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Improved Beckner's inequality for axially symmetric functions on $\mathbb{S}^n$

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ABSTRACT

In this article we present various uniqueness and existence results for Q-curvature type equations with a Paneitz operator on $\mathbb{S}^n$ in axially symmetric function spaces. In particular, we show uniqueness results for $n = 6, 8$ and improve the best constant of Beckner's inequality in these dimensions for axially symmetric functions under the constraint that their centers of mass are at the origin. As a consequence, the associated Szegö type inequality is also proven for axially symmetric functions which is similar to the first of a series of inequalities following from the Szegö limit theorem.

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1. Introduction and statement of results

In this paper, we consider the constant Q-curvature type equation

$$\alpha P_n u + (n-1)!\left(1 - \frac{e^{nu}}{\int_{\mathbb{S}^n} e^{nu} dw}\right) = 0, \quad \text{on } \mathbb{S}^n, \tag{1.1}$$

where $\mathbb{S}^n$ is the n-dimensional sphere and

$$P_n = \begin{cases} \prod_{k=0}^{\frac{n-2}{2}}(-\Delta + k(n-k-1)), & \text{for } n \text{ even;} \\ \left(-\Delta + \left(\frac{n-1}{2}\right)^2\right)^{1/2} \prod_{k=0}^{\frac{n-3}{2}}(-\Delta + k(n-k-1)), & \text{for } n \text{ odd} \end{cases}$$

is the Paneitz operator on $\mathbb{S}^n$ and α is a positive constant. The volume form dw is normalized so that $\int_{\mathbb{S}^n} dw = 1$.

The corresponding functional is defined in $H^{\frac{n}{2}}(\mathbb{S}^n)$ by

$$J_\alpha(u) = \frac{\alpha}{2}\int_{\mathbb{S}^n} (P_n u) u dw + (n-1)! \int_{\mathbb{S}^n} u dw - \frac{(n-1)!}{n} \ln \int_{\mathbb{S}^n} e^{nu} dw. \tag{1.2}$$

If $\alpha = 1$, (1.1) corresponds to the constant Q-curvature equation on $\mathbb{S}^n$. It is shown in [1] that the following Beckner's inequality, a higher order Moser-Trudinger type inequality, holds

$$J_1(u) \geq 0, \quad u \in H^{\frac{n}{2}}(\mathbb{S}^n). \tag{1.3}$$

Furthermore, J_1 is invariant under the conformal transformation

$$u(\xi) \to u(\tau\xi) + \frac{1}{n}\ln(|\det(d\tau)(\xi)|)$$

where τ is an element of the conformal group of $\mathbb{S}^n$ and $|\det(\cdot)|$ is the modulus of the corresponding Jacobian determinant. Equality in (1.3) is only attained at functions of the form

$$u(\xi) = -\ln(1 - \zeta \cdot \xi) + C, \quad C \in \mathbb{R},$$

where $\zeta \in B^{n+1} := \{\xi \in \mathbb{R}^{n+1}, |\xi| < 1\}$. (See also [7].) In particular, (1.1) with $\alpha = 1$ has a family of axially symmetric solutions

$$u(\xi) = -\ln(1 - a\xi_1), \quad \xi \in \mathbb{S}^n \text{ for } a \in (-1, 1).$$

On the other hand, an improved Aubin-type inequality has been shown in [7, Lemma 4.6]: for any $\alpha > 1/2$, there exists a constant $C(\alpha) \geq 0$ such that $J_\alpha(u) \geq -C(\alpha)$ provided that u belongs to the set of functions with center of mass at the origin

$$\mathfrak{L} = \{v \in H^{\frac{n}{2}}(\mathbb{S}^n) : \int\limits_{\mathbb{S}^n} e^{nv}\xi_j dw = 0;\ j = 1, 2, \cdots, n+1\}.$$

This gives rise to the existence of a minimizer u_0 of J_α in $\mathfrak{L}$, and u_0 satisfies the corresponding Euler-Lagrange equation

$$\alpha P_n u + (n-1)!\left(1 - \frac{e^{nu}}{\int_{\mathbb{S}^n} e^{nu} dw}\right) = \sum_{i=1}^{n+1} a_i \xi_i e^{nu}, \quad \text{on} \quad \mathbb{S}^n \tag{1.4}$$

for some constants $a_i, i = 1, 2, \cdots n+1$. Furthermore, by exploiting the invariance of J_1 under the conformal transformation, [7, Remarks (3) (ii) for Cor. 5.4] implies that the following Kazdan-Warner condition

$$\int\limits_{\mathbb{S}^n} \langle \nabla Q, \nabla \xi_i \rangle e^{nu} dw = 0, \quad i = 1, 2, \cdots n+1 \tag{1.5}$$

is also applicable for the prescribing Q-curvature equation

$$P_n u + (n-1)! - Q e^{nu} = 0, \quad \xi \in \mathbb{S}^n.$$

It is an immediate consequence that $a_i = 0$, $i = 1, 2, \cdots n+1$ in (1.4). (See [32], proof of Theorem 2.6.) This argument is reminiscent of that in [6, Cor. 2.1] on the constant Gaussian curvature type equation, or the mean field equation on $\mathbb{S}^2$,

$$-\alpha \Delta u + \left(1 - \frac{e^{2u}}{\int_{\mathbb{S}^2} e^{2u} dw}\right) = 0, \quad \xi \in \mathbb{S}^2. \tag{1.6}$$

For (1.6), there is a vast literature. See, e.g., [6], [16] and references therein. Moreover, interested reader is referred to [3,8,10,11,13,19–21,24–26,29,30,33] for literature on equations that have conformal structure.

In what follows, we shall consider axially symmetric functions that are only dependent on ξ_1. The first aim of this article is to discuss the classification of axially symmetric solutions for (1.1) at the critical parameter $\alpha = \frac{1}{n+1}$. We have the following theorem.

Theorem 1.1. *If $\alpha = \frac{1}{n+1}$ and u is an axially symmetric solution to (1.1) with $n \geq 2$, then u must be constant.*

The rest of this paper focuses on (1.1) with n even in the axially symmetric setting. We shall show that (1.1) admits only constant solutions when α belongs to some suitable subinterval in $(1/2, 1)$ for $n = 6, 8$. As a consequence we obtain an improved Aubin-type inequality for axially symmetric functions in $\mathfrak{L}$. Note that the case $n = 4$ has been considered in [17] and similar results are obtained.

Considering solutions axially symmetric about ξ_1-axis and denoting ξ_1 by x, we can reduce (1.1) to

$$\alpha\,(-1)^{\frac{n}{2}}\,[(1-x^2)^{\frac{n}{2}}u']^{(n-1)}+(n-1)!-\frac{(n-1)!\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\gamma}e^{nu}=0, \tag{1.7}$$

where $\gamma=\int_{-1}^{1}(1-x^2)^{\frac{n-2}{2}}e^{nu}$.

One can refer to Section 3, for the detailed derivation of (1.7). By direct computations, we see that the corresponding functional $I_\alpha(u)$ in $H^{\frac{n}{2}}(-1,1)$ can be expressed as follows

$$I_\alpha(u)=(-1)^{\frac{n}{2}}\,\frac{\alpha}{2}\int_{-1}^{1}(1-x^2)^{\frac{n-2}{2}}[(1-x^2)^{\frac{n}{2}}u']^{(n-1)}u+(n-1)!\int_{-1}^{1}(1-x^2)^{\frac{n-2}{2}}u$$
$$-\frac{(n-1)!\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)}{n\Gamma\left(\frac{n+1}{2}\right)}\ln\left(\frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)}\int_{-1}^{1}(1-x^2)^{\frac{n-2}{2}}e^{nu}\right),$$

where $H^{\frac{n}{2}}(-1,1)$ is defined as the restriction of $H^{\frac{n}{2}}(\mathbb{S}^n)$ in the set of functions axially symmetric about ξ_1-axis and $\xi_1=x$. Moreover, the set $\mathfrak{L}$ is replaced by

$$\mathfrak{L}_r=\left\{u\in H^2(\mathbb{S}^4):u=u(x)\text{ and }\int_{-1}^{1}x(1-x^2)^{\frac{n-2}{2}}e^{nu}=0\right\}. \tag{1.8}$$

Let

$$\alpha^{(6)}=\frac{115+\sqrt{2851}}{273}\approx 0.6168\text{ and }\alpha^{(8)}=\frac{19}{23}\approx 0.8261.$$

Now we state the main results.

Theorem 1.2. *Let $n=6$ or 8. If $\alpha^{(n)}\le\alpha<1$, then (1.7) admits only constant solutions. As an immediate consequence, we have*

$$\inf_{u\in\mathfrak{L}_r}I_\alpha(u)=0.$$

We believe that $J_{1/2}(u)\ge 0$ for $u\in\mathfrak{L}$, given the similar inequality for $\mathbb{S}^2$ as shown in [16].

Next we define the following first momentum functionals on $H^{\frac{n}{2}}(\mathbb{S}^n)$

$$\mathcal{J}_\alpha(u)=\frac{\alpha}{2}\int_{\mathbb{S}^n}(P_nu)udw+(n-1)!\int_{\mathbb{S}^n}udw$$
$$-\frac{(n-1)!}{2n}\ln\left(\left(\int_{\mathbb{S}^n}e^{nu}dw\right)^2-\sum_{i=1}^{n+1}(\int_{\mathbb{S}^n}e^{nu}\xi_idw)^2\right).$$

Note that $\mathcal{J}_\alpha(u)=J_\alpha(u)$ when $u\in\mathfrak{L}$.

As a consequence of Theorem 1.2, we have the following sharp inequality on $\mathbb{S}^n$ for axially symmetric functions reminding us of the infinite inequalities arising from the Szegö limit theorem [4,5,15].

Theorem 1.3. *Let $n = 6$ or 8, then*

$$\mathcal{J}_{\frac{n}{n+1}}(u) \geq 0, \quad \forall u \in \{u \in H^{\frac{n}{2}}(\mathbb{S}^n) : u(\xi) = u(\xi_1)\}.$$

Using a bifurcation approach and Theorem 1.1–1.2, we can also show the existence of non constant axially symmetric solution for $\alpha \in (\frac{1}{n+1}, \frac{1}{2})$.

Theorem 1.4. *For $n \geq 2$, there exists a non constant solution u_α to (1.7) for $\alpha \in (\frac{1}{n+1}, \frac{1}{2})$. Moreover, there exists a sequence $\alpha_m \in (\frac{1}{n+1}, \alpha^{(n)})$ and a sequence of non constant solutions $u_{\alpha_m}, m = 1, 2, \cdots$ to (1.7) such that $\alpha_m \to \frac{1}{2}$, $\int_{-1}^{1}(1-x^2)^{\frac{n-2}{2}} e^{nu_{\alpha_m}} = \frac{\sqrt{\pi}\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})}$ and $\|u_{\alpha_m}\|_{L^\infty([-1,1])} \to \infty$ as $m \to \infty$.*

We also establish the following proposition concerning the centers of mass and first order momenta of solutions to (1.1).

Proposition 1.5. *If u solves (1.1), then*

$$\int\limits_{\mathbb{S}^n} e^{nu}\xi_i dw = 0 \quad \text{and} \int\limits_{\mathbb{S}^n} u\xi_i dw = 0, \quad i = 1, 2, \cdots, n+1,$$

whenever $\alpha \neq 1$.

The remainder of this paper is organized as follows. First, we give some preliminaries and validate Theorem 1.1 and Proposition 1.5 in the study of the case $n \geq 2$ in Section 2. Section 3 is devoted to the case $n = 6$ or 8 and the proof of Theorems 1.2–1.3. In Section 4, we carry out a bifurcation analysis of (1.7) and its equivalent form, and prove Theorem 1.4 based on Theorems 1.1 and 1.2.

2. Preliminaries and classification of $\alpha = \frac{1}{n+1}$

In this section, we state several preliminaries which will be needed in the proof of our main results.

Note that the eigenfunctions associated with the Paneitz operator coincide with those associated with the Laplacian. It is natural to introduce Gegenbauer polynomials, see [14, 8.93], which can be considered as a family of generalized Legendre polynomials.

Let us first introduce the Gegenbauer polynomials (see [14, 8.93]). Recall that

$$C_k^{\frac{n-1}{2}}(x) = \left(\frac{-1}{2}\right)^k \frac{\Gamma(k+n-1)\,\Gamma\left(\frac{n}{2}\right)}{k!\Gamma(n-1)\,\Gamma\left(k+\frac{n}{2}\right)}(1-x^2)^{-\frac{n-2}{2}} \frac{d^k}{dx^k}(1-x^2)^{k+\frac{n-2}{2}} \tag{2.1}$$

is called Gegenbauer polynomial of order $\frac{n-1}{2}$ and degree k. Then $C_k^{\frac{n-1}{2}}$ satisfies

$$(1-x^2)(C_k^{\frac{n-1}{2}})'' - nx(C_k^{\frac{n-1}{2}})' + \bar{\lambda}_k C_k^{\frac{n-1}{2}} = 0, \quad k = 0, 1, \cdots, \tag{2.2}$$

where $\bar{\lambda}_k = k(k+n-1)$.

