QUASICONFORMAL MAPS WITH THIN DILATATIONS

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Abstract. We give an estimate that quantifies the fact that a normalized quasiconformal map whose dilatation is non-zero only on a set of small area approximates the identity uniformly on the whole plane. The precise statement is motivated by applications of the author's quasiconformal folding method for constructing entire functions; in particular an application to constructing transcendental wandering domains given by Fagella, Godillon and Jarque [7].

1. Introduction

A quasiconformal mapping $F : \mathbb{C} \to \mathbb{C}$ is a homeomorphism that is absolutely continuous on almost all horizontal and vertical lines, and whose partial derivatives satisfy $F_{\overline{z}} = \mu F_z$ for some complex valued, measurable function μ with $\|\mu\|_\infty = k < 1$, called the complex dilatation of f. For the basic properties of quasiconformal maps, see Ahlfors' book [1].

If $\mu = 0$ then F is a conformal homeomorphism of $\mathbb{C}$ to itself, and hence it is linear. If the complex dilatation μ is small, then we expect F to be close to linear. There are at least two reasonable senses in which we can ask μ to be small: that $\|\mu\|_\infty$ is small or that the set $\{z : \mu(z) \neq 0\}$ is small. In this note we consider the latter possibility.

To be more precise, we say a measurable set $E \subset \mathbb{C}$ is (ϵ, h)-thin if $\epsilon > 0$ and

$$\text{area}(E \cap D(z, 1)) \leq \epsilon h(|z|)$$

for all $z \in \mathbb{C}$, where $h : [0, \infty) \to [0, \pi]$ is a bounded, decreasing function, such that

$$\int_0^\infty h(r) r^n dr < \infty,$$

for every $n > 1$. If $a > 0$, the function $h(r) = \exp(-ar)$ satisfies this condition, and this example suffices for many applications.

Date: July 2014, revised Jan 2021.

1991 *Mathematics Subject Classification.* Primary: 30C62 Secondary: 37F30 .

Key words and phrases. quasiconformal maps, conformal modulus, quasiconformal folding, Pompeiu's formula, holomorphic dynamics.

The author is partially supported by NSF Grant DMS 1906259.

Recall that a quasiconformal map $F : \mathbb{C} \to \mathbb{C}$ is often normalized by post-composing by a conformal linear map in one of two ways. First, we can assume $F(0) = 0$ and $F(1) = 1$. We call this the **2-point normalization**. Second, if the dilatation of F is supported on a bounded set, then F is conformal in a neighborhood of ∞ and then we can choose R large and post-compose with a linear conformal map so that

$$|F(z) - z| = O(\frac{1}{|z|}),$$

for $|z| > R/2$. We say that such an F is normalized at ∞. This is also called the **hydrodynamical normalization** of F. We will first prove an estimate for the hydrodynamical normalization and then deduce one for the 2-point normalization.

Theorem 1.1. *Suppose $F : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal, and $E = \{z : \mu(z) \neq 0\}$ is bounded (so F is conformal near ∞) and F is normalized so*

$$|F(z) - z| \leq M/|z|,$$

near ∞. Assume E is (ϵ, h)-thin. Then for all $z \in \mathbb{C}$,

$$|F(z) - z| \leq \frac{\epsilon^\beta}{|z| + 1},$$

where $\beta > 0$ depends only on K and h. In particular, as $\epsilon \to 0$, F converges uniformly to the identity on the whole plane.

From this we will deduce the following version for the 2-point normalization. This estimate is stated as Theorem 2.5 in [7] by Fagella, Godillon and Jarque, based on "personal communication" with the author and the main goal of this paper is to provide a concrete citation for this result.

Corollary 1.2. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal, $f(0) = 0$, $f(1) = 1$, and $E = \{z : \mu(z) \neq 0\}$ is (ϵ, h)-thin. Then*

$$(1 - C\epsilon^\beta)|z - w| - C\epsilon^\beta \leq |f(z) - f(w)| \leq (1 + C\epsilon^\beta)|z - w| + C\epsilon^\beta, \tag{1.1}$$

where C and β only depend on $\|\mu\|_\infty$ and h.

