# On the Energy or Power Efficiency of a Linear System

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**Abstract.** A linear time-invariant (LTI) system is usually characterized by its gains, which are ratios of various output and input signal magnitudes. When the main function of the system is energy conversion, one is more interested in the power or energy gain or loss between input and output terminals. Such a "gain" represents the *efficiency* of the system as an energy conversion device. In this paper, we show that the power or energy "gain" of an LTI system can be determined as a nonlinear function of three signal gains of the system. This results in an explicit formula for the power (energy) gain in terms of the system parameters. This can then be optimized over the design parameters to maximize the gain and thus the system energy efficiency. A detailed DC circuit example is included for illustration. The energy efficiency of an LTI system operating in a steady state in response to a sinusoidal signal is also determined.

**Keywords.** Energy, Power, Gain, Linear Systems, Energy Efficiency.

## 1. Introduction

A linear system is usually described in terms of its gains relating to various input and output signals [1],[2]. These gains may be constants as in a static linear time-invariant (LTI) system, such as a DC circuit, or transfer functions as in a dynamic system. When the system is to be evaluated in terms of its efficiency as an energy conversion device, it is important to know the "energy gain" or "power gain" of the system. In this paper, we introduce this latter type of "gain" and compute it for general static or dynamic LTI systems in terms of the system transfer functions. An illustrative example of a DC circuit is included. We also derive the energy efficiency of an LTI dynamic system operating in the sinusoidal steady-state at a single frequency. Although problems similar to this were treated by Kalman [3], we were unable to find explicit treatments of the type of results presented here related to "energy efficiency". On the other hand, the results presented here may have important applications in control system design (see [4]-[11]) where energy efficiency is important, such as PID control of air-conditioning systems and renewable energy systems (see [12],[13])

## 2. Main Results

*A. Power and Energy Gain of Static LTI Systems*

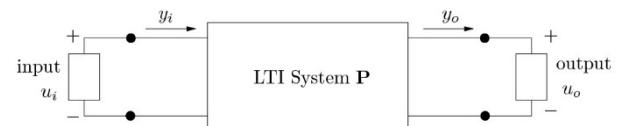Consider the following linear time-invariant (LTI) static system shown in Figure 1.



Figure 1. An LTI static system with input and output terminals

We identify the input terminal, denoted $i$, and the output terminal, denoted $o$, and the independent input variable $u_i$ and the dependent output variables $u_o$, $y_i$, and $y_o$. In this general representation, the $u_i$, $u_o$ are terminal variables and the $y_i$, $y_o$ are flow variables. For example, in a DC or AC circuit, the $u_i$, $u_o$ may represent voltages and $y_o$, $y_i$ may denote currents. In a hydraulic system, $u_o$, $u_i$ might denote pressure and $y_o$, $y_i$ flow rates. In a mechanical system, the $u$'s might denote force or torque and the $y$'s might denote linear or angular velocity.

With this background, the *instantaneous* input power at terminal $i$ is:

$$P_i(t) := u_i(t)y_i(t) \qquad (1)$$

and the instantaneous output power at terminal $o$ is:

$$P_o(t) = u_o(t)y_o(t) \qquad (2)$$

Since the system in Figure 1 is a linear time-invariant *static* system, we must have:

$$u_o(t) = K_o u_i(t) \qquad (3)$$
$$y_o(t) = L_o u_i(t) \qquad (4)$$

and

$$y_i(t) = L_i u_i(t). \qquad (5)$$

Therefore, the instantaneous power output is
$$P_o(t) = K_o L_o u_i^2(t) \qquad (6)$$
and the instantaneous power output is
$$P_i(t) = L_i u_i^2(t) \qquad (7)$$

This leads to the following result:

**Theorem 1**
$$\frac{P_o(t)}{P_i(t)} = \frac{K_o L_o}{L_i} =: K_P \qquad (8)$$

*Proof:* Follows from (1) – (7).

**Remark 1:** In (8), $K_P$ is the power gain of the system in Figure 1, regarded as a power converter from input to output.

