New Conditions for the Stability and Instability of Feedback Systems

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***Abstract*—In this paper we derive some new and useful conditions on the open loop transfer function, necessary for closed loop stability. The derivation is based on imposing the sign-invariance of the closed loop characteristic polynomial evaluated on the real axis. Although this is a simple necessary condition for closed loop stability, it leads to fairly stringent requirements on the open loop transfer function. Also, the results can be stated only in terms of the numbers of real poles and zeros of the open loop system even though the system may have complex poles and zeros. Using the same approach new necessary conditions for stabilizability by P, PI and PID controllers as well as by arbitrary controllers are also presented.**

***Index Terms*—stability, instability, Nyquist plot**

I. Introduction

The stability of a feedback system is usually verified using the Nyquist plot of the open loop system or its transfer function (see [1], [2], [3], [4], [5], and references therein). This is an evaluation of the transfer function on the imaginary axis and provides a complete answer to the question of closed loop stability of a given feedback system based on the frequency response of the open loop system.

In many situations, it may be useful to know simple conditions on the open loop system that guarantees closed loop stability or instability. This may aid in the design of low order controllers of prescribed structure such as lag or lead first order controllers, Proportional Integral (PI) controllers or Proportional Integral Derivative (PID) controllers (see [6],[7], [8], [9] [10], [11], [12], and references therein).

In this paper, we are able to obtain some useful and fairly detailed structural conditions, required to be satisfied by the open loop transfer function to guarantee stability or instability of the closed loop system. Remarkably, these are obtained fairly simply by systematically imposing the requirement of sign-invariance of the characteristic polynomial evaluated on the *real axis*, a necessary condition for closed loop stability. Also the conditions obtained are only in terms of the numbers of real axis poles and zeros of the plant, even though the plant may have complex poles and zeros. Finally the conditions are simple to check and insightful as they involve only the numbers of real axis poles and zeros to the right of each real right half plane pole or zero of the open loop plant. Several illustrative examples are included.

II. Main Results

Consider the system in Fig. 1

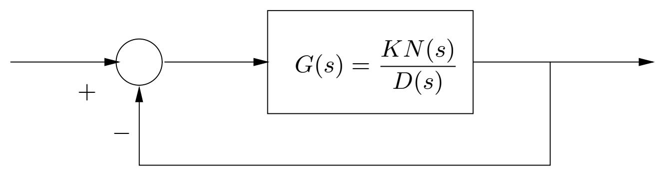

Fig. 1. Scalar feedback system

The characteristic polynomial of the feedback system is

$$\delta(s) = D(s) + KN(s). \quad (1)$$

The feedback system is stable if and only if all roots of $\delta(s)$ lie in $\mathbb{C}^-$, the open left-half plane, or $\delta(s)$ is Hurwitz.

To proceed, we assume without loss of generality that $D(s)$ and $N(s)$ are monic polynomials, that is, have leading coefficient equals $+1$. Let z_i^+, $i \in \mathbf{q} := [1, 2, \cdots, q]$ and p_j^+ $j \in \mathbf{p} := [1, 2, \cdots, p]$ denote the finite, positive real, distinct, zeros and poles, respectively, of $G(s)$.

Lemma 1:

(a) If $K > 0$, a necessary condition for closed-loop stability is:

$$D(z_i^+), N(p_j^+), \quad \text{for} \quad i \in \mathbf{q}, j \in \mathbf{p} \quad (2)$$

are all nonzero and have the same sign (positive or negative).

(b) If $K < 0$, necessary conditions for closed-loop stability are:

(i) $D(z_i^+)$ for $i \in \mathbf{q}$ are nonzero and have the same sign.

(ii) $N(p_j^+)$ for $j \in \mathbf{p}$ are nonzero and have the same sign.

(iii) $D(z_i^+), i \in \mathbf{q}$ and $N(p_j^+), j \in \mathbf{p}$ have opposite signs.

Proof: If $\delta(s)$ is Hurwitz, then

$$\delta(\sigma) \neq 0 \qquad \text{for all} \quad \sigma \geq 0 \quad (3)$$

and

$$D(\sigma) + KN(\sigma) \neq 0, \qquad \text{for all} \quad \sigma \geq 0. \quad (4)$$

Therefore,

$$D(\sigma) + KN(\sigma) > 0 \qquad \text{for all} \quad \sigma \geq 0 \tag{5}$$

or

$$D(\sigma) + KN(\sigma) < 0 \qquad \text{for all} \quad \sigma \geq 0. \tag{6}$$

The conclusions in (a) and (b) now follow by successively setting $\sigma = z_i^+$ for $i \in \mathbf{q}$ and $\sigma = p_j^+$ for $j \in \mathbf{p}$ in (5) and (6). ■

The above condition can be interpreted in terms of the real poles and zeros of the system as shown below.