Similar estimates are known, e.g., compare to the well known result of Teichmüller and Wittich (e.g., Theorem 7.3.1 of [9], [13], [14]) or estimates of Dyn'kin [6]. The version stated above is intended for specific applications to holomorphic dynamics involving the author's quasiconformal folding technique of constructing entire functions, introduced in [3]. Given an infinite tree T in the plane satisfying certain geometric conditions, this method constructs a quasiregular function g on the whole plane that is holomorphic outside a (usually small) neighborhood U of the tree. Then $f = g \circ \varphi^{-1}$ is entire where φ is the quasiconformal map whose dilatation is given by $\mu = g_{\overline{z}}/g_z$ and is supported in U; such a φ exists by the measurable Riemann mapping theorem. In applications, g is usually constructed to have certain properties and we want $f = g \circ \varphi^{-1}$ to have the same or similar properties. Thus we usually want φ to be close to the identity. In many applications of quasiconformal folding, the neighborhood U can be chosen to be very small, e.g., it often is (ϵ, e^{-r}) thin, which is why the estimates above are helpful. Quasiconformal folding has been used to construct various examples in complex analysis and holomorphic dynamics, e.g., [2], [4], [5], [8], [10], [11], [12], [15]. Often estimating the correction map φ is the hardest part of applying the folding method, and these papers sometimes use weaker versions of the estimates given here, or leave some details to the reader. The goal of this note is to provide a complete proof of the estimates needed in many applications of the folding theorem. The paper [7] uses Corollary 1.2 as part of a construction of two entire functions, neither of which has a wandering domain, but whose composition does have a wandering domain. That paper also provides addition information about the wandering domains constructed in my paper [3].

I thank Xavier Jarque for helpful comments on a draft of this paper that clarified the notation and several of the arguments. I also thank the two anonymous referees for their thoughtful comments and numerous suggestions to improve the paper. One of the referees suggested the results in this paper might extend to higher dimensions. This seems reasonable, and the parts of the proof concerning modulus and L^p estimates should extend, but it is not obvious (to the author) how to generalize the arguments using Pompeiu's formula or properties of holomorphic functions. We leave this interesting question open for future investigation.

2. Quasiconformal maps and conformal modulus

Here we review a few basic facts about quasiconformal maps and conformal modulus that we will need. All these results can be found in Ahlfors' book [1].

Lemma 2.1 (Shapes of quasicircles)**.** *For each $K \geq 1$ there is a $C = C(K) < \infty$ so that the following holds. If $F : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal and γ is a circle centered at a point $w \in \mathbb{C}$, then there is an $r > 0$ so that $F(\gamma) \subset \{z : r \leq |z - F(w)| \leq Cr\}$.*

Theorem 2.2 (Bojarski's theorem). *If $1 \le K < \infty$, there is a $p = p(K) > 2$ and $A, B < \infty$ so that the following holds. If $F : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal, and $Q \subset \mathbb{C}$ is a square, then*

$$(\frac{1}{\text{area}(Q)} \int_Q |F_z|^p dxdy)^{1/p} \le A(\frac{1}{\text{area}(Q)} \int_Q |F_z|^2 dxdy)^{1/2} \le B \frac{\text{diam}(F(Q))}{\text{diam}(Q)}$$

Lemma 2.3 (Pompeiu's formula). *If Ω has a piecewise C^1 boundary and F is quasiconformal on Ω, then*

$$F(w) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{F(z)}{z-w} dz - \frac{1}{\pi} \int_\Omega \frac{F_{\overline{z}}}{z-w} dxdy. \tag{2.1}$$

Suppose Ω is a planar domain and suppose Γ is a path family in Ω, i.e., a collection of locally rectifiable curves in Ω. A non-negative Borel function ρ is called admissible for Γ if $\int_\gamma \rho ds \ge 1$ for every curve $\gamma \in \Gamma$. The modulus of Γ (also called conformal modulus) is the infimum of $\int_\Omega \rho^2 dxdy$ over all admissible ρ for Γ and is denoted $\text{mod}(\Gamma)$. The reciprocal of the modulus is called the extremal length of Γ. A quasiconformal map F of Ω with complex dilatation satisfying $\|\mu\|_\infty = k < 1$ has the property that it can change conformal modulus of a path family in Ω by at most a factor of $K = (k+1)/(k-1)$.

