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A Note on Realization Theory

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Abstract— **Realization theory as developed hitherto, deals with the following question: How can an arbitrary transfer function be constructed using standard building blocks. The answer is well known to control engineers, is useful, and consists of integrators, multipliers and summers, because the latter components are standard building blocks and can be mass produced, as integrated circuits.If the transfer function to be realized is improper, differentiators would be required as additional building blocks. Since differentiators amplify high frequency noise they are generally avoided and control theory avoids building improper transfer functions. In general integrators are also susceptible to low frequency noise and may be unsuitable in environments where low frequency noise is present. This suggests that it may be interesting to consider as a standard building block, a system which does not amplify low or high frequencies. It turns out that a first order filter does precisely that, namely the high and low frequency gains are constant, independent of frequency. In the following note, we ask the question: Can an arbitrary transfer function be constructed using first order filters, summers, and multipliers? We show that the answer is "yes" regardless of whether the transfer function is proper or improper, and show how such a realization may be constructed for arbitrary linear systems, proper or improper. Indeed we show that almost any first order filter can be used as a building block. We also show through simulations that the first order implementation of a differentiator has superior noise rejection compared to a pure differentiator.**

I. Introduction

An linear time invariant (LTI) system can be realized with integrators if it is strictly proper (see [1], [2], [3], [4], [5] and references therein). If not, use of differentiators is apparently necessary if it is improper. However, the use of differentiators are often prohibited in practice due to the presence of high frequency noise in the system. Consider the frequency responses of an integrator and a differentiator shown in Fig. 1. By the same token, an integrator also magnifies low frequency response.

In order to avoid high gains in low frequencies for an integrator and high gains in high frequencies for a differentiator, we consider a first order system and its frequency response shown in Figs. 2 and 3.

The gain of a first order filter, remarkably, is *constant at both low and high frequencies*. If used as a building block we would expect such a filter would be less susceptible to low or high frequency noise.

In this paper, we show how an integrator *and* a differentiator can be implemented by an arbitrary first order filter. As a consequence, any transfer function, proper or improper can be implemented exactly by using first order filters as building blocks. We also show that such implementations have less susceptibility to high frequency noise compared to differentiators.

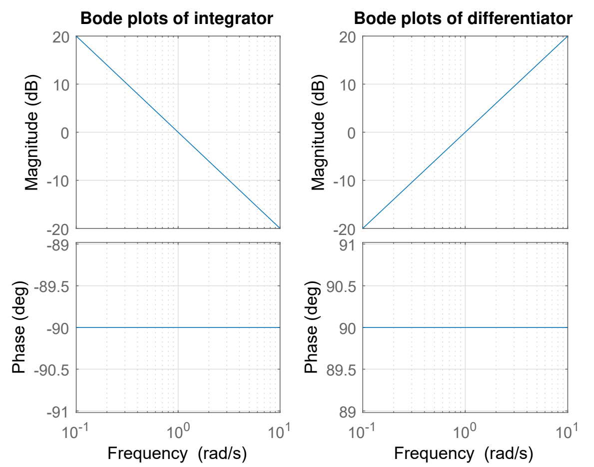

Fig. 1. Frequency responses of an integrator $\frac{1}{s}$ and differentiator s

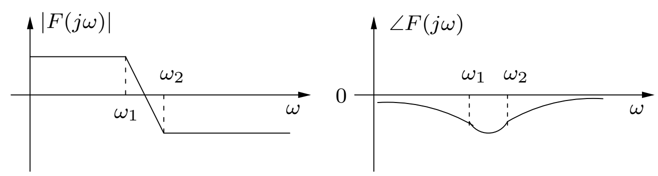

Fig. 2. Lag First Order Filter: $\dfrac{as+b}{s+c} = F(s)$

II. Realization of an Integrator with $F(s)$

Consider an arbitrary first order filter $F(s)$:

$$F(s) = \frac{as+b}{s+c} =: z^{-1}.. \tag{1}$$

The low frequency gain $\frac{b}{c}$ and high frequency gain a are adjustable by choosing the parameters a, b, c. In the following we assume that $F(s)$ is given and fixed.

