



USING RANDOM WALKS TO ESTABLISH WAVELIKE BEHAVIOR IN A LINEAR FPUT SYSTEM WITH RANDOM COEFFICIENTS

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ABSTRACT. We consider a linear Fermi-Pasta-Ulam-Tsingou lattice with random spatially varying material coefficients. Using the methods of stochastic homogenization we show that solutions with long wave initial data converge in an appropriate sense to solutions of a wave equation. The convergence is strong and both almost sure and in expectation, but the rate is quite slow. The technique combines energy estimates with powerful classical results about random walks, specifically the law of the iterated logarithm.

1. Introduction. We prove an almost sure convergence result for solutions of the following one-dimensional random polymer linear Fermi-Pasta-Ulam-Tsingou (FPUT) lattice in the long wave limit:

$$m(j)\ddot{u}(j) = k(j)[u(j+1) - u(j)] - k(j-1)[u(j) - u(j-1)]. \quad (1.1)$$

Here $j \in \mathbf{Z}$, $u = u(j, t) \in \mathbf{R}$ and $t \in \mathbf{R}$. We choose the coefficients $m(j)$ (which we refer to as “the masses”) to be independent and identically distributed (i.i.d.) random variables contained almost surely in some intervals $[a_m, b_m] \subset \mathbf{R}^+$ with standard deviation σ_m . We similarly take the coefficients $1/k(j)$ (“the springs”) to be i.i.d. with support in $[a_k, b_k] \subset \mathbf{R}^+$ and deviation σ_k . This system is well-understood when these coefficients are either constant or periodic with respect to j [8], but for the random problem most of what is known is formal or numerical [6, 9].

For initial conditions whose wavelength is $O(1/\epsilon)$, with ϵ a small positive number, we prove that the ℓ^2 norm of the difference between true solutions and appropriately scaled solutions to the wave equation is at most $O\left(\sqrt{\log \log(1/\epsilon)}\right)$ for times of $O(1/\epsilon)$ for almost every realization. While such an absolute error diverges as $\epsilon \rightarrow 0^+$, it happens that this is enough to establish an almost sure convergence result within the “coarse-graining” setting used in [8] to study the (multi-dimensional) periodic problem. In addition to the almost sure convergence, we are able to provide estimates on the mean of the error in terms of σ_m and σ_k and prove convergence in mean.

The articles [1, 5] study the nonlinear FPUT lattice with periodic coefficients. These show that soliton-like solutions exist for very large time scales using Korteweg-de Vries (KdV) approximations. The authors of [5] used the so called multiscale

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method of homogenization, a by-now classical tool with a long history in PDE for deriving effective equations, see [2]. In this paper, we carry out a very similar approach in deriving and proving the results; however, our expansions only result in an effective wave equation, not the KdV equation. In our setting, since the coefficients are random, it is necessary to average over the entire lattice. The law of large numbers implies this average is equal to the expectation, so the speed of the approximate solution depends on the expectation of the random variables. The probability theory hinges upon classical but extremely powerful asymptotic analysis of random walks, namely the law of the iterated logarithm, as well as basic martingale theory.

We denote a doubly infinite sequence $\{f(j)\}_{j \in \mathbf{Z}}$ by f . Let $S^\pm$ be the shift operators which act on sequences f as

$$(S^\pm f)(j) := f(j \pm 1),$$

and the operators δ^+ and δ^- , the left and right difference operators, are

$$\begin{aligned} (\delta^+ f)(j) &:= f(j+1) - f(j) \\ (\delta^- f)(j) &:= f(j) - f(j-1). \end{aligned}$$

Defining

$$\begin{aligned} r &:= \delta^+ u \\ p &:= \dot{u}, \end{aligned}$$

we convert our second order equation (1.1) to the system

$$\begin{aligned} \dot{r} &= \delta^+ p \\ \dot{p} &= \frac{1}{m} \delta^-(kr). \end{aligned} \tag{1.2}$$

For the remainder of the paper, we work with (1.2).

Here is the idea of our ultimate result. Suppose that the initial conditions for (1.2) have the following long wave form:

$$r(j, 0) = \Phi(\epsilon j)/k(j) \quad \text{and} \quad p(j, 0) = \Psi(\epsilon j)$$

where $\Phi, \Psi : \mathbf{R} \rightarrow \mathbf{R}$ are suitably smooth and of somewhat rapid decay. Then the solution p of (1.2) has

$$\mathcal{L}[p](X/\epsilon, \tau/\epsilon) \longrightarrow P_0(X, \tau)$$

as $\epsilon \rightarrow 0^+$ where P_0 solves the wave equation $\partial_\tau^2 P_0 = c^2 \partial_X^2 P_0$. The operator $\mathcal{L}$ interpolates the sequence p into a function on $\mathbf{R}$. It is defined below, as is the wave speed c . The convergence is strong in $L^2(\mathbf{R})$ and is both almost sure and in expectation. A similar convergence holds for r .

The paper is organized as follows. We carry out the multiscale expansion in Section 2 and derive effective equations and approximate solutions. In Section 3 we dive into the analysis of various smooth, rapidly decaying functions which are sampled at integers and multiplied componentwise by random walks. These estimates are necessary to control the error and here is where most the probability theory is needed. In Section 4 we provide the rigorous estimates of the error. We introduce coarse-graining and prove the convergence results in Section 5. In Section 6 we provide numerical simulations as evidence that our estimates are good ones *i.e.* they are not vast overestimates.

2. Homogenization and derivation of the effective wave equation. In this section we homogenize the equation following closely what is done in [5]. First, we define “residuals”, which quantify how close some function is to a true solution. For any functions $\tilde{r}(j, t)$ and $\tilde{p}(j, t)$ put

$$\begin{aligned} \text{Res}_1(\tilde{r}, \tilde{p}) &:= \delta^+ \tilde{p} - \partial_t \tilde{r} \\ \text{Res}_2(\tilde{r}, \tilde{p}) &:= \frac{1}{m} \delta^-(k\tilde{r}) - \partial_t \tilde{p}. \end{aligned} \quad (2.1)$$

We look for approximate long wave solutions of the form

$$\begin{aligned} \tilde{r}(j, t) &= \tilde{r}_\epsilon(j, t) := R(j, \epsilon j, \epsilon t) \\ \tilde{p}(j, t) &= \tilde{p}_\epsilon(j, t) := P(j, \epsilon j, \epsilon t), \end{aligned} \quad (2.2)$$

where $R = R(j, X, \tau)$ and $P = P(j, X, \tau)$ are maps

$$\mathbf{Z} \times \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}.$$

In the periodic-coefficient problem studied in [5], it was necessary to assume that these functions are periodic in the $\mathbf{Z}$ slot, but this needs to be exchanged in the random case. Here, we make a “sublinear growth” assumption that makes averaging possible:

$$\lim_{|j| \rightarrow \infty} R(j, X, \tau)/j = \lim_{|j| \rightarrow \infty} P(j, X, \tau)/j = 0. \quad (2.3)$$

The following lemma is crucial to the derivation of the effective equations.

Lemma 2.1. *Fix $g = \{g(j)\}_{j \in \mathbf{Z}}$. There exists an $f = \{f(j)\}_{j \in \mathbf{Z}}$ satisfying both*

$$(\delta^\pm f)(j) = g(j)$$

and

$$\lim_{|j| \rightarrow \infty} f(j)/j \rightarrow 0 \quad (2.4)$$

if and only if

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^{N-1} g(i) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N g(-i) = 0.$$

Proof. We only give a proof for “ δ^+ ”.

$\Rightarrow$ Since $g(j) = f(j+1) - f(j)$ we get

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^{N-1} g(i) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^{N-1} [f(i+1) - f(i)] = \lim_{N \rightarrow \infty} \frac{f(N) - f(0)}{N} = 0$$

by assumption (2.4). Proof of the second equality follows the same reasoning.

$\Leftarrow$ It is easily checked that

$$f(j) = \sum_{k=0}^{j-1} g(k) \quad \text{and} \quad f(-j) = -\sum_{k=1}^j g(-k) \quad (2.5)$$

for $j > 0$ solves $\delta^+ f = g$. Then

$$\lim_{j \rightarrow \infty} \frac{f(j)}{j} = \lim_{j \rightarrow \infty} \frac{\sum_{i=0}^{j-1} g(i)}{j} = 0.$$

It is likewise seen that

$$\lim_{j \rightarrow \infty} \frac{f(-j)}{-j} = 0$$

by using the formula for $f(-j)$. $\square$

Now we continue with the homogenization procedure. We must understand how $\delta^\pm$ act on functions of the type (2.2). The following expansions are found in [5]. If $u(j) = U(j, \epsilon j)$, then

$$\delta^\pm u(j) = \sum_{n \geq 0} \epsilon^n \delta_n^\pm U$$

where

$$\delta_0^\pm := \check{\delta}^\pm \text{ and } \delta_n^\pm := \frac{(\pm 1)^{n+1}}{n!} \check{S}^\pm \partial_X^n.$$

Here $\check{\delta}^\pm$ and $\check{S}^\pm$ act only on the first slot; they are analogous to partial derivatives with respect to j . Precisely,

$$\begin{aligned} \check{S}^+(U)(j, X) &:= U(j+1, X), \\ \check{S}^-(U)(j, X) &:= U(j-1, X), \\ \check{\delta}^+(U)(j, X) &:= U(j+1, X) - U(j, X), \\ \check{\delta}^-(U)(j, X) &:= U(j, X) - U(j-1, X). \end{aligned}$$

Let

$$(E_M^\pm u)(j) := (\delta^\pm u)(j) - \sum_{n=0}^M \epsilon^n (\delta_n^\pm U)(j, \epsilon j)$$

be the error made by truncating the series expansion of $\delta^\pm u$ after M terms. Thus the lowest power of ϵ we see in the error term is ϵ^{M+1} .

