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Four end solutions of a free boundary problem ☆

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A B S T R A C T

In this paper we study the structure and classification of four end solutions to a free boundary problem on the plane. These four end solutions are also characterized by having Morse index 1.

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1. Introduction

In this paper we study four end solutions to the following free boundary problem on the plane $\mathbb{R}^2$,

$$\begin{cases} \Delta u = 0, & \text{in } \Omega := \{-1 < u < 1\}, \\ u = \pm 1, & \text{outside } \Omega, \\ |\nabla u| = 1, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Throughout this paper solutions to this free boundary problem are understood in the classical sense, that is, Ω is a smooth domain on $\mathbb{R}^2$, $u \in C^2(\overline{\Omega})$, where both the equation and the boundary conditions in (1.1) hold pointwisely. We will also assume Ω to be connected, because the solution restricted to each connected component can be viewed as a solution of (1.1).

Equation (1.1) arises as the Euler-Lagrange equation of the functional

$$\int \left(|\nabla u|^2 + \chi_{\{-1<u<1\}} \right). \tag{1.2}$$

The second variation of this energy functional is

$$\mathcal{Q}(\varphi, \varphi) := \int_\Omega |\nabla \varphi|^2 - \int_{\partial\Omega} H\varphi^2,$$

where H is the mean curvature of $\partial\Omega$ with respect to the inward unit normal vector. (We have $H \geq 0$ by such a choice, see Proposition 3.2.) The linearized problem of (1.1) is

$$\begin{cases} \Delta \varphi = 0, & \text{in } \Omega, \\ \varphi_\nu = H\varphi, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Here ν denotes the unit outward normal vector of $\partial\Omega$. The eigenvalue problem associated to $\mathcal{Q}$ is

$$\begin{cases} -\Delta \varphi = \lambda\varphi, & \text{in } \Omega, \\ \varphi_\nu = H\varphi, & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

The Morse index of a solution is defined to be the number of negative eigenvalues of this problem (counting multiplicity), or equivalently, the maximal dimension of the negative space for the quadratic form $\mathcal{Q}$. If the Morse index is 0, the solution is stable.

A special solution to problem (1.1) is $\Omega = \{(x, y) \in \mathbb{R}^2 : |x| < 1, y \in \mathbb{R}\}$ and $u(x, y) = x$ in Ω. This is a solution depending only on one direction. We call this solution a one dimensional solution or a two end solution. The one dimensional solution is unique up to rotation and translation.

In this paper we are interested in solutions of (1.1) with four ends, which depend on both variables.

Definition 1.1 *(Solutions with finite ends).* A solution to (1.1) is said to have finite ends, if there exists an $R > 0$ such that $\Omega \setminus B_R(0)$ has only finitely many connected components and each connected component has the form

$$\left\{X \in \mathbb{R}^2 : f^-(X \cdot e_\alpha) + t_\alpha \le X \cdot e_\alpha^\perp \le f^+(X \cdot e_\alpha) + t_\alpha\right\},$$

where e_α is a unit vector and t_α is a constant, $f^\pm$ are convex (concave) functions defined on $[R, +\infty)$ satisfying

$$\lim_{t \to +\infty} f^\pm(t) = \pm 1$$

respectively.

Each connected component of $\Omega \setminus B_R(0)$ is called an end. The unit vector e_α is called the asymptotic direction of this end. In each end, the solution is in fact locally close to the one dimensional solution (see Proposition 3.7 below).

One may wonder if the definition of solutions with finite ends could be weakened by dropping the asymptotic behavior condition in Definition 1.1 and being replaced instead by assuming $\Omega \setminus B_R$ for large R have a finite number of unbounded connected components or $R^n \setminus \Omega$ have a finite number of unbounded connected components. We believe it is possible, however, it may be very technical to prove so. To make the main ideas of the paper more transparent, we focus on the solutions with the more restrictive condition, which are satisfied if we assume the solution has a finite Morse index.

For Allen-Cahn equation

$$\Delta u = u^3 - u,$$

solutions with finitely many ends, four end solutions in particular, have been studied extensively in [5,6,10,11,13–15]. By now the following are known.

- Gui [10]: After a translation and rotation, u is evenly symmetric with respect to the x and y axis. Changing the sign of u if it is necessary, in the first quadrant it holds that

$$\frac{\partial u}{\partial x} < 0, \quad \frac{\partial u}{\partial y} > 0.$$

- Kowalczyk-Liu-Pacard [13,14]: Let $\mathcal{M} \subset L^\infty(\mathbb{R}^2)$ be the set of evenly symmetric four end solutions of the Allen-Cahn equation. Then $\mathcal{M}$ is a smooth manifold diffeomorphic to $\mathbb{R}$.
- Kowalczyk-Liu-Pacard [13,14]: Let $\alpha \in (0, \pi/2)$ be the angle between the end in the first quadrant and the x axis. Then $\alpha : \mathcal{M} \mapsto (0, \pi/2)$ is a smooth proper map.
- Gui-Liu-Wei [11]: Via a sophisticated variational approach, it is shown that for each $\alpha \in (0, \pi/2)$, there exists a four end solution of the Allen-Cahn equation with angle α whose Morse index is 1.
- Wang-Wei [22]: Every Morse index 1 solution of the Allen-Cahn equation has four ends.

However, there is still an important problem remaining open: the uniqueness of four end solutions with given angle α.

In this paper all of these issues will be considered for four end solutions to (1.1). A complete classification as well as a Morse index characterization will be established. We note that for the elliptic sine-Gordon equation

$$-\Delta u = \sin u, \quad |u| < \pi,$$

Liu and Wei [17] obtained a complete classification for all finite end solutions in $\mathbb{R}^2$. However, our treatment will be very different.

The paper is organized as follows. In Section 2 main results of this paper are stated. We present some preliminary results in Section 3. In Section 4, we prove the even symmetry of four end solutions. In Section 5 an important notion, quasi Gauss map, is introduced, where it is used to prove the uniqueness of solutions. A nondegeneracy property is established in Section 6. The structure of moduli space is studied in Section 7. Then we prove the existence of four end solutions in Section 9, after establishing the existence of a special four end solution, the saddle solution, in Section 8. Finally in Section 10 we find a Morse index characterization of four end solutions.

2. Main results

The main results of this paper are the following.

Theorem 2.1 *(Even symmetry). A solution of* (1.1) *with four ends, after a translation and a rotation, is evenly symmetric with respect to the x and y axis.*

Theorem 2.2 *(Geometric properties). Suppose u is a solution of* (1.1) *with four ends. Then*

(i) *There are exactly two connected components of $\{u = 1\}$ and $\{u = -1\}$ respectively.*
(ii) *Ω is simply connected.*

(iii) *Each connected component of $\{u = \pm 1\}$ is strictly convex and unbounded. The opening angle of every connected component of $\{u = \pm 1\}$, i.e., the angle formed by any two neighboring asymptotic directions, is strictly positive.*

We note that $-u$ is also a solution if u is a solution, we may focus only on one solution and assume u has one whole component of $\{u = 1\}$ contained in the upper half plane. By the even symmetry of u, we can define

Definition 2.3. Suppose u is an evenly symmetric, four end solution. Take the end with its asymptotic direction e in the first quadrant. The angle between e and the positive x axis is denoted by α.

By the definition of α and Theorem 2.2 (iii), the value of α lies in $(0, \pi/2)$. In this setting, we may say $-u$ has an angle α lies in $(\pi/2, \pi)$.

Theorem 2.4 *(Existence and uniqueness). For each $\alpha \in (0, \pi/2)$, there exists a unique evenly symmetric, four end solution of (1.1) with angle α.*

Since $-u_{\pi/2-\alpha}(y, x)$ is also a four end solution with angle α, the uniqueness implies that

Corollary 2.5. *For any $\alpha \in (0, \pi/2)$, $u_\alpha(x, y) = -u_{\pi/2-\alpha}(y, x)$. In particular, $u_{\pi/4}$ is oddly symmetric with respect to $\{y = x\}$ and $\{y = -x\}$, and may be called a saddle solution.*

For each $\alpha \in (0, \pi/2)$, u_α is nondegenerate in the following sense.

Theorem 2.6 *(Nondegeneracy). Suppose u is a four end solution to (1.1) and $\varphi \in L^\infty(\overline{\Omega}) \cap C^2(\overline{\Omega})$ is a solution to the linearized equation (1.3), then there exist two constants a and b such that*

$$\varphi \equiv a u_x + b u_y \quad \text{in } \overline{\Omega}.$$

Remark 2.7. There do exist two other linearly independent kernels which grow linearly along the asymptotic directions at infinity. One is the rotational derivative $\phi = y u_x - x u_y$, the other one is obtained by differentiating u_α in α.

Definition 2.8 *(Moduli space).* Let $\mathcal{M} \subset Lip_b(\mathbb{R}^2)$ be the set

$$\{u_\alpha : \text{ evenly symmetric, four end solution with angle } \alpha\}.$$

In the above $Lip_b(\mathbb{R}^2)$ denotes the space of bounded, Lipschitz continuous functions on $\mathbb{R}^2$.

By Theorem 2.4 we obtain

Theorem 2.9 *(Structure of moduli space). $\mathcal{M}$ is an embedded curve diffeomorphic to $(0, \pi/2)$.*

Let us define the following two copies of one dimensional solutions,

$$u_0(x,y) = \begin{cases} y-1, & \text{if } 0 \le y \le 2, \\ -y-1, & \text{if } -2 \le y \le 0, \\ 1, & \text{otherwise} \end{cases} \tag{2.1}$$

and

$$u_{\frac{\pi}{2}}(x,y) = \begin{cases} -x+1, & \text{if } 0 \le x \le 2, \\ x+1, & \text{if } -2 \le x \le 0. \\ -1, & \text{otherwise .} \end{cases} \tag{2.2}$$

For the behavior of u_α as $\alpha \to 0$ or $\pi/2$, we have

Theorem 2.10 *(Boundary behavior in moduli space). u_α converges to u_0 uniformly on any compact set of $\mathbb{R}^2$ as $\alpha \to 0$; similarly, u_α converges to $u_{\frac{\pi}{2}}$ uniformly on any compact set of $\mathbb{R}^2$ as $\alpha \to \pi/2$.*

Finally, we establish the following Morse index characterization of four end solutions.

Theorem 2.11 *(Morse index characterization). A solution of (1.1) has four ends if and only if its Morse index is 1.*

For simplicity of the presentation of the paper, we shall focus on the four end solutions. The existence and properties of $2k$-end solutions will be discussed in future work. We note that the finiteness of number of ends of a solution should be equivalent to the finiteness of Morse index, and it is proven in [21] that finite Morse index of a solution implies finite number of ends of the solution while the two-end solution and four-end solution have Morse index 0 and 1 respectively as shown in this paper. It is also shown in Section 10 of this paper that a $2k$-end solution must have Morse index at least $\lceil k/2 \rceil$. The equivalence of general $2k$-end solution and finite Morse index shall be addressed in future work. It is interesting to point out that the corresponding equivalent results for Allen-Cahn equation have been obtained in [13] and [22].

3. Preliminary

In this section we collect several basic results on solutions of (1.1) in $\mathbb{R}^2$ as well as some technical results needed in this paper. In this section, it is only assumed that u is a solution of (1.1) with finite ends.

The following two propositions are Proposition 2.1 and Proposition 2.4 in [21].

Proposition 3.1 *(Modica inequality). In Ω, $|\nabla u|^2 \le 1$, where the inequality is strict unless u is one dimensional.*

Proposition 3.2 *(Convexity). Every connected component of Ω^c is convex. Moreover, it is strictly convex unless u is one dimensional.*

The following monotonicity formula is a consequence of the Modica inequality and the Pohazaev identity, which is similar to the case of Allen-Cahn equation, see also [23, Proposition 2.4] for a proof in a similar setting in the presence of free boundaries.

Proposition 3.3 *(Monotonicity formula). For any $X \in \mathbb{R}^2$,*

$$E(R;X) := \frac{1}{R} \int\limits_{B_R(X)} \left[|\nabla u|^2 + \chi_{\{-1<u<1\}} \right]$$

is non-decreasing in R.

The following two results can be obtained as in [10], or by a blowing down analysis, using a Hutchinson-Tonegawa type theory as presented in [23, Section 3].

