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Orbital Stability of Internal Waves

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Abstract: This paper studies the nonlinear stability of capillary-gravity waves propagating along the interface dividing two immiscible fluid layers of finite depth. The motion in both regions is governed by the incompressible and irrotational Euler equations, with the density of each fluid being constant but distinct. A diverse collection of small-amplitude solitary wave solutions for this system have been constructed by several authors in the case of strong surface tension (as measured by the Bond number) and slightly subcritical Froude number. We prove that all of these waves are (conditionally) orbitally stable in the natural energy space. Moreover, the trivial solution is shown to be conditionally stable when the Bond and Froude numbers lie in a certain unbounded parameter region. For the near critical surface tension regime, we prove that one can infer conditional orbital stability or orbital instability of small-amplitude traveling waves solutions to the full Euler system from considerations of a dispersive PDE model equation. These results are obtained by reformulating the problem as an infinite-dimensional Hamiltonian system, then applying a version of the Grillakis–Shatah–Strauss method recently introduced in Varholm et al. (Commun Pure Appl Math 73:2634–2684, 2020). A key part of the analysis consists of computing the spectrum of the linearized augmented Hamiltonian at a shear flow or small-amplitude wave. For this, we generalize an idea used by Mielke (R Soc Lond Philos Trans Ser A Math Phys Eng Sci 360:2337–2358, 2002) to treat capillary-gravity water waves beneath vacuum.

Contents

1. Introduction

We consider the classical problem of determining the evolution of a free boundary dividing two superposed incompressible, inviscid, and immiscible fluids under the influence of gravity. This situation arises in countless applications, with a particularly important example being internal waves propagating along a pycnocline or thermocline in the ocean. Recent years have seen enormous progress made in understanding the Cauchy problem for this system, and there is now a robust (local) well-posedness theory. In parallel, a large body of work has established the existence of myriad traveling wave solutions. Far less is known about the stability of these waves. While many authors have addressed the spectral or linear stability of interfacial waves, nonlinear results are mostly limited to dispersive model equations such as Korteweg–de Vries (KdV). In this paper, we prove a number of theorems on the (conditional) orbital stability of small-amplitude traveling wave solutions to the full system when the surface tension is strong in a sense to be quantified shortly.

Mathematically, the problem is formulated as follows. Fix Cartesian coordinates (x, y) so that the wave propagates in the x-direction with gravity acting in the negative y-direction. Because we are most interested in the motion of the boundary, we suppose that the fluid domain is confined to a channel with rigid walls at heights $y = \pm d_\pm$ for fixed $d_\pm \in (0, \infty)$. At each time $t \geq 0$, the interface $\mathscr{S} = \mathscr{S}(t)$ is taken to be the graph of an unknown smooth function $\eta = \eta(t, x)$. For small-amplitude waves, this choice incurs no loss of generality. Then, the upper layer inhabits the (time-dependent) set

$$\Omega_+ = \Omega_+(t) := \left\{(x, y) \in \mathbb{R}^2 : \eta(t, x) < y < d_+\right\},$$

while the lower layer is given by

$$\Omega_- = \Omega_-(t) := \left\{(x, y) \in \mathbb{R}^2 : -d_- < y < \eta(t, x)\right\}.$$

We write $\Omega(t) := \Omega_+(t) \cup \Omega_-(t)$ to denote the fluid domain. Our focus will be on spatially localized waves for which $\eta(t, \cdot)$ decays at infinity. See Fig. 1 for an illustration.

Assuming that the flow in each region is irrotational and incompressible, the velocity field in $\Omega_\pm(t)$ is then given by $\nabla\Phi_\pm$, for some function $\Phi_\pm = \Phi_\pm(t, x, y)$ called the

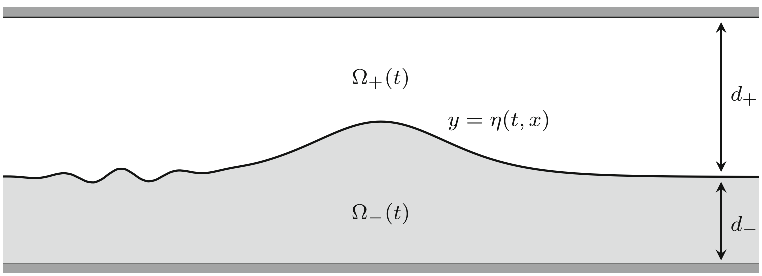

Fig. 1. Configuration of the internal wave system. The unshaded fluid region $\Omega_+(t)$ has density ρ_+ while the darker shaded region $\Omega_-(t)$ below is of density $\rho_- \geq \rho_+$. Their interface $\mathscr{S}(t)$ is a free boundary given by the graph of $\eta = \eta(t, x)$. In the far field, the widths of the upper and lower layer limit to d_+ and d_-, respectively

velocity potential. We take the density in $\Omega_\pm(t)$ to be constant and denote it by $\rho_\pm > 0$. In order to ensure that heavier fluid elements do not lie above lighter elements, it is required that $\rho_+ \leq \rho_-$. The case $\rho_+ = 0$ formally corresponds to a single fluid beneath vacuum. All of our analysis extends to this regime with only superficial modifications to the arguments.

The evolution of the system is governed by the incompressible irrotational Euler equations with a free boundary. In the bulk, the conservation of momentum has the simple expression

$$\Delta \Phi_\pm = 0 \qquad \text{in } \Omega_\pm(t). \tag{1.1a}$$

On both the rigid and moving boundary components, we have the kinematic condition

$$\begin{cases} \partial_t \eta = -\eta' \partial_x \Phi_- + \partial_y \Phi_- = -\eta' \partial_x \Phi_+ + \partial_y \Phi_+ & \text{on } \{y = \eta(t, x)\} \\ \partial_y \Phi_\pm = 0 & \text{on } \{y = \pm d_\pm\}, \end{cases} \tag{1.1b}$$

while on $\mathscr{S}(t)$ the dynamic or Bernoulli condition is imposed:

$$[\![\rho \partial_t \Phi + \frac{1}{2} \rho |\nabla \Phi|^2 + g \rho \eta]\!] + \sigma \left(\frac{\eta'}{\sqrt{1 + (\eta')^2}} \right)' = 0 \qquad \text{on } \{y = \eta(t, x)\}. \tag{1.1c}$$

Here $[\![\cdot]\!] := (\cdot)_+ - (\cdot)_-$ denotes the jump of a quantity over the interface, $g > 0$ is the gravitational constant, and $\sigma > 0$ is the coefficient of surface tension. The last term on the right-hand side above is the signed curvature of the interface and represents the influence of capillary effects. In (1.1b), we are enforcing the continuity of the normal velocity across the interface, while (1.1c) arises from the Young–Laplace law for the pressure jump. Also, here and in what follows we will mostly adhere to the convention that primes denote x-derivatives of functions depending on (t, x), while ∂_x is reserved for functions of (t, x, y) or in defining operators.

Rather than work with the full velocity potential $\Phi_\pm$, which is defined on a moving domain, it is advantageous to consider its restriction to the free boundary:

$$\varphi_\pm = \varphi_\pm(t, x) := \Phi_\pm(t, x, \eta(t, x)).$$

Through the use of nonlocal operators, it is possible to reformulate (1.1) in terms of the surface variables $(\eta, \varphi_+, \varphi_-)$; see Sect. 3.1.

1.1. Informal statement of results. *Traveling* or *steady* solutions of (1.1) are waves of permanent configuration that appear independent of time when viewed in a moving reference frame. Specifically, they exhibit the ansatz

$$\eta(t, x) = \eta^c(x - ct), \qquad \varphi_\pm(t, x) = \varphi_\pm^c(x - ct),$$

for some traveling wave profile $(\eta^c, \varphi_+^c, \varphi_-^c)$ and wave speed $c \in \mathbb{R}$.

In the gravity wave case $\sigma = 0$, it is known that there exist *solitary waves* [4,11,34,45], for which η^c decays as $|x| \to \infty$; *periodic waves* [4,5], for which η^c is periodic in x; and *fronts* [5,20,21,44,45], for which η^c has distinct limits upstream and downstream. Without surface tension, however, the dynamical problem is ill-posed [39], so to study stability we always take $\sigma > 0$. Rigorous existence results for small-amplitude periodic waves (including those with vorticity) were obtained in this regime by Le [41]. Solitary

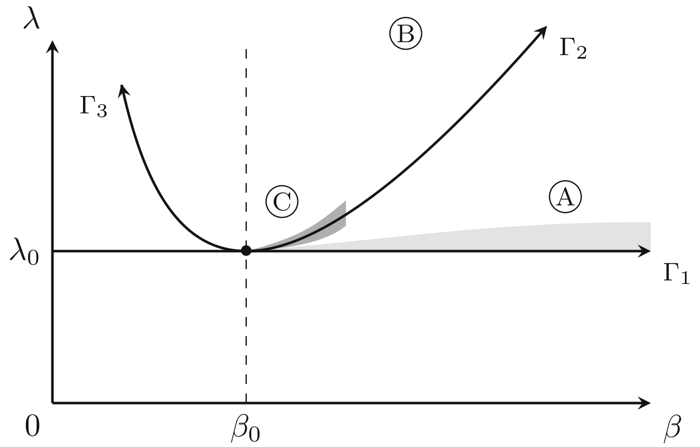

Fig. 2. Bifurcation diagram for internal capillary-gravity waves. Region A is the lighter shaded area that lies above Γ_1 and to the right of Γ_2; this is where one has monotone solitary waves. Region B consists of all (β, λ) lying above Γ_1 and to the right of Γ_3. Finally, Region C is the darker shaded set neighboring Γ_2. Explicit parameterizations for these curves can be found in (3.39) and (3.49)

internal capillary-gravity waves were constructed by Kirrmann [37] and Nilsson [47]; the stability of these solutions is the main subject of the present paper. We also note that analytical and numerical investigations of this regime have been performed by Laget and Dias [38].

The existence and qualitative properties of traveling internal waves are determined by four dimensionless parameters. The primary two are the Bond number β and inverse square Froude number λ given by

$$\beta := \frac{\sigma}{d_+\rho_-c^2}, \qquad \lambda := -\frac{g[\![\rho]\!]d_+}{\rho_-c^2}. \tag{1.2}$$

The Bond number measures the strength of the surface tension, while λ describes the balance between kinetic and potential energy. One can think of the Froude number $1/\sqrt{\lambda}$ as a non-dimensionalized wave speed, hence large λ corresponds roughly to slow moving waves.

The dispersion relation for internal capillary-gravity waves (rescaled to dimensionless variables) is given by

$$\sum_{\pm} \frac{\rho_\pm}{\rho_-}\xi \coth\left(\frac{d_\pm}{d_+}\xi\right) = \lambda + \beta\xi^2. \tag{1.3}$$

This results from linearizing the problem at the trivial solution $(\eta, \varphi_+, \varphi_-) = (0, 0, 0)$, then looking for eigenvalues of the form $i\xi$. If ξ is a root to (1.3), the linearized problem admits a plane wave solution with $\eta = \exp(i\xi(x - ct))$. After some algebra, it can be shown that there are three bifurcation curves Γ_1, Γ_2, Γ_3 that organize the (β, λ)-plane into regions where the configuration of the spectrum near the imaginary axis is qualitatively the same; see Fig. 2. They meet at the point (β_0, λ_0), which is given by

$$\beta_0 := \frac{1}{3}\left(\frac{\rho_+}{\rho_-} + \frac{d_-}{d_+}\right), \qquad \lambda_0 := \frac{\rho_+}{\rho_-} + \frac{d_+}{d_-}, \tag{1.4}$$

and there we find that $\xi = 0$ is a root of (1.3) with multiplicity 4. We say that β_0 is the critical Bond number separating the weak and strong surface tension regimes.

In this regard, the internal wave system is quite similar to that of water waves beneath vacuum; see, for instance, [2,16,17,25,30,32,33]. However, there are two additional parameters to consider: the ratios of the fluid densities ϱ and far-field layer heights h, defined by

$$\varrho := \frac{\rho_+}{\rho_-}, \qquad h := \frac{d_-}{d_+}. \tag{1.5}$$

These are specific to the two-fluid problem and allow for a surprisingly rich variety of traveling waves. For example, it has been proved by Nilsson [47] and Kirrmann [37] that for (β, λ) in the Region A illustrated in Fig. 2, there exist six qualitatively distinct types of small-amplitude waves. When $\varrho - 1/h^2$ is negative and $O(1)$ as $\lambda \searrow \lambda_0$, they find waves of depression (that is, $\eta < 0$) that are to leading order KdV solitons. These are the only kind of wave possible in the corresponding parameter regime for the one-fluid case, which is consistent with simply taking $\rho_+ = 0$. On the other hand, when $\varrho - 1/h^2 > 0$, there are internal waves of elevation ($\eta > 0$) whose interface is a perturbed KdV soliton. Moreover, in the regime $|\varrho - 1/h^2| \eqsim |\lambda - \lambda_0|^{1/2} \ll 1$, they construct traveling waves that are Gardner solitons to leading order. This furnishes four types of solutions, with waves of depression and elevation for both signs of $\varrho - 1/h^2$. A fuller account is given in Sect. 3.5.

Our first theorem, stated informally for the time being, establishes the nonlinear stability of all these waves in the orbital sense.

Theorem 1.1 (Strong surface tension). *Every sufficiently small-amplitude solitary internal wave $(\eta^c, \varphi_+^c, \varphi_-^c)$ with (β, λ) in Region A and $0 < \lambda - \lambda_0 \ll 1$ is conditionally orbitally stable in the following sense. For all $R > 0$ and $r > 0$, there exists $r_0 > 0$ such that, if $(\eta, \varphi_+, \varphi_-)$ is any solution defined on a time interval $[0, t_0)$ that obeys the bound*

$$\sup_{t \in [0,t_0)} \left(\|\eta(t)\|_{H^{3+}} + \|\varphi_+(t)\|_{\dot H^{\frac{5}{2}+} \cap \dot H^{\frac{1}{2}}} + \|\varphi_-(t)\|_{\dot H^{\frac{5}{2}+} \cap \dot H^{\frac{1}{2}}} \right) < R, \tag{1.6}$$

and for which the initial data satisfies

$$\|\eta(0) - \eta^c\|_{H^1} + \|\varphi_+(0) - \varphi_+^c\|_{\dot H^{\frac{1}{2}}} + \|\varphi_-(0) - \varphi_-^c\|_{\dot H^{\frac{1}{2}}} < r_0, \tag{1.7}$$

then

$$\sup_{t \in [0,t_0)} \inf_{s \in \mathbb{R}} \Big(\|\eta(t, \cdot - s) - \eta^c\|_{H^1} + \|\varphi_+(t, \cdot - s) - \varphi_+^c\|_{\dot H^{\frac{1}{2}}} + \|\varphi_-(t, \cdot - s) - \varphi_-^c\|_{\dot H^{\frac{1}{2}}} \Big) < r. \tag{1.8}$$

Here $H^s(\mathbb{R})$ and $\dot H^s(\mathbb{R})$ are the standard inhomogeneous and homogeneous Sobolev spaces respectively.

Remark 1.2. The bound in (1.8) controls the distance between $(\eta, \varphi_+, \varphi_-)$ and the family of translates of the steady wave. This is natural given that the underlying system (1.1) is translation invariant, and indeed it is necessary even for model equations such as KdV. Local well-posedness for the Cauchy problem at the level of regularity represented by the norm in (1.6) has been proved by Shatah and Zeng [52,53]. On the other hand, we will show in Sect. 3.3 that the lower regularity norm in (1.7) and (1.8) is equivalent to the physical energy. We also emphasize that because r is independent of t_0, this result is much stronger than continuity of the data-to-solution map. For a global-in-time solution, it gives orbital stability in the classical sense.

Our next result concerns uniform flows for which the interface is perfectly flat and the velocity is purely horizontal with the same constant value c in both layers. In a reference frame moving with the wave, it therefore appears quiescent. While linear stability criteria for this regime are classical (see, for example, [26]), as far as we are aware, this is the first nonlinear stability result.

Theorem 1.3 (Uniform flow). *The laminar solution* $(\eta^c, \varphi_+^c, \varphi_-^c) = (0, 0, 0)$ *is conditionally stable in the sense of Theorem 1.1 provided that* (β, λ) *lies in Region B.*

Lastly, we consider the critical surface tension case where (β, λ) lies in Region C near (β_0, λ_0); see Fig. 2. It is well-established that in this regime, the dynamics of sufficiently shallow waves are captured by a fifth-order nonlinear dispersive PDE similar to the Kawahara equation [10,36]. For spatially localized traveling waves, one can then integrate to obtain a fourth-order ODE

$$Z'''' - 2(1 + \delta)Z'' + Z - Z^2 = 0, \tag{1.9}$$

where we have scaled out all but the non-dimensional parameter $\delta = \delta_c$, which is determined explicitly by the wave speed via (3.46). The ODE (1.9) boasts an extraordinarily large variety of solutions that are homoclinic to 0 (see, for example, [17]). For this paper, we focus on the family $\{Z_\delta\}$ of "primary homoclinic" orbits that are even, unimodal, and exponentially localized. These primary homoclinic solutions have been rigorously constructed for $\delta \geq 0$ and $-1 \ll \delta < 0$, and numerically observed to persist as $\delta \searrow -2$. On the other hand, it is proved in [12] via variational methods that homoclinic solutions exist for $\delta > -2$. Nilsson [47] shows that for every $|\delta_c| \ll 1$, there exists a traveling wave $(\eta^c, \varphi_+^c, \varphi_-^c)$ solution to (1.1) with η^c given to leading order by a rescaling of Z_{δ_c}. The next result states that the orbital stability or instability of these solutions to the full internal wave problem can be determined by considerations of the far simpler model equation (1.9).

Theorem 1.4 (Critical surface tension). *Let* $\{Z_\delta\}$ *be the family of primary homoclinic solutions to* (1.9) *and suppose that* (β, λ) *lies in Region C with* $|\delta_c| \ll 1$. *Then the corresponding traveling wave solution* $(\eta^{c_*}, \varphi_+^{c_*}, \varphi_-^{c_*})$ *to* (1.1) *is conditionally orbitally stable provided that the function*

$$c \mapsto \operatorname{sgn} c \int_{\mathbb{R}} Z_{\delta_c}^2 \, dx \tag{1.10}$$

is strictly increasing at c_**, and it is orbitally unstable if this function is strictly decreasing there.*

We remark that this theorem is new even for the one-fluid case. Physically, the integral in (1.10) represents the momentum carried by the wave; whether it is increasing or decreasing as a function of δ has been investigated by many authors but remains open in the present case. Under conditions analogous to Theorem 1.4, Levandosky [42,43] proves a nonlinear stability/instability result for ground state solutions to a family of fifth-order dispersive PDEs that includes the Kawahara equation. Treating Kawahara orbits as constrained minimizers, Posukhovskyi–Stefanov [48] establish a criterion for spectral stability. On the other hand, for $\delta = 1/6$, the primary homoclinic solution to (1.9) has the explicit formula

$$Z_{\frac{1}{6}} = \frac{35}{24}\operatorname{sech}^4\left(\frac{\sqrt{6}}{12}\,\cdot\right),$$

and by exploiting this, (1.10) can be evaluated directly for various choices of the dimensional parameters [1,24,35]. Although this value of the parameter δ falls outside range considered in [47], the hydrodynamic relevance of $Z_{1/6}$ has been justified in [54]. Numerical evidence in [31] suggests that stability holds for the Kawahara equation with $\delta > 0$, but analytical results are not currently available. Through Theorem 1.4, progress on this question for the model equation can immediately be translated to (1.1).

1.2. Idea of the proof. It is well known that the internal wave problem (1.1) can be formulated as an abstract Hamiltonian system of the general form

$$\partial_t u = J\mathrm{D}E(u),$$

where $u = u(t)$ is an unknown related to $(\eta, \varphi_+, \varphi_-)$, the Poisson map J is a skew-adjoint operator, and E is a conserved energy functional. The translation invariance of the system gives rise to a second conserved quantity, the momentum P. A traveling wave solution with wave speed c is in fact a critical point of the augmented Hamiltonian $E_c := E - cP$.

It is therefore natural to adopt a constrained variational viewpoint, attempting to show that the waves are minimizers of the energy on level sets of the momentum. A serious challenge that arises in many applications, including the present one, is that $\mathrm{D}^2 E_c$ has an unstable direction as well as a 0 eigenvalue due to translation invariance. This situation can lead to either stability or instability, and a deft use of the conserved quantities is necessary to discern which occurs for the waves in question. Benjamin [7] pioneered this approach in his study of the orbital stability of KdV solitons. A systematic and greatly expanded version was later developed by Grillakis, Shatah, and Strauss [27]. Now called the GSS method, it is one of the primary tools in nonlinear stability theory for Hamiltonian systems.

Historically, though, GSS has not been especially successful in treating the full water wave problem. Indeed, (1.1) exhibits a host of features that make it highly resistant to naïve applications of systematic methods. For example, the theory in [27] requires that J be an isomorphism, which does not hold here as we show in Sect. 3.3. It is also formulated under the hypothesis that the Cauchy problem is globally well-posed in the natural energy space. At present, (1.1) is only known to be locally well-posed and this assumes considerably more smoothness. Because the water wave problem is quasilinear, it is not expected to generate a flow on the energy space. Worse still, the corresponding functional E is not even differentiable at this level of regularity.

Seeking to address these issues, Varholm et al. [55] obtained a variant of the GSS method that weakens the above hypotheses. In place of the bijectivity of J, it essentially requires only that J is injective with dense range. The functional analytic framework is also designed to accommodate the gap in regularity between the energy space and the smoothness needed for local well-posedness. In this paper, we use the relaxed GSS method to attack the water wave problem directly and prove Theorem 1.1 and Theorem 1.4. A simpler, self-contained argument suffices for Theorem 1.3 as the augmented linearized Hamiltonian has no unstable directions in that case.

The most challenging step in this procedure is computing the spectrum of the linearized augmented Hamiltonian at a traveling wave. For this, we generalize a technique introduced by Mielke [46] in his work on solitary capillary-gravity waves in a single finite-depth fluid and with strong surface tension. Briefly, this involves using the kinematic condition to eliminate $\varphi_{\pm}$ and obtain an auxiliary functional acting only on η. Conjugating by a rescaling operator, a delicate argument shows that for sufficiently small-amplitude waves, the spectrum coincides to leading order with the linearization of a dispersive model equation (steady KdV or Gardner in the setting of Theorem 1.1 and steady Kawahara for Theorem 1.4). Here it is important to note that these calculations are substantially more difficult in the internal wave setting than for a single fluid: the nonlocal operators introduced in the Hamiltonian reformulation are more complicated, and they must be expanded to higher order. On the other hand, Mielke proves conditional orbital stability using an ad hoc modification of the GSS method. Because we have at our disposal the general theory from [55], we are able to streamline this part of the argument.

Let us also mention an alternative variational approach to proving nonlinear stability of water waves due to Buffoni. Roughly speaking, this consists of a penalization scheme followed by a concentration compactness argument to directly construct traveling waves as constrained minimizers of the energy with fixed momentum. In some circumstances, one can then apply a soft analysis argument of Cazenave and Lions [18] to infer so-called (conditional) *energetic stability*, meaning that the *set* of constrained minimizers is stable in the energy norm. This differs from the orbital stability we obtain unless one also has uniqueness of the minimizer up to translation, which is typically not available. Through this variational method, Buffoni proved the existence and stability (in the above sense) of solitary waves in the single-fluid case with strong surface tension [13]. He also gave partial results concerning waves with weak surface tension and in infinite depth [14,15]. Pushing significantly further the technique, Groves and Wahlén [28,29] subsequently obtained complete versions of these theorems, and also treated the case of constant vorticity [30].

1.3. Plan of the article. In Sect. 2, we briefly recall the hypotheses and conclusions of the general theory in [55] to better motivate the analysis in the rest of the paper.

In Sect. 3, we turn to application at hand, reformulating the internal wave problem (1.1) as an abstract Hamiltonian system in the style of Benjamin and Bridges [8]. A number of hypotheses necessary to apply the general theory in [55] are then be verified. We also recall the existence theory due to Nilsson [47], recasting it within the Hamiltonian framework of the present paper.

Section 4 is devoted to computing the spectrum of the linearized augmented Hamiltonian at a uniform flow or small-amplitude traveling wave. As mentioned above, our calculation is patterned on the basic approach of Mielke [46], but with many additional challenges owing to the more complicated physical setting.

The main results are then proved in Sect. 5. Thanks to the general theory, this requires us only to determine whether the so-called moment of instability, a scalar-valued function of the wave speed, is strictly convex or concave. This is accomplished by exploiting a long-wave rescaling and the leading-order form of the waves known from the existence theory.

Finally, "Appendix A" contains some elementary calculations that play an essential part in the spectral computation.

2. General Theory

For the convenience of the reader, this section gives a self-contained presentation of the relaxed GSS method introduced in [55] that will be the main abstract tool for the proof of Theorems 1.1 and 1.4. We do not need the full strength of this result, however, because our Poisson map will be state independent and the symmetry group is linear rather than affine. We are therefore able to simplify the statement in several places.