We further assume that our approximate solutions R and P themselves have expansions in ϵ :

$$R(j, X, \tau) = R_0(j, X, \tau) + \epsilon R_1(j, X, \tau) \quad \text{and} \quad P(j, X, \tau) = P_0(j, X, \tau) + \epsilon P_1(j, X, \tau). \quad (2.6)$$

Of course $R_i(j, X, \tau)$ and $P_i(j, X, \tau)$ meet (2.4). Using the above expansion, we directly compute $\text{Res}_1(\tilde{r}_\epsilon, \tilde{p}_\epsilon)$:

$$\begin{aligned} \text{Res}_1(\tilde{r}_\epsilon, \tilde{p}_\epsilon) &= \delta_0^+ P_0 + \epsilon \delta_1^+ P_0 + E_1^+(P_0) \\ &\quad + \epsilon \delta_0^+ P_1 + \epsilon^2 \delta_1^+ P_1 + \epsilon E_1^+(P_1) \\ &\quad - \epsilon \partial_\tau R_0 - \epsilon^2 \partial_\tau R_1. \end{aligned} \quad (2.7)$$

Here we have used the expansion for δ^+ . Similarly

$$\begin{aligned} \text{Res}_2(\tilde{r}_\epsilon, \tilde{p}_\epsilon) &= \frac{1}{m} (\delta_0^- k R_0 + \epsilon \delta_1^- k R_0 + E_1^-(k R_0) \\ &\quad + \epsilon \delta_0^- k R_1 + \epsilon^2 \delta_1^- k R_1 + \epsilon E_1^-(k R_1) \\ &\quad - \epsilon \partial_\tau P_0 - \epsilon^2 \partial_\tau P_1). \end{aligned} \quad (2.8)$$

Next set

$$Q_i := k R_i. \quad (2.9)$$

We choose P_0, P_1, Q_0 and Q_1 so that the $O(1)$ and $O(\epsilon)$ terms in (2.7) and (2.8) vanish. We get

$$\begin{aligned} \check{\delta}^+ P_0 &= 0 \\ \frac{1}{m} \check{\delta}^- Q_0 &= 0 \end{aligned} \quad (O(1))$$

and

$$\begin{aligned}\check{\delta}^+ P_1 &= \frac{1}{k} \partial_\tau Q_0 - S^+ \partial_X P_0 \\ \check{\delta}^- Q_1 &= m \partial_\tau P_0 - S^- \partial_X Q_0.\end{aligned}\tag{O(\epsilon)}$$

From (O(1)) we learn that P_0 and Q_0 do not depend on j , i.e.

$$P_0(j, X, \tau) = \bar{P}_0(X, \tau) \text{ and } Q_0(j, X, \tau) = \bar{Q}_0(X, \tau).\tag{2.10}$$

If there are to be solutions P_1 and Q_1 to (O(\epsilon)) which satisfy (2.4), Lemma 2.1 tells us we must have

$$\begin{aligned}\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{N-1} \left[\frac{1}{k(j)} \partial_\tau \bar{Q}_0 - \partial_X \bar{P}_0 \right] &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^N \left[\frac{1}{k(-j)} \partial_\tau \bar{Q}_0 - \partial_X \bar{P}_0 \right] = 0 \\ \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{N-1} [m(j) \partial_\tau \bar{P}_0 - \partial_X \bar{Q}_0] &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^N [m(-j) \partial_\tau \bar{P}_0 - \partial_X \bar{Q}_0] = 0.\end{aligned}\tag{2.11}$$

Since $\bar{P}_0$ and $\bar{Q}_0$ do not depend upon j these can be rewritten as

$$\left[\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{N-1} \frac{1}{k(j)} \right] \partial_\tau \bar{Q}_0 = \lim_{N \rightarrow \infty} \left[\frac{1}{N} \sum_{j=1}^N \frac{1}{k(-j)} \right] \partial_\tau \bar{Q}_0 = \partial_X \bar{P}_0$$

and

$$\left[\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{N-1} m(j) \right] \partial_\tau \bar{P}_0 = \left[\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^N m(-j) \right] \partial_\tau \bar{P}_0 = \partial_X \bar{Q}_0.$$

The strong law of large numbers tells us that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{N-1} m(j) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^N m(-j) = \mathbb{E}[m] =: \bar{m}\tag{2.12}$$

and

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{N-1} \frac{1}{k(j)} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^{N-1} \frac{1}{k(-j)} = \mathbb{E} \left[\frac{1}{k} \right] =: \frac{1}{\bar{k}}\tag{2.13}$$

almost surely, since m and k are sequences of i.i.d. random variables. To be clear $\mathbb{E}[\cdot]$ is the expectation of a random variable. And so we find that

$$\begin{aligned}\partial_\tau \bar{Q}_0 &= \tilde{k} \partial_X \bar{P}_0 \\ \partial_\tau \bar{P}_0 &= \frac{1}{\tilde{m}} \partial_X \bar{Q}_0.\end{aligned}\tag{2.14}$$

From this, out pops the effective wave equation

$$\partial_\tau^2 \bar{Q}_0 = c^2 \partial_X^2 \bar{Q}_0$$

with wave-speed

$$c := \sqrt{\tilde{k}/\tilde{m}}.$$

We can use d'Alemberts formula to get $\bar{Q}_0$ and subsequently find $\bar{P}_0$ from its relation to $\bar{Q}_0$:

$$\begin{aligned}\bar{Q}_0(X, \tau) &= A(X - c\tau) + B(X + c\tau) \\ \bar{P}_0(X, \tau) &= \frac{1}{\sqrt{\tilde{k}\tilde{m}}} (-A(X - c\tau) + B(X + c\tau)).\end{aligned}\tag{2.15}$$

The functions A and B will ultimately be determined by the initial conditions for (1.2) in a fashion that is consistent with (2.2).

At this point we have computed the effective wave equation but we must also determine the full form of P_1 and Q_1 . Using (2.10) and (2.14) in $(O(\epsilon))$ we get

$$\begin{aligned}\delta^+ P_1 &= \left(\frac{\tilde{k}}{k} - 1 \right) \partial_X \bar{P}_0 \\ \delta^- Q_1 &= \left(\frac{m}{\bar{m}} - 1 \right) \partial_X \bar{Q}_0.\end{aligned}$$

Define χ_m and χ_k as the solutions to

$$\delta^+ \chi_k = \frac{\tilde{k}}{k} - 1 \text{ and } \delta^- \chi_m = \frac{m}{\bar{m}} - 1.$$

Using formula (2.5) in Lemma 2.1, we can solve explicitly for χ_k and χ_m . They are

$$\begin{aligned}\chi_k(j) &= \sum_{i=0}^{j-1} \left[\frac{\tilde{k}}{k(i)} - 1 \right] \text{ and } \chi_k(-j) = \sum_{i=1}^j \left[1 - \frac{\tilde{k}}{k(i)} \right] \\ \chi_m(j) &= \sum_{i=0}^{j-1} \left[\frac{m(i)}{\bar{m}} - 1 \right] \text{ and } \chi_m(-j) = \sum_{i=1}^j \left[1 - \frac{m(i)}{\bar{m}} \right].\end{aligned}\tag{2.16}$$

Observe that $\frac{\tilde{k}}{k} - 1$ and $\frac{m}{\bar{m}} - 1$ are mean zero random variables and as such χ_k and χ_m are classical random walks. The expression for Q_1 and P_1 can be given in terms of χ_k and χ_m :

$$\begin{aligned}Q_1(j, X, \tau) &= \chi_m(j) \partial_X \bar{Q}_0(X, \tau) \\ P_1(j, X, \tau) &= \chi_k(j) \partial_X \bar{P}_0(X, \tau).\end{aligned}\tag{2.17}$$

We need to know estimates for the norm of P_1 and Q_1 so that we can estimate the residuals. Results are given in the next section. Here is an important preview of what we find: the growth rates for random walks ultimately imply that the terms ϵP_1 and ϵR_1 in (2.6) are, despite appearances, not actually $O(\epsilon)$; we show below that they are in fact $O(\sqrt{\log \log(1/\epsilon)})$. This in turn implies that the residuals are not as small as the above derivation would lead one to believe; the formal calculation presented above predicts $O(\epsilon^2)$ but they turn out to be $O(\epsilon \sqrt{\log \log(1/\epsilon)})$. This is the main technical complication in this article and the key difference between the random problem we study here and the periodic or constant coefficient problems studied in [8].

Before moving on, we now spell out our long wave approximation in detail. Putting together (2.2), (2.6), (2.9), (2.10), (2.15) and (2.17) we see that

$$\begin{aligned}\tilde{r}_\epsilon(j, t) &= \frac{1}{k(j)} (A(\epsilon(j - ct)) + B(\epsilon(j + ct))) \\ &\quad + \epsilon \frac{\chi_m(j)}{k(j)} (A'(\epsilon(j - ct)) + B'(\epsilon(j + ct))) \\ \tilde{p}_\epsilon(j, t) &= \frac{1}{\sqrt{k\bar{m}}} (-A(\epsilon(j - ct)) + B(\epsilon(j + ct))) \\ &\quad + \epsilon \frac{\chi_k(j)}{\sqrt{k\bar{m}}} (-A'(\epsilon(j - ct)) + B'(\epsilon(j + ct))).\end{aligned}\tag{2.18}$$

3. Probabilistic estimates. In this section we provide tools which will allow us to compute the ℓ^2 norms of the residuals for all $|t| \leq T_0/\epsilon$. The first subsection deals with almost sure and realization dependent estimates by making use of the law of the iterated logarithm (LIL). The second subsection provides estimates on the expectation of the norms using martingale inequalities.

3.1. Almost sure estimates. One can find the statement of the LIL in [3] and more details can be found in [4]. Here we present the theorem in a form convenient to us.

Theorem 3.1. (*The Law of the Iterated Logarithm*) Suppose $y(j)$ ($j \in \mathbf{Z}$) are i.i.d random variables with mean zero and $\mathbb{E}[y^2] = \sigma^2$. Define the (two-sided) random walk χ via

$$\chi(j) := \sum_{i=0}^{j-1} y(i) \quad \text{and} \quad \chi(-j) := \sum_{i=1}^j y(-i) \quad (3.1)$$

for $j > 0$ and $\chi(0) = 0$.

Then

$$\limsup_{|j| \rightarrow \pm\infty} \frac{\pm\chi(j)}{\sqrt{2|j| \log \log(|j|)}} \stackrel{a.s.}{=} \sigma.$$

The LIL is an extremely sharp description of a random walk. It says that, with a probability of one, the magnitude of $\chi(j)$ exceeds the curve $\sigma\sqrt{2j \log \log(j)}$ (by any fixed amount) only a finite number of times but comes arbitrarily near it an infinite number of times. Here is how we use the LIL:

Corollary 3.2. For almost every realization of $\{k(j)\}$ and $\{m(j)\}$ there is a finite positive constant $C_\omega = C_\omega(k, m)$ for which

$$|\chi_k(j)| + |\chi_m(j)| \leq C_\omega \sqrt{|j| \log \log(|j| + e)}$$

for all $j \in \mathbf{Z}$.

Remark 1. The constant C_ω is almost surely finite by the LIL, but it may be extremely large. There is no way to determine its magnitude except in very special circumstances. Note, however, it does not depend on ϵ .

Remark 2. In this paper, we use a small modification of the usual “big C ” notation. If a constant in an estimate depends on the particular realization of the coefficients we mark it as “ C_ω .” If it does not, we omit the subscript ω . All such constants C_ω are always almost surely finite. No such constants will ever depend on ϵ .

Proof. (Corollary 3.2) We have assumed that $1/k(j)$ and $m(j)$ are i.i.d. and as such

$$y_k(j) = \frac{\tilde{k}}{k(j)} - 1$$

and

$$y_m(j) = \frac{m(j)}{\tilde{m}} - 1$$

satisfy the hypotheses of the LIL. Thus that result implies for almost every realization of $\{k(j)\}$ there is a natural number N_k for which

$$|\chi_k(j)| \leq 2\tilde{k}\sigma_k \sqrt{2|j| \log \log(|j|)}$$

when $|j| \geq N_k$. Then put

$$C_k := \max \left\{ 2\tilde{k}\sigma_k\sqrt{2}, \max_{0 < |j| \leq N_k} \frac{\chi_k(j)}{\sqrt{|j| \log \log(|j|)}} \right\}.$$

It follows that $|\chi_k(j)| \leq C_k \sqrt{|j| \log \log(|j|)} \leq C_k \sqrt{|j| \log \log(|j| + e)}$ for all j . The same argument can be used to estimate χ_m . $\square$

Given the growth rate in the LIL, we introduce a new norm fashioned to absorb it:

$$\|F\|_{H_{LIL}^s} := \sum_{i=0}^s \|(1 + |\cdot| \log \log(|\cdot| + e))^{1/2} F^{(i)}\|_{L^2}.$$

The space H_{LIL}^s will be the completion of L^2 with respect to this norm. Similarly, we also introduce

$$\|F\|_{H_{sr}^s} := \sum_{i=0}^s \|(1 + |\cdot|)^{1/2} F^{(i)}\|_{L^2}$$

and the space H_{sr}^s . Note that $\|F\|_{H^s} \leq \|F\|_{H_{sr}^s} \leq C\|F\|_{H_{LIL}^s}$ where H^s is the usual L^2 -based Sobolev space of functions $\mathbf{R} \rightarrow \mathbf{R}$ which are weakly s -times differentiable.