If Ω is a topological annulus in the plane with boundary components γ_1, γ_2 that are closed Jordan curves, then $\text{mod}(\Omega)$ refers to the modulus of the path family in Ω that separates the boundary components. This is the same as the extremal length of the path family that connects the boundary components (also called the extremal distance between the boundary components). If $A(a, b) \equiv \{z : a < |z| < b\}$ then it is standard fact that $\text{mod}(A) = 2\pi/\log \frac{b}{a}$. Let

$$K_F = \frac{|F_z| + |F_{\overline{z}}|}{|F_z| - |F_{\overline{z}}|},$$

$$J_F = |F_z|^2 - |F_{\overline{z}}|^2 = (|F_z| - |F_{\overline{z}}|)(|F_z| + |F_{\overline{z}}|),$$

denote the distortion and Jacobian functions of F respectively. Note that $K_F \ge 1$ and F is conformal if and only if $K_F \equiv 1$. If $K_F \le K$, then F can distort the modulus of an annulus by a factor of most K, and hence for a map between round annuli, the ratio of radii changes by a most a power K. In the rest of this section we show that better estimates are possible of $K_F \le K$ everywhere, but $K_F \approx 1$ "most places". In what follows $z = x + iy = re^{i\theta}$ and area measure is denoted by $dxdy$ or $rdrd\theta$.

Lemma 2.4. *Suppose F is a K-quasiconformal map from $A_m = A(1, e^m)$ onto $A_M = A(1, e^M)$. Then*

$$M \geq m - \frac{1}{2\pi} \int_{A(1,e^m)} (K_F(z) - 1) \frac{dxdy}{r^2}.$$

Proof. Let Γ_M be the path family connecting the boundary components of A_M. If ρ is admissible for A_m, then $\widetilde{\rho}(F(z)) = \rho(z)/(|F_z| - |F_{\overline{z}}|)$ is admissible for A_M, hence it is one of the metrics in the infimum defining $\mathrm{mod}(\Gamma_M)$. Therefore

$$\begin{aligned} \mathrm{mod}(F(\Gamma_m)) &\leq \int_{A_m} \rho(z)^2 \frac{1}{(|F_z| - |F_{\overline{z}}|)^2} J_F dxdy \\ &= \int_{A_m} \rho(z)^2 \frac{1}{(|F_z| - |F_{\overline{z}}|)^2} (|F_z|^2 - |F_{\overline{z}}|^2) dxdy \\ &= \int_{A_m} \rho(z)^2 \frac{|F_z| + |F_{\overline{z}}|}{|F_z| - |F_{\overline{z}}|} dxdy \\ &= \int_{A_m} \rho(z)^2 K_F(z) dxdy. \end{aligned}$$

Applying this with the admissible metric $\rho(z) = \frac{1}{m|z|}$, we get

$$\begin{aligned} \frac{2\pi}{M} = \mathrm{mod}(F(\Gamma_m)) &\leq \frac{1}{m^2} \int_{A_m} \frac{K_F(z)}{|z|^2} dxdy \\ &= \frac{1}{m^2} \left[\int_{A_m} \frac{K_F(z) - 1}{|z|^2} dxdy + \int_{A_m} \frac{1}{|z|^2} dxdy \right] \\ &= \frac{1}{m^2} \int_{A_m} \frac{K_F(z) - 1}{|z|^2} dxdy + \frac{2\pi}{m}. \end{aligned}$$

Rearranging gives

$$m - M \leq \frac{M}{2\pi m} \int_{A_m} \frac{K_F(z) - 1}{|z|^2} dxdy,$$

or

$$M \geq m - \frac{M}{2\pi m} \int_{A_m} \frac{K_F(z) - 1}{|z|^2} dxdy.$$

Since $K_F \geq 1$, the integral is non-negative. So if $M > m$, the lemma is trivially true. If $M \leq m$, the inequality above becomes

$$M \geq m - \frac{1}{2\pi} \int_{A_m} \frac{K_F(z) - 1}{|z|^2} dxdy.$$

Thus in either case the lemma holds. □

Lemma 2.5. *Suppose F is a K-quasiconformal map from $A_m = A(1, e^m)$ to $A_M = A(1, e^M)$. Then*
$$M \leq m + \frac{1}{2\pi} \int_{A_m} (K_F - 1) \frac{dxdy}{r^2}.$$

