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CLASSROOM NOTE

Roadmap to glory: scaffolding real analysis for deeper learning

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ABSTRACT

In this paper, we explain changes made in an introductory Real Analysis classroom that address common challenges and that may allow for all levels of students to meet a high standard of learning and written work. We aim for this goal by fusing elements of a hybrid course with scaffolded collaborative work to improve student learning. The hybrid structure allows students to engage new material at their pace, while the scaffolding allows for more challenging coursework and models for students some typical arguments in analysis. As a result, students that initially submitted proofs needing significant improvement in coherence and organization show positive gains in self-confidence and are enabled to progress quickly to submitting mathematical writing of a high calibre. Our case study occurred at a large institution serving many underrepresented and non-traditional student populations. The approach may be effective at schools with similar goals and challenges.

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1. Introduction

Real Analysis is an upper-level undergraduate, proof-based course that is critical for students who will continue beyond undergraduate studies in mathematics. Topics cover limits of sequences, series and functions, topology of the reals, the derivative and the integral. One could argue that students do not learn substantially new results in Real Analysis I, but rather learn techniques that can serve as a lens to understand deeply the major results of Calculus in a manner that is logically rigorous (Shannon, 2018). In this way, Real Analysis is similar to a figure-drawing course: every student already knows what a human figure looks like, but to capture it in a drawing requires looking at it in a completely different way. Thus, many of the learning objectives of the analysis course are about understanding the theoretical building blocks of calculus and the logical rigour necessary to recognize it as a valid system. This is followed by expectations to use this new, rigorous language to prove results and solve problems in the language of a mathematician.

Yet for mathematics majors, the Real Analysis course is often a barrier that prevents students from continuing with upper-level mathematics (Mullen, 2012). At our university, only the second semester of Real Analysis requires the first as a prerequisite. Furthermore,

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most math majors at our institution are also enrolled in teacher-training programmes that require only the first semester of Real Analysis. Hence, for many of our students, there is no required course that explicitly requires Real Analysis as a prerequisite. A significant challenge for instructors is to teach the course at a level that sufficiently prepares students for graduate programmes, while maintaining accessibility for all. By accessibility, we mean providing students of all skill levels with a pathway to success in the course. There is a need for a supportive and accessible approach that allows students to gain the necessary skills or maturity that they may have missed in previous, proof-based courses, and in which students have a clear path for continuous improvement that does not entail them hitting their heads harder against the wall. In the latter case, the Real Analysis course becomes a roadblock to advancing in the math degree. At our own institution, we have made a troubling observation: a slow but steady decrease in math majors over the last five years, with some students citing experiences in upper-level courses like Real Analysis, Topology, and Partial Differential Equations as a major impediment. Since our math program prioritizes student learning and retention, a positive experience in the Real Analysis course is important for all mathematics majors. Furthermore, many math majors are planning to pursue careers in math teaching. Their experience and engagement with Real Analysis will carry over into classes they will teach.

With the above in mind, the aim of our Real Analysis course is to prepare students who can communicate concise, logically rigorous proofs, who have confidence to solve challenging problems and who share in the excitement of mathematical discovery. Yet, we find there are significant obstacles to reaching this goal. One of these seems to concern the mode of instruction of upper-level courses and came to our attention through data collected from students approaching graduation. These data come from a general mathematical knowledge exam that includes topics selected from courses across the major, that students who are nearing completion of their degrees must take. On this exam, the lowest item-level scores are often found in the Real Analysis section, which reflects that concepts and skills are not being retained for this course in the major at the same level as other courses. It is possible that this result and the decline in math majors is related to a sink-or-swim mentality perpetuated in advanced courses like Real Analysis. Although the teaching methods of previous courses may also be an issue, the approach discussed here is our rethinking of how we teach Analysis.

In our experience, teaching a primarily lecture-based Real Analysis course has involved little student participation. A small but 'elite' group of students worked very hard and did well in the class while the majority of students typically dropped the course or earned a 'D' or less. Students struggled – often not productively – with homework problems that were intended to give them fruitful practice with the concepts. Some students asked for help on assignments while others persisted on their own and thought of asking for help as illegitimate or an indication of weakness. An outcome of the lack of progress made toward the challenge posed to them was reinforcement of students' negative self-concepts and impostor syndrome with respect to mathematics, as indicated by students' own comments to the instructors in which they later questioned their own 'fit' with the math major.

Additional challenges faced by Real Analysis students at our institution are socio-economic, since the institution is located in two counties that have among the highest poverty rates in Texas and in the US. Because of this, students typically work multiple jobs and have significant family responsibilities. Whereas students in advanced courses at other institutions may already have resources like time, study-groups, and finances in place, these are not as widely available to our students. Consequently, our Real Analysis course is often offered in the evening and attended by many non-traditional students.