Now we unveil the two main estimates we need to provide almost sure control of the residuals.

Lemma 3.3. *For any $T_0 > 0$ and almost every realization of $\{k(j)\}$ and $\{m(j)\}$ there is a finite positive constant $C_\omega = C_\omega(k, m, T_0)$ for which $\epsilon \in (0, 1/4)$ implies*

$$\sup_{|t| \leq T_0/\epsilon} \|\chi(\cdot)F(\epsilon(\cdot - ct))\|_{\ell^2} \leq C_\omega \epsilon^{-1} \sqrt{\log \log(\epsilon^{-1})} \|F\|_{H_{LIL}^1} \quad (3.2)$$

and

$$\sup_{|t| \leq T_0/\epsilon} \|\chi(\cdot)\delta^\pm F(\epsilon(\cdot - ct))\|_{\ell^2} \leq C_\omega \sqrt{\log \log(\epsilon^{-1})} \|F\|_{H_{LIL}^2}. \quad (3.3)$$

In the above χ is either χ_k or χ_m .

To prove these we need some calculus estimates.

Lemma 3.4. *For all $\epsilon \in (0, 1/4)$, and $a, b \in \mathbf{R}$*

$$|a + b| \log \log(|a + b| + e) \leq |a| \log \log(2|a| + e) + |b| \log \log(2|b| + e)$$

and

$$\log \log(|x| + e) \leq \log(2 \log(\epsilon|x| + e)) + \log \log(\epsilon^{-1} + e).$$

Proof. The first inequality follows from the fact that $|x| \log \log(|x| + e)$ is a convex function.

We will show the second inequality in two steps. First we show that

$$\log \log(|x| + e) \leq \log(2 \log(\epsilon|x| + \epsilon^{-1} + e)).$$

Since $\log$ is monotonic, this inequality follows from

$$|x| + e \leq (\epsilon|x| + \epsilon^{-1} + e)^2,$$

which is trivial.

Now we show that

$$\log(2 \log(\epsilon|x| + \epsilon^{-1} + e)) \leq \log(2 \log(\epsilon|x| + e)) + \log \log(\epsilon^{-1} + e).$$

Note that at $x = 0$, equality holds. For $x \geq 0$ we have that

$$\frac{d}{dx} \log(\epsilon x + \epsilon^{-1} + e) = \frac{\epsilon}{\epsilon x + \epsilon^{-1} + e}$$

and

$$\frac{d}{dx} \log(\epsilon|x| + e) \log(\epsilon^{-1} + e) = \frac{\epsilon \log(\epsilon^{-1} + e)}{\epsilon x + e}.$$

Since

$$\frac{\epsilon}{\epsilon x + \epsilon^{-1} + e} \leq \frac{\epsilon}{\epsilon x + e} \leq \frac{\epsilon \log(\epsilon^{-1} + e)}{\epsilon x + e},$$

we see that $\log(\epsilon x + \epsilon^{-1} + e)$ grows more slowly than $\log(\epsilon x + e) \log(\epsilon^{-1} + e)$. Since both functions are even, we get by symmetry that

$$2 \log(\epsilon|x| + \epsilon^{-1} + e) \leq 2 \log(\epsilon|x| + e) \log(\epsilon^{-1} + e).$$

Taking log of both sides, we get the desired result. $\square$

Now we can prove our key estimates.

Proof. (Lemma 3.3) Take χ to be χ_k or χ_m and fix $T_0 > 0$. Using Corollary 3.2

$$\begin{aligned} \|\chi(\cdot)F(\epsilon(\cdot - ct))\|_{\ell^2} &= \left(\sum_{j \in \mathbf{Z}} \chi(j)^2 F(\epsilon(j - ct))^2 \right)^{1/2} \\ &\leq C_\omega \left(\sum_{j \in \mathbf{Z}} |j| \log \log(|j| + e) F(\epsilon(j - ct))^2 \right)^{1/2}. \end{aligned}$$

The constant C_ω here depends upon the realization and any estimate below will depend on the realization because of this step only.

Using the first inequality in Lemma 3.4 with $a = j - ct$ and $b = ct$ and the triangle inequality we get

$$\begin{aligned} \|\chi(\cdot)F(\epsilon(\cdot - ct))\|_{\ell^2} &\leq C_\omega \left(\sum_{j \in \mathbf{Z}} |j - ct| \log \log(2|j - ct| + e) F(\epsilon(j - ct))^2 \right)^{1/2} \\ &\quad + C_\omega \sqrt{|t| \log \log(2c|t| + e)} \|F(\epsilon(\cdot - ct))\|_{\ell^2}. \end{aligned}$$

Call the two terms on the right I and II . We estimate II first.

Lemma 4.3 from [5] shows that

$$\|F(\epsilon(\cdot - ct))\|_{\ell^2} \leq C\epsilon^{-1/2} \|F(\cdot - ct)\|_{H^1} = C\epsilon^{-1/2} \|F\|_{H^1}$$

and so

$$II \leq C_\omega \epsilon^{-1/2} \sqrt{|t| \log \log(2c|t| + e)} \|F\|_{H^1}.$$

Then

$$\sup_{|t| \leq T_0/\epsilon} II \leq C_\omega \epsilon^{-1} \sqrt{\log \log(2cT_0\epsilon^{-1} + e)} \|F\|_{H^1}.$$

Routine features of the logarithm show that $\log \log(2cT_0\epsilon^{-1} + e) \leq C \log \log(1/\epsilon)$ when $\epsilon \in (0, 1/4)$ and so we have

$$\sup_{|t| \leq T_0/\epsilon} II \leq C_\omega \epsilon^{-1} \sqrt{\log \log(1/\epsilon)} \|F\|_{H^1}.$$

As for I , using the second inequality in Lemma 3.4 with $|x| = 2|j - ct|$ followed by the triangle inequality gets us:

$$I \leq C_\omega \left(\sum_{j \in \mathbf{Z}} |j - ct| \log(2 \log(2\epsilon|j - ct| + e)) F(\epsilon(j - ct))^2 \right)^{1/2} \\ + C_\omega \left(\sum_{j \in \mathbf{Z}} |j - ct| \log \log(\epsilon^{-1} + e) F(\epsilon(j - ct))^2 \right)^{1/2}.$$

Then we multiply by $\sqrt{\epsilon/\epsilon}$ and do some algebra to get:

$$I \leq C_\omega \epsilon^{-1/2} \|\sqrt{\epsilon}|\cdot - ct| \log(2 \log(2\epsilon|\cdot - ct| + e)) F(\epsilon(\cdot - ct))\|_{\ell^2} \\ + C_\omega \epsilon^{-1/2} \sqrt{\log \log(\epsilon^{-1} + e)} \|\sqrt{\epsilon}|\cdot - ct| F(\epsilon(\cdot - ct))\|_{\ell^2}.$$

Applying Lemma 4.3 from [5] tells us that

$$\|\sqrt{\epsilon}|\cdot - ct| \log(2 \log(2\epsilon|\cdot - ct| + e)) F(\epsilon(\cdot - ct))\|_{\ell^2} \leq C \epsilon^{-1/2} \|F\|_{H_{LL}^1}$$

and

$$\|\sqrt{\epsilon}|\cdot - ct| F(\epsilon(\cdot - ct))\|_{\ell^2} \leq C \epsilon^{-1/2} \|F\|_{H_{sr}^1}$$

and so we have,

$$I \leq C_\omega \epsilon^{-1} \sqrt{\log \log(\epsilon^{-1} + e)} \|F\|_{H_{LL}^1} \leq C_\omega \epsilon^{-1} \sqrt{\log \log(\epsilon^{-1})} \|F\|_{H_{LL}^1},$$

where we have absorbed the additional plus e inside the iterated logarithm into the constant C_ω . Note that the right hand side does not depend on t and so $\sup_{|t| \leq T_0/\epsilon} I \leq C_\omega \epsilon^{-1} \sqrt{\log \log(\epsilon^{-1})}$ and all together we have shown (3.2).

It happens that (3.3) follows almost immediately from (3.2) with some operator trickery. For functions $G : \mathbf{R} \rightarrow \mathbf{R}$ and $\epsilon \neq 0$ define the operator $\mathcal{A}_\epsilon$ via

$$(\mathcal{A}_\epsilon G)(X) := \frac{1}{\epsilon} \int_X^{X+\epsilon} G(s) ds.$$

We have

$$\|\mathcal{A}_\epsilon G\|_H \leq C \|G\|_H \quad (3.4)$$

where H may be H^s , H_{LL}^s or H_{sr}^s . Here comes the argument. First we use Jensen's inequality to get:

$$\|w(\cdot) \mathcal{A}_\epsilon G\|_{L^2}^2 = \int_{-\infty}^{\infty} w(X)^2 \left(\frac{1}{\epsilon} \int_X^{X+\epsilon} G(s) ds \right)^2 dX \\ \leq \int_{-\infty}^{\infty} w(X)^2 \frac{1}{\epsilon} \int_X^{X+\epsilon} G(s)^2 ds dX.$$

In the above $w(X)$ is a weight function. If we change the order of integration we get

$$\|w(\cdot) \mathcal{A}_\epsilon G\|_{L^2}^2 = \int_{-\infty}^{\infty} G(s)^2 \frac{1}{\epsilon} \int_{s-\epsilon}^s w(X)^2 dX ds.$$

Let $b_\epsilon(s) := \frac{1}{\epsilon w(s)^2} \int_{s-\epsilon}^s w(X)^2 dX$ so we have

$$\|w(\cdot) \mathcal{A}_\epsilon G\|_{L^2}^2 = \int_{-\infty}^{\infty} w(s)^2 G(s)^2 b_\epsilon(s) dX ds \leq \|b_\epsilon\|_{L^\infty} \|w(\cdot) G\|_{L^2}^2.$$

If $w(X) = 1$, $w(X) = \sqrt{1 + |X|}$ or $w(X) = \sqrt{1 + |X| \log \log(|X| + e)}$ it is easy to use the mean value theorem to show $\|b_\epsilon\|_{L^\infty} \leq C$ when $\epsilon \in (0, 1/4)$. With this, the last displayed inequality implies (3.4).