After some calculations, it is easy to see from [14] that

$$|(C_k^{\frac{n-1}{2}})'| \leq \frac{\Gamma(k+n)}{n\Gamma(n-1)\Gamma(k)} \tag{2.3}$$

and

$$\int_{-1}^{1}(1-x^2)^{\frac{n-2}{2}} C_k^{\frac{n-1}{2}}(x) C_s^{\frac{n-1}{2}}(x) = \begin{cases} \frac{\pi(k+n-2)!}{2^{(n-2)}k!(k+\frac{n-1}{2})[\Gamma(\frac{n-1}{2})]^2} := A_n \frac{(k+n-2)!}{k!(k+\frac{n-1}{2})} & k = s; \\ 0 & k \neq s. \end{cases} \tag{2.4}$$

Furthermore, we know that

$$P_n C_k^{\frac{n-1}{2}} = \lambda_k C_k^{\frac{n-1}{2}}, \tag{2.5}$$

where

$$\lambda_k = \prod_{s=0}^{n-1}(k+s) = \frac{\Gamma(n+k)}{\Gamma(k)}. \tag{2.6}$$

Indeed, for n even,

$$\begin{aligned}\lambda_k &= \prod_{s=0}^{\frac{n-2}{2}}[k(k+n-1)+s(n-s-1)] = \prod_{s=0}^{\frac{n-2}{2}}(k+s)(k+n-1-s) \\ &= \prod_{s=0}^{n-1}(k+s).\end{aligned}$$

The final formula also works when n is odd.

We now prove Proposition 1.5. Since (1.1) is invariant under addition by a constant, we can normalize u so that $\int_{\mathbb{S}^n} e^{nu} dw = 1$. Then, (1.1) can be written as

$$\alpha P_n u = (n-1)!(e^{nu}-1), \quad \xi \in \mathbb{S}^n. \tag{2.7}$$

As in [6,22], we can multiply (2.7) by ξ_i, $i = 1, 2, \cdots, n+1$ and integrate to get

$$\alpha \int_{\mathbb{S}^n} (P_n u)\xi_i dw = (n-1)! \int_{\mathbb{S}^n} e^{nu}\xi_i dw.$$

It is easy to see from (2.2) and (2.5) that

$$-\Delta\xi_i = \bar{\lambda}_1\xi_i \text{ and } P_n\xi_i = \lambda_1\xi_i, \quad i = 1, 2, \cdots, n+1.$$

We further have

$$n\alpha \int_{\mathbb{S}^n} u\xi_i dw = \int_{\mathbb{S}^n} e^{nu}\xi_i dw.$$

On the other hand, let

$$Q = \frac{(n-1)!}{\alpha} + (n-1)!\left(1 - \frac{1}{\alpha}\right)e^{-nu}.$$

Then (2.7) can be reduced to

$$P_n u + (n-1)! - Qe^{nu} = 0 \tag{2.8}$$

As stated in the Introduction, the Kazdan-Warner condition (1.5) holds. It follows from (1.5) that

$$0 = n!\left(\frac{1}{\alpha} - 1\right)\int_{\mathbb{S}^n} \langle\nabla u, \nabla\xi_i\rangle dw = -n!\left(\frac{1}{\alpha} - 1\right)\int_{\mathbb{S}^n} u\Delta\xi_i dw = nn!\left(\frac{1}{\alpha} - 1\right)\int_{\mathbb{S}^n} u\xi_i dw.$$

Therefore,

$$\int_{\mathbb{S}^n} u\xi_i dw = 0 \quad \text{ and } \quad \int_{\mathbb{S}^n} e^{nu}\xi_i dw = 0 \quad i = 1, 2, \cdots, n+1$$

whenever $\alpha \neq 1$. Proposition 1.5 has been proven.

Throughout this paper, we assume that u is axially symmetric w.r.t. ξ_1-axis, i.e., $u = u(\xi_1)$ for $u \in \mathcal{C}^\infty(\mathbb{S}^n)$. We may drop the subscript for simplicity to write

$$u = u(x), \quad x \in (-1, 1).$$

Next, we shall prove the uniqueness of axially symmetric solutions when $\alpha = \frac{1}{n+1}$ in (1.1) for all $n \geq 2$.

Let

$$u = \sum_{k=0}^{\infty} a_k C_k^{\frac{n-1}{2}}(x). \tag{2.9}$$

As previously discussed, we can get

$$P_n u = \sum_{k=0}^{\infty} \lambda_k a_k C_k^{\frac{n-1}{2}}(x), \tag{2.10}$$

Now we assume that u is solution for (2.7) and define

$$G(x) = (1-x^2)u'. \tag{2.11}$$

One has

$$G = \sum_{k=0}^{\infty} b_k C_k^{\frac{n-1}{2}}(x). \tag{2.12}$$

By the recursive relations of $C_k^{\frac{n-1}{2}}(x)$ ([14, 8.939])

$$\begin{aligned}(1-x^2)(C_k^{\frac{n-1}{2}}(x))' &= 2(n-1)C_{k-1}^{\frac{n-1}{2}}(x) - kxC_k^{\frac{n-1}{2}}(x)\\ &= (k+n-1)xC_k^{\frac{n-1}{2}}(x) - (k+1)C_{k+1}^{\frac{n-1}{2}}(x),\end{aligned}$$

we have

$$(1-x^2)(C_k^{\frac{n-1}{2}}(x))' = \frac{(k+n-1)(k+n-2)}{2k+n-1}C_{k-1}^{\frac{n-1}{2}}(x) - \frac{k(k+1)}{2k+n-1}C_{k-1}^{\frac{n-1}{2}}(x), \quad \text{for } k \geq 1. \tag{2.13}$$

Therefore, we see from (2.12) and (2.13) that

$$b_k = \begin{cases} \frac{(k+n)(k+n-1)}{2k+n+1} - \frac{k(k-1)}{2}a_{k-1} & \text{for } k \geq 1;\\ \frac{n(n-1)}{n+1}a_1 & \text{for } k = 0. \end{cases} \tag{2.14}$$

Differentiate (2.7) w.r.t. x and multiply both sides by $(1-x^2)$ to get

$$(1-x^2)(P_n u)' = \frac{n!}{\alpha}e^{nu}(1-x^2)u'.$$

Replacing e^{nu} by $\frac{\alpha}{(n-1)!}P_n u + 1$, we derive that

$$(1-x^2)(P_n u)' = nP_n uG + \frac{n!}{\alpha}G. \tag{2.15}$$

Inspired by Osgood, Phillips and Sarnak [27], we shall compare the coefficients in front of $C_k^{\frac{n-1}{2}}(x)$ in both sides of (2.15). It is worthy pointing out that 1-d case is solved by comparing Fourier coefficients in [31].

Proof of Theorem 1.1. We first compare the coefficients before $C_0^{\frac{n-1}{2}}(x)$. By (2.10) and (2.14), we see that

$$(1-x)^2(P_n u)' = \frac{n(n-1)}{n+1}\lambda_1 a_1 C_0^{\frac{n-1}{2}}(x)$$
$$+\sum_{k=1}^{\infty}\left[\frac{(k+n)(k+n-1)\lambda_{k+1}}{2k+n+1}a_{k+1} - \frac{k(k-1)\lambda_{k-1}}{2k+n-3}a_{k-1}\right]C_k^{\frac{n-1}{2}}(x). \tag{2.16}$$

On the other hand, multiplying (2.15) by $(1-x^2)^{\frac{n-2}{2}}$ and integrating, we obtain

$$\frac{2n(n-2)!n!}{n+1}A_n a_1 = n\int_{\mathbb{S}^n} P_n uG + \frac{2n(n-2)!n!}{n+1}A_n a_1.$$

Equivalently, we have

$$\frac{2(n-2)!n!}{n+1}\left(1-\frac{1}{\alpha}\right)a_1 = \int_{\mathbb{S}^n} P_n uG. \tag{2.17}$$

It remains to compute $\int_{\mathbb{S}^n} P_n uG$. It follows from (2.10), (2.12), (2.14) and (2.4) that

$$\int_{\mathbb{S}^n} P_n uG = \int_{-1}^{1}(1-x^2)^{\frac{n-2}{2}}\left(\sum_{k=0}^{\infty}\lambda_k a_k C_k^{\frac{n-1}{2}}(x)\right)\sum_{k=0}^{\infty}b_k C_k^{\frac{n-1}{2}}(x)$$
$$= A_n\sum_{k=1}^{\infty}\frac{(k+n-2)!}{k!(k+\frac{n-1}{2})}\lambda_k a_k b_k$$
$$= 2A_n\sum_{k=1}^{\infty}\lambda_k\left[\frac{(k+n)!}{k!(2k+n-1)(2k+n+1)}a_{k+1}a_k\right.$$
$$\left. - \frac{(k-1)(k+n-2)!}{(k-1)!(2k+n-3)(2k+n-1)}a_{k-1}a_k\right]$$
$$= 2A_n\sum_{k=1}^{\infty}\left[\frac{(k+n-1)!\lambda_{k+1}}{(k-1)!(2k+n-1)(2k+n+1)}a_{k+1}a_k\right.$$
$$\left. - \frac{(k-1)(k+n-2)!\lambda_k}{(k-1)!(2k+n-3)(2k+n-1)}a_{k-1}a_k\right]$$
$$= 0.$$

By (2.17), we conclude that

$$\text{if } \alpha \neq 1, \text{ then } a_1 = 0 \text{ and so } b_0 = 0. \tag{2.18}$$

Then we compare the coefficients in front of $C_1^{\frac{n-1}{2}}(x)$ in (2.15). More precisely,

$$\int_{-1}^{1}(1-x^2)^{\frac{n}{2}}(P_nu)'C_1^{\frac{n-1}{2}} = n\int_{-1}^{1}(1-x^2)^{\frac{n-2}{2}}(P_nu)'P_nuGC_1^{\frac{n-1}{2}} + \frac{n!}{\alpha}\int_{-1}^{1}(1-x^2)^{\frac{n-2}{2}}GC_1^{\frac{n-1}{2}}. \tag{2.19}$$

From (2.16), we deduce that

$$\int_{-1}^{1}(1-x^2)^{\frac{n}{2}}(P_nu)'C_1^{\frac{n-1}{2}}(x) = \frac{2n!(n+1)!}{n+3}A_na_2. \tag{2.20}$$

For the second term of RHS of (2.19), we have

$$\frac{n!}{\alpha}\int_{-1}^{1}(1-x^2)^{\frac{n-2}{2}}GC_1^{\frac{n-1}{2}} = \frac{2(n!)^2}{\alpha(n+3)}A_na_2. \tag{2.21}$$

For the first term of RHS of (2.19), after integration by part, we obtain

$$\begin{aligned}\int_{-1}^{1}(1-x^2)^{\frac{n-2}{2}}P_nuGC_1^{\frac{n-1}{2}}(x) &= -\frac{n-1}{n}\int_{-1}^{1}P_nuGd((1-x^2)^{\frac{n}{2}})\\ &= \frac{n-1}{n}\int_{-1}^{1}(1-x^2)^{\frac{n-2}{2}}\left[(1-x^2)(P_nu)'G+(1-x^2)G'P_nu\right]dx\\ &:= \frac{n-1}{n}(I+II).\end{aligned}$$

By (2.9)–(2.13), we find

$$\begin{aligned}(1-x^2)G' &= \sum_{k=1}^{\infty}b_k\left[\frac{(k+n-1)(k+n-2)}{2k+n-1}C_{k-1}^{\frac{n-1}{2}}(x)-\frac{k(k+1)}{2k+n-1}C_{k-1}^{\frac{n-1}{2}}(x)\right]\\ &= \frac{n(n-1)}{n+1}b_1C_0^{\frac{n-1}{2}}(x)+\sum_{k=1}^{\infty}\left(\frac{(k+n)(k+n-1)}{2k+n-1}b_{k+1}-\frac{k(k-1)}{2k+n-3}b_{k-1}\right)\\ &\quad\times C_k^{\frac{n-1}{2}}(x).\end{aligned} \tag{2.22}$$

After some computations, we deduce from (2.10), (2.4), (2.16) and (2.22),

$$
\begin{aligned}
I+II &= \frac{2(n+1)!n!}{n+3}A_n a_2 b_1 - \frac{2n!n!}{n+3}A_n a_2 b_1 \\
&\quad +2A_n \sum_{k=2}^{\infty} a_{k+1} b_k \lambda_{k+1} \left[\frac{(k+n)!}{(2k+n+1)(2k+n-1)k!}\right. \\
&\quad \left. - \frac{(k+n-1)!}{(2k+n-1)(2k+n+1)k!}\right] \\
&\quad +2A_n \sum_{k=2}^{\infty} a_k b_{k+1} \lambda_k \left[\frac{(k+n)!}{(2k+n+1)(2k+n-1)k!}\right. \\
&\quad \left. - \frac{(k+n-1)!}{(2k+n-1)(2k+n+1)(k-1)!}\right] \\
&= \frac{2n(n!)^2}{n+3}A_n a_2 b_1 + 2nA_n \sum_{k=2}^{\infty} \frac{(k+n-1)!\lambda_k}{(2k+n+1)(2k+n-1)k!} \\
&\quad \times \left(\frac{k+n}{k} a_{k+1} b_k + a_k b_{k+1}\right) \\
&= \frac{2n(n-1)n!(n+1)!}{(n+3)(n+5)} A_n a_2^2 \\
&\quad + 2n(n-1)A_n \sum_{k=2}^{\infty} \frac{\lambda_k (k+n)(k+n)!}{(2k+n+1)(2k+n-1)(2k+n-3)kk!} a_{k+1}^2 \\
&\quad -2n(n-1)A_n \sum_{k=2}^{\infty} \frac{\lambda_k (k+n+1)!}{(2k+n+1)(2k+n-1)(2k+n-3)(k+1)!} a_k a_{k+2} \\
&= 2n(n-1)A_n \sum_{k=2}^{\infty} \frac{(k+n-1)!}{(2k+n+1)(2k+n-1)(k-1)!} \\
&\quad \times \left(\frac{(k+n-1)\lambda_{k-1}}{(2k+n-3)(k-1)} a_k^2 - \frac{(k+n+1)(k+n)\lambda_k}{k(2k+n-3)(k-1)} a_k a_{k+2}\right) \\
&= n(n-1)A_n \sum_{k=0}^{\infty} \frac{(k+n-1)!\prod_{s=1}^{n-1}(k+s)}{(k+1)(2k+n+1)(k+1)!} b_{k+1}^2 .
\end{aligned}
$$

Therefore,

$$
\int_{-1}^{1} (1-x^2)^{\frac{n-2}{2}} P_n u = (n-1)^2 A_n \sum_{k=0}^{\infty} \frac{(k+n-1)!\prod_{s=1}^{n-1}(k+s)}{(k+1)(2k+n+1)(k+1)!} b_{k+1}^2 . \tag{2.23}
$$

Combining (2.20)–(2.23), we have

$$
\frac{2(n!)^2}{(n-1)^2(n+3)} \left(n+1-\frac{1}{\alpha}\right) a_2 = \sum_{k=0}^{\infty} \frac{(k+n-1)!\prod_{s=1}^{n-1}(k+s)}{(k+1)(2k+n+1)(k+1)!} b_{k+1}^2 .
$$

It is easy to see that if $\alpha = \frac{1}{n+1}$ then $b_k = 0$ for $k \geq 1$. Thus, $G \equiv 0$ and $u \equiv C$. $\square$

3. The case: n is even

In this section, we shall show some results for (1.1) with $n \geq 6$ even, which can be regarded as the generalization of recent results in [17] for $n = 4$.