In this paper, we describe changes that we are making in the Real Analysis classroom that address some of the above challenges and that will allow beginning students to do high-quality work immediately while maintaining strong standards for all students' learning. What follows should be considered not as a didactical engineering project, but as an experiment in teaching Real Analysis that shows promise for improving student outcomes. Although we want students to spend some time personally wrestling with the concepts, we do not want them to stay in this state. Thus, we consider how we can keep the bar high while making effective use of students' time. The goal of the approach described here is not to eliminate time spent outside of the classroom, but rather to focus it, and in doing so decrease the overall time commitment needed to be successful. The guided learning (Billett, 2012) course design described in this paper allows all students, independent of their background and preparation, to begin writing proofs. It also lowers the threshold for students to begin writing good proofs. Through in-class engagement, students gain a persistent feedback mechanism that keeps them on track and allows them to get to the finish line. Additionally, the suggested structure is designed to serve a student population that does not have uniform access to academic and technical resources. In the following sections of this paper, we discuss the rationale for the innovations that we have made in Real Analysis and then demonstrate the method through a discussion of resources developed for the course. Finally, we close the paper with observed impacts on students, evidence of the method's effectiveness and suggestions for instructors.

2. Course rationale and course design

In this section, we explain the theoretical considerations for changing the way our Real Analysis course has been taught. As implied above, Real Analysis has a different aesthetic from other courses. For example, epsilon-delta convergence proofs are a repeated motif and not likely encountered previously by most students. Many techniques that must be exposited are difficult to motivate until they have been presented. A traditional lecture-based approach to teaching, in which the instructor explains at length with very little time for group work or student-led discussion, can lead to great confusion and may appear from the students' perspective as a long chain of disparate tricks. So, how can we bring students up to speed quickly and keep them engaged in mathematical work that is much more abstract and challenging than work required in previous courses, that mainly focused on computation? One approach might be to lay a foundation of basic mathematical logic and proof techniques useful in analysis at the very beginning, similar to Seager's (2020) 'Analysis Boot Camp' implemented in the first two weeks of the semester. Indeed, a well-designed 'intro to proof' or 'intro to advanced math' course frequently serves to do just this, to prepare students for Real Analysis (Petrilli, 2019). However, in our experience students continue to struggle with the synthesis of a large amount of technical material and the need for creative approaches to solutions that a course in real analysis entails.

Beyond helping students to obtain the prerequisite knowledge for analysis, keeping students engaged and confident that they can achieve throughout the course is often a serious challenge in a lecture-style course. To address the problem of low student engagement in

Real Analysis, Mullen (2012) made the move away from a purely lecture-style course and began to incorporate pre-reading assignments, warm-up problems and significant class time for questioning in the course. At the heart of such efforts is the goal of engaging students in *active learning*. To this end also Pawlaschyk and Wegner (2019) changed the nature of the homework given in Real Analysis, assigning more open-ended and discovery-oriented questions in the hope that students would be encouraged to actively generate and then prove or refute their own conjectures. Going even further, Dumitraşcu (2009), Shannon (2016, 2018), and Reinholz (2020) all addressed the problem of low engagement in Real Analysis lectures head-on by totally restructuring class-time in a way that incorporates significant group-work with peers in the classroom. These researchers all allocated significant classroom time for students to work in groups on pre-designed tasks or worksheets, by presenting board work, or by participating in carefully structured discussions.

Our innovations in Real Analysis presented in this paper are most similar to these latter efforts in so far as they strategically use grouping structures to promote active learning in the classroom. This has been shown to specifically support the advancement of underrepresented students in science and mathematics (Theobald et al., 2020). As instructors we warmly reflected on our own experiences of taking Real Analysis and where the bulk of the comprehension took place: it all happened in the study groups that we formed with peers *outside of class*! It was there that we broke down each homework problem into the main ideas, identifying which theorems or definitions we would need to deploy to connect A to B, and to sketch out a map of the proofs. It was there, communicating in the group sessions, that the actual learning occurred. For this reason, the first essential element of our redesigned Analysis course is *community*.

2.1. Essential course design principles

In redesigning our Real Analysis course, the three principles of *community, intervention, and in-class assignments* are seen as essential for supporting students' engagement and progress.