2.1. Assumptions. As we shall see in Sect. 3, it will be necessary to work with a scale of spaces

$$\mathbb{W} \hookrightarrow \mathbb{V} \hookrightarrow \mathbb{X},$$

where $\mathbb{X}$ is a real Hilbert space, while $\mathbb{V}$ and $\mathbb{W}$ are reflexive Banach spaces. (For our application, in fact all three of these will be Hilbert spaces, but that is not necessary.) The inner product on $\mathbb{X}$ will be denoted by $(\cdot\,,\,\cdot)_{\mathbb{X}}$, and the corresponding norm by $\|\cdot\|_{\mathbb{X}}$. Likewise, let $\|\cdot\|_{\mathbb{V}}$ and $\|\cdot\|_{\mathbb{W}}$ be the norms for $\mathbb{V}$ and $\mathbb{W}$, respectively. We write $\mathbb{X}^*$ for the (continuous) dual of $\mathbb{X}$, which is naturally isomorphic to $\mathbb{X}$ via the mapping $I : \mathbb{X} \to \mathbb{X}^*$ taking $u \in \mathbb{X}$ to $(u, \cdot)_{\mathbb{X}} \in \mathbb{X}^*$. We will not make this identification here, but rather use I explicitly. On the other hand, we will simply identify $\mathbb{X}^{**}$ with $\mathbb{X}$, and likewise for $\mathbb{V}$ and $\mathbb{W}$. The pairing of $\mathbb{X}$ and $\mathbb{X}^*$ we denote by $\langle\cdot\,,\,\cdot\rangle_{\mathbb{X}^*\times\mathbb{X}}$, while $\langle\cdot\,,\,\cdot\rangle_{\mathbb{W}^*\times\mathbb{W}}$ is the pairing between $\mathbb{W}^*$ and $\mathbb{W}$; when there is no risk of confusion, we will omit the subscript.

Intuitively, $\mathbb{X}$ is the energy space for the system under consideration. This is where the Hamiltonian structure will be formulated, and is the natural setting for analyzing the spectrum. On the other hand, $\mathbb{V}$ is a space where the conserved quantities are smooth. Finally, we think of $\mathbb{W}$ as a "well-posedness space", with the norm coming from higher-order energy estimates used to prove that the Cauchy problem is at least locally well-posed in time. The norm on $\mathbb{W}$ also plays the secondary role of allowing us to get control over $\mathbb{V}$ via interpolation. More precisely, we require the following:

Assumption 1 (*Spaces*). Let $\mathbb{X}$, $\mathbb{V}$, and $\mathbb{W}$ be given as above. Assume that there exist constants $\theta \in (0, 1]$ and $C > 0$ such that

$$\|u\|_{\mathbb{V}}^3 \le C\|u\|_{\mathbb{X}}^{2+\theta}\|u\|_{\mathbb{W}}^{1-\theta} \tag{2.1}$$

for all $u \in \mathbb{W}$.

It is often necessary to restrict attention to some smaller subset of these spaces in order to ensure that the problem is well-defined. For the internal wave problem that we will consider, it is necessary that the interface remains away from the upper and lower rigid boundaries. For that reason, we introduce an open set $\mathcal{O} \subset \mathbb{X}$, where solutions must live.

We endow $\mathbb{X}$ with symplectic structure in the form of a Poisson map $J : \mathrm{Dom}\, J \subset \mathbb{X}^* \to \mathbb{X}$ which is required to satisfy the following hypotheses.

Assumption 2 (*Poisson map*).

(i) The domain $\mathrm{Dom}\, J$ is dense in $\mathbb{X}^*$.
(ii) J is injective.
(iii) J is skew-adjoint in the sense that

$$\langle Jv,\ w\rangle = -\langle v,\ Jw\rangle \qquad \text{for all } v, w \in \mathrm{Dom}\, J.$$

We are interested in abstract Hamiltonian systems taking the form

$$\partial_t = JDE(u), \qquad u|_{t=0} = u_0, \tag{2.2}$$

where $E \in C^3(\mathcal{O} \cap \mathbb{V}; \mathbb{R})$ is the *energy functional*. In addition to the energy, we suppose that there is a second conserved quantity $P \in C^3(\mathcal{O} \cap \mathbb{V}; \mathbb{R})$, which for our application will be the (linear horizontal) *momentum*. In order to state what it means to be a solution of (2.2), and to work with it in a meaningful way, we need to be able to view the Fréchet derivatives $\mathrm{D}E(u)$ and $\mathrm{D}P(u)$ as elements of $\mathbb{X}^*$ rather than $\mathbb{V}^*$.

Assumption 3 (*Derivative extension*). There exist mappings $\nabla E, \nabla P \in C^0(\mathcal{O} \cap \mathbb{V}; \mathbb{X}^*)$ such that $\nabla E(u)$ and $\nabla P(u)$ are extensions of $DE(u)$ and $DP(u)$, respectively, for every $u \in \mathcal{O} \cap \mathbb{V}$.

We then say that $u \in C^0([0, t_0); \mathcal{O} \cap \mathbb{W})$ is a solution of (2.2) on the interval $[0, t_0)$ if

$$\partial_t \langle u(t),\ w \rangle = -\langle \nabla E(u(t)),\ Jw \rangle \qquad \text{for all } w \in \operatorname{Dom} J, \tag{2.3}$$

is satisfied in the distributional sense on $(0, t_0)$, the initial condition $u(0) = u_0$ is satisfied, and both E and P are conserved.

As we have mentioned, the internal wave system is invariant under translation in the x-direction. More generally, we can consider the situation where the system (2.2) is invariant with respect to a symmetry group. Specifically, we assume that there exists a one-parameter family of linear maps $T(s)\colon \mathbb{X} \to \mathbb{X}$ having the properties described below.

Assumption 4 (*Symmetry group*). The symmetry group $T(\cdot)$ satisfies the following.

(i) The neighborhood $\mathcal{O}$, the subspaces $\mathbb{V}$ and $\mathbb{W}$, and $I^{-1} \operatorname{Dom} J$ are all invariant under the symmetry group.
(ii) T comprises a flow on $\mathbb{X}$ in the sense that $T(0) = \mathrm{Id}_{\mathbb{X}}$ and $T(s+r) = T(s)T(r)$ for all $s, r \in \mathbb{R}$. Moreover, $T(s)$ is unitary on $\mathbb{X}$ and an isometry on $\mathbb{V}$ and $\mathbb{W}$ for all $s \in \mathbb{R}$.
(iii) The symmetry group commutes with the Poisson map in the sense that

$$JIT(\cdot) = T(\cdot)JI.$$

(iv) The infinitesimal generator of T is the linear mapping

$$T'(0)u = \lim_{s\to 0}(s^{-1}(T(s)u - u)),$$

with dense domain $\operatorname{Dom} T'(0) \subset \mathbb{X}$ consisting of all $u \in \mathbb{X}$ such that the limit exists in $\mathbb{X}$. Similarly, we may speak of the dense subspaces $\operatorname{Dom} T'(0)|_{\mathbb{V}} \subset \mathbb{V}$ and $\operatorname{Dom} T'(0)|_{\mathbb{W}} \subset \mathbb{W}$ on which the limit exists in $\mathbb{V}$ and $\mathbb{W}$, respectively.
We assume that $\nabla P(u) \in \operatorname{Dom} J$ for every $u \in \operatorname{Dom} T'(0)|_{\mathbb{V}} \cap \mathcal{O}$, and that

$$T'(0)u = J\nabla P(u) \tag{2.4}$$

for all such u.
(v) The subspace $\operatorname{Dom} T'(0)|_{\mathbb{W}} \cap \operatorname{Rng} J$ is dense in $\mathbb{X}$.
(vi) For all $u \in \mathcal{O} \cap \mathbb{V}$, the energy is conserved by flow of the symmetry group:

$$E(u) = E(T(s)u), \qquad \text{for all } s \in \mathbb{R}. \tag{2.5}$$

We say that $u \in C^1(\mathbb{R}; \mathcal{O} \cap \mathbb{W})$ is a *bound state* of the Hamiltonian system (2.2) provided that it is a solution of the form

$$u(t) = T(ct)U_c,$$

for some $c \in \mathbb{R}$ and $U_c \in \mathcal{O} \cap \mathbb{W}$. We will also refer to U_c itself as a bound state. For the internal wave problem where T represents translation, these are exactly the traveling waves we wish to study.

Assumption 5 (*Bound states*). There exists a one-parameter family of bound state solutions $\{U_c : c \in \mathscr{I}\}$, where $\mathscr{I} \subset \mathbb{R}$ is a non-empty open interval, to the Hamiltonian system (2.2). The family enjoys the following properties.

(i) The mapping $c \in \mathscr{I} \mapsto U_c \in \mathcal{O} \cap \mathbb{W}$ is C^1.
(ii) For all $c \in \mathscr{I}$,

$$U_c \in \operatorname{Dom} T'''(0) \cap \operatorname{Dom} JIT'(0), \qquad U_c,\ JIT'(0)U_c \in \operatorname{Dom} T'(0)|_{\mathbb{W}}.$$

(iii) The non-degeneracy condition $T'(0)U_c \neq 0$ holds for every $c \in \mathscr{I}$.
(iv) It holds that $\liminf_{|s|\to\infty} \|T(s)U_c - U_c\|_{\mathbb{X}} > 0$.

For a fixed parameter c, we define the *augmented Hamiltonian* to be the functional $E_c \in C^3(\mathbb{V} \cap \mathcal{O}; \mathbb{R})$ given by

$$E_c(u) := E(u) - cP(u).$$

One can confirm from the previous assumptions that in fact U_c is necessarily a critical point of E_c. Due to this observation, we can think of each bound state U_c as being a critical point of the energy with the constraint of a fixed momentum, with the wave speed c arising naturally as a Lagrange multiplier.

As mentioned in the introduction, it is often the case that the bound states sit at a saddle point of the energy with Morse index 1. That is, the second derivative of the augmented Hamiltonian at U_c has a single simple negative (real) eigenvalue, a 0 eigenvalue generated by the symmetry group, and the rest of the spectrum lies along the positive real axis; bounded uniformly away from the origin. This is the basic setting of the problem considered in [27], and what we will prove is the case for the internal waves considered later in the paper. We therefore make the following hypotheses about the configuration of the spectrum for the general theory.

Assumption 6 (*Spectrum*). The operator $\mathrm{D}^2E_c(U_c) \in \operatorname{Lin}(\mathbb{V}, \mathbb{V}^*)$ extends uniquely to a bounded linear operator $H_c : \mathbb{X} \to \mathbb{X}^*$ such that:

(i) $I^{-1}H_c$ is self-adjoint on $\mathbb{X}$.
(ii) The spectrum of $I^{-1}H_c$ satisfies

$$\operatorname{spec} I^{-1}H_c = \{-\mu_c^2,\ 0\} \cup \Sigma_c, \tag{2.6}$$

where $-\mu_c^2 < 0$ is a simple eigenvalue corresponding to a unit eigenvector χ_c, 0 is a simple eigenvalue generated by T, and $\Sigma_c \subset (0, \infty)$ is bounded away from 0.

2.2. Stability and instability theorems. Assuming that all of the hypotheses from the previous subsection hold, we now state the main stability and instability theorems from [55]. For a fixed bound state U_c and radius $r > 0$, we define the tubular neighborhoods

$$\mathcal{U}_r^{\mathbb{X}} := \{u \in \mathcal{O} : \inf_{s \in \mathbb{R}} \|u - T(s)U_c\|_{\mathbb{X}} < r\},$$
$$\mathcal{U}_r^{\mathbb{W}} := \{u \in \mathcal{O} \cap \mathbb{W} : \inf_{s \in \mathbb{R}} \|u - T(s)U_c\|_{\mathbb{W}} < r\}.$$

Similarly, for any $R > 0$, let $\mathcal{B}_R^{\mathbb{W}}$ denote the intersection of $\mathcal{O}$ with the ball of radius R centered at the origin in $\mathbb{W}$. Then U_c is said to be *conditionally orbitally stable* provided that for all $r > 0$ and $R > 0$, there exists $r_0 > 0$ such that if $u : [0, t_0) \to \mathcal{B}_R^{\mathbb{W}}$ is a solution to (3.31) with $u(0) \in \mathcal{U}_{r_0}^{\mathbb{X}}$, then $u(t) \in \mathcal{U}_r^{\mathbb{X}}$ for all $t \in [0, t_0)$. Here conditional refers to the fact that stability only holds provided we know the solution exists, and that its growth in $\mathbb{W}$ is controllable.

The *moment of instability* is the scalar-valued function $d = d(c)$ that results from evaluating the augmented Hamiltonian along the family of bound states:

$$d(c) := E_c(U_c) = E(U_c) - cP(U_c). \tag{2.7}$$

The main appeal of the GSS method lies in its ability to characterize the orbital stability of bound states in terms of the sign of d''. Under the relaxed hypotheses, we have by [55, Theorem 2.4] the following stability criterion.

Theorem 2.1 (Stability). *Suppose that the above assumptions hold. If $d''(c) > 0$, then the bound state U_c is conditionally orbitally stable.*

In order to state an instability result, we need to know that (2.2) can be solved at least locally around the U_c-orbit. That is, we require one more assumption.

Assumption 7 (*Local existence*). There exists $\nu_0 > 0$ and $t_0 > 0$ such that for all initial data $u_0 \in \mathcal{U}_{\nu_0}^{\mathbb{W}}$, there exists a unique solution to (2.2) on the interval $[0, t_0)$.

Note that ill-posedness of the Cauchy problem can itself be interpreted as a form of instability. Supposing that Assumption 7 holds, we say that U_c is *orbitally unstable* provided that there exists $\nu_0 > 0$ such that, for all $0 < \nu < \nu_0$ there exists initial data in $\mathcal{U}_\nu^{\mathbb{W}}$ for which the corresponding solution exits $\mathcal{U}_{\nu_0}^{\mathbb{W}}$ in finite time. Observe that this is not conditional.

Now, from [55, Theorem 2.6], we have the following orbital instability result.

Theorem 2.2 (Instability). *If $d''(c) < 0$ and Assumption 7 is satisfied, then the bound state U_c is orbitally unstable.*

3. Hamiltonian Formulation for Internal Waves

3.1. Nonlocal operators and surface variables. Following the classical Zakharov–Craig–Sulem idea, we will reformulate the interface Euler equations (1.1) as a nonlocal problem in terms of quantities restricted to the free boundary $\mathscr{S}(t)$. A similar approach was taken by Benjamin and Bridges [8] and Craig and Groves [22] in their treatments of this system.

Recall that we have defined

$$\varphi_\pm(t, x) := \Phi_\pm(t, x, \eta(t, x)),$$

to be the traces of the velocity potentials for the upper and lower regions. The velocity field can then be recovered by means of the Dirichlet–Neumann operator in $\Omega_\pm(t)$. For a fixed η, this is the mapping given by

$$G_\pm(\eta) f_\pm := \langle \eta' \rangle \big(N_\pm \cdot \nabla \mathcal{H}_\pm(\eta) f \big)|_{\mathscr{S}} \tag{3.1}$$

where $N_\pm$ is the unit outward normal to $\Omega_\pm$ along $\mathscr{S}$, we are making use of the Japanese bracket notation $\langle \cdot \rangle := \sqrt{1 + |\cdot|^2}$, and $\mathcal{H}_\pm(\eta) f$ is the harmonic extension of f to $\Omega_\pm$. Specifically, in view of the kinematic conditions (1.1b) on the rigid boundaries, we take $\mathcal{H}_\pm(\eta) f$ to be the unique solution to

$$\begin{cases} \Delta \mathcal{H}_\pm(\eta) f = 0 & \text{in } \Omega_\pm \\ \mathcal{H}_\pm(\eta) f = f & \text{on } \{y = \eta\} \\ \partial_y \mathcal{H}_\pm(\eta) f = 0 & \text{on } \{y = \pm d_\pm\}. \end{cases} \tag{3.2}$$

Dirichlet–Neumann operators are a standard tool in the study of water waves; for a general reference, see [40] or [51]. In particular, for any real numbers $k_0 > 1/2$ and $k \in [1/2 - k_0, 1/2 + k_0]$, and profile $\eta \in H^{k_0+1/2}(\mathbb{R})$ with $-d_- < \eta < d_+$, we have that $G_\pm(\eta)$ is an isomorphism $\dot{H}^k(\mathbb{R}) \to \dot{H}^{k-1}(\mathbb{R})$, where $\dot{H}^k$ denotes the usual homogeneous Sobolev space of order k. Similarly, $\mathcal{H}_\pm(\eta)$ is bounded as a mapping $H^k(\mathbb{R}) \to H^{k+1/2}(\Omega_\pm)$ and $\dot{H}^k(\mathbb{R}) \to \dot{H}^{k+1/2}(\Omega_\pm)$. Our analysis relies on the fact that the Dirichlet–Neumann operator depends smoothly on η. Indeed, $\eta \mapsto G_\pm(\eta)$ is real analytic as a function from $H^{k_0+1/2}(\mathbb{R}) \to \mathrm{Lin}(\dot{H}^k(\mathbb{R}), \dot{H}^{k-1}(\mathbb{R}))$, and at $\eta = 0$, it is the Fourier multiplier $G_\pm(0) = |\partial_x| \coth(d_\pm |\partial_x|)$. Note also that $G_\pm(\eta)$ is self-adjoint $\dot{H}^{1/2}(\mathbb{R}) \to \dot{H}^{-1/2}(\mathbb{R})$ and positive definite.

Because $N_+ + N_- = 0$, the continuity of the normal velocity over the interface is equivalent to

$$G_+(\eta)\varphi_+ + G_-(\eta)\varphi_- = 0. \tag{3.3}$$

Thus the kinematic condition (1.1b) on $\mathscr{S}(t)$ can be expressed as

$$\partial_t \eta = \mp G_\pm(\eta)\varphi_\pm. \tag{3.4}$$

Note that the kinematic condition on $\{y = \pm d_\pm\}$ is encoded in the definition of $\mathcal{H}$.

Rather than work with $\varphi_\pm$, we consider the quantity

$$\psi := -[\![\rho \Phi]\!] = \rho_- \varphi_- - \rho_+ \varphi_+. \tag{3.5}$$

Using (3.3), we can recover both φ_+ and φ_- from ψ. Indeed, we compute that

$$\begin{aligned} -G_-(\eta)\psi &= \rho_+ G_-(\eta)\varphi_+ - \rho_- G_-(\eta)\varphi_- \\ &= \rho_+ G_-(\eta)\varphi_+ + \rho_- G_+(\eta)\varphi_+ = B(\eta)\varphi_+, \end{aligned}$$

where

$$B(\eta) := \rho_+ G_-(\eta) + \rho_- G_+(\eta). \tag{3.6}$$

By the above discussion, we have that $B(\eta)$ is bounded and linear $H^k(\mathbb{R}) \to H^{k-1}(\mathbb{R})$ and $\dot{H}^k(\mathbb{R}) \to \dot{H}^{k-1}(\mathbb{R})$, for all $\eta \in H^{k_0+1/2}(\mathbb{R})$ and with k_0, k given as before. One can readily confirm, moreover, that $B(\eta)$ is an isomorphism $\dot{H}^k(\mathbb{R}) \to \dot{H}^{k-1}(\mathbb{R})$. Thus, repeating the same computation with signs reversed leads to the identity

$$\varphi_\pm = \mp B(\eta)^{-1} G_\mp(\eta)\psi. \tag{3.7}$$

The kinematic condition (3.4) can then be recast as

$$\partial_t \eta = A(\eta)\psi \tag{3.8}$$

for the operator

$$A(\eta) := G_-(\eta)B(\eta)^{-1}G_+(\eta). \tag{3.9}$$

It is simple to show that these operators commute, and hence we can alternatively write

$$A(\eta) = G_+(\eta)B(\eta)^{-1}G_-(\eta).$$

To reformulate the Bernoulli condition (1.1c) requires being able to reconstruct the full gradient $\nabla\Phi_\pm$ restricted to the interface from the surface variables. For this, we simply observe that

$$\varphi_\pm' = (\partial_x \Phi_\pm)|_{y=\eta} + \eta'(\partial_y \Phi_\pm)|_{y=\eta},$$

which together with the definition of $G_\pm(\eta)$ in (3.1) leads to the useful identities

$$\begin{aligned} \begin{pmatrix} \varphi_\pm' \\ G_\pm(\eta)\varphi_\pm \end{pmatrix} &= \begin{pmatrix} 1 & \eta' \\ \pm\eta' & \mp 1 \end{pmatrix} (\nabla\Phi_\pm)|_{\mathscr{S}}, \\ (\nabla\Phi_\pm)|_{\mathscr{S}} &= \frac{1}{1+(\eta')^2} \begin{pmatrix} 1 & \pm\eta' \\ \eta' & \mp 1 \end{pmatrix} \begin{pmatrix} \varphi_\pm' \\ G_\pm(\eta)\varphi_\pm \end{pmatrix}. \end{aligned} \tag{3.10}$$

Now, observe that simply by definition

$$-\partial_t \psi = \rho_+ \partial_t \varphi_+ - \rho_- \partial_t \varphi_- = [\![\rho \partial_t \Phi]\!] + (\partial_t \eta)[\![\rho \partial_y \Phi]\!].$$

Thus (1.1c) can be rewritten as

$$\partial_t \psi = \frac{1}{2}[\![\rho |\nabla \Phi|^2]\!] - (\partial_t \eta)[\![\rho \partial_y \Phi]\!] + g[\![\rho]\!]\eta + \sigma \left(\frac{\eta'}{\langle \eta' \rangle} \right)'. \tag{3.11}$$

In view of (3.10), this gives a formulation of the Bernoulli condition involving only the surface variables η and ψ.

3.2. Functional analytic setting. Let us now define the function spaces in which the internal wave problem will be posed. Following the approach outlined above, we wish to recast the system in terms of the unknown $u := (\eta, \psi)$. It is convenient to introduce a scale of spaces describing the spatial regularity of u: for each $k \geq 1/2$, let

$$\mathbb{X}^k = \mathbb{X}_1^k \times \mathbb{X}_2^k := H^{k+\frac{1}{2}}(\mathbb{R}) \times \left(\dot{H}^k(\mathbb{R}) \cap \dot{H}^{\frac{1}{2}}(\mathbb{R}) \right). \tag{3.12}$$

In what follows, we will frequently use the shorthand $\mathbb{X}^{k+}$ (and likewise H^{k+}) to denote $\mathbb{X}^{k+\varepsilon}$ for any $0 < \varepsilon \ll 1$ that is fixed and then suppressed.

Remark 3.1. Observe that $H^r(\mathbb{R}) \cap \dot{H}^s(\mathbb{R})$ is dense in both $H^r(\mathbb{R})$ and $\dot{H}^s(\mathbb{R})$ for all $r, s \in \mathbb{R}$; see, for example, [55, Lemma A.1]. This will turn out to be crucial when we turn to verifying Assumption 4(v).

We will work in a trio of nested Banach spaces $\mathbb{W} \hookrightarrow \mathbb{V} \hookrightarrow \mathbb{X}$. The largest, $\mathbb{X}$, we call the *energy space*. Specifically, we take

$$\mathbb{X} := \mathbb{X}^{\frac{1}{2}} = H^1(\mathbb{R}) \times \dot{H}^{\frac{1}{2}}(\mathbb{R}). \tag{3.13}$$

Its dual is

$$\mathbb{X}^* = H^{-1}(\mathbb{R}) \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}),$$

and we let $I = (1 - \partial_x^2, |\partial_x|)$ denote the natural isomorphism $\mathbb{X} \to \mathbb{X}^*$. In particular, when $u \in \mathbb{X}$, the velocity field $\nabla\Phi_\pm \in L^2(\Omega_\pm)$. As we will see below, this ensures that the kinetic energy is indeed finite. Likewise, the H^1 norm of η is equivalent to the excess potential energy relative to the undisturbed state.

However, observe that $u \mapsto G_\pm(\eta)$ is not smooth with domain $\mathbb{X}$, since we must have that η is at least Lipschitz continuous and also bounded away from the rigid boundaries at $y = \pm d_\pm$. This leads us to introduce the space

$$\mathbb{V} := \mathbb{X}^{1+} = H^{\frac{3}{2}+}(\mathbb{R}) \times \left(\dot{H}^{1+}(\mathbb{R}) \cap \dot{H}^{\frac{1}{2}}(\mathbb{R})\right), \tag{3.14}$$

and neighborhood

$$\mathcal{O} := \{(\eta, \psi) \in \mathbb{X} : -d_- < \eta < d_+\}.$$

Note that $H^{3/2+}(\mathbb{R}) \hookrightarrow W^{1,\infty}(\mathbb{R})$, so $u \in \mathbb{V}$ does indeed imply that η has the requisite Lipschitz continuity.

Lastly, because the Cauchy problem is not likely to be well-posed in $\mathbb{V}$, we consider the even smoother space

$$\mathbb{W} := \mathbb{X}^{\frac{5}{2}+} = H^{3+}(\mathbb{R}) \times \left(\dot{H}^{\frac{5}{2}+}(\mathbb{R}) \cap \dot{H}^{\frac{1}{2}}(\mathbb{R})\right). \tag{3.15}$$

Local well-posedness at this level of regularity was proved by Shatah and Zeng [53], for example.

Before continuing, we record the fact these spaces have the following embedding property that corresponds to Assumption 1.

Lemma 3.2 (Spaces). *Let the spaces $\mathbb{X}$, $\mathbb{V}$, and $\mathbb{W}$ be defined by* (3.13)*,* (3.14)*, and* (3.15)*, respectively. There exists a constant $C > 0$ and $\theta \in (0, \frac{1}{4})$ such that*

$$\|u\|_{\mathbb{V}}^3 \le C\|u\|_{\mathbb{X}}^{2+\theta}\|u\|_{\mathbb{W}}^{1-\theta} \qquad \textit{for all } u \in \mathbb{W}.$$

Proof. This can be quickly verified using from the definitions of $\mathbb{X}$, $\mathbb{V}$, and $\mathbb{W}$ and the Gagliardo–Nirenberg interpolation inequality. □

Observe that this inequality ensures that small cubic terms in $\mathbb{V}$ are dominated by quadratic terms in $\mathbb{X}$ on bounded sets in $\mathbb{W}$, which is needed in the general theory when Taylor expanding functionals that are smooth with domain $\mathbb{V} \cap \mathcal{O}$. A similar argument appears in the proof of Theorem 5.1.