**Remark 2:** The energy gain of the system may also be defined. Consider the energy over a finite time interval, and let the input and output be defined over $[0, T]$. By definition:
$$E_i = \int_0^T P_i(t) dt \qquad (9)$$
$$E_o := \int_0^T P_o(t) dt \qquad (10)$$
The following result establishes *the energy gain* of the system.

**Theorem 2**
$$\frac{E_o}{E_i} =: K_E = K_P \qquad (11)$$
*Proof:* Substituting (8) into (9) and (10), we have
$$\begin{aligned} E_o &= \int_0^T K_P P_i(t) dt \\ &= K_P \int_0^T P_i(t) dt \qquad (12) \\ &= K_P E_i. \end{aligned}$$

**Remark 3:** (11) shows that $K_P$ is also the energy gain of the system in Figure 1.

**Remark 4:** In general, $K_P$ may be positive or negative and $|K_P|$ may be less than equal to or greater than unity. The role of the system in Figure 1, is accordingly that of an *energy* amplifier, absorber or attenuator.

**Remark 5:** Note that even though the system in Figure 1 is linear, $K_P$ is a *nonlinear* function of the system gains, $K_o$, $L_o$, and $L_i$.

In general, the system in Figure 1 may have some adjustable parameters, denoted by the vector $\mathbf{q}$. Thus,
$$K_E(\mathbf{p}) = \frac{K_o(\mathbf{q}) L_o(\mathbf{q})}{L_i(\mathbf{q})} \qquad (13)$$
and a design problem of importance is
$$\sup_{\mathbf{p} \in \Omega} K_E(\mathbf{q}) \qquad (14)$$
which optimizes the *energy efficiency* of the system.

In the following, a simple DC circuit is solved to illustrate these ideas.
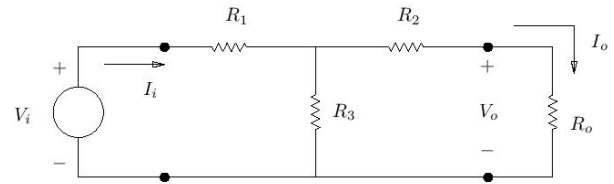
*B. An Example: DC Circuit*



Figure 2.  A DC Circuit

Input Power: $\qquad\qquad V_i I_i =: P_i \qquad (15a)$
Output Power: $\qquad\qquad V_o I_o =: P_o \qquad (15b)$
Power Gain: $\qquad\qquad \dfrac{P_o}{P_i} =: K_P \qquad (16)$

**Calculation of $K_P$**
System equations:
$$\begin{bmatrix} R_1 & R_0 + R_2 \\ R_1 + R_3 & -R_3 \end{bmatrix} \begin{bmatrix} I_i \\ I_o \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} V_i \qquad (17)$$

$$I_i = \frac{\begin{vmatrix} 1 & R_0 + R_2 \\ 1 & -R_3 \end{vmatrix}}{|A|} V_i = \frac{|A_i|}{|A|} V_i \qquad (18)$$

$$I_o = \frac{\begin{vmatrix} R_1 & 1 \\ R_1 + R_3 & 1 \end{vmatrix}}{|A|} V_i = \frac{|A_o|}{|A|} V_i \qquad (19)$$

$$V_o = R_o I_o = \frac{R_o |A_o|}{|A|} V_i \qquad (20)$$

Therefore,
$$P_o = V_o I_o = R_o I_o^2 = R_o \frac{|A_o|^2}{|A|^2} V_i^2 \qquad (21)$$

$$P_i = V_i I_i = \frac{|A_i|}{|A|} V_i^2 \qquad (22)$$

and
$$\begin{aligned} K_P &= \frac{P_o}{P_i} \\ &= \frac{R_o |A_o|^2}{|A_i||A|} \qquad (23) \\ &= \frac{R_o R_3^2}{(R_1 R_3 + R_1 R_o + R_1 R_2 + R_3 R_o + R_2 R_2)(R_o + R_2 + R_3)} \end{aligned}$$