- *Community*. While some students may do well in Real Analysis on their own, in our experience, forming learning communities leads to wider success in the course. As we have said, collaborating in study groups is vital. The importance of collaboration is often emphasized in lecture-based courses but not made an integral part of the course. One innovation of the present structure is that it moves the study group into the class. Just as getting together and working on material in graduate school is the standard way to get things done, we are trying to cultivate this culture at the undergraduate level and among non-traditional students for whom finding time and space for arranging study groups outside of class may be difficult or impossible.
- *Guided learning/targeted intervention*. In our experience with the lecture approach, students often did not ask for help even though instructors may have extended the invitation. Additionally, struggling students relied too much on their peers and submitted assignments that were very similar in content and verbiage, and not entirely their own. Consequently, the lecture approach led to course evaluations indicating that some students felt disengaged or lacked an operational knowledge of Real Analysis, dutifully writing down lecture notes, yet without opportunity for meaningful personal

engagement in class or easy access to the instructor *at the time when that intervention was most needed*. To address this lack, our innovation is designed to promote the instructor's ability to intervene in students' work in a timely manner. Although students may not consistently raise their hands, the instructor is able to cultivate discussion by asking 'what are you working on' and helping students to work through their reasoning or directing them to revisit a definition.

- *In-class assignment*. In his writing about a peer-tutoring intervention implemented at UC Berkeley that was aimed at overcoming the pattern of low achievement of certain students in the calculus sequence, Treisman (1992) highlighted the importance of the actual task completed:

> Most visitors to the program thought that the heart of our project was group learning. They were impressed by the enthusiasm of the students and the intensity of their interactions as they collectively attacked challenging problems. But the real core was the problem sets which drove the group interaction. One of the greatest challenges that we faced and still face today was figuring out suitable mathematical tasks for the students that not only would help them to crystallize their emerging understanding of the calculus, but that also would show them the beauty of the subject. (Treisman, 1992, p. 368)

For Treisman's group, peer interaction driven by dynamic problem sets was found to be the solution for students to have success in calculus. Essential to this peer interaction was the careful design of learning tasks. Similarly, in our innovation, students work together in class on an assignment that requires them to integrate relevant results from the lectures and the textbook readings and serves as a 'roadmap' for them to enter into the topics of the course. In practice, our student groups are self-selected, and can lead to groups of mixed abilities or preparation. Typically, this has benefited the group as stronger students explain the concepts, crystalizing the ideas more deeply for them, while weaker students get the benefit of something akin to a peer mentoring interaction.

With the above principles in mind, the idea behind our approach is to bring to life highly technical material that we expect will help students to develop a deep understanding in a short amount of time. This approach is aimed at keeping the course challenging and expectations high, while making effective use of students' time and resources. The following structural elements support this goal.

2.2. Structural course design elements

Central structural elements of the redesigned Real Analysis course include organizing the course into *Modules* implemented through a *Flipped Design* that makes usage of readings, lecture videos and comprehension quizzes to convey important concepts that students will then apply in the study groups. Students are then prepared to complete the *Weekly Written Homework*.

- *Modules*. The Real Analysis course is organized into 15 weekly Modules, making the expectations and weekly schedule clear and easy to follow. Each weekly module includes: a reading assignment, two sets of lecture notes and corresponding lecture videos, two online quizzes to assess basic understanding of the lecture results and definitions, followed by the weekly homework assignment.

- *Flipped design*. The course is 'flipped' so that almost all direct instruction and delivery of new concepts – the entire lecture – happens outside of the classroom through video lectures and comprehension quizzes.

A significant barrier to success in Real Analysis is fluency in its language, understanding not just the various definitions, but the roles that the epsilons and deltas play in the definition, for example. In a traditional lecture, by the time students parse through the math jargon and shorthand of a definition, the professor is already brandishing the new term in a proof. By moving the lecture to a video, students can watch it at their own pace and according to their own schedule, pausing to decipher a concept or idea before moving on. Video lectures follow the lecture notes that are provided to students and are designed to highlight major concepts seen in the reading assignments.

In the videos, we augment the definitions with plain language interpretations and animations, connecting the new, rigorous way of defining a concept with its portrayal in introductory courses like calculus. For example, this lecture video offers the following definition of convergence, which brings the definition to life through associating the phrases of the technical definition with more familiar, everyday language.

Definition 2.1: A sequence converges to a limit if it arrives, and remains forever after, *arbitrarily close* to that limit. Stated more technically, a sequence $\{a_n\}_{n=1}^{\infty}$ converges to a limit $L \in \mathbb{R}$ if and only if, for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that, for all $n \geq N$: $|a_n - L| < \epsilon$.