Proof. If we cut A_m with a radial slit and let $G = \log(F)$, then G maps A_m to a generalized quadrilateral with two vertical sides on $V_0 = \{x = 0\}$ and $V_M = \{x = M\}$. This quadrilateral has area $2\pi M$. Each radial segment in A_m maps to a curve connecting V_0 and V_m, so the image has length that least M. So if we integrate over the radial segments in A_m, we get
$$M \leq \int_1^{e^m} (|G_z| + |G_{\overline{z}}|) dr$$
so integrating over all angles and using $r dr d\theta = dxdy$ gives
$$2\pi M \leq \int_0^{2\pi} \int_1^{\exp(m)} (|G_z| + |G_{\overline{z}}|) dr d\theta \leq \int_{A_m} (|G_z| + |G_{\overline{z}}|) \frac{dxdy}{r}.$$
Thus by Cauchy-Schwarz,
$$\begin{aligned} (2\pi M)^2 &\leq \left(\int_{A_m} (|G_z| + |G_{\overline{z}}|)(|G_z| - |G_{\overline{z}}|) dxdy \right) \left(\int_{A_m} \frac{|G_z| + |G_{\overline{z}}|}{|G_z| - |G_{\overline{z}}|} \frac{dxdy}{r^2} \right) \\ &\leq \left(\int_{A_m} J_G dxdy \right) \left(\int_{A_m} K_G \frac{dxdy}{r^2} \right) \\ &\leq 2\pi M \left(\int_{A_m} K_F \frac{dxdy}{r^2} \right), \end{aligned}$$

where in the last line we have used the facts that $G(A_m)$ has area $2\pi M$ and $K_G = K_F$ (since $\log z$ is conformal on the slit annulus). Thus
$$\begin{aligned} M &\leq \frac{1}{2\pi} \int_{A_m} 1 + (K_F(z) - 1) \frac{dxdy}{r^2} \\ &= m + \frac{1}{2\pi} \int_{A_m} (K_F(z) - 1) \frac{dxdy}{r^2}. \end{aligned}$$
□

The following simply combines the last two results.

Corollary 2.6. *Suppose F is a K-quasiconformal map from $A_m = A(1, e^m)$ to $A_M = A(1, e^M)$. Then*
$$M = m + O\left(\frac{1}{2\pi} \int_{A_m} \frac{K_F(z) - 1}{r^2} dxdy. \right)$$

A special case of this is:

Corollary 2.7. *Suppose F is a K-quasiconformal map from $A_m = A(1, e^m)$ to $A_M = A(1, e^M)$. Suppose μ is the dilatation of F, that $E = \{z : \mu(z) \neq 0\}$ and that $E_k = E \cap \{e^{k-1} < |z| < e^k\}$. If we choose an integer n so that $m \leq 2^n$, then*

$$M = m + O\left((K-1)\sum_{k=1}^{n} e^{-2k}\text{area}(E_k)\right).$$

3. Dilatations with thin support

Next we apply these estimates to quasiconformal maps with dilatations that have small support in a precise sense.

Lemma 3.1. *Suppose F is a K-quasiconformal map with dilatation μ, that μ has bounded support, and that F has the hydrodynamical normalization at ∞. Let $E = \{z : \mu(z) \neq 0\}$ and suppose for some $t > 0$, E satisfies*

$$\int_{E\setminus D(w,t)} \frac{dxdy}{|z-w|^2} \leq a,$$

for every $w \in \mathbb{C}$. Then there is a $C = C(K, a) = O(e^{O(Ka)}) < \infty$, depending only on K and a, so that for every $w \in \mathbb{C}$ and $r \geq t$,

$$\frac{1}{C} \leq \frac{\text{diam}(F(D(w,r))}{r} \leq C. \tag{3.1}$$

Proof. We need only prove this for $r = t$ since for $r > t$, we can simply apply the lemma after setting $t = r$ (the integral just gets smaller). Moreover, the mapping $G(z) = F(tx)/t$, satisfies the same estimates as F, but with t replaced by 1. If we prove the lemma for G, then it follows for F, so it suffices to assume $t = 1$.