Proposition 3.4 *(Energy quantization). Suppose u has $2k$ ends, $k \ge 1$. For any $X \in \mathbb{R}^2$,*

$$\lim_{R \to +\infty} E(R;X) = 4k.$$

Proposition 3.5 *(Balancing condition). Suppose u has $2k$ ends, $k \ge 1$, and e_i is the asymptotic direction of these ends, $1 \le i \le 2k$. Then*

$$\sum_{i=1}^{2k} e_i = 0.$$

The proofs of Propositions 3.3-3.5 will be given in the Appendix A.

For four end solutions ($k = 2$), by this balancing condition and noting that each e_i is a unit vector, after a rotation, the asymptotic directions have the form

$$\begin{aligned} e_1 &= (\cos\alpha, \sin\alpha), \quad e_2 = (\cos\alpha, -\sin\alpha), \\ e_3 &= (-\cos\alpha, \sin\alpha), \quad e_4 = (-\cos\alpha, -\sin\alpha), \end{aligned} \tag{3.1}$$

where $\alpha \in [0, \pi/2]$ is the angle defined in Definition 2.3. This configuration is evenly symmetric with respect to the x and y axis.

From now on we assume the following lemma can be proved by using De Giorgi type result, i.e. the characterization of one dimensional solutions (see [21, Lemma 3.4]).

Lemma 3.6. *For any $X_i \in \Omega$, $|X_i| \to +\infty$, there exists a subsequence of*

$$u_i(X) := u(X_i + X)$$

converging to a one dimensional solution in C^1 sense on any compact set of $\mathbb{R}^2$.

The following result is also taken from [21], see Lemma 4.5 and Section 5 therein.

Proposition 3.7 *(Refined asymptotics). There exist two positive constants C and μ so that the following holds. Suppose u is a solution with finite ends. Then for every end of Ω, there exists a ray L (which we assume to be the positive x axis) so that outside a compact set this end has the form*

$$\left\{(x, y) : f^-(x) < y < f^+(x)\right\},$$

where $f^\pm$ are convex (concave) functions, satisfying

$$|f^\pm(x) \mp 1| \le Ce^{-\mu x}, \qquad \text{as } x \to +\infty.$$

Moreover, as $x \to +\infty$,

$$u(x, y) \to \begin{cases} 1, & y \ge 1, \\ y, & |y| \le 1 \\ -1, & y \le -1 \end{cases}$$

uniformly in $\mathbb{R}$.

Next, we recall two results on nodal sets of solutions to (1.4).

Proposition 3.8. *Let φ be a solution of (1.4). Then the nodal set $\{\varphi = 0\} \cap \Omega$ consists of a singular set of isolated points and a family of smooth embedded curves with their end points lying in the singular set, $\partial\Omega$ or at infinity.*

Proof. Results on the structure of nodal sets in the interior are classical, see [3]. For nodal curves near boundary, see [8, Appendix B]. □

For the nodal set of directional derivatives $u_e := e \cdot \nabla u$, because it satisfies the linearized equation (1.3), we can say something more.

Lemma 3.9. *For any $X \in \{u_e = 0\} \cap \partial\Omega$, there is only one smooth curve belonging to the nodal set of $\{u_e = 0\}$ emanating from X. Moreover, this curve intersects $\partial\Omega$ orthogonally.*

Proof. Without loss of generality, assume $X = \mathbf{0}$, $e = (1,0)$, and there holds locally around $X = \mathbf{0}$,

$$\Omega = \{(x,y) : y > f(x)\},$$

where f is a smooth concave function satisfying $f(0) = f'(0) = 0$.

By Proposition 3.2, we have from the strict convexity of Ω^c that

$$u_{xx}(\mathbf{0}) = -H(\mathbf{0}) < 0.$$

After extending u to Ω^c in a smooth way, we can use the implicit function theorem to deduce that locally $\{u_x = 0\}$ is a single smooth curve. A differentiation of the free boundary condition in (1.1) also shows that $u_{xy}(\mathbf{0}) = 0$, so this nodal curve is orthogonal to $\partial\Omega$ at $\mathbf{0}$. □

It is also useful to note that, by the free boundary condition in (1.1), $u_e(X) = 0$ if and only if e is the tangent vector of $\partial\Omega$ at X.

Finally we present a technical lemma on Liouville property for elliptic equations in $\mathbb{R}^2$.

Lemma 3.10. *Suppose $\mathcal{D}$ is a domain (bounded or unbounded) with piecewise smooth boundary, $\sigma \in C(\overline{\mathcal{D}})$ is positive. Assume $\varphi \in C^1(\overline{\mathcal{D}})$ satisfies weakly (in distributional sense)*

$$\begin{cases} \varphi \cdot div\left(\sigma^2 \nabla\varphi\right) \geq 0, & in\ \mathcal{D} \\ \varphi = 0, & on\ \partial\mathcal{D}. \end{cases} \tag{3.2}$$

If $\sigma\varphi \in L^\infty(\mathcal{D})$, then $\varphi \equiv 0$ in $\mathcal{D}$.

Proof. If $\mathcal{D}$ is bounded, this follows directly from the maximum principle.

If $\mathcal{D}$ is unbounded, we use the method of [7] (see also [2]). For any $\eta \in C_0^\infty(\mathbb{R}^2)$, testing (3.2) with η^2 we get

$$\int_{\mathcal{D}} \sigma^2 |\nabla\varphi|^2 \eta^2 \leq 4\|\sigma\varphi\|^2_{L^\infty(\mathcal{D})} \int_{\mathcal{D}} |\nabla\eta|^2. \tag{3.3}$$

For each $R > 1$, take η to be the standard log cut-off function

$$\eta(X) := \begin{cases} 1, & |X| \leq R, \\ 2 - \frac{\log|X|}{\log R}, & R \leq |X| \leq R^2 \\ 0, & |X| \geq R^2. \end{cases} \tag{3.4}$$

Substituting this function into (3.3) and then letting $R \to +\infty$, we obtain

$$\int_{\mathcal{D}} \sigma^2 |\nabla \varphi|^2 = 0.$$

Since $\varphi = 0$ on $\partial\mathcal{D}$, $\varphi \equiv 0$ in $\mathcal{D}$. $\square$

4. Even symmetry: the method of moving planes

From now on u denotes a solution of (1.1) with four ends. Assume its four asymptotic directions are given as in (3.1). In this section, we use the method of moving planes to prove the even symmetry of u in x and y. We mainly follow the treatment in [10], with one distinct point where Serrin's method in [19] is applied to treat the case when two free boundaries touch tangentially on the boundary.

There are two unbounded connected components of $\{u = 1\}$ (or $\{u = -1\}$), denoted by $D_i^{\pm}$ $(i = 1, 2)$ respectively. They are given by

$$D_1^+ = \{y > f_+(x)\}, \qquad D_2^+ = \{y < f_-(x)\},$$

$$D_1^- = \{x > g_+(y)\}, \qquad D_2^- = \{x < g_-(y)\}.$$

Here f_+ and g_+ are convex functions, f_- and g_- are concave functions, satisfying $f_+ > f_-$ and $g_+ > g_-$.

First we show that

Lemma 4.1. $0 < \alpha < \pi/2$.

Proof. Assume for example, $\alpha = 0$. Because f_+ is convex, this implies that

$$\lim_{x \to \pm\infty} f_+'(x) = 0.$$

Using convexity once again we deduce that $f_+ \equiv \text{const}$. Then by Proposition 3.2, u is one dimensional, this is a contradiction. $\square$

Denote $k = \tan\alpha > 0$. By Proposition 3.7, there exist four constants A_i, $1 \le i \le 4$ such that, the end of $\{-1 < u < 1\}$ in the i-th quadrant has the asymptotical expansion

$$\left\{ \pm kx + A_i - \frac{1}{\cos\alpha} + o(1) < y < \pm kx + A_i + \frac{1}{\cos\alpha} + o(1) \right\}, \tag{4.1}$$

where we take the positive sign in the first and third quadrant and the negative sign in the second and fourth quadrant.

By (4.1), we have

$$
\begin{cases}
f_+(x) = kx + A_1 + \dfrac{1}{\cos\alpha} + o(1), & \text{as } x \to +\infty, \\
f_+(x) = -kx + A_2 + \dfrac{1}{\cos\alpha} + o(1), & \text{as } x \to -\infty, \\
f_-(x) = -kx + A_4 - \dfrac{1}{\cos\alpha} + o(1), & \text{as } x \to +\infty, \\
f_-(x) = kx + A_3 - \dfrac{1}{\cos\alpha} + o(1), & \text{as } x \to -\infty.
\end{cases}
\tag{4.2}
$$

Similarly, if we write the two inverse functions of $g_+(y)$ for $x \geq x_2$ as $y = g_{+,1}(x)$ and $g_{+,2}(x)$ with $y = g_{+,1}(x) \geq g_{+,2}(x), x > x_2$ and $g_{+,1}(x_2) = g_{+,2}(x_2)$, and write the two inverse functions of $g_-(y)$ for $x \leq x_1$ as $y = g_{-,1}(x)$ and $g_{-,2}(x)$ with $y = g_{-,1}(x) \geq g_{-,2}(x), x < x_1$ and $g_{-,1}(x_1) = g_{-,2}(x_1)$, then we have

$$
\begin{cases}
g_{+,1}(x) = kx + A_1 - \dfrac{1}{\cos\alpha} + o(1), & \text{as } x \to +\infty, \\
g_{+,2}(x) = -kx + A_4 + \dfrac{1}{\cos\alpha} + o(1), & \text{as } x \to +\infty, \\
g_{-,1}(x) = -kx + A_2 - \dfrac{1}{\cos\alpha} + o(1), & \text{as } x \to -\infty, \\
g_{-,2}(x) = kx + A_3 + \dfrac{1}{\cos\alpha} + o(1), & \text{as } x \to -\infty.
\end{cases}
\tag{4.3}
$$

Lemma 4.2. *The four constants $A_1, ..., A_4$ satisfy*

$$
A_1 + A_4 = A_2 + A_3. \tag{4.4}
$$

Proof. Define

$$
\mathcal{H}(x) := \int_{-\infty}^{+\infty} y \left[\chi_\Omega(x,y) + u_y(x,y)^2 - u_x(x,y)^2\right] dy.
$$

In fact, this integration is only on a finite interval because the integrands equal 0 in $\{y > f_+(x)\} \cup \{y < f_-(x)\}$.

Similar to [9], we have the following Hamiltonian identities after some computation

$$
\frac{d^2}{dx^2}\mathcal{H}(x) = 0. \tag{4.5}
$$

Therefore $\mathcal{H}$ is a linear function. By the expansion in (4.2) (in particular, $f_+ + f_-$ are bounded as $x \to \pm\infty$) and the asymptotic behavior of u along each end from Proposition 3.7, we see that $\mathcal{H}(x)$ is bounded as $x \to \pm\infty$. Hence $\mathcal{H}(x)$ is a constant function, in particular,

$$\lim_{x\to+\infty}\mathcal{H}(x)=\lim_{x\to-\infty}\mathcal{H}(x).$$

Using the fact that u looks like one dimensional solutions at infinity (Proposition 3.7) once again, we get a positive constant $C(\alpha)$ depending only on α such that

$$\lim_{x\to+\infty}\mathcal{H}(x)=C(\alpha)\left(A_1+A_4\right),\quad \lim_{x\to-\infty}\mathcal{H}(x)=C(\alpha)\left(A_2+A_3\right).$$

The identity in (4.4) follows. $\square$

With this relation in hand, after the translation $(x,y)\mapsto(x-\frac{A_1-A_2}{2k},y+\frac{A_1+A_4}{2})$, we may assume for some constant A,

$$A_1=A_2=A,\quad A_3=A_4=-A.$$

Then the four asymptotic rays in (4.2) and (4.3) are evenly symmetric with respect to the x and y axis.

With these preliminaries now we come to

Proof of Theorem 2.1. We only prove the even symmetry in x. For each $\lambda\in\mathbb{R}$, define

$$u_\lambda(x,y):=u(2\lambda-x,y),\quad \mathcal{D}_\lambda:=\{x>\lambda\}.$$

Step 1. If λ is sufficiently large, then $u_\lambda\geq u$ in $\mathcal{D}_\lambda$.

Indeed, let

$$\Omega_\lambda:=\{u>-1\}\cap\mathcal{D}_\lambda\setminus\{u_\lambda=1\}$$

It is obvious that $u_\lambda\geq u$ in

$$\mathcal{D}_\lambda\setminus\Omega_\lambda=\left(\{u=-1\}\cup\{u_\lambda=1\}\right)\cap\mathcal{D}_\lambda.$$

By the expansions in (4.2) and (4.3), if λ is sufficiently large, then Ω_λ is a bounded set and $u_\lambda\geq u$ on $\partial\Omega_\lambda$. Hence by the maximum principle, $u_\lambda\geq u$ in Ω_λ.