As seen in the video, we use animations and graphs frequently to illustrate ideas. Often, we illustrate the entire concept of a proof using an animation before we go through a formal, written proof. This offers a conceptual and intuitive development of the proof before the formal proof is given.

After viewing the lecture, students take a short online comprehension quiz that assesses a basic understanding of the lecture. This also allows us to reinforce ideas that will come up repeatedly throughout the course. For example, it is crucial that students understand that the phrase 'x in an ε-neighbourhood of 1' means exactly the same as 'the absolute difference between x and 1 is strictly less than ε', or that 'x is strictly in between $1-\varepsilon$ and $1+\varepsilon$'. The understanding that is assessed in the quizzes is crucial to writing proofs in a precise way and applying these definitions to solve problems and explain the theoretical structure of differential calculus. Struggling on a quiz is a sign that students should stop and seek feedback from the instructor or their study group before tackling the homework. Each in-class session gives students ample opportunity for this. Class time is spent working in groups on an activity relevant to the previously viewed videos. The in-class activities are designed to provide a framework for completing the homework.

- *Weekly written homework*. Weekly homework assignments constitute a regular formative assessment that allows students to gauge their progress in the course each week. As such, it is one of the most important aspects of the course. The written homework gives students' an opportunity to demonstrate what they have learned throughout the week in the lecture videos and to crystalize in-class group work. Students are also informed that understanding the homework assignment thoroughly is crucial to preparation for the course exams.

As an aid to students in structuring their time to work through each weekly module, we recommend that they do the following (assuming Monday/Wednesday class meetings):

- Sunday-Monday. Read Homework and assigned portions of the textbook, Watch Lecture, Take Lecture Quiz, Attend Class.
- Tuesday. Begin homework problems covered in Monday's lecture. Ask questions during office hours.
- Wednesday. Watch Lecture, Take Quiz, Attend Class.
- Thursday. Finish working through each homework problem. Ask questions during office hours.
- Friday. Finalize turn-in draft of homework.

The flipped design of this course introduces students to new concepts in an environment that fits their academic and work schedules. Notice that the schedule above is not required but only suggested; actual time required outside of class is limited and the structure is offered as an aid to support students' finding time to complete the work in the midst of their often-busy work schedules. Attending class then gives them a regularly scheduled biweekly study session in which they can work through new ideas with their peers, with timely interventions from the instructor. The result is to scaffold Real Analysis learning in such a way that, when students engage with the weekly homework assignment, the essential concepts and skills are all in place and much of the challenging work has already been done within the learning community. Thus, what happens in the classroom – the labour in the study group – is critical.

3. Roadmaps to glory

Real Analysis typically requires a substantial amount of conceptual brainstorming and time outside of the classroom. Often, the most successful students are those that can put together productive study groups where students can develop the analytical intuition necessary to perform well. However as mentioned earlier, we have found that for a large portion of our student population, this has not been feasible. One of the driving ideas of our flipped classroom is to extend the study group into the classroom and curate/focus the discussion to maximize the efficiency of the course time.

Putting the lecture before class time brings us closer to accomplishing this goal. In order to ensure that students watch the lecture, there is a brief online quiz meant to assess a basic understanding of the lecture contents. During class time, which consists of two 75-minute sessions per week, there are typically 6–7 student groups, each of size 3 or 4 students. With classes of this size, only one instructor is needed. The in-class group work, which we call *Adventures in Real Analysis*, is meant to deepen the understanding of the material covered in the lecture and to serve as 'Roadmap' that guides students into the concepts. The homework, then, is the main week-to-week assessment of students' grasp of course concepts. The problems in this homework are very challenging, often results that would typically be covered in the lecture of a traditional course. Assigning such challenging homework is tenable since a subset of the homework problems are scaffolded in the in-class group assignments. Thus, the scaffolding in the group work serve as a kind of interactive lecture that students must think through in their groups, rather than receive as a dictation to be digested later.

In this section of the paper, we illustrate our implementation of the flipped design and in-class study sessions, with emphasis on the latter. Hence, we first briefly show how video lectures, seen by students outside of class time, use animation to initially define concepts and give students an intuitive understanding of them. Following this, we exemplify two usages of in-class Adventures in Real Analysis assignments – Road Maps – that we have developed to scaffold students' successful writing of challenging proofs in their homework assignments.