A. Integrator

In this subsection we show that an integrator can be built using $F(s)$ as a building block. Consider the representation

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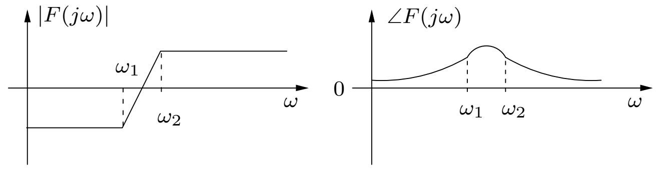

Fig. 3. Lead First Order Filter: $\frac{as+b}{s+c} = F(s)$

of $F(s)$ in (1) above and observe that:

$$\begin{aligned}
\frac{1}{s} = \frac{az-1}{c-bz} &= \frac{-\frac{a}{b}\left(z-\frac{c}{b}\right) - \frac{ac}{b^2} + \frac{1}{b}}{z-\frac{c}{b}} \\
&= \frac{\overbrace{\left(-\frac{ac}{b^2}+\frac{1}{b}\right)}^{\alpha}}{z \underbrace{-\frac{c}{b}}_{\beta}} \underbrace{-\frac{a}{b}}_{\delta} \\
&= \frac{\alpha}{z+\beta} + \delta =: g(z)
\end{aligned} \tag{2}$$

Since $g(z)$ can be realized by $z^{-1} = F(s)$ the integrator can be realized by $F(s)$!

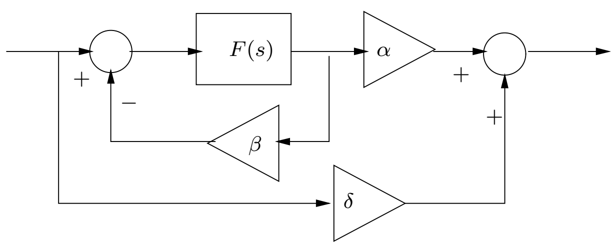

Fig. 4. Realization of an integrator $I(s) = \frac{1}{s}$ by $F(s)$

B. Verification of Integrator Implementation with $F(s)$.

In this subsection we verify that the realization shown above actually represents an integrator.

$$\begin{aligned}
I(s) &= \frac{\alpha F(s)}{1+\beta F(s)} + \delta \\
&= \frac{\left(\frac{b-ac}{b^2}\right)\left(\frac{as+b}{s+c}\right)}{1-\frac{c}{b}\left(\frac{as+b}{s+c}\right)} - \frac{a}{b} \\
&= \frac{abs+b^2-a^c s+acb}{b^2 s+b^2 c-abcs-b^2 c} - \frac{a}{b} \\
&= \frac{b^3-ab^2 c}{\left(b^2-ab^2 c\right)s} = \frac{1}{s}.
\end{aligned} \tag{3}$$

III. Realization of a Differentiator with $F(s)$

In this section we show that a differentiator can indeed also be built using first order filters.

A. Differentiator

From the relationship in (1) it follows that:

$$\begin{aligned}
s = \frac{c-bz}{az-1} &= \frac{\frac{c}{a}-\frac{b}{a}z}{z-\frac{1}{a}} \\
&= \frac{-\frac{b}{a}\left(z-\frac{1}{a}\right) - \frac{b}{a^2} + \frac{c}{a}}{z-\frac{1}{a}} \\
&= \frac{\overbrace{\left(\frac{c}{a}-\frac{b}{a^2}\right)}^{\gamma}}{z \underbrace{-\frac{1}{a}}_{\lambda}} + \underbrace{\left(-\frac{b}{a}\right)}_{\mu} \\
&= \frac{\gamma}{z+\lambda} + \mu =: h(z)
\end{aligned} \tag{4}$$

Since $h(z)$ can be realized by z^{-1}, multipliers and summers, we can indeed realize a differentiator using the the first order filter $F(s)$. In the following subsection we verify that a differentiator can actually be realized by the first order filter $F(s)$.