Therefore, the claim follows.

Step 2. Now the following constant is well defined:

$$\Lambda:=\inf\{\lambda:\quad u_{\lambda'}\geq u\ \text{ in }\ \mathcal{D}_{\lambda'},\ \lambda'\geq\lambda\}.$$

We claim that $\Lambda=0$.

Assume by the contrary that $\Lambda>0$. By the expansions in (4.2) and (4.3), when $x\gg1$ and $y>0$,

$$\{u_\Lambda = 0\} = \{(x, y) : y = kx + A - 2k\Lambda + o(1)\}$$

lies below

$$\{u = 0\} = \{(x, y) : y = kx + A + o(1)\}.$$

Combining this fact with the strong maximum principle and Hopf lemma, we deduce that the free boundaries $\partial\{-1 < u_\Lambda < 1\} \cap \mathcal{D}_\Lambda$ and $\partial\{-1 < u < 1\} \cap \mathcal{D}_\Lambda$ do not touch.

By definition, there exists a sequence $\lambda_i \le \Lambda$ and $\lambda_i \to \Lambda$ such that

$$\inf_{\mathcal{D}_{\lambda_i}} (u_{\lambda_i} - u) < 0. \tag{4.6}$$

Because $\Lambda > 0$, by the expansions in (4.2) and (4.3), when $x \gg 1$ and $y > 0$,

$$\{u_{\lambda_i} = 0\} = \{(x, y) : y = kx + A - 2k\lambda_i + o(1)\}$$

still lies below (with a fixed distance)

$$\{u = 0\} = \{(x, y) : y = kx + A + o(1)\}.$$

A similar phenomenon can be seen in $\{(x, y) : x \gg 1, y < 0\}$. Then by Proposition 3.7, we find a constant R (depending only on Λ) such that

$$u_{\lambda_i} \ge u \quad \text{in} \quad \{x > R\}.$$

This inequality also holds trivially in $\mathcal{D}_{\lambda_i} \cap \{|x| < R, |y| > R\}$, perhaps after enlarging R, because we always have $u_{\lambda_i} = u = 1$ in $\mathcal{D}_{\lambda_i} \cap \{|x| < R, |y| > R\}$.

Therefore the infimum in (4.6) is a minimum. Because $u_{\lambda_i} = u$ on $\{x = \lambda_i\}$, it is attained at a point $X_i \in \mathcal{D}_{\lambda_i}$. By the above discussion, X_i lies in a fixed compact set. Assume they converge to a limit point X_Λ. Then by continuity and recalling that $u_\Lambda \ge u$ in $\mathcal{D}_\Lambda$, we get

$$u(X_\Lambda) = u_\Lambda(X_\Lambda).$$

There are three cases depending on the position of X_Λ.

Case 1. $X_\Lambda \in \{x > \Lambda\}$.

In this case, either $u(X_\Lambda) = u_\Lambda(X_\Lambda) = 1$ or $u(X_\Lambda) = u_\Lambda(X_\Lambda) = -1$. Without loss of generality, assume it is the first case. Then X_Λ is an interior point of $\{u_\Lambda = 1\}$. By continuity, for all i large, X_i is an interior point of $\{u_{\lambda_i} = 1\}$. In particular,

$$u_{\lambda_i}(X_i) = u(X_i).$$

This is a contradiction with (4.6).

Case 2. $X_\Lambda \in \{x=\Lambda\} \cap \{-1<u<1\}$.

In this case, X_Λ is an interior point of $\{-1<u<1\}$ and $\{-1<u_\Lambda<1\}$. By continuity, for all i large, X_i lies in the interior of $\{-1<u<1\} \cap \mathcal{D}_{\lambda_i}$ and $\{-1<u_{\lambda_i}<1\} \cap \mathcal{D}_{\lambda_i}$. Hence

$$\nabla u(X_i) - \nabla u_{\lambda_i}(X_i) = 0.$$

Passing to the limit, we deduce that

$$\partial_x \left(u_\Lambda - u\right)\left(X_\Lambda\right) = 0.$$

By the Hopf lemma, $u \equiv u_\Lambda$ in $\mathcal{D}_\Lambda$. This is a contradiction.

Case 3. $X_\Lambda \in \{x=\Lambda\} \cap \partial\{-1<u<1\}$.

Without loss of generality, assume $X_\Lambda \in \{x=\Lambda\} \cap \partial\{u=1\}$. We claim that the vertical line $\{x=\Lambda\}$ is normal to $\partial\{u=1\}$ at X_Λ. (Recall that $\partial\{u=1\}$ is a smooth curve.) If this claim is true, we can follow the same argument of Serrin in [19] (in particular, the second order Hopf lemma [19, Lemma 1] therein) to get a contradiction.

To prove the claim, denote $X_\Lambda := (\Lambda, y_\Lambda)$ and $X_i := (x_i, y_i)$. Assume in a small neighborhood of X_Λ, $\{u=1\} = \{y > f(x)\}$, where f is a smooth, convex function. First it is impossible that $f'(\Lambda) < 0$, because otherwise the reflection of a part of $\partial\{u = 1\} \cap \{x < \Lambda\}$ would lie above $\{\partial\{u=1\} \cap \{x > \Lambda\}$, which violates the assumption that $u_\Lambda \geq u$. If $f'(\Lambda) > 0$, because $u(2\lambda_i - x_i, y_i) < u(x_i, y_i)$, we must have $u(2\lambda_i - x_i, y_i) < 1$ and

$$u_x(X_\Lambda) = \lim_{i \to +\infty} u_x(2\lambda_i - x_i, y_i) \geq 0.$$

This is a contradiction, because by the free boundary condition and the above assumption on $f'(\Lambda)$,

$$u_x(X_\Lambda) = -\frac{f'(\Lambda)}{\sqrt{1+f'(\Lambda)^2}} < 0.$$

In conclusion, the only possibility is that $f'(\Lambda) = 0$.

Step 3. $\Lambda = 0$ implies that

$$u(-x,y) \geq u(x,y) \quad \text{in} \quad \{x>0\}.$$

We can repeat the moving plane procedure from the other direction, which leads to

$$u(-x, y) \geq u(x, y) \quad \text{in} \quad \{x < 0\}.$$

Combining these two inequalities together, we get

$$u(x, y) = u(-x, y). \quad \square$$

Because $u_\lambda \geq u$ in $\{x > \lambda\}$ for any $\lambda > 0$, by the strong maximum principle we obtain

Corollary 4.3. *Suppose u is a four end solution to* (1.1)*.*

(i) *In $\{x > 0\}$, $u_x < 0$ and in $\{x < 0\}$, $u_x > 0$.*
(ii) *In $\{y > 0\}$, $u_y > 0$ and in $\{y < 0\}$, $u_y < 0$.*

Finally, we show that $D_i^\pm$ $(i = 1, 2)$ are the only components of Ω^c, that is, there is no bounded component of Ω^c. Together with Lemma 4.1, this finishes the proof of Theorem 2.2.

Lemma 4.4.

(i) *There is no bounded component of Ω^c.*
(ii) *There is only one critical point of u in Ω, which is the origin and it is nondegenerate and of saddle type.*

Proof. (i) Assume there is a bounded component of $\{u = 1\}$. Take a point (x_0, y_0) in this component. Because $u(x_0, f^+(x_0)) = u(x_0, f^-(x_0)) = 1$, there exists a $y_1 \neq 0$ such that $(x_0, y_1) \in \Omega$ and $u_y(x_0, y_1) = 0$. This is a contradiction with Corollary 4.3.

(ii) By Corollary 4.3, we have

- $\nabla u(X) = 0$ if and only if $X = 0$;
- because $u_x = 0$ on $\{x = 0\}$, a differentiation in y shows that $u_{xy}(0) = 0$;
- because $u_x < 0$ in $\{x > 0\}$, by the Hopf Lemma, $u_{xx}(0) < 0$, and similarly, $u_{yy}(0) > 0$.

Hence 0 is a nondegenerate critical point and it is of saddle type. $\square$

5. Quasi Gauss map and uniqueness

In this section we introduce the quasi Gauss map and use it to prove the uniqueness part of Theorem 2.4. We will use the complex notation $z = x + iy$, where i is the imaginary root corresponding to the vector $(0, 1)$.

Since u is harmonic, the map $G := u_x - iu_y$ is holomorphic. Our main tool in this section is the following result.

Proposition 5.1. *G is a biholomorphism between Ω and $B_1(0)$.*

Proof. By Proposition 3.1, G maps Ω into $B_1(0)$. Moreover, since $|\nabla u| = 1$ on $\partial\Omega$, we have

$$u_x\left(x, f_+(x)\right) = -\frac{f_+'(x)}{\sqrt{1+f_+'(x)^2}}, \quad u_y\left(x, f_+(x)\right) = \frac{1}{\sqrt{1+f_+'(x)^2}}, \tag{5.1}$$

and similar identities hold on $\{y = f_-(x)\}$ and $\{x = g_\pm(y)\}$. Using the expansion (4.2) and (4.3) (which also hold in C^1 sense by Lemma 3.6), $G(\partial\Omega)$ is the set $\mathbb{S}^1 \setminus \{\pm e^{i(\frac{\pi}{2}-\alpha)}, \pm e^{-i(\frac{\pi}{2}-\alpha)}\}$. In particular, $\overline{\nabla u(\partial\Omega)} = \mathbb{S}^1$.

By (5.1), ∇u is homotopic to the vector field $(-x, y)$ on $\partial\Omega$. Therefore, the topological degree $\deg(\nabla u, \partial\Omega) = -1$. Hence for any $e \in B_1(0)$, there exists a $z \in \Omega$ such that $G(z) = e$, that is, G is surjective.

Next, for any $z \in G^{-1}(e)$, because u_z is holomorphic, the index of ∇u at z is a negative integer. Then by the Poincare-Hopf index formula, we get a contradiction with the fact that $\deg(\nabla u, \partial\Omega) = -1$ unless there is only one point in $G^{-1}(e)$. □

Remark 5.2. The map G corresponds to the Gauss map for minimal surfaces. For minimal surfaces, the image of Gauss map lies in the unit sphere, while here (and more generally, for many semilinear elliptic equations) the image of G is the unit ball. This is different from the correspondence established in [20], where the one-to-one correspondence between certain minimal surfaces and the solutions to the one phase free boundary problem in $\mathbb{R}^2$ is proven. Indeed, the nature of the problem discussed in this paper is very different from the one in [20] despite of similarity in the equation: the free boundary here consists of two components at both $u = 1$ and $u = -1$ while the one phase problem only deals with the free boundary at $u = 0$. Also we are looking at the geometry of entire solutions which have four ends structure and exhibit minimal surface behavior without adding one additional dimension.

Let $F : B_1(0) \to \Omega$ be the inverse of G. Denote

$$v(z) := u(F(z)).$$

It is a harmonic function in $B_1(0)$, satisfying the boundary condition

$$\begin{cases} v(e^{i\theta}) = 1, & \text{for } \theta \in \left(\frac{\pi}{2} - \alpha, \frac{\pi}{2} + \alpha\right) \cup \left(\frac{3\pi}{2} - \alpha, \frac{3\pi}{2} + \alpha\right), \\ v(e^{i\theta}) = -1, & \text{for } \theta \in \left(\frac{\pi}{2} + \alpha, \frac{3\pi}{2} - \alpha\right) \cup \left(-\frac{\pi}{2} + \alpha, \frac{\pi}{2} - \alpha\right). \end{cases} \tag{5.2}$$

By Poisson representation formula, v is uniquely determined by α.

Since $u(z) = v(G(z))$, taking derivative in z gives

$$G(z) = u_z(z) = v_z(G(z))G'(z). \tag{5.3}$$

Because F is the inverse of G, this identity is equivalent to

$$F'(z) = \frac{v_z(z)}{z}. \tag{5.4}$$

Using this quasi Gauss map, now we prove the uniqueness part of Theorem 2.4.

Proof of Theorem 2.4: Uniqueness. Suppose there are two even, four end solutions u_1 and u_2 with angle α. Define G_1, G_2 and their inverse F_1, F_2 as above. Then $u_1 \circ F_1 \equiv u_2 \circ F_2$ in $B_1(0)$, which implies that $F_1 \equiv F_2$ because both F_1' and F_2' are given by (5.4) and they satisfy $F_1(0) = F_2(0) = 0$. Hence $F_1(B_1(0)) = F_2(B_1(0))$, and $u_1 \equiv u_2$ in this domain, because they have the same boundary value. □

6. Nondegeneracy

This section is devoted to the proof of Theorem 2.6. In fact, for applications in Section 10, we prove something more.