3.1. Animating the Bolzano-Weierstrass theorem

The Bolzano-Weierstrass Theorem is one of the most fundamental and frequently used results in the Real Analysis course. It asserts that any infinite and bounded set of real numbers has an accumulation point. The proof of the Bolzano-Weierstrass Theorem, to an initiated analyst, is straightforward to understand, but requires some digestion for the uninitiated. The main conceptual tool in the proof is that the interval containing the set can be repeatedly bisected, with at least one of the halves created containing infinitely many elements of the set. In our course, the Bolzano-Weierstrass Theorem is introduced outside of class time in the video-lecture component. Giving the lecture in a video format allows us to seamlessly animate this process in a way that is intuitive. Figure 1 illustrates how the videos bring to life the repeated bisection of intervals that is central to the theorem.

The goal of the animation above is to make the result obvious, to generate in students a 'Well, of course, that's true' reaction. The intuition gleaned from the animation is then translated into a formal proof, thus modelling for students how intuitive ideas are translated into rigorous, mathematical explanations.

3.2. Adventures in real analysis

In the in-class daily adventure (Figure 2) that follows the above video lecture, student groups first encounter a set of straightforward exercises before getting to their main result. An important study skill that is not obvious to all students is that problems involving key definitions and results from the lecture should be begun by finding that definition or result in the lecture notes. Hence, we model this study skill by often inserting problems like the first one on the page: What does it mean for a number to be an accumulation point? If the definition can be found and recited, then it can be interpreted in answering problems 2 and 3 from the daily adventure. These problems are very straightforward, and are not in the homework sets, but serve as important check-points to ensure a basic understanding before proceeding to the main problem. The Bolzano-Weierstrass Theorem requires two conditions of a set of real numbers to come into effect: infinite and bounded. Hence, the fourth problem asks the student to confirm that the bounded condition is required by coming up with an example.

Finally, in problem 5 we come to our main exercise of the day, to prove that the set of numbers in the sequence $\{\sin(n)\}_{n=1}^{\infty}$ has an accumulation point. From problem 4, the student should be tipped off to the fact that this result can be obtained by showing the two

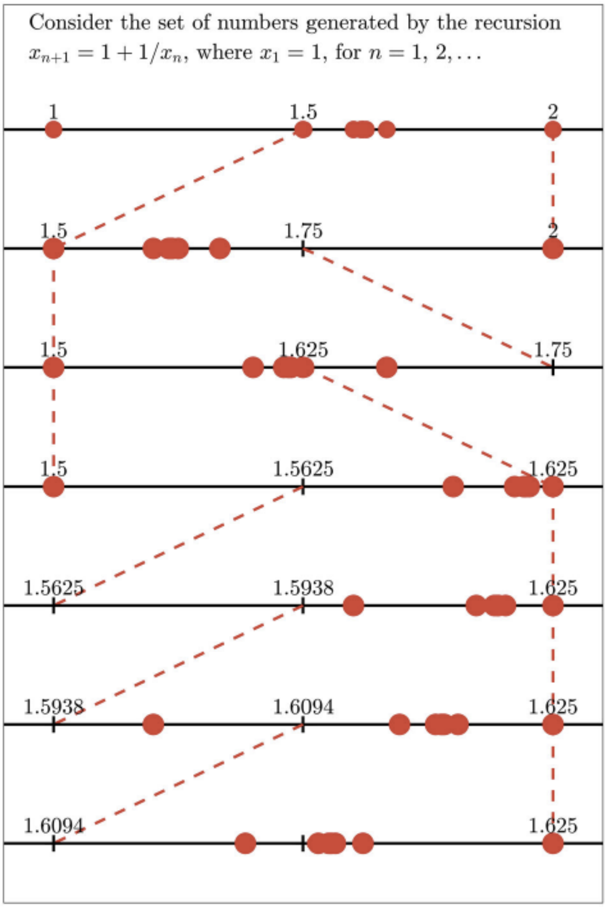

Figure 1. Animating the Bolzano-Weierstrass theorem. See animation here.

conditions in the hypothesis of the Bolzano-Weierstrass Theorem to be true. Items (a) and (b) of the Road Map address these two conditions while problem (c) asks students to invoke the theorem. Certainly, some more advanced students, or those with access to a productive study group, could have come to this conclusion without a road map. But by focusing the class time on key activities and curating the discussions of the groups, more students can be brought along to this result, and with stronger understanding.

3.3. *Another adventure in real analysis*

The following Adventure in Real Analysis (Figure 3) asks students to prove that continuous functions on compact domains are uniformly continuous. This is a fundamental result of real analysis, useful in building Riemann integration and beyond in topology. The proof is challenging since it requires an extensive application of open covers – a new conceptual construction for almost everyone sitting in a first Real Analysis course. Proving the result to a passive class during lecture is usually not sufficient to get these ideas across, ideas that are really important and useful in analysis and later in topology. In a traditional Real

Adventures in Real Analysis • Week 4, Day 2

1. Let $S \subseteq \mathbb{R}$. What does it mean to say that the number $x \in \mathbb{R}$ is an accumulation point of S?