Continuing on in the proof of (3.3), the fundamental theorem of calculus tells us that $F(X + \epsilon) - F(X) = \epsilon(\mathcal{A}_\epsilon F')(X)$. Thus:

$$(\delta^+ F)(\epsilon(j - ct)) = F(\epsilon(j - ct) + \epsilon) - F(\epsilon(j - ct)) = \epsilon(\mathcal{A}_\epsilon F')(\epsilon(j - ct)).$$

In which case we see that

$$\|\chi(\cdot)\delta^+ F(\epsilon(\cdot - ct))\|_{\ell^2} = \epsilon\|\chi(\cdot)(\mathcal{A}_\epsilon F')(\epsilon(\cdot - ct))\|_{\ell^2}.$$

We have produced an extra factor of ϵ . Using (3.2) and (3.4)

$$\|\chi(\cdot)\delta^+ F(\epsilon(\cdot - ct))\|_{\ell^2} \leq C_\omega \sqrt{\log \log(\epsilon^{-1})} \|\mathcal{A}_\epsilon F'\|_{H_{LIL}^1} \leq C_\omega \sqrt{\log \log(\epsilon^{-1})} \|F\|_{H_{LIL}^2}.$$

That is (3.3) and does it for this proof. $\square$

Now we can prove:

Proposition 3.5. *Fix $A, B \in H_{LIL}^3$ and take $\tilde{r}_\epsilon$ and $\tilde{p}_\epsilon$ as in (2.18). Fix $T_0 > 0$. Then for almost every realization of $\{k(j)\}$ and $\{m(j)\}$ there is a finite positive constant $C_\omega = C_\omega(k, m, T_0, \|A\|_{H_{LIL}^3}, \|B\|_{H_{LIL}^3})$ for which $\epsilon \in (0, 1/4)$ implies*

$$\sup_{|t| \leq T_0/\epsilon} (\|\text{Res}_1(\tilde{r}_\epsilon, \tilde{p}_\epsilon)\|_{\ell^2} + \|\text{Res}_1(\tilde{r}_\epsilon, \tilde{p}_\epsilon)\|_{\ell^2}) \leq C_\omega \epsilon \sqrt{\log \log(1/\epsilon)}. \quad (3.5)$$

Proof. We prove the estimate for the piece involving Res_1 as the other part is all but identical. A tedious calculation (plugging (2.18) into (2.1) and using the product rule for finite differences) shows that

$$\begin{aligned} \text{Res}_1(\tilde{r}_\epsilon, \tilde{p}_\epsilon) &= \frac{1}{\sqrt{k\tilde{m}}} (-\delta^+[A(\epsilon(j - ct))] + \epsilon A'(\epsilon(j - ct))) \\ &\quad + \frac{1}{\sqrt{k\tilde{m}}} (\delta^+[B(\epsilon(j + ct))] - \epsilon B'(\epsilon(j + ct))) \\ &\quad + \frac{c\epsilon^2 \chi_m(j)}{k(j)} A'(\epsilon(j - ct)) + \frac{c\epsilon^2 \chi_m(j)}{k(j)} B'(\epsilon(j + ct)) \\ &\quad - \frac{\epsilon \chi_k(j + 1)}{\sqrt{k\tilde{m}}} \delta^+[A'(\epsilon(j - ct))] - \frac{\epsilon \chi_k(j + 1)}{\sqrt{k\tilde{m}}} \delta^+[B'(\epsilon(j + ct))]. \end{aligned} \quad (3.6)$$

The terms in the first two lines are fully deterministic and estimable using Lemma 4.3 of [5]. Specifically the ℓ^2 norm of each is controlled by

$$C\epsilon^{3/2} (\|A\|_{H^2} + \|B\|_{H^2})$$

for $|t| \leq T_0/\epsilon$. This is dominated by the right hand side of (3.5). Using (3.2) we see that the ℓ^2 norm in the third line is controlled by

$$\epsilon^2 \left(C_\omega \epsilon^{-1} \sqrt{\log \log(1/\epsilon)} \left(\|A'\|_{H_{LIL}^1} + \|B'\|_{H_{LIL}^1} \right) \right)$$

for $|t| \leq T_0/\epsilon$. Again this is dominated by the right hand side of (3.5). Similarly we use (3.3) to handle the terms in the last line, which are controlled by

$$\epsilon \left(C_\omega \sqrt{\log \log(1/\epsilon)} \left(\|A'\|_{H_{LIL}^2} + \|B'\|_{H_{LIL}^2} \right) \right).$$

It is here we see why H_{LIL}^3 is needed in (3.5). $\square$

Remark 3. We quickly note that if the springs and masses vary periodically, one finds that $\chi_m(j)$ and $\chi_k(j)$ are in ℓ^∞ and then this proof would demonstrate the size of the residuals is bounded by $C\epsilon^{1/2}$.

3.2. Boundedness in mean. The almost sure boundedness does not provide us with any kind of description for the ω dependent constant C_ω . In this section we estimate the error in mean, finding estimates in terms of σ_m and σ_k .

Lemma 3.6. *Let $y(j)$ and $\chi(j)$ be as in Theorem 3.1 and $n \in \mathbf{Z}^+ \cup \{0\}$. Consider the process*

$$W_j(n) := \chi(j+n) - \chi(j).$$

Then, for every j , $W_j(n)$ is a martingale in the variable n and, for any $N > 0$,

$$\mathbb{E}[\max_{0 \leq n \leq N} (W_j(n))^2] \leq 4N\sigma^2. \quad (3.7)$$

Proof. From the definition of χ , we have

$$W_j(n) = y(j) + \cdots + y(j+n-1). \quad (3.8)$$

Then

$$\mathbb{E}[|W_j(n)|] \leq C|n|.$$

Conditioning upon $W_j(n)$ gives

$$\mathbb{E}[W_j(n+1)|W_j(n)] = \mathbb{E}[y(j+n) + W_j(n)|W_j(n)] = W_j(n).$$

This proves that W_j is a martingale. From the basic theory of martingales this tells us $|W_j|^2$ is a submartingale. It follows from the L^p maximum inequality, see [3], and a direct computation using (3.8)

$$\mathbb{E}\left[\max_{0 \leq n \leq N} (W_j(n))^2\right] \leq 4\mathbb{E}[|W_j(N)|^2] = 4N\sigma^2.$$

□

Remark 4. We can define a similar process $W_j(n) := \chi(j-n) - \chi(j)$ which would have exactly the same properties but with a different version of (3.8) i.e.

$$W_j(n) = y(j-n) + \cdots + y(j-1).$$

This symmetry allows us to handle positive and negative times with the same argument.

We use the following corollary in the results that follow.

Corollary 3.7.

$$\chi_k(j+n) - \chi_k(j) \quad \text{and} \quad \chi_m(j+n) - \chi_m(j)$$

are martingales in n with

$$\begin{aligned} \mathbb{E}\left[\max_{0 \leq n \leq N} (\chi_k(j+n) - \chi_k(j))^2\right] &\leq 4N\tilde{k}\sigma_k^2 \quad \text{and} \\ \mathbb{E}\left[\max_{0 \leq n \leq N} (\chi_m(j+n) - \chi_m(j))^2\right] &\leq 4N\frac{\sigma_m^2}{\tilde{m}}. \end{aligned}$$

We have now gotten the necessary probability out of the way to prove the following lemma, analogous to Lemma 3.3, but in expectation.

Lemma 3.8. *For any $T_0 > 0$ and $\epsilon \in (0, 1/4)$ the following inequalities hold*

$$\mathbb{E} \left[\sup_{|t| \leq T_0/\epsilon} \|\chi(\cdot) F(\epsilon(\cdot - ct))\|_{\ell^2} \right] \leq 2\sqrt{2}\epsilon^{-1} \sigma \max\{2\sqrt{|c|T_0}, 1\} \|F\|_{H_{sr}^2} \quad (3.9)$$

and

$$\mathbb{E} \left[\sup_{|t| \leq T_0/\epsilon} \|\chi(\cdot) \delta^\pm F(\epsilon(\cdot - ct))\|_{\ell^2} \right] \leq 3\sqrt{2}\sigma \max\{2\sqrt{|c|T_0}, 1\} \|F\|_{H_{sr}^3}. \quad (3.10)$$

In the above χ is either χ_k or χ_m with σ either $\sigma_k \sqrt{k}$ or $\frac{\sigma_m}{\sqrt{m}}$ respectively.

Proof. Without loss of generality (see Remark 4) let $t \in \mathbf{R}^+ \cup \{0\}$. Write $ct = [ct] + \alpha$ where $\alpha \in [0, 1)$. Let $n \in \mathbf{Z}$ in the following. We start with the inequality

$$\sup_{0 \leq t \leq T_0/\epsilon} \|\chi(\cdot) F(\epsilon \cdot - \epsilon ct)\|_{\ell^2}^2 \leq \sup_{0 \leq n \leq cT_0/\epsilon, \alpha \in [0, 1)} \sum_{j \in \mathbf{Z}} \chi(j)^2 F(\epsilon j - \epsilon n - \epsilon \alpha)^2.$$

The inequality is due to the fact that for any $t \in [0, [T_0/\epsilon] + 1)$ there exists an $n \in [0, cT_0/\epsilon]$ and $\alpha \in [0, 1)$ such that $n + \alpha = ct$, which is a slightly greater range for t than we initially cared about. Using the Mean Value Theorem, we have that

$$F(\epsilon j - \epsilon n - \epsilon \alpha) = F(\epsilon j - \epsilon n) - \epsilon \alpha F'(x_j)$$

where $x_j \in (\epsilon j - \epsilon n - \epsilon \alpha, \epsilon j - \epsilon n)$. Substituting this in and using the basic inequality $(a + b)^2 \leq 2(a^2 + b^2)$, we get

$$\begin{aligned} & \sup_{0 \leq t \leq T_0/\epsilon} \|\chi(\cdot) F(\epsilon \cdot - \epsilon ct)\|_{\ell^2}^2 \\ & \leq \sup_{0 \leq n \leq cT_0/\epsilon, \alpha \in [0, 1)} \sum_{j \in \mathbf{Z}} \chi(j)^2 (F(\epsilon j - \epsilon n) - \epsilon \alpha F'(x_j))^2 \\ & \leq \sup_{0 \leq n \leq cT_0/\epsilon, \alpha \in [0, 1)} 2 \sum_{j \in \mathbf{Z}} \chi(j)^2 F(\epsilon j - \epsilon n)^2 + \chi(j)^2 (\epsilon \alpha F'(x_j))^2. \end{aligned}$$

We look at the two terms

$$\begin{aligned} I &= \mathbb{E} \left[\sup_{0 \leq n \leq cT_0/\epsilon, \alpha \in [0, 1)} 2 \sum_{j \in \mathbf{Z}} \chi(j)^2 F(\epsilon j - \epsilon n)^2 \right] \\ II &= \mathbb{E} \left[\sup_{0 \leq n \leq cT_0/\epsilon, \alpha \in [0, 1)} 2 \sum_{j \in \mathbf{Z}} \chi(j)^2 (\epsilon \alpha F'(x_j))^2 \right] \end{aligned}$$

separately. Starting with I , we find from a change of indices, that

$$\begin{aligned} I &= \mathbb{E} \left[\sup_{0 \leq n \leq cT_0/\epsilon} 2 \sum_{j \in \mathbf{Z}} \chi(j + n)^2 F(\epsilon j)^2 \right] \\ &= \mathbb{E} \left[\sup_{0 \leq n \leq cT_0/\epsilon} 2 \sum_{j \in \mathbf{Z}} (\chi(j + n) - \chi(j) + \chi(j))^2 F(\epsilon j)^2 \right] \end{aligned} \quad (3.11)$$

Using the same basic inequality as above we get

$$I \leq \mathbb{E} \left[\sup_{0 \leq n \leq cT_0/\epsilon} 4 \sum_{j \in \mathbf{Z}} (\chi(j)^2 + (\chi(j + n) - \chi(j))^2) F(\epsilon j)^2 \right]. \quad (3.12)$$

The supremum sees only the term with n , and Fubini's theorem allows the expected value to pass through the sum. And so

$$I \leq 4 \sum_{j \in \mathbf{Z}} \left(\mathbb{E} [\chi(j)^2] + \mathbb{E} \left[\sup_{0 \leq n \leq cT_0/\epsilon} (\chi(j+n) - \chi(j))^2 \right] \right) F(\epsilon j)^2.$$

A direct computation on the first term using the definition of χ and using Corollary 3.7 on the second term we find

$$I \leq 4 \sum_{j \in \mathbf{Z}} (\sigma^2 |j| + 4\sigma^2 cT_0 \epsilon^{-1}) F(\epsilon j)^2. \quad (3.13)$$