Let $\theta_i,\ i = 1, 2, \ldots, n$ denote the usual angular coordinates on the sphere with

$$\theta_n \in [0, 2\pi] \quad \text{and } \theta_i \in [0, \pi],\ i = 1, 2, \ldots, n-1$$

and define $x = \cos\theta_1$. Then the metric tensor is

$$g = \begin{pmatrix} (1-x^2)^{-1} & 0 & 0 & \cdots & 0 \\ 0 & 1-x^2 & 0 & \cdots & 0 \\ 0 & 0 & (1-x^2)\sin^2\theta_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (1-x^2)\sin^2\theta_2 \ldots \sin^2\theta_{n-1} \end{pmatrix} \tag{3.1}$$

In what follows, we shall consider axially symmetric functions which only depend on x. For such functions, we have

$$\int_{S^n} dw = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2\pi^{\frac{n+1}{2}}} \int_{-1}^{1}\int_0^{\pi}\int_0^{\pi}\cdots\int_0^{2\pi} (1-x^2)^{\frac{n-2}{2}} \sin^{(n-2)}\theta_2 \cdots\cdots \sin\theta_{n-1} d\theta_n \cdots d\theta_2 dx. \tag{3.2}$$

Note that

$$\int_0^{\frac{\pi}{2}} sin^{s-1}\theta d\theta = 2^{s-2} B\left(\frac{s}{2}, \frac{s}{2}\right) = 2^{-1} B\left(\frac{1}{2}, \frac{s}{2}\right).$$

Then for $k = 2, 3, \ldots, n-1$,

$$\int_0^{\pi} sin^{n-k}\theta_k d\theta_k = B\left(\frac{1}{2}, \frac{n-k+1}{2}\right) = \frac{\sqrt{\pi}\Gamma\left(\frac{n+1-k}{2}\right)}{\Gamma\left(\frac{n+2-k}{2}\right)}.$$

We further have

$$\begin{aligned} \int_{S^n} dw &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n-1}{2}}} \prod_{k=2}^{n-1} \frac{\sqrt{\pi}\Gamma\left(\frac{n+1-k}{2}\right)}{\Gamma\left(\frac{n+2-k}{2}\right)} \int_{-1}^{1} (1-x^2)^{\frac{n-2}{2}} dx \\ &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)} \int_{-1}^{1} (1-x^2)^{\frac{n-2}{2}} dx \end{aligned}$$

$$= \frac{(n-1)!}{2^{n-1}[(n/2-1)!]^2} \int_{-1}^{1} (1-x^2)^{\frac{n-2}{2}} dx,$$

for even n. Moreover,

$$\Delta u = |g|^{-\frac{1}{2}} \frac{\partial}{\partial x} \left(|g|^{\frac{1}{2}} g^{11} \frac{\partial u}{\partial x} \right) = (1-x^2)^{-\frac{n-2}{2}} \frac{\partial}{\partial x} \left[(1-x^2)^{\frac{n}{2}} \frac{\partial u}{\partial x} \right]$$
$$= (1-x^2)u'' - nxu'$$

and

$$P_n u = (-1)^{\frac{n}{2}} \left[(1-x^2)^{\frac{n}{2}} u' \right]^{(n-1)} = (-1)^{\frac{n}{2}} \left[(1-x^2)^{\frac{n-2}{2}} \Delta u \right]^{(n-2)} \tag{3.3}$$

for $u = u(x)$. Hence, we can transform the original equation (1.1) on $\mathbb{S}^n$ into an ODE (1.7).

In the following, we assume that $\alpha < 1$.

Let G be defined as (2.11). In view of equation (1.7), we drive

$$\alpha(-1)^{\frac{n}{2}} ((1-x^2)^{\frac{n-2}{2}} G)^{(n-1)} + (n-1)! - \frac{(n-1)!\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\gamma} e^{nu} = 0. \tag{3.4}$$

By differentiating (3.4), we further have

$$(-1)^{\frac{n}{2}} (1-x^2)^{\frac{n}{2}} [(1-x^2)^{\frac{n-2}{2}} G]^{(n)} - \frac{n!}{\alpha} (1-x^2)^{\frac{n-2}{2}} G$$
$$- (-1)^{\frac{n}{2}} n (1-x^2)^{\frac{n-2}{2}} G [(1-x^2)^{\frac{n-2}{2}} G]^{(n-1)} = 0. \tag{3.5}$$

For simplicity, let

$$\hat{C}_k^{\frac{n-1}{2}} = \frac{k!\Gamma(n-1)}{\Gamma(k+n-1)} C_k^{\frac{n-1}{2}} \quad \text{and} \quad d_k = \frac{\Gamma(k+n-1)}{k!\Gamma(n-1)} b_k \tag{3.6}$$

and drop the hat and the index $\frac{n-1}{2}$ in the notation in later discussion. In the following, we will study the above C_k and d_k.

Combining (2.12) and (2.18), we have the following decomposition using the orthogonal polynomials $C_k, k \geq 1$:

$$G = \sum_{k=1}^{\infty} d_k C_k(x). \tag{3.7}$$

Recall that

$$\bar{\lambda}_k = k(k+n-1) \text{ and } \lambda_k = \frac{\Gamma(n+k)}{\Gamma(k)}.$$

Note that (2.3) and (2.4) respectively become

$$|C_k'(x)| \le \frac{\bar{\lambda}_k}{n}, \quad \forall x \in (-1,1); \tag{3.8}$$

and

$$\int_{-1}^{1} (1-x^2)^{\frac{n-2}{2}} C_k C_l = \frac{2^{n-1}[(n/2-1)!]^2 \bar{\lambda}_k}{(n+2k-1)\lambda_k} \delta_{kl}. \tag{3.9}$$

Define $d_1 = \beta$ and

$$t_k^2 = \begin{cases} d_k^2 \int_{-1}^1 (1-x^2)^{\frac{n-2}{2}} C_k^2, & \text{for } k \ge 2; \\ \beta^2 \int_{-1}^1 (1-x^2)^{\frac{n-2}{2}} C_1^2, & \text{for } k = 1. \end{cases}$$

It follows from (2.2) and (3.3) that

$$\left[(1-x^2)^{\frac{n}{2}} C_k'\right]' = -\bar{\lambda}_k (1-x^2)^{\frac{n-2}{2}} C_k$$

and

$$[(1-x^2)^{\frac{n-2}{2}} C_k]^{(n-2)} = (-1)^{\frac{n-2}{2}} \frac{\lambda_k}{\bar{\lambda}_k} C_k.$$

After direct calculations, we obtain the following decompositions.

$$\int_{-1}^{1} (1-x^2)^{\frac{n-2}{2}} G^2 = \sum_{k=1}^{\infty} t_k^2, \tag{3.10}$$

$$\int_{-1}^{1} (1-x^2)^{\frac{n}{2}} (G')^2 = \sum_{k=1}^{\infty} \bar{\lambda}_k t_k^2, \tag{3.11}$$

$$\int_{-1}^{1} \left| [(1-x^2)^{\frac{n-2}{2}} G]^{(\frac{n-2}{2})} \right|^2 = \sum_{k=1}^{\infty} \frac{\lambda_k}{\bar{\lambda}_k} t_k^2 = \sum_{k=1}^{\infty} \frac{\Gamma(n+k-1)}{\Gamma(k+1)} t_k^2. \tag{3.12}$$

Next, we shall state some important integral identities which will be used frequently in the proof of the main results.

Lemma 3.1. *We establish the following equalities for G.*

$$\int_{-1}^{1} (1-x^2)^{\frac{n-2}{2}} C_1 G = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)}{2\Gamma\left(\frac{n+3}{2}\right)} \beta, \tag{3.13}$$

$$\int_{-1}^{1}(1-x^2)^{\frac{n}{2}}\frac{e^{nu}}{\gamma}=\frac{n}{n+1}(1-\alpha\beta), \tag{3.14}$$

$$\int_{-1}^{1}(1-x^2)^{\frac{n-2}{2}}C_kG=-\frac{2^{n-1}[(n/2-1)!]^2}{\alpha\lambda_k}\int_{-1}^{1}\frac{e^{nu}}{\gamma}(1-x^2)^{\frac{n}{2}}C_k', \quad k\geq 2, \tag{3.15}$$

$$\int_{-1}^{1}\left|[(1-x^2)^{\frac{n-2}{2}}G]^{(\frac{n-2}{2})}\right|^2=\frac{\sqrt{\pi}(n-2)!\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+3}{2}\right)}\left(n+1-\frac{1}{\alpha}\right)\beta. \tag{3.16}$$

Proof. A direct calculation shows that

$$\int_{-1}^{1}(1-x^2)^{\frac{n-2}{2}}C_1G=\beta\int_{-1}^{1}(1-x^2)^{\frac{n-2}{2}}x^2=\frac{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)}{2\Gamma\left(\frac{n+3}{2}\right)}\beta.$$

Then (3.13) follows.

To prove (3.14) and (3.15), multiplying (3.4) by $\int_{-1}^{x}(1-s^2)^{\frac{n-2}{2}}C_k(s)ds$ with $k\geq 1$ and integrating over $[-1,1]$, we have

$$\begin{aligned}&\int_{-1}^{1}\int_{-1}^{x}(1-s^2)^{\frac{n-2}{2}}C_k(s)\Big[\alpha(-1)^{\frac{n}{2}}((1-x^2)^{\frac{n-2}{2}}G)^{(n-1)}+(n-1)!\\&-\frac{2^{n-1}[(n/2-1)!]^2}{\gamma}e^{nu}\Big]=0.\end{aligned}$$

After integrating by parts, we obtain

$$\begin{aligned}&(-1)^{\frac{n}{2}}\alpha\int_{-1}^{1}\int_{-1}^{x}(1-s^2)^{\frac{n-2}{2}}C_k(s)((1-x^2)^{\frac{n-2}{2}}G)^{(n-1)}\\&=(-1)^{\frac{n+2}{2}}\alpha\int_{-1}^{1}((1-x^2)^{\frac{n-2}{2}}G)^{(n-2)}(1-x^2)^{\frac{n-2}{2}}C_k(x)\\&=\frac{\alpha\Gamma(n+k-1)}{\Gamma(k+1)}\int_{-1}^{1}(1-x^2)^{\frac{n-2}{2}}C_k(x)G.\end{aligned} \tag{3.17}$$

Furthermore,

$$\begin{aligned}\int_{-1}^{1}\int_{-1}^{x}(1-s^2)^{\frac{n-2}{2}}C_k(s) &= \left(x\int_{-1}^{x}(1-s^2)^{\frac{n-2}{2}}C_k(s)\right)\Big|_{-1}^{1} - \int_{-1}^{1}(1-x^2)^{\frac{n-2}{2}}xC_k \\ &= -\frac{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)}{2\Gamma\left(\frac{n+3}{2}\right)}\delta_{1k}.\end{aligned} \tag{3.18}$$

By (2.2) we find that

$$\int_{-1}^{x}(1-s^2)^{\frac{n-2}{2}}C_k(s) = -\frac{1}{\bar{\lambda}_k}(1-x^2)^{\frac{n}{2}}C_k'(x). \tag{3.19}$$

Let $k=1$, then from (3.17)–(3.19) we deduce that

$$\frac{\sqrt{\pi}(n-1)!\Gamma\left(\frac{n}{2}\right)}{\bar{\lambda}_1\Gamma\left(\frac{n+1}{2}\right)}\int_{-1}^{1}(1-x^2)^{\frac{n}{2}}\frac{e^{nu}}{\gamma} = \frac{\sqrt{\pi}(n-1)!\Gamma\left(\frac{n}{2}\right)}{2\Gamma\left(\frac{n+3}{2}\right)}(1-\alpha\beta).$$

This leads to (3.14).