Proposition 6.1. *Suppose u is a four end solution to (1.1) and $\varphi \in L^\infty(\overline{\Omega}) \cap C^2(\overline{\Omega})$ is a solution of the eigenvalue problem (1.4), where $\lambda \le 0$. Then we have*

(i) *if $\lambda < 0$, either $\varphi \equiv 0$ or $\varphi > 0$ in $\overline{\Omega}$;*
(ii) *if $\lambda = 0$, $\varphi = au_x + bu_y$ in Ω for two constants a and b.*

This proposition follows from the following two lemmas.

Lemma 6.2. *Assume the condition of Proposition 6.1.*

- *If $\lambda < 0$, φ is even in x and y.*
- *If $\lambda = 0$, there exist two constants a and b such that $\varphi - au_x - bu_y$ is even in x and y.*

Proof. Let $\widetilde{\varphi}(x, y) := \varphi(x, y) - \varphi(-x, y)$. Note that $\widetilde{\varphi}$ is an odd function of x and $\tilde{\varphi} = 0$ in $\Omega \cap \{x = 0\}$.

Although $u_x = 0$ on $\{x = 0\} \cap \Omega$, by Corollary 4.3, u_x has definite signs on the two sides of $\{x = 0\} \cap \Omega$, hence by the Hopf lemma as well as Proposition 3.2, $u_{xx} < 0$ strictly on $\{x = 0\} \cap \overline{\Omega}$. This then implies that $\widetilde{\varphi}/u_x$ is well defined and it is smooth in $\overline{\Omega}$.

It can be directly checked that

$$\begin{cases} -\operatorname{div}\left(u_x^2 \nabla \dfrac{\widetilde{\varphi}}{u_x}\right) = \lambda \widetilde{\varphi} u_x, & \text{in } \Omega \cap \{x \neq 0\}, \\ \partial_\nu \dfrac{\widetilde{\varphi}}{u_x} = 0, \quad \text{on } \partial\Omega \cap \{x \neq 0\}. \end{cases} \tag{6.1}$$

For any $\eta \in C_0^\infty(\mathbb{R}^2)$, multiplying (6.1) by $\frac{\widetilde{\varphi}}{u_x}\eta^2$, integrating in $\Omega \cap \{x > 0\}$ and $\Omega \cap \{x < 0\}$ respectively, and then adding these two equalities, we obtain

$$\int_\Omega u_x^2 \left|\nabla \frac{\widetilde{\varphi}}{u_x}\right|^2 \eta^2 \leq 2 \left(\int_{\Omega \cap \{\nabla \eta \neq 0\}} u_x^2 \left|\nabla \frac{\widetilde{\varphi}}{u_x}\right|^2 \eta^2 \right)^{\frac{1}{2}} \left(\int_{\Omega \cap \{\nabla \eta \neq 0\}} \widetilde{\varphi}^2 |\nabla \eta|^2 \right)^{\frac{1}{2}}.$$

Taking η to be the standard log cut-off function as in the proof of Lemma 3.10, we deduce that $\frac{\widetilde{\varphi}}{u_x}$ is constant in $\Omega \cap \{x > 0\}$ and $\Omega \cap \{x < 0\}$. By continuity, we get a constant a such that $\widetilde{\varphi} \equiv 2au_x$ in Ω.

Similarly setting $\hat{\varphi}(x, y) = \frac{1}{2}[(\varphi(x, y) + \varphi(-x, y)) - (\varphi(x, -y) + \varphi(-x, -y))]$, an odd function of y, we can prove that there exists a constant b such that $\hat{\varphi} \equiv 2bu_y$ in Ω. Note that

$$\begin{aligned} &2\varphi(x, y) - [\widetilde{\varphi}(x, y) + \hat{\varphi}(x, y)] \\ &= \frac{1}{2}[(\varphi(x, y) + \varphi(-x, y)) + (\varphi(x, -y) + \varphi(-x, -y))] \\ &=: \check{\varphi}(x, y), \end{aligned}$$

where $\check{\varphi}$ is even with respect to both the x and y variables.

If $\lambda < 0$, substituting the equality $\widetilde{\varphi} \equiv 2au_x$ into (6.1) we get $a = 0$. Hence φ is even in x. Similar argument gives $b = 0$, which yields that φ is also even in y.

If $\lambda = 0$, the relation $\frac{1}{2}\check{\varphi} = \varphi - \frac{1}{2}[\widetilde{\varphi} + \hat{\varphi}] = \varphi - (au_x + bu_y)$ gives the desired result. □

Lemma 6.3. *Suppose u is a four end solution to (1.1) and $\varphi \in L^\infty(\overline{\Omega}) \cap C^2(\overline{\Omega})$ is a solution of the eigenvalue problem (1.4), where $\lambda \leq 0$. If φ is even in x and y, then either $\varphi > 0$ or $\varphi \equiv 0$ in $\overline{\Omega}$.*

Proof. We set the nodal set of φ

$$\mathcal{N} := \{(x, y) \in \bar{\Omega} : \varphi(x, y) = 0\}.$$

If $\mathcal{N} = \bar{\Omega}$, then $\varphi \equiv 0$ in $\bar{\Omega}$. If $\mathcal{N} = \emptyset$, then $\varphi > 0$ in $\bar{\Omega}$ (perhaps after changing the sign of φ).

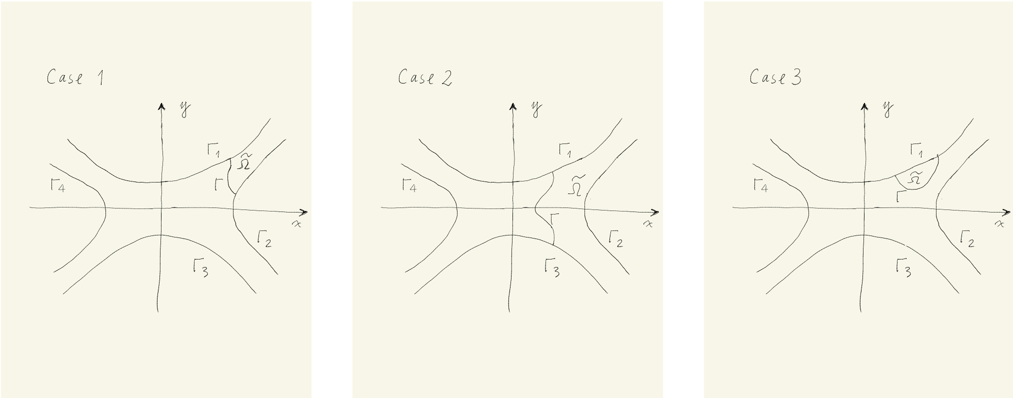


Fig. 1. Three Cases.

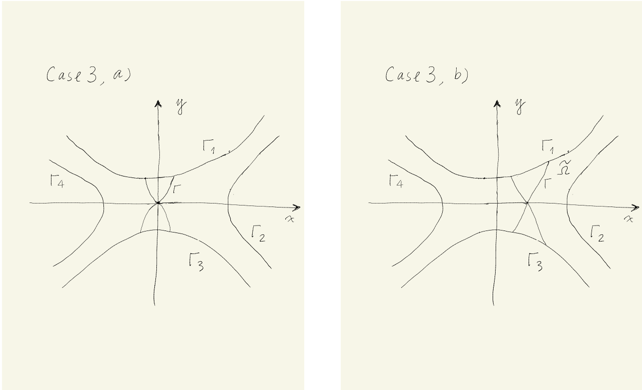
If $\mathcal{N}$ is not an empty set, from Proposition 3.8 we know that $\mathcal{N}$ consists of a singular set of isolated points and a family of smooth embedded curves with their end points lying in the singular set, $\partial\Omega$ or at infinity.

By Lemma 3.10, there is no nodal domain of φ disjoint from $\partial\Omega$. Hence we only need to rule out the case that the piecewise smooth nodal curves of φ have their end points lying in $\partial\Omega$.

There are three cases. Denote the four parts $\{y = f_+(x)\}, \{x = g_+(y)\}, \{y = f_-(x)\}, \{x = g_-(x)\}$ of $\partial\Omega$ as Γ_i $(i = 1, 2, 3, 4)$ respectively. See Fig. 1.

Case 1: There exists a nodal curve Γ connecting Γ_1 and Γ_2. If Γ intersects y-axis, by the even symmetry of φ in x and y, we know that there exists a nodal curve $\tilde{\Gamma}$ connecting Γ_1 to Γ_1 and being above the x-axis and symmetric about y-axis. We leave the discussion of this special sub case to Case 3 below. So we assume in Case 1 without loss of generality that Γ does not intersect y-axis. Note that there exists other nodal curve $\hat{\Gamma}$ connecting Γ_2 and Γ_3 by the even symmetry of φ in y and we can assume that Γ_1 is above x-axis if it is connected with $\hat{\Gamma}$. We denote the unbounded domain lying at the upper right side of Γ and being enclosed by Γ and part of $\partial\Omega$ as $\tilde{\Omega}$. We note that $\tilde{\Omega}$ is located on the right side of Γ and should not contain a portion of y-axis (since it is the unbounded part of Γ) except that it may touch y-axis at the origin, while the latter will be discussed in Case 3. Hence, the argument from Lemma 6.2 works. Similar argument as in Lemma 6.2 gives $\varphi = au_x$ in $\tilde{\Omega}$. By the unique continuation theorem, $\varphi = au_x$ in Ω. Since φ is even in x and y, $a = 0$, which yields $\varphi \equiv 0$ in $\tilde{\Omega}$. We obtain a contradiction.

Case 2: A nodal curve Γ connects Γ_1 and Γ_3 directly. (The case that a nodal curve connects Γ_2 and Γ_4 directly can be proved with the same method.) As in Case 1, we can assume that Γ does not intersect with y-axis, and denote the unbounded domain lying at the right side of Γ and being enclosed by Γ and $\partial\Omega$ as $\tilde{\Omega}$. We also note that $\tilde{\Omega}$ is located on the right side of the Γ and should not intersect the y-axis (since it is the unbounded part). Similar argument as in Case 1 shows that this case is also impossible.



(A) (B)

Fig. 2. Sub Cases: Intersection with x-axis.

Case 3: A nodal curve Γ starts from Γ_1 and returns back to Γ_1. (The case that a nodal curve starts from Γ_2 and return back to Γ_2 can be proved by the same method.) Denote the bounded domain enclosed by Γ and Γ_1 as $\tilde{\Omega}$. We also note that $\tilde{\Omega}$ should not contain a portion of the x-axis in view of the even symmetry of φ in y and can be chosen as above x-axis by the reflection of Γ about x-axis, although it may touch x-axis at some points. Indeed, we may assume that $\tilde{\Omega}$ can only touch x-axis at the origin, since otherwise we can reduce the case to Case 2 by replacing Γ properly (for example by picking Γ as the right branch of the nodal curve symmetric about x-axis, see Fig. 2 (B)). Hence we only need to focus on the worst scenario when the origin is the intersection point of Γ with x-axis (and y-axis). It is well known that the nodal set of the eigenfunction φ behaves locally like that of a harmonic polynomial, i.e., finitely many lines forming equal angles at the origin. See, for example, Theorem 2.5 in [4]. By the even symmetry of φ with respect to both axes, we know that the nodal set consists of at least two lines and it does not coincide with the axes. Hence we may choose Γ to be symmetric about y-axis and there exists no nodal curve inside $\tilde{\Omega}$ (see Fig. 2 (A)). In particular, we have $|x| \leq Cy$ when $(x, y) \in \tilde{\Omega}$ for some constant $C > 0$.