According to Lemma 4.3 and 4.4 from [5], I is dominated by

$$8\epsilon^{-2} \sigma^2 \max\{4cT_0, 1\} \|F\|_{H_{sr}^1}^2. \quad (3.14)$$

Now we turn our attention to II . We can eliminate the α dependence by taking $\alpha = 1$ *i.e.* choose $\tilde{x}_j$ such that

$$F'(\tilde{x}_j) = \max_{x \in [\epsilon j - \epsilon n - \epsilon, \epsilon j - \epsilon n]} F'(x).$$

Then

$$(\epsilon \alpha F'(x_j))^2 \leq (\epsilon F'(\tilde{x}_j))^2.$$

Shifting the index by n we get

$$II \leq \mathbb{E} \left[\sup_{0 \leq n \leq cT_0/\epsilon} 2 \sum_{j \in \mathbf{Z}} \chi(j+n)^2 F'(\tilde{x}_{j+n})^2 \right]$$

where $\tilde{x}_{j+n} \in [\epsilon j - \epsilon, \epsilon j]$ does not depend on n . We therefore may relabel $\tilde{x}_j = \tilde{x}_{j+n}$. We use the same steps here as we used from (3.12) to (3.13).

$$II \leq 4 \sum_{j \in \mathbf{Z}} (\sigma^2 |j| + 4\sigma^2 cT_0 \epsilon^{-1}) \epsilon^2 F'(\epsilon \tilde{x}_j)^2.$$

Again, by Lemma 4.3 from [5], II is dominated by

$$8\sigma^2 \max\{4cT_0, 1\} \|F\|_{H_{sr}^2}^2. \quad (3.15)$$

By (3.14) and (3.15) we have

$$\mathbb{E} \left[\sup_{0 \leq t \leq T_0/\epsilon} \|\chi(\cdot) F(\epsilon \cdot - \epsilon ct)\|^2 \right] \leq 8\epsilon^{-2} \sigma^2 \max\{4cT_0, 1\} \|F\|_{H_{sr}^2}^2.$$

An standard application of Jensen's inequality yields

$$\mathbb{E} \left[\sup_{0 \leq t \leq T_0/\epsilon} \|\chi(\cdot) F(\epsilon \cdot - \epsilon ct)\| \right] \leq 2\sqrt{2} \epsilon^{-1} \sigma \max\{2\sqrt{cT_0}, 1\} \|F\|_{H_{sr}^3}.$$

This proves (3.9).

The exact same trickery that was used in Lemma 3.3 works to prove (3.10). Using (3.9) and then (3.4)

$$\mathbb{E} \left[\sup_{0 \leq t \leq T_0/\epsilon} \|\chi(\cdot) \delta^+ F(\epsilon(\cdot - ct))\|_{\ell^2} \right] \leq 3\sqrt{2} \sigma \max\{2\sqrt{cT_0}, 1\} \|F\|_{H_{sr}^3}.$$

This shows (3.10). □

Remark 5. The functions in this subsection are required to be once more differentiable than the functions in the previous subsection, due to the use of the Mean Value Theorem in the beginning of the proof of the previous lemma.

Now we can prove:

Proposition 3.9. Fix $A, B \in H_{sr}^4$ and take $\tilde{r}_\epsilon$ and $\tilde{p}_\epsilon$ as in (2.18). Fix $T_0 > 0$. For there exists a positive constant $C(k, \bar{m}, a_k, b_k, a_m, b_m, T_0, \|A\|_{H_{sr}^4}, \|B\|_{H_{sr}^4})$ for which $\epsilon \in (0, 1/4)$ implies

$$\mathbb{E} \left[\sup_{|t| \leq T_0/\epsilon} (\|\text{Res}_1(\tilde{r}_\epsilon, \tilde{p}_\epsilon)\|_{\ell^2} + \|\text{Res}_2(\tilde{r}_\epsilon, \tilde{p}_\epsilon)\|_{\ell^2}) \right] \leq C\epsilon \left(\epsilon^{1/2} + \max\{\sigma_m, \sigma_k\} \right). \quad (3.16)$$

Proof. The proof begins the same way as the proof for Proposition 3.5 except now we take expectation of (3.6). Since the first two lines of (3.6) are deterministic, using Lemma 4.3 in [5], they are controlled by

$$\epsilon^{3/2} \frac{\sqrt{2}}{\sqrt{k\bar{m}}} (\|A\|_{H^2} + \|B\|_{H^2}).$$

Next use (3.9) to control the third line with

$$\frac{2\sqrt{2}\epsilon\sigma_m c \max\{2\sqrt{|c|T_0}, 1\}}{a_k\sqrt{\bar{m}}} (\|A'\|_{H_{sr}^2} + \|B'\|_{H_{sr}^2}),$$

which is dominated by (3.16). We use (3.10) to estimate the fourth line:

$$\frac{3\sqrt{2}\epsilon\sigma_k \max\{2\sqrt{|c|T_0}, 1\}}{\sqrt{\bar{m}}} (\|A'\|_{H_{sr}^3} + \|B'\|_{H_{sr}^3}).$$

As before, the estimate for Res_2 follows a parallel argument and is omitted. $\square$

4. Error estimates. In this section we prove rigorous estimates using “energy” arguments, similar to [5, 1, 10].

4.1. The energy argument. Let r and p be a true solution to (1.2) and take $\tilde{r}_\epsilon$ and $\tilde{p}_\epsilon$ as in (2.18). Define *error functions* η and ξ implicitly by

$$r = \tilde{r}_\epsilon + \frac{\eta}{k} \quad \text{and} \quad p = \tilde{p}_\epsilon + \xi. \quad (4.1)$$

It is our goal to determine the size in ℓ^2 of η and ξ during the period $|t| \leq T_0/\epsilon$. To that end, insert (4.1) into (1.2) to find that

$$\begin{aligned} \frac{\dot{\eta}}{k} &= \delta^+ \xi + \text{Res}_1 \\ m\dot{\xi} &= \delta^- \eta + \text{Res}_2 \end{aligned} \quad (4.2)$$

where $\text{Res}_1 = \text{Res}_1(\tilde{r}_\epsilon, \tilde{p}_\epsilon)$ and $\text{Res}_2 = \text{Res}_2(\tilde{r}_\epsilon, \tilde{p}_\epsilon)$ as in (2.1).

Next define the *energy* to be

$$H(t) := \frac{1}{2} \sum_{j \in \mathbb{Z}} [k(j)^{-1} \eta^2(j, t) + m(j) \xi^2(j, t)].$$

Since we have assumed that the $k(j)$ and $m(j)$ are drawn from distributions with support in $[a_k, b_k] \subset \mathbf{R}^+$ and $[a_m, b_m] \subset \mathbf{R}^+$, respectively, a short calculation shows that $\sqrt{H}$ is equivalent to $\|\eta, \xi\|_{\ell^2 \times \ell^2}$ and the constants of equivalence depend only on a_k, a_m, b_k and b_m . That is to say, the equivalence is realization independent.

Time differentiation of H gives

$$\dot{H} = \sum_{j \in \mathbf{Z}} \left[k^{-1} \eta \dot{\eta} + m \xi \dot{\xi} \right].$$

Using (4.2)

$$\dot{H} = \sum_{j \in \mathbf{Z}} \left[\eta (\delta^+ \xi + \text{Res}_1) + \xi (\delta^- \eta + \text{Res}_2) \right].$$

Summing by parts:

$$\dot{H} = \sum_{j \in \mathbf{Z}} [\eta \text{Res}_1 + \xi \text{Res}_2].$$

Cauchy-Schwarz implies that

$$\dot{H} \leq \|\text{Res}_1, \text{Res}_2\|_{\ell^2 \times \ell^2} \|\eta, \xi\|_{\ell^2 \times \ell^2}.$$

Then we use the equivalence of $\sqrt{H}$ and $\|\eta, \xi\|_{\ell^2 \times \ell^2}$ to get:

$$\dot{H} \leq C \|\text{Res}_1, \text{Res}_2\|_{\ell^2 \times \ell^2} \sqrt{H}.$$

Set

$$\Gamma_\epsilon := \sup_{|t| \leq T_0/\epsilon} \|\text{Res}_1, \text{Res}_2\|_{\ell^2 \times \ell^2},$$

so $\dot{H}/\sqrt{H} \leq C\Gamma_\epsilon$. We integrate from 0 to t

$$2\sqrt{H(t)} \leq 2\sqrt{H(0)} + C\Gamma_\epsilon t.$$

And so, for $t \leq T_0/\epsilon$, we have

$$\sqrt{H(t)} \leq \sqrt{H(0)} + C\Gamma_\epsilon T_0 \epsilon^{-1}.$$

If we use the equivalence of the $\sqrt{H}$ and $\|\eta, \xi\|_{\ell^2 \times \ell^2}$ once again, we find that we have proven

$$\sup_{|t| \leq T_0/\epsilon} \|\eta(t), \xi(t)\|_{\ell^2 \times \ell^2} \leq C \|\eta(0), \xi(0)\|_{\ell^2 \times \ell^2} + C\Gamma_\epsilon \epsilon^{-1}. \quad (4.3)$$

A key feature of the above inequality is that the only place where the specific realization of the springs and masses enters is through Γ_ϵ .

4.2. Almost sure error estimates. We can now prove our first main theorem, which is about almost sure estimation of the absolute error:

Theorem 4.1. *Fix $\Phi, \Psi \in H_{LIL}^3$ and $T_0 > 0$. Let r and p be the solution of (1.2) with initial data*

$$r(j, 0) = \Phi(\epsilon j)/k(j) \quad \text{and} \quad p(j, 0) = \Psi(\epsilon j).$$

For almost every realization of $\{k(j)\}$ and $\{m(j)\}$ there is a finite positive constant

$$C_\omega = C_\omega(k, m, a_k, b_k, a_m, b_m, \|\Phi\|_{H_{LIL}^3}, \|\Psi\|_{H_{LIL}^3})$$

for which $\epsilon \in (0, 1/4)$ implies

$$\sup_{|t| \leq T_0/\epsilon} \left\| r(\cdot, t) - \frac{1}{k(\cdot)} (A(\epsilon(\cdot - ct)) + B(\epsilon(\cdot + ct))) \right\|_{\ell^2} \leq C_\omega \sqrt{\log \log(1/\epsilon)}$$

and

$$\sup_{|t| \leq T_0/\epsilon} \left\| p(\cdot, t) - \frac{1}{\sqrt{k\bar{m}}} (-A(\epsilon(\cdot - ct)) + B(\epsilon(\cdot + ct))) \right\|_{\ell^2} \leq C_\omega \sqrt{\log \log(1/\epsilon)}.$$

In the above

$$A(X) := \frac{1}{2}\Phi(X) - \frac{\sqrt{k\tilde{m}}}{2}\Psi(X) \quad \text{and} \quad B(X) := \frac{1}{2}\Phi(X) + \frac{\sqrt{k\tilde{m}}}{2}\Psi(X).$$

Remark 6. In the case where the masses and springs vary periodically instead of randomly, the size of the error decreases to $C\epsilon^{1/2}$; in fact the proof we supply in a moment together with Remark 3 suffices to demonstrate this. It is this extra wiggle room in the error in this case which opens the door to longer time scales and KdV-like approximations.