Lemma 2:

(a) $D(z_i^+), N(p_j^+)$ for $i \in \mathbf{q}, j \in \mathbf{p}$ are nonzero and have the same sign if and only if the numbers of real axis poles counting multiplicities to the right of every zero z_i^+, and the numbers of real axis zeros counting multiplicities, to the right of every pole p_j^+, are all *even* or *all* odd.

(b) $D(z_i^+), N(p_j^+)$ for $i \in \mathbf{q}, j \in \mathbf{p}$ are nonzero and have opposite signs if and only if the numbers of real axis poles counting multiplicities, to the right of every zero z_i^+ are all even (odd) and the numbers of real axis zeros counting multiplicities, to the right of every pole p_j^+ are all odd (even).

Proof: First, observe that $N(s), D(s)$ admit the monic factorizations,

$$N(s) = N_c(s)N_r^-(s)N_r^+(s) \tag{7a}$$
$$D(s) = D_c(s)D_r^-(s)D_r^+(s) \tag{7b}$$

where the zeros of $N_c(s)(D_c(s))$ are complex, the zeros of $N_r^-(s)(D_r^-(s))$ are real and negative, and the zeros of $N_r^+(s)(D_r^+(s))$ are real and nonnegative. Then the sign of $N(\sigma)(D(\sigma))$ is identical to the sign of $N_r^+(\sigma)(D_r^+(\sigma))$ for all $\sigma \geq 0$, since

$$N_c(\sigma)N_r^-(\sigma) > 0 \tag{8a}$$
$$D_c(\sigma)D_r^-(\sigma) > 0. \tag{8b}$$

Hence,

$$\text{sign}\left[N(p_j^+)\right] = \text{sign}\left[N_r^+(p_j^+)\right] \tag{9a}$$
$$\text{sign}\left[D(z_i^+)\right] = \text{sign}\left[D_r^+(z_i^+)\right]. \tag{9b}$$

The proof is completed by observing that

(i) $N_r^+(p_j^+) > 0$ if and only if the number of real zeros to the right of p_j^+ is even.

(ii) $N_r^+(p_j^+) < 0$ if and only if the number of real zeros to the right of p_j^+ is odd,

and similarly

(iii) $D_r^+(z_i^+) > 0$ if and only if the number of poles to the right of z_i^+ is even.

(iv) $D_r^+(z_i^+) < 0$ if and only if the number of zeros to the right of z_i^+ is odd.

■

Example 1: In the following examples, we denote poles by ”X” and zeros by ”○”. In each example, we list the numbers of real RHP zeros (poles) to the right of each real RHP pole (zero) and state the conclusion obtained from the above Lemmas. Note that violation of the necessary conditions for stability are sufficient conditions for instability.