Now we define $\tilde{\Omega}_\delta := \{(x, y) \in \tilde{\Omega}, x^2 + y^2 > \delta\}$ for any $\delta > 0$ sufficiently small. Then we have

$$0 = -\int\limits_{\tilde{\Omega}_\delta} \operatorname{div}\left(u_y^2 \nabla \frac{\varphi}{u_y}\right) \frac{\varphi}{u_y} = \int\limits_{\tilde{\Omega}_\delta} u_y^2 \left|\nabla \frac{\varphi}{u_y}\right|^2 + \frac{1}{\delta} \int\limits_{\tilde{\Omega} \cap \partial B_\delta} (\varphi \nabla \varphi - \frac{\varphi^2}{u_y} \nabla u_y) \cdot (x, y) ds.$$

Letting $\delta \to 0$, we obtain

$$\int\limits_{\tilde{\Omega}} u_y^2 \left|\nabla \frac{\varphi}{u_y}\right|^2 = 0$$

which gives $\varphi = au_y$ in $\tilde{\Omega}$, and a contradiction follows similarly. We note that here even when Γ touches the x-axis at the origin, $\frac{\varphi}{u_y}$ is still bounded since that $u_{yy} < 0$ strictly on $\{y = 0\} \cap \Omega$, hence the boundary integral above tends to 0 as δ goes to 0. □

7. Moduli space

Theorem 2.4 implies that $\alpha \mapsto u_\alpha$, $\alpha \in (0, \pi/2)$ is a parametrization of $\mathcal{M}$. (That this map is surjective will be proven in Section 9.) In this section, we study the global structure of $\mathcal{M}$, including the closedness of $\mathcal{M}$ and its boundary behavior.

We need a technical result on the distance between $\partial\{u = 1\}$ and $\partial\{u = -1\}$.

Lemma 7.1. *There exists a universal constant C such that, for any solution u of (1.1) in $\mathbb{R}^2$ and $X \in \partial\{u = 1\}$, $dist(X, \partial\{u = -1\}) \le C$.*

Proof. Assume by the contrary, there exist a sequence of solutions u_i of (1.1) in $\mathbb{R}^2$, and $X_i \in \partial\{u_i = 1\}$ such that

$$\mathrm{dist}(X_i, \partial\{u_i = -1\}) \ge i.$$

Let $v_i(X) := 1 - u_i(X_i + X)$, which satisfies $0 \le v_i \le 2$ and $\Delta v_i = 0$ in $\{v_i > 0\} \cap B_i(0)$. (Note that $v_i < 2$ in $B_i(0)$.)

By the Lipschitz bound in Proposition 3.1, we can assume (after passing to a subsequence) v_i converges to a limit v_∞ uniformly on any compact set of $\mathbb{R}^2$. Because $v_\infty \ge 0$ and $\Delta v_\infty = 0$ in $\{v_\infty > 0\}$, v_∞ is subharmonic in the entire space. Since $v_\infty \le 2$, by the Liouville theorem

$$v_\infty \equiv v_\infty(0) = 0.$$

In particular,

$$\int_{B_1(0)} \Delta v_\infty = 0. \tag{7.1}$$

On the other hand, since $X_i \in \partial\{u_i = 1\}$, $|\nabla u_i| = 1$ on $\partial\{u_i = 1\}$ and $\partial\{u_i = 1\} \cap B_{1/2}(X_i)$ is a convex curve with end points in $\partial B_{1/2}(X_i)$ and it also contains X_i, for all i large,

$$\int_{B_{1/2}(0)} \Delta v_i = \mathcal{H}^1(\partial\{u_i = 1\} \cap B_{1/2}(0)) \ge 1.$$

Here $\mathcal{H}^1(\partial\{u_i = 1\} \cap B_{1/2}(0))$ denotes the length of the curve $\partial\{u_i = 1\} \cap B_{1/2}(0)$. Passing to the limit we obtain a contradiction with (7.1). □

Proposition 7.2 *(Closedness of $\mathcal{M}$). Given a sequence $\alpha_i \in (0,\pi/2)$ and a sequence of four end, evenly symmetric solutions u_{α_i} with angle α_i, if*

$$\lim_{i\to+\infty} \alpha_i = \alpha_0 \in (0,\pi/2),$$

then u_{α_i} converges to u_{α_0} uniformly on any compact set of $\mathbb{R}^2$, where u_{α_0} is the four end, evenly symmetric solution with angle α_0.

Proof. Because $|u_{\alpha_i}| \le 1$ and $|\nabla u_{\alpha_i}| \le 1$ in $\mathbb{R}^2$, passing to a subsequence we get a limit u.

Recall that $\partial\{u_{\alpha_i} = 1\} \cap \{y > 0\}$ has the form

$$y = f_{\alpha_i}(x),$$

and $\partial\{u_{\alpha_i} = -1\} \cap \{x > 0\}$ has the form

$$x = g_{\alpha_i}(y),$$

where both f_{α_i} and g_{α_i} are positive convex even functions. Moreover,

$$\lim_{x\to+\infty} f'_{\alpha_i}(x) = \tan\alpha_i, \quad \lim_{y\to+\infty} g'_{\alpha_i}(y) = \frac{1}{\tan\alpha_i}.$$

By Lemma 7.1, both $f_{\alpha_i}(0)$ and $g_{\alpha_i}(0)$ remain bounded as $\alpha_i \to \alpha_0$. From these facts we deduce that, after passing to a subsequence, f_{α_i} and g_{α_i} converges uniformly on any compact set of $\mathbb{R}$ to two limits f and g respectively, where f and g are nonnegative convex even functions.

Because $|\nabla u_{\alpha_i}| \le 1$, the distance between $\{y = f_{\alpha_i}(x)\}$ and $\{x = g_{\alpha_i}(y)\}$ is larger than 2. Hence $\{y = f(x)\}$ and $\{x = g(y)\}$ do not touch.

As in [9], the following Hamiltonian identity for u_{α_i} holds,

$$\int_{-f_{\alpha_i}(0)}^{f_{\alpha_i}(0)} \left[-\left|\frac{\partial u_{\alpha_i}}{\partial x}(0,y)\right|^2 + \left|\frac{\partial u_{\alpha_i}}{\partial y}(0,y)\right|^2 + 1\right] dy = \frac{4}{\cos\alpha_i}. \tag{7.2}$$

Substituting $|\nabla u_{\alpha_i}| \le 1$ into this identity we get

$$f_{\alpha_i}(0) \ge \frac{1}{\cos\alpha_i}.$$

Passing to the limit we get

$$f(0) \ge \frac{1}{\cos\alpha_0}.$$

Recall that 0 is the global minimal point of f, because it is a C^2 even convex function. Therefore $f > 0$ strictly. In other words, the two components of $\partial\{u = 1\}$ do not touch, too.

By the regularity theory on free boundaries in [1], the C^4 norms of f_{α_i} and g_{α_i} are uniformly bounded. Hence they also converge in C^3 and $f, g \in C^4(\mathbb{R})$. By standard elliptic estimates, $\|u_{\alpha_i}\|_{C^3(\overline{\Omega_{\alpha_i}})}$ are uniformly bounded, where

$$\Omega_{\alpha_i} = \{(x, y) : -f_{\alpha_i}(x) < y < f_{\alpha_i}(x), -g_{\alpha_i}(y) < x < g_{\alpha_i}(y)\}.$$

Then it is readily verified that u is an evenly symmetric solution of (1.1), where

$$\{-1 < u < 1\} = \{(x, y) : -f(x) < y < f(x), -g(y) < x < g(y)\}.$$

We have shown that u has four ends. Assume its angle is α. By the above smooth convergence of f_{α_i} and u_{α_i}, passing to the limit in (7.2) gives

$$\int_{-f(0)}^{f(0)} \left[-\left|\frac{\partial u}{\partial x}(0, y)\right|^2 + \left|\frac{\partial u}{\partial y}(0, y)\right|^2 + 1\right] dy = \frac{4}{\cos \alpha_0}.$$

Since Hamiltonian identity also holds for u, we must have $\alpha = \alpha_0$. Therefore u is the evenly symmetric, four end solution with angle α_0. $\square$

Finally, we study the boundary behavior of the moduli space, that is, the behavior of u_α when $\alpha \to 0$ or $\pi/2$.

Proof of Theorem 2.10. We only need to consider the case when $\alpha \to 0$ since the case $\alpha \to \pi/2$ is similar. As in Proposition 7.2, after passing to a subsequence, we can assume u_i converges to a limit u_0 uniformly on any compact set of $\mathbb{R}^2$.

Recall that $\partial\{u_i = 1\} \cap \{y > 0\} = \{y = f_i(x)\}$, and $\partial\{u_i = -1\} \cap \{x > 0\} = \{x = g_i(y)\}$, where both f_i and g_i are positive convex even functions. Moreover,

$$\lim_{x \to +\infty} f_i'(x) = \tan \alpha_i \to 0, \quad \lim_{y \to +\infty} g_i'(y) = \frac{1}{\tan \alpha_i} \to +\infty. \tag{7.3}$$

By Lemma 7.1, both $f_i(0)$ and $g_i(0)$ remain bounded as $\alpha_i \to 0$. From these we deduce that, after passing to a subsequence, f_i converges uniformly on any compact set of $\mathbb{R}$ to a limit f. By the convexity of f_i and (7.3), $f_i' \to 0$ uniformly on $\mathbb{R}$. Thus $f \equiv a$ for some constant $a \geq 0$.

When $x > g_i(0)$ and $y > 0$, there exists a concave function h_i such that $\{x = g_i(y)\} = \{y = h_i(x)\}$. After subtracting a subsequence, assume $g_i(0) \to b$ for some constant $b \geq 0$. Assume h_i converges to h uniformly on any compact set of $(b, +\infty)$. It is clear that h is continuous on $[b, +\infty)$ and $h(b) = 0$.

Because $|\nabla u_i| \leq 1$, the distance between $\{y = f_i(x)\}$ and $\{x = g_i(y)\}$ is larger than 2. For any $X_i \in \{y = f_i(x)\}$, if i is large enough, $v_i := 1 - u_i$ is a classical solution of the one phase problem

$$\begin{cases} \Delta v_i = 0, & \text{in } \{v_i > 0\} \cap B_{a/2}(X_i), \\ |\nabla v_i| = 1, & \text{on } \partial\{v_i > 0\} \cap B_{a/2}(X_i). \end{cases} \tag{7.4}$$

Furthermore, the free boundary $\partial\{v_i > 0\} \cap B_{a/2}(X_i)$ is the graph of a function. Since these functions converge to constant functions uniformly, the regularity theory in [1] applies, which says $\partial\{v_i > 0\} \cap B_{a/2}(X_i)$ are uniformly bounded in C^4. Then standard elliptic estimates lead to a uniform bound on the $C^3(\overline{\{v_i > 0\}} \cap B_{a/4}(X_i)$ norm of v_i. By pulling back via a diffeomorphism, we have that $v_i \to v_\infty$ in C^2 sense in $B_{a/4}(X_\infty)$, where v_∞ is still a classical solution of (7.4).

Since the curves in $\partial\{v_\infty > 0\}$ are flat, by Proposition 3.2, v_∞ is a one dimensional solution. Coming back to u_0, we get

$$u_0(x,y) = \begin{cases} y - a + 1, & \text{if } a - 2 \leq y \leq a, \\ -y - a + 1, & \text{if } -a \leq y \leq -a + 2, \\ 1, & \text{if } y \geq 2 \text{ or } y \leq -a. \end{cases}$$

Since $\Delta u_0 = 0$ in the open set $\{-1 < u_0 < 1\}$, by unique continuation principle u_0 must have the form as given in (2.1). In particular, $a = 2$ and $b = 0$. $\square$

Remark 7.3. For any $x > 0$, locally around $(x, 0)$, $u_i + 1$ converges to a two copy solution of (7.4).

At the origin $(0, 0)$, denote $r_i := g_i(0)$, which goes to 0 as $\alpha_i \to 0$. Define

$$v_i(x,y) := \frac{1}{r_i}\left[1 + u_i(r_i x, r_i y)\right].$$

Then v_i converges to the hairpin solution of the one phase problem (7.4) constructed by Hauswirth, Hélein and Pacard [12]. In high dimensions, the same phenomena have been observed in [16].

8. Saddle solution

In this section we prove

Proposition 8.1. *There exists an evenly symmetric, four end solution with angle $\alpha = \pi/4$.*

By the uniqueness part of Theorem 2.4, this solution is unique, hence it is oddly symmetric with respect to the line $\{x = y\}$. In fact, its nodal set $\{u = 0\}$ is exactly $\{x = y\} \cup \{x = -y\}$. This is called the saddle solution.