3.3. Hamiltonian structure. Benjamin and Bridges [8] established that the internal wave problem (1.1) has a (canonical) Hamiltonian formulation in terms of the state variable u by adapting the well-known Zakharov–Craig–Sulem formulation for the single-fluid case. In this section, we will recall the system obtained in [8] while verifying that it satisfies a number of the hypotheses of the general theory.

The kinetic energy carried by the wave is given by

$$\begin{aligned} K &= \frac{1}{2}\int_{\Omega_+(t)} \rho_+|\nabla\Phi_+|^2\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\int_{\Omega_-(t)} \rho_-|\nabla\Phi_-|^2\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{2}\int_{\mathbb{R}} \rho_+\varphi_+ G_+(\eta)\varphi_+\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} \rho_-\varphi_- G_-(\eta)\varphi_-\,\mathrm{d}x. \end{aligned}$$

Using (3.3) and (3.7), this can be rewritten as

$$K = \frac{1}{2}\int_{\mathbb{R}} \psi G_-(\eta)B(\eta)^{-1}G_+(\eta)\psi\,\mathrm{d}x.$$

Thus, we can view K as the $C^\infty(\mathcal{O}\cap\mathbb{V},\mathbb{R})$ functional acting on u given by

$$K(u) := \frac{1}{2}\int_{\mathbb{R}} \psi A(\eta)\,\psi\,\mathrm{d}x, \tag{3.16}$$

where recall $A(\eta)$ was defined in (3.9). Likewise, the potential energy for the system is described by the functional

$$V(u) := -\frac{1}{2}\int_{\mathbb{R}} g[\![\rho]\!]\eta^2\,\mathrm{d}x + \sigma\int_{\mathbb{R}}\left(\sqrt{1+(\eta')^2}-1\right)\mathrm{d}x.$$

The total energy is thus

$$\begin{aligned} E(u) &:= K(u) + V(u) \\ &= \frac{1}{2}\int_{\mathbb{R}} \psi A(\eta)\,\psi\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}} g[\![\rho]\!]\eta^2\,\mathrm{d}x + \sigma\int_{\mathbb{R}}\left(\sqrt{1+(\eta')^2}-1\right)\mathrm{d}x. \end{aligned} \tag{3.17}$$

By our choice of spaces, $E \in C^\infty(\mathcal{O}\cap\mathbb{V};\mathbb{R})$. We claim, moreover, that $\mathrm{D}E(u)$ can be extended to a mapping defined on the entire dual space $\mathbb{X}^*$. This rather technical fact is necessary in order to reformulate the problem as a Hamiltonian system.

Before addressing this question, we pause to record the following crucial formulas for the Fréchet derivatives of the nonlocal operators $G_\pm(\eta)$ and $A(\eta)$. As it is quite simple, the proof of this lemma is given in "Appendix A".

Lemma 3.3 (First derivatives). *Let $(\eta,\psi)\in\mathcal{O}\cap\mathbb{V}$, $\dot\eta\in\mathbb{V}_1$, and $\xi\in\mathbb{V}_2$ be given.*

(a) *The Fréchet derivative of $G_\pm(\eta)$ admits the representation formula*

$$\int_{\mathbb{R}} \xi\,\langle \mathrm{D}G_\pm(\eta)\dot\eta,\ \psi\rangle\,\mathrm{d}x = \int_{\mathbb{R}}\left(a_1^\pm(\eta,\psi)\xi' + a_2^\pm(\eta,\psi)G_\pm(\eta)\xi\right)\dot\eta\,\mathrm{d}x, \tag{3.18}$$

with

$$\begin{aligned} a_1^\pm(\eta,\psi) &:= \frac{1}{1+(\eta')^2}\left(\mp\psi' - \eta' G_\pm(\eta)\psi\right) \\ a_2^\pm(\eta,\psi) &:= \frac{1}{1+(\eta')^2}\left(\pm G_\pm(\eta)\psi - \eta'\psi'\right). \end{aligned} \tag{3.19}$$

(b) *The Fréchet derivative of* $A(\eta)$ *admits the representation formula*

$$\begin{aligned}\int_{\mathbb{R}} \xi \, \langle \mathrm{D}A(\eta)\dot{\eta}, \psi \rangle \, \mathrm{d}x &= \sum_{\pm} \rho_\pm \int_{\mathbb{R}} \left(a_1^\pm(\eta, A(\eta)G_\pm(\eta)^{-1}\psi) \left(A(\eta)G_\pm(\eta)^{-1}\xi \right)' \right) \dot{\eta} \, \mathrm{d}x \\ &\quad + \sum_{\pm} \rho_\pm \int_{\mathbb{R}} \left(a_2^\pm(\eta, A(\eta)G_\pm(\eta)^{-1}\psi) A(\eta)\xi \right) \dot{\eta} \, \mathrm{d}x. \end{aligned} \tag{3.20}$$

Remark 3.4. Observe that by (3.10), $a_1^\pm(\eta, \psi) = \mp(\partial_x \mathcal{H}_\pm(\eta)\psi)|_{\mathscr{S}}$ while $a_2^\pm(\eta, \psi) = -(\partial_y \mathcal{H}_\pm(\eta)\psi)|_{\mathscr{S}}$. In particular, this means that both are linear in ψ.

Likewise, concise formulas for the second variations of the nonlocal operators is an essential ingredient for the spectral analysis taken up in Sect. 4. First, we record following elementary second derivative formula for the Dirichlet–Neumann operators $G_\pm$. Here we use notation similar to that in [46,55].

Lemma 3.5 (Second derivative of $G_\pm$). *For all* $u = (\eta, \psi) \in \mathcal{O} \cap \mathbb{V}$ *and* $\dot{\eta} \in \mathbb{V}_1$*, it holds that*

$$\begin{aligned}\int_{\mathbb{R}} \psi \left\langle \mathrm{D}^2 G_\pm(\eta)[\dot{\eta}, \dot{\eta}], \, \psi \right\rangle \mathrm{d}x \\ = \int_{\mathbb{R}} \left(a_4^\pm(u)\dot{\eta}^2 + 2a_2^\pm(u)\dot{\eta}G_\pm(\eta) \left(a_2^\pm(u)\dot{\eta} \right) \right) \mathrm{d}x, \end{aligned} \tag{3.21}$$

where

$$a_4^\pm(u) := -2a_1^\pm(u)' a_2^\pm(u), \tag{3.22}$$

and $a_1^\pm$, $a_2^\pm$ *are given by* (3.19).

Proof. This is a straightforward though quite long calculation. □

Far more involved is the second derivative of $A(\eta)$, a formula for which is given in the next lemma. As the proof is rather long but not difficult, we delay it to "Appendix A".

Lemma 3.6 (Second derivative of A). *For all* $u = (\eta, \psi) \in \mathcal{O} \cap \mathbb{V}$ *and* $\dot{\eta} \in \mathbb{V}_1$*, it holds that*

$$\begin{aligned}\int_{\mathbb{R}} \psi \left\langle \mathrm{D}^2 A(\eta)[\dot{\eta}, \dot{\eta}], \, \psi \right\rangle \mathrm{d}x = \int_{\mathbb{R}} \Big(a_4(u)\dot{\eta} + 2\sum_{\pm} \rho_\pm a_2^\pm(\eta, \theta_\pm) G_\pm(\eta) \left(a_2^\pm(\eta, \theta_\pm)\dot{\eta} \right) \\ -2\mathscr{M}(u)\dot{\eta} + 2\mathscr{N}(u)\dot{\eta} \Big) \dot{\eta} \, \mathrm{d}x, \end{aligned} \tag{3.23}$$

where we define the functions

$$\theta_\pm(u) := G_\pm(\eta)^{-1} A(\eta)\psi, \qquad a_4(u) := \sum_{\pm} \rho_\pm a_4^\pm(\eta, \theta_\pm), \tag{3.24}$$

and linear operators

$$\mathscr{L}_\pm(u)\dot\eta := -G_\pm(\eta)^{-1}\left(a_1^\pm(\eta,\theta_\pm)\dot\eta\right)' + a_2^\pm(\eta,\theta_\pm)\dot\eta, \qquad \mathscr{L}(u) := \sum_\pm \rho_\pm \mathscr{L}_\pm(u) \tag{3.25}$$

$$\mathscr{M}(u)\dot\eta := \sum_\pm \rho_\pm\left(a_1^\pm(\eta,\theta_\pm)(\mathscr{L}_\pm(u)\dot\eta)' + a_2^\pm(\eta,\theta_\pm)G_\pm(\eta)\mathscr{L}_\pm(u)\dot\eta\right) \tag{3.26}$$

$$\mathscr{N}(u)\dot\eta := \sum_\pm \rho_\pm\left(a_1^\pm(\eta,\theta_\pm)\left(A(\eta)G_\pm(\eta)^{-1}\mathscr{L}(u)\dot\eta\right)' + a_2^\pm(\eta,\theta_\pm)A(\eta)\mathscr{L}(u)\dot\eta\right). \tag{3.27}$$

Remark 3.7. Formally setting $\rho_+ = 0$ and $\rho_- = 1$ recovers the standard one-fluid model with normalized density. We can see from (A.1) that this would imply $A(\eta) = G_-(\eta)$, and so (3.23) must agree with the second variation formula (3.21). Indeed, one can verify directly that $\theta_- = \psi$, so that

$$\mathscr{L}_-(u) = -G_-(\eta)^{-1}\partial_x a_1^-(u) + a_2^-(u), \qquad a_4(u) = a_4^-(u),$$

and hence

$$\mathscr{N}(u) = a_1^-(u)\partial_x\mathscr{L}_-(u) + a_2^-(u)G_-(\eta)\mathscr{L}_-(u) = \mathscr{M}(u),$$

giving back the one-fluid formula in [46, Proposition 2.1].

We are now able to prove that $\mathrm{D}E(u)$ extends to $\mathbb{X}^*$ when the base point u has sufficient regularity.

Lemma 3.8 (Energy extension). *There exists a mapping $\nabla E \in C^\infty(\mathcal{O}\cap\mathbb{V};\mathbb{X}^*)$ such that*

$$\langle\nabla E(u), v\rangle_{\mathbb{X}^*\times\mathbb{X}} = \mathrm{D}E(u)v \qquad \textit{for all } u\in\mathcal{O}\cap\mathbb{V},\ v\in\mathbb{V}.$$

Proof. Let $u = (\eta,\psi)\in\mathcal{O}\cap\mathbb{V}$ and $\dot u = (\dot\eta,\dot\psi)\in\mathbb{V}$ be given. Then from the definition of E in (3.17) and the self-adjointness of $A(\eta)$, we compute that

$$\mathrm{D}E(u)\dot u = \frac{1}{2}\int_{\mathbb{R}}\psi\langle\mathrm{D}A(\eta)\dot\eta,\ \psi\rangle\,\mathrm{d}x + \int_{\mathbb{R}}\dot\psi A(\eta)\psi\,\mathrm{d}x - \int_{\mathbb{R}}\left(g[\![\rho]\!]\eta + \sigma\left(\frac{\eta'}{\langle\eta'\rangle}\right)'\right)\dot\eta\,\mathrm{d}x.$$

The latter two terms on the right-hand side certainly correspond to an element of $\mathbb{X}^*$ acting on $\dot u$. To see the same is true for the first term, we make use of the representation formula (3.20) to write

$$\int_{\mathbb{R}}\psi\langle\mathrm{D}A(\eta)\dot\eta,\ \psi\rangle\,\mathrm{d}x = \sum_\pm\rho_\pm\int_{\mathbb{R}} a_1^\pm(\eta,\theta_\pm)\theta_\pm'\dot\eta\,\mathrm{d}x + \sum_\pm\rho_\pm\int_{\mathbb{R}}\left(a_2^\pm(\eta,\theta_\pm)A(\eta)\psi\right)\dot\eta\,\mathrm{d}x,$$

for $a_1^\pm$ and $a_2^\pm$ given by (3.19) and $\theta_\pm := A(\eta)G_\pm(\eta)^{-1}\psi$. Since $u\in\mathcal{O}\cap\mathbb{V}$, it is easy to check that

$$A(\eta)\psi,\ a_1^\pm(\eta,\theta_\pm),\ a_2^\pm(\eta,\theta_\pm)\in L^2(\mathbb{R}),\quad \theta_\pm\in H^1(\mathbb{R}),$$

and hence the extension $\nabla E(u)$ can be defined explicitly as

$$\langle \nabla E(u),\ v\rangle_{\mathbb{X}^*\times\mathbb{X}} = (E'(u), v)_{L^2},$$

where the L^2 gradient $E'(u) = (E'_\eta(u), E'_\psi(u))$ takes the form

$$E'_\eta(u) := \frac{1}{2}\sum_{\pm} \rho_\pm \left(a_1^\pm(\eta,\theta_\pm)\theta'_\pm + a_2^\pm(\eta,\theta_\pm)A(\eta)\psi\right) - g[\![\rho]\!]\eta - \sigma\left(\frac{\eta'}{\langle\eta'\rangle}\right)',$$
$$E'_\psi(u) := A(\eta)\psi. \tag{3.28}$$

This completes the proof. □

Remark 3.9. Throughout the paper, we use the notational convention that, for a C^1 functional $F(\mathbb{V};\mathbb{R})$ and $u\in\mathbb{V}$, $\mathrm{D}F(u)\in\mathbb{V}^*$ is the Fréchet derivative at u, $F'(u)$ is the L^2 gradient, and $\nabla F(u)$ is an extension of $\mathrm{D}F(u)$ to $\mathbb{X}^*$ (should such an extension exist).

The energy space $\mathbb{X}$ will be endowed with symplectic structure through the prescription of the Poisson map

$$J := \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix} : \operatorname{Dom} J \subset \mathbb{X}^* \to \mathbb{X} \tag{3.29}$$

with domain

$$\operatorname{Dom} J := \left(H^{-1}(\mathbb{R})\cap \dot H^{\frac{1}{2}}(\mathbb{R})\right)\times\left(H^{1}(\mathbb{R})\cap \dot H^{-\frac{1}{2}}(\mathbb{R})\right). \tag{3.30}$$

While J appears relatively anodyne at first glance, the difference in regularity and homogeneity between $\mathbb{X}_1$ and $\mathbb{X}_2$ means that it is not bijective. This unpleasant fact is one of the major barriers to applying the classical GSS method [27] to the system. The next lemma shows, however, that J satisfies the weaker requirements of [55, Assumption 2].

Lemma 3.10 (Poisson map). *The Poisson map J defined by* (3.29) *satisfies Assumption* 2

Proof. Part (i) is a consequence of Remark 3.1, while (ii) and (iii) are obvious by definition. □

Theorem 3.11 (Hamiltonian formulation). *Consider the abstract Hamiltonian system*

$$\partial_t u = J\mathrm{D}E(u), \quad u|_{t=0} = u_0 \tag{3.31}$$

where $u_0\in\mathcal{O}\cap\mathbb{W}$ is the initial data, J is the canonical symplectic matrix (3.29)*, and the energy E is defined in* (3.17)*. Then $u\in C^0([0,t_0);\mathcal{O}\cap\mathbb{W})$ is a (weak) solution to* (3.31) *provided if and only if the corresponding $(\eta,\Phi_\pm)$ solves the Eulerian internal wave problem* (1.1).

Proof. As this formulation of the problem was previously obtained by Benjamin and Bridges [8], we provide a sketch of the argument for completeness. Suppose that $u(t) = (\eta(t),\psi(t))\in C^0([0,t_0);\mathcal{O}\cap\mathbb{W})$ is a weak solution to the Hamiltonian system (3.31). Recalling (3.7), we have that $\Phi_\pm := \mp\mathcal{H}(\eta)G_\pm(\eta)^{-1}A(\eta)\psi\in\dot H^{3+}\cap\dot H^1$ is the velocity potential in $\Omega_\pm$ and satisfies (1.1a). The definition of the harmonic extension operator

$\mathcal{H}(\eta)$ in (3.2) also ensures the kinematic condition holds on $\{y = \pm d_\pm\}$. Moreover, from the expression for $E'(u)$ obtained in (3.28), we see that

$$\partial_t \eta = E'_\psi(u) = A(\eta)\psi,$$

in the distributional sense. This is precisely (3.8) and hence corresponds to the kinematic condition on the internal interface (1.1b).

We claim that the Bernoulli condition (1.1c) is equivalent to

$$\partial_t \psi = -E'_\eta(u).$$

interpreted again in the distributional sense. Observe that, due to Remark 3.4 and the identity (3.7), many of the quantities occurring in $E'_\eta(u)$ have physical significance:

$$\begin{aligned}&\theta_\pm = A(\eta)G_\pm(\eta)^{-1}\psi = \mp\varphi_\pm, \quad a_1^\pm(\eta,\theta_\pm) = (\partial_x\Phi_\pm)|_{\mathscr{S}},\\ &a_2^\pm(\eta,\theta_\pm) = \pm(\partial_y\Phi_\pm)|_{\mathscr{S}},\end{aligned}$$

and hence,

$$\begin{aligned}E'_\eta(u) &= \frac{1}{2}\sum_\pm \rho_\pm\left(\mp(\partial_x\Phi_\pm)|_{\mathscr{S}}\varphi'_\pm \pm (\partial_y\Phi_\pm)|_{\mathscr{S}}A(\eta)\psi\right) - g[\![\rho]\!]\eta - \sigma\left(\frac{\eta'}{\langle\eta'\rangle}\right)'\\ &= -\frac{1}{2}[\![\rho|\nabla\Phi|^2]\!] + (\partial_t\eta)[\![\rho\partial_y\Phi]\!] - g[\![\rho]\!]\eta - \sigma\left(\frac{\eta'}{\langle\eta'\rangle}\right)',\end{aligned}$$

where in the second line we have used the kinematic condition (3.8) and the identities (3.10). Comparing this to equivalent statement of the Bernoulli condition in (3.11), we see that the proof is indeed complete. □

3.4. The symmetry group and the momentum. The internal wave problem is invariant under translations in the x-direction, which formally should be associated to the conservation of (horizontal linear) momentum; see, for example, [9]. To put this on firmer ground, we introduce the one-parameter symmetry group

$$T(s)u := u(\cdot - s) \qquad \text{for all } u \in \mathbb{X}. \tag{3.32}$$

Now, letting

$$P_\pm := \pm\int_{\mathbb{R}} \rho_\pm \eta'\varphi_\pm \,\mathrm{d}x$$

represent the momentum in $\Omega_\pm$, we have that the total momentum carried by the wave is

$$P(u) := P_+(u) + P_-(u) = -\int_{\mathbb{R}} \eta'\psi\,\mathrm{d}x, \tag{3.33}$$

which defines a $C^\infty(\mathcal{O}\cap\mathbb{V};\mathbb{R})$ functional. The next lemma establishes that T exhibits the necessary properties for the abstract theory in [55] and P is indeed generated by the translation invariance in the sense that (3.35) holds.

Lemma 3.12 (Conserved quantities and symmetry). *The energy E, momentum P, and translation symmetry group T given above satisfy Assumptions 3 and 4. In particular, the infinitesimal generator of $T|_{\mathbb{X}^k}$ is the unbounded linear operator*

$$T'(0)|_{\mathbb{X}^k} : \operatorname{Dom} T'(0) \subset \mathbb{X}^k \to \mathbb{X}^k \qquad u \mapsto -\partial_x u \tag{3.34}$$

with (dense) domain $\operatorname{Dom} T'(0)|_{\mathbb{X}^k} := \mathbb{X}^{k+1}$, *and*

$$T'(0)u = J\nabla P(u) \qquad \text{for all } u \in \mathcal{O} \cap \operatorname{Dom} T'(0). \tag{3.35}$$

Proof. Regarding Assumption 3, we already confirmed that the energy can be extended in Lemma 3.8. The existence of the extension ∇P is obvious from the formulas for the derivative $\mathrm{D}P$. In particular, for $u = (\eta, \psi) \in \mathcal{O} \cap \mathbb{V}$ and $\dot{u} = (\dot{\eta}, \dot{\psi}) \in \mathbb{V}$, we have

$$\mathrm{D}P(u)\dot{u} = \int_{\mathbb{R}} \psi'\dot{\eta}\, \mathrm{d}x - \int_{\mathbb{R}} \eta'\dot{\psi}\, \mathrm{d}x =: \langle \nabla P(u),\ \dot{u}\rangle_{\mathbb{X}^*\times\mathbb{X}}. \tag{3.36}$$

The right-hand side above clearly defines an element of $\mathbb{X}^*$ that depends continuously on u. In particular, it has the explicit L^2 gradient

$$P'(u) = (P'_\eta(u), P'_\psi(u)), \qquad P'_\eta(u) := \psi', \quad P'_\psi(u) := -\eta'. \tag{3.37}$$

From this it is also clear that $\nabla P(u) \in \operatorname{Dom} J$ for $u \in \mathcal{O} \cap \mathbb{V}$. Noting that $\operatorname{Dom} T'(0) = \mathbb{X}^{3/2} \subset \mathbb{V}$, the identity (3.35) now follows from the definitions of J in (3.29) and $T'(0)$ in (3.34).

Most of the statements in Assumption 4 are simple to confirm, so we omit the details. However, part (v) merits closer consideration since its conclusion is the key assumption in [55] that replaces the hypothesis that J is bijective in the standard GSS approach. First note that

$$\operatorname{Rng} J = \left(H^1(\mathbb{R}) \cap \dot{H}^{-\frac{1}{2}}(\mathbb{R})\right) \times \left(H^{-1}(\mathbb{R}) \cap \dot{H}^{\frac{1}{2}}(\mathbb{R})\right),$$

and hence by part (3.34) we have that

$$\operatorname{Dom} T'(0)|_{\mathbb{W}} \cap \operatorname{Rng} J = \left(H^{4+}(\mathbb{R}) \cap \dot{H}^{-\frac{1}{2}}(\mathbb{R})\right) \times \left(H^{-1}(\mathbb{R}) \cap \dot{H}^{\frac{1}{2}}(\mathbb{R}) \cap \dot{H}^{\frac{7}{2}+}(\mathbb{R})\right).$$

This is indeed dense in $\mathbb{X}$ due to Remark 3.1. □

3.5. Traveling waves. In Hamiltonian language, a traveling internal wave is a solution to (3.31) taking the form

$$u(t) = T(ct)U, \tag{3.38}$$

for some wave speed $c \in \mathbb{R}$ and time-independent bound state $U \in \mathcal{O} \cap \mathbb{W}$. Let us now discuss in somewhat finer detail the existence theory obtained by Nilsson in [47].

Recall that we have defined the dimensionless parameters β, λ, ϱ, and h in (1.2) and (1.5). Let $\mathscr{T} := \{z \in \mathbb{C} : \operatorname{Re} z \in (-r, r)\}$ be a thin slab centered on the imaginary axis. For $r > 0$ sufficiently small, we have by the dispersion relation (1.3) that there exist three curves in the (β, λ)-plane along which the spectrum of the linearized problem in $\mathscr{T}$ crosses the real or imaginary axis.

Consider first the curve Γ_1, which is simply the line $\lambda = \lambda_0$. Immediately below it and to the right of $\beta = \beta_0$, the spectrum in $\mathscr{T}$ consists of a pair of oppositely signed real eigenvalues and a complex conjugate pair on the imaginary axis. Passing through Γ_1, the imaginary eigenvalues collide at the origin then move along the real axis. This same 0^2 resonance is associated with transition from periodic solutions to solitons in the steady KdV equation, for example. On the curve

$$\Gamma_2 := \{(\beta(\xi), \lambda(\xi)) : \xi \in [0, \infty)\},$$

where

$$\begin{aligned} \beta(\xi) &:= \sum_{\pm} \frac{\rho_\pm}{\rho_-} \frac{d_+}{d_\pm} \left(\frac{-\sin(\frac{d_\pm}{d_+}\xi)\cos(\frac{d_\pm}{d_+}\xi) + \frac{d_\pm}{d_+}\xi}{2\xi \sin^2(\frac{d_\pm}{d_+}\xi)} \right), \\ \lambda(\xi) &:= \beta(\xi)^2 + \xi \sum_{\pm} \frac{\rho_\pm}{\rho_-} \coth(\frac{d_+}{d_\pm}\xi), \end{aligned} \tag{3.39}$$

the spectrum in $\mathscr{T}$ consists of two real eigenvalues with multiplicity 2. In the region bounded by Γ_1 and Γ_2, there are two pairs of oppositely signed simple real eigenvalues.

Nilsson's approach is to fix $\beta > \beta_0$ and treat λ as a bifurcation parameter with $0 < \lambda - \lambda_0 \ll 1$. This ensures that (β, λ) remains in the Region A depicted in Fig. 2, which is the narrow open set bounded below by Γ_1 and lying beneath Γ_2. Because he opts to non-dimensionalize the system at the outset, translating his result to our setting involves introducing some heavy notation, but this will be pared down soon.