By the normalization assumption we can choose $R > 100$ so large that $|F(z) - z| \leq 1/2$, for $|z| > R/8$. Thus if $|w| > R/4$, the circle of radius 1 around w is mapped to a set of diameter at least 1 and at most 3. Therefore we may assume $|w| \leq R/4$. Fix such a w. Then the circle of radius R around w lies in $\{|z| > R/2\}$, where we know $F(z)$ is close to z.

Let $m = \log R$, so $R = e^m$, and consider the annulus $A = \{z : 1 < |z - w| < e^m\}$. $F(A)$ is a topological annulus and can be conformally mapped to $A_M = \{1 < |z| < e^M\}$ for some $M > 1$. By Corollary 2.6,

$$M = m + O\left(\frac{1}{2\pi}\int_{A_m} \frac{K_f - 1}{|z-w|^2}dxdy\right).$$

By our assumptions, this becomes

$$M = m + O\left(\frac{K-1}{2\pi}\int_{A_m} \mathbf{1}_E(z)\frac{dxdy}{|z-w|^2}\right) = m + O(Ka),$$

where $\mathbf{1}_E$ denotes the indicator function of E (the function that is one on E and zero off E) and we have used the fact that E has finite planar area and $|z-w|^{-1} \leq 1$ on A_m (recall w is the center of the annulus and the inner radius is at least 1.).

By Lemma 2.1, the boundary components of $F(A_m)$ are closed curves that are each contained in annuli of bounded modulus, depending only on K. Each annulus has boundary components are two concentric circles. Thus $F(A_m)$ is contained in a topological annulus A' with circular boundaries γ_1, γ_2 (not necessarily concentric) whose diameters are comparable to the diameters of the boundary components of $F(A_m)$. By monotonicity of modulus, the modulus of the annulus A' (denoted $M'/2\pi$) is larger than the modulus $M/2\pi$ of $F(A)$, hence $M' \geq M$. Moreover, we claim

$$M' \leq \log\frac{\text{diam}(\gamma_2)}{\text{diam}(\gamma_1)}.$$

This is well known to hold with equality if the circles γ_1, γ_2 are concentric. If they are not, then we can apply a Möbius transformation that maps the outer circle, γ_2, to itself and moves the inner circle, γ_1 to circle concentric with γ_2. This makes the Euclidean diameter of γ_1 larger and preserves the modulus between the circles, and this proves the claimed inequality. Thus

$$M \leq M' \leq \log\frac{\text{diam}(\gamma_2)}{\text{diam}(\gamma_1)},$$

or

$$\text{diam}(\gamma_1) \leq \text{diam}(\gamma_2)\cdot e^{-M} = \text{diam}(\gamma_2)\cdot e^{-m+O(KA)}.$$

Since $|F(z)-z| \leq 1/2$ on $\{|z| = R\}$ we know $\text{diam}(\gamma_2) \simeq R = e^m$. Using this and the fact that $M = m + O(Ka)$ gives

$$\text{diam}(F(\{|z-w| = 1\})) \simeq \text{diam}(\gamma_1) = O(e^{Ka}).$$

This is the right-hand side of (3.1).

To the other side of (3.1), we choose γ_1, γ_2 to be circles that bound an annulus inside $F(A_m)$, again with diameters comparable to the diameters of the corresponding components of $\partial F(A_m)$. We then use monotonicity again, and argue as before, but now we note that since F is close to the identity for $|z| > R/2$, the curve γ_1 is not too close to γ_2, i.e., the distance between them is comparable to R. Thus in the argument above, where we moved γ_1 be be concentric with γ_2, its Euclidean diameter was only changed by a bounded factor. Thus

$$\operatorname{diam}(\gamma_1) \gtrsim \operatorname{diam}(\gamma_2) \cdot e^{-M} = \operatorname{diam}(\gamma_2) \cdot e^{-m-O(Ka)} \gtrsim e^{-O(Ka)}.$$

This proves the lemma. □

If F is as above, then Theorem 2.2 says there is a $p = p(K) > 2$ so that $\|F_z\|_p$ is uniformly bounded on every unit radius disk. Thus if a region Y can be covered by n such disks, then

$$\|F_z \chi_Y\|_p = O(n^{1/p}) \tag{3.2}$$

with a uniform constant. If Y is a disk of radius $r \geq 1$, it can be covered by $O(r^2)$ unit disks, so we get the following.