When $k\geq 2$, (3.15) follows from (3.17)–(3.19). Here we employ the fact that

$$\frac{\Gamma(n+k-1)}{\Gamma(k+1)} = \frac{\lambda_k}{\bar{\lambda}_k} \text{ and } 2^{n-1}[(n/2-1)!]^2 = \frac{(n-1)!\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}.$$

For (3.16), multiplying (3.5) by x and integrating from -1 to 1, we obtain

$$\begin{aligned}&\int_{-1}^{1}\Big[(-1)^{\frac{n}{2}}x(1-x^2)^{\frac{n}{2}}[(1-x^2)^{\frac{n-2}{2}}G]^{(n)} - \frac{n!}{\alpha}x(1-x^2)^{\frac{n-2}{2}}G \\ &- (-1)^{\frac{n}{2}}nx(1-x^2)^{\frac{n-2}{2}}G[(1-x^2)^{\frac{n-2}{2}}G]^{(n-1)}\Big] = 0.\end{aligned} \tag{3.20}$$

By integrating by parts, we find

$$\begin{aligned}\int_{-1}^{1}(-1)^{\frac{n}{2}}x(1-x^2)^{\frac{n}{2}}[(1-x^2)^{\frac{n-2}{2}}G]^{(n)} &= \int_{-1}^{1}(-1)^{\frac{n}{2}}[x(1-x^2)^{\frac{n}{2}}]^{(n)}[(1-x^2)^{\frac{n-2}{2}}G] \\ &= (n+1)!\int_{-1}^{1}x(1-x^2)^{\frac{n-2}{2}}G.\end{aligned} \tag{3.21}$$

Furthermore,

$$(-1)^{\frac{n}{2}} n \int_{-1}^{1} [x(1-x^2)^{\frac{n-2}{2}} G][(1-x^2)^{\frac{n-2}{2}} G]^{(n-1)}$$

$$= n \int_{-1}^{1} [(1-x^2)^{\frac{n-2}{2}} G]^{(\frac{n-2}{2})} \left[x((1-x^2)^{\frac{n-2}{2}} G)^{(\frac{n}{2})} + \frac{n}{2}((1-x^2)^{\frac{n-2}{2}} G)^{(\frac{n-2}{2})} \right] \tag{3.22}$$

$$= \frac{n(n-1)}{2} \int_{-1}^{1} |[(1-x^2)^{\frac{n-2}{2}} G]^{(\frac{n}{2}-1)}|^2.$$

It follows from (3.20)–(3.22) that

$$n! \left(n + 1 - \frac{1}{\alpha} \right) \int_{-1}^{1} x(1-x^2)^{\frac{n-2}{2}} G = \frac{n(n-1)}{2} \int_{-1}^{1} |[(1-x^2)^{\frac{n-2}{2}} G]^{(\frac{n}{2}-1)}|^2,$$

which, joint with (3.13), implies (3.16). □

Multiplying (3.5) by G and integrating over $[-1, 1]$, we have

$$\begin{aligned} &\int_{-1}^{1} (-1)^{\frac{n}{2}} (1-x^2)^{\frac{n}{2}} G[(1-x^2)^{\frac{n-2}{2}} G]^{(n)} - \frac{n!}{\alpha} \int_{-1}^{1} (1-x^2)^{\frac{n-2}{2}} G^2 \\ &\qquad - (-1)^{\frac{n}{2}} n \int_{-1}^{1} (1-x^2)^{\frac{n-2}{2}} G^2 [(1-x^2)^{\frac{n-2}{2}} G]^{(n-1)} = 0. \end{aligned} \tag{3.23}$$

For the first term,

$$\begin{aligned} &\int_{-1}^{1} (-1)^{\frac{n}{2}} (1-x^2)^{\frac{n}{2}} G[(1-x^2)^{\frac{n-2}{2}} G]^{(n)} \\ &\quad = (-1)^{\frac{n}{2}} \int_{-1}^{1} [(1-x^2)^{\frac{n-2}{2}} G][(1-x^2)^{\frac{n}{2}} G']^{(n-1)} \\ &\qquad + (-1)^{\frac{n}{2}} n \int_{-1}^{1} [x(1-x^2)^{\frac{n-2}{2}} G][(1-x^2)^{\frac{n-2}{2}} G]^{(n-1)} \\ &\quad = \lfloor G \rfloor^2 + \frac{n(n-1)}{2} \int_{-1}^{1} |[(1-x^2)^{\frac{n-2}{2}} G]^{(\frac{n}{2}-1)}|^2, \end{aligned}$$

where

$$\lfloor G\rfloor^2 = \frac{2^{n-1}[(n/2-1)!]^2}{(n-1)!}\int_{\mathbb{S}^n}(P_nG)Gdw = (-1)^{\frac{n}{2}}\int_{-1}^{1}(1-x^2)^{\frac{n-2}{2}}[(1-x^2)^{\frac{n}{2}}G']^{(n-1)}G. \tag{3.24}$$

However, the last term is very sophisticated, and we will consider it for the cases $n = 6$ and $n = 8$ below. More precisely, after some complicated computations, we derive that for $n = 6$,

$$\int_{-1}^{1}(1-x^2)^2G^2[(1-x^2)^2G]^{(5)} = -5\int_{-1}^{1}(1-x^2)^4G'(G'')^2 - \frac{80}{3}\int_{-1}^{1}(1-x^2)^3(G')^3, \tag{3.25}$$

and for $n = 8$,

$$\begin{aligned}\int_{-1}^{1}(1-x^2)^3G^2[(1-x^2)^3G]^{(7)} &= 1260\int_{-1}^{1}(1-x^2)^4(G')^3 + 252\int_{-1}^{1}(1-x^2)^5G'(G'')^2\\ +7\int_{-1}^{1}(1-x^2)^6G'(G^{(3)})^2 &- 21\int_{-1}^{1}[(1-x^2)G]^{(3)}(1-x^2)^5(G'')^2.\end{aligned} \tag{3.26}$$

The details of the computation are postponed to Appendices A and B.

In view of the above relations, (3.23) for $n = 6$ becomes

$$\begin{aligned}&\lfloor G\rfloor^2 + 15\int_{-1}^{1}|[(1-x^2)^2G]''|^2 - \frac{6!}{\alpha}\int_{-1}^{1}(1-x^2)^2G^2\\ &- 30\int_{-1}^{1}(1-x^2)^4G'(G'')^2 - 160\int_{-1}^{1}(1-x^2)^3(G')^3 = 0.\end{aligned} \tag{3.27}$$

Similarly, (3.23) for $n = 8$ is equivalent to

$$\begin{aligned}&\lfloor G\rfloor^2 + 28\int_{-1}^{1}|[(1-x^2)^3G]^{(3)}|^2 - \frac{8!}{\alpha}\int_{-1}^{1}(1-x^2)^3G^2\\ &- 56\Big[180\int_{-1}^{1}(1-x^2)^4(G')^3 + 36\int_{-1}^{1}(1-x^2)^5G'(G'')^2\\ &+\int_{-1}^{1}(1-x^2)^6G'(G^{(3)})^2 - 3\int_{-1}^{1}[(1-x^2)G]^{(3)}(1-x^2)^5(G'')^2\Big] = 0.\end{aligned} \tag{3.28}$$

Next we shall show a family of important gradient estimates in $\mathbb{S}^n$, which generalizes a similar result in $\mathbb{S}^4$.

Lemma 3.2. *For all $x \in [-1,1]$, we have*

$$G_j := (-1)^j[(1-x^2)^j G]^{(2j+1)} \le \frac{(2j+1)!}{\alpha}, \quad x \in (-1,1), \quad 0 \le j \le \frac{n}{2} - 1. \tag{3.29}$$

Proof. We will first prove the result for the case $n = 6$.

Since

$$\begin{aligned}\left[(1-x^2)^2 G\right]^{(5)} &= \left[(1-x^2)((1-x^2)G)'' - 4x((1-x^2)G)' - 2(1-x^2)G\right]^{(3)} \\ &= \Big[(1-x^2)((1-x^2)G)^{(3)} - 6x((1-x^2)G)'' - 6((1-x^2)G)'\Big]'' \\ &= -(1-x^2)G_1'' + 10xG_1' + 20G_1.\end{aligned}$$

Therefore, by (3.4) we have

$$-(1-x^2)G_1'' + 10xG_1' + 20G_1 \le \frac{5!}{\alpha}. \tag{3.30}$$

Let $M_1 = \max_{x\in[-1,1]} G_1'(x)$.

Case 1:

$$M_1 = \lim_{x_k \to 1} G_1'(x_k) \quad \text{for some } x_k \in (-1,1).$$

As in [17], let $r = |x'| = \sqrt{1-x^2}$, then we write

$$G(x) = \bar{G}(r), \quad G_1(x) = \bar{G}_1(r) \text{ and } u(x) = \bar{u}(r) \quad \text{for } r \in [0,1) \text{ and } x \in (0,1],$$

and $\bar{u}(r)$ can be extended evenly so that $\bar{u}(r) \in \mathcal{C}^\infty(-1,1)$. Hence,

$$\begin{aligned}G_1(x) = \bar{G}_1(r) &= \frac{d^3}{dx^3}(-r^2\bar{G}(r)) \\ &= \frac{d^3}{dx^3}[r^3\sqrt{1-r^2}\bar{u}_r] := \frac{d^3}{dx^3}[g(r^2)],\end{aligned} \tag{3.31}$$

where $g(s),\ s \in (-1,1)$ is a $\mathcal{C}^\infty$ function.

Let $g_0(s) = g(s)$ and

$$g_i(s) = -2g_{i-1}'(s)\sqrt{1-s}, \quad i = 1,2,3.$$

Then $g_i(s) \in \mathcal{C}^\infty(-1,1)$ for $i = 1,2,3$. Differentiate $g(r^2)$ with respect to x:

$$\frac{d}{dx}g(r^2) = -2g'(r^2)\sqrt{1-r^2} = g_1(r^2).$$

Similarly, we have

$$\frac{d^i}{dx^i} g(r^2) = g_i(r^2), \quad i = 2, 3.$$

Therefore, $G_1(x) = \bar{G}_1(r) \in \mathcal{C}^\infty(-1, 1)$ and is even. We now can write

$$\bar{G}_1(r) = c_1 + c_2 r^2 + c_3 r^4 = O(r^6) \quad \text{near } r = 0. \tag{3.32}$$

Then $c_2 \leq 0$, since $M_1 = G_1(1) = \bar{G}_1(0)$. Note that near $r = 0$,

$$\begin{aligned} xG_1'(x) &= \sqrt{1 - r^2} \frac{d\bar{G}_1'(x)}{dr} \frac{dr}{dx} \\ &= -2c_2 + O(r^2) \end{aligned}$$

and

$$(1 - x^2) G_1''(x) = (-2(c_2 - 4c_3) + O(r^2)) r^2.$$

It follows from (3.30) that

$$G_1 \leq \frac{5!}{20\alpha} = \frac{3!}{\alpha}, \quad \forall x \in [-1, 1]. \tag{3.33}$$

Case 2: if

$$M_1 = \lim_{x_k \to -1} G_1'(x_k) \quad \text{ for some } x_k \in (-1, 1).$$

Then it is similar to the case 1 to show (3.33).

Case 3: Let $M_1 = G_1(x_0)$ for some $x_0 \in (-1, 1)$. Then

$$G_1'(x_0) = 0 \quad \text{and} \quad G''(x_0) \leq 0.$$

(3.33) immediately follows from (3.30).

We see from (3.33) that

$$((1 - x^2) G)^{(3)} = -6G' - 6xG'' + (1 - x^2) G''' \geq -\frac{6}{\alpha}, \quad \forall x \in [-1, 1].$$

Repeating the previous arguments, we can conclude

$$G' \leq \frac{1}{\alpha}, \quad \forall x \in [-1, 1].$$

In general, we can start with $G_j, j = \frac{n}{2} - 2$.

In view of (3.4), one directly has

$$(-1)^{\frac{n}{2}}((1-x^2)^{\frac{n-2}{2}}G)^{(n-1)} \leq \frac{(n-1)!}{\alpha}.$$

Since

$$(-1)^{\frac{n}{2}}((1-x^2)^{\frac{n-2}{2}}G)^{(n-1)} = -(1-x^2)G''_{\frac{n}{2}-2} + 2(n-1)G'_{\frac{n}{2}-2} + (n-1)(n-2)G_{\frac{n}{2}-2},$$

we can follow the same arguments above to obtain (3.29) for $j = \frac{n}{2}-2$. Then we can apply mathematical induction on j from $\frac{n}{2}-2$ to 0 to conclude (3.29) for all $0 \leq j \leq \frac{n}{2}-1$. $\square$

3.1. The case: $n = 6$

Throughout the rest of this subsection, we will focus on $n = 6$ and the equation

$$\alpha P_6 u + 5!\left(1 - \frac{e^{6u}}{\int_{\mathbb{S}^6} e^{6u} dw}\right) = 0, \tag{3.34}$$

where $P_6 = -\Delta(-\Delta+4)(-\Delta+6)$. As previously introduced,

$$P_6 u = -[(1-x^2)^3 u']^{(5)} = -[(1-x^2)^2 \Delta u]^{(4)}, \tag{3.35}$$

and equation (3.34) is reduced to

$$-\alpha[(1-x^2)^3 u']^{(5)} + 5! - \frac{2^7}{\gamma} e^{6u} = 0. \tag{3.36}$$

In view of (3.10)–(3.12), we derive that

$$\int_{-1}^{1} (1-x^2)^4 (G'')^2 = \sum_{k=1}^{\infty} \bar{\lambda}_k(\bar{\lambda}_k - 6) t_k^2. \tag{3.37}$$

Moreover, (3.12) can be rewritten as

$$\int_{-1}^{1} |[(1-x^2)^2 G]''|^2 = \sum_{k=1}^{\infty} (\bar{\lambda}_k + 4)(\bar{\lambda}_k + 6) t_k^2. \tag{3.38}$$

Lemma 3.3. *Let $n = 6$, using the semi-norm $\lfloor G \rfloor$ defined in (3.24), we have the following estimate:*

$$\lfloor G \rfloor^2 \leq \left(\frac{30}{\alpha} - 15\right) \int_{-1}^{1} |[(1-x^2)^2 G]''|^2 - \frac{320}{\alpha} \int_{-1}^{1} (1-x^2)^3 (G')^2. \tag{3.39}$$

Proof. By (3.10), (3.37), (3.38) and Lemma 3.2, we get

$$\begin{aligned}
&\lfloor G \rfloor^2 + 15 \int_{-1}^{1} |[(1-x^2)^2 G]''|^2 \\
&\leq \frac{1}{\alpha} \int_{-1}^{1} \left[6!(1-x^2)^2 G^2 + 30(1-x^2)^4 (G'')^2 + 160(1-x^2)^3 (G')^2 \right] \\
&= \frac{1}{\alpha} \int_{-1}^{1} \left[30 \left| [(1-x^2)^2 G]'' \right|^2 - 320(1-x^2)^3 (G')^2 \right].
\end{aligned}$$