Theorem 3.13 (Nilsson [47]). *Let $\{\Pi_\varepsilon = (\rho_{\pm\varepsilon}, d_{\pm\varepsilon}, \sigma_\varepsilon, c_\varepsilon) : 0 < \varepsilon \ll 1\}$ be a smooth curve in the dimensional parameter space such that the corresponding Bond number is fixed to $\beta > \beta_0$ and $\lambda = \lambda_0 + \varepsilon^2$.*

(a) *Suppose that $\varrho_\varepsilon - 1/h_\varepsilon^2 = O(1)$ as $\varepsilon \searrow 0$. Then for any $k > 1/2$, there exists a smooth curve*

$$\mathscr{C}_\beta^{\mathrm{A}} = \{u_{\varepsilon;\,\beta}^{\mathrm{A}} : 0 < \varepsilon \ll 1\} \subset \mathbb{X}^k$$

so that $u_{\varepsilon;\,\beta}^{\mathrm{A}}$ is a traveling internal wave for the parameter values Π_ε. Along this curve, the free surface profile has leading-order form

$$\eta_{\varepsilon;\,\beta}^{\mathrm{A}} = \frac{\varepsilon^2 d_+}{\varrho - 1/h^2} \operatorname{sech}^2\left(\frac{\varepsilon\,\cdot}{2d_+\sqrt{\beta - \beta_0}} \right) + O(\varepsilon^3) \quad \text{in } \mathbb{X}_1^k \text{ as } \varepsilon \searrow 0. \tag{3.40}$$

(b) *Suppose instead that $\varrho_\varepsilon - 1/h_\varepsilon^2 = \kappa\varepsilon$ for a fixed $\kappa \neq 0$. Then for any $k > 1/2$ there exists two smooth curves*

$$\mathscr{C}_{\beta,\kappa,\pm}^{\mathrm{A}} = \{u_{\varepsilon;\,\beta,\kappa,\pm}^{\mathrm{A}} : 0 < \varepsilon \ll 1\} \subset \mathbb{X}^k$$

so that $u_{\varepsilon;\,\beta,\kappa,\pm}^{\mathrm{A}}$ is a traveling internal wave for the parameter values Π_ε. Along $\mathscr{C}_{\beta,\kappa,\pm}^{\mathrm{A}}$, the free surface profile has leading-order form

$$\eta_{\varepsilon;\,\beta,\kappa,\pm}^{\mathrm{A}} = \frac{2\varepsilon d_+}{\kappa \pm \sqrt{\kappa^2 + 4(\varrho + 1/h^3)}} \operatorname{sech}\left(\frac{\varepsilon\,\cdot}{d_+\sqrt{\beta - \beta_0}} \right) + O(\varepsilon^3) \quad \text{in } \mathbb{X}_1^k \text{ as } \varepsilon \searrow 0. \tag{3.41}$$

Remark 3.14. The above solutions are obtained using a center manifold reduction at the point $(\beta, \lambda_0) \in \Gamma_1$. For the scaling regime of part (a), the reduced equation is a perturbation of steady KdV. This gives rise to waves with the classical sech^2 asymptotics in (3.40). However, when $\varrho - 1/h^2 \ll 1$, cubic terms enter at leading order, and so one instead obtains an equation of Gardner or mKdV-KdV type. An important consequence of this construction is that the $O(\varepsilon^3)$ remainder terms in (3.40) and (3.41) are exponentially decaying and exhibit the same scaling of the spatial variable as the leading-order part. Note also that the regularity of the solutions is not stated by Nilsson, but follows from a standard bootstrapping argument.

Theorem 3.13 fixes β but allows the dimensional parameters to vary. While convenient for proving existence, this choice is not ideal for stability analysis: two waves on one of these curves may not necessarily solve the same physical problem. The general theory in [27,55] instead asks for a family of bound states parameterized by c, with the remaining dimensional parameters held constant. Given a choice of parameters $(\rho_{\pm *}, d_{\pm *}, \sigma_*, c_*)$, we therefore let

$$(\beta_c, \lambda_c) := \left(\frac{\sigma_*}{d_{+*}\rho_{-*}c^2}, -\frac{g[\![\rho_*]\!]d_{+*}}{\rho_{-*}c^2} \right), \quad \varepsilon_c^{\mathrm{A}} := \sqrt{\lambda_c - \lambda_0} \qquad \text{for } |c - c_*| \ll 1. \tag{3.42}$$

The first of these parameterizes a segment of the straight line joining (β_*, λ_*) to the origin in the (β, λ)-plane, while the second expresses the bifurcation parameter ε from Theorem 3.13 in terms of c.

The next two corollaries convert Theorem 3.13 to statements on bound states indexed by c. In particular, they prove that Assumption 5 is satisfied.

Corollary 3.15. (KdV bound states) *Let $(\rho_{\pm *}, d_{\pm *}, \sigma_*, c_*)$ be given so that $\varrho_* - 1/h_*^2 \neq 0$ and the corresponding non-dimensional parameters (β_*, λ_*) lies in Region A. There exists an open interval $\mathscr{I} \ni c_*$ and a family of bound states $\{U_c^{\mathrm{A}}\}_{c \in \mathscr{I}} \subset \mathcal{O} \cap \mathbb{W}$ having the non-dimensional parameter values (β_c, λ_c) given by (3.42). The free surface profile is*

$$\eta_c^{\mathrm{A}} := \eta^{\mathrm{A}}_{\varepsilon_c^{\mathrm{A}};\, \beta_c} \qquad \textit{for } c \in \mathscr{I}.$$

Moreover, $\{U_c^{\mathrm{A}}\}$ satisfies Assumption 5.

Proof. Let $(\rho_{\pm *}, d_{\pm *}, \sigma_*, c_*)$ be given as above and assume that the corresponding (β_*, λ_*) satisfy $\beta_* > \beta_0$ and $0 < \lambda_* - \lambda_0 \ll 1$. Then for all $|c - c_*| \ll 1$, the dimensional parameters meet the hypotheses of Theorem 3.13(a), and so may simply take $U_c^{\mathrm{A}} := u_{\varepsilon_c^{\mathrm{A}};\, \beta_c}$ for β_c and $\varepsilon_c^{\mathrm{A}}$ defined according to (3.42).

The free surface profile from (3.40) is constructed as a solution to a second-order ODE that is a homoclinic to 0. It can be verified directly that the origin is a saddle point, and hence $\eta^{\mathrm{A}}_{\varepsilon;\beta}$ is exponentially localized, with uniform decay rate on compact subsets of parameter space. Moreover, due to the translation invariance, the profile is of class C^∞. In particular, it is clearly an element of $\mathbb{X}_1^k$ for all $k \geq 1/2$. Solving the kinematic condition, we see that the corresponding $\psi = \psi^{\mathrm{A}}_{\varepsilon;\beta}$ is likewise smooth and an element of $\mathbb{X}_2^k$ for all $k \geq 1/2$. Part(i) now follows from the smooth dependence of $u_{\varepsilon;\beta}$ on (ε, β). Part(ii) certainly holds in view of the (arbitrarily high) regularity of the bound states. Finally, parts (iii) and (iv) are obvious given the form of η_c^{A}. □

An identical argument applied to the family of waves in Theorem 3.13(b) yields the following.

Corollary 3.16 (Gardner bound states). *Let $(\rho_{\pm *}, d_{\pm *}, \sigma_*, c_*)$ be given so that the corresponding (β_*, λ_*) lies in Region A and $|\varrho_* - 1/h_*^2| \eqsim |\lambda_* - \lambda_0|^{1/2}$. There exists an open interval $\mathscr{I} \ni c_*$ and two families of bound states $\{U_c^{\mathrm{A}\pm}\}_{c\in\mathscr{I}} \subset \mathcal{O} \cap \mathbb{W}$ having the non-dimensional parameter values $(\beta_c^{\mathrm{A}}, \lambda_c^{\mathrm{A}})$ given by (3.42) and with the remaining parameters fixed. They satisfy Assumption 5 and the corresponding free surface is given by*

$$\eta_c^{\mathrm{A}\pm} := \eta^{\mathrm{A}}_{\varepsilon_c^{\mathrm{A}};\, \beta_c, \kappa_c^{\mathrm{A}}, \pm} \qquad \text{for } \kappa_c^{\mathrm{A}} := \frac{1}{\varepsilon_c^{\mathrm{A}}} \left(\varrho_* - \frac{1}{h_*^2} \right), \quad c \in \mathscr{I}.$$

Consider now the situation where (β, λ) is contained in Region C, which is a neighborhood of the curve Γ_2. Nilsson uses a center manifold reduction method to construct traveling waves, this time bifurcating from the point (β_0, λ_0). Setting $\gamma := (\varrho + h)/45$, one can show using the parameterization of Γ_2 that for all $\delta \in \mathbb{R}$, the point

$$\beta = \beta_0 + 2(1+\delta)\gamma\varepsilon^2, \qquad \lambda = \lambda_0 + \gamma\varepsilon^4 \tag{3.43}$$

is contained in Region C for all $0 < \varepsilon \ll 1$. When $\delta > 0$, it lies below Γ_2 and for $\delta < 0$, it lies above.

At $\varepsilon = 0$, this gives the critical parameter value (β_0, λ_0) where we recall that 0 is an eigenvalue of multiplicity 4. The resulting reduced equation on the center manifold thus has a four-dimensional phase space. When $\varrho - 1/h^2 = O(1)$ as $\varepsilon \searrow 0$, after performing a rescaling and truncation, we obtain the ODE

$$Z'''' - 2(1+\delta)Z'' + Z - \frac{3}{2}\gamma^{-3/2}\left(\varrho - \frac{1}{h^2}\right) Z^2 = 0. \tag{3.44}$$

This equation arises in the study of capillary-gravity waves beneath vacuum in the critical surface tension regime as well as a modeling the buckling of elastic struts [3]. Analysis in [16,19] shows that, at $\delta = 0$, there is a *primary homoclinic* solution Z_0 to (3.44) that is unimodal, even, and exponentially localized. Moreover, there is a smooth one-parameter family of homoclinic orbits $\{Z_\delta\}_\delta$ defined for $\delta \geq 0$ and $-1 \ll \delta < 0$ that bifurcates from Z_0. These solutions are *transversely constructed*, in that the stable and unstable manifolds of the zero equilibrium of (1.9) intersect transversely at $Z = Z_\delta(0)$ at the zero level set of the Hamiltonian energy. For $\delta \geq 0$, we have that Z_δ is the unique (up to translation) homoclinic solution to (1.9) that is positive, even and monotone for $x > 0$ (see [3]). When $-1 \ll \delta < 0$, uniqueness is not known and Z_δ has exponentially decaying oscillatory tails. In addition to the primary homoclinic orbits, there exists a "plethora" of other solutions to (3.44) that take the form of multisolitons; see [16,23]. Because these are multimodal, they are unlikely to be amenable to analysis through the general theory in [55] and so we will not consider them here.

On the other hand, if $\varrho - 1/h^2 = \kappa\varepsilon^2$, for some $\kappa \neq 0$, then upon rescaling and truncating to leading order, the reduced equation on the center manifold takes the form

$$Z'''' - 2(1+\delta)Z'' + Z - \frac{3}{2}\gamma^{-3/2}\kappa Z^2 - 4\gamma^{-2}\left(\varrho + \frac{1}{h^3} + \frac{2(\varrho-1)^2}{225\gamma}\right) Z^3 = 0. \tag{3.45}$$

In [47, Appendix B], it is shown that, at $\delta = 0$, this ODE has both a positive and negative primary homoclinic solution, which we denote by $Z_{0;\kappa,\pm}$. As in the non-resonant case, these are exponentially localized, unique up to translation (for the fixed sign), and because they are transversely constructed, they persists for $|\delta| \ll 1$. Let the corresponding families be denoted $\{Z_{\delta;\kappa,\pm}\}$.

We now state Nilsson's results for this case reformulated in the style of Corollaries 3.15 and 3.16. Let $(\rho_{\pm *}, d_{\pm *}, \sigma_*, c_*)$ be given so that the corresponding (β_*, λ_*) lies in Region C. In view of (3.43), we define

$$\varepsilon_c^{\mathrm{C}} := \left(\frac{\lambda_c - \lambda_0}{\gamma_*}\right)^{1/4}, \qquad \delta_c := \frac{\beta_c - \beta_0}{2\gamma_*(\varepsilon_c^{\mathrm{C}})^2} - 1 \qquad \text{for } |c - c_*| \ll 1, \tag{3.46}$$

with (β_c, λ_c) given in (3.42). The existence of bound states is then summarized in the following lemma.

Lemma 3.17. (Kawahara bound states) *Let $(\rho_{\pm *}, d_{\pm *}, \sigma_*, c_*)$ be given so that $\varrho_* - 1/h_*^2 \neq 0$ and the corresponding non-dimensional parameters (β_*, λ_*) lie in Region C with $0 < \varepsilon_{c_*}^{\mathrm{C}} \ll 1$.*

(a) *There exists an open interval $\mathscr{I} \ni c_*$ and a family of bound states $\{U_c^{\mathrm{C}}\}_{c \in \mathscr{I}} \subset \mathcal{O} \cap \mathbb{W}$ having the non-dimensional parameter values (β_c, λ_c) and satisfying Assumption 5. The corresponding free surface profile takes the form*

$$\eta_c^{\mathrm{C}} = \varepsilon^4 d_+ \sqrt{\gamma} Z_\delta\left(\frac{\varepsilon\,\cdot}{d_+}\right) + O\left(\varepsilon^5\right) \qquad \text{in } \mathbb{X}_1^k \tag{3.47}$$

with $\varepsilon = \varepsilon_c^{\mathrm{C}}$ and $\delta = \delta_c^{\mathrm{C}}$ given by (3.46), $d_+ = d_{+*}$, and $\gamma = \gamma_*$.

(b) *Suppose that $|\varrho_* - 1/h_*^2| \eqsim (\epsilon_{c_*}^{\mathrm{C}})^2 \ll 1$. Then there exists an open interval $\mathscr{I} \ni c_*$ and two families of bound states $\{U_c^{\mathrm{C}\pm}\}_{c \in \mathscr{I}} \subset \mathcal{O} \cap \mathbb{W}$ having the non-dimensional parameter values (β_c, λ_c) and satisfying Assumption 5. The corresponding free surface profile takes the form*

$$\begin{gathered}\eta_c^{\mathrm{C}\pm} = \varepsilon^2 d_+ \sqrt{\gamma} Z_{\delta;\,\kappa,\pm}\left(\frac{\varepsilon\,\cdot}{d_+}\right) + O\left(\varepsilon^3\right) \quad \text{in } \mathbb{X}_1^k \\ \text{for } \kappa = \kappa_c^{\mathrm{C}} := \frac{1}{\varepsilon_c^2}\left(\varrho_* - \frac{1}{h_*^2}\right),\end{gathered} \tag{3.48}$$

with $\varepsilon = \varepsilon_c^{\mathrm{C}}$ and $\delta = \delta_c^{\mathrm{C}}$ given by (3.46), $d_+ = d_{+*}$, and $\gamma = \gamma_*$.

Remark 3.18. While we will carry out many of the calculations for both families $\{U_c^{\mathrm{C}}\}$ and $\{U_c^{\mathrm{C}\pm}\}$, we only obtain a stability result for the former. In the latter case, we find that the rescaled linearized augmented potential does not converge precisely to the linearization of (3.44), which obstructs the spectral analysis in the next section; see Lemma 4.7(b).

We conclude this section by noting that Nilsson also proves the existence of many types of traveling waves with (β, λ) in a neighborhood of the bifurcation curve

$$\Gamma_3 = \{(\beta(i\xi), \lambda(i\xi)) : \xi \in [0, \infty)\}, \tag{3.49}$$

with $(\beta(\xi), \lambda(\xi))$ given by (3.39). The stability of these solutions will be the subject of a forthcoming work.

4. Spectral Analysis

Observe that if $u(t) = T(ct)U$ is a traveling wave for the bound state $U \in \mathcal{O} \cap \mathbb{W}$ and wave speed $c \in \mathbb{R}$, then necessarily by (3.31), Lemma 3.12, and Assumption 4(vi) we have

$$\frac{\mathrm{d}u}{\mathrm{d}t} = cT'(0)U = J\mathrm{D}E(U).$$

Combining this with (3.35), we obtain the steady equation

$$\mathrm{D}E(U) = c\mathrm{D}P(U),$$

where we have used that J is injective. This motivates us to consider the *augmented Hamiltonian*, which for a fixed c is the functional $E_c \in C^\infty(\mathcal{O} \cap \mathbb{V}; \mathbb{R})$ given by

$$E_c(u) := E(u) - cP(u).$$

The above calculation shows that bound states are critical points of E_c. It also suggests that one can construct such solutions as constrained extrema of the energy on level sets of the momentum, with the wave speed a Lagrange multiplier. In order to exploit this connection, we must first understand better the second derivative of E_c.

With that in mind, this section is devoted to the quite difficult task of computing the spectrum of the linearized augmented Hamiltonian at either a shear flow or small-amplitude internal capillary-gravity wave. Here we will follow the general approach of Mielke [46], which was also the basis for the calculation in [55]. The strategy has two steps. First, via the kinematic condition ψ is eliminated in favor of η. Making this substitution in the definition of E_c gives the so-called augmented potential $V_c^{\mathrm{aug}} = V_c^{\mathrm{aug}}(\eta)$, which proves to be much more amenable to analysis. In particular, we show in Sect. 4.1 that its second variation at a critical point is characterized by a certain second-order nonlocal differential operator $Q_c(\eta)$. As one might predict, $Q_c(0)$ is a Fourier multiplier whose symbol is related directly to the dispersion relation (1.3).

This is enough to characterize the continuous spectrum of $\mathrm{D}^2 V_c^{\mathrm{aug}}(\eta)$ when η is sufficiently small amplitude; see Lemma 4.4. Determining the discrete spectrum, however, requires considerably more effort. Following Mielke, the second step is to conjugate $Q_c(\eta)$ with a rescaling S_ε informed by the asymptotics of η discussed in Sect. 3.5. Briefly put, the idea here is to show that linearization and scaling almost commute. It is well known that in the shallow water regime, the internal wave system can be modeled by nonlinear dispersive PDEs such as KdV or Gardner. We seek to prove that imposing this scaling on the linearized operator $Q_c(\eta)$ via conjugation by S_ε will, to leading order, coincide with the linearization of the corresponding model equation. After a delicate calculation, we do indeed find that in the long-wave limit $\varepsilon \searrow 0$, the rescaled operator $S_\varepsilon^{-1} Q_c(\eta) S_\varepsilon$ converges (in an appropriate sense) to the linearized steady KdV or Gardner equation in the case of Region A, and to the linearization of (3.44) or (3.45) in the case of Region C. This is the subject of Sect. 4.2. In Sect. 4.3, we prove that the spectrum of $\mathrm{D}^2 V_c^{\mathrm{aug}}$ is qualitatively the same as that of this limiting rescaled operator.

Lastly, in Sect. 4.4 we take the information about the spectrum of $\mathrm{D}^2 V_c^{\mathrm{aug}}$ and translate it back to that of $\mathrm{D}^2 E_c$. For U_c one of the family of bound states described in Sect. 3.5, we confirm that $\mathrm{D}^2 E_c(U_c)$ extends to a self-adjoint operator on $\mathbb{X}$ that has Morse index 1 as required by Assumption 6.

4.1. The augmented potential and its derivatives. If $u_* = (\eta_*, \psi_*)$ is a critical point of E_c, then in particular $\mathrm{D}_\psi E(u_*) = c\mathrm{D}_\psi P(u_*)$. Because V is independent of ψ and $A(\eta)$ is self-adjoint, we see that

$$\mathrm{D}_\psi E(u)\dot{\psi} = \int_{\mathbb{R}} \dot{\psi} A(\eta)\psi \,\mathrm{d}x.$$

Combining this with (3.36) we find that ψ_* can be uniquely determined from η_* via

$$\psi_*(\eta) := -cA(\eta)^{-1}\eta'. \tag{4.1}$$

Note that ψ_* also depends on c, but in this section the wave speed will be fixed, so there is no harm in suppressing it. In fact it will turn out to be easier to work with φ_* rather than ψ_*. So we recall from (3.5) and (3.4) that

$$\psi_* = \rho_-\varphi_{*-} - \rho_+\varphi_{*+}, \qquad \varphi_{*\pm} = \pm cG_\pm(\eta)^{-1}\eta'. \tag{4.2}$$

When there is no risk of confusion, we will drop the $*$ subscripts to declutter the notation.

Recall from Remark 3.4 that the coefficients $a_1^\pm$ and $a_2^\pm$ that arise in the first derivative formula (3.18) for $G_\pm(\eta)$ can be alternatively be expressed as

$$a_1^\pm(\eta,\phi) := \mp(\partial_x \mathcal{H}_\pm(\eta)\phi)|_{\mathscr{S}}, \qquad a_2^\pm := -(\partial_y \mathcal{H}_\pm(\eta)\phi)|_{\mathscr{S}}.$$

Therefore, when they are evaluated at $\phi = \varphi_\pm$, they give (up to a sign) the trace of the velocity field on $\mathscr{S}$. Following [55, Section 6], we introduce the related functions

$$b_1^\pm := \mp a_1^\pm(\eta, \varphi_\pm) - c, \qquad b_2^\pm := -a_2^\pm(\eta,\varphi_\pm). \tag{4.3}$$

This way, $(b_1^\pm, b_2^\pm)$ represents the *relative velocity* in $\Omega_\pm$ restricted to the interface. Consequently, for $\eta \in \mathbb{W}_1$, we have from (4.2) that $b_1^\pm, b_2^\pm \in H^{2+}(\mathbb{R})$. Notice also that, because u represents a traveling wave, the kinematic condition (3.8) gives

$$b_2^\pm = \eta' b_1^\pm. \tag{4.4}$$

Differentiating (4.2), we find that

$$\begin{aligned}
\mathrm{D}\psi_*(\eta)\dot{\eta} &= \rho_-\mathrm{D}\varphi_-(\eta)\dot{\eta} - \rho_+\mathrm{D}\varphi_+(\eta)\dot{\eta},\\
\mathrm{D}\varphi_\pm(\eta)\dot{\eta} &= \pm\langle c\mathrm{D}(G_\pm(\eta)^{-1})\dot{\eta},\ \eta'\rangle \pm cG_\pm(\eta)^{-1}\dot{\eta}'.
\end{aligned}$$

On the other hand,

$$\langle \mathrm{D}(G_\pm(\eta)^{-1})\dot{\eta},\ \eta'\rangle = -G_\pm(\eta)^{-1}\langle \mathrm{D}G_\pm(\eta)\dot{\eta},\ G_\pm(\eta)^{-1}\eta'\rangle,$$

and so we may infer from Lemma 3.3 and (4.2) that

$$\begin{aligned}
G_\pm(\eta)\langle \mathrm{D}(G_\pm(\eta)^{-1})\dot{\eta},\ c\eta'\rangle &= (a_1^\pm(\eta, \pm\varphi_\pm)\dot{\eta})' - G_\pm(\eta)(a_2^\pm(\eta,\pm\varphi_\pm)\dot{\eta})\\
&= \pm(a_1^\pm(\eta,\varphi_\pm)\dot{\eta})' \mp G_\pm(\eta)(a_2^\pm(\eta,\varphi_\pm)\dot{\eta}).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{D}\varphi_\pm(\eta)\dot{\eta} &= G_\pm(\eta)^{-1}(a_1^\pm(\eta,\varphi_\pm)\dot{\eta})' - a_2^\pm(\eta,\varphi_\pm)\dot{\eta} \pm cG_\pm(\eta)^{-1}\dot{\eta}'\\
&= \mp G_\pm(\eta)^{-1}(b_1^\pm\dot{\eta})' + b_2^\pm\dot{\eta},
\end{aligned}$$

and hence

$$\mathrm{D}\psi_*(\eta)\dot{\eta} = \underbrace{\sum_{\pm} \rho_\pm G_\pm(\eta)^{-1}(b_1^\pm \dot{\eta})'}_{=:\,\mathcal{S}\dot{\eta}} - \underbrace{\sum_{\pm} \pm\rho_\pm b_2^\pm \dot{\eta}}_{=:\,\mathcal{T}\dot{\eta}}. \tag{4.5}$$

Now, let the *augmented potential* be the functional $V_c^{\mathrm{aug}} \in C^\infty(\mathcal{O}\cap\mathbb{V};\mathbb{R})$ given by

$$V_c^{\mathrm{aug}}(\eta) := E_c(\eta, \psi_*(\eta)) = \min_{\psi} E_c(\eta, \psi). \tag{4.6}$$

While it is not immediately obvious, for small-amplitude waves the spectrum of $\mathrm{D}^2 E_c$ can be determined from that of $\mathrm{D}^2 V_c^{\mathrm{aug}}$. We therefore devote the remainder of this subsection to studying the second variation of V_c^{aug}. In particular, we will derive an analytically tractable quadratic form representation defined in terms of physical quantities.

Lemma 4.1 (Second derivative of V_c^{aug}). *For all $(\eta, \psi_*(\eta)) \in \mathcal{O}\cap\mathbb{V}$ and $\dot{\eta} \in \mathbb{V}_1$, it holds that*

$$\mathrm{D}^2 V_c^{\mathrm{aug}}(\eta)[\dot{\eta}, \dot{\eta}] = \mathrm{D}_\eta^2 E_c(\eta, \psi_*(\eta))[\dot{\eta}, \dot{\eta}] - \int_{\mathbb{R}} (\mathcal{S}-\mathcal{T})\dot{\eta} A(\eta)(\mathcal{S}-\mathcal{T})\dot{\eta}\,\mathrm{d}x \tag{4.7}$$

where $\mathcal{S}$ and $\mathcal{T}$ are defined in (4.5).