Proof. Form $\tilde{r}_\epsilon$ and $\tilde{p}_\epsilon$ from the functions A and B as specified in (2.18) and η and ξ as in (4.1). A bit of algebra shows that

$$\eta(j, 0) = \epsilon \chi_m(j) (A'(\epsilon j) + B'(\epsilon j)) \quad \text{and} \quad \xi(j, 0) = \epsilon \frac{\chi_k(j)}{\sqrt{k\tilde{m}}} (-A'(\epsilon j) + B'(\epsilon j)). \quad (4.4)$$

Using (3.2) in a very crude way, we see that almost surely

$$\|\eta(0), \xi(0)\|_{\ell^2 \times \ell^2} \leq C_\omega \sqrt{\log \log(1/\epsilon)}$$

with the constant depending on $\|A\|_{H_{LIL}^2}$ and $\|B\|_{H_{LIL}^2}$. We estimated Γ_ϵ in Proposition 3.5 and found that $\Gamma_\epsilon \leq C_\omega \epsilon \sqrt{\log \log(1/\epsilon)}$ when $\epsilon \in (0, 1/4)$ almost surely. Therefore (4.3) gives

$$\sup_{|t| \leq T_0/\epsilon} \|\eta(t), \xi(t)\|_{\ell^2 \times \ell^2} \leq C_\omega \sqrt{\log \log(1/\epsilon)}.$$

To finish the proof we note that the triangle inequality tells us

$$\begin{aligned} & \left\| r(\cdot, t) - \frac{1}{k(\cdot)} (A(\epsilon(\cdot - ct)) + B(\epsilon(\cdot + ct))) \right\|_{\ell^2} \\ & \leq \|r(t) - \tilde{r}_\epsilon(t)\|_{\ell^2} + \left\| \tilde{r}_\epsilon(\cdot, t) - \frac{1}{k(\cdot)} (A(\epsilon(\cdot - ct)) + B(\epsilon(\cdot + ct))) \right\|_{\ell^2} \\ & \leq C\|\eta(t)\|_{\ell^2} + C\epsilon\|\chi_m(\cdot)A'(\epsilon(\cdot - ct))\| + C\epsilon\|\chi_m(\cdot)B'(\epsilon(\cdot - ct))\|_{\ell^2}. \end{aligned}$$

The terms involving A and B can be estimated using (3.2) by $C_\omega \sqrt{\log \log(1/\epsilon)}$ so we find

$$\sup_{|t| \leq T_0/\epsilon} \left\| r(\cdot, t) - \frac{1}{k(\cdot)} (A(\epsilon(\cdot - ct)) + B(\epsilon(\cdot + ct))) \right\|_{\ell^2} \leq C_\omega \sqrt{\log \log(1/\epsilon)}.$$

The remaining estimate in the Theorem 4.1 is shown by a parallel argument and is omitted. $\square$

It may seem like the estimates in Theorem 4.1 are utterly useless since the size of the error diverges as $\epsilon \rightarrow 0^+$. But the error in that theorem is the *absolute* error; the *relative* error does in fact vanish in the limit.

Corollary 4.2. *Under the same conditions as in Theorem 4.1 we almost surely have*

$$\lim_{\epsilon \rightarrow 0^+} \sup_{|t| \leq T_0/\epsilon} \frac{\left\| r(\cdot, t) - \frac{1}{k(\cdot)} (A(\epsilon(\cdot - ct)) + B(\epsilon(\cdot + ct))) \right\|_{\ell^2}}{\|r(t)\|_{\ell^2}} = 0$$

and

$$\lim_{\epsilon \rightarrow 0^+} \sup_{|t| \leq T_0/\epsilon} \frac{\left\| p(\cdot, t) - \frac{1}{\sqrt{k\tilde{m}}} (-A(\epsilon(\cdot - ct)) + B(\epsilon(\cdot + ct))) \right\|_{\ell^2}}{\|p(t)\|_{\ell^2}} = 0.$$

Proof. The reverse triangle inequality gives

$$\begin{aligned} \|r(t)\|_{\ell^2} &\geq \left\| \frac{1}{k(\cdot)} (A(\epsilon(\cdot - ct)) + B(\epsilon(\cdot + ct))) \right\|_{\ell^2} \\ &\quad - \left\| r(\cdot, t) - \frac{1}{k(\cdot)} (A(\epsilon(\cdot - ct)) + B(\epsilon(\cdot + ct))) \right\|_{\ell^2}. \end{aligned}$$

Using Lemma 4.3 from [5] for the first term and Theorem 4.1 for the second we obtain

$$\|r(t)\|_{\ell^2} \geq C\epsilon^{-1/2} - C_\omega \sqrt{\log \log(1/\epsilon)}$$

for all $|t| \leq T_0/\epsilon$. This is positive for ϵ small enough and so we get the first limit in the corollary by dividing the absolute error for r in Theorem 4.1 by this estimate and taking the limit. The second limit is analogous. $\square$

4.3. Error estimate in mean. We can now prove our second main theorem, which is an estimate of the mean of the error.

Theorem 4.3. Fix $\Phi, \Psi \in H_{sr}^4$ and $T_0 > 0$. Let r and p be the solution of (1.2) with initial data

$$r(j, 0) = \Phi(\epsilon j)/k(j) \quad \text{and} \quad p(j, 0) = \phi(\epsilon j).$$

There exists a positive constant $C(\tilde{k}, \tilde{m}, a_k, b_k, a_m, b_m, T_0, \|A\|_{H_{sr}^4}, \|B\|_{H_{sr}^4})$ for which $\epsilon \in (0, 1/4)$ implies

$$\mathbb{E} \left[\sup_{|t| \leq T_0/\epsilon} \left\| r(\cdot, t) - \frac{1}{k(\cdot)} (A(\epsilon(\cdot - ct)) + B(\epsilon(\cdot + ct))) \right\|_{\ell^2} \right] \leq C \left(\epsilon^{1/2} + \max\{\sigma_m, \sigma_k\} \right)$$

and

$$\begin{aligned} &\mathbb{E} \left[\sup_{|t| \leq T_0/\epsilon} \left\| p(\cdot, t) - \frac{1}{\sqrt{k\tilde{m}}} (-A(\epsilon(\cdot - ct)) + B(\epsilon(\cdot + ct))) \right\|_{\ell^2} \right] \\ &\leq C \left(\epsilon^{1/2} + \max\{\sigma_m, \sigma_k\} \right). \end{aligned}$$

In the above

$$A(X) := \frac{1}{2}\Phi(X) - \frac{\sqrt{k\tilde{m}}}{2}\Psi(X) \quad \text{and} \quad B(X) := \frac{1}{2}\Phi(X) + \frac{\sqrt{k\tilde{m}}}{2}\Psi(X).$$

Proof. Begin as in the proof of Theorem 4.1. Using (3.9) on (4.4)

$$\|\eta(0), \xi(0)\|_{\ell^2 \times \ell^2} \leq C \max\{\sigma_m, \sigma_k\}$$

with constant C depending on $\|A\|_{H_{sr}^3}$ and $\|B\|_{H_{sr}^3}$. Proposition 3.9 gives us

$$\mathbb{E} [\Gamma_\epsilon] \leq C\epsilon \left(\epsilon^{1/2} + \max\{\sigma_m, \sigma_k\} \right)$$

when $\epsilon \in (0, 1/4)$. Therefore (4.3) gives

$$\mathbb{E} \left[\sup_{|t| \leq T_0/\epsilon} \|\eta(t), \xi(t)\|_{\ell^2 \times \ell^2} \right] \leq C \left(\epsilon^{1/2} + \max\{\sigma_m, \sigma_k\} \right).$$

To finish the proof we note that the triangle inequality tells us

$$\begin{aligned}
& \mathbb{E} \left[\sup_{|t| \leq T_0/\epsilon} \left\| r(\cdot, t) - \frac{1}{k(\cdot)} (A(\epsilon(\cdot - ct)) + B(\epsilon(\cdot + ct))) \right\|_{\ell^2} \right] \\
& \leq \mathbb{E} \left[\sup_{|t| \leq T_0} \|r(t) - \tilde{r}_\epsilon(t)\|_{\ell^2} \right] \\
& \quad + \mathbb{E} \left[\sup_{|t| \leq T_0} \left\| \tilde{r}_\epsilon(\cdot, t) - \frac{1}{k(\cdot)} (A(\epsilon(\cdot - ct)) + B(\epsilon(\cdot + ct))) \right\|_{\ell^2} \right] \\
& \leq \mathbb{E} \left[\sup_{|t| \leq T_0} C \|\eta(t)\|_{\ell^2} \right] \\
& \quad + \mathbb{E} \left[\sup_{|t| \leq T_0} C \epsilon \|\chi_m(\cdot) A'(\epsilon(\cdot - ct))\| \right] + \mathbb{E} \left[\sup_{|t| \leq T_0} C \epsilon \|\chi_m(\cdot) B'(\epsilon(\cdot - ct))\|_{\ell^2} \right].
\end{aligned}$$

The C depends on a_k, b_k, a_m , and b_m , which are fixed, so we may pull it out of the expected value. The terms that involve A and B can be estimated using (3.9) by $C \max\{\sigma_k, \sigma_m\}$ so we find

$$\begin{aligned}
& \mathbb{E} \left[\sup_{|t| \leq T_0/\epsilon} \left\| r(\cdot, t) - \frac{1}{k(\cdot)} (A(\epsilon(\cdot - ct)) + B(\epsilon(\cdot + ct))) \right\|_{\ell^2} \right] \\
& \leq C \left(\epsilon^{1/2} + \max\{\sigma_k, \sigma_m\} \right).
\end{aligned}$$

The remaining estimate in the Theorem 4.3 is shown by a parallel argument and is omitted. $\square$

5. Coarse-graining. We now prove strong convergence results using the ideas of coarse-graining from [8]. We need quite a few tools. Letting $f : \mathbf{Z} \rightarrow \mathbf{R}$ and $g, u, v : \mathbf{R} \rightarrow \mathbf{R}$ define

$$\begin{aligned}
F[f](\kappa) &:= \frac{1}{2\pi} \sum_{j \in \mathbf{Z}} e^{-ij\kappa} f(j) \\
F^{-1}[g](j) &:= \int_{-\pi}^{\pi} g(\kappa) e^{i\kappa j} \\
\mathcal{F}[u](\xi) &:= \frac{1}{2\pi} \int_{\mathbf{R}} u(x) e^{-i\xi x} dx \\
\mathcal{F}^{-1}[v](x) &:= \int_{\mathbf{R}} v(\xi) e^{i\xi x} d\xi \\
\theta_\phi(\kappa) &:= \begin{cases} 1 & \kappa \in (-\phi, \phi) \\ 0 & \text{else} \end{cases} \\
\mathcal{L}[f](x) &:= \mathcal{F}^{-1}[\theta_\pi(\cdot) F[f](\cdot)](x) \\
\mathcal{S}[u](j) &:= u(j).
\end{aligned}$$

These are, in order, the Fourier transform for sequences, its inverse, the Fourier transform of functions $\mathbf{R} \rightarrow \mathbf{R}$, its inverse, the indicator function of $(-\phi, \phi)$, a “low pass” interpolation operator, and a sampling operator. To be clear, in the above $x, \xi, \kappa \in \mathbf{R}$ and $j \in \mathbf{Z}$, always.

The operator $\mathcal{L}$ converts a sequence f defined on $\mathbf{Z}$ to a new function defined on $\mathbf{R}$. The sampling function $\mathcal{S}$ returns a sequence from a function defined on $\mathbf{R}$. It is an easy exercise to show that $\mathcal{S}\mathcal{L}[f](j) = f(j)$ so it is clear $\mathcal{L}$ is an interpolation operator. Another essential property is the following.