2. Does the set $S = \{1, 2, 3, 4\}$ have an accumulation point. Why or why not?

3. What is the set of all accumulation points of the open interval $(0, 1)$?

4. Does every infinite set have an accumulation point? If your answer is in the negative, give a counterexample. (Hint: Read the Bolzano-Weierstrass Theorem).

5. Prove that the set of values in the sequence $\{\sin(n)\}_{n=1}^{\infty}$, has an accumulation point.

   A Road Map to Glory:

   (a) Use the fact that π is irrational and $\sin(x) = \sin(y)$ if and only if

   $$y = x + 2k\pi \qquad \text{or} \qquad y = (2k+1)\pi - x$$

   for some integer k, to show that the set of sequence values is infinite (by showing that the sequence never repeats).

   (b) Is the set of sequence values $\{\sin(n) \mid n \in \mathbb{N}\}$ bounded? Explain why or why not.

   (c) Have you now, with the aid of a theorem from this weeks lecture, shown that the set $\{\sin(n) \mid n \in \mathbb{N}\}$ has an accumulation point? Explain how.

Figure 2. A roadmap into the Bolzano-Weierstrass theorem.

Analysis course, it would be very difficult to leave this result as a homework exercise. But with the structure of lecture videos watched at the students' pace, followed by scaffolded group work, it becomes a feasible student exercise.

Part (a) of the *Road Map to Glory* asks them to recall the definition of continuity, and to structure the definition in terms relevant to the task at hand. Guiding students to the right structure is accomplished by offering the definition with blanks to be filled in by students. This also reminds students of the basic study skill that a problem on continuity should be begun by recalling and understanding the definition of continuity. Since our road map should lead us from mere continuity to uniform continuity, the dependence and independence of the δ parameter on x is central to understanding the difference in the two types of continuity. The student is prompted to grapple with this idea in part (b).

Part (c) then brings us to our first delicate maneuver. An open cover of our domain must be established in a way that connects to the continuity criteria established in part (a). For each $x \in D$, there is an open set containing D, and the family of these sets satisfies the definition of an open cover. This is the part that students will typically struggle with. Since lecture time is occupied with group work, we are free to intervene with any group that appears to be stuck in any place longer than is healthy. In such interventions we often

Adventures in Real Analysis • Week 10, Day 2

Today we will prove that if a function f is continuous on a compact domain, then it must be uniformly continuous.

A Road Map To Glory: but first, let $\varepsilon > 0$.

(a) Let's name the compact domain of f, D. **Fill in the blanks:**
For each $x \in D$, there exists a $\delta(x) > 0$ for which the following is true:

$$\text{If } y \in D \text{ and } |x - y| < \underline{\quad},$$

$$\text{then } |\underline{\quad} - \underline{\quad}| < \frac{\varepsilon}{2}$$

(b) Explain briefly why the δ value above depends on x.

(c) Prove that $D \subset \bigcup_{x \in D} \left(x - \frac{\delta(x)}{2},\ x + \frac{\delta(x)}{2} \right)$.

Note: This means that the family of sets $\left\{ \left(x - \frac{\delta(x)}{2},\ x + \frac{\delta(x)}{2} \right) \right\}_{x \in D}$ *is an open cover of* D.

(d)
- Since D is compact, what property do we know to be true about an open cover of D?
- Use this fact to explain why there exists a finite number of elements $\{x_1, x_2, \ldots, x_n\}$ of D for which (**Fill in the blanks:**)

$$D \underline{\quad} \bigcup_{j=1}^{n} \left(\underline{\qquad\qquad\qquad} \right)$$

(e) Let[1] $\delta = \min \left\{ \frac{\delta(x_1)}{2}, \frac{\delta(x_2)}{2}, \ldots, \frac{\delta(x_n)}{2} \right\} > 0$ and suppose that $x,\ y \in D$ and

$$|x - y| < \delta.$$

Prove that there exists a x_k from the set $\{x_1, x_2, \ldots, x_n\}$ for which

$$|x - x_k| < \delta(x_k) \qquad \text{and} \qquad |y - x_k| < \delta(x_k)$$

Hint: Explain why there exists a x_k *for which* $x \in \delta(x_k)/2$*-neighborhood of* x *and then add zero strategically:* $|y - x_k| = |y - x + x - x_k|$

(f) From your result in (a), what does this prove about the differences

$$|f(x) - f(x_k)| < \underline{\quad} \qquad \text{and} \qquad |f(y) - f(x_k)| < \underline{\quad}$$

(g) Prove that this implies that

$$|f(x) - f(y)| < \varepsilon.$$

[1]The reuse of δ as a constant here, and a function earlier, is somewhat of an abuse of notation. But the key idea is that we can go from expressing δ as a function of the variable x to a single δ to rule over all x in the domain.