So (3.39) holds. □

Proposition 3.4. *If $\frac{2}{3} < \alpha < 1$, any axially symmetric solution to (3.34) must be constant.*

Proof. We only need to show that either $\alpha \leq \frac{2}{3}$ or that G and hence u is identically 0. Indeed, when $\alpha > 2/3$, it follows from (3.39) that

$$\lfloor G \rfloor^2 + 15 \int_{-1}^{1} |[(1-x^2)^2 G]''|^2 < \frac{3}{2} \sum_{k=1}^{\infty} (30\bar{\lambda}_k^2 - 20\bar{\lambda}_k + 720) t_k^2.$$

Equivalently,

$$\begin{aligned}
0 &> \sum_{k=1}^{\infty} \left[(\bar{\lambda}_k + 15)(\bar{\lambda}_k + 4)(\bar{\lambda}_k + 6) - \frac{3}{2}(30\bar{\lambda}_k^2 - 20\bar{\lambda}_k + 720) \right] t_k^2 \\
&= \sum_{k=1}^{\infty} (\bar{\lambda}_k - 6)(\bar{\lambda}_k^2 - 14\bar{\lambda}_k + 120) t_k^2.
\end{aligned}$$

In view of (2.2) the assertion holds. □

For $n = 6$, (3.13) and (3.16) become

$$\int_{-1}^{1} (1-x^2)^2 C_1 G = \frac{16}{105} \beta \tag{3.40}$$

and

$$\int_{-1}^{1} |[(1-x^2)^2 G]''|^2 = \frac{256}{35} \left(7 - \frac{1}{\alpha} \right) \beta. \tag{3.41}$$

Inspired by [18] and [12], our basic strategy is to assume $\beta \neq 0$, and show that it leads to a contradiction with the range of α. It is fairly easy to see from (3.41) that

$$\text{if } \beta = 0, \text{ then } \nabla u = 0, \text{ which shows that } u \text{ is a constant.} \tag{3.42}$$

In what follows, suppose that $\beta \neq 0$ and

$$\frac{3}{5} < \alpha \leq \frac{2}{3}. \tag{3.43}$$

Then it is easy to see from (3.40) and (3.14) that

$$0 < \beta < \frac{1}{\alpha}. \tag{3.44}$$

Proof of Theorem 1.2 for $n = 6$. We first define the following quantity

$$D := \sum_{k=3}^{\infty} \left[\bar{\lambda}_k(\bar{\lambda}_k + 4)(\bar{\lambda}_k + 6) - \left(14 + \frac{112}{9\alpha}\right)(\bar{\lambda}_k + 4)(\bar{\lambda}_k + 6) + \frac{320}{\alpha}\bar{\lambda}_k \right] t_k^2. \tag{3.45}$$

We now give the upper bound of D. From (3.39) we see that

$$\begin{aligned}
D &= \lfloor G \rfloor^2 - \left(14 + \frac{112}{9\alpha}\right) \int_{-1}^{1} |[(1-x^2)^2 G]''|^2 + \frac{320}{\alpha} \int_{-1}^{1} (1-x^2)^3 (G')^2 \\
&\quad - \left(\frac{1280}{3\alpha} - 960\right) \beta^2 \int_{-1}^{1} (1-x^2)^2 C_1^2 \\
&\leq \left(\frac{158}{9\alpha} - 29\right) \int_{-1}^{1} |[(1-x^2)^2 G]''|^2 + \frac{16}{21}\left(192 - \frac{256}{3\alpha}\right)\beta^2 \\
&\leq \frac{16\beta}{7}\left[\frac{16}{5}\left(7 - \frac{1}{\alpha}\right)\left(\frac{158}{9\alpha} - 29\right) + \left(64 - \frac{256}{9\alpha}\right)\beta\right].
\end{aligned} \tag{3.46}$$

It is easy to check $D > 0$. Combining (3.44) and (3.46), we have

$$\frac{16}{5}\left(7 - \frac{1}{\alpha}\right)\left(\frac{158}{9\alpha} - 29\right) + \left(64 - \frac{256}{9\alpha}\right)\frac{1}{\alpha} > 0.$$

Therefore,

$$\alpha < \bar{\alpha} := \frac{221 + \sqrt{13345}}{522} < \frac{2}{3}. \tag{3.47}$$

To estimate the refined lower bound of D, we define

$$g(t) = t - \left(14 + \frac{112}{9\alpha}\right) + \frac{320}{\alpha}\frac{t}{(t+4)(t+6)}.$$

Then it is easy to see that

$$g'(t) > 0 \text{ for } t \geq \bar{\lambda}_3 (= 24).$$

Thus,

$$\begin{aligned} D &= \sum_{k=3}^{\infty}\left[\bar{\lambda}_k - \left(14 + \frac{112}{9\alpha}\right) + \frac{320}{\alpha}\frac{\bar{\lambda}_k}{(\bar{\lambda}_k+4)(\bar{\lambda}_k+6)}\right](\bar{\lambda}_k+4)(\bar{\lambda}_k+6)t_k^2 \\ &\geq \left[\bar{\lambda}_3 - \left(14 + \frac{112}{9\alpha}\right) + \frac{320}{\alpha}\frac{\bar{\lambda}_3}{(\bar{\lambda}_3+4)(\bar{\lambda}_3+6)}\right]\sum_{k=3}^{\infty}(\bar{\lambda}_k+4)(\bar{\lambda}_k+6)t_k^2 \\ &= \left(10 - \frac{208}{63\alpha}\right)\sum_{k=3}^{\infty}(\bar{\lambda}_k+4)(\bar{\lambda}_k+6)t_k^2. \end{aligned}$$

On the other hand, we derive from (3.8), (3.9) and (3.15) that

$$\begin{aligned} t_k^2 = d_k^2\int_{-1}^{1}(1-x^2)^2 C_k^2 &= \left(\int_{-1}^{1}(1-x^2)^2 C_k^2\right)^{-1}\left[\frac{128}{\alpha\lambda_k}\int_{-1}^{1}\frac{e^{6u}}{\gamma}(1-x^2)^3 C_k'\right]^2 \\ &\leq \frac{(2k+5)(\bar{\lambda}_k+4)(\bar{\lambda}_k+6)}{128}\left[\frac{8}{\alpha\lambda_k}\frac{\bar{\lambda}_k}{7}(1-\alpha\beta)\right]^2 \\ &= \frac{128(2k+5)}{49(\bar{\lambda}_k+4)(\bar{\lambda}_k+6)}\left(\frac{1}{\alpha}-\beta\right)^2, \quad k \geq 2. \end{aligned} \tag{3.48}$$

In particular,

$$\begin{aligned} d_k^2 &\leq \left(\int_{-1}^{1}(1-x^2)^2 C_k^2\right)^{-1}\frac{128(2k+5)}{49(\bar{\lambda}_k+4)(\bar{\lambda}_k+6)}\left(\frac{1}{\alpha}-\beta\right)^2 \\ &\leq \left[\frac{2k+5}{7}\left(\frac{1}{\alpha}-\beta\right)\right]^2, \quad k \geq 2. \end{aligned} \tag{3.49}$$

By (3.40), (3.41) and (3.48), we get the lower bound of D:

$$
\begin{aligned}
D &\geq \left(10-\frac{208}{63\alpha}\right)\sum_{k=3}^{\infty}(\bar{\lambda}_k+4)(\bar{\lambda}_k+6)t_k^2 \\
&= \left(10-\frac{208}{63\alpha}\right)\left[\int_{-1}^{1}|[(1-x^2)^2G]''|^2-\frac{128}{7}\beta^2-\frac{1152}{49}\left(\frac{1}{\alpha}-\beta\right)^2\right] \\
&\geq \frac{16}{7}\left(10-\frac{208}{63\alpha}\right)\left[\frac{16\beta}{5}\left(7-\frac{1}{\alpha}\right)-8\beta^2-\frac{72}{7}\left(\frac{1}{\alpha}-\beta\right)^2\right].
\end{aligned}
\tag{3.50}
$$

By both (3.46) and (3.50), we see that

$$
\begin{aligned}
&\frac{16\beta}{7}\left[\frac{16}{5}\left(7-\frac{1}{\alpha}\right)\left(\frac{158}{9\alpha}-29\right)+\left(64-\frac{256}{9\alpha}\right)\beta\right] \\
&\geq \frac{16}{7}\left(10-\frac{208}{63\alpha}\right)\left[\frac{16\beta}{5}\left(7-\frac{1}{\alpha}\right)-8\beta^2-\frac{72}{7}\left(\frac{1}{\alpha}-\beta\right)^2\right].
\end{aligned}
$$

A straightforward computation shows that

$$
\begin{aligned}
0 &\leq \frac{16\beta}{5}\left(7-\frac{1}{\alpha}\right)\left[\frac{158}{9\alpha}-29-\left(10-\frac{208}{63\alpha}\right)\right]+\beta^2\left[64-\frac{256}{9\alpha}+8\left(10-\frac{208}{63\alpha}\right)\right] \\
&+\frac{72}{7}\left(10-\frac{208}{63\alpha}\right)\left(\frac{1}{\alpha}-\beta\right)^2 \\
&= \frac{16\beta}{5}\left(7-\frac{1}{\alpha}\right)\left(\frac{146}{7\alpha}-39\right)+48\beta^2\left(3-\frac{8}{7\alpha}\right)+\frac{72}{7}\left(10-\frac{208}{63\alpha}\right)\left(\frac{1}{\alpha}-\beta\right)^2.
\end{aligned}
$$

We further have

$$
\begin{aligned}
&\beta\left[\frac{2}{5}\left(7-\frac{1}{\alpha}\right)\left(\frac{146}{7\alpha}-39\right)+\frac{1}{\alpha}\left(18-\frac{48}{7\alpha}\right)\right] \\
&\geq \left(\frac{1}{\alpha}-\beta\right)\left[\left(18-\frac{48}{7\alpha}\right)\beta-\left(\frac{90}{7}-\frac{208}{49\alpha}\right)\left(\frac{1}{\alpha}-\beta\right)\right] \\
&:= \left(\frac{1}{\alpha}-\beta\right)I.
\end{aligned}
\tag{3.51}
$$

We want to show that the term I is nonnegative. Thus, we need to obtain the lower bound of β. Note that the argument exploited in [17] is not applicable to this case. Precisely, following [17], (3.46) implies that

$$
\left(64-\frac{256}{9\alpha}\right)\beta > \frac{16}{5}\left(7-\frac{1}{\alpha}\right)\left(29-\frac{158}{9\alpha}\right).
$$

However, the term $29-\frac{158}{9\alpha}$ is positive when $\alpha > \frac{158}{261} \approx 0.6054$. Hence, we choose a suitable $\underline{\alpha} > \frac{158}{261}$ so that

$$I > 0, \qquad \text{for } \alpha \in (\underline{\alpha}, \bar{\alpha}). \tag{3.52}$$

Then $\underline{\alpha}$ satisfies that

$$\begin{aligned} I &\geq \left(18 - \frac{48}{7\underline{\alpha}}\right)\beta + \left(\frac{208}{49\bar{\alpha}} - \frac{90}{7}\right)\left(\frac{1}{\alpha} - \beta\right) \\ &\geq \left(18 + \frac{90}{7} - \frac{48}{7\underline{\alpha}} - \frac{208}{49\bar{\alpha}}\right)\beta + \left(\frac{208}{49\bar{\alpha}} - \frac{90}{7}\right)\frac{1}{\underline{\alpha}} \\ &\geq \left(\frac{216}{7} - \frac{48}{7\underline{\alpha}} - \frac{208}{49\bar{\alpha}}\right)\left(7 - \frac{1}{\underline{\alpha}}\right)\left(29 - \frac{158}{9\underline{\alpha}}\right)\frac{9\bar{\alpha}}{180\bar{\alpha} - 80} + \left(\frac{208}{49\bar{\alpha}} - \frac{90}{7}\right)\frac{1}{\underline{\alpha}} \\ &\geq 0, \end{aligned}$$

which indicates $\underline{\alpha} \geq 0.61488$. So we take $\underline{\alpha} = 0.61488$. For $\alpha \in (\underline{\alpha}, \bar{\alpha})$, it follows from (3.51) that

$$\frac{2}{5}\left(7 - \frac{1}{\alpha}\right)\left(\frac{146}{7\alpha} - 39\right) + \frac{1}{\alpha}\left(18 - \frac{48}{7\alpha}\right) > 0.$$

Hence we find

$$\alpha < \alpha^{(6)} := \frac{115 + \sqrt{2851}}{273} \approx 0.61683. \tag{3.53}$$

Theorem 1.2 for $n = 6$ is proven. □

3.2. The case: $n = 8$

This subsection focuses on the case $n = 8$. As we have addressed,

$$P_8 = -\Delta(-\Delta + 6)(-\Delta + 10)(-\Delta + 12), \tag{3.54}$$

and equation (1.1) for $n = 8$ becomes

$$\alpha[(1 - x^2)^4 u']^{(7)} + 7! - \frac{9 * 2^9}{\gamma} e^{8u} = 0. \tag{3.55}$$

In view of (3.10)–(3.12), we derive that

$$\int_{-1}^{1} (1 - x^2)^5 (G'')^2 = \sum_{k=1}^{\infty} \bar{\lambda}_k (\bar{\lambda}_k - 8) t_k^2 \tag{3.56}$$

and

$$\int_{-1}^{1}(1-x^2)^6(G^{(3)})^2=\sum_{k=1}^{\infty}\bar{\lambda}_k(\bar{\lambda}_k^2-26\bar{\lambda}_k+144)t_k^2. \tag{3.57}$$

Moreover, (3.12) can be rewritten as

$$\int_{-1}^{1}|[(1-x^2)^3G]^{(3)}|^2=\sum_{k=1}^{\infty}\tilde{\lambda}_k t_k^2, \tag{3.58}$$

where

$$\tilde{\lambda}_k=(\bar{\lambda}_k+6)(\bar{\lambda}_k+10)(\bar{\lambda}_k+12). \tag{3.59}$$

Lemma 3.5. *Let $n=8$, then we have the following estimate:*

$$\lfloor G\rfloor^2\le 28\left(\frac{2}{\alpha}-1\right)\int_{-1}^{1}|[(1-x^2)^3G]^{(3)}|^2-\frac{20160}{\alpha}\int_{-1}^{1}(1-x^2)^4(G')^2, \tag{3.60}$$

where $\lfloor G\rfloor$ is defined in (3.24).