The existence of such a solution will be proved by finding a minimizer in the first quadrant with Dirichlet condition and then taking successive reflections across the axes. For any $R > 1$, consider the square

$$Q_R := \{(x, y) : 0 < x < R,\ 0 < y < R\}.$$

Take φ_R to be the function on ∂Q_R, defined as

$$\varphi_R = \begin{cases} 0, & \text{if } x = 0 \quad \text{or } y = 0, \\ 1, & \text{if } x = R, 1 \le y \le R \quad \text{or } y = R, 1 \le x \le R, \\ x, & \text{if } y = R, \quad 0 < x < 1, \\ y, & \text{if } x = R, \quad 0 < y < 1. \end{cases}$$

Let $u_R \in H^1(Q_R)$ be a minimizer of the functional

$$\mathcal{J}_{Q_R}(u) := \int\limits_{Q_R} \left(|\nabla u|^2 + \chi_{\{u<1\}}\right), \tag{8.1}$$

with Dirichlet data φ_R on ∂Q_R. The existence of such a minimizer can be proved as in [1]. Moreover, the results in [1] imply that

(1) $0 \le u_R \le 1$;
(2) there exists a constant C independent of R such that $|\nabla u_R| \le C$ in $\overline{Q}_{R-1}$;
(3) $\partial\{u_R = 1\}$ consist of smooth curves.

By constructing a suitable competitor, we get the following energy bound on u_R:

Lemma 8.2. *For any $R > 1$,*

$$\int\limits_{Q_R} \left(|\nabla u_R|^2 + \chi_{\{u_R<1\}}\right) \le 4R.$$

Proof. Take a test function w_R in Q_R, defined as

$$w_R = \begin{cases} xy, & \text{if } 0 \le x \le 1, \quad 0 \le y \le 1, \\ 1, & \text{if } 1 \le x \le R, \quad 1 \le y \le R, \\ x, & \text{if } 1 \le y \le R, \quad 0 < x < 1, \\ y, & \text{if } 1 \le x \le R, \quad 0 < y < 1. \end{cases}$$

It is easy to obtain the estimate

$$\int\limits_{Q_R} \left(|\nabla w_R|^2 + \chi_{\{w_R<1\}}\right) \le 4R.$$

The lemma follows by the minimality of u_R. □

Extend u_R by successive odd reflections to $D_R := \{|x| < R, |y| < R\}$, which is still denoted by u_R. It is clear that u_R is a solution of (1.1) in D_R with a proper domain Ω_R. By the monotonicity formula (Proposition 3.3), we have

Lemma 8.3. *For any $r \in (0, R)$,*

$$\int_{B_r(0)} \left(|\nabla u_R|^2 + \chi_{\{-1<u_R<1\}}\right) \leq 16r.$$

As $R \to +\infty$, by the Lipschitz bound on u_R in D_{R-1}, we can assume that u_R converges to a limit u uniformly on any compact set of $\mathbb{R}^2$.

By definition, $u_R = 0$ on $\{xy = 0\}$. Because $|\nabla u_R| \leq C$ in D_{R-1}, the distance between $\{xy = 0\}$ and $\partial\{-1 < u_R < 1\}$ in D_{R-1} is not less than $1/C$. Then by the regularity theory for the free boundary in [1] (in combination with the compactness for minimizers of (8.1), see [1, Section 4.7]), $\partial\{-1 < u_R < 1\}$ are uniformly bounded and converge in $C^{1,1/2}_{loc}$-norm as R goes to infinity.

Let

$$d_R := \mathrm{dist}(0, \partial\{-1 < u_R < 1\}).$$

By Lemma 8.3,

$$\pi d_R^2 \leq \int_{B_{d_R}(0)} \left[|\nabla u_R|^2 + \chi_{\{-1<u_R<1\}}\right] \leq 16 d_R.$$

Hence $d_R \leq 16/\pi$. Therefore the free boundary $\partial\{-1 < u_R < 1\}$ cannot escape to infinity. In particular, $\{u = 1\}$ is nonempty and u is not identically 0.

To describe the geometry of u, we need a monotonicity property for u.

Lemma 8.4. *In $\{0 < u < 1\} \cap \{x > 0, y > 0\}$, $u_x > 0$ and $u_y > 0$.*

Proof. We only prove the claim for u_x. Because u_x is harmonic in $\{0 < u < 1\} \cap \{x > 0, y > 0\}$, by the strong maximum principle, we need only to show that $u_x \geq 0$. Assume by the contrary, $\{u_x < 0\} \cap \{0 < u < 1\} \cap \{x > 0, y > 0\}$ is non-empty. By Proposition 3.8, there exists a point $X \in \{u_x = 0\} \cap \{0 < u < 1\} \cap \{x > 0, y > 0\}$ and a ball $B_\rho(X) \subset \{0 < u < 1\} \cap \{x > 0, y > 0\}$ such that $\nabla u_x \neq 0$ in $B_\rho(X)$, i.e. X is a regular point of the nodal set $\{u_x = 0\}$.

Because $u = 0$ on $\{xy = 0\}$ and $u > 0$ in the first quadrant, $u_x \geq 0$ on $\{xy = 0\}$. (Note that we have shown that $\{xy = 0\}$ is contained in $\{-1 < u < 1\}$.) For any $\eta \in C_0^1(\mathbb{R}^2)$, because u_x satisfies the linearized equation (1.3), testing it with $u_x^- \eta^2$ and integrating by parts, we obtain

$$\int_{\{x>0,y>0\}} |\nabla(u_x^-\eta)|^2 - \int_{\{x>0,y>0\}\cap\partial\{u=1\}} H(u_x^-\eta)^2 = \int_{\{x>0,y>0\}} (u_x^-)^2|\nabla\eta|^2. \tag{8.2}$$

Define a function ϕ in the following way: $\phi \equiv u_x^-$ outside $B_\rho(X)$, and it is the harmonic function in $B_\rho(X)$ with Dirichlet boundary value u_x^- on $\partial B_\rho(X)$. By this choice of $B_\rho(X)$, we know that u_x^- is not a harmonic function in $B_\rho(X)$ and hence does not minimize the Dirichlet integral. Therefore there exists a constant $\delta > 0$ (depending only on ρ and the choice of $B_\rho(X)$ above) such that

$$\int_{B_\rho(X)} |\nabla\phi|^2 \le \int_{B_\rho(X)} |\nabla u_x^-|^2 - \delta. \tag{8.3}$$

Combining this inequality with (8.2), we get

$$\int_{\{x>0,y>0\}} |\nabla(\phi\eta)|^2 - \int_{\{x>0,y>0\}\cap\partial\{u=1\}} H(\phi\eta)^2 \le \int_{\{x>0,y>0\}} \phi^2|\nabla\eta|^2 - \delta. \tag{8.4}$$

Since $\phi \in L^\infty(\mathbb{R}^2)$, we can choose η to be a suitable log cut off function (see (3.4)) so that the first term in the right hand side of (8.4) is as small as we wish, just as in the proof of Lemma 3.10. Note that once $B_\rho(X)$ is chosen and δ is fixed, the choice of η can be achieved by choosing R sufficiently large (depending on δ). Hence we get

$$\int_{\{x>0,y>0\}} |\nabla(\phi\eta)|^2 - \int_{\{x>0,y>0\}\cap\partial\{u=1\}} H(\phi\eta)^2 < 0.$$

This is a contradiction because u is stable in the first quadrant. □

By this monotonicity of u, we obtain

Corollary 8.5. *There exists a decreasing function f defined on an interval $(T, +\infty)$, such that*

$$\partial\{u > 1\} \cap \{x > 0, y > 0\} = \{(x, y) : y = f(x), x > T\}.$$

By this corollary, we see that the free boundary of u consists of $\{y = f(x)\}$ and its reflections with respect to the x and y axis. Therefore u is a solution with four ends. After a rotation of angle $\pi/4$, we get the four end solution with angle $\pi/4$. The proof of Proposition 8.1 is thus complete.

9. Existence in the general case

In this section, we prove the existence part in Theorem 2.4. We will mainly rely on the quasi Gauss map introduced in Section 5.

Given $\alpha \in (0, \pi/2)$, take v_α to be the (unique) harmonic function in $B_1(0)$ with boundary value as in (5.2). Because v_α is even in x and y, $\nabla v_\alpha(0) = 0$. Hence $v_{\alpha,z}$ is a holomorphic function in B_1 satisfying $v_{\alpha,z}(0) = 0$. Then $v_{\alpha,z}/z$ is also holomorphic in B_1. Let F_α be its primitive function satisfying $F_\alpha(0) = 0$. Note that F_α is evenly symmetric in x and y.

The main result of this section is

Proposition 9.1. *For each $\alpha \in (0, \pi/2)$, F_α is injective on B_1.*

By the following lemma, the existence part of Theorem 2.4 will follow from Proposition 9.1.

Lemma 9.2. *There exists an evenly symmetric, four end solution with angle α if and only if F_α is injective on B_1.*

Proof. Denote $\Omega_\alpha := F_\alpha(B_1)$, which is an evenly symmetric, open domain in the complex plane, because F_α is even and it is an open map. If F_α is injective, then it is a biholomorphism between B_1 and Ω_α. Moreover, by its definition, F_α extends continuously to $\partial B_1 \setminus \{e^{i(\frac{\pi}{2}-\alpha)}, e^{i(\frac{\pi}{2}+\alpha)}, e^{i(\frac{3\pi}{2}-\alpha)}, e^{i(\frac{3\pi}{2}+\alpha)}\}$, so there are four ends of Ω_α, represented by the image of F_α of these four points.

Let $G_\alpha := F_\alpha^{-1}$ and $u_\alpha := v_\alpha \circ G_\alpha$. By (5.3) and (5.4), u_α satisfies (1.1) with $\Omega_\alpha = \{-1 < u_\alpha < 1\}$. Hence u_α is an evenly symmetric, four end solution. Because v_α is a solution of (5.2), the angle of u_α is exactly α.

Conversely, by the discussion in Section 5, if u_α is an evenly symmetric, four end solution with angle α, then the holomorphic map F_α defined therein is injective on B_1. $\square$

Proof of Proposition 9.1. Let $\mathcal{I} \subset (0, \pi/2)$ be the set of those α such that F_α is injective on B_1. By Proposition 8.1, $\pi/4 \in \mathcal{I}$, hence it is non-empty. By Proposition 7.2, $\mathcal{I}$ is closed. Therefore we need only to prove the openness of $\mathcal{I}$, that is,

Claim. *If for some $\alpha_0 \in (0, \pi/2)$, F_{α_0} is injective on B_1, then there exists an $\varepsilon > 0$ such that for any $\alpha \in (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)$, F_α is injective on B_1.*

First by the Poisson representation formula for harmonic functions in B_1, there exists a small radius $\rho_1 > 0$ and $\varepsilon_1 > 0$ such that, for any $\alpha \in (\alpha_0 - \varepsilon_1, \alpha_0 + \varepsilon_1)$ and $z \in B_1 \cap B_{\rho_1}(e^{i(\frac{\pi}{2}-\alpha_0)})$,

$$F_\alpha'(z) = \frac{a}{z - e^{i(\frac{\pi}{2}-\alpha)}} + O(1),$$

where a is a complex constant. Then there exists another complex constant b such that in the same domain,

$$F_\alpha(z) = a \log\left(z - e^{i(\frac{\pi}{2}-\alpha)}\right) + b + O\left(|z - e^{i(\frac{\pi}{2}-\alpha)}|\right), \tag{9.1}$$

where we take the principal branch for the log function.

Let $\rho_2 = \rho_1/2$. In the closure of

$$\mathcal{D}_{\rho_2} := B_1 \setminus \left[B_{\rho_2}(e^{i(\frac{\pi}{2}-\alpha_0)}) \cup B_{\rho_2}(e^{i(\frac{\pi}{2}+\alpha_0)}) \cup B_{\rho_2}(e^{i(\frac{3\pi}{2}-\alpha_0)}) \cup B_{\rho_2}(e^{i(\frac{3\pi}{2}+\alpha_0)})\right],$$

F_α depends continuously on α. Hence by our assumption on F_{α_0}, there exists an $\varepsilon_2 > 0$ such that for any $\alpha \in (\alpha_0 - \varepsilon_2, \alpha_0 + \varepsilon_2)$, F_α is injective in this domain. In particular, $z = 0$ is the only zero of F_α in this domain and it is simple.