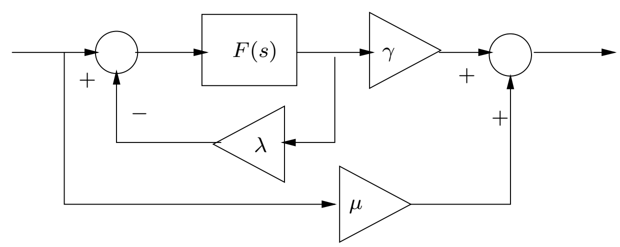

Fig. 5. Realization of a differentiator $D(s)$ using $F(s)$

B. Verification of the realization of a Differentiator using $F(s)$.

$$\begin{aligned}
D(s) &= \frac{\gamma F(s)}{1+\lambda F(s)} + \mu \\
&= \frac{\overbrace{\left(\frac{ca-b}{a^2}\right)}^{\gamma}\overbrace{\left(\frac{as+b}{s+c}\right)}^{F(s)}}{1+\underbrace{\left(-\frac{1}{a}\right)}_{\lambda}\underbrace{\left(\frac{as+b}{s+c}\right)}_{F(s)}} + \underbrace{\left(-\frac{b}{a}\right)}_{\mu} \\
&= \frac{(ca-b)(as+b)}{a^2(s+c)-a^2 s-ab} - \frac{b}{a} \\
&= \frac{s\left(ca^3-a^2 b\right)}{ca^3-a^2 b} = s
\end{aligned} \tag{5}$$

IV. Realization of an Arbitrary $G(s)$ with a First Order Filter $F(s)$

An arbitrary transfer function $G(s)$ can always be decomposed into the sum of a strictly proper part $S(s)$ and a polynomial part $P(s)$:

$$G(s) = \underbrace{S(s)}_{\text{strictly proper}} + \underbrace{P(s)}_{\text{polynomial}} \tag{6}$$

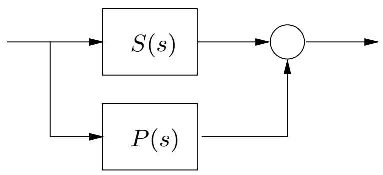

Fig. 6. $G(s)$

$S(s)$ can be realized by integrators and $P(s)$ can be realized by differentiators. Therefore each of them, and thus $G(s)$, can be realized with $F(s)$ as shown below.

A. Realization of $S(s)$ using integrators

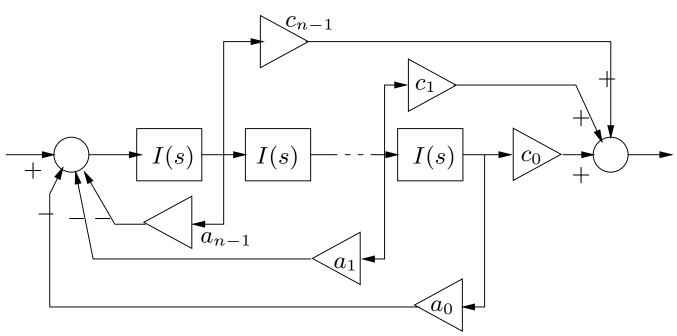

Fig. 7. Realization of $S(s)$ using $F(s)$

Next we consider the realization of the polynomial part $P(s)$ using $F(s)$.