Proof. Starting from the definition of V_c^{aug} in (4.6), we see that

$$\mathrm{D}V_c^{\mathrm{aug}}(\eta)\dot{\eta} = \mathrm{D}_\eta E_c(u_*)\dot{\eta} + \mathrm{D}_\psi E_c(u_*)\mathrm{D}\psi_*(\eta)\dot{\eta} = \mathrm{D}_\eta E_c(u_*)\dot{\eta},$$

where $u_* = u_*(\eta) := (\eta, \psi_*(\eta))$. Note that the last equality follows from the fact that u_* is a critical point of E_c for all η. Differentiating again in η gives

$$\begin{aligned}\mathrm{D}^2 V_c^{\mathrm{aug}}(\eta)[\dot{\eta}, \dot{\eta}] &= \mathrm{D}_\eta^2 E_c(u_*)[\dot{\eta}, \dot{\eta}] + \mathrm{D}_\psi \mathrm{D}_\eta E_c(u_*)[D\psi_*(\eta)\dot{\eta}, \dot{\eta}] \\ &= \mathrm{D}_\eta^2 E_c(u_*)[\dot{\eta}, \dot{\eta}] - \mathrm{D}_\psi^2 E_c(u_*)[\mathrm{D}\psi_*(\eta)\dot{\eta},\ \mathrm{D}\psi_*(\eta)\dot{\eta}].\end{aligned}$$

The potential energy is independent of ψ and the momentum is linear in ψ. Thus,

$$\begin{aligned}\mathrm{D}_\psi^2 E_c(u_*)[\mathrm{D}\psi_*(\eta)\dot{\eta},\ \mathrm{D}\psi_*(\eta)\dot{\eta}] &= \mathrm{D}_\psi^2 K(u_*)[\mathrm{D}\psi_*(\eta)\dot{\eta},\ \mathrm{D}\psi_*(\eta)\dot{\eta}] \\ &= \int_{\mathbb{R}} \mathrm{D}\psi_*(\eta)\dot{\eta} A(\eta)\mathrm{D}\psi_*(\eta)\dot{\eta}\,\mathrm{d}x,\end{aligned}$$

which, from (4.5), implies (4.7). □

Lemma 4.2. (Quadratic form) *For all $(\eta, \psi_*(\eta)) \in \mathcal{O}\cap\mathbb{V}$ and $c \in \mathbb{R}$, there is a self-adjoint linear operator $Q_c(\eta) \in \mathrm{Lin}(\mathbb{X}_1; \mathbb{X}_1^*)$ such that*

$$\mathrm{D}^2 V_c^{\mathrm{aug}}(\eta)[\dot{\eta}, \dot{\zeta}] = \langle Q_c(\eta)\dot{\eta},\ \dot{\zeta}\rangle_{\mathbb{X}_1^*\times\mathbb{X}_1} \tag{4.8}$$

for all $\dot{\eta}, \dot{\zeta} \in \mathbb{V}_1$. It is given explicitly by

$$\begin{aligned} Q_c(\eta)\dot{\eta} = &-\left(\sigma \frac{\dot{\eta}'}{\langle \eta' \rangle^3}\right)' - \left(g[\![\rho]\!] + \sum_{\pm} \pm\rho_\pm b_1^\pm (b_2^\pm)'\right)\dot{\eta} \\ &+ \sum_{\pm} \rho_\pm b_1^\pm \left(G_\pm(\eta)^{-1}(b_1^\pm \dot{\eta})'\right)'. \end{aligned} \tag{4.9}$$

Remark 4.3. Taking $\rho_+ = 0$ and $\rho_- = 1$ recovers the one-fluid problem, and it is straightforward to see that formula (4.9) agrees with computation in [46, Theorem 3.3].

Proof. We continue to write $u_* := (\eta, \psi_*(\eta))$. Since the momentum is linear in η, we see that

$$\begin{aligned} \mathrm{D}_\eta^2 E_c(u_*)[\dot\eta, \dot\eta] &= \mathrm{D}_\eta^2 K(u_*)[\dot\eta, \dot\eta] + \mathrm{D}_\eta^2 V(u_*)[\dot\eta, \dot\eta] \\ &= \frac{1}{2}\int_{\mathbb{R}} \psi_* \langle \mathrm{D}^2 A(\eta)[\dot\eta, \dot\eta],\ \psi_*\rangle \,\mathrm{d}x - \int_{\mathbb{R}} g[\![\rho]\!]\dot\eta^2 \,\mathrm{d}x + \int_{\mathbb{R}} \sigma \frac{(\dot\eta')^2}{\langle \eta' \rangle^3}\,\mathrm{d}x. \end{aligned} \tag{4.10}$$

The latter two terms on the right-hand side above are already in the desired form. But, to understand the first requires the formula for the second variation of $A(\eta)$ derived in Lemma 3.6.

In particular, notice that when $\theta_\pm$ defined in (3.24) is evaluated at u_*, it simplifies to

$$\theta_\pm(u_*) = -cG_\pm(\eta)^{-1}\eta' = \mp\varphi_\pm,$$

and $a_1^\pm(\eta, \theta_\pm) = b_1^\pm + c$, $a_2^\pm(\eta, \theta_\pm) = \pm b_2^\pm$. We further define

$$\mathcal{S}_\pm(\eta)\xi := G_\pm(\eta)^{-1}(b_1^\pm \xi)', \quad \mathcal{T}_\pm(\eta)\xi := \pm b_2^\pm \xi,$$

so that $\mathcal{S}(\eta) = \sum_\pm \rho_\pm \mathcal{S}_\pm(\eta)$ and $\mathcal{T}(\eta) = \sum_\pm \rho_\pm \mathcal{T}_\pm(\eta)$. Making these substitution, we find from the second derivative formula (3.23) that

$$\begin{aligned} \frac{1}{2}\int_{\mathbb{R}} \psi_* \langle \mathrm{D}^2 A(\eta)[\dot\eta, \dot\eta],\ \psi_*\rangle \,dx &= \sum_\pm \rho_\pm \int_{\mathbb{R}} \Big(\mp (b_1^\pm)' b_2^\pm \dot\eta^2 + \mathcal{T}_\pm \dot\eta G_\pm(\eta) \mathcal{T}_\pm \dot\eta\Big)\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}} (-\dot\eta \mathscr{M}(u_*)\dot\eta + \dot\eta \mathscr{N}(u_*)\dot\eta)\,\mathrm{d}x. \end{aligned}$$

Let us next look more closely at the two terms on the second line above. Observe first that

$$\begin{aligned} \mathscr{L}_\pm(u_*)\dot\eta &= -G_\pm(\eta)^{-1}\left((b_1^\pm + c)\dot\eta\right)' \pm b_2^\pm \dot\eta = -\left(\mathcal{T}_\pm - \mathcal{S}_\pm + cG_\pm(\eta)^{-1}\partial_x\right)\dot\eta \\ \mathscr{L}(u_*) &= \mathcal{T} - \mathcal{S} - cA(\eta)^{-1}\partial_x, \end{aligned} \tag{4.11}$$

where the second line follows from the first and (A.1). Because $\mathscr{L}_\pm$ and $\mathscr{L}$ will be evaluated at u_* throughout the calculation, we will suppress their arguments in the interests of readability. Using (4.11), we see that the operator $\mathscr{M}$ defined in (3.26) at the critical point satisfies

$$\begin{aligned} \int_{\mathbb{R}} \dot\eta \mathscr{M}\dot\eta \,\mathrm{d}x &= \sum_\pm \rho_\pm \int_{\mathbb{R}} \Big((b_1^\pm + c)(\mathscr{L}_\pm\dot\eta)' \pm b_2^\pm G_\pm(\eta)\mathscr{L}_\pm\dot\eta\Big)\dot\eta\,\mathrm{d}x \\ &= \sum_\pm \rho_\pm \int_{\mathbb{R}} \Big(-\left((b_1^\pm + c)\dot\eta\right)' \pm G_\pm(\eta) b_2^\pm \dot\eta\Big)\mathscr{L}_\pm\dot\eta\,\mathrm{d}x \\ &= \sum_\pm \rho_\pm \int_{\mathbb{R}} \mathscr{L}_\pm\dot\eta G_\pm(\eta)\mathscr{L}_\pm\dot\eta\,\mathrm{d}x, \end{aligned}$$

where again we are abbreviating $\mathscr{M} = \mathscr{M}(u_*)$. Substituting in the expression (4.11) and expanding yields

$$\int_{\mathbb{R}} \dot\eta \mathscr{M} \dot\eta \,\mathrm{d}x = \sum_{\pm} \rho_\pm \int_{\mathbb{R}} \Big(\mathcal{S}_\pm \dot\eta G_\pm(\eta) \mathcal{S}_\pm \dot\eta - 2\mathcal{S}_\pm \dot\eta G_\pm(\eta) \mathcal{T}_\pm \dot\eta + \mathcal{T}_\pm \dot\eta G_\pm(\eta) \mathcal{T}_\pm \dot\eta \Big) \,\mathrm{d}x$$
$$+ \int_{\mathbb{R}} \Big(c^2 \dot\eta' A(\eta)^{-1} \dot\eta' + 2c\dot\eta'(\mathcal{S} - \mathcal{T})\dot\eta \Big) \,\mathrm{d}x.$$

For later use, we compute

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{S}_\pm \dot\eta G_\pm(\eta) \mathcal{T}_\pm \dot\eta \,\mathrm{d}x &= \pm \int_{\mathbb{R}} G_\pm(\eta)^{-1} \big(b_1^\pm \dot\eta\big)' G_\pm(\eta)(b_2^\pm \dot\eta) \,\mathrm{d}x \\ &= \pm \int_{\mathbb{R}} \big(b_1^\pm \dot\eta\big)' (b_2^\pm \dot\eta) \,\mathrm{d}x \\ &= \pm \frac{1}{2} \int_{\mathbb{R}} \Big((b_1^\pm)' b_2^\pm - b_1^\pm (b_2^\pm)' \Big) \dot\eta^2 \,\mathrm{d}x.\end{aligned}$$

Finally, in view of (3.27) and the formula for $\mathscr{L}$ in (4.11), we have that $\mathscr{N} = \mathscr{N}(u_*)$ satisfies

$$\begin{aligned}\int_{\mathbb{R}} \dot\eta \mathscr{N} \dot\eta \,\mathrm{d}x &= \sum_{\pm} \rho_\pm \int_{\mathbb{R}} \Big((b_1^\pm + c)(A(\eta) G_\pm(\eta)^{-1} \mathscr{L} \dot\eta)' \pm b_2^\pm A(\eta) \mathscr{L} \dot\eta \Big) \dot\eta \,\mathrm{d}x \\ &= \sum_{\pm} \rho_\pm \int_{\mathbb{R}} \Big((b_1^\pm + c)(A(\eta) G_\pm(\eta)^{-1} \big(\mathcal{T} - \mathcal{S} - cA(\eta)^{-1}\partial_x)\dot\eta\big)' \\ &\qquad \pm b_2^\pm A(\eta)(\mathcal{T} - \mathcal{S} - cA(\eta)^{-1}\partial_x)\dot\eta \Big) \dot\eta \,\mathrm{d}x.\end{aligned}$$

Recalling that $A(\eta)$ and $G_\pm(\eta)^{-1}$ commute, continuing to simplify the right-hand side we obtain

$$\begin{aligned}\int_{\mathbb{R}} \dot\eta \mathscr{N} \dot\eta \,\mathrm{d}x &= -\sum_{\pm} \rho_\pm \int_{\mathbb{R}} G_\pm(\eta)^{-1} \big((b_1^\pm + c)\dot\eta\big)' A(\eta)(\mathcal{T} - \mathcal{S} - cA(\eta)^{-1}\partial_x)\dot\eta \,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}} \mathcal{T}\dot\eta A(\eta)(\mathcal{T} - \mathcal{S} - cA(\eta)^{-1}\partial_x)\dot\eta \,\mathrm{d}x \\ &= \int_{\mathbb{R}} (\mathcal{T} - \mathcal{S} - cA(\eta)^{-1}\partial_x)\dot\eta A(\eta)(\mathcal{T} - \mathcal{S} - cA(\eta)^{-1}\partial_x)\dot\eta \,\mathrm{d}x \\ &= \int_{\mathbb{R}} \Big(\mathrm{D}\psi_*(\eta)\dot\eta A(\eta) \mathrm{D}\psi_*(\eta)\dot\eta + 2c\dot\eta'(\mathcal{S} - \mathcal{T})\dot\eta + c^2 \dot\eta' A^{-1} \dot\eta' \Big) \,\mathrm{d}x.\end{aligned}$$

Putting the above together and using Lemma 4.1, (4.10) and Lemma 3.6 we obtain

$$\begin{aligned}D^2 V_c^{\mathrm{aug}}(\eta)[\dot\eta, \dot\eta] &= \mathrm{D}_\eta^2 E_c(u_*)[\dot\eta, \dot\eta] - \int_{\mathbb{R}} \mathrm{D}\psi_*(\eta)\dot\eta A(\eta) \mathrm{D}\psi_*(\eta)\dot\eta \,\mathrm{d}x \\ &= \int_{\mathbb{R}} \left(\frac{1}{2} \psi_* \langle \mathrm{D}^2 A(\eta)[\dot\eta, \dot\eta],\ \psi_* \rangle - g[\![\rho]\!] \dot\eta^2 + \sigma \frac{(\dot\eta')^2}{\langle \eta' \rangle^3} \right) \mathrm{d}x \\ &\quad - \int_{\mathbb{R}} \mathrm{D}\psi_*(\eta)\dot\eta A(\eta) \mathrm{D}\psi_*(\eta)\dot\eta \,\mathrm{d}x\end{aligned}$$

$$= \int_{\mathbb{R}} \left(\sigma \frac{(\dot{\eta}')^2}{\langle \eta' \rangle^3} - \Big(g[\![\rho]\!] + \sum_{\pm} \pm \rho_\pm b_1^\pm (b_2^\pm)' \Big) \dot{\eta}^2 \right.$$
$$\left. - \sum_{\pm} \rho_\pm \mathcal{S}_\pm \dot{\eta} G_\pm(\eta) \mathcal{S}_\pm \dot{\eta} \right) \mathrm{d}x,$$

which leads to the formula $Q_c(\eta)$ claimed in (4.9). □

Following [46, Theorem 3.5], we can determine the continuous spectrum of $Q_c(\eta)$ as follows.

Lemma 4.4. (Continuous spectrum) *Let $u = (\eta, \psi) \in \mathcal{O} \cap \mathbb{V}$ be given. Then the operator $Q_c(\eta)$ defined in (4.9) is self-adjoint on $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$. The continuous spectrum of $Q_c(\eta)$ is the same as the one of $Q_c(0)$, which is $[\nu_*, +\infty)$, where*

$$\nu_* := \begin{cases} -g[\![\rho]\!] \left(1 - \dfrac{\lambda_0^2}{\lambda^2} \right), & \textit{for } \beta \geq \beta_0, \\ -g[\![\rho]\!] \left[1 - \dfrac{1}{\lambda^2} \max\limits_{\xi \in \mathbb{R}} \left(\sum\limits_{\pm} \dfrac{\rho_\pm}{\rho_-} d_+ \xi \coth(d_\pm \xi) - \beta d_+^2 \xi^2 \right) \right], & \textit{for } \beta < \beta_0. \end{cases} \tag{4.12}$$

Proof. The domain and the self-adjointness of $Q_c(\eta)$ follows from the regularity of η. The continuous spectrum of $Q_c(\eta)$ coincides with that of $Q_c(0)$ because $\eta(x) \to 0$ as $|x| \to \infty$. A direct computation yields that the Fourier symbol of $Q_c(0)$ is given by

$$\mathfrak{q}_c(\xi) := -g[\![\rho]\!] \left[1 - \frac{1}{\lambda^2} \left(\sum_{\pm} \frac{\rho_\pm}{\rho_-} d_+ \xi \coth(d_\pm \xi) - \beta d_+^2 \xi^2 \right) \right],$$

which leads to the conclusion of the lemma. □

Remark 4.5. Observe that the symbol $\mathfrak{q}_c$ above recovers the dispersion relation in that $d_+\xi$ is a root of (1.3) if and only if $\mathfrak{q}_c(\xi) = 0$.

4.2. Rescaled operator. We now execute the second step in the plan outlined at the start of the section, namely using a long-wave rescaling to discern the leading-order form of the operator $Q_c(\eta)$ in the small-amplitude limit along the families of waves discussed in Sect. 3.5. Because we wish to exploit the fact that (β, λ) is close to the curve Γ_1 or Γ_2, it is more convenient to perform these calculations working with the parameterization in [47]. With that in mind, let $\{\Pi_\varepsilon\}$ be a smooth curve in the dimensional parameter space. For Region A, we assume that the corresponding $\beta > \beta_0$ is fixed and $\lambda = \lambda_0 + \varepsilon^2$, whereas for Region C, (β, λ) are given by (3.43) with δ fixed. To avoid cluttered notation, the dependence of $(\rho_\pm, d_\pm, \sigma, c)$ on ε will be suppressed when there is no risk of confusion. Recall that the corresponding curves of traveling waves are denoted $\mathscr{C}_\beta^{\mathrm{A}}$, $\mathscr{C}_{\beta,\kappa,\pm}^{\mathrm{A}}$, $\mathscr{C}_{\beta,\delta}^{\mathrm{C}}$, and $\mathscr{C}_{\beta,\delta,\kappa,\pm}^{\mathrm{C}}$.

The main character in this analysis is the scaling operator

$$S_\varepsilon f := f\left(\frac{\varepsilon \,\cdot}{d_+} \right).$$

Clearly S_ε is a bounded isomorphism on $H^k(\mathbb{R})$ for all $k \geq 0$ with $\|S_\varepsilon\|_{\mathrm{Lin}(H^k)} = O(\varepsilon^{-1})$. Note that ∂_x and S_ε satisfy the following commutation identities.

$$\partial_x S_\varepsilon = \frac{\varepsilon}{d_+} S_\varepsilon \partial_x, \qquad \partial_x S_\varepsilon^{-1} = \frac{d_+}{\varepsilon} S_\varepsilon^{-1} \partial_x,$$

In particular, this shows that $\partial_x S_\varepsilon$ and $\partial_x S_\varepsilon^{-1}$ are uniformly bounded in $\mathrm{Lin}(H^{k+1}, H^k)$ for any k.

From the existence theory in Sect. 3.5, the traveling wave profiles can be written

$$\eta_\varepsilon =: \varepsilon^m d_+ S_\varepsilon \left(\widetilde{\eta} + \widetilde{r}_\varepsilon\right), \qquad \widetilde{r}_\varepsilon = O(\varepsilon) \quad \text{in } \mathbb{W}_1 \text{ as } \varepsilon \searrow 0, \tag{4.13}$$

with

$$m := \begin{cases} 2 & \text{for } \mathscr{C}^{\mathrm{A}}_\beta \text{ and } \mathscr{C}^{\mathrm{C}}_{\beta,\delta,\kappa,\pm}, \\ 1 & \text{for } \mathscr{C}^{\mathrm{A}}_{\beta,\kappa,\pm}, \\ 4 & \text{for } \mathscr{C}^{\mathrm{C}}_{\beta,\delta}. \end{cases}$$

Note that in (4.13) we are continuing the practice of omitting superscripts and subscripts when they can be inferred from context. Thus, from (3.40) and (3.41) it follows that in Region A, $\tilde{\eta}$ is a scaled KdV or Gardner soliton, while in Region C it is given by Z_δ or $Z_{\delta,\kappa,\pm}$. From the commutation identities, we then have that $\eta'_\varepsilon = \varepsilon^{m+1} S_\varepsilon \left(\widetilde{\eta}' + \widetilde{r}'_\varepsilon\right)$.

Abusing notation somewhat, let Q_ε be the operator resulting from evaluating Q_c at the parameter values Π_ε:

$$\begin{aligned} Q_\varepsilon(\eta_\varepsilon) := -\partial_x \left(\frac{\sigma}{\langle \eta'_\varepsilon \rangle^3} \partial_x \right) - \left(g[\![\rho]\!] + \sum_\pm \pm \rho_\pm b^\pm_{1\varepsilon} (b^\pm_{2\varepsilon})' \right) \\ + \sum_\pm \rho_\pm b^\pm_{1\varepsilon} \partial_x G_\pm(\eta_\varepsilon)^{-1} \partial_x b^\pm_{1\varepsilon} \end{aligned} \tag{4.14}$$

where $b^\pm_{i\varepsilon} = b^\pm_i(\eta_\varepsilon)$ is a multiplication operator and η_ε is from one of the families $\mathscr{C}^{\mathrm{A}}_\beta$, $\mathscr{C}^{\mathrm{A}}_{\beta,\kappa,\pm}$, $\mathscr{C}^{\mathrm{C}}_{\beta,\delta}$, or $\mathscr{C}^{\mathrm{C}}_{\beta,\delta,\kappa,\pm}$. Note that again the dependence of many quantities on ε is being suppressed. Our interest is the rescaled operator:

$$\widetilde{Q}_\varepsilon(\eta_\varepsilon) := \frac{1}{\varepsilon^n} \frac{d_+}{c^2 \rho_-} S_\varepsilon^{-1} Q_\varepsilon(\eta_\varepsilon) S_\varepsilon, \tag{4.15}$$

where $n = 2$ in Region A and $n = 4$ in Region C. Conjugating by S_ε imposes a long-wave scaling that will, in the limit $\varepsilon \searrow 0$, converge to the linearized operator for the corresponding dispersive model equation. We are also non-dimensionalizing the problem in order to simplify the resulting expressions.

Lemma 4.6 (Expansion of $\widetilde{Q}_\varepsilon$). *The operator $\widetilde{Q}_\varepsilon$ defined in* (4.15) *admits the expansion*

$$\widetilde{Q}_\varepsilon(\eta_\varepsilon) = \widetilde{Q}_\varepsilon(0) + \widetilde{R}_\varepsilon,$$

where in Region A

$$\widetilde{R}_\varepsilon = \begin{cases} -3\left(\varrho - \dfrac{1}{h^2}\right)\widetilde{\eta} + O(\varepsilon^2) & \textit{for } \mathscr{C}^{\mathrm{A}}_\beta \\ -3\kappa\widetilde{\eta} - 6\left(\varrho + \dfrac{1}{h^3}\right)\widetilde{\eta}^2 + O(\varepsilon) & \textit{for } \mathscr{C}^{\mathrm{A}}_{\beta,\kappa,\pm}, \end{cases} \tag{4.16}$$

in $\mathrm{Lin}(H^{k+2}, H^k)$*, and in Region C*

$$\widetilde{R}_\varepsilon = \begin{cases} -3\left(\varrho - \dfrac{1}{h^2}\right)\widetilde{\eta} + O(\varepsilon^2) & \text{for } \mathscr{C}^{\mathrm{C}}_{\beta,\delta} \\ -3\kappa\widetilde{\eta} - 6\left(\varrho + \dfrac{1}{h^3}\right)\widetilde{\eta}^2 + (1-\varrho)\left(\partial_x(\widetilde{\eta}\partial_x) + \widetilde{\eta}''\right) + O(\varepsilon^2) & \text{for } \mathscr{C}^{\mathrm{C}}_{\beta,\delta,\kappa,\pm}, \end{cases} \tag{4.17}$$

in $\mathrm{Lin}(H^{k+2}, H^k)$.