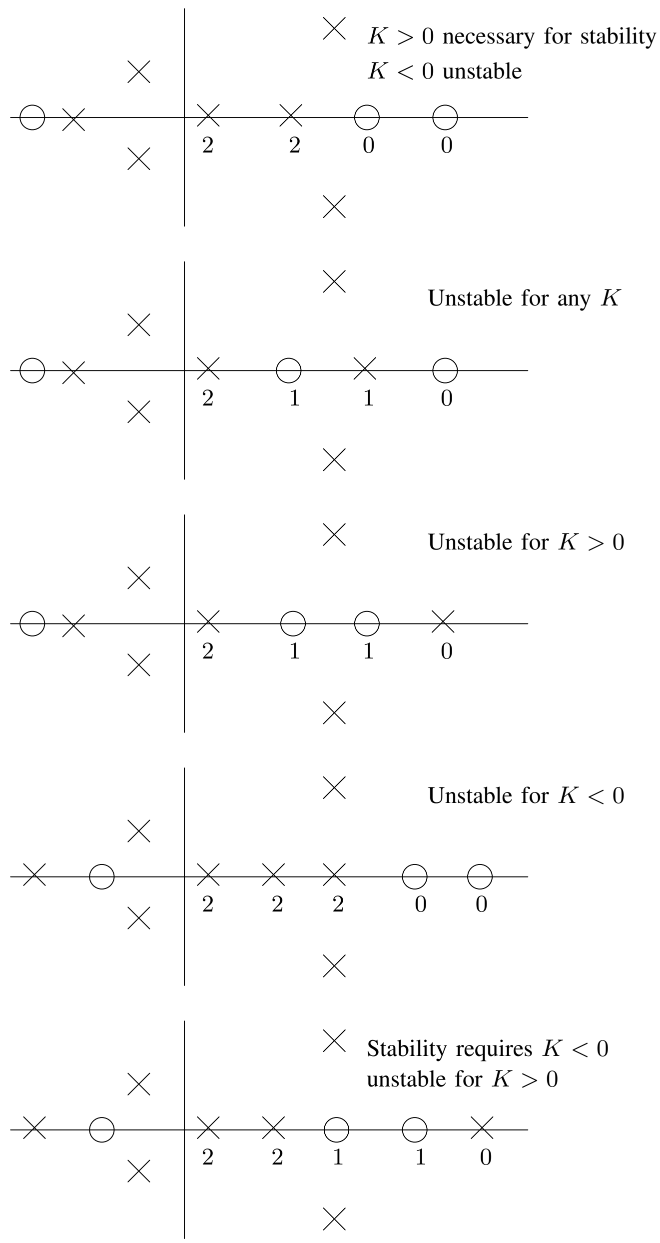

Fig. 2. Pole-zero patterns (Example 1)

Remark 2.1: The conditions given in Lemma 1 are necessary for stability, but are not sufficient. As an example, consider the pole-zero pattern shown in Fig. 3.

The closed loop is *not* stable if $z_1 > \frac{6}{5}$ for any K, even though condition (a) of Lemma 1 is satisfied. On the other hand, the pattern is sufficient to predict instability for $K < 0$.

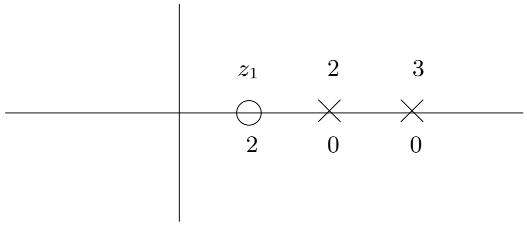

Fig. 3. Pole-zero pattern (Remark 2.1)

III. Stabilizability by 3-Term Controllers

In this section, we first develop some new necessary conditions for a plant to be stabilizable by integral (I), proportional-integral (PI) or proportional ntegral derivative (PID) controllers.

Consider the plant, augmented by an integrator

$$\frac{G(s)}{s} = \frac{N(s)}{sD(s)} =: \bar{G}(s) \qquad (10)$$

and let S_i denote the open intervals between successive real axis distinct RHP poles, called *pole segments*, and let Q_j denote the open intervals between successive real axis distinct RHP zeros called *zero segments* of the augmented plant in (10). A segment $S_i(Q_j)$ is said to be of *odd order (even order)* if the number of zeros (poles) lying on it, counting multiplicities is odd (even).

Theorem 1: A necessary condition for stabilizability,

(a) by an integral controller, is the absence of odd order pole segments, and odd order zero segments;
(b) by a PI controller, is the absence of any odd order zero segments and the presence of no more than one odd order pole segment;
(c) by a PID controller, is the absence of any odd order zero segments, and the presence of no more than two odd order pole segments.

The proof depends on the following lemmas.

Lemma 3: Suppose $S_i(Q_j)$ is an odd order pole (zero) segment. Then

$$\left\{-\frac{1}{\bar{G}(\sigma)} \; : \; \sigma \in S_i\right\} = \mathbb{R} \quad \text{(the real axis)} \qquad (11)$$

and

$$\left\{-\frac{1}{\bar{G}(\sigma)} \; : \; \sigma \in Q_j\right\} = \mathbb{R} \quad \text{(the real axis).} \qquad (12)$$

Proof: The graph of $-\frac{1}{\bar{G}(\sigma)}$ transitions from $+\infty$ $(-\infty)$ or $-\infty$ $(+\infty)$ over an odd order pole or zero segment. ■
This leads to the following conclusion.

Lemma 4: The equation

$$1 + K\bar{G}(\sigma) = 0 \qquad (13)$$

has at least one real root σ_i in each odd order pole and in each odd order zero segment for each real K.

Proof: Follows from Lemma 1 and the fact that (13) can be written as

$$K = -\frac{1}{\bar{G}(\sigma)}. \qquad (14)$$

■

Proof: (Theorem 1)

(i) Stabilization of the plant by an integral controller is equivalent to stabilization of the augmented plant $\bar{G}(s)$ by a gain. Therefore, the result follows from Lemma 2.
(ii) A PI controller adds at most one RHP real zero to $\bar{G}(s)$ and therefore can convert at most one odd order pole segment to an even order one.
(iii) A PID controller adds at most two RHP real zeros to $\bar{G}(s)$ and therefore can convert at most two odd order pole segments to even order ones.