Lemma 5.1. *Let f be a sequence in ℓ^2 . Then*

$$\|f\|_{\ell^2} = 2\pi \|\mathcal{L}[f]\|_{L^2(\mathbf{R})}.$$

Proof. By Plancherel's theorem for Fourier series:

$$\|F[f]\|_{L^2(-\pi, \pi)} = \frac{1}{2\pi} \|f\|_{\ell^2}.$$

Then by Plancherel's theorem for the Fourier transform:

$$\|\mathcal{F}^{-1}[\theta_\pi F[f]]\|_{L^2(\mathbf{R})} = \|\theta_\pi F[f]\|_{L^2(\mathbf{R})} = \|F[f]\|_{L^2(-\pi, \pi)}$$

completing the proof. $\square$

We need one more lemma before we can state the strong convergence results. It states that the more frequently you sample a function, the more its interpolation looks like the original function.

Lemma 5.2. *Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be continuous and in H^s with $s > 0$. Put $f_\epsilon(x) := f(\epsilon x)$. Then*

$$\lim_{\epsilon \rightarrow 0^+} \|\mathcal{L}\mathcal{S}[f_\epsilon](\cdot/\epsilon) - f\|_{L^2} = 0.$$

Proof. From their definitions we have

$$\mathcal{L}\mathcal{S}[f_\epsilon](x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{j \in \mathbf{Z}} e^{-i\kappa j} f(\epsilon j) \right) e^{i\kappa x} d\kappa = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{j \in \mathbf{Z}} e^{-i \frac{\kappa}{\epsilon} \epsilon j} f(\epsilon j) \right) e^{i\kappa x} d\kappa.$$

Changing variables with $u = \kappa/\epsilon$ we get

$$\mathcal{L}\mathcal{S}[f_\epsilon](x) = \frac{1}{2\pi} \int_{-\pi/\epsilon}^{\pi/\epsilon} \left(\sum_{j \in \mathbf{Z}} \epsilon e^{-iu\epsilon j} f(\epsilon j) \right) e^{iu\epsilon x} du.$$

Exchanging the sum and integral and then computing the integral gives

$$\mathcal{L}\mathcal{S}[f_\epsilon](x) = \sum_{j \in \mathbf{Z}} f(\epsilon j) \operatorname{sinc}(x - j).$$

This sinc is the normalized sinc function, $\frac{\sin \pi x}{\pi x}$.

Now put $\tilde{f}_\epsilon(X) := \mathcal{F}^{-1}[\theta_{\pi/\epsilon} \mathcal{F}[f]](X)$. $\tilde{f}_\epsilon$ is a band-limited approximation of f . Using Plancherel's theorem

$$\|f - \tilde{f}_\epsilon\|_{L^2}^2 = \|\mathcal{F}[f] - \theta_{\pi/\epsilon} \mathcal{F}[f]\|_{L^2}^2 = \int_{|\kappa| > \pi/\epsilon} |\mathcal{F}[f](\kappa)|^2 d\kappa.$$

Since $f \in H^s$ we have

$$\begin{aligned} \|f - \tilde{f}_\epsilon\|_{L^2}^2 &= \int_{|\kappa| > \pi/\epsilon} \frac{1}{|\kappa|^{2s}} |\kappa|^{2s} |\mathcal{F}[f](\kappa)|^2 d\kappa \\ &\leq \sup_{|\kappa| > \pi/\epsilon} \left(\frac{1}{|\kappa|^{2s}} \right) \int_{|\kappa| > \pi/\epsilon} |\kappa|^{2s} |\mathcal{F}[f](\kappa)|^2 d\kappa \\ &\leq C \epsilon^{2s} \|f\|_{H^s}^2. \end{aligned}$$

Since $s > 0$ we see that $\lim_{\epsilon \rightarrow 0^+} \|f - \tilde{f}_\epsilon\|_{L^2} = 0$.

Since $\tilde{f}_\epsilon$ is band-limited, it is exactly equal to its cardinal series, see [7]:

$$\tilde{f}_\epsilon(X) = \sum_{j \in \mathbf{Z}} f(\epsilon j) \operatorname{sinc}(X/\epsilon - j).$$

But this is exactly equal to $\mathcal{LS}[f_\epsilon](X/\epsilon)$. Therefore we have shown that

$$\lim_{\epsilon \rightarrow 0^+} \|\mathcal{LS}(f_\epsilon)(\cdot/\epsilon) - f(\cdot)\|_{L^2} = 0.$$

□

Remark 7. We assumed that f is continuous in Lemma 5.2 so that the sampling map $\mathcal{S}$ is well-defined upon it. In fact, using the same sorts of calculations as in the proof it is possible to show that $\mathcal{LS}[f](x) = \mathcal{F}^{-1}[\theta_\pi(\cdot)\mathcal{F}[f](\cdot)](x)$ for continuous functions. Since the right hand side of this formula makes sense for functions which are merely L^2 , we can extend $\mathcal{LS}$ to such functions without the need of continuity. This line of reasoning allows us to eliminate the continuity condition from the lemma (though we still require $s > 0$).

Here is our first coarse-graining result:

Theorem 5.3. Fix $\Phi, \Psi \in H_{LIL}^3$ and $T_0 > 0$. Let r and p be the solution of (1.2) with initial data

$$r(j, 0) = \Phi(\epsilon j)/k(j) \quad \text{and} \quad p(j, 0) = \Psi(\epsilon j).$$

Put

$$Q_\epsilon(X, \tau) = \mathcal{L}[kr(\cdot, \tau/\epsilon)](X/\epsilon) \quad \text{and} \quad P_\epsilon(X, \tau) = \mathcal{L}[p(\cdot, \tau/\epsilon)](X/\epsilon).$$

Suppose that $Q_0(X, \tau)$ and $P_0(X, \tau)$ solve (2.14) with initial data $Q_0(X, 0) = \Phi(X)$ and $P_0(X, 0) = \Psi(X)$. Then, almost surely,

$$\lim_{\epsilon \rightarrow 0^+} \sup_{|\tau| \leq T_0} (\|Q_\epsilon(\cdot, \tau) - Q_0(\cdot, \tau)\|_{L^2} + \|P_\epsilon(\cdot, \tau) - P_0(\cdot, \tau)\|_{L^2}) = 0.$$

Proof. We show the limit for $\|P_\epsilon - P_0\|_{L^2}$ as the other is all but identical. By the triangle inequality we have

$$\begin{aligned} & \|P_\epsilon(\cdot, \tau) - P_0(\cdot, \tau)\|_{L^2} \\ & \leq \|P_\epsilon(\cdot, \tau) - \mathcal{LS}[P_0(\epsilon, \tau)](\cdot/\epsilon)\|_{L^2} + \|\mathcal{LS}[P_0(\epsilon, \tau)](\cdot/\epsilon) - P_0(\cdot, \tau)\|_{L^2}. \end{aligned}$$

The second term vanishes as $\epsilon \rightarrow 0^+$ by virtue of Lemma 5.2. (In fact, given (2.15) one sees that this convergence happens uniformly for all $\tau \in \mathbf{R}$.)

For the first term we do a change of variables $X = \epsilon x$ and $\tau = \epsilon t$ to get

$$\|P_\epsilon(\cdot, \tau) - \mathcal{LS}[P_0(\epsilon, \tau)](\cdot/\epsilon)\|_{L^2} = \sqrt{\epsilon} \|P_\epsilon(\epsilon, \epsilon t) - \mathcal{LS}[P_0(\epsilon, \epsilon t)](\cdot)\|_{L^2}.$$

Then we use the definition of P_ϵ and Lemma 5.1 to get

$$\|P_\epsilon(\cdot, \tau) - \mathcal{LS}[P_0(\epsilon, \tau)](\cdot/\epsilon)\|_{L^2} = \frac{1}{2\pi} \sqrt{\epsilon} \|p(\cdot, t) - \mathcal{S}[P_0(\epsilon, \epsilon t)]\|_{\ell^2}.$$

Using (2.15) and the formulas relating Φ and Ψ to A and B in Theorem 4.1 we see

$$\mathcal{S}[P_0(\epsilon, \epsilon t)](j) = \frac{1}{\sqrt{k\bar{m}}} (-A(\epsilon(j - ct)) + B(\epsilon(j + ct))).$$

Thus we can use the final estimate in Theorem 4.1 to get

$$\begin{aligned} \sup_{|\tau| \leq T_0} \|P_\epsilon(\cdot, \tau) - \mathcal{LS}[P_0(\epsilon, \tau)](\cdot/\epsilon)\|_{L^2} &\leq \sup_{|t| \leq T_0/\epsilon} \frac{1}{2\pi} \sqrt{\epsilon} \|p(\cdot, t) - \mathcal{S}[P_0(\epsilon, \epsilon t)]\|_{\ell^2} \\ &\leq C_\omega \sqrt{\epsilon \log \log(1/\epsilon)}. \end{aligned} \quad (5.1)$$

The right hand side goes to zero as $\epsilon \rightarrow 0^+$ and we are done. $\square$

We have a similar result but the convergence is in mean:

Theorem 5.4. Fix $\Phi, \Psi \in H_{sr}^4$ and $T_0 > 0$. Let r and p be the solution of (1.2) with initial data

$$r(j, 0) = \Phi(\epsilon j)/k(j) \quad \text{and} \quad p(j, 0) = \Psi(\epsilon j).$$

Put

$$Q_\epsilon(X, \tau) = \mathcal{L}[kr(\cdot, \tau/\epsilon)](X/\epsilon) \quad \text{and} \quad P_\epsilon(X, \tau) = \mathcal{L}[p(\cdot, \tau/\epsilon)](X/\epsilon).$$

Suppose that $Q_0(X, \tau)$ and $P_0(X, \tau)$ solve (2.14) with initial data $Q_0(X, 0) = \Phi(X)$ and $P_0(X, 0) = \Psi(X)$. Then

$$\lim_{\epsilon \rightarrow 0^+} \mathbb{E} \left[\sup_{|\tau| \leq T_0} (\|Q_\epsilon(\cdot, \tau) - Q_0(\cdot, \tau)\|_{L^2} + \|P_\epsilon(\cdot, \tau) - P_0(\cdot, \tau)\|_{L^2}) \right] = 0.$$

Proof. As before, we start with the triangle inequality

$$\begin{aligned} &\mathbb{E} \left[\sup_{|\tau| \leq T_0} \|P_\epsilon(\cdot, \tau) - P_0(\cdot, \tau)\|_{L^2} \right] \\ &\leq \mathbb{E} \left[\sup_{|\tau| \leq T_0} \|P_\epsilon(\cdot, \tau) - \mathcal{LS}[P_0(\epsilon, \tau)](\cdot/\epsilon)\|_{L^2} \right] \\ &\quad + \mathbb{E} \left[\sup_{|\tau| \leq T_0} \|\mathcal{LS}[P_0(\epsilon, \tau)](\cdot/\epsilon) - P_0(\cdot, \tau)\|_{L^2} \right]. \end{aligned}$$

The expected value does not see the second term, so it vanishes as $\epsilon \rightarrow 0^+$ by virtue of Lemma 5.2. The same steps are valid up through (5.1) only now we take expectation and use Theorem 4.3

$$\mathbb{E} \left[\sup_{|\tau| \leq T_0} \|P_\epsilon(\cdot, \tau) - \mathcal{LS}[P_0(\epsilon, \tau)](\cdot/\epsilon)\|_{L^2} \right] \leq C\sqrt{\epsilon} \max\{\sigma_k, \sigma_m\}.$$

The right hand side vanishes as $\epsilon \rightarrow 0^+$. $\square$

6. Simulations. We present various numerical data supporting our results. In our experiments, the springs k are picked to be constant, and the probability distribution of the masses m such that $\bar{m} = 1$. If the springs had also been chosen randomly, the results would have looked the same, but it is computationally less expensive to keep them constant. We choose initial conditions

$$r(j) = e^{-(\epsilon j)^2} \quad \text{and} \quad p(j) = -e^{-(\epsilon j)^2}.$$

From these

$$A(X) = e^{-X^2} \quad \text{and} \quad B(X) = 0.$$

We numerically integrate (1.2) to get $r(j, t)$ and use this to calculate the relative error which we call ρ

$$\rho := \sup_{0 \leq t \leq T_0/\epsilon} \frac{\|r(\cdot, t) - A(\epsilon(\cdot - ct))\|_{\ell^2}}{\|r(t)\|_{\ell^2}}.$$

According to Corollary 4.2, for some C_ω , ρ will vanish to 0 at least as fast as $C_\omega \sqrt{\epsilon \log \log(1/\epsilon)}$. Seeing the $\sqrt{\log \log(\epsilon)}$ is numerically challenging and we make no claim that we do here. However, if it were to show up in the numerical calculations, it would be best to factor it out, so we calculate

$$\frac{\rho}{\sqrt{\log \log(1/\epsilon)}}.$$

Now this should vanish at a rate no slower than $C_\omega \sqrt{\epsilon}$, which on a log-log plot, should look like a straight line with a slope of 1/2. Anything with a slope greater than 1/2 is vanishing at a faster rate.