B. Realization of $P(s)$ using $F(s)$

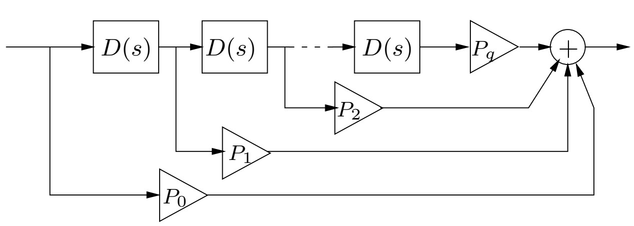

Fig. 8. Realization of $P(s) = P_0 + P_1 s + \cdots + P_q s^q$ using differentiators

Example 1: Consider the transfer function that is improper:

$$G(s) = \frac{s^3 + s^2 + 2s + 1}{s^2 + s + 1} = \underbrace{\frac{s+1}{s^2+s+1}}_{S(s)} + \underbrace{s}_{P(s)} \tag{7}$$

The realization of the given $G(s)$ with first order blocks is shown below:

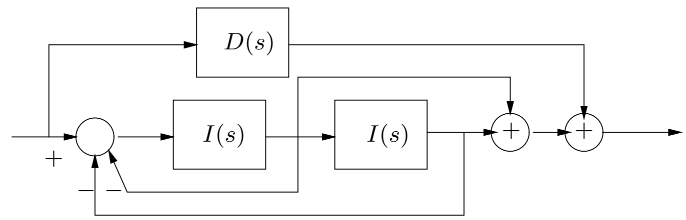

Fig. 9. Realization of $G(s)$ with first order building blocks (Example 1)

V. Testing a Pure Differentiator and its 1st Order Implementation

In this section, we gave some simulation results that compare the behavior of the pure differentiator and it first order implementation.

Example 2: With the Simulink model given in Fig. 10, we fist consider an input without noise:

$$r(t) = \sin t + A\sin(100t), \quad \text{for} \quad A = 0. \tag{8}$$

Consider the first order block $F(s)$ with $a = 1, b = 1, c = 3$:

$$F(s) = \frac{s+1}{s+3}. \tag{9}$$

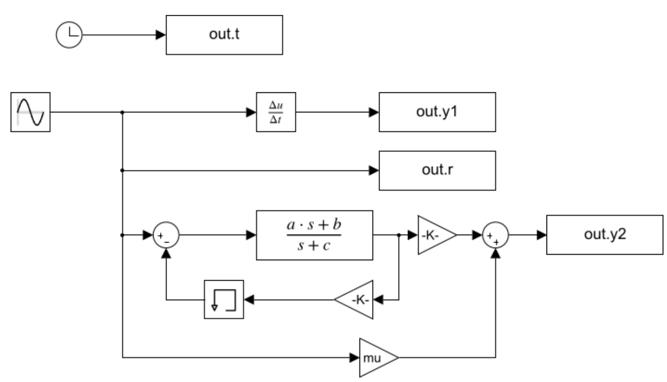

Fig. 10. Simulink Model (Example 2)

Fig. 11 shows that the response of the first order implementation of the differentiator is identical to that of the standard differentiator after a short transient period due to the memory block inserted to run the simulation.

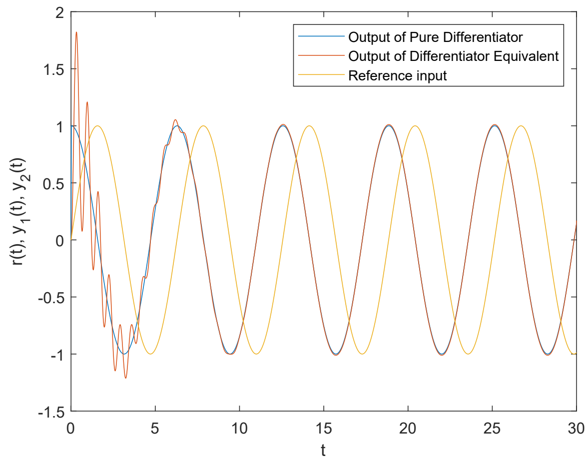

Fig. 11. Outputs for the Input without noise (Example 2)

Example 3: We now display some simulations to test the noise susceptibility [6] of the first order implementation $D(s)$ of a differentiator s. With the sinusoidal input corrupted with small high frequency noise,

$$r(t) = \sin t + A\sin(100t) \quad \text{for} \quad A = 0.01. \tag{10}$$

we have the following responses. The Simulink model used is depicted in Fig. 12.