Figure 3. A roadmap into the uniform continuity of continuous functions.

remind students of the fundamentals of set theory that they would have seen in a previous introductory course to mathematical proofs, helping them to break this problem into smaller pieces of fundamental set theory. In part (d) we invoke the compactness of the function's domain, and what this property implies about open covers. This step requires a solid grasp of the finite subcover property of the compactness condition, even if most of what is asked is a recitation of the definition of a subcover. These steps are big conceptual leaps for most students, and the scaffolding allows us to actively challenge the students to confront the steps with their working group.

In part (e), we near the end of our road map. We make the leap from finite subcovers of compact sets to arguing that we can remove the dependence of x on δ that will bring about the uniform continuity result. It is doubtful that by directly lecturing this result to students they could achieve the level of understanding needed to navigate the scaffolding of the road map that we have laid out. Even though the scaffolding lays out much of the path for the student, the active learning and discussion in the group are necessary to progress from step to step of the scaffolding. The step in (e) asks the groups to grasp what finite subcovers say about the topology of a set. The last steps, (f) and (g) connects these dots back to our beginnings in (a).

Note that each step of this road map isolates a different concept: continuity, open covers, finite subcovers, compactness, set theory, etc. In the past, we have noticed that group work assignments like this – even when a group completed their road map – would still leave some students in the weeds, lost to where the forest of their proof was. To support their making that larger connection, we began to highlight in red which pieces of their journey formed the proof required in the related homework problem. The proof still needs to be assembled, which requires understanding the relevance of the red portions. This turned out to be helpful to students when they transferred their group work to final drafts of their homework submission.

4. Observed impacts of this approach/evidence of effectiveness

Before we delve into the overall impacts of this approach, we offer some reflection on the second *Adventure in Analysis*. As mentioned previously, this would not be left as an assignment in a more traditional structure. However, by assigning it as homework and providing scaffolding during class time, a majority of students were able to give a correct proof in the homework. Of course, many of the correct responses had the same structure as offered in the scaffolding, but this is a success in our view. In this way, the scaffolded group assignment acts as an interactive – *active* being the key part – lecture. While a path to a proof is proffered, completing and organizing the proof requires understanding.

Although the authors have observed positive impacts for many students, the approach does not work immediately and for all students. As with any approach, the degree of success depends on the extent and quality of student and faculty engagement. The observed impacts on student learning are often significant. Within the structured approach, some students go from submitting weak proofs to writing at a high level not characteristic of beginning proof writers. From the perspective of the teacher, it is as if a switch is flipped. Students start with the submission of proofs having little coherence and organization and progress to submitting writing that is clear and conveys deep understanding. For many students this transition is relatively quick, often a few weeks, for others it is more gradual. It is possible that the change in confidence and self-efficacy is cultivated and reinforced by the structure of the course. Students see that mathematics is something within their grasp. The community aspect allows students to see that they are on a level playing field with their peers, and that what is being asked of them is challenging for everyone, but attainable. The end results are promising. We see more high-quality proofs on exams than in lecture courses, especially the midterms and finals, in the sense that they are better written and more logically sound. Student course evaluations indicate the approach cultivates learners' confidence and forces them to increase their engagement with the material. This is anecdotal evidence that the training wheels (scaffolding) are working – that student can ride the bike. One reason for this result may be attributable to the fact that the training wheels that are being provided are not procedural. We avoid telling students how to do something and instead give them tools to think through the ideas on their own. Ideally, the activities move results typically covered in lecture to results that are discovered by students both within and outside of class. By deriving results themselves within a well-structured framework, students retain not just the essential results, but the reasoning that goes into a proof.

Course evaluations reflect students' perceptions of the approach. A standard course evaluation question to which students in the newly designed course have most strongly agreed was that they had 'taken an active role in their own learning'. At least among this group of students, a striking feature in their own minds was related to their personal sense of *agency* in the learning, which is probably an indication that the method really did promote active learning, a desirable outcome and a specific goal of this innovation, as well as a proven way to narrow achievement gaps for underrepresented students in math and science (Theobald et al., 2020). Written student evaluations underscore students' increased sense of agency. Active engagement during the class allows learners to gain insight into motivating ideas by building key results from more fundamental principles. This translates to increased self-efficacy. Students have written that their ability to explain a solution to their peers helps them confirm the validity of their argument and gives them confidence that they have integrated what they have learned into a coherent and correct solution. This indicates that the approach effectively contributes to increases in student mathematical maturity and self-efficacy.