We move onto the figures after one aside on the methods of integration used. Since the total energy of the system is conserved, it is worth performing experiments with a symplectic integrator. A six-step version of Yoshida's method, see [11], was initially used, as well as the standard four-step Runge-Kutta method. As it turns out, these methods produce negligible differences for the time scales studied, so most of the experiments below all use only the four-step Runge-Kutta for the sake of computational efficiency.

Moving on, Figure 1 gives some numerical validations of our relative error results, since the slope produced by the log-log plot is greater than 1/2. In this case, the realization of masses is the same for each ϵ . Figure 2 repeats the experiment in Figure 1 40 times, displaying the results as a series of box plots. A sample size of 40 was used because significantly larger sample sizes would require using more computing power since simulations for small values of ϵ are computationally demanding due to the long time scales. Figure 2, suggests the slope in Figure 1 is not a statistical anomaly.

It is worth noting that the most important tool of our analysis is χ_m . For instance, it allows one to carry out similar analysis with many different kinds of sequences of masses. If the average of the masses exists and one knows the growth rate of χ_m , then one can find an upper bound on the error. For example, in Figure 4, we use a sequence of masses such that $\chi(j)$ grows like $\sqrt{j}$. In particular, using two types of masses m_1 and m_2 , the following pattern works

$$m_1, m_2, m_1, m_1, m_2, m_2, m_1, m_1, m_1, m_2, m_2, m_2, \dots$$

We conjecture without providing arguments that if χ_m grows like $|j|^p$, analysis would show that the relative error is bounded by an ϵ^{1-p} order term, which in this case coincides with the numerical results seen in Figure 4.

There are also hints in our work, see for example 3.9, that for fixed ϵ and small σ_m , the mean of the error should be close to that of the system where masses and springs are taken to be constant average. Evidence for this is seen in Figure 5. When σ_m is smallest, then error from 40 trials, is concentrated around the error in the case of the system being constant coefficient, which was numerically calculated to be roughly 0.126. In conclusion, the simulations are a strong affirmation of our analytic results and that our bounds are at least close to optimal.

Relative Error for Fixed Random Masses

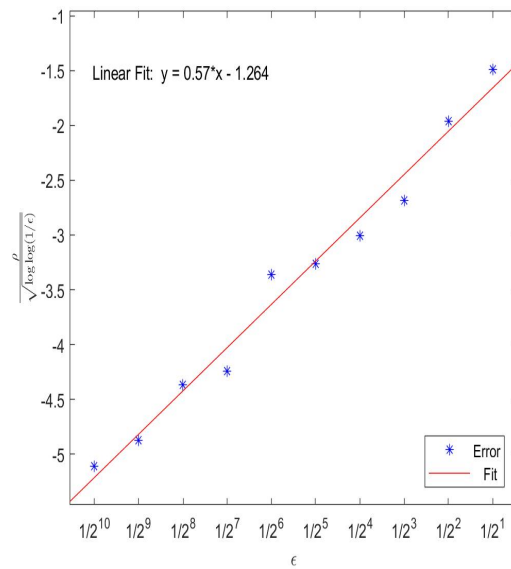


Figure 1. Figure 1 is a log-log plot of the relative error ρ divided by $\sqrt{\log \log(1/\epsilon)}$.

40 Random Experiments

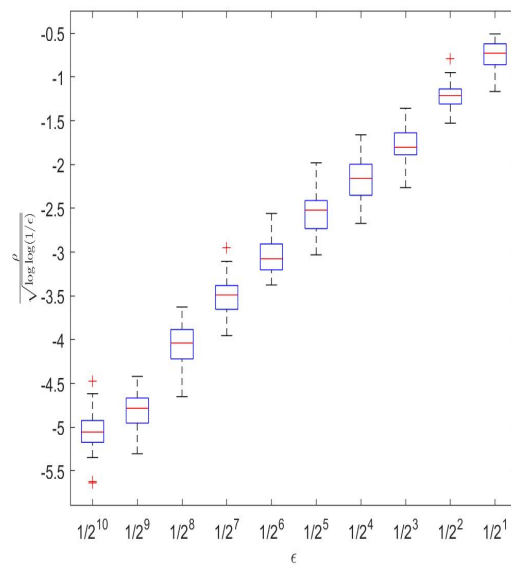


Figure 2. Figure 2 is 10 box plots of 40 different realization of masses at 10 various epsilons. It is also log-log.

Periodic Masses

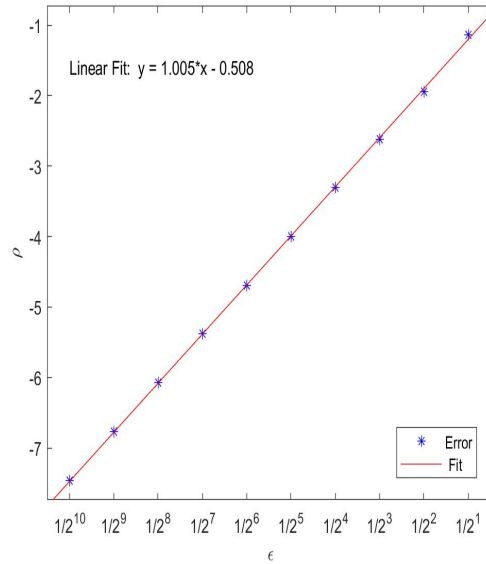


Figure 3. Figure 3 is a log-log plot of the relative error masses chosen periodically

$\chi(j)$ Grows like $\sqrt{j}$

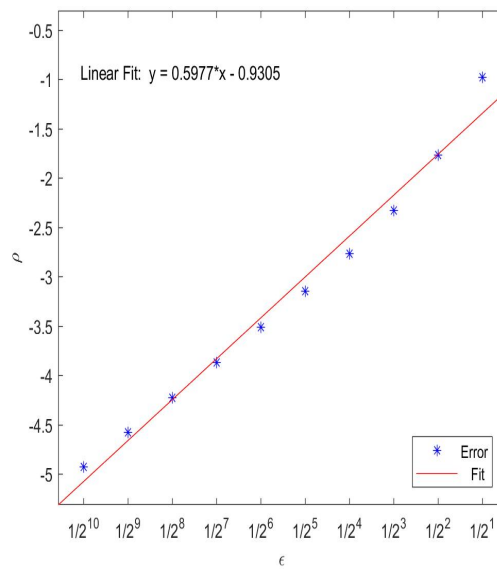


Figure 4. In Figure 4 masses are chosen so that $\chi(j)$ will grow like $\sqrt{j}$.

7. Conclusion. Our results are significant in several important ways regarding the description of approximate waves in the random polymer linear FPUT system. We have proven from first principles that that solutions to the wave equation are good approximate solutions to the system studied here. We showed that the absolute

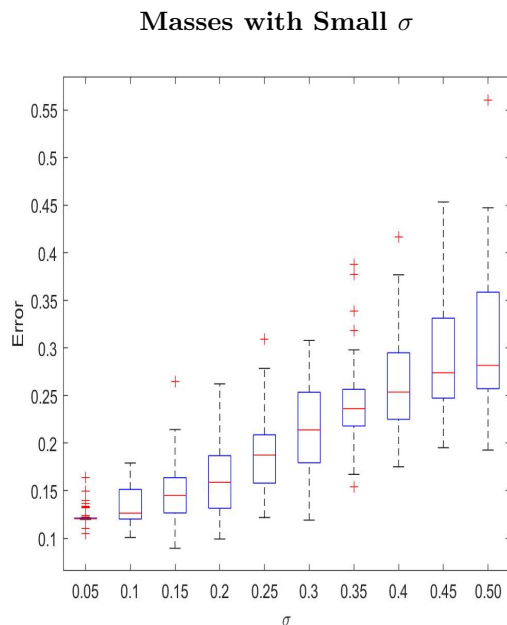


Figure 5. In Figure 5 ϵ is fixed and small while σ is varied and the absolute error is measured. When σ is smallest, the data is concentrated near the error for the constant coefficient case.

error only grows at most like $O(\log \log(1/\epsilon))$ almost surely and is constant in mean, but also small in mean if the masses and springs have small deviation. Using an interpolation operator with strong analytic properties we were able to show that the interpolated approximate solutions converged to interpolated true solutions in a relative sense a.s. and in expectation. Such results provide a rigorous justification for claiming that the relative error is made arbitrarily small by taking ϵ to be small.

The advantage of our method comes from the use of the random walk in capturing the build up of error. Since random walks of independent variables are well studied and sharp asymptotic estimates are known, we were able to use the random walk to its full extent. Although it remains unproven if the error we achieved is sharp, the numerical results suggest it is close, and it seems nothing more about the asymptotics of the random walk, at least in the almost sure sense, could be used to prove sharper bounds. It also remains unclear if the random walk is an intrinsic part of the mechanics of the problem or if it is only a useful fiction for modeling the error. To what extent could it be further exploited here and in other models that have similar dynamics

With this work we have laid the foundation for a couple of questions. First, can the error term be modeled by a random variable independent of ϵ with a nice probability distribution such as a Gaussian. There is also the question as to whether the results can be extended to higher dimensions. Probably most interesting is to determine what happens on longer time scales. The nonlinear periodic problem is known to be well-approximated by KdV equations for times times proportional to $1/\epsilon^3$ [1, 5] but it is not clear how to extend our work here to these longer time scales (where for this problem we would expect the approximating equation to be something like Airy's equation instead of KdV). This is mainly because one needs

to make sense of $\lim_{n \rightarrow \infty} \sum_{|j| \leq n} \chi(j)/n$, which, even if one optimistically replaces $\chi(j)$ with $\sqrt{|j|}$, will diverge. This raises the question: is it possible to find an effective equation describing the the dynamics for longer times and will these descriptions be statistical or is there room to achieve anything more definite, like the high probability and almost sure results constructed here?

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