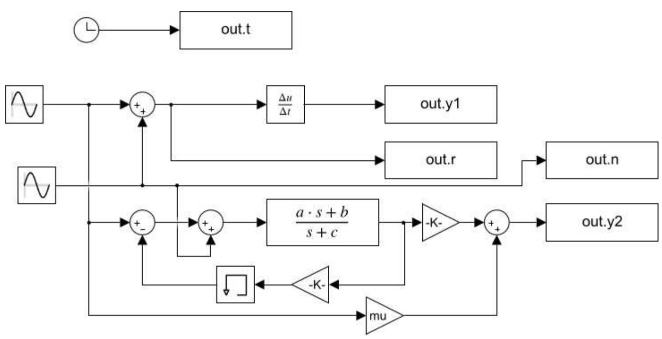

Fig. 12. Simulink Model (Example 3)

The input and the noise are shown in Figs. 13 and 14. The outputs of the respective systems are shown in Figs. 15, and 16.

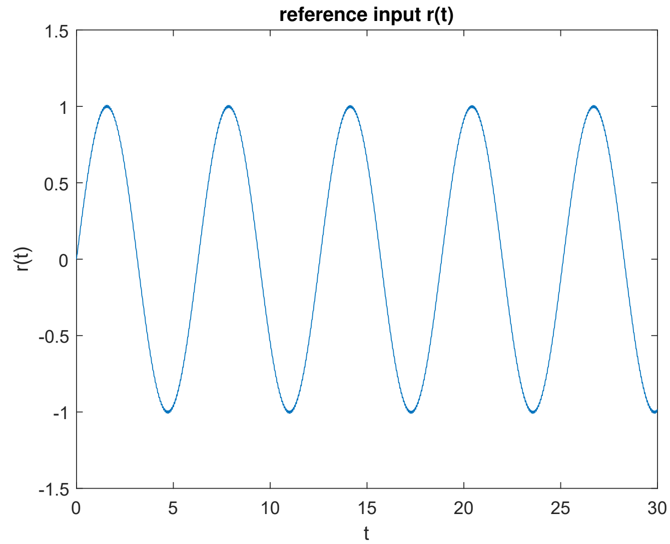

Fig. 13. Reference input $r(t) = \sin t$ (Example 3)

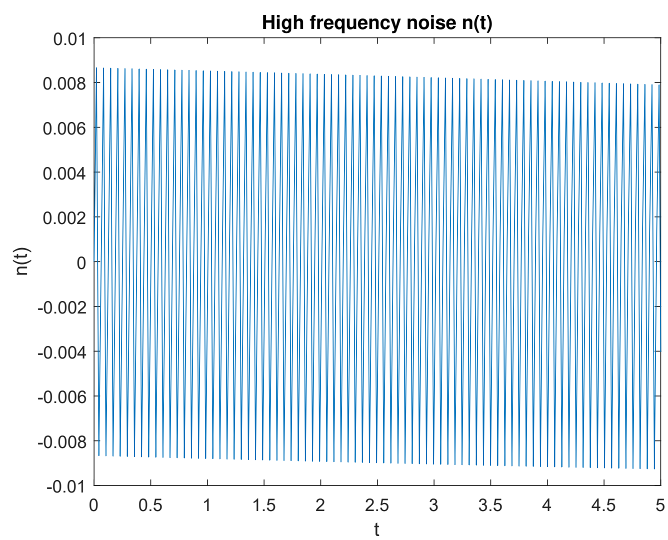

Fig. 14. High frequency noise $n(t) = 0.01\sin(100t)$ (Example 3)

Example 4: Finally, we consider a ramp input signal with small high frequency noise:

$$r(t) = t + A\sin(100t) \quad \text{for} \quad A = 0.01 \tag{11}$$

and it is shown in Fig. 17. The resulting outputs are shown in Figs. 18 and 19. These results show that first order implementations may be worth exploring as effective implementation blocks for differentiators and perhaps even integrators.