In the following, we consider the problem of maximizing $K_P$ over some design variables to optimize energy output. In the following we consider the problem of maximizing $K_P$ over some design variables to optimize energy output. For illustration, we fixed $R_1 = R_2 = 0.1$. Figure 3 show $K_P$ values over $0.1 \le R_3 \le 5$ at some selected values of $R_o$.
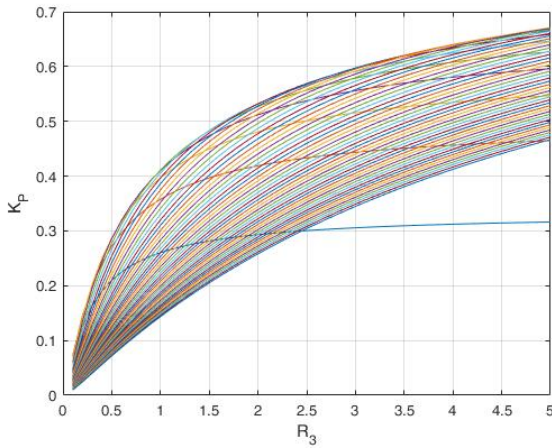
Figure 3. $R_3$ vs. $K_P$ with selected values of $R_o \in [0.1, 5]$ with $R_1 = R_2 = 0.1$

Figure 4 shows that

$$K_P(R_o) = \sup_{R_3 \in [0.1,5]} K_P(R_0, R_3)$$

with $R_1 = R_2 = 0.1$. A 3D version of this is also shown in Figure 5.
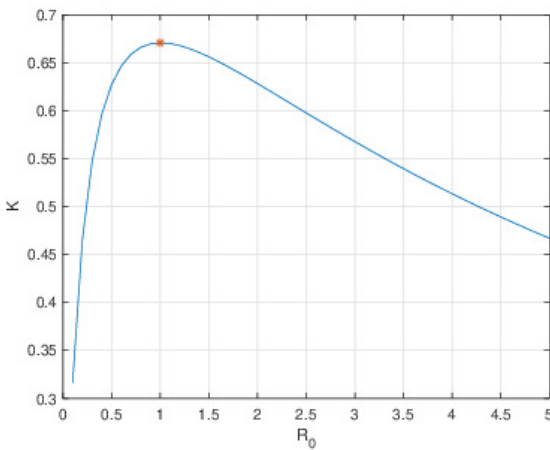


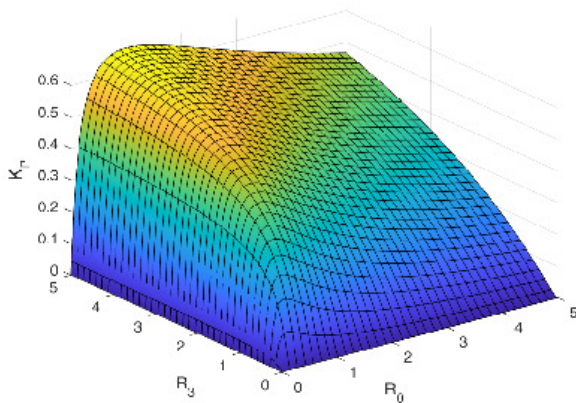Figure 4. $R_o$ vs. $\sup_{R_3 \in [0.1,5]} K_P(R_0, R_3)$



Figure 5. $K_P$ vs. $(R_o, R_3)$

## C. Power or Energy Gain of Dynamic Systems
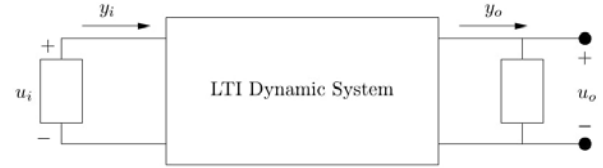
Consider the linear time-invariant dynamic system in Figure 6:



