\sqrt{1+\epsilon^3(\Phi_\xi)^2}\right) + \left(1+\frac{\epsilon^3}{2}(U_\xi)^2\right)} \\ &\qquad + (\alpha-1)\frac{\epsilon^3(\Phi_\xi)^2U_{\xi\xi}}{1+\epsilon^3(\Phi_\xi)^2} + 10\alpha^2(\alpha+3)\frac{\epsilon^3\Phi_\xi\Phi_{\xi\xi}\Phi_{\xi\xi\xi}}{(1+\epsilon^3(\Phi_\xi)^2)^3} + 3\alpha^2(\alpha+3)\frac{\epsilon^3(\Phi_{\xi\xi})^3}{(1+\epsilon^3(\Phi_\xi)^2)^3} \end{aligned}$$

$$- 18\alpha^2(\alpha+3)\frac{\epsilon^6(\Phi_{\xi\xi})^3(\Phi_\xi)^2}{(1+\epsilon^3(\Phi_\xi)^2)^4} - \left(1+\frac{1}{2}\alpha^2\right)\frac{\epsilon(\Phi_{\xi\xi})^2}{(1+\epsilon^3(\Phi_\xi)^2)^{\frac{5}{2}}}$$
$$- \left(2\alpha+5\alpha^2-\frac{1}{3}\alpha^3\right)\frac{\epsilon^3(\Phi_{\xi\xi})^3}{(1+\epsilon^3(\Phi_\xi)^2)^4} - \epsilon(\epsilon+3)^2\frac{\Phi_{\xi\xi\xi\xi}}{(1+\epsilon^3(\Phi_\xi)^2)^2}$$

with the initial condition $v(\xi,0)=0$. The presence of at least one ϵ in the right-hand side of this relation, as well as the H^4 bounds for both $U(\xi,\tau)$ and $\Phi(\xi,\tau)$, makes the right-hand side very convenient. For future calculations we give the right-hand side a name, say, $\epsilon F(\xi,\tau)$. It is not very difficult to see that for any time $\tau\in[0,\tau_*]$ we have

$$\|F\|_{L^2} \le C. \tag{4.14}$$

Now we find the inner product of the above equation with v,

$$\begin{aligned}(4.15)\quad &\frac{1}{2}\partial_\tau\|v\|^2_{L^2} + \alpha^2(\alpha+3)\int v\cdot\left[\frac{v_{\xi\xi\xi\xi}}{(1+\epsilon^3(\Phi_\xi)^2)^2}\right]d\xi\\ &= -\int v\cdot\left[\frac{v_\xi\cdot(\Phi_\xi+U_\xi)}{\left(\sqrt{1+\epsilon^3(\Phi_\xi)^2}\right)+\left(1+\frac{\epsilon^3}{2}(U_\xi)^2\right)}\right]d\xi - \int v\cdot\left[\frac{v_{\xi\xi}}{1+\epsilon^3(\Phi_\xi)^2}\right]d\xi + \epsilon\int v\cdot F(\xi,t)d\xi.\end{aligned}$$

Then,

$$\begin{aligned}\int v\cdot\left[\frac{v_{\xi\xi\xi\xi}}{(1+\epsilon^3(\Phi_\xi)^2)^2}\right]d\xi &= \int\frac{(v_{\xi\xi})^2}{(1+\epsilon^3(\Phi_\xi)^2)^2}d\xi\\ -4\epsilon^3\int v_{\xi\xi}&\left[\frac{2v_\xi\Phi_\xi\Phi_{\xi\xi}+v_\xi\Phi_\xi\Phi_{\xi\xi}+v(\Phi_{\xi\xi})^2+v\Phi_\xi\Phi_{\xi\xi\xi}}{(1+\epsilon^3(\Phi_\xi)^2)^3}\right]d\xi + 24\epsilon^6\int v_{\xi\xi}\left[\frac{v(\Phi_\xi)^2(\Phi_{\xi\xi})^2}{(1+\epsilon^3(\Phi_\xi)^2)^4}\right]d\xi.\end{aligned}$$

Considering the relation (4.4), we can present a lower bound for the major part of this equality, i.e.,

$$\int\frac{(v_{\xi\xi})^2}{(1+\epsilon^3(\Phi_\xi)^2)^2}d\xi \ge \|v_{\xi\xi}\|^2_{L^2}. \tag{4.16}$$