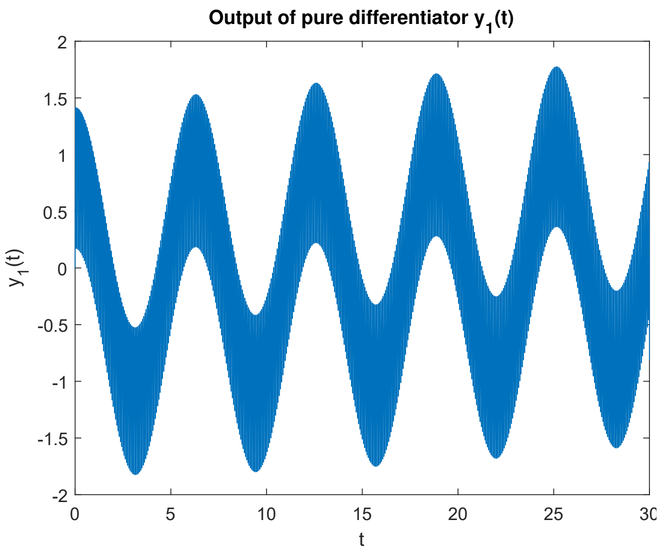

Fig. 15. Output of pure differentiator $y_1(t)$ (Example 3)

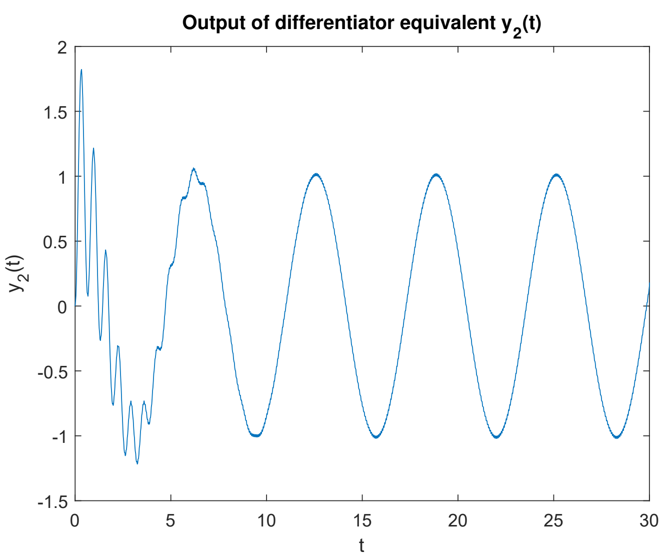

Fig. 16. Output of differentiator-equivalent $y_2(t)$ (Example 3)

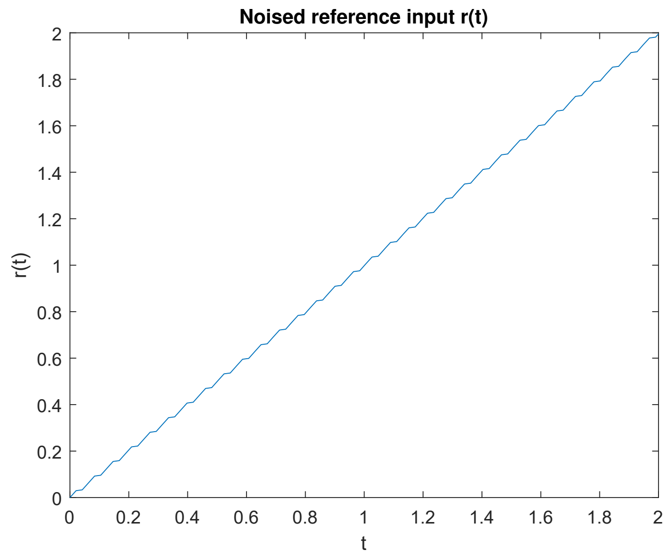

Fig. 17. Input corrupted with high frequency noise $r(t) = t + 0.01\sin(100t)$ (Example 4)