Finally, by letting $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$, we deduce that for any $\alpha \in (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)$, $z = 0$ is the only zero of F_α in B_1 and it is simple. By the argument principle, this implies that for all ρ sufficiently small,

$$\frac{1}{2\pi i} \int\limits_{\partial \mathcal{D}_\rho} \frac{F'_\alpha(z)}{F_\alpha(z)} dz = 1.$$

For any $w \in \Omega_\alpha$, in view of (9.1), for any sufficiently small ρ, by the homotopy invariance,

$$\frac{1}{2\pi i} \int\limits_{\partial \mathcal{D}_\rho} \frac{F'_\alpha(z)}{F_\alpha(z) - w} dz = \frac{1}{2\pi i} \int\limits_{\partial \mathcal{D}_\rho} \frac{F'_\alpha(z)}{F_\alpha(z)} dz = 1.$$

Thus there exists only one $z \in B_1$ satisfying $F_\alpha(z) = w$. $\square$

10. Morse index characterization

10.1. Morse index of four end solutions is 1

Let u be a solution with four ends, which is evenly symmetric with respect to the x and y axis.

By the Stable De Giorgi theorem [21, Theorem 1.2], u cannot be stable, so the Morse index solution of u is at least 1. By the standard elliptic theory, for any $R > 0$ large enough, there exists a $\lambda_{1,R} < 0$ and the associate positive first eigenfunction $\varphi_R \in H^1_0(B_R(0))$. The first eigenfunction is unique up to the multiplication of constants.

Let $\lambda_{2,R}$ be the second eigenvalue. Take a second eigenfunction $\psi_R \in H^1_0(B_R(0))$. It must change sign in $\Omega \cap B_R(0)$. If $\lambda_{2,R} < 0$, by Proposition 6.1, we get a contradiction. Hence we must have $\lambda_{2,R} \geq 0$. Therefore for any R large, the index of the quadratic form $\mathcal{Q}$ in $H^1_0(B_R)$ is exactly 1. Hence the Morse index of u is exactly 1.

10.2. Morse index of solutions with more than four ends is larger than 1

Suppose u has $2k$ ends with $k \geq 3$. Take an R large so that $\Omega \setminus B_R(0) = \cup_{i=1}^{2k} D_i$, $1 \leq i \leq 2k$. Denote the asymptotic directions of these ends by $e_1, \cdots, e_{2k}$, which are in anticlockwise order but not necessarily distinct.

Take a direction e so that it does not equal e_i, $\forall i = 1, \cdots, 2k$. The directional derivative u_e satisfies the linearized equation (1.3). By Proposition 3.7 and Lemma 3.6, perhaps after taking a larger R, for each $i = 1, \cdots, 2k$, u_e has a fixed sign in D_i. Moreover, the sign of u_e in every two adjacent ends is different, except those two pairs lying on different sides of the line $\{te, t \in \mathbb{R}\}$.

Recall that the nodal set of u_e consists of finitely many singular points and finitely many smooth curves, with the end points of these curves lying in this singular set or $\partial\Omega$. Because u_e has a fixed sign in each D_i, the nodal set of u_e is contained in a compact set.

Using the nodal domains of u_e we build a planar graph, with each vertex point representing one end of u_e and two vertex points connected by an edge if they belong to the same nodal domain.

Lemma 10.1. *The number of nodal domains of u_e is not less than k.*

Proof. By the previous analysis, we can assume for some $R > 0$ large, $\{u_e \neq 0\} \setminus B_R(0)$ consists of $2k$ connected components $D_i, i = 1, 2, \cdots 2k$. Among them, there are two pairs of adjacent sets which indeed belong to the same nodal domain of u_e. We may denote by $\mathcal{C}_i^{\pm}$, $1 \leq i \leq k-1$ these sets by combining each of the pairs into one set so that $u_e > 0$ in $\mathcal{C}_i^+$ and $u_e < 0$ in $\mathcal{C}_i^-$. In other words, each $\mathcal{C}_i^{\pm}$ contains exactly one connected component of $\Omega \setminus B_R(0)$, except two of which are composed of two adjacent connected components of $\Omega \setminus B_R(0)$ respectively.

Without loss of generality, in the following we assume $\mathcal{C}_1^+, \mathcal{C}_1^-, \cdots$ are arranged in anti-clockwise direction. $\mathcal{C}_i^{\pm}$ form a graph with each of $\mathcal{C}_1^+, \mathcal{C}_1^-, \cdots$ being a vertex point and two vertex points are connected by an edge if they belong to the same nodal domain. Denote this graph by $\mathcal{G}$.

We claim that the number of connected components in $\mathcal{G}$ is at least k. This can be seen by performing the following surgery on a sequence of graphs, starting from an initial special graph with k connected components and all $\mathcal{C}_i^-$ being connected and all $\mathcal{C}_i^+$ being disconnected, and terminating when the graph has the same connectivity of $\mathcal{C}_i^+$ as the real planar graph $\mathcal{G}$ constructed above.

Step 1. Consider the special case that each $\mathcal{C}_i^+$ belongs to a distinct nodal domain of u_e, and all of $\mathcal{C}_i^-$ belong to a single nodal domain of u_e. In other words, we assume $\mathcal{C}_i^+$ are disconnected vertex points of an initial graph $\mathcal{G}_0$, while all $\mathcal{C}_i^-$ are all

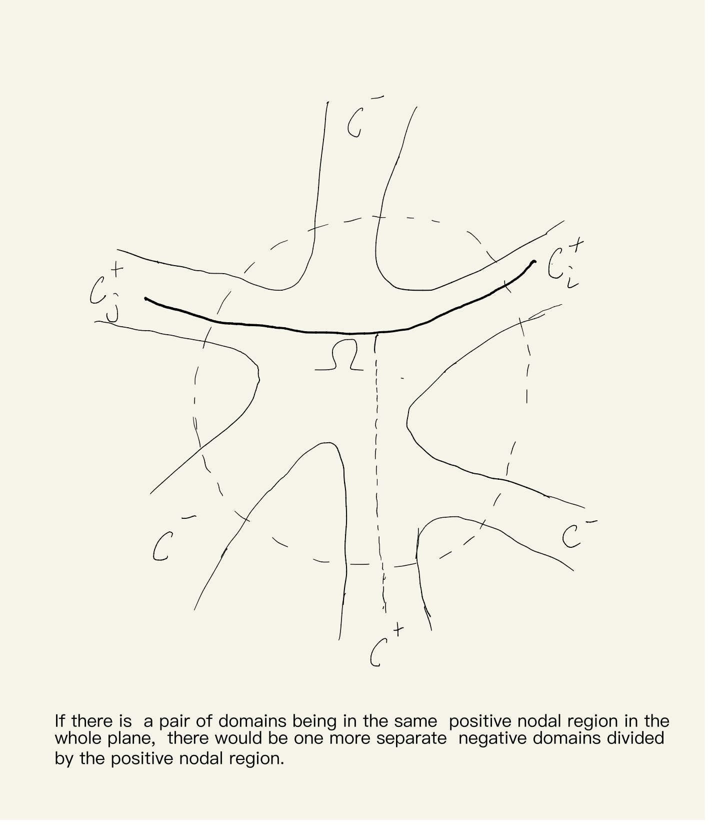


Fig. 3. The Surgery Process.

connected. Obviously, in this special case, the number of connected components of $\mathcal{G}_0$ is exactly k.

Step 2. If there exist two $\mathcal{C}_i^+$ belonging to the same nodal domain of u_e, then we start with the smallest pair (i, j) with $i < j$ (in the standard order of integer pairs) by connecting $\mathcal{C}_i^+$ and $\mathcal{C}_j^+$ in the graph $\mathcal{G}_0$ with an edge. To illustrate the connectivity, we may use a Jordan curve to connect the ends $\mathcal{C}_i^+$ and $\mathcal{C}_j^+$ to indicate that they belong to the same nodal domain, as in Fig. 3. By the previous analysis, this Jordan curve separates the plane into at least two connected components, and there are connected component of $\cup_{i=1}^{k-1}\mathcal{C}_i^-$ on both sides of this curve. Correspondingly, this process splits the connected subgraph with vertex points $\cup_{i=1}^{k-1}\mathcal{C}_i^-$ into two disconnected sub graphs. We keep all edges of these two sub graphs of $\mathcal{G}_0$, resulting a new graph $\mathcal{G}_1$ with the least number of possible connected components. It is clear that the total number of connected components in $\mathcal{G}_1$ is at least k.

Step 3. Repeat Step 2 for the graph $\mathcal{G}_m$ to obtain the graph $\mathcal{G}_{m+1}$, until we reach the graph $\mathcal{G}_N$ so that $\mathcal{G}_N$ has the same connectivity of subgraph $\cup_{i=1}^{k-1}\mathcal{C}_i^+$ as the graph $\mathcal{G}$.

In the above procedure, each time we eliminate one connected component of $\mathcal{G}_m$ by connecting a pair of $\mathcal{C}^+$ according to their being in the nodal domain of u_e, we produce at least one more connected component in $\mathcal{G}_m$ for $\mathcal{C}^-$. Therefore the number of connected components in the resulting graph $\mathcal{G}_{m+1}$ is non-decreasing. Since the initial number of connected component of $\mathcal{G}_0$ is k, the number of connected components in $\mathcal{G}_N$ is at least k. Note that the graph $\mathcal{G}$ has at most as many edges as $\mathcal{G}_N$, and hence must have at least k connected components. This implies that there are at least k nodal domains of u_e. □

Let $\lceil k/2 \rceil$ be the first integer number not smaller than $k/2$. Note that $\mathcal{G}$ can be naturally divided into two sub graphs $\mathcal{G}^\pm$, with $u_e > 0$ in each nodal domain represented by a connected component of $\mathcal{G}^+$ while $u_e < 0$ in each nodal domain represented by a connected component of $\mathcal{G}^-$. By the strong maximum principle, each nodal domain in $\mathcal{G}^+$ has a part of regular boundary belonging to $\mathcal{G}^-$, and vice versa. Thus we arrive at the following conclusion:

Lemma 10.2. *There exist at least $\lceil k/2 \rceil$ nodal domains of u_e, $\mathcal{C}_\alpha$, $1 \le \alpha \le \lceil k/2 \rceil$, so that each $\mathcal{C}_\alpha$ has a part of regular boundary not contained in $\cup_{1\le\beta\neq\alpha\le\lceil k/2\rceil}\overline{\mathcal{C}_\beta}$.*

Now for each $\alpha = 1, \cdots, \lceil k/2 \rceil$, take a regular curve $\Gamma_\alpha \subset \partial D_\alpha$ satisfying the previous lemma. Let

$$\varphi_\alpha = \begin{cases} |u_e|, & \text{in } D_\alpha, \\ 0, & \text{outside } D_\alpha, \end{cases}$$

which is a continuous subsolution of the linearized problem (1.3). By definition, for $1 \le \alpha \neq \beta \le \lceil k/2 \rceil$, φ_α and φ_β have almost disjoint supports (i.e. at most the boundaries $\partial\{\varphi_\alpha > 0\}$ and $\partial\{\varphi_\beta > 0\}$ could intersect).

Take a point $X_\alpha \in \Gamma_\alpha$ and an $r_\alpha > 0$ so that $B_{r_\alpha}(X_\alpha) \subset \Omega$ does not intersect $\cup_{1\le\beta\neq\alpha\le\lceil k/2\rceil}\overline{D_\beta}$. Let $\bar{\varphi}_\alpha$ be the solution of

$$\begin{cases} \Delta\bar{\varphi}_\alpha = 0, & \text{in } B_{r_\alpha}(X_\alpha), \\ \bar{\varphi}_\alpha = \varphi_\alpha, & \text{on } \partial B_{r_\alpha}(X_\alpha). \end{cases}$$

Because $\varphi_\alpha = 0$ in the open set $B_{r_\alpha}(X_\alpha) \setminus \overline{D_\alpha}$, there exists a constant $\delta_\alpha > 0$ such that

$$\int_{B_{r_\alpha}(X_\alpha)} |\nabla\bar{\varphi}_\alpha|^2 \le \int_{B_{r_\alpha}(X_\alpha)} |\nabla\varphi_\alpha|^2 - \delta_\alpha. \tag{10.1}$$

Extend $\bar{\varphi}_\alpha$ outside $B_{r_\alpha}(X_\alpha)$ to be φ_α. Defined in this way, for $1 \le \alpha \neq \beta \le \lceil k/2 \rceil$, $\bar{\varphi}_\alpha$ and $\bar{\varphi}_\beta$ still have disjoint supports.