Proof. Looking at its definition in (4.14), we see that $Q_\varepsilon(\eta_\varepsilon)$ is the sum of a second-order differential operator (call it the surface tension term), a multiplication operator (the potential term), and a first-order nonlocal operator (the nonlocal term). Rescaling the surface tension term yields

$$-\frac{1}{\varepsilon^n}\frac{d_+}{c^2\rho_-}S_\varepsilon^{-1}\partial_x\left(\frac{\sigma}{\langle\eta_\varepsilon'\rangle^3}\partial_x\right)S_\varepsilon = -\varepsilon^{2-n}\partial_x\left(\frac{\beta}{\langle\varepsilon^{m+1}(\widetilde{\eta}' + \widetilde{r}_\varepsilon')\rangle^3}\partial_x\right). \tag{4.18}$$

To understand the contribution of the potential term to $\widetilde{Q}_\varepsilon(\eta_\varepsilon)$, we first denote the non-dimensionalized and rescaled relative velocity field

$$b_{1\varepsilon}^\pm =: cS_\varepsilon\widetilde{b}_1^\pm, \qquad b_{2\varepsilon}^\pm =: cS_\varepsilon\widetilde{b}_2^\pm. \tag{4.19}$$

From the kinematic boundary condition (4.4) we then have that $\widetilde{b}_2^\pm = \varepsilon^{k+1}\widetilde{\eta}'\widetilde{b}_1^\pm$. Hence

$$-\frac{1}{\varepsilon^n}\frac{d_+}{c^2\rho_-}S_\varepsilon^{-1}\left(g[\![\rho]\!] + \sum_\pm \pm\rho_\pm b_{1\varepsilon}^\pm(b_{2\varepsilon}^\pm)'\right)S_\varepsilon = \frac{\lambda}{\varepsilon^n} - \varepsilon^{m-n+2}\sum_\pm \pm\frac{\rho_\pm}{\rho_-}\widetilde{b}_1^\pm(\widetilde{\eta}'\widetilde{b}_1^\pm)'.$$

The rescaling of the nonlocal term in $Q_\varepsilon(\eta_\varepsilon)$ will require the most effort to expand. Towards that end, we define the operator $\widetilde{\mathcal{M}}_\varepsilon^\pm(\eta_\varepsilon) \in \mathrm{Lin}(H^{k+2}, H^{k+1})$ by

$$\widetilde{\mathcal{M}}_\varepsilon^\pm(\eta_\varepsilon) := \frac{d_+}{\varepsilon^n}S_\varepsilon^{-1}\partial_x G_\pm(\eta_\varepsilon)^{-1}\partial_x S_\varepsilon. \tag{4.20}$$

In particular, this means that

$$\widetilde{Q}_\varepsilon(0) = \frac{1}{\varepsilon^n}\left(-\varepsilon^2\beta\partial_x^2 + \lambda + \sum_\pm \frac{\rho_\pm}{\rho_-}\varepsilon^n\widetilde{\mathcal{M}}_\varepsilon^\pm(0)\right). \tag{4.21}$$

Now, using the above calculations, we will analyze the difference operator

$$\begin{aligned}\widetilde{R}_\varepsilon &:= \widetilde{Q}_\varepsilon(\eta_\varepsilon) - \widetilde{Q}_\varepsilon(0) \\ &= -\beta\varepsilon^{2-n}\partial_x\left[\left(\frac{1}{\langle\varepsilon^{m+1}(\widetilde{\eta}' + \widetilde{r}_\varepsilon')\rangle^3} - 1\right)\partial_x\right] - \varepsilon^{m-n+2}\sum_\pm \pm\frac{\rho_\pm}{\rho_-}\widetilde{b}_1^\pm(\widetilde{\eta}'\widetilde{b}_1^\pm)' \\ &\quad + \sum_\pm \frac{\rho_\pm}{\rho_-}\left(\widetilde{b}_1^\pm\widetilde{\mathcal{M}}_\varepsilon^\pm(\eta_\varepsilon)\widetilde{b}_1^\pm - \widetilde{\mathcal{M}}_\varepsilon^\pm(0)\right). \end{aligned} \tag{4.22}$$

In view of (4.13) and (4.18), the first term on the right-hand side above is higher order:

$$-\beta\varepsilon^{2-n}\partial_x\left[\left(\frac{1}{\langle\varepsilon^{m+1}(\widetilde{\eta}' + \widetilde{r}_\varepsilon')\rangle^3} - 1\right)\partial_x\right] = O(\varepsilon^{2m-n+4}) \qquad \text{in } \mathrm{Lin}(H^{k+2}, H^k). \tag{4.23}$$

Consider the remaining two terms in (4.22). Notice that for any $f \in H^{k+2}$ we have

$$\mathcal{F}\left(\widetilde{\mathcal{M}}_\varepsilon^\pm(0) f\right)(\xi) = \frac{d_+}{\varepsilon^n}\frac{\varepsilon}{d_+}\mathcal{F}\left(\partial_x G_\pm(0)^{-1}\partial_x S_\varepsilon f\right)\left(\frac{\varepsilon}{d_+}\xi\right) = \frac{d_+}{\varepsilon^n}\mathfrak{m}_\pm\left(\frac{\varepsilon}{d_+}\xi\right)\widehat{f}(\xi)$$

where $\mathfrak{m}_\pm(\xi) := -\xi \coth(d_\pm \xi)$ is the symbol for $\partial_x G_\pm(0)^{-1}\partial_x$. Thus $\widetilde{\mathcal{M}}_\varepsilon^\pm(0)$ is indeed a Fourier multiplier and its symbol is given by

$$\widetilde{\mathfrak{m}}_\varepsilon^\pm(\xi) := -\frac{1}{\varepsilon^n}\frac{\varepsilon\xi}{\tanh(d_\pm \varepsilon\xi/d_+)}. \tag{4.24}$$

As an immediate consequence, it follows that

$$\left\| \varepsilon^n \widetilde{\mathcal{M}}_\varepsilon^\pm(0) + \frac{d_+}{d_\pm}\right\|_{\mathrm{Lin}(H^{k+2},H^k)} \leq \left\| \frac{1}{\langle \cdot \rangle^2}\left(\varepsilon^n \widetilde{\mathfrak{m}}_\varepsilon^\pm + \frac{d_+}{d_\pm}\right)\right\|_{L^\infty} \lesssim \varepsilon^2. \tag{4.25}$$

In other words, $\varepsilon^n \widetilde{\mathcal{M}}_\varepsilon^\pm(0)$ is to leading order the multiplication operator $-d_+/d_\pm$ in $\mathrm{Lin}(H^{k+2}, H^k)$.

To estimate the scaled relative velocity, we observe that by (4.2)–(4.3) and (4.19), it holds that

$$\widetilde{b}_1^\pm = \frac{1}{c} S_\varepsilon^{-1}\left(\partial_x \Phi_{\varepsilon\pm}|_{\mathscr{S}} - c\right),$$

where, as usual, $\Phi_{\varepsilon\pm}$ denotes the velocity potential. But expanding the Dirichlet–Neumann operator, we find that

$$\begin{aligned}
\varphi_\pm' &= \pm c\partial_x\left(G_\pm(\eta_\varepsilon)^{-1}\partial_x \eta_\varepsilon\right) \\
&= \pm c\partial_x\left[G_\pm(0)^{-1}\eta_\varepsilon' + \left\langle \mathrm{D}G_\pm(0)^{-1}\eta_\varepsilon, \eta_\varepsilon'\right\rangle\right] + O(\varepsilon^{3m}) && \text{in } H^k, \\
(\partial_x \Phi_{\varepsilon\pm})|_{\mathscr{S}} &= \frac{1}{1+(\eta_\varepsilon')^2}\left(\varphi_\pm' \pm \eta_\varepsilon' G_\pm(\eta_\varepsilon)\varphi_\pm\right) = \frac{1}{1+(\eta_\varepsilon')^2}\left(\varphi_\pm' \pm (\eta_\varepsilon')^2\right) \\
&= \pm c\partial_x\left[G_\pm(0)^{-1}\eta_\varepsilon' + \left\langle \mathrm{D}G_\pm(0)^{-1}\eta_\varepsilon, \eta_\varepsilon'\right\rangle\right] + O(\varepsilon^{3m}) && \text{in } H^k.
\end{aligned}$$

We can compute $\mathrm{D}G_\pm(0)^{-1}$ as

$$\left\langle \mathrm{D}G_\pm(0)^{-1}\eta_\varepsilon, f\right\rangle = -G_\pm(0)^{-1}\left\langle \mathrm{D}G_\pm(0)\eta_\varepsilon, G_\pm(0)^{-1} f\right\rangle,$$

and from Lemma 3.3, we see that

$$\begin{aligned}
\left\langle \mathrm{D}G_\pm(0)\eta_\varepsilon, G_\pm(0)^{-1}\partial_x S_\varepsilon f\right\rangle &= \pm\partial_x S_\varepsilon\left[\left(S_\varepsilon^{-1}\partial_x G_\pm(0)^{-1}\partial_x S_\varepsilon f\right) d_+\varepsilon^m \widetilde{\eta}\right] \\
&\quad \pm G_\pm(0) S_\varepsilon\left(\frac{\varepsilon}{d_+}(\partial_x f) d_+ \varepsilon^m \widetilde{\eta}\right) \\
&= \pm\partial_x S_\varepsilon \varepsilon^{m+n}\left(\widetilde{\mathcal{M}}_\varepsilon^\pm(0) f\right)\widetilde{\eta} \pm \varepsilon^{m+1} G_\pm(0) S_\varepsilon(\widetilde{\eta}\partial_x f).
\end{aligned} \tag{4.26}$$

Therefore

$$\begin{aligned}
\widetilde{b}_1^{\pm} &= S_\varepsilon^{-1}\big[\pm\partial_x\left(G_\pm(0)^{-1}\eta_\varepsilon' + \langle \mathrm{D}G_\pm(0)^{-1}\eta_\varepsilon, \eta_\varepsilon'\rangle\right) - 1\big] + O(\varepsilon^{3m}) \\
&= \pm\varepsilon^m d_+\left[S_\varepsilon^{-1}\partial_x G_\pm(0)^{-1}\partial_x(S_\varepsilon\widetilde{\eta}) - S_\varepsilon^{-1}\partial_x G_\pm(0)^{-1}\left\langle \mathrm{D}G_\pm(0)\eta_\varepsilon, G_\pm(0)^{-1}\partial_x S_\varepsilon\widetilde{\eta}\right\rangle\right] \\
&\quad - 1 + O(\varepsilon^{3m}) \\
&= \pm\varepsilon^{m+n}\widetilde{\mathcal{M}}_\varepsilon^{\pm}(0)\widetilde{\eta} - 1 + O(\varepsilon^{3m}) \\
&\quad \mp \varepsilon^m d_+ S_\varepsilon^{-1}\partial_x G_\pm(0)^{-1}\left[\pm\varepsilon^{m+n}\partial_x S_\varepsilon\left(\widetilde{\mathcal{M}}_\varepsilon^{\pm}(0)\widetilde{\eta}\right)\widetilde{\eta} \pm \varepsilon^{m+1} G_\pm(0) S_\varepsilon(\widetilde{\eta}\widetilde{\eta}')\right] \\
&= \pm\varepsilon^{m+n}\widetilde{\mathcal{M}}_\varepsilon^{\pm}(0)\widetilde{\eta} - 1 - \varepsilon^{2m+2n}\widetilde{\mathcal{M}}_\varepsilon^{\pm}(0)\left(\left(\widetilde{\mathcal{M}}_\varepsilon^{\pm}(0)\widetilde{\eta}\right)\widetilde{\eta}\right) - \varepsilon^{2m+2}\partial_x(\widetilde{\eta}\widetilde{\eta}') + O(\varepsilon^{3m}) \\
&= -1 \mp \varepsilon^m \frac{d_+}{d_\pm}\widetilde{\eta} - \varepsilon^{2m}\frac{d_+^2}{d_\pm^2}\widetilde{\eta}^2 + O(\varepsilon^{m+2}) \qquad \text{in } H^k.
\end{aligned} \tag{4.27}$$

Hence for the second term on the right-hand side of (4.22) we have

$$\begin{aligned}
&-\varepsilon^{m-n+2}\sum_\pm \pm\frac{\rho_\pm}{\rho_-}\widetilde{b}_1^{\pm}(\widetilde{\eta}'\widetilde{b}_1^{\pm})' \\
&\quad = \varepsilon^{m-n+2}(1-\varrho)\widetilde{\eta}'' + \varepsilon^{2m-n+2}\left(\varrho + \frac{1}{h}\right)\left[2\widetilde{\eta}\widetilde{\eta}'' + (\widetilde{\eta}')^2\right] \\
&\quad\quad + O(\varepsilon^{3m-n+2}) \qquad \text{in } \mathrm{Lin}(H^{k+2}, H^k).
\end{aligned} \tag{4.28}$$

Using the expansion (4.27) for $\tilde{b}_1^{\pm}$ also furnishes the estimate

$$\begin{aligned}
\widetilde{b}_1^{\pm}\widetilde{\mathcal{M}}_\varepsilon^{\pm}(\eta_\varepsilon)\widetilde{b}_1^{\pm} &= \widetilde{\mathcal{M}}_\varepsilon^{\pm}(\eta_\varepsilon) \pm \varepsilon^m\frac{d_+}{d_\pm}\left[\widetilde{\eta}\widetilde{\mathcal{M}}_\varepsilon^{\pm}(\eta_\varepsilon) + \widetilde{\mathcal{M}}_\varepsilon^{\pm}(\eta_\varepsilon)\widetilde{\eta}\right] \\
&\quad + \varepsilon^{2m}\frac{d_+^2}{d_\pm^2}\left[\widetilde{\eta}\widetilde{\mathcal{M}}_\varepsilon^{\pm}(\eta_\varepsilon)\widetilde{\eta} + \widetilde{\eta}^2\widetilde{\mathcal{M}}_\varepsilon^{\pm}(\eta_\varepsilon) + \widetilde{\mathcal{M}}_\varepsilon^{\pm}(\eta_\varepsilon)\widetilde{\eta}^2\right] + O(\varepsilon^{3m-n})
\end{aligned} \tag{4.29}$$

in $\mathrm{Lin}(H^{k+2}, H^k)$. On the other hand, from the definition of $\widetilde{\mathcal{M}}_\varepsilon^{\pm}$ in (4.20) it follows that for all $f \in H^{k+2}$ with $\|f\|_{H^{k+2}} = 1$,

$$\begin{aligned}
\left(\widetilde{\mathcal{M}}_\varepsilon^{\pm}(\eta_\varepsilon) - \widetilde{\mathcal{M}}_\varepsilon^{\pm}(0)\right) f &= \frac{d_+}{\varepsilon^n} S_\varepsilon^{-1}\partial_x\left(G_\pm(\eta_\varepsilon)^{-1} - G_\pm(0)^{-1}\right)\partial_x S_\varepsilon f \\
&= \frac{d_+}{\varepsilon^n} S_\varepsilon^{-1}\partial_x\left\langle \mathrm{D}G_\pm(0)^{-1}\eta_\varepsilon, \partial_x S_\varepsilon f\right\rangle \\
&\quad + \frac{d_+}{2\varepsilon^n} S_\varepsilon^{-1}\partial_x\left\langle \mathrm{D}^2 G_\pm(0)^{-1}[\eta_\varepsilon, \eta_\varepsilon], \partial_x S_\varepsilon f\right\rangle \\
&\quad + O(\varepsilon^{3m-n}) \quad \text{in } H^k.
\end{aligned} \tag{4.30}$$

Explicit calculation yields

$$\begin{aligned}
&\left\langle \mathrm{D}^2 G_\pm(0)^{-1}[\eta_\varepsilon, \eta_\varepsilon], f\right\rangle \\
&\quad = -G_\pm(0)^{-1}\left\langle \mathrm{D}^2 G_\pm(0)[\eta_\varepsilon, \eta_\varepsilon], G_\pm(0)^{-1} f\right\rangle \\
&\quad\quad + 2G_\pm(0)^{-1}\left\langle \mathrm{D}G_\pm(0)\eta_\varepsilon, G_\pm(0)^{-1}\left\langle \mathrm{D}G_\pm(0)\eta_\varepsilon, G_\pm(0)^{-1} f\right\rangle\right\rangle.
\end{aligned}$$

From (4.26) we have

$$\begin{aligned}
&\langle \mathrm{D}G_\pm(0)\eta_\varepsilon,\ \ G_\pm(0)^{-1}\left\langle \mathrm{D}G_\pm(0)\eta_\varepsilon, G_\pm(0)^{-1}\partial_x S_\varepsilon f\right\rangle\Big\rangle \\
&\qquad = \partial_x S_\varepsilon\left[\left(S_\varepsilon^{-1}\partial_x G_\pm(0)^{-1}\partial_x S_\varepsilon \varepsilon^{m+n}\left(\widetilde{\mathcal{M}}_\varepsilon^\pm(0)f\right)\widetilde{\eta}\right)d_+\varepsilon^m\widetilde{\eta}\right] + O(\varepsilon^{2m+1}) \\
&\qquad = \varepsilon^{2(m+n)}\partial_x S_\varepsilon \widetilde{\mathcal{M}}_\varepsilon^\pm(0)\left(\left(\widetilde{\mathcal{M}}_\varepsilon^\pm(0)f\right)\widetilde{\eta}\right)\widetilde{\eta} + O(\varepsilon^{2m+1}) \qquad \text{in } H^k.
\end{aligned}$$

Likewise, Lemma 3.5 allows us to estimate

$$\left\langle \mathrm{D}^2 G_\pm(0)[\eta_\varepsilon, \eta_\varepsilon],\ G_\pm(0)^{-1} f\right\rangle = O(\varepsilon^{2m+1}) \qquad \text{in } H^k.$$

Substituting the above into (4.30) yields

$$\begin{aligned}
\left(\widetilde{\mathcal{M}}_\varepsilon^\pm(\eta_\varepsilon) - \widetilde{\mathcal{M}}_\varepsilon^\pm(0)\right) f &= \mp \frac{d_+}{\varepsilon^n} S_\varepsilon^{-1}\partial_x G_\pm(0)^{-1}\partial_x S_\varepsilon\left(\varepsilon^{m+n}\widetilde{\mathcal{M}}_\varepsilon^\pm(0)f\right)\widetilde{\eta} \\
&\quad + \frac{d_+}{\varepsilon^n} S_\varepsilon^{-1}\partial_x G_\pm(0)^{-1}\partial_x S_\varepsilon\left(\varepsilon^{2(m+n)}\widetilde{\mathcal{M}}_\varepsilon^\pm(0)\left(\widetilde{\mathcal{M}}_\varepsilon^\pm(0)f\right)\widetilde{\eta}\right)\widetilde{\eta} \\
&\quad \mp \frac{d_+}{\varepsilon^n} S_\varepsilon^{-1}\partial_x S_\varepsilon\left(\varepsilon^{m+1}\widetilde{\eta}\partial_x f\right) + O(\varepsilon^{2m-n+1}) \\
&= \mp\,\varepsilon^{m+n}\widetilde{\mathcal{M}}_\varepsilon^\pm(0)\left(\left(\widetilde{\mathcal{M}}_\varepsilon^\pm(0)f\right)\widetilde{\eta}\right) \mp \varepsilon^{m-n+2}\partial_x\left(\widetilde{\eta}\partial_x f\right) \\
&\quad + \varepsilon^{2(m+n)}\widetilde{\mathcal{M}}_\varepsilon^\pm(0)\left(\widetilde{\mathcal{M}}_\varepsilon^\pm(0)\left(\widetilde{\mathcal{M}}_\varepsilon^\pm(0)f\right)\widetilde{\eta}\right)\widetilde{\eta} + O(\varepsilon^{2m-n+1}) \\
&= \mp\,\varepsilon^{m-n}\frac{d_+^2}{d_\pm^2}\widetilde{\eta}f \mp \varepsilon^{m-n+2}\partial_x\left(\widetilde{\eta}\partial_x f\right) - \varepsilon^{2m-n}\frac{d_+^3}{d_\pm^3}\widetilde{\eta}^2 f \\
&\quad + O(\varepsilon^{2m-n+1}),
\end{aligned} \tag{4.31}$$

in H^k. Using this, the previous estimate (4.29) becomes

$$\begin{aligned}
\widetilde{b}_1^\pm\widetilde{\mathcal{M}}_\varepsilon^\pm(\eta_\varepsilon)\widetilde{b}_1^\pm f = {}& \widetilde{\mathcal{M}}_\varepsilon^\pm(0)f \pm \varepsilon^m\frac{d_+}{d_\pm}\widetilde{\eta}\widetilde{\mathcal{M}}_\varepsilon^\pm(0)f \pm \varepsilon^m\frac{d_+}{d_\pm}\widetilde{\mathcal{M}}_\varepsilon^\pm(0)\widetilde{\eta}f \\
& + \varepsilon^{2m}\frac{d_+^2}{d_\pm^2}\left[\widetilde{\eta}\widetilde{\mathcal{M}}_\varepsilon^\pm(0)\widetilde{\eta}f + \widetilde{\eta}^2\widetilde{\mathcal{M}}_\varepsilon^\pm(0)f + \widetilde{\mathcal{M}}_\varepsilon^\pm(0)\widetilde{\eta}^2 f\right] \\
& \mp \varepsilon^{m-n}\frac{d_+^2}{d_\pm^2}\widetilde{\eta}f \mp \varepsilon^{m-n+2}\partial_x\left(\widetilde{\eta}\partial_x f\right) - 3\varepsilon^{2m-n}\frac{d_+^3}{d_\pm^3}\widetilde{\eta}^2 f \\
& - \varepsilon^{2m-n+2}\frac{d_+}{d_\pm}\left[\widetilde{\eta}\partial_x(\widetilde{\eta}\partial_x f) + \partial_x(\widetilde{\eta}\partial_x(\widetilde{\eta}f))\right] + O(\varepsilon^{2m-n+1}),
\end{aligned}$$

in H^k. We can simplify further by applying (4.25), which results in

$$\begin{aligned}
\widetilde{b}_1^\pm\widetilde{\mathcal{M}}_\varepsilon^\pm(\eta_\varepsilon)\widetilde{b}_1^\pm f = {}& \widetilde{\mathcal{M}}_\varepsilon^\pm(0)f \mp 3\varepsilon^{m-n}\frac{d_+^2}{d_\pm^2}\widetilde{\eta}f - 6\varepsilon^{2m-n}\frac{d_+^3}{d_\pm^3}\widetilde{\eta}^2 f \mp \varepsilon^{m-n+2}\partial_x\left(\widetilde{\eta}\partial_x f\right) \\
& + O(\varepsilon^{2m-n+1}) \qquad \text{in } H^k.
\end{aligned}$$

Therefore in computing the third term on the right-hand side of (4.22) we find

$$\begin{aligned}
&\sum_{\pm} \frac{\rho_\pm}{\rho_-} \left(\widetilde{b}_1^{\pm} \widetilde{\mathcal{M}}_\varepsilon^{\pm}(\eta_\varepsilon) \widetilde{b}_1^{\pm} - \widetilde{\mathcal{M}}_\varepsilon^{\pm}(0) \right) f \\
&= 3\varepsilon^{m-n} \sum_{\pm} \frac{\mp\rho_\pm}{\rho_-} \frac{d_+^2}{d_\pm^2} \widetilde{\eta} f - 6\varepsilon^{2m-n} \sum_{\pm} \frac{\rho_\pm}{\rho_-} \frac{d_+^3}{d_\pm^3} \widetilde{\eta}^2 f \\
&\quad + \varepsilon^{m-n+2} \sum_{\pm} \frac{\mp\rho_\pm}{\rho_-} \partial_x \left(\widetilde{\eta} \partial_x f \right) + O(\varepsilon^{2m-n+1}) \\
&= -3\varepsilon^{m-n} \left(\varrho - \frac{1}{h^2} \right) \widetilde{\eta} f - 6\varepsilon^{2m-n} \left(\varrho + \frac{1}{h^3} \right) \widetilde{\eta}^2 f \\
&\quad + \varepsilon^{m-n+2} (1-\varrho) \partial_x \left(\widetilde{\eta} \partial_x f \right) + O(\varepsilon^{2m-n+1}),
\end{aligned}$$

in H^k. Taken together with (4.23) and (4.28), this yields the claimed expansion for $\widetilde{R}_\varepsilon$. □

Let us now look more closely at the leading-order part of $\widetilde{Q}_\varepsilon(\eta_\varepsilon)$, which by the above lemma is the Fourier multiplier $\widetilde{Q}_\varepsilon(0)$. Analyzing its symbol will allow us to infer that it has a point-wise limit as $\varepsilon \searrow 0$. Near the critical Bond number, however, there is a degeneracy that causes the limiting operator to be fourth order. Combining this with the previous result, we obtain the following.

Lemma 4.7 (Limiting rescaled operator). *Consider the rescaled operator $\widetilde{Q}_\varepsilon(\eta_\varepsilon)$ given by* (4.15).