■

Example 2: We display some pole-zero patterns of $\bar{G}(s)$ (see Fig. 4) and apply the theorem to them to derive conclusions about I,PI or PID stabilizability.

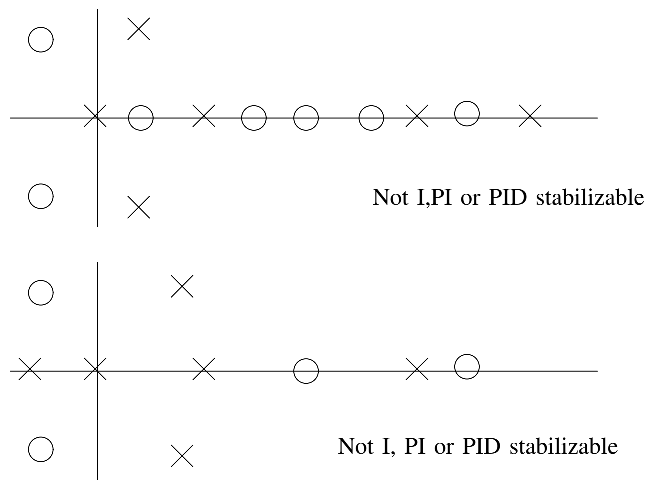

Fig. 4. Not I,PI or PID stabilizable (Example 2)

Example 3: In this example, we display a pole-zero pattern of $\bar{G}(s)$ (see Fig. 5) and apply the theorem to them to derive conclusions about P or PI stabilizability.

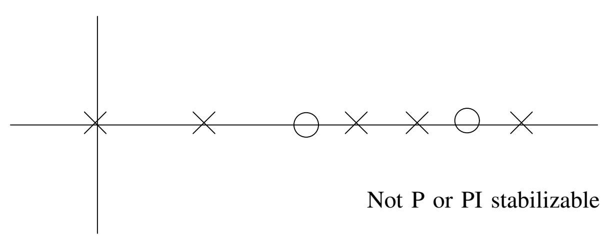

Fig. 5. Not P or PI stabilizable (Example 3)

By applying Lemma 1 to a plant under PID control, we obtain some new linear programming conditions to be satisfied by the controller parameters, as shown below.

Under PID control, the augmented plant to be considered is:

$$\frac{K(s)N(s)}{sD(s)} = G(s) \tag{15}$$

where

$$K(s) = K_i + sK_p + s^2K_d. \tag{16}$$

Let $z_i^+ (p_j^+)$ denote the real axis RHP zeros (poles) of $\frac{N(s)}{sD(s)}$.

Lemma 5:

(a) If $D(z_i^+)$ are positive, a necessary condition for PID stabilization is

$$K(p_j^+)N(p_j^+) > 0, \quad \text{for} \quad j \in \mathbf{p} \tag{17}$$

or

$$\text{sign}\left[K(p_j^+)\right] = \text{sign}\left[N(p_j^+)\right], \quad \text{for} \quad j \in \mathbf{p}. \tag{18}$$

(b) If $D(z_i^+)$ are negative, a necessary condition for PID stabilizability is

$$K(p_j^+)N(p_j^+) < 0, \quad \text{for} \quad j \in \mathbf{p} \tag{19}$$

or

$$\text{sign}\left[K(p_j^+)\right] = -\text{sign}\left[N(p_j^+)\right], \quad \text{for} \quad j \in \mathbf{p}. \tag{20}$$

Proof: The proof follows immediately on applying Lemma 1 to the $G(s)$ in (15). ■

Example 4: Consider the plant

$$\frac{s-3}{s-2} \quad \text{with} \quad z_1^+ = 3, \; p_1^+ = 2 \tag{21}$$

and the augmented plant

$$\begin{aligned} G(s) &= \frac{K(s)(s-3)}{s(s-2)} \\ &= \frac{\left(K_i + sK_p + s^2K_d\right)(s-3)}{s(s-2)}. \end{aligned} \tag{22}$$

Since

$$sD(s)|_{s=z_1^+} = 3(1) = 3 > 0, \tag{23}$$

we have the necessary conditions

$$\begin{aligned} K(s)N(s)|_{s=0} &= -3K_i > 0 \\ K(s)N(s)|_{s=2} &= -(K_i + 2K_p + 4K_d) > 0 \end{aligned} \tag{24}$$

For the case of lead-lag controllers, similar results can be easily stated. The proof is similar to the case of PID controllers and omitted here.

Theorem 2: A necessary condition for stabilizability with lead-lag controllers of the form

$$C(s) = \frac{K(s-z_0)}{(s-p_0)} \tag{25}$$

is presence of no more than one odd order pole segment and no more than one odd order zero segment.