Proof. By (3.28), Lemma 3.2 and (3.57)–(3.58), we get

$$\begin{aligned}
&\lfloor G\rfloor^2+28\int_{-1}^{1}|[(1-x^2)^3G]^{(3)}|^2\\
&\le\frac{56}{\alpha}\int_{-1}^{1}\left[6!(1-x^2)^3G^2+180(1-x^2)^4(G')^2+54(1-x^2)^5(G'')^2+(1-x^2)^6(G^{(3)})^2\right]\\
&=\frac{56}{\alpha}\sum_{k=1}^{\infty}\left[720+180\bar{\lambda}_k+54\bar{\lambda}_k(\bar{\lambda}_k-8)+\bar{\lambda}_k(\bar{\lambda}_k^2-26\bar{\lambda}_k+144)\right]t_k^2\\
&=\frac{56}{\alpha}\sum_{k=1}^{\infty}\left[(\lambda_k+6)(\lambda_k+10)(\lambda_k+12)-360\bar{\lambda}_k\right]t_k^2\\
&=\frac{56}{\alpha}\int_{-1}^{1}\left[|[(1-x^2)^3G]^{(3)}|^2-360(1-x^2)^4(G')^2\right].
\end{aligned}$$

The proof is complete. □

Proposition 3.6. *If $\frac{21}{25}\le\alpha<1$, any axially symmetric solution to (1.1) must be constant.*

Proof. As in the proof of Proposition 3.4, when $\alpha > \frac{21}{25}$, it follows from (3.60) that

$$\lfloor G \rfloor^2 + 28 \int_{-1}^{1} |[(1-x^2)^3 G]^{(3)}|^2 = \frac{56}{\alpha} \sum_{k=1}^{\infty} \left[\bar{\lambda}_k^3 + 28\bar{\lambda}_k^2 - 108\bar{\lambda}_k + 720\right] t_k^2$$

$$< \frac{200}{3} \sum_{k=1}^{\infty} \left[\bar{\lambda}_k^3 + 28\bar{\lambda}_k^2 - 108\bar{\lambda}_k + 720\right] t_k^2.$$

Equivalently,

$$0 > \sum_{k=1}^{\infty} \left[(\bar{\lambda}_k + 28)\tilde{\lambda}_k - \frac{200}{3}(\bar{\lambda}_k^3 + 28\bar{\lambda}_k^2 - 108\bar{\lambda}_k + 720)\right] t_k^2$$

$$= \sum_{k=1}^{\infty} \left(\bar{\lambda}_k^4 - \frac{32}{3}\bar{\lambda}_k^3 - \frac{2492}{3}\bar{\lambda}_k^2 - 14976\bar{\lambda}_k - 27840\right) t_k^2 > 0,$$

since $\bar{\lambda}_k \geq 8$ and the equation

$$t^4 - \frac{32}{3}t^3 - \frac{2492}{3}t^2 - 14976t - 27840 = 0$$

has two real solutions $t_1 \approx -31.6$ or $t_2 \approx 2.1$. This is a contradiction, which completes the proof. □

Remark 3.1. We note that it fails to get the same result as that in Proposition 3.4. The reason is the following: when $\alpha > 2/3$, it follows from (3.60) that

$$\lfloor G \rfloor^2 + 28 \int_{-1}^{1} |[(1-x^2)^3 G]^{(3)}|^2 = \frac{56}{\alpha} \sum_{k=1}^{\infty} \left[\bar{\lambda}_k^3 + 28\bar{\lambda}_k^2 - 108\bar{\lambda}_k + 720\right] t_k^2$$

$$< 84 \sum_{k=1}^{\infty} \left[\bar{\lambda}_k^3 + 28\bar{\lambda}_k^2 - 108\bar{\lambda}_k + 720\right] t_k^2.$$

Equivalently,

$$0 > \sum_{k=1}^{\infty} \left[(\bar{\lambda}_k + 28)(\bar{\lambda}_k + 6)(\bar{\lambda}_k + 10)(\bar{\lambda}_k + 12) - 84(\bar{\lambda}_k^3 + 28\bar{\lambda}_k^2 - 108\bar{\lambda}_k + 720)\right] t_k^2$$

$$= \sum_{k=1}^{\infty} (\bar{\lambda}_k - 8)(\bar{\lambda}_k^3 - 20\bar{\lambda}_k^2 - 1476\bar{\lambda}_k + 5040)t_k^2.$$

There is no contradiction, since the equation

$$t^3 - 20t^2 - 1476t + 5040 = 0$$

has the following real solutions

$$t_1 \approx -31.6, \quad t_2 \approx 3.3 \quad t_3 \approx 48.4.$$

Recall that (3.13) and (3.16) for $n = 8$ are reduced to

$$\int_{-1}^{1} (1-x^2)^3 C_1 G = \frac{32}{315}\beta \tag{3.61}$$

and

$$\int_{-1}^{1} |[(1-x^2)^3 G]^{(3)}|^2 = \frac{1024}{7}\left(9 - \frac{1}{\alpha}\right)\beta. \tag{3.62}$$

As in the proof of Theorem 1.2 for $n = 6$, in what follows, we assume that

$$\beta \neq 0, \quad \frac{2}{3} < \alpha < \frac{21}{25} \tag{3.63}$$

and note

$$0 < \beta < \frac{1}{\alpha}.$$

Proof of Theorem 1.2 for $n = 8$. Define

$$E := \sum_{k=3}^{\infty} \left[\bar{\lambda}_k \tilde{\lambda}_k - \left(18 + \frac{18}{\alpha}\right)\tilde{\lambda}_k + \frac{20160}{\alpha}\bar{\lambda}_k\right] t_k^2. \tag{3.64}$$

We now present the upper bound of E. From (3.60)–(3.62) we derive

$$\begin{aligned} E &= \lfloor G \rceil^2 - \left(\frac{18}{\alpha} + 18\right) \int_{-1}^{1} |[(1-x^2)^3 G]^{(3)}|^2 + \frac{20160}{\alpha} \int_{-1}^{1} (1-x^2)^4 (G')^2 \\ &\quad - 7!\left(\frac{14}{\alpha} - 10\right)\beta^2 \int_{-1}^{1} (1-x^2)^2 C_1^2 \\ &\leq \left(\frac{38}{\alpha} - 46\right) \int_{-1}^{1} |[(1-x^2)^3 G]^{(3)}|^2 - 1024\left(\frac{7}{\alpha} - 5\right)\beta^2 \\ &= \frac{1024\beta}{7}\left[\left(\frac{38}{\alpha} - 46\right)\left(9 - \frac{1}{\alpha}\right) + 7\left(5 - \frac{7}{\alpha}\right)\beta\right]. \end{aligned} \tag{3.65}$$

To estimate the lower bound of D, define

$$\bar{g}(t) = t - \left(18 + \frac{18}{\alpha}\right) + \frac{20160}{\alpha} \frac{t}{(t+6)(t+10)(t+12)}.$$

Differentiating $\bar{g}(t)$, we have

$$\bar{g}'(t) = 1 - \frac{20160}{\alpha} \frac{2\left(t^3 + 14t^2 - 360\right)}{(t+6)^2(t+10)^2(t+12)^2}.$$

After some calculations, we deduce that $\bar{g}''(t) > 0$ for $t > \bar{\lambda}_3 (= 30)$. Thus, for $t \geq 30$,

$$\bar{g}'(t) \geq \bar{g}'(30) = 1 - \frac{109}{252\alpha} > 0$$

due to (3.63). We further have

$$\begin{aligned} E &= \sum_{k=3}^{\infty} \left[\bar{\lambda}_k - \left(18 + \frac{18}{\alpha}\right) + \frac{20160\bar{\lambda}_k}{\alpha\tilde{\lambda}_k}\right] \tilde{\lambda}_k t_k^2 \\ &\geq \left(12 - \frac{8}{\alpha}\right) \sum_{k=3}^{\infty} \tilde{\lambda}_k t_k^2 \\ &> 0, \end{aligned} \tag{3.66}$$

since $\alpha > \frac{2}{3}$. We conclude from (3.63), (3.65) and (3.66) that

$$0 < 7\left(\frac{7}{\alpha} - 5\right)\beta < \left(\frac{38}{\alpha} - 46\right)\left(9 - \frac{1}{\alpha}\right)$$

and so

$$\alpha < \alpha^{(8)} := \frac{19}{23} < \frac{21}{25}. \quad \square \tag{3.67}$$

Remark 3.2. We wish to obtain for $n = 8$ a similar inequality as (3.51) for $n = 6$ so that the range of α can be more refined. However, the similar approach seems not working as shown below.

As in (3.48), we find

$$\tilde{\lambda}_k t_k^2 \leq \frac{512(2k+7)}{9}\left(\frac{1}{\alpha} - \beta\right)^2, \quad k \geq 2. \tag{3.68}$$

It follows from (3.61), (3.62) and (3.68) that

$$
\begin{aligned}
E &\geq \left(12-\frac{8}{\alpha}\right)\sum_{k=3}^{\infty}\tilde{\lambda}_k t_k^2 \\
&\geq \left(12-\frac{8}{\alpha}\right)\left[\int_{-1}^{1}|[(1-x^2)^3G]^{(3)}|^2-512\beta^2-\frac{512*11}{9}\left(\frac{1}{\alpha}-\beta\right)^2\right] \\
&= 512\left(12-\frac{8}{\alpha}\right)\left[\frac{2\beta}{7}\left(9-\frac{1}{\alpha}\right)-\beta^2-\frac{11}{9}\left(\frac{1}{\alpha}-\beta\right)^2\right].
\end{aligned}
\tag{3.69}
$$

Combining both (3.65) and (3.69), we conclude that

$$
\begin{aligned}
&\frac{2\beta}{7}\left[\left(\frac{38}{\alpha}-46\right)\left(9-\frac{1}{\alpha}\right)+7\left(5-\frac{7}{\alpha}\right)\beta\right] \\
&\geq \left(12-\frac{8}{\alpha}\right)\left[\frac{2\beta}{7}\left(9-\frac{1}{\alpha}\right)-\beta^2-\frac{11}{9}\left(\frac{1}{\alpha}-\beta\right)^2\right].
\end{aligned}
$$

Equivalently,

$$
0\leq \frac{2\beta}{7}\left(9-\frac{1}{\alpha}\right)\left(\frac{46}{\alpha}-58\right)+22\beta^2\left(1-\frac{1}{\alpha}\right)+\frac{11}{9}\left(12-\frac{8}{\alpha}\right)\left(\frac{1}{\alpha}-\beta\right)^2.
$$

Furthermore,

$$
\begin{aligned}
&\beta\left[\frac{2}{7}\left(9-\frac{1}{\alpha}\right)\left(\frac{46}{\alpha}-58\right)+\frac{22}{\alpha}\left(1-\frac{1}{\alpha}\right)\right] \\
&\geq \left(\frac{1}{\alpha}-\beta\right)\left[22\beta\left(1-\frac{1}{\alpha}\right)-\frac{11}{9}\left(12-\frac{8}{\alpha}\right)\left(\frac{1}{\alpha}-\beta\right)\right].
\end{aligned}
\tag{3.70}
$$

However, the r.h.s. of (3.70) is negative. Thus, the previous argument for $n=6$ is not applicable here.

Theorem 1.2 follows from (3.42), (3.53) and (3.67).

Next we shall show Theorem 1.3.

Proof of Theorem 1.3. Following [7], we define $\phi_{P,t}, P\in\mathbb{S}^n, t>0$ to be $\phi_{P,t}(\xi)=\tilde{\xi}:=\pi_P^{-1}(ty)$, where $y=\pi_P(\xi)$ is the stereographic project of $\mathbb{S}^n$ from P as the north pole to the equatorial plane. In particular, we denote $\phi_t=\phi_{P_0,t}$ where $P_0=(1,0,\cdots 0)$.

Given $u\in H^{\frac{n}{2}}(\mathbb{S}^n)$ and $t>0$, let

$$
v(\xi)=u(\phi_t(\xi))+\frac{n+1}{n^2}\ln|\det(d\phi_t)|,\quad \xi\in\mathbb{S}^n.
$$

We first claim that $\mathcal{J}_{\frac{n}{n+1}}$ owns the following invariance property:

$$
\mathcal{J}_{\frac{n}{n+1}}(u)=\mathcal{J}_{\frac{n}{n+1}}(v),\quad \forall u\in H^{\frac{n}{2}}(\mathbb{S}^n),\ t>0. \tag{3.71}
$$

The proof has been carried out in detail for the case $n = 4$ in [17, Pro. 3.4]. The same argument there works for general n with slight modifications; so we will skip the proof here. We further see that for any $u \in H^{\frac{n}{2}}(\mathbb{S}^n)$, there is a $\phi_{P,t}$ such that

$$v(\xi) = u(\phi_{P,t}(\xi)) + \frac{n+1}{n^2} \ln |\det(d\phi_{P,t})|, \quad \xi \in \mathbb{S}^n$$

belongs to $\mathfrak{L}$.