For any R large, take η_R to be a standard log cut-off function. Multiplying the equation for u_e by $\varphi_\alpha \eta_R^2$ and integrating by parts leads to

$$\int_\Omega |\nabla(\varphi_\alpha \eta_R)|^2 - \int_{\partial\Omega} H\varphi_\alpha^2\eta_R^2 \le \frac{C}{\log R}, \quad \alpha = 1, \cdots, \lceil k/2 \rceil.$$

Combining this with (10.1), we see, once R is large enough, for each $\alpha = 1, \cdots, \lceil k/2 \rceil$,

$$\int_\Omega |\nabla(\bar\varphi_\alpha \eta_R)|^2 - \int_{\partial\Omega} H\bar\varphi_\alpha^2\eta_R^2 \le \frac{C}{\log R} - \delta_\alpha < 0.$$

Because $\bar\varphi_\alpha \eta_R$ are orthogonal in $L^2(\mathbb{R}^2)$, this implies that the Morse index of u is at least $\lceil k/2 \rceil$.

Remark 10.3. In a recent paper of Mantoulidis [18], he established a similar lower bound for the Morse index of finite end solutions to the Allen-Cahn equation.

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Appendix A. Proof of several technical results

We will give the proofs of Propositions 3.3-3.5 as well as the Hamiltonian identity (4.5) in this appendix.

A.1. Proof of Proposition 3.3

For any solution u of (1.1) and any $X^0 \in \mathbb{R}^2$, we have the Pohazaev identity

$$\int_{B_R(X^0)} 4F(u) = \int_{\partial B_R(X^0)} [|\nabla u|^2 + 2F(u)](X - X^0)\cdot\nu - 2\int_{\partial B_R(X^0)} (\nabla u\cdot\nu)(\nabla u\cdot(X - X^0)),$$

where $F(u) = \frac{1}{2}\chi_{\{-1<u<1\}}$ and $\nu(X) = \frac{X-X^0}{R}$ for $X \in \partial B_R(X^0)$, and so

$$\int_{B_R(X^0)} 4F(u) = R\int_{\partial B_R(X^0)} [|\nabla u|^2 + 2F(u)] - 2R\int_{\partial B_R(X^0)} (\nabla u\cdot\nu)^2. \tag{A.1}$$

Recall that $E(R; X^0) = \frac{1}{R}\int_{B_R(X^0)} \left[|\nabla u|^2 + \chi_{\{-1<u<1\}}\right]$. Simple computation shows that

$$\frac{dE(R,X^0)}{dR} = -\frac{1}{R^2}\int\limits_{B_R(X^0)} [|\nabla u|^2 + 2F(u)] + \frac{1}{R}\int\limits_{\partial B_R(X^0)} [|\nabla u|^2 + 2F(u)].$$

From this equality and (A.1), we obtain

$$\frac{dE}{dR} = \frac{2}{R}\int\limits_{\partial B_R(X^0)} (\nabla u \cdot \nu)^2 + \frac{1}{R^2}\int\limits_{B_R(X^0)} [2F(u) - |\nabla u|^2].$$

From the Modica inequality (see Proposition 3.1), we deduce that $2F(u) - |\nabla u|^2 = 1 - |\nabla u|^2 \geq 0$ in $B_R(X^0) \cap \Omega$. In the other domain $B_R(X^0)\backslash\Omega$, one has $2F(u) - |\nabla u|^2 = 0$, since $\chi_{\{-1<u<1\}} = |\nabla u| \equiv 0$ in this domain. Therefore

$$\frac{dE}{dR} \geq \frac{2}{R}\int\limits_{\partial B_R(X^0)} (\nabla u \cdot \nu)^2 > 0,$$

which gives the desired result of this proposition. □

A.2. Proof of Proposition 3.4

Suppose u is a solution with $2k$ ends. We first show that for any $X \in \mathbb{R}^2$ there exists a positive constant C such that

$$E(R, X) \leq C \quad \text{for any} \quad R > 0. \tag{A.2}$$

From Proposition 3.7, for some end of Ω, we may suppose that there exists two positive constants c and μ so that outside a compact set this end has the form

$$\left\{(x,y) : f^-(x) < y < f^+(x)\right\},$$

where f^+, f^- are convex and concave functions respectively, satisfying

$$|f^{\pm}(x) \mp 1| \leq ce^{-\mu x}, \qquad \text{as } x \to +\infty.$$

For some $R_0 > 0$ we define $\Omega^1 := \{(x,y) \in \Omega : x > R_0\}$. Let

$$\rho(x) := \int\limits_{f^-(x)}^{f^+(x)} [1 + (u_y^2 - u_x^2)]dy.$$

It is easy to see that

$$
\begin{aligned}
&|\rho'(x)| \\
&= |2(u_y u_x)|_{y=f^-(x)}^{y=f^+(x)} + [1+(u_y^2-u_x^2)]|_{y=f^+(x)}(f^+)'(x) - [1+(u_y^2-u_x^2)]|_{y=f^-(x)}(f^-)'(x)| \\
&\le ce^{-\mu x}, \ \forall x > R_0.
\end{aligned}
$$

Hence we have

$$|\rho(R_1)-\rho(R_2)| \le ce^{-\mu R_1}, \ \forall R_2 \ge R_1 > R_0.$$

In particular, we have

$$|\rho(x)| \le c, \ \forall x > R_0.$$

This and the definition of ρ give

$$\int\limits_{B_R(X)\cap\Omega^1} [1+(u_y^2-u_x^2)] \le CR.$$

From the Modica inequality, we have $u_y^2 \le 1+(u_y^2-u_x^2)$, and so

$$\int\limits_{B_R(X)\cap\Omega^1} u_y^2 \le CR.$$

Now we choose another Cartesian coordinates (x', y') so that the x'-axis is a small rotation of x-axis. Then we can obtain

$$\int\limits_{B_R(X)\cap\Omega^1} u_{y'}^2 dx'dy' = \int\limits_{B_R(X)\cap\Omega^1} u_y^2 dxdy \le CR.$$

Therefore we obtain

$$
\begin{aligned}
\int\limits_{B_R(X)\cap\Omega^1} (1+|\nabla u|^2)dxdy &\le \int\limits_{B_R(X)\cap\Omega^1} [1+(u_y^2-u_x^2)]dxdy + C\int\limits_{B_R(X)\cap\Omega^1} (u_y^2+u_{y'}^2)dxdy \\
&\le CR.
\end{aligned}
\tag{A.3}
$$

Similarly we can define $2k-1$ domains $\Omega^i \subset \Omega (i = 2, 3, \ldots, 2k)$ contain the rest $2k-1$ ends outside a compact set respectively. Denote Ω^0 as a bounded domain in Ω such that $\Omega = \cup_{i=0}^{2k}\Omega^i$. Repeat the above argument $2k-1$ times, we obtain

$$\int\limits_{B_R(X)\cap\Omega^i} (1+|\nabla u|^2) \le CR, \ i = 2, \ldots, 2k. \tag{A.4}$$

Note that

$$\int_{B_R(X)\cap\Omega^0} (1+|\nabla u|^2) \leq \tilde{C}, \tag{A.5}$$

where $\tilde{C}$ is independent of R. Plainly we have

$$\int_{B_R(X)\setminus\Omega} (2F(u)+|\nabla u|^2) = 0, \tag{A.6}$$

where $F(u) = \frac{1}{2}\chi_{\{-1<u<1\}}$ is given in the proof of Proposition 3.3. Hence for large R, from (A.3)-(A.6), we obtain (A.2).

From (A.2) and Proposition 3.3, we know that the limit $\lim_{R\to+\infty} E(R,X)$ exists.

Proposition 3.7 tells us that the nodal sets of u are asymptotically straight lines. Applying the Hutchinson-Tonegawa theory, we obtain

$$\lim_{R\to+\infty} E(R,X) = 2k\mathbf{e}.$$

Here $\mathbf{e} := \frac{1}{2}\int_{-\infty}^{\infty}[|g'(x)|^2 + \chi_{\{-1<g(x)<1\}}]dx$, where g is the one dimensional solution given in Section 1. Namely

$$g(x) = \begin{cases} -1, & \text{if } x < -1, \\ x, & \text{if } -1 < x < 1, \\ 1, & \text{if } x > 1. \end{cases}$$

Simple computation shows that $\mathbf{e} = 2$. So $\lim_{R\to+\infty} E(R,X) = 4k$. □

A.3. Proof of Proposition 3.5

Denote the asymptotic direction of the ends of u by $e_i = (\cos\theta_i, \sin\theta_i), 1 \leq i \leq 2k$ with $0 < \theta_i < \theta_{i+1} < 2\pi, 1 \leq i \leq 2k-1$. Applying Hamiltonian identity as in the proof of Theorem 1.3 in [10], we obtain

$$\sum_{i=1}^{2k} \sin(\theta_i + \theta) = 0$$

for almost all θ. The desired result of this proposition follows. □

A.4. Proof of the Hamiltonian identity (4.5)

We divide the proof of this identity into two steps.

Lemma A.1.

$$\frac{d}{dx}\mathcal{H}(x) = -2\int_{-\infty}^{+\infty} u_x u_y dy. \tag{A.7}$$

Proof. It is clear that the right hand side of (A.7) is continuous in x. (We need only to verify this at those points where the number of connected components of $\{-1 < u < 1\} \cap (\{x\} \times \mathbb{R})$ changes.)

We will perform the calculation by assuming u is a classical solution of (1.1) in $\{|x| < 1\}$, with the free boundary given by $\{y = f_\pm(x)\}$. The general case follows from this calculation by first showing the identity for one sided derivatives and then using the continuity of the right hand side of (A.7).

Under this simplified setting, we find that

$$\begin{aligned}\frac{d}{dx}\mathcal{H}(x) = 2&\int_{f_-(x)}^{f_+(x)} y\left(u_y u_{yx} - u_x u_{xx}\right) dy \\ &+ f_+(x)\left[1 + u_y(x, f_+(x))^2 - u_x(x, f_+(x))^2\right] f_+'(x) \\ &- f_-(x)\left[1 + u_y(x, f_-(x))^2 - u_x(x, f_-(x))^2\right] f_-'(x) \\ &= 2\int_{f_-(x)}^{f_+(x)} y\left(u_y u_{yx} + u_x u_{yy}\right) dy \\ &+ 2u_y(x, f_+(x))^2 f_+(x) f_+'(x) - 2u_y(x, f_-(x))^2 f_-(x) f_-'(x) \\ &= -2\int_{f_-(x)}^{f_+(x)} u_x u_y dy \\ &+ 2u_y(x, f_+(x))u_x(x, f_+(x) f_+(x) - 2u_y(x, f_+(x))u_x(x, f_+(x)) f_-(x) \\ &+ 2u_y(x, f_+(x))^2 f_+(x) f_+'(x) - 2u_y(x, f_-(x))^2 f_-(x) f_-'(x) \\ &= -2\int_{f_-(x)}^{f_+(x)} u_x u_y dy.\end{aligned}$$

In the above, we have used the following three facts: (i) $\Delta u = 0$ in $\{-1 < u < 1\}$; (ii) the free boundary condition; (iii) the identity $u_x + u_y f_+' = 0$ on $\{y = f_+(x)\}$ and a similar one on $\{y = f_-(x)\}$. □

Lemma A.2.

$$\frac{d}{dx}\int_{-\infty}^{+\infty} u_x u_y dy = 0.$$

Proof. As in the proof of the previous lemma, we still assume u is a classical solution of (1.1) in $\{|x| < 1\}$, with the free boundary given by $\{y = f_\pm(x)\}$. Under this simplified setting, we find that

$$\begin{aligned}
\frac{d}{dx}\int_{f_-(x)}^{f_+(x)} u_x u_y dx &= \int_{f_-(x)}^{f_+(x)} (u_x u_{yx} + u_y u_{xx})\, dy \\
&+ u_y(x, f_+(x))u_x(x, f_+(x))f'_+(x) - u_y(x, f_-(x))u_x(x, f_-(x))f'_-(x) \\
&= \int_{f_-(x)}^{f_+(x)} (u_x u_{yx} - u_y u_{yy})\, dy \\
&+ u_y(x, f_+(x))u_x(x, f_+(x))f'_+(x) - u_y(x, f_-(x))u_x(x, f_-(x))f'_-(x) \\
&= \frac{u_x(x, f_+(x)^2 - u_y(x, f_+(x))^2}{2} - \frac{u_x(x, f_-(x))^2 - u_y(x, f_-(x))^2}{2} \\
&+ u_y(x, f_+(x))u_x(x, f_+(x))f'_+(x) - u_y(x, f_-(x))u_x(x, f_-(x))f'_-(x) \\
&= -\frac{1}{2}|\nabla u(x, f_+(x))|^2 + \frac{1}{2}|\nabla u(x, f_-(x))|^2 \\
&= 0.
\end{aligned}$$

In the above, we have used exactly the same three facts as in the proof of the previous lemma. □

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