(a) *Suppose that $\beta > \beta_0$ and $\lambda = \lambda_0 + \varepsilon^2$ lies in Region A. Then for any $k > 1/2$ and $\zeta \in H^{k+2}$,*

$$\| \widetilde{Q}_\varepsilon(\eta_\varepsilon)\zeta - \widetilde{Q}_0 \zeta \|_{H^k} \longrightarrow 0 \qquad \text{as } \varepsilon \searrow 0,$$

where the operator $\tilde{Q}_0 \in \mathrm{Lin}(H^{k+2}, H^k)$ is given by

$$\widetilde{Q}_0 = \begin{cases} -(\beta-\beta_0)\partial_x^2 + 1 - 3\left(\varrho - \dfrac{1}{h^2}\right)\widetilde{\eta} & \text{for } \mathscr{C}_\beta^{\mathrm{A}} \\[2ex] -(\beta-\beta_0)\partial_x^2 + 1 - 3\kappa\widetilde{\eta} - 6\left(\varrho + \dfrac{1}{h^3}\right)\widetilde{\eta}^2 & \text{for } \mathscr{C}_{\beta;\kappa,\pm}^{\mathrm{A}}. \end{cases}$$

(b) *Suppose that (β, λ) lie in Region C and are given by* (3.43) *for a fixed $\delta < 0$. Then for any $k > 1/2$ and $\zeta \in H^{k+4}$,*

$$\| \widetilde{Q}_\varepsilon(\eta_\varepsilon)\zeta - \widetilde{Q}_0 \zeta \|_{H^k} \longrightarrow 0 \qquad \text{as } \varepsilon \searrow 0,$$

where the operator $\widetilde{Q}_0 \in \mathrm{Lin}(H^{k+4}, H^k)$ is given by

$$\widetilde{Q}_0 = \begin{cases} \gamma\partial_x^4 - 2(1+\delta)\gamma\partial_x^2 + \gamma - 3\left(\varrho - \dfrac{1}{h^2}\right)\widetilde{\eta} & \text{for } \mathscr{C}_{\beta,\delta}^{\mathrm{C}} \\[2ex] \gamma\partial_x^4 - 2(1+\delta)\gamma\partial_x^2 + \gamma - 3\kappa\widetilde{\eta} - 6\left(\varrho + \dfrac{1}{h^3}\right)\widetilde{\eta}^2 + (1-\varrho)\left(\partial_x(\widetilde{\eta}\partial_x) + \widetilde{\eta}''\right) & \text{for } \mathscr{C}_{\beta;\kappa,\delta,\pm}^{\mathrm{C}}. \end{cases}$$

Proof. Fix $k > 1/2$. Recall that $\widetilde{Q}_\varepsilon(\eta_\varepsilon) = \widetilde{Q}_\varepsilon(0) + \widetilde{R}_\varepsilon$, where $\widetilde{Q}_\varepsilon(0)$ is given in (4.21). We have already seen in Lemma 4.6 that $\widetilde{R}_\varepsilon$ has a uniform limit in $\mathrm{Lin}(H^{k+2}, H^k)$ as $\varepsilon \searrow 0$. From (4.24), it is clear that $\widetilde{Q}_\varepsilon(0)$ is a Fourier multiplier: for all $f \in H^{k+2}$,

$$\mathcal{F}\left(\widetilde{Q}_\varepsilon(0) f\right)(\xi) = \frac{1}{\varepsilon^n}\left(\varepsilon^2\beta\xi^2 + \lambda - \sum_\pm \frac{\rho_\pm}{\rho_-}\varepsilon\xi \coth\left(\frac{d_\pm}{d_+}\varepsilon\xi\right)\right)\widehat{f}(\xi) =: \widetilde{\mathfrak{q}}_\varepsilon(\xi)\widehat{f}(\xi).$$

Consider the point-wise limit of the symbol $\widetilde{\mathfrak{q}}_\varepsilon$ as $\varepsilon \searrow 0$. Here it is important to keep in mind that the dimensional parameters are moving along the curve $\{\Pi_\varepsilon\}$ and $\lambda \searrow \lambda_0$ in this limit. Therefore, we write

$$\begin{aligned}\widetilde{\mathfrak{q}}_\varepsilon(\xi) &= \varepsilon^{2-n}(\beta - \beta_0)\xi^2 + \frac{\lambda - \lambda_0}{\varepsilon^n} + \frac{1}{\varepsilon^n}\left(\beta_0(\varepsilon\xi)^2 + \lambda_0 - \sum_\pm \frac{\rho_\pm}{\rho_-}\varepsilon\xi \coth\left(\frac{d_\pm}{d_+}\varepsilon\xi\right)\right)\\ &=: \varepsilon^{2-n}(\beta - \beta_0)\xi^2 + \frac{\lambda - \lambda_0}{\varepsilon^n} + \frac{\mathfrak{r}(\varepsilon\xi)}{\varepsilon^n}.\end{aligned}$$

Taylor expanding $\mathfrak{r}$ near $\widetilde{\xi} := \varepsilon\xi = 0$ yields that

$$\mathfrak{r}(\widetilde{\xi}) = \beta_0\widetilde{\xi}^2 + \lambda_0 - \sum_\pm \frac{\rho_\pm}{\rho_-}\widetilde{\xi}\coth\left(\frac{d_\pm}{d_+}\widetilde{\xi}\right) = \gamma\widetilde{\xi}^4 + O(\widetilde{\xi}^6) \quad \text{as } \widetilde{\xi} \to 0. \tag{4.32}$$

For Region A, we have $n = 2$ and $\lambda = \lambda_0 + \varepsilon^2$, and hence for each fixed $\xi \in \mathbb{R}$,

$$\widetilde{\mathfrak{q}}_\varepsilon(\xi) \longrightarrow (\beta - \beta_0)\xi^2 + 1 \qquad \text{as } \varepsilon \searrow 0.$$

On the other hand, in Region C we have $n = 4$ with (β, λ) given by (3.43). Again, fixing ξ we then have that the limiting symbol is

$$\widetilde{\mathfrak{q}}_\varepsilon(\xi) \longrightarrow \gamma\xi^4 + 2(1+\delta)\gamma\xi^2 + \gamma \qquad \text{as } \varepsilon \searrow 0.$$

Combining these expressions for the limiting symbol with the asymptotics of $\widetilde{R}_\varepsilon$ from (4.16) and (4.17), the formulas for $\widetilde{Q}_0$ in (a) and (b) now follow. □

4.3. Spectrum of the linearized augmented potential. Using the limiting behavior derived above, we will now characterize the spectrum of the $Q_\varepsilon(\eta_\varepsilon)$. It is worth reiterating that an essential challenge in this analysis is that the operator converges point-wise to $Q_0(0)$ whose essential spectrum is $[0, \infty)$. It is for this reason that we introduced the rescaled operator $\widetilde{Q}_\varepsilon(\eta_\varepsilon)$, since by Lemma 4.7 converges (again only point-wise) to $\widetilde{Q}_0$, which has a gap between the positive essential spectrum and 0.

Spectral analysis in Region A We start by deriving the spectral properties of $Q_\varepsilon(\eta_\varepsilon)$ for the strong surface tension waves with parameters (β, λ) in Region A.

Lemma 4.8. *In the setting of Lemma 4.7 (a), the limiting rescaled operator $\widetilde{Q}_0$ satisfies*

$$\operatorname{ess\,spec}\widetilde{Q}_0 = [1, \infty), \qquad \operatorname{spec}\widetilde{Q}_0 = \{-\widetilde{\nu}^2,\ 0\} \cup \widetilde{\Lambda} \tag{4.33}$$

where the first two eigenvalues $-\widetilde{\nu}^2 < 0$ and 0 are both simple with corresponding eigenfuctions $\widetilde{\phi}_1$ and $\widetilde{\phi}_2 = \widetilde{\eta}'$, respectively; and there exists $\nu_* > 0$ such that $\widetilde{\Lambda} \subset [\nu_*, \infty)$.

Proof. This is a classical result on linear Schrödinger operators, and can be found, for example, in [6]. The fact that $-\widetilde{\nu}^2$ and 0 are all simple follows from the theory of ODEs: the Wronskian of two L^2 solutions to the eigenvalue problem $\widetilde{Q}_0 f = \widetilde{\nu} f$ is necessarily 0. □

Using a similar argument as [46, Theorem 4.3], we then have the following result.

Theorem 4.9 (Spectrum in Region A). *Let the assumptions of Lemma 4.7 (a) hold. For each $a \in (0, \nu_*)$ there exists some $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the operator $Q_\varepsilon(\eta_\varepsilon)$ satisfies*

$$\operatorname{ess\,spec} Q_\varepsilon(\eta_\varepsilon) \subset [\varepsilon^2 c^2 \rho_-/d_+, \infty), \qquad \operatorname{spec} Q_\varepsilon(\eta_\varepsilon) = \{-\nu^2,\ 0\} \cup \Lambda,$$

where $\Lambda \subset [a\varepsilon^2 c^2 \rho_-/d_+, \infty)$, and

$$\nu^2 = \frac{\varepsilon^2 c^2 \rho_-}{d_+} \widetilde{\nu}^2 + o(\varepsilon^2) \qquad as\ \ \varepsilon \searrow 0.$$

The first two eigenvalues $\nu_1 := -\nu^2 < 0$ and $\nu_2 := 0$ are simple with the associated eigenfunctions taking the form $\phi_i = S_\varepsilon \widetilde{\phi}_i + o(1)$ in H^k as $\varepsilon \searrow 0$.

Proof. From Lemma 4.6 we see that it suffices to prove that the operator

$$\mathcal{Q}_\varepsilon := \widetilde{Q}_\varepsilon(0) + \widetilde{R}_0,$$

with $\widetilde{R}_0$ defined by (4.16) with $\varepsilon = 0$, has exactly two simple eigenvalues lying in $(-\infty, a)$ that converge to $\widetilde{\nu}_i$ respectively for $i = 1, 2$. It is clear that $\mathcal{Q}_\varepsilon$ is self-adjoint. Note that $\mathcal{Q}_\varepsilon$ may not have 0 as an exact eigenvalue, but this does hold for $\widetilde{Q}_\varepsilon(\eta_\varepsilon)$.

Firstly, from Lemma 4.7 (a) it follows that

$$\| (\mathcal{Q}_\varepsilon - \widetilde{\nu}_i) \widetilde{\phi}_i \|_{H^k} \le C\varepsilon^2 \|\widetilde{\phi}_i\|_{H^k}. \tag{4.34}$$

Therefore $\mathcal{Q}_\varepsilon$ admits spectral values close to $\widetilde{\nu}_i$ with $O(\varepsilon^2)$ distance.

Now we consider a sequence $\{(\nu_{\varepsilon_j}, \phi_{\varepsilon_j})\}$ of eigenpairs of $\mathcal{Q}_{\varepsilon_j}$ with $\nu_{\varepsilon_j} \in (-\infty, a)$ and $\varepsilon_j \searrow 0$ as $j \to \infty$. Our goal is to prove the compactness of the eigenpair sequence and confirm that the limit must be an eigenpair of $\widetilde{Q}_0$.

We normalize so that $\|\phi_{\varepsilon_j}\|_{H^k} = 1$. Note that $\|\widetilde{\eta}\|_{W^{N,\infty}} \le C_N$ for any $N \ge 0$. Moreover from the proof of Lemma 4.6 we see that $\widetilde{Q}_\varepsilon(0) - 1$ is positive semi-definite. From this we know that the spectrum of $\mathcal{Q}_\varepsilon$ is bounded below: $\operatorname{spec} \mathcal{Q}_\varepsilon \subset [1 - C_k, \infty)$. Since $\widetilde{\eta}$ decays exponentially, we have that $\operatorname{ess\,spec} \mathcal{Q}_\varepsilon = [1, \infty)$. Thus $\operatorname{spec} \mathcal{Q}_\varepsilon \cap [1 - C_k, a]$ consists of discrete eigenvalues of finite multiplicity. By definition,

$$\left(\widetilde{Q}_{\varepsilon_j}(0) - \nu_{\varepsilon_j}\right) \phi_{\varepsilon_j} = -\widetilde{R}_0 \phi_{\varepsilon_j}. \tag{4.35}$$

Since $\nu_{\varepsilon_j} \in [1 - C_k, a]$, from the proof of Lemma 4.6, the Fourier symbol of operator on the left-hand side is

$$\widetilde{\mathfrak{q}}_{\varepsilon_j}(\xi) - \nu_{\varepsilon_j} \ge \widetilde{\mathfrak{q}}_0(\xi) - a = (\beta - \beta_0)\xi^2 + 1 - a \ge \delta_*(1 + \xi^2)$$

for some $\delta_* > 0$ independent of ε_j. This uniform ellipticity property allows us via bootstrapping to obtain the bound $\|\phi_{\varepsilon_j}\|_{H^{k+4}} \le C_*$ from some universal constant $C_* > 0$.

To obtain compactness of the sequence $\{\phi_{\varepsilon_j}\}$ in H^{k+2}, we proceed to prove a uniform decay estimate. Given an exponential weight $w := \cosh(\alpha\,\cdot)$ for some $\alpha > 0$, we see that for any Schwartz function f,

$$\mathcal{F}\left[w\left(\widetilde{Q}_{\varepsilon_j}(0) - \nu_{\varepsilon_j}\right)^{-1} f\right](\xi) = \frac{1}{2}\left[\frac{\widehat{f}(\xi + i\alpha)}{\widetilde{\mathfrak{q}}_{\varepsilon_j}(\xi + i\alpha) - \nu_{\varepsilon_j}} + \frac{\widehat{f}(\xi - i\alpha)}{\widetilde{\mathfrak{q}}_{\varepsilon_j}(\xi - i\alpha) - \nu_{\varepsilon_j}}\right].$$

Taking $\alpha^2 < (1-a)/(\beta - \beta_0)$ it follows that

$$\sup_{|\operatorname{Im}\xi| \le \alpha} \left|\frac{1}{\widetilde{\mathfrak{q}}_{\varepsilon_j}(\xi \pm i\alpha) - \nu_{\varepsilon_j}}\right| \le C^*,$$

for some $C^* > 0$. Therefore

$$\left\|\left(\widetilde{Q}_{\varepsilon_j}(0) - \nu_{\varepsilon_j}\right)^{-1}\right\|_{\mathrm{Lin}(L^2_w)} \le C^*,$$

where $L^2_w := \{f \in L^2 : wf \in L^2\}$ is the weighted L^2 space corresponding to w. Hence from (4.35),

$$\|\phi_{\varepsilon_j}\|_{L^2_w} \le C^* \|\widetilde{R}_0 \phi_{\varepsilon_j}\|_{L^2_w} \le C^* \|w\widetilde{R}_0\|_{L^\infty} \|\phi_{\varepsilon_j}\|_{L^2} \le C^* \|w\widetilde{R}_0\|_{L^\infty} \lesssim 1.$$

Thus $\{\phi_{\varepsilon_j}\}$ is bounded in $H^{k+4} \cap L^2_w$, which is compactly embedded in H^{k+2}. Hence up to a subsequence, as $j \to \infty$, $\nu_{\varepsilon_j} \to \nu_* \in (-\infty, a]$ and $\phi_{\varepsilon_j} \to \phi_*$ in H^{k+2} with $\|\phi_*\|_{H^k} = 1$. Moreover, $\widetilde{Q}_0 \phi_* = \nu_* \phi_*$, which indicates that $\phi_* = \widetilde{\phi}_i$ for some $i = 1, 2$.

Finally we check the convergence of the corresponding spectral projections. Set $\mathcal{P}_\varepsilon$ to be the spectral projection for $\mathcal{Q}_\varepsilon$ associated with the interval $[1 - C_k, a]$. From (4.34), there exists $\varepsilon_0 > 0$ such that $\dim \operatorname{Rng} \mathcal{P}_\varepsilon \ge 2$ for $\varepsilon \in (0, \varepsilon_0)$. Also $\mathcal{P}_\varepsilon = \sum_{i=1}^{N_\varepsilon} \langle\,\cdot\,, \phi_{i,\varepsilon}\rangle \phi_{i,\varepsilon}$ for some finite integer N_ε and orthonormal eigenbasis $\{\phi_{i,\varepsilon}\}_{i=1}^{N_\varepsilon}$. Were there a sequence $\varepsilon_j \searrow 0$ such that $N_{\varepsilon_j} \ge 3$, then it would contradict the above convergence result. Therefore, for all ε sufficiently small, it must be that $N_\varepsilon = 2$. We can then conclude that $\phi_{i,\varepsilon} \to \widetilde{\phi}_i$ in H^k. □

Spectral analysis in Region C The same argument can also be applied to the near critical surface tension waves with (β, λ) in Region C. On the solution curve $\mathscr{C}^{\mathrm{C}}_{\beta,\delta}$, we have that $\widetilde{\eta}$ satisfies

$$\gamma \partial_x^4 \widetilde{\eta} - 2(1+\delta)\gamma \partial_x^2 \widetilde{\eta} + \gamma \widetilde{\eta} - \frac{3}{2}\left(\varrho - \frac{1}{h^2}\right)\widetilde{\eta}^2 = 0. \tag{4.36}$$

Direct computation shows that the Green's function of $[\partial_x^4 - 2(1+\delta)\partial_x^2 + 1]^{-1}$ decays like $e^{-s|x|}$ as $x \to \pm\infty$, where

$$s := \begin{cases} \sqrt{1 + \delta - \sqrt{\delta(2+\delta)}}, & \delta \ge 0, \\ \sqrt{|\delta|/2}, & \delta < 0, \end{cases} \tag{4.37}$$

indicating that $\widetilde{\eta}$ is exponentially localized. Therefore, invoking the Weyl theorem on continuous spectrum (see, for example, [49]), we know that

$$\operatorname{ess\,spec} \widetilde{Q}_0 = [\gamma, \infty) \quad \text{when } \delta > -2. \tag{4.38}$$

Note that the operator $\widetilde{Q}_0$ is self-adjoint in $L^2(\mathbb{R})$ with domain $H^{k+4}(\mathbb{R})$. Therefore, its spectrum is confined to the real line. Standard ODE theory shows that any eigenvalue of $\widetilde{Q}_0$ has geometric multiplicity ≤ 2.

By setting $Z := \frac{3}{2\gamma}\left(\varrho - \frac{1}{h^2}\right)\widetilde{\eta}$, equation (4.36) becomes (1.9)

$$Z'''' - 2(1+\delta)Z'' + Z - Z^2 = 0,$$

which leads us to study

$$Q_\delta := \partial_x^4 - 2(1+\delta)\partial_x^2 + 1 - 2Z_\delta \tag{4.39}$$

viewed as an unbounded operator on $L^2(\mathbb{R})$ with domain $H^4(\mathbb{R})$. While more exotic than the Schrödinger operator encountered in Region A, the spectral properties of this Q_δ for $\delta > -2$ have been studied by Sandstede [50]. We quote an important result of his below.

Lemma 4.10 (Sandstede [50]). *Let $\delta > -2$ and Z_δ be a homoclinic solution of (1.9), and consider the linearized operator Q_δ given by (4.39).*

(i) *Q_δ has at least one negative eigenvalue.*
(ii) *If Z_δ is transversely constructed, then zero is a simple eigenvalue of Q_δ. Moreover, when δ is varied, the number of negative eigenvalues remains constant until Z_δ ceases to be transversely constructed.*
(iii) *In particular, for $\delta \geq 0$ or $-1 \ll \delta < 0$ and consider Z_δ being a transversely constructed primary homoclinic orbit. Then Q_δ has exactly one negative eigenvalue. That is, the spectrum of Q_δ takes the form*

$$\operatorname{ess\,spec} Q_\delta = [1, \infty), \qquad \operatorname{spec} Q_\delta = \{-\widetilde{\nu}^2,\ 0\} \cup \widetilde{\Lambda}$$

where $-\widetilde{\nu}^2 < 0$ and 0 are both simple with corresponding eigenfuctions $\widetilde{\phi}_1$ and $\widetilde{\phi}_2 = Z_\delta'$, respectively; and there exists $\nu_* > 0$ such that $\widetilde{\Lambda} \subset [\nu_*, \infty)$.

With these provisions, we obtain the following theorem of the spectrum of the augmented potential in Region C. The proof is very similar to the one for Theorem 4.9, and hence we omit it.

Theorem 4.11 (Spectrum in Region C). *Let the assumptions of Lemma 4.7 (b) hold. Let $\widetilde{\nu}$, ν_* and $\widetilde{\phi}_{1,2}$ given as in Lemma 4.10 (iii). Then for each $a \in (0, \nu_*)$ there exists some $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the operator $Q_\varepsilon(\eta_\varepsilon)$ satisfies*

$$\operatorname{ess\,spec} Q_\varepsilon(\eta_\varepsilon) \subset [\gamma\varepsilon^4 c^2\rho_-/d_+, \infty), \qquad \operatorname{spec} Q_\varepsilon(\eta_\varepsilon) = \{-\nu^2,\ 0\} \cup \Lambda,$$

where $\Lambda \subset [a\varepsilon^4 c^2\rho_-/d_+, \infty)$, and

$$\nu^2 = \frac{\varepsilon^4 c^2 \rho_-}{d_+}\widetilde{\nu}^2 + o(\varepsilon^4) \qquad \text{as } \varepsilon \searrow 0.$$

The first two eigenvalues $\nu_1 := -\nu^2$ and $\nu_2 := 0$ are simple with the associated eigenfunctions taking the form $\phi_i = S_\varepsilon\widetilde{\phi}_i + o(1)$ in H^k as $\varepsilon \searrow 0$.

4.4. Spectrum of the linearized augmented Hamiltonian.

Lemma 4.12 (Extension of $\mathrm{D}^2 E_c$). *Let $\{U_c\}$ be one of the family of bound states $\{U_c^{\mathrm{A}}\}$, $\{U_c^{\mathrm{A}\pm}\}$, or $\{U_c^{\mathrm{C}}\}$ given by Corollaries 3.15, 3.16 or Lemma 3.17(a), respectively. Then $\mathrm{D}^2 E_c(U_c)$ extends uniquely to a bounded linear operator $H_c : \mathbb{X} \to \mathbb{X}^*$ such that*

$$\mathrm{D}^2 E_c(U_c)[\dot{u},\ \dot{v}] = \langle H_c \dot{u},\ \dot{v}\rangle_{\mathbb{X}^*\times\mathbb{X}} \qquad \text{for all } \dot{u},\ \dot{v} \in \mathbb{V},$$

and $I^{-1} H_c$ is self-adjoint on $\mathbb{X}$.

Proof. It suffices to consider the diagonal, so let a bound state $U_c = (\eta_c, \psi_c)$ and $\dot{u} = (\dot{\eta}, \dot{\psi}) \in \mathbb{V}$ be given. By Lemmas 4.1 and 4.2, we have that

$$\begin{aligned}
\mathrm{D}^2 E_c(U_c)[\dot{u}, \dot{u}] =& \mathrm{D}^2 V_c^{\mathrm{aug}}(\eta_c)[\dot{\eta}, \dot{\eta}] + \int_{\mathbb{R}} (\mathcal{T}_c - \mathcal{S}_c)\dot{\eta} A(\eta_c)(\mathcal{T}_c - \mathcal{S}_c)\dot{\eta}\, \mathrm{d}x \\
&+ 2\mathrm{D}_\psi \mathrm{D}_\eta E_c(U_c)[\dot{\eta}, \dot{\psi}] + \mathrm{D}^2_\psi E_c(U_c)[\dot{\psi}, \dot{\psi}] \\
=& \langle Q_c(\eta_c)\dot{\eta}, \dot{\eta}\rangle_{\mathbb{X}^*\times\mathbb{X}} + \int_{\mathbb{R}} (\mathcal{T}_c - \mathcal{S}_c)\dot{\eta} A(\eta_c)(\mathcal{T}_c - \mathcal{S}_c)\dot{\eta}\, \mathrm{d}x \\
&+ 2\int_{\mathbb{R}} \dot{\psi} \langle \mathrm{D}A(\eta_c)\dot{\eta}, \psi_c\rangle\, \mathrm{d}x + 2c\int_{\mathbb{R}} \dot{\eta}'\dot{\psi}\, \mathrm{d}x + \int_{\mathbb{R}} \dot{\psi} A(\eta_c)\dot{\psi}\, \mathrm{d}x,
\end{aligned}$$

where we write $\mathcal{S}_c$ and $\mathcal{T}_c$ to indicate that these operators are being evaluated at U_c. The first derivative formula in Lemma 3.3 then gives

$$\begin{aligned}
\mathrm{D}^2 E_c(U_c)[\dot{u}, \dot{u}] = & \langle Q_c(\eta_c)\dot{\eta}, \dot{\eta}\rangle_{\mathbb{X}^*\times\mathbb{X}} + \int_{\mathbb{R}} (\mathcal{T}_c - \mathcal{S}_c)\dot{\eta} A(\eta_c)(\mathcal{T}_c - \mathcal{S}_c)\dot{\eta}\, \mathrm{d}x \\
&+ \int_{\mathbb{R}} \dot{\psi} A(\eta_c)\dot{\psi}\, \mathrm{d}x \\
&+ 2c\int_{\mathbb{R}} \dot{\eta}'\dot{\psi}\, \mathrm{d}x + 2\sum_{\pm} \rho_\pm \int_{\mathbb{R}} \Big((b_{1c}^{\pm} + c)\left(G_\pm(\eta_c)^{-1} A(\eta_c)\dot{\psi}\right)' \\
&\pm b_{2c}^{\pm} A(\eta_c)\dot{\psi}\Big)\, \dot{\eta}\, \mathrm{d}x,
\end{aligned}$$

where $(b_{1c}^{\pm}, b_{2c}^{\pm})$ is the relative velocity determined by U_c via (4.3). Recalling the definitions of $\mathcal{S}$ and $\mathcal{T}$ in (4.5), this can be expressed quite concisely as:

$$\mathrm{D}^2 E_c(U_c)[\dot{u}, \dot{u}] = \langle Q_c(\eta_c)\dot{\eta}, \dot{\eta}\rangle_{\mathbb{X}^*\times\mathbb{X}} + \int_{\mathbb{R}} \left((\mathcal{T}_c - \mathcal{S}_c)\dot{\eta} + \dot{\psi}\right) A(\eta_c) \left((\mathcal{T}_c - \mathcal{S}_c)\dot{\eta} + \dot{\psi}\right) \mathrm{d}x. \tag{4.40}$$

It is then clear that $\mathrm{D}^2 E_c(U_c)$ extend to an element of $\mathbb{X}^*$. □

We can now state and prove the main result of this section, which characterizes the spectrum.

Theorem 4.13. (Spectrum) *Let $\{U_c\}$ be one of the family of bound states $\{U_c^{\mathrm{A}}\}$, $\{U_c^{\mathrm{A}\pm}\}$, or $\{U_c^{\mathrm{C}}\}$ given by Corollaries 3.15, 3.16 or Lemma 3.17(a), respectively. Then*

$$\operatorname{spec} I^{-1} H_c = \{-\mu_c^2,\ 0\} \cup \Sigma_c,$$

where $-\mu_c^2 < 0$ is a simple eigenvalue corresponding to a unique eigenvector χ_c; 0 is a simple eigenvalue generated by T; and $\Sigma_c \subset (0, \infty)$ is bounded uniformly away from 0.

Proof. This follows from the structure of $I^{-1}H_c$ and a soft analysis argument as in [46, Proposition 5.3]. Due either to Theorem 4.9 or Theorem 4.11, the operator

$$Q_c(\eta_c) + (\alpha - \nu_c^2)\langle\,\cdot\,, \phi_{1c}\rangle\phi_{1c} + \alpha\langle\,\cdot\,, \eta_c'\rangle\eta_c'$$

is positive definite for all $\alpha > 0$, where $-\nu_c^2$ is the negative eigenvalue of $Q_c(\eta_c)$ and ϕ_{1c} is the corresponding eigenfunction. As $A(\eta_c)$ is itself positive definite, from (4.40) we obtain the estimate

$$\langle H_c u,\, u\rangle_{\mathbb{X}^*\times\mathbb{X}} + (\alpha - \nu_c^2)\langle I^{-1}(\phi_{1c}, 0), u\rangle^2_{\mathbb{X}^*\times\mathbb{X}} + \alpha\langle I^{-1}(\eta_c', 0),\, u\rangle^2_{\mathbb{X}^*\times\mathbb{X}} \gtrsim_c \|u\|^2_{\mathbb{X}},$$

for all $u \in \mathbb{X}$. Thus $I^{-1}H_c$ is positive definite on a codimension 2 subspace.