In conclusion, Theorem 1.3 follows immediately from Theorem 1.2 and (3.71). □

We note that a similar but more general Szegö type inequality for $u \in H^1(\mathbb{S}^2)$ is proven in [4] using a variational method under a mass center constraint, in combination with the improved Moser-Trudinger inequality in [16]. For the classical Szegö limit theorem, please see [15]. Interested reader is referred to [2] for a generalization of Szegö limit theorem on arbitrary Riemann surfaces.

4. Bifurcation

In this section we shall obtain results on bifurcation curves to (1.7) in general for $\alpha > 0$ and in particular for $\alpha \in (\frac{1}{n+1}, \frac{1}{2})$. We shall first apply the standard bifurcation theory to analyze the local bifurcation diagram. Let us recall the following general theorem.

Theorem 4.1. ([9, Theorem 1.7]) *Let X, Y be Hilbert spaces, V a neighborhood of 0 in X and $F : (-1, 1) \times V \to Y$ a map with the following properties:*

(1) $F(t, 0) = 0$ *for any t;*
(2) $\partial_t F$, $\partial_x F$ *and* $\partial^2_{t,x} F$ *exist and are continuous;*
(3) $\ker(\partial_x F(0, 0)) = span\{w_0\}$ *and* $Y/\mathcal{R}(\partial_x F(0, 0))$ *are one-dimensional;*
(4) $\partial^2_{t,x} F(0, 0)w_0 \notin \mathcal{R}(\partial_x F(0, 0))$.

If Z is any complement of $\ker(\partial_x F(0, 0))$ in X. Then there exists $\varepsilon_0 > 0$, a neighborhood of $(0, 0)$ in $U \subset (-1, 1) \times X$ and continuously differentiable maps $\eta : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ and $z : (-\varepsilon_0, \varepsilon_0) \to Z$ such that $\eta(0) = 0$, $z(0) = 0$ and

$$F^{-1}(0) \cap U \setminus ((-1, 1) \times \{0\}) = \{(\eta(\varepsilon), \varepsilon w_0 + \varepsilon z(\varepsilon)) \mid \varepsilon \in (-\varepsilon_0, \varepsilon_0)\}.$$

Recall that the shape of the above local bifurcating branch can be further described by the following theorem (see, e.g., [23, I.6]):

Theorem 4.2. *In the setting of Theorem 4.1, let $\psi \neq 0 \in Y^{-1}$ satisfy*

$$\mathcal{R}(\partial_x F(0, 0)) = \{y \in Y \mid \langle \psi, y \rangle = 0\},$$

where Y^{-1} is the dual space of Y. Then we have

$$\eta'(0) = -\frac{\langle \partial^2_{x,x} F(0,0)[w_0, w_0], \psi \rangle}{2\langle \partial^2_{t,x} F(0,0) w_0, \psi \rangle}.$$

Furthermore, the bifurcation is transcritical provided that $\eta'(0) \neq 0$.

Note that critical points of $I_\alpha(u)$ satisfy

$$(-1)^{\frac{n}{2}} \left[(1-x^2)^{\frac{n}{2}} u'\right]^{(n-1)} + \rho (1-x^2)^{\frac{n-2}{2}} \left(1 - \frac{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \frac{e^{nu}}{\int_{-1}^{1} (1-x^2)^{\frac{n-2}{2}} e^{nu}}\right) = 0,$$
$$x \in (-1, 1), \tag{4.1}$$

where $\rho = \frac{(n-1)!}{\alpha}$.

Let

$$\mathcal{V} = \left\{ u \in H^n(\mathbb{S}^n) : u = u(x), \int\limits_{\mathbb{S}^n} u dw = 0 \right\};$$
$$\mathcal{W} = \left\{ u \in L^2(\mathbb{S}^n) : u = u(x), \int\limits_{\mathbb{S}^n} u dw = 0 \right\}$$

and define a nonlinear operator $\mathcal{T} : \mathbb{R} \times \mathcal{V} \to \mathcal{W}$ as

$$\mathcal{T}(\rho, u) = P_n u + \rho \left(1 - \frac{e^{nu}}{\int_{\mathbb{S}^n} e^{nu} dw}\right).$$

Obviously, the operator $\mathcal{T}$ is well defined. After direct computations, one has

$$\partial_u \mathcal{T}(\rho, 0)\phi = P_n \phi - n\rho\phi.$$

Define

$$\mathcal{F}(\rho, u) = u + \rho P_n^{-1} \left(1 - \frac{e^{nu}}{\int_{\mathbb{S}^n} e^{nu} dw}\right).$$

Let $\mathcal{S}$ denote the closure of the set of nontrivial solutions of

$$\mathcal{F}(\rho, u) = 0. \tag{4.2}$$

It is clear that (4.2) and (4.1) are equivalent.

Let λ_k and $C_k^{\frac{n-1}{2}}$ be given in (2.5). Then by similar arguments as in [17, Lemma 5.3], we have

Theorem 4.3. *Let $\rho_k = \frac{\lambda_k}{n}$ for $k = 1, 2, 3, \ldots,$ the points $(\rho_k, 0)$ are bifurcation points for the curve of solutions $(\rho, 0)$. In particular, there exists $\varepsilon_0 > 0$ and continuously* *differentiable functions* $\rho_k : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ *and* $\psi_k : (-\varepsilon_0, \varepsilon_0) \to \{C_k^{\frac{n-1}{2}}\}^{\perp}$ *such that* $\rho_k(0) = \rho_k$, $\psi_k(0) = 0$ *and every nontrivial solution of* (4.1) *in a small neighborhood of* $(\rho_k, 0)$ *is of the form*

$$(\rho_k(\varepsilon), \varepsilon C_k^{\frac{n-1}{2}} + \varepsilon\psi_k(\varepsilon)).$$

In particular, when $k = 2$*, the bifurcation point* $(\rho_2, 0) = (\frac{(n+1)!}{n}, 0)$ *is a transcritical bifurcation point. Indeed, we have*

$$\rho_2'(0) = -\frac{(n+1)!}{2} \frac{\int_{-1}^{1}(1-x^2)^{\frac{n-2}{2}}(C_k^{\frac{n-1}{2}})^3}{\int_{-1}^{1}(1-x^2)^{\frac{n-2}{2}}(C_k^{\frac{n-1}{2}})^2} = -\frac{(n+1)!(n-1)^2}{n(n+5)} \neq 0.$$

Corollary 4.4. *Let* $\alpha_k = \frac{n!}{\lambda_k}$ *for* $k = 1, 2, 3, \ldots$*, the points* $(\alpha_k, 0)$ *are bifurcation points for the curve of solutions* $(\alpha, 0)$ *of* (1.7)*. Moreover, when* $k = 2$*, the bifurcation point* $(\frac{1}{n+1}, 0)$ *is a transcritical bifurcation point.*

Remark 4.1. When $k = 1$, the bifurcation leads to the family of solutions $u = -\ln(1 - ax), a \in (-1, 1)$ and $\rho = (n-1)!$. It is clear that $(\rho_k, 0)$ is not a transcritical bifurcation point for k odd since $C_k^{\frac{n-1}{2}}$ is an odd function and $\rho'(0) = 0$ in this case. It should be true that $(\rho_k, 0)$ is a transcritical bifurcation point for k even, we only need to check if $\int_{-1}^{1}(1-x^2)^{\frac{n-2}{2}}(C_k^{\frac{n-1}{2}})^3 \neq 0$ in this case, which can be confirmed for small k numerically. However, in this paper we only need to use the transcriticality of $(\rho_2, 0)$.

In order to analyze the global bifurcation diagram, we employ a global bifurcation theorem via degree arguments (see [23,28]) and also exploit special properties of solutions to (4.1).

First, we recall a global bifurcation result (see [23, Theorem II.5.8]).

Proposition 4.5. *In Theorem 4.3, the bifurcation at* $(\rho_k, 0)$ *is global and satisfies the Rabinowitz alternative, i.e., a global continuum of solutions to* (4.1) *either goes to infinity in* $R \times \mathcal{W}$ *or meets the trivial solution curve at* $(\rho_m, 0)$ *for some* $m \geq 1$ *and* $m \neq k$*.*

Next we state and prove the following more specific global bifurcation result regarding (4.1).

Theorem 4.6. *1) For* $k \geq 2$*, there exists a global continuum of solutions* $\mathcal{B}_k^+ \subset \mathcal{S} \setminus \{(\rho, 0), \rho \in \mathbb{R}\}$ *of* (4.1) *which coincides in a small neighborhood of* $(\rho_k, 0)$ *with*

$$\{(\rho_k(\varepsilon), \varepsilon C_k^{\frac{n-1}{2}} + \varepsilon\psi_k(\varepsilon)), \varepsilon < 0\}.$$

$\mathcal{B}_k^+$ *is contained in* $\mathcal{N}_2 := \{(\rho, u) : \rho > \frac{(n+1)!}{n},\ u \in L^2(-1, 1)\}$ *and is uniformly bounded in* $L^2(-1, 1)$ *for* ρ *in any fixed finite interval* $[\rho_m, \rho_M] \subset (\frac{(n+1)!}{n}, \infty)$*. Furthermore,* $\mathcal{B}_k^+$

satisfies the improved Rabinowitz alternative, i.e., either $\mathcal{B}_k^+$ *extends in* ρ *to infinity or meets the trivial solution curve at* $(\rho_m, 0)$ *for some* $m \geq 2$.

2) Similarly, for $k \geq 2$, *there exists a global continuum of solutions* $\mathcal{B}_k^-$ *which coincides in a small neighborhood of* $(\rho_k, 0)$ *with* $\{(\rho_k(\varepsilon), \varepsilon C_k^{\frac{n-1}{2}} + \varepsilon\psi_k(\varepsilon)), \varepsilon > 0\}$. *When* $k \geq 3$, $\mathcal{B}_k^-$ *is contained in* $\mathcal{N}_2$ *and satisfies the boundedness for* ρ *in any fixed finite interval* $[\rho_m, \rho_M] \subset (2(n-1)!, \infty)$. *Furthermore, the improved Rabinowitz alternative holds.*

3) Moreover, $\mathcal{B}_k^+ = \{u : u(x) = v(-x), v \in \mathcal{B}_k^-\}$ *when* k *is odd.*

4) The global continuum of solutions $\mathcal{B}_2^-$ *of* (4.1) *must be contained in the set*

$$\mathcal{N}_1 := \left\{(\rho, u) : \rho \in \left(\frac{(n-1)!}{\alpha^{(n)}}, \frac{(n+1)!}{n}\right) \supset (2(n-1)!, \frac{(n+1)!}{n}), \, u \in L^2(-1,1)\right\}.$$

Furthermore, $\mathcal{B}_2^-$ *is unbounded in* $L^\infty([-1,1])$, *and there exists a sequence of* $(\rho^{(t)}, u^{(t)}) \in \mathcal{B}_2^-, t = 1, 2, \cdots$ *such that* $\rho^{(t)} \to 2(n-1)!$ *and* $\|u^{(t)}\|_{L^\infty([-1,1])} \to \infty$. *As an immediate consequence, there is a nontrivial solution to* (4.1) *for any* $\rho \in (2(n-1)!, \frac{(n+1)!}{n})$.

Proof. The proof is similar to that of the case $n = 4$ in [17]. So we omit it. $\square$

Proof of Theorem 1.4. Theorem 1.4 follows immediately from Theorem 4.6. This leads to the existence of a nontrivial solution to (1.7) for $\alpha \in (\frac{1}{n+1}, \frac{1}{2})$. $\square$

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Appendix A. Proof of (3.25)

In this appendix, we compute the term $\int_{-1}^{1} (1-x^2)^2 G^2 [(1-x^2)^2 G]^{(5)}$.

First,

$$\int_{-1}^{1} (1-x^2)^2 G^2 [(1-x^2)^2 G]^{(5)} = \int_{-1}^{1} (1-x^2)^4 G^2 G^{(5)} - 20 \int_{-1}^{1} x(1-x^2)^3 G^2 G^{(4)}$$

$$-40 \int_{-1}^{1} (1-3x^2)(1-x^2)^2 G^2 G^{(3)} + 240 \int_{-1}^{1} x(1-x^2)^2 G^2 G'' + 120 \int_{-1}^{1} (1-x^2)^2 G^2 G'$$

$$:= \sum_{i=1}^{5} I_i. \tag{A.1}$$

Let

$$G_5=\int_{-1}^{1}(1-x^2)^4(G^3)^{(5)}=3\int_{-1}^{1}(1-x^2)^4G^2G^{(5)}+30\int_{-1}^{1}(1-x^2)^4GG'G^{(4)}$$

$$+60\int_{-1}^{1}(1-x^2)^4GG''G^{(3)}+60\int_{-1}^{1}(1-x^2)^4(G')^2G^{(3)}+90\int_{-1}^{1}(1-x^2)^4G'(G'')^2$$

$$:=\sum_{i=1}^{5}G_5i,$$

$$G_4=\int_{-1}^{1}x(1-x^2)^3(G^3)^{(4)}=3\int_{-1}^{1}x(1-x^2)^3G^2G^{(4)}+24\int_{-1}^{1}x(1-x^2)^3GG'G^{(3)}$$

$$+18\int_{-1}^{1}x(1-x^2)^3G(G'')^2+36\int_{-1}^{1}x(1-x^2)^3(G')^2G''$$

$$:=\sum_{i=1}^{4}G_4i,$$

$X_1=(1-7x^2)(1-x^2)^2$, $X_2=(1-3x^2)(1-x^2)^2$ and

$$G_3^{(j)}=\int_{-1}^{1}X_j(G^3)^{(3)}=3\int_{-1}^{1}X_jG^2G^{(3)}+18\int_{-1}^{1}X_jGG'G''+6\int_{-1}^{1}X_j(G')^3$$

$$:=\sum_{i=1}^{3}G_3^{(j)}i,\quad j=1,2.$$

Here, we neglect the coefficients before $G_5i,\ G_4i$ and G_3i. For example, $G_54=\int_{-1}^{1}(1-x^2)^4(G')^2G^{(3)}$.