Therefore the energy estimate (4.15) turns into

$$\begin{aligned}\frac{1}{2}\partial_\tau\|v\|^2_{L^2} + \alpha^2(\alpha+3)\|v_{\xi\xi}\|^2_{L^2} &= -\int v\cdot\left[\frac{v_\xi\cdot(\Phi_\xi+U_\xi)}{\left(\sqrt{1+\epsilon^3(\Phi_\xi)^2}\right)+\left(1+\frac{\epsilon^3}{2}(U_\xi)^2\right)}\right]d\xi\\ -\int v\cdot\left[\frac{v_{\xi\xi}}{1+\epsilon^3(\Phi_\xi)^2}\right]d\xi &+ 4\alpha^3(\alpha+3)\epsilon^3\int v_{\xi\xi}\left[\frac{2v_\xi\Phi_\xi\Phi_{\xi\xi}+v_\xi\Phi_\xi\Phi_{\xi\xi}+v(\Phi_{\xi\xi})^2+v\Phi_\xi\Phi_{\xi\xi\xi}}{(1+\epsilon^3(\Phi_\xi)^2)^3}\right]d\xi\\ +24\alpha^3(\alpha+3)\epsilon^6\int v_{\xi\xi}&\left[\frac{v(\Phi_\xi)^2(\Phi_{\xi\xi})^2}{(1+\epsilon^3(\Phi_\xi)^2)^4}\right]d\xi + \epsilon\int v\cdot F(\xi,t)d\xi = K_1+K_2+K_3+K_4+K_5.\end{aligned}$$

Now we find proper bounds for the right-hand side of this relation.

Estimate for K_1. Considering the relation $vv_\xi = \frac{1}{2}\partial_\xi(v^2)$ we have

$$|K_1| \le \left| \int v \cdot \left[\frac{v_\xi \cdot (\Phi_\xi + U_\xi)}{(1+\sqrt{\epsilon^3(\Phi_\xi)^2}) + \left(1+\frac{\epsilon^3}{2}(U_\xi)^2\right)} \right] d\xi \right|$$
$$= \left| \int v^2 \cdot \partial_\xi \left[\frac{(\Phi_\xi + U_\xi)}{(1+\sqrt{\epsilon^3(\Phi_\xi)^2}) + \left(1+\frac{\epsilon^3}{2}(U_\xi)^2\right)} \right] d\xi \right|$$
$$\le \|v\|_{L^2}^2 \left\| \frac{\Phi_\xi + U_\xi}{(\sqrt{1+\epsilon^3(\Phi_\xi)^2}) + \left(1+\frac{\epsilon^3}{2}(U_\xi)^2\right)} \right\|_{L^\infty} < C\|v\|_{L^2}^2.$$

Estimate for K_2.

$$|K_2| \le \left| \int v \cdot \left[\frac{v_{\xi\xi}}{1+\epsilon^3(\Phi_\xi)^2} \right] d\xi \right| \le C\|v\|_{L^2}^2 + \frac{1}{100}\|v_{\xi\xi}\|_{L^2}^2.$$

Estimate for K_3.

$$|K_3| \le C\epsilon^3 \left| \int v_{\xi\xi} \left[\frac{2v_\xi\Phi_\xi\Phi_{\xi\xi} + v_\xi\Phi_\xi\Phi_{\xi\xi} + v(\Phi_{\xi\xi})^2 + v\Phi_\xi\Phi_{\xi\xi\xi}}{(1+\epsilon^3(\Phi_\xi)^2)^3} \right] d\xi \right|$$
$$\le C\epsilon^3 \|v_{\xi\xi}\|_{L^2}(\|v\|_{L^2} + \|v_\xi\|_{L^2}) \le \frac{1}{100}\|v_{\xi\xi}\|_{L^2}^2 + C\epsilon^6\|v\|_{L^2}^2 + C\epsilon^6.$$

Note that all the terms $\partial_\xi^s \Phi$, $1 \le s \le 3$, are bounded (since $\Phi \in H^4$).

Estimate for K_4.

$$|K_4| \le C\epsilon^6 \left| \int v_{\xi\xi} \left[\frac{v(\Phi_\xi)^2(\Phi_{\xi\xi})^2}{(1+\epsilon^3(\Phi_\xi)^2)^4} \right] d\xi \right| \le C\epsilon^6 \|v_{\xi\xi}\|_{L^2}\|v\|_{L^2} \le \frac{1}{100}\|v_{\xi\xi}\|_{L^2}^2 + C\epsilon^{12}\|v\|_{L^2}^2.$$

Estimate for K_5.

$$|K_5| \le C\epsilon \left| \int v \cdot F(\xi,\tau) d\xi \right| \le C\epsilon \|F(\cdot,\tau)\|_{L^2}\|v\|_{L^2} \le C\|v\|_{L^2}^2 + C\epsilon^2.$$

Overall, the energy estimate (4.15) turns into

$$\partial_\tau \|v\|_{L^2}^2 + C\|v_{\xi\xi}\|_{L^2}^2 \le C_1\|v\|_{L^2}^2 + C_2\epsilon^2,$$

or

$$\partial_\tau \|v\|_{L^2}^2 \le C_1\|v\|_{L^2}^2 + C_2\epsilon^2.$$

Then we take the integral from both sides,

$$\|v\|_{L^2}^2 \le e^{C_1\tau}\|v(0)\|_{L^2}^2 + C_2\epsilon^2 \int_0^\tau e^{C_0(\tau-s)} ds = e^{C_1\tau}\|v(0)\|_{L^2}^2 + \frac{C_2\epsilon^2}{C_1}\left[e^{C_1\tau} - 1\right].$$

Finally, we restrict ourselves to $\tau < \tau_0$, $\tau_0 = O(1)$, as well as $\|v(0)\|_{L^2} \le \epsilon$, and complete the proof. ■

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