Figure 6. An LTI dynamic system

Taking Laplace transforms, we have

$$\begin{aligned} u_o(s) &= H_o(s)u_i(s) \\ y_o(s) &= L_o(s)u_i(s) \\ y_i(s) &= L_i(s)u_i(s) \end{aligned} \tag{24}$$

where $H_o(s)$, $L_o(s)$, $L_i(s)$ denote transfer functions. To discuss power and energy relations, we assume that the input signal is a sinusoidal signal with frequency $\omega$ and the system in Figure 6 is operating in the sinusoidal steady-state. Let

$$\begin{aligned} H_o(j\omega) &= H_o(\omega)e^{j\theta_o}(\omega) \\ L_o(j\omega) &= L_o(\omega)e^{j\varphi_o}(\omega) \\ L_i(j\omega) &= L_i(\omega)e^{j\varphi_i}(\omega). \end{aligned} \tag{25}$$

The average input and output real power are:
$$P_i(\omega) = \mathrm{Re}[u_i(j\omega)y_i^*(j\omega)] \tag{26}$$
$$P_i(\omega) = \mathrm{Re}[u_i(j\omega)y_i^*(j\omega)] \tag{27}$$

where Re[.] denotes the real part of [.]. Without loss of generality, let

$$u_i(j\omega) = u_i(\omega)e^{j0} \tag{28}$$

where $u_i(\omega)$ denotes the root mean square (RMS) value of the amplitude of the sinusoidal input. Then

$$\begin{aligned} y_o(j\omega) &= L_o(j\omega)u_i(j\omega) \\ u_o(j\omega) &= H_o(j\omega)u_i(j\omega) \\ y_i(j\omega) &= L_i(j\omega)u_i(j\omega). \end{aligned} \tag{29}$$

Then, the average output power is:

$$\begin{aligned} P_o(\omega) &= \mathrm{Re}\big[|H_o(\omega)||L_o(\omega)|e^{j(\theta_o - \varphi_o)}\big]|u_i(\omega)|^2 \\ &= H_o(\omega)L_o(\omega)cos(\theta_o - \varphi_o)|u_i(\omega)|^2 \end{aligned} \tag{30}$$

and the average input power, is:

$$\begin{aligned} P_i(\omega) &= \mathrm{Re}\big[|L_i(\omega)|e^{-j\varphi(\omega)}\big]|u_i(\omega)|^2 \\ &= L_i(\omega)cos\varphi_i(\omega)|u_i(\omega)|^2 \end{aligned} \tag{31}$$

This leads to the following result on the average "power gain" of the system.

**Theorem 3.** The average power gain of the system in Figure 6, operating in steady-state at the fixed frequency $\omega$, is:

$$\begin{aligned} K_P(\omega) &:= \frac{P_o(\omega)}{P_i(\omega)} \\ &= \frac{H_o(\omega)L_o(\omega)cos(\theta_o - \varphi_o)}{L_i(\omega)cos\varphi_i} \end{aligned} \tag{32}$$

*Proof.* Follows from (24) – (31) above.

Let $E_o$ and $E_i$ denote the output and input energies of the system, over a predefined time period, say $0$ to $T$, Then:

$$E_o(\omega) = \int_0^T P_o(\omega)dt = P_o(\omega)T$$

$$E_i(\omega) = \int_0^T P_i(\omega)dt = P_o(\omega)T. \qquad (33)$$

The following result on the "energy gain" of a system can this be enunciated:

**Corollary 1.** Then the energy gain or efficiency of the system in Figure 6 is:

$$K_E(\omega) := \frac{E_o(\omega)}{E_i(\omega)} = K_P(\omega). \qquad (34)$$

*Proof.* Follows from (32) – (33) above.

As before the problem of optimizing $K_E(\omega)$ over some design parameters represents a problem of interest for maximizing the energy efficiency of the system.

## 3. Systems with Arbitrary Inputs

In renewable energy systems, the source inputs such as wind, solar heat or ocean waves are often sporadic and irregular. In such cases, it is useful to characterize the input frequency content by taking the Fourier Transform of a typical input. This allows for the generalization of the results of the last section, as derived below.