On the other hand, we know that $T'(0)U_c$ is in the kernel of H_c, and by (4.40) we have that

$$\langle H_c u,\, u\rangle_{\mathbb{X}^*\times\mathbb{X}} = \langle Q_c(\eta_c)\phi_{1c}, \phi_{1c}\rangle_{\mathbb{X}^*\times\mathbb{X}} = -\nu_c^2 < 0 \qquad \text{for } u = (\phi_{1c}, (\mathcal{S}_c - \mathcal{T}_c)\phi_{1c}).$$

Thus $I^{-1}H_c$ has a one-dimensional kernel generated by $T'(0)U_c$, a one-dimensional negative definite subspace, and it is positive definite in the orthogonal complement. The claimed spectral properties of $I^{-1}H_c$ are now easily confirmed. □

5. Proof of the Main Results

5.1. Stability of uniform flows. We begin with the simpler case of the trivial solution $U_c = (0, 0)$, corresponding to a laminar flow with (the same) constant purely horizontal velocity in each layer. For (β, λ) in Region B, we then have by Lemma 4.4 that $I^{-1}H_c$ is positive definite. Let us now state and prove a rigorous version of Theorem 1.3. Because $U_c = 0$, the tubular neighborhoods above simply become balls in the appropriate spaces, and hence conditional orbital stability is equivalent to conditional stability.

Theorem 5.1 (Stability of uniform flows). *Let $U_c = (0, 0)$ be the trivial bound state for the internal wave problem (3.31) with wave speed $c \in \mathbb{R}$. Then U_c is conditionally stable if the corresponding (β, λ) lies in Region B.*

Proof. Because E_c is $C^\infty(\mathbb{V}; \mathbb{R})$, $E_c(U_c) = 0$, and $\mathrm{D}E_c(U_c) = 0$, Taylor expanding it at U_c gives

$$E_c(u) = \frac{1}{2}\langle H_c u,\, u\rangle + O(\|u\|^3_{\mathbb{V}}).$$

For (β, λ) in Region B, we have by Lemmas 4.4 and 4.12 that $I^{-1}H_c$ is positive definite on $\mathbb{X}$. On the other hand, the cubic term above can be controlled via Lemma 3.2:

$$\|u\|^3_{\mathbb{V}} \lesssim \|u\|^{1-\theta}_{\mathbb{W}}\|u\|^{2+\theta}_{\mathbb{X}} \le r^\theta R^{1-\theta}\|u\|^2_{\mathbb{X}} \qquad \text{for all } u \in \mathcal{U}^{\mathbb{X}}_r \cap \mathcal{B}^{\mathbb{W}}_R.$$

Thus, for $r > 0$ sufficiently small, it holds that

$$E_c(u) \ge \alpha\|u\|^2_{\mathbb{X}} \qquad \text{for all } u \in \mathcal{U}^{\mathbb{X}}_r \cap \mathcal{B}^{\mathbb{W}}_R, \tag{5.1}$$

for some $\alpha = \alpha(r, R) > 0$.

Now, seeking a contradiction, suppose that U_c is not conditionally stable. Thus there exists $R > 0$, $r > 0$, and a sequence of initial data $\{u_0^n\} \subset \mathcal{O} \cap \mathbb{W}$ with $u_0^n \to 0$ in $\mathbb{X}$ but for which the corresponding solution $u_n : [0, t_0^n) \to \mathcal{B}_R^{\mathbb{W}}$ exits $\mathcal{U}_r^{\mathbb{X}}$ in finite time:

$$\|u_n(\tau_n)\|_{\mathbb{X}} = r \qquad \text{for some } \tau_n \in (0, t_0^n).$$

Let τ_n be the first such time and, if necessary, shrink r so that (5.1) holds. Together with the conservation of energy and momentum, this ensures that

$$E_c(u_0^n) = E_c(u_n(\tau_n)) \geq \alpha r^2 \qquad \text{for all } n \geq 1. \tag{5.2}$$

Because $\{u_0^n\} \subset \mathcal{B}_R^{\mathbb{W}}$ and $u_0^n \to 0$ in $\mathbb{X}$, Lemma 3.2 forces $u_0^n \to 0$ in $\mathbb{V}$. But then, the continuity of E and P would imply that $E_c(u_0^n)$ also vanishes in the limit. As this is in obvious contradiction with (5.2), the proof is complete. □

5.2. Stability for strong surface tension. Next, we turn to the more complicated situation where the wave in question is small-amplitude but nontrivial. Consider first the strong surface tension case corresponding to the waves in Region A. In Theorem 4.13, it was shown that $I^{-1}H_c$ has a negative direction in this regime, and so we will use the energy-momentum approach to show stability. Having laid the groundwork for this argument in the previous sections, we are prepared to state and prove a precise version of Theorem 1.1.

Theorem 5.2 (Stability for strong surface tension). *For all c such that* $0 < \lambda_c - \lambda_0 \ll 1$, *the bound states* U_c^{A} *and* $U_c^{\mathrm{A}\pm}$ *given by Corollaries 3.15 and 3.16 are conditionally orbitally stable.*

Proof. Let U_c stand for both U_c^{A} and $U_c^{\mathrm{A}\pm}$, as the first stage of the proof is identical in either case. In Sect. 3, we confirmed that Assumptions 1—5, and Assumption 6 was verified in Theorem 4.13. By Theorem 2.1, to prove that U_{c_*} is conditionally orbitally stable we need only show that $d''(c_*) > 0$, where recall that d is moment of instability defined by

$$d(c) := E_c(U_c) = E(U_c) - cP(U_c). \tag{5.3}$$

Because U_c is a critical point of E_c, differentiating the above equation gives

$$d'(c) = -P(U_c). \tag{5.4}$$

Thus we must confirm that $c \mapsto -P(U_c)$ is strictly increasing at $c = c_*$.

The definition of the momentum (3.33) and kinematic condition (4.1) yield the explicit formula

$$d'(c) = \int_{\mathbb{R}} \eta_c' \psi_c \, \mathrm{d}x = c \int_{\mathbb{R}} \eta_c \partial_x A(\eta_c)^{-1} \eta_c' \, \mathrm{d}x.$$

As in Sect. 4.2, we will exploit a long-wave rescaling to analyze this quantity. Recycling notation, let us redefine the scaling operator to be

$$S_c f := f\left(\frac{\varepsilon_c \,\cdot}{d_+\sqrt{\beta_c - \beta_0}}\right), \tag{5.5}$$

where $\varepsilon_c = \varepsilon_c^{\mathrm{A}}$ and β_c are given by (3.42). Likewise, the asymptotics for the free surface profile established in (3.40) and (3.41) permits us to write

$$\eta_c =: \varepsilon_c^m d_+ S_c\,(\widetilde{\eta} + \widetilde{r}_c) \qquad \text{for } \widetilde{r}_c = O(\varepsilon_c) \quad \text{in } H^k,$$

with $m = 2$ for U_c^{A} and $m = 1$ for $U_c^{\mathrm{A}\pm}$. Using the rescaling, we compute that

$$\begin{aligned} d'(c) &= c\varepsilon_c^{2m} d_+^2 \int_{\mathbb{R}} (S_c(\widetilde{\eta} + \widetilde{r}_c))\partial_x A(\eta_c)^{-1}\partial_x S_c(\widetilde{\eta} + \widetilde{r}_c)\,\mathrm{d}x \\ &= c\varepsilon_c^{2m-1} d_+^3 \sqrt{\beta_c - \beta_0} \int_{\mathbb{R}} (\widetilde{\eta} + \widetilde{r}_c) S_c^{-1}\partial_x A(\eta_c)^{-1}\partial_x S_c(\widetilde{\eta} + \widetilde{r}_c)\,\mathrm{d}x \\ &= c\varepsilon_c^{2m-1} d_+^3 \sqrt{\beta_c - \beta_0} \sum_{\pm} \rho_\pm \int_{\mathbb{R}} (\widetilde{\eta} + \widetilde{r}_c) S_c^{-1}\partial_x G_\pm(\eta_c)^{-1}\partial_x S_c(\widetilde{\eta} + \widetilde{r}_c)\,\mathrm{d}x, \end{aligned}$$

where the last line follows from (A.1). Similar to (4.20), let us define

$$\widetilde{\mathcal{M}}_c^\pm(\eta_c) := d_+ S_c^{-1}\partial_x G_\pm(\eta_c)^{-1}\partial_x S_c.$$

Arguing as in Lemma 4.6, we then find that

$$\left\| \widetilde{\mathcal{M}}_c^\pm(0) + \frac{d_+}{d_\pm} \right\|_{\mathrm{Lin}(H^2,L^2)} \lesssim \varepsilon_c^2, \qquad \| \widetilde{\mathcal{M}}_c^\pm(\eta_c) - \widetilde{\mathcal{M}}_c^\pm(0) \|_{\mathrm{Lin}(H^2,L^2)} \lesssim \varepsilon_c^m,$$

and hence

$$\begin{aligned} d'(c) &= c\varepsilon_c^{2m-1} d_+^2 \sqrt{\beta_c - \beta_0} \sum_{\pm} \rho_\pm \int_{\mathbb{R}} \widetilde{\eta}\widetilde{\mathcal{M}}_c^\pm(0)\widetilde{\eta}\,\mathrm{d}x + O(\varepsilon_c^{3m-1}) \qquad &&\text{in } C^1(\mathscr{I}) \\ &= -c\varepsilon_c^{2m-1} d_+^2 \sqrt{\beta_c - \beta_0} \sum_{\pm} \rho_\pm \frac{d_+}{d_\pm} \int_{\mathbb{R}} \widetilde{\eta}^2\,\mathrm{d}x + O(\varepsilon_c^{3m-1}) \qquad &&\text{in } C^1(\mathscr{I}), \end{aligned} \tag{5.6}$$

where recall that $\mathscr{I}$ is a sufficiently small interval containing c_*.

Now, observe that $\varepsilon_c, \beta_c > 0$, and from (3.42),

$$c \mapsto \varepsilon_c \text{ and } c \mapsto c\beta_c \operatorname{sgn} c \text{ are both positive and } \begin{cases} \text{strictly decreasing for } c > 0, \text{ and} \\ \text{strictly increasing for } c < 0, \end{cases}$$

Therefore $c \mapsto -c\varepsilon_c^{2m-1}(\beta_c - \beta_0)^{1/2}$ is strictly increasing. This completes the proof for the family $\{U_c^{\mathrm{A}}\}$, as $\widetilde{\eta}$ is independent of c in that case.

The argument for $\{U_c^{\mathrm{A}\pm}\}$ is only slightly more complicated. Recall that by (3.41),

$$\widetilde{\eta} = \widetilde{\eta}_c^{\mathrm{A}\pm}(x) = \frac{1}{\kappa_c \pm \sqrt{\kappa_c^2 + 4(\varrho + h)}\cosh x}, \tag{5.7}$$

with $\kappa_c = \kappa_c^{\mathrm{A}}$ defined as in Corollary 3.16. Since we are in fact computing $\int_{\mathbb{R}} \widetilde{\eta}^2\,\mathrm{d}x$, it is sufficient to assume that $\kappa_c > 0$. Then, clearly $\widetilde{\eta}_c^{\mathrm{A}+} > 0$ and $c \mapsto \kappa_c \operatorname{sgn} c$ is increasing, so we again have by (5.6) and the argument in the previous paragraph that d' is strictly increasing. Finally, $\widetilde{\eta}_c^{\mathrm{A}-}$ is a wave of depression and an explicit computation using (5.7) gives

$$\int_{\mathbb{R}} (\widetilde{\eta}_c^{\mathrm{A}-})^2 \, \mathrm{d}x = \frac{\kappa_c \tan^{-1}\left(\frac{\kappa_c + \sqrt{\kappa_c^2 + 4(\varrho+h)}}{4(\varrho+h)}\right)}{2(\varrho+h)^{3/2}} - \frac{1}{2(\varrho+h)}.$$

It is easily seen that the right-hand side above is strictly increasing in c for $c > 0$ and strictly decreasing for $c < 0$. The proof is therefore complete. $\square$

5.3. Stability for near critical surface tension. Consider now the families of bound states $\{U_c^{\mathrm{C}}\}$ that correspond to traveling waves in Region C. Recall from Sect. 3.5, that to leading order, the corresponding free surface profiles are rescalings of the family of primary homoclinic orbits $\{Z_\delta\}$ of the ODEs (3.44). To unify the presentation, we will write Z_c as shorthand for $Z_{\delta_c^{\mathrm{C}}}$.

The next theorem shows that under the hypothesis of Theorem 4.11, the orbital stability/instability of these waves can be inferred purely from properties of the primary homoclinic orbits.

Theorem 5.3. (Stability for critical surface tension) *Consider the family of traveling waves $\{U_c^{\mathrm{C}}\}$ given in Lemma 3.17(a) and assume that the hypothesis of Theorem 4.11 holds. For all c_* with $0 < \lambda_{c_*} - \lambda_0 \ll 1$, the corresponding wave is conditionally orbitally stable provided that the function*

$$c \mapsto \operatorname{sgn} c \int_{\mathbb{R}} Z_c^2 \, \mathrm{d}x \quad \textit{is strictly increasing at } c_*, \tag{5.8}$$

and it is orbitally unstable if this function is strictly decreasing there.

Proof. Throughout the argument, we abbreviate $\{U_c\}$ for $\{U_c^{\mathrm{C}}\}$ and $\varepsilon_c = \varepsilon_c^{\mathrm{C}}$. We have already proved in Theorem 4.13 that the spectral hypothesis Assumption 6 holds for $I^{-1}H_c$. As in the previous subsection, we may therefore apply Theorem 2.1 to conclude that U_{c_*} is conditionally orbitally stable provided that $d''(c_*) > 0$, where d is the moment of instability (5.3). On the other hand, because the Cauchy problem is locally well-posed, Assumption 7 is satisfied, and so Theorem 2.2 tells us that U_{c_*} is orbitally unstable if $d''(c_*) < 0$.

From Lemma 3.17, we know that free surface profile takes the form

$$\eta_c = \varepsilon_c^4 d_+ S_c\,(Z_c + \widetilde{r}_c) \qquad \text{with } \widetilde{r}_c = O(\varepsilon_c) \text{ in } H^k \text{ as } \varepsilon \searrow 0,$$

where we have redefined the scaling operator to be $S_c f := f(\varepsilon \,\cdot\, /d_+)$. The same argument as in the proof of Theorem 5.2 reveals that

$$d'(c) = -c\varepsilon_c^7 d_+^2 \gamma \sum_{\pm} \rho_\pm \frac{d_+}{d_\pm} \int_{\mathbb{R}} Z_c^2 \, \mathrm{d}x + O(\varepsilon_c^{11}) \quad \text{in } C^1(\mathscr{I}).$$

From the definition of ε_c in (3.46),

$$c \mapsto \varepsilon_c \text{ and } c \mapsto c\varepsilon_c^2 \operatorname{sgn} c \text{ are both positive and } \begin{cases} \text{strictly decreasing for } c > 0, \text{ and} \\ \text{strictly increasing for } c < 0, \end{cases}$$

Thus $c \mapsto -c\varepsilon_c^7$ is strictly increasing. Therefore $d'(c)$ is strictly increasing at c_* when (5.8) is satisfied. $\square$

We remark that (5.8) is stated in terms of the wave speed c, but to compare it to results on dispersive model equations of Kawahara type (1.9) it is natural to consider the related function $\delta \mapsto \int Z_\delta^2 \,\mathrm{d}x$. Looking carefully at its definition in (3.46), we see that $c \mapsto \delta_c^{\mathrm{C}}$ can be both increasing or decreasing depending on the various physical parameters.

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Declarations

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Appendix A. Elementary Identities

Here we give the proofs of the elementary first and second derivative formulas for the nonlocal operators $G_\pm(\eta)$ and $A(\eta)$.

Proof of Lemma 3.3. The formula (3.18) for $\mathrm{D}G_\pm(\eta)$ can be derived using the same method as the standard one-fluid case. To obtain (3.20), it is easier to first consider the derivative of

$$A(\eta)^{-1} = G_+(\eta)^{-1} B(\eta) G_-(\eta)^{-1} = \rho_+ G_+(\eta)^{-1} + \rho_- G_-(\eta)^{-1}. \tag{A.1}$$

Then,

$$\begin{aligned}
\langle \mathrm{D}A(\eta)\dot{\eta},\ \psi\rangle &= -A(\eta)\left\langle \mathrm{D}(A(\eta)^{-1})\dot{\eta},\, A(\eta)\psi\right\rangle \\
&= \sum_{\pm} \rho_\pm A(\eta) G_\pm(\eta)^{-1}\left\langle \mathrm{D}G_\pm(\eta)\dot{\eta},\, G_\pm(\eta)^{-1}A(\eta)\psi\right\rangle.
\end{aligned}$$

Using the self-adjointness of $G_\pm(\eta)$ and the formula (3.18) for $\mathrm{D}G_\pm(\eta)$, this leads immediately to (3.20). □

Proof of Lemma 3.6. As in the proof of Lemma 3.3, we start by considering the corresponding formula for $A(\eta)^{-1}$. Recalling (A.1), we see that

$$\begin{aligned}
D^2(A(\eta)^{-1})[\dot{\eta},\dot{\eta}] &= \sum_{\pm} \rho_\pm \mathrm{D}^2(G_\pm(\eta)^{-1})[\dot{\eta},\dot{\eta}] \\
&= -\sum_{\pm} \rho_\pm G_\pm(\eta)^{-1}\Big(\mathrm{D}^2 G_\pm(\eta)[\dot{\eta},\dot{\eta}] \\
&\quad -2\mathrm{D}G_\pm(\eta)[\dot{\eta}]G_\pm(\eta)^{-1}\mathrm{D}G_\pm(\eta)[\dot{\eta}]\Big)\, G_\pm(\eta)^{-1}.
\end{aligned}$$

On the other hand, we have the elementary identity

$$\mathrm{D}^2 A(\eta)[\dot{\eta},\dot{\eta}] = -A(\eta)\mathrm{D}^2(A(\eta)^{-1})[\dot{\eta},\dot{\eta}]A(\eta) + 2\mathrm{D}A(\eta)[\dot{\eta}]A(\eta)^{-1}\mathrm{D}A(\eta)[\dot{\eta}]. \tag{A.2}$$

Together, these will furnish a representation formula for the second variation of $A(\eta)^{-1}$ once we have fully expanded these expressions using (3.18) and (3.21).

Consider each of the terms on the right-hand side of (A.2). For the first, we have

$$
\begin{aligned}
-\int_{\mathbb{R}} \psi A(\eta) \mathrm{D}^2(A(\eta)^{-1})[\dot{\eta},\dot{\eta}] A(\eta)\psi \,\mathrm{d}x &= \sum_{\pm} \rho_{\pm} \int_{\mathbb{R}} \theta_{\pm} \mathrm{D}^2 G_{\pm}(\eta)[\dot{\eta},\dot{\eta}]\theta_{\pm}\,\mathrm{d}x \\
&\quad - 2\sum_{\pm} \rho_{\pm} \int_{\mathbb{R}} \theta_{\pm} \mathrm{D}G_{\pm}(\eta)[\dot{\eta}] G_{\pm}(\eta)^{-1} \mathrm{D}G_{\pm}(\eta)[\dot{\eta}]\theta_{\pm}\,\mathrm{d}x,
\end{aligned}
$$

where recall that $\theta_{\pm} = \theta_{\pm}(\eta,\psi)$ is given by (3.24). Throughout the remainder of the proof, $a_i^{\pm}$ will always be evaluated at $(\eta,\theta_{\pm})$, so we suppress the arguments for readability. By the first variation (3.18) and second variation (3.21) formulas for $G_{\pm}(\eta)$, this becomes

$$
\begin{aligned}
-\int_{\mathbb{R}} \psi A(\eta) \mathrm{D}^2(A(\eta)^{-1})[\dot{\eta},\dot{\eta}] A(\eta)\psi \,\mathrm{d}x &= \sum_{\pm} \rho_{\pm} \int_{\mathbb{R}} \left(a_4^{\pm}\dot{\eta}^2 + 2a_2^{\pm}\dot{\eta} G_{\pm}(\eta)\left(a_2^{\pm}\dot{\eta}\right)\right)\mathrm{d}x \\
&\quad - 2\sum_{\pm} \rho_{\pm} \int_{\mathbb{R}} a_1^{\pm} \left(G_{\pm}(\eta)^{-1} \mathrm{D}G_{\pm}(\eta)[\dot{\eta}]\theta_{\pm}\right)' \dot{\eta}\,\mathrm{d}x \\
&\quad - 2\sum_{\pm} \rho_{\pm} \int_{\mathbb{R}} a_2^{\pm} \left(\mathrm{D}G_{\pm}(\eta)[\dot{\eta}]\theta_{\pm}\right) \dot{\eta}\,\mathrm{d}x \\
&= \sum_{\pm} \rho_{\pm} \int_{\mathbb{R}} \left(a_4^{\pm}\dot{\eta}^2 + 2a_2^{\pm}\dot{\eta} G_{\pm}(\eta)\left(a_2^{\pm}\dot{\eta}\right)\right)\mathrm{d}x \\
&\quad - 2\sum_{\pm} \rho_{\pm} \int_{\mathbb{R}} \mathscr{L}_{\pm}[\dot{\eta}] \mathrm{D}G_{\pm}(\eta)[\dot{\eta}]\theta_{\pm}\,\mathrm{d}x,
\end{aligned}
$$

for the linear operator $\mathscr{L}_{\pm}$ given by (3.25). Using (3.18) once more allows us to simplify this to

$$
\begin{aligned}
-\int_{\mathbb{R}} \psi A(\eta) \mathrm{D}^2(A(\eta)^{-1})[\dot{\eta},\dot{\eta}] A(\eta)\psi \,\mathrm{d}x &= \sum_{\pm} \rho_{\pm} \int_{\mathbb{R}} \left(a_4^{\pm}\dot{\eta} + 2a_2^{\pm} G_{\pm}(\eta)\left(a_2^{\pm}\dot{\eta}\right)\right)\dot{\eta}\,\mathrm{d}x \\
&\quad - 2\sum_{\pm} \rho_{\pm} \int_{\mathbb{R}} \left(a_1^{\pm}\mathscr{L}_{\pm}[\dot{\eta}]' + a_2^{\pm} G_{\pm}(\eta)\mathscr{L}_{\pm}[\dot{\eta}]\right)\dot{\eta}\,\mathrm{d}x.
\end{aligned}
$$

So finally we have

$$
\begin{aligned}
&-\int_{\mathbb{R}} \psi A(\eta) \mathrm{D}^2(A(\eta)^{-1})[\dot{\eta},\dot{\eta}] A(\eta)\psi \,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}} \left(a_4\dot{\eta} + 2\sum_{\pm} \rho_{\pm} a_2^{\pm} G_{\pm}(\eta)\left(a_2^{\pm}\dot{\eta}\right) - 2\mathscr{M}\dot{\eta}\right)\dot{\eta}\,\mathrm{d}x
\end{aligned}
\tag{A.3}
$$

where recall $a_4 = a_4(\eta,\psi)$ and $\mathscr{M} = \mathscr{M}(\eta,\psi)$ were defined in (3.24) and (3.26), respectively.

Likewise, the second in term on the right-hand side of (A.2) can be treated as follows. Using (3.20), we calculate that

$$\begin{aligned}\int_{\mathbb{R}} \psi \mathrm{D}A(\eta)[\dot{\eta}]A(\eta)^{-1}\mathrm{D}A(\eta)[\dot{\eta}]\psi \,\mathrm{d}x &= \sum_{\pm} \rho_{\pm} \int_{\mathbb{R}} \Big(a_1^{\pm} \Big(G_{\pm}(\eta)^{-1} \mathrm{D}A(\eta)[\dot{\eta}]\psi \Big)' \\ &\quad + a_2^{\pm} A(\eta) \mathrm{D}A(\eta)[\dot{\eta}]\psi \Big) \dot{\eta} \,\mathrm{d}x \\ &= \sum_{\pm} \rho_{\pm} \int_{\mathbb{R}} \mathscr{L}_{\pm}[\dot{\eta}] \mathrm{D}A(\eta)[\dot{\eta}]\psi \,\mathrm{d}x = \int_{\mathbb{R}} \mathscr{L}[\dot{\eta}] \mathrm{D}A(\eta)[\dot{\eta}]\psi \,\mathrm{d}x.\end{aligned}$$

Applying (3.20) once more then yields

$$\begin{aligned}\int_{\mathbb{R}} \psi \mathrm{D}A(\eta)[\dot{\eta}]A(\eta)^{-1}\mathrm{D}A(\eta)[\dot{\eta}]\psi \,\mathrm{d}x &= \sum_{\pm} \rho_{\pm} \int_{\mathbb{R}} \Big(a_1^{\pm} \Big(A(\eta) G_{\pm}(\eta)^{-1} \mathscr{L}[\dot{\eta}] \Big)' \\ &\quad + a_2^{\pm} A(\eta) \mathscr{L}[\dot{\eta}] \Big) \dot{\eta} \,\mathrm{d}x \\ &= \int_{\mathbb{R}} \dot{\eta} \mathscr{N} \dot{\eta} \,\mathrm{d}x,\end{aligned} \tag{A.4}$$

with $\mathscr{N} = \mathscr{N}(\eta, \psi)$ defined in (3.27). Combining this with (A.2) and (A.3) gives the formula (3.23), completing the proof. □

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