IV. Stabilizability by Arbitrary Controllers

We now consider the controller

$$C(s) = \frac{K_1(s)}{K_2(s)} \tag{26}$$

applied to the plant

$$P(s) = \frac{N(s)}{D(s)}. \tag{27}$$

The characteristic polynomial of the closed-loop system is

$$\delta(s) = K_2(s)D(s) + K_1(s)N(s). \tag{28}$$

Assume that $D(s)$ and $N(s)$ are monic with real axis RHP roots p_j^+ for $j \in \mathbf{p}$ and z_i^+ for $i \in \mathbf{q}$, respectively.

Lemma 6: A necessary condition for $C(s)$ in (26) to stabilize $P(s)$ in (27) is that it satisfy:

$$\begin{aligned} \text{sign}\left[K_2(z_i^+)D(z_i^+)\right] = \text{sign}\left[K_1(p_j^+)N(p_j^+)\right], \\ \text{for} \quad i \in \mathbf{q}, \; j \in \mathbf{p} \end{aligned} \tag{29}$$

or equivalently, either

$$\text{sign}\left[K_2(z_i^+)\right] = \text{sign}\left[D(z_i^+)\right], \quad \text{for} \quad i \in \mathbf{q} \tag{30a}$$

$$\text{sign}\left[K_1(p_j^+)\right] = \text{sign}\left[N(p_i^+)\right], \quad \text{for} \quad j \in \mathbf{p} \tag{30b}$$

or

$$\text{sign}\left[K_2(z_i^+)\right] = -\text{sign}\left[D(z_i^+)\right], \quad \text{for} \quad i \in \mathbf{q} \tag{31a}$$

$$\text{sign}\left[K_1(p_j^+)\right] = -\text{sign}\left[N(p_i^+)\right], \quad \text{for} \quad j \in \mathbf{p} \tag{31b}$$

Proof: Condition (29) follows immediately from Lemma 1. That (29) is equivalent to (30) or (31) is straightforward. ■

Example 5: Consider the plant

$$P(s) = \frac{(s+1)^2(s-1)(s-3)}{s(s^2+2s+3)(s-2)} \tag{32}$$

with the controller

$$C(s) = \frac{\alpha_1 s + \alpha_0}{s + \beta_0}. \tag{33}$$

Then

$$z_i^+ = 1, 3, \quad p_j^+ = 0, 2 \tag{34}$$

and

$$\begin{aligned} D(1) < 0 &\quad (\text{sign}[D(1)] = -1) \\ D(3) > 0 &\quad (\text{sign}[D(3)] = +1) \\ N(0) > 0 &\quad (\text{sign}[N(0)] = +1) \\ N(2) < 0 &\quad (\text{sign}[N(2)] = -1). \end{aligned} \tag{35}$$

Therefore, we have the necessary conditions:

$$\begin{cases} 1 + \beta_0 & < 0 \\ 3 + \beta_0 & > 0 \\ \alpha_0 & > 0 \\ 2\alpha_1 + \alpha_0 & < 0 \end{cases} \tag{36}$$

or

$$\begin{cases} 1+\beta_0 & >0 \\ 3+\beta_0 & <0 \\ \alpha_0 & <0 \\ 2\alpha_1+\alpha_0 & >0 \end{cases} \quad (37)$$

to be satisfied by the controller parameters.
Consider now a plant with n_1 odd order pole segments and n_2 odd order zero segments.

Lemma 7: A necessary condition for $C(s)$ in (26) to stabilize $P(s)$ in (27) is that it has at least n_1 zeros, with at least one each located in the odd order RHP pole segments and n_2 poles, with at least one each located in the odd order RHP zero segments.

Proof: The controller must necessarily convert each odd order zero or pole segment to an even order one. ■

V. Concluding Remarks

Some simple necessary conditions for stability of a feedback system are derived in terms of the real RHP poles and zeros of the open-loop system. The violation of these conditions are then sufficient conditions for instability. Using these, we also derive some necessary conditions for stabilizability of an LTI plant by PID controllers.

The results given here are applicable to stable and minimum-phase controllers since the addition of real LHP and complex poles and zeros by the controller to $G(s)$ does not affect the signs of $D(z_i^+)$ and $N(p_j^+)$.

The necessary conditions given in this paper may have an interpretation in terms of the Nyquist Criterion and the Hermite-Bieler Theorem of Robust Control (see [16], [17], and references therein). They also may have connections to the strong stabilizability condition given in [13], [14]. Some of the ideas of the proof used here are similar to that in [15]

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