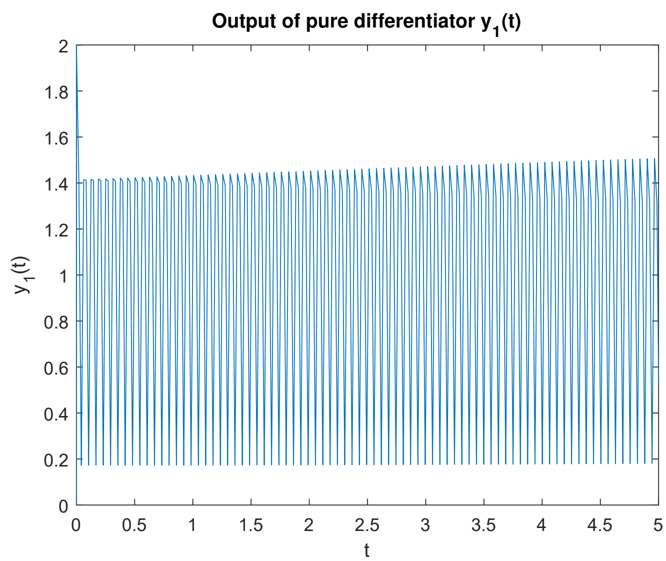

Fig. 18. Output of pure differentiator $y_1(t)$ (Example 4)

VI. Concluding Remarks

In this paper we have shown that an integrator as well as a differentiator may be implemented by using a given first order filter as a building block. Simulations show that such implementations may have superior noise rejection properties. Since the filter parameters a, b, c are almost arbitrary they may be used to further improve the properties of this implementation. These and related topics deserve further research.

References

[1] T. Kailath, *Linear Systems*, Prentice-Hall, 1980.
[2] N. S. Nice, *Control Systems Engineering*, Wiley, 2020.
[3] R. C. Dorf and R. H. Bishop, *Modern Control Systems (12th Edition)*, Pearson, 2011.
[4] P. J. Antsaklis, A. Astolfi, "Realizations in Linear Systems Theory," in: J. Baillieul J. and T. Samad (eds), *Encyclopedia of Systems and Control*, Springer, London, 2014. https://doi.org/10.1007/978-1-4471-5102-9_193-1
[5] W. A. Wolovich, *Automatic Control Systems: Basic Analysis and Design*, Saunders College Publishing, 1993.
[6] L. Thiele, "Design of sensitivity and round-off noise optimal state-space discrete systems," *International Journal of Circuits Theory and Applications*, vol. 12, issue 1, pp. 39 - 46, 1984.

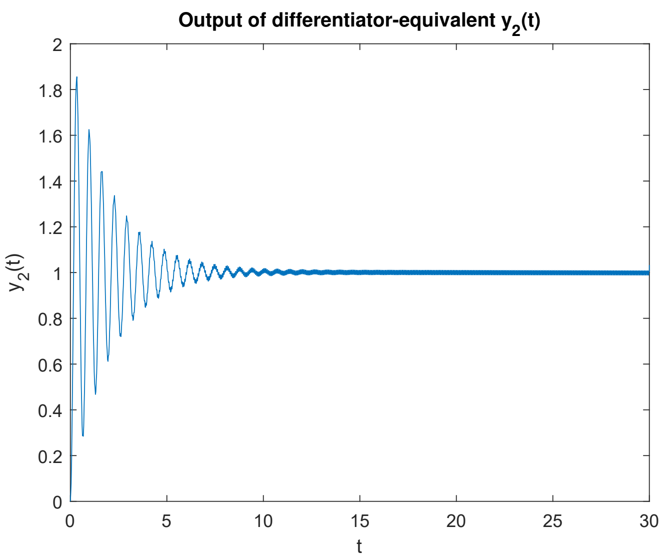

Fig. 19. Output of first order differentiator-equivalent $y_2(t)$ (Example 4)