Referring to Figure 6, let $u_i^*(t)$ for $t \geq 0$ denote a typical input. Then its Fourier transform $U_i(j\omega)$ is:

$$U_i(j\omega) := F[u_i^*(t)] := A_i(\omega)e^{i\theta_i(\omega)}, \; for \; \omega \epsilon \Omega \qquad (35)$$

where $\Omega$ is the frequency band of the inputs.
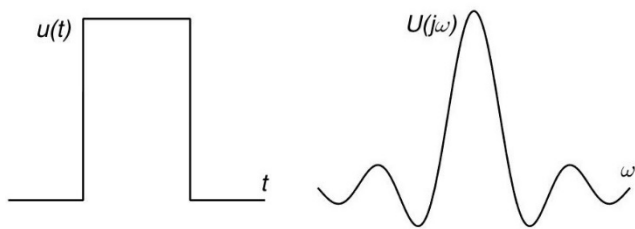


Figure 7. A typical input and its Fourier transform

Therefore,

$$y_o(j\omega) = L_o(j\omega)U_i(j\omega)$$

$$= L_o(\omega)A_i(\omega)e^{j(\phi_o(\omega)+\theta_i(\omega))} \qquad (36)$$

and similarly,

$$u_o(j\omega) = H_o(\omega)A_i(\omega)e^{j(\theta_o(\omega)+\theta_i(\omega))} \qquad (37)$$

$$y_i(j\omega) = L_i(\omega)A_i(\omega)e^{j(\phi_i(\omega)+\theta_i(\omega))} \qquad (38)$$

for $\omega \epsilon \Omega$.

Now the power output and input of the system may be written down as:

$$P_o = \int_\Omega \text{Re}[U_o(j\omega)y_o^*(j\omega)]d\omega \qquad (39)$$

$$P_i = \int_\Omega \text{Re}[U_i(j\omega)y_i^*(j\omega)]d\omega \qquad (40)$$

From (36) – (40), we obtain:

$$P_o = \int_\Omega H_o(\omega)L_o(\omega)A_i^2(\omega)\cos(\theta_o(\omega) - \phi_o(\omega))d\omega \qquad (41)$$

and

$$P_i = \int_\Omega L_i(\omega)A_i^2(\omega)\cos(\phi_i(\omega))d\omega \qquad (42)$$

From (41) and (42), the power gain or energy gain can be determined as before:

$$K_P = \frac{P_o}{P_i} = K_E. \qquad (43)$$

In practice, the Fourier transform in (35) should be determined by considering several inputs and averaging them.

## 4. Discussion and Future Research

The energy efficiency introduced here could be an important performance index to evaluate the efficacy of energy conversion systems. These could include renewable energy systems such as wind energy, solar energy, energy from ocean waves as well as electromechanical systems, and thermal energy.

For dynamic systems, we derived the energy efficiency for a single frequency sinusoidal input. For systems containing a discrete set of frequencies or even a continuum of frequencies, the energy gain could be averaged over a prescribed frequency band taking into account the Fourier spectrum of the input signal. Although these derivations are based on the assumption of a "linear" energy conversion system the basic ideas can be extended to nonlinear systems as well. Another important area of research would be the extension of these concepts to the case where there are multiple independent inputs and outputs. In each of these cases, the problem of interest would be the optimization of the energy efficiency over a set of design parameters varying in prescribed intervals.

## 5. Concluding Remarks

In this paper, we have formulated the definition of the energy or power gain of a linear time-invariant static or dynamic system. This is distinct from the $L_2$ gain of a system. An example has been included to demonstrate how such an energy efficiency measure could be optimized over some parameters for a DC circuit. The power gain of a dynamic system operating at a fixed frequency was also defined and determined in terms of

system transfer functions. Further work is needed to extend these ideas to systems with arbitrary inputs and also to nonlinear and multi-input, multi-output systems. These would have useful applications in the design of energy conversion systems.

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