Corollary 3.2. *If F satisfies the conditions of Lemma 3.1, $r \geq 1$, and $p = p(K) > 2$ is as above, then $\|F_z \cdot \mathbf{1}_{D(z,r)}\|_p = O(r^{2/p})$ uniformly for all $z \in \mathbb{C}$.*

Proof of Theorem 1.1. Suppose the support of μ is contained in $D(0, R)$. The main idea is to use the Pompeiu formula

$$F(w) = \frac{1}{2\pi i} \int_{|z|=r} \frac{F(z)}{z - w} dz - \frac{1}{\pi} \int_{|z|<r} \frac{F_{\overline{z}}}{z - w} dxdy. \tag{3.3}$$

Because of our assumptions on F, the first integral is

$$\frac{1}{2\pi i} \int_{|z|=r} \frac{z + O(1/|z|)}{z - w} dz = w + O(1/r).$$

Since the left-hand side of (3.3) and the second integral are both constant for $r > R$, we see that the first integral must equal w for all $r > R$. Thus

$$F(w) = w - \frac{1}{\pi} \int_{|z|<r} \frac{F_{\overline{z}}}{z - w} dxdy = w - \frac{1}{\pi} \int_{|z|<r} \frac{\mu F_z}{z - w} dxdy.$$

Since $|F_{\overline{z}}| = |\mu F_z| \leq k|F_z|$, we get

$$|F(w) - w| \leq \frac{k}{\pi} \int_E |\frac{F_z}{z - w}| dxdy.$$

where $k = \|\mu\|_\infty$.

We have assumed that $F_{\overline{z}}$ is supported on $D(0, R)$. Hence $(F(w)-w)/w$ is bounded and holomorphic on $\{|w| > R\}$, so by the maximum principle it attains its maximum on $\{|w| = R\}$. Therefore it suffices to prove the desired bound on $\{|w| \leq R\}$.

So assume $|w| \leq R$. Let $r = \max(1, |w|/2)$. We will estimate the integral

$$\int_E |\frac{F_z}{z-w}| dxdy,$$

by cutting $D(0,R)$ into three pieces:

$$\begin{aligned} D_1 &= \{z : |z-w| \leq 1\} \\ A_r &= \{z : 1 \leq |z-w| \leq r\} \\ X &= D(0,R) \setminus (D_1 \cup A_r) = D(0,R) \setminus D(w,r), \end{aligned}$$

and show integral over each piece is $O(\epsilon^\beta/|w|)$, where $\beta = \beta(K) > 0$.

First consider D_1. With p as in Corollary 2.2, the L^p norm of F_z over D_1 is uniformly bounded, so using Hölder's inequality with the conjugate exponents p, q satisfying $\frac{1}{p} + \frac{1}{q} = 1$, we get

$$\int_{D_1} |\frac{F_z}{z-w}| dxdy = O\left(\| \frac{\mathbf{1}_{E \cap D(w,1)}}{|z-w|} \|_q \right). \tag{3.4}$$

In general, if we fix the area of a set Y, the integral $\int_Y dxdy/|z|$ is maximized when Y is a disk around the origin. Thus the integral above is bounded by the integral we obtain by replacing E by a disk of the same area around w. Since $E \cap D(w,1)$ has area at most $h(|w|) \leq h(r)$, we can take a disk of radius $s \simeq \sqrt{h(r)}$. Hence the L^q norm on the right side of (3.4) is bounded above by (using polar coordinates and recalling $1 < q < 2$)

$$O\left(\left[\int_0^s r^{-q} r dr \right]^{1/q} \right) = O\left(s^{(2-q)/q} \right) = O\left((\epsilon h(r))^{\frac{1}{q} - \frac{1}{2}} \right).$$

Since h tends to zero faster than any polynomial $|z|^{-d}$, we get $h(r)^{\frac{1}{q}-\frac{1}{2}} = o(r^{-d(\frac{1}{q}-\frac{1}{2})})$ for any d, and we can choose d so that $h(r) = O(1/r)$. Since r was chosen so $r \gtrsim |w|$, this also gives $h(r)^{\frac{1}{q}-\frac{1}{2}} = O(1/|w|)$. Thus

$$\int_{D_1} |\frac{F_z}{z-w}| dxdy = O\left(\frac{\epsilon^{\frac{1}{q}-\frac{1}{2}}}{|w|} \right).$$

This is the desired estimate with $\beta = \frac{1}{q} - \frac{1}{2} = (2-q)/2q > 0$.