Since a large number of students taking the course intend to teach mathematics at the secondary level, the impact of the approach may extend well beyond what happens in a single semester. Student evaluations and anecdotal feedback indicate that students see value in the community of learners. Our hope is that future teachers use the experience to craft similar dynamic discussions with their students. The ultimate outcome will be a significant change in the way mathematics is perceived, especially with respect to its accessibility of so-called upper-level mathematics. Feedback indicates students finish the course with a sense of ownership of the results and confidence in their ability to work through mathematics at any level. Our hope is that the course cultivates a lasting conviction that problem-solving is an exciting, accessible, and collaborative human endeavour. To illustrate the impact of our course structure on students' experience, we end this section with comments received from different semesters of our innovative Real Analysis course, all from different students.

- 'I consider this class one of the toughest classes needed for my degree and yet I enjoyed the class, and learned. The worksheet method during class helped in the learning process'.
- 'Class was setup in an unorthodox way but interesting way of peer learning during class time'.
- 'Any student can tell that Dr. _____ spent copious hours restructuring the class'.
- 'Dr. ____ wants his students to understand analysis, and the way that the class is structured helps students achieve that'.
- 'This is the second time I take the course and it was so much easier with your structure of the class it's a lot easier to understand what's going on'.

- 'This class was the most fun I had all semester. I thought that the quiz format and adventure format made it impossible to ignore the work for the class, which is brilliant for an online format. Furthermore, these made the homework assignments approachable rather than terrifying'.
- 'This course was my favourite of the semester. I respond well to organized group work and doing problems on the lessons we've covered prior to class. Because if I can teach it to a fellow classmate, then I know I've integrated what I've learned'.

5. Conclusion

Students' experience in a first course in real analysis has significant bearing on retention and degree completion. Moreover, the way upper-level courses such as Real Analysis are taught has key broader impacts that can influence the representation of women and minorities in mathematics. Such courses may give students their first practical contact with research-level mathematics. Mathematics is often thought of as a contact sport that requires persistence and grit, but the contact must be compassionate, constructive, and structured to allow learners to advance and remediate with abundant feedback. Progress in mathematics is largely the outcome of sustained encouragement and collaborative efforts rather than an aggressive individual pursuit. This is the primary lesson we hope to convey to students through the course innovations.

The collaborative aspect that is vital for student learning is also important in the development of the course. Elements of the Real Analysis course described here were revised and re-worked by several faculty members over a number of years. Its present incarnation is the result of many discussions and honest reflection about the student experience. As with any good pedagogical effort, it is a work in progress. The course material has been shared with faculty teaching the course and is constantly refined within the community of instructors. Our current efforts are focused on improving classroom activities, aligning the scaffolding to perfect the mix of support and challenge, and by jazzing up videos with improved illustrations and interactivity. Another effort is to consider whether a more purposeful construction of student groups leads to better overall outcomes.

Adaptation of active learning methods is an effective evidence-based strategy to increase student learning (Theobald et al., 2020). We plan on mirroring the approach discussed here in other core upper-level mathematics courses such as Abstract Algebra, Advanced Linear Algebra, Topology, Number Theory, and other 'gateway' courses to advanced mathematics. There is an initial time investment required to adapt a course to this format. Animated and still illustrations were created using Matlab's coding environment. The lecture notes were typeset with LaTex, and the lecture videos were recorded and edited using Camtasia video editing software. Some elements of the approach taken in Real Analysis may need to be altered to optimize learning in other courses. Furthermore, significant faculty involvement is needed to create lasting change in the way these other courses are taught. Faculty involved in the Real Analysis revamp speak to colleagues informally and formally about their work in teaching seminars.

Conclusions about the effectiveness of the methods discussed here are limited since we have not done a rigorous quantitative study of the impact of the revised course structure on student success in Real Analysis and courses that follow it. A more rigorous longitudinal study is needed to determine the long-term impacts of this kind of instruction. It would also be of interest to determine the impact of this kind of instruction on the subsequent evolution of teachers. The limited numerical data available to us indicates that the approach leads to more uniform pass rates in Real Analysis when compared to the highly variable rates we have observed in lecture-based Real Analysis courses. The uniformity may also be due to the small community of faculty teaching the course with the new methods. The faculty involved with the revamp are committed to continue sharing the material and encouraging other faculty to adopt similar techniques.

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