Next consider the integral over A_r:

$$\begin{aligned}
\int_{A_r} \left|\frac{F_z}{z-w}\right| dxdy &= \int_{A_r} \mathbf{1}_E(z)|F_z|dxdy \\
&= \left(\int_{A_r} \mathbf{1}_E(z)^q dxdy\right)^{1/q} \left(\int_{A_r} |F_z|^p dxdy\right)^{1/p} \\
&= O\left(\text{area}(E \cap A_r)\right)^{1/q} \cdot \|F_z \mathbf{1}_{A_r}\|_p \\
&= O\left((\epsilon r^2 h(r))^{1/q}\right) \cdot r^{2/p} \\
&= O\left(\frac{\epsilon^{1/q}}{|w|}\right),
\end{aligned}$$

again since h decays faster than any power.

Finally, write $X = \cup_{k=1}^R X_k$ where $X_k = X \cap \{z : k-1 \leq |z| < k\}$. Then since each X_k can be covered by $O(k)$ unit disks, (3.2) and $\frac{1}{q} + \frac{1}{p} = 1$ imply

$$\begin{aligned}
\int_{X_k} \mathbf{1}_E(z)|F_z|dxdy &= \left(\int_{A_k} \mathbf{1}_E(z)^q dxdy\right)^{1/q} \left(\int_{A_k} |F_z|^p dxdy\right)^{1/p} \\
&= \left(\text{area}(E \cap A_k)\right)^{1/q} \left(\int_{A_k} |F_z|^p dxdy\right)^{1/p} \\
&= (\epsilon k h(k))^{1/q} \cdot O\left(k^{1/p}\right) \\
&= O\left(\epsilon^{1/q} h(k))^{1/q} k^{1/q+1/p}\right) \\
&= O\left(\epsilon^{1/q} h(k)^{1/q} k\right) = O\left(\epsilon^{1/q} k^{-2}\right),
\end{aligned}$$

again since h decays faster than any power. Summing over k gives the desired estimate. This proves the theorem with $\beta = (2-q)/2q > 0$. □

The proof given above shows that the conclusion of Theorem 1.1 still holds if $\int_0^\infty h(r) r^n dr < \infty$ for some (large) finite n that depends on K (in particular, it depends on the value $p > 2$ so that $F_z \in L^p$ in Bojarski's theorem). Similarly, we can assume less if we simply want a uniform bound on $|F(w) - w|$, rather than the $O(1/|z|)$ estimate above. We leave these generalizations to the reader.

Proof of Corollary 1.2. First we note that it suffices to prove this with the additional assumption that μ has bounded support, for a general quasiconformal f is the pointwise limit of such maps (truncate μ_f, apply the measurable Riemann mapping theorem and show the truncated maps converge uniformly on compact subsets to f).

So assume $\mu = \mu_f$ has bounded support, say inside the disk $D(0, R)$. Then f is conformal outside $D(0, R)$, so we can post-compose by a conformal linear map L

to get a quasiconformal map so that $|F(z) - z| \leq C/|z|$, outside $D(0, 2R)$ with a constant that does not depend on F (this follows from the distortion theorem for conformal maps). We apply Theorem 1.1 to get $|F(z) - z| \leq C\epsilon^\beta$, for all z with constants C, β that depend only on k. Note that

$$f(z) = \frac{F(z) - F(0)}{F(1) - F(0)},$$

and that $|F(1) - F(0) - 1| \leq C\epsilon^\beta$, so we get

$$|f(z) - f(w)| \;=\; |\frac{F(z) - F(w)}{F(1) - F(0)}| = \frac{|z - w| + O(\epsilon^\beta)}{1 + O(\epsilon^\beta)},$$

and this implies (1.1). □

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