After integration by parts,

$$\int_{-1}^{1}(1-x^2)^4G^2G^{(5)}=-\int_{-1}^{1}[(1-x^2)^4G^2]'G^{(4)}=8G_41-2G_52;$$

$$\int_{-1}^{1}(1-x^2)^4GG'G^{(4)}=8G_42-(G_53+G_54);$$

$$\int_{-1}^{1}(1-x^2)^4GG''G^{(3)}=4G_43-\frac{1}{2}G_55;$$

$$\int_{-1}^{1}(1-x^2)^4(G')^2G^{(3)} = 8G_44 - 2G_55. \tag{A.2}$$

Similarly,

$$\begin{aligned}
\int_{-1}^{1} x(1-x^2)^3G^2G^{(4)} &= -G_3^{(1)}1 - 2G_42; \\
\int_{-1}^{1} x(1-x^2)^3GG'G^{(3)} &= -G_3^{(1)}2 - G_43 - G_44 \\
\int_{-1}^{1} x(1-x^2)^3G(G'')^2 &= -G_3^{(1)}2 - G_42 - G_44 \\
\int_{-1}^{1} x(1-x^2)^3(G')^2G'' &= -\frac{1}{3}G_3^{(1)}3
\end{aligned} \tag{A.3}$$

Then

$$\begin{aligned}
&I_1 + I_2 + a_1G_5 + a_2G_4 \\
&= (1+3a_1)G_51 + 30a_1G_52 + 60a_1(G_53 + G_54) + 90a_1G_55 \\
&\quad + (3a_2 - 20)G_41 + 24a_2G_42 + 18a_2G_43 + 36a_2G_44 \\
&= (24a_1 - 2)G_52 + ... + (24a_1 + 3a_2 - 12)G_41 + ... \\
&= (24a_1 - 2)[8G_42 - (G_53 + G_54)] + 60a_1(G_53 + G_54) + 90a_1G_55 \\
&\quad + (24a_1 + 3a_2 - 12)G_41 + ... \\
&= (36a_1 + 2)(G_53 + G_54) + 90a_1G_55 + (24a_1 + 3a_2 - 12)G_41 \\
&\quad + (24a_2 + 192a_1 - 16)G_42 + ... \\
&= -5G_55 + (24a_1 + 3a_2 - 12)G_41 + (24a_2 + 192a_1 - 16)G_42 \\
&\quad + (144a_1 + 18a_2 + 8)(G_43 + 2G_44).
\end{aligned}$$

Thus,

$$\begin{aligned}
&I_1 + I_2 + a_1G_5 + a_2G_4 + 5G_55 \\
&= (24a_1 + 3a_2 - 12)G_41 + (24a_2 + 192a_1 - 16)G_42 + (144a_1 + 18a_2 + 8)(G_43 + 2G_44) \\
&= -(24a_1 + 3a_2 - 12)G_3^{(1)}1 + (144a_1 + 18a_2 + 8)(G_42 + G_43 + 2G_44) \\
&= -(24a_1 + 3a_2 - 12)G_3^{(1)}1 - (144a_1 + 18a_2 + 8)G_3^{(1)}2 + (144a_1 + 18a_2 + 8)G_44.
\end{aligned}$$

Let $t = 24a_1 + 3a_2$, then we have

$$\begin{aligned}
&I_1 + I_2 + a_1 G_5 + a_2 G_4 + 5G_5 5 + \frac{t-12}{3} G_3^{(1)} \\
&= 6(t-12)G_3^{(1)} 2 - (6t+8)G_3^{(1)} 2 + 2(t-12)G_3^{(1)} 3 - \frac{6t+8}{3} G_3^{(1)} 3 \\
&= -80 G_3^{(1)} 2 - \frac{80}{3} G_3^{(1)} 3.
\end{aligned}$$

Notice that

$$\begin{aligned}
&\int_{-1}^{1} X_j G^2 G^{(3)} = -\int_{-1}^{1} X_j' G^2 G'' - 2\int_{-1}^{1} X_j G G' G'' = -\int_{-1}^{1} X_j' G^2 G'' - 2G_3^{(j)} 2 \\
&\int_{-1}^{1} X_j G G' G'' = -\frac{1}{2}\int_{-1}^{1} X_j G (G')^2 - \frac{1}{2} G_3^{(j)} 3.
\end{aligned} \tag{A.4}$$

Then

$$I_3 - 80 G_3^{(1)} 2 - \frac{80}{3} G_3^{(1)} 3 = 40\int_{-1}^{1} X_2' G^2 G'' + 80\int_{-1}^{1} [X_2 - X_1] G G' G'' - \frac{80}{3} G_3^{(1)} 3$$

$$= 40\int_{-1}^{1} X_2' G^2 G'' + 320\int_{-1}^{1} x^2(1-x^2)^2 G G' G'' - \frac{80}{3} G_3^{(1)} 3.$$

Thus,

$$I_3 + I_4 - 80 G_3^{(1)} 2 - \frac{80}{3} G_3^{(1)} 3 = I_4 + \int_{-1}^{1} 40 X_2' G^2 G'' + 320\int_{-1}^{1} x^2(1-x^2)^2 G G' G''$$

$$-\frac{80}{3} G_3^{(1)} 3$$

$$= 80\int_{-1}^{1} x(1-x^2)^2[3(1-x^2) - (9x^2-5)] G^2 G'' - 320\int_{-1}^{1} x(1-x^2)(1-3x^2) G (G')^2$$

$$-\int_{-1}^{1} (1-x^2)^2 \left[160x^2 - \frac{80}{3}(1-7x^2)\right] (G')^3$$

$$= 160\int_{-1}^{1} x(1-x^2)(3x^2-1)(G^2 G'' + 2G(G')^2) - \frac{80}{3}\int_{-1}^{1} (1-x^2)^3 (G')^3$$

$$= \frac{160}{3}\int_{-1}^{1} x(1-x^2)(3x^2-1)(G^3)'' - \frac{80}{3}\int_{-1}^{1}(1-x^2)^3(G')^3.$$

It is easy to see that

$$\begin{aligned}\sum_{i=1}^{5} I_i &= -a_1\int_{-1}^{1}(1-x^2)^4(G^3)^{(5)} - a_2\int_{-1}^{1} x(1-x^2)^3(G^3)^{(4)}\\
&\quad - \frac{t-12}{3}\int_{-1}^{1}(1-7x^2)(1-x^2)^2(G^3)^{(3)}\\
&\quad - \frac{160}{3}\int_{-1}^{1} x(1-x^2)(3x^2-1)(G^3)'' + 40\int_{-1}^{1}(1-x^2)^2(G^3)'\\
&\quad - 5\int_{-1}^{1}(1-x^2)^4G'(G'')^2 - \frac{80}{3}\int_{-1}^{1}(1-x^2)^3(G')^3\\
&= -5\int_{-1}^{1}(1-x^2)^4G'(G'')^2 - \frac{80}{3}\int_{-1}^{1}(1-x^2)^3(G')^3.\end{aligned}$$

Appendix B. Proof of (3.26)

Compute $\int_{-1}^{1}(1-x^2)^3G^2[(1-x^2)^3G]^{(7)}$. First,

$$\begin{aligned}&\int_{-1}^{1}(1-x^2)^3G^2[(1-x^2)^3G]^{(7)} = \int_{-1}^{1}(1-x^2)^6G^2G^{(7)} - 42\int_{-1}^{1} x(1-x^2)^5G^2G^{(6)}\\
&-126\int_{-1}^{1}(1-x^2)^4\left(1-5x^2\right)G^2G^{(5)} + 840\int_{-1}^{1} x(1-x^2)^3\left(3-5x^2\right)G^2G^{(4)}\\
&+2520\int_{-1}^{1}(1-x^2)^3\left(1-5x^2\right)G^2G^{(3)} - 3\times 7!\int_{-1}^{1} x(1-x^2)^3G^2G'' - 7!\int_{-1}^{1}(1-x^2)^3G^2G'\\
&:= \sum_{i=1}^{7} I_i \qquad \text{(B.1)}\end{aligned}$$

Let

$$
\begin{aligned}
G_7 = \int_{-1}^{1} (1-x^2)^6 (G^3)^{(7)} &= 3\int_{-1}^{1} (1-x^2)^6 G^2 G^{(7)} + 42\int_{-1}^{1} (1-x^2)^6 GG'G^{(6)} \\
&+ 126\int_{-1}^{1} (1-x^2)^6 GG''G^{(5)} + 126\int_{-1}^{1} (1-x^2)^6 (G')^2 G^{(5)} + 210\int_{-1}^{1} (1-x^2)^6 GG^{(3)}G^{(4)} \\
&+ 630\int_{-1}^{1} (1-x^2)^6 G'G''G^{(4)} + 630\int_{-1}^{1} (1-x^2)^6 (G'')^2 G^{(3)} + 420\int_{-1}^{1} (1-x^2)^6 G'(G^{(3)})^2 \\
&:= \sum_{i=1}^{8} G_7 i;
\end{aligned}
$$

and

$$
\begin{aligned}
G_6 = \int_{-1}^{1} x(1-x^2)^5 (G^3)^{(4)} &= 3\int_{-1}^{1} x(1-x^2)^5 G^2 G^{(6)} + 36\int_{-1}^{1} x(1-x^2)^5 GG'G^{(5)} \\
&+ 90\int_{-1}^{1} x(1-x^2)^5 GG''G^{(4)} + 90\int_{-1}^{1} x(1-x^2)^5 (G')^2 G^{(4)} + 60\int_{-1}^{1} x(1-x^2)^5 G(G^{(3)})^2 \\
&+ 360\int_{-1}^{1} x(1-x^2)^5 G'G''G^{(3)} + 90\int_{-1}^{1} x(1-x^2)^5 (G'')^3 \\
&:= \sum_{i=1}^{7} G_6 i.
\end{aligned}
$$

We consider these functions

$$
\begin{aligned}
&X_5^{(1)} = [x(1-x^2)^5]' = (1-x^2)^4(1-11x^2), \quad X_5^{(2)} = (1-x^2)^4(1-5x^2); \\
&X_4^{(s)} = \left[X_5^{(j)}\right]', \ s=1,2; \qquad X_4^{(3)} = x(1-x^2)^3(3-5x^2); \\
&X_3^{(s)} = \left[X_4^{(j)}\right]', \ s=1,2,3; \qquad X_3^{(4)} = (1-x^2)^3(1-5x^2).
\end{aligned} \tag{B.2}
$$

Then define

$$
G_5^{(j)} = \int_{-1}^{1} X_5^{(j)} (G^3)^{(5)} = 3\int_{-1}^{1} X_5^{(j)} G^2 G^{(5)} + 30\int_{-1}^{1} X_5^{(j)} GG'G^{(4)}
$$

$$
\begin{aligned}
&+60\int_{-1}^{1} X_5^{(j)} G G'' G^{(3)} + 60\int_{-1}^{1} X_5^{(j)} (G')^2 G^{(3)} + 90\int_{-1}^{1} X_5^{(j)} G' (G'')^2 \\
:= &\sum_{i=1}^{5} G_5^{(j)} i, \quad j = 1, 2; \\
G_4^{(j)} = &\int_{-1}^{1} X_4^{(j)} (G^3)^{(4)} = 3\int_{-1}^{1} X_4^{(j)} G^2 G^{(4)} + 24\int_{-1}^{1} X_4^{(j)} G G' G^{(3)} \\
&+ 18\int_{-1}^{1} X_4^{(j)} G (G'')^2 + 36\int_{-1}^{1} X_4^{(j)} (G')^2 G'' \\
&:= \sum_{i=1}^{4} G_4 i, \quad j = 1, 2, 3;
\end{aligned}
$$

and

$$
\begin{aligned}
G_3^{(j)} &= \int_{-1}^{1} X_3^{(j)} (G^3)^{(3)} = 3\int_{-1}^{1} X_3^{(j)} G^2 G^{(3)} + 18\int_{-1}^{1} X_3^{(j)} G G' G'' + 6\int_{-1}^{1} X_3^{(j)} (G')^3 \\
&:= \sum_{i=1}^{3} G_3^{(j)} i, \quad j = 1, 2, 3, 4.
\end{aligned}
$$

As in Appendix A, we neglect the coefficients before $G_7 i, G_6 i, \ldots G_3^{(j)} i$.

After integration by parts, one has

$$
\begin{aligned}
&G_7 1 = \int_{-1}^{1} (1-x^2)^6 G^2 G^{(7)} = -\int_{-1}^{1} [(1-x^2)^6 G^2]' G^{(4)} = 12 G_6 1 - 2 G_7 2; \\
&G_7 2 = 12 G_6 2 - (G_7 3 + G_7 4); \quad G_7 3 = 12 G_6 3 - (G_7 5 + G_7 6); \\
&G_7 4 = 12 G_6 4 - 2 G_7 6; \quad G_7 5 = 6 G_6 5 - \frac{1}{2} G_7 8; \\
&G_7 6 = 12 G_6 6 - (G_7 7 + G_7 8); \quad G_7 7 = 4 G_6 7;
\end{aligned}
\tag{B.3}
$$

Similarly,

$$
\begin{aligned}
&- G_6 1 = G_5^{(1)} 1 + 2 G_6 2; \qquad -G_6 2 = G_5^{(1)} 2 + (G_6 3 + G_6 4); \\
&- G_6 3 = G_5^{(1)} 3 + (G_6 5 + G_6 6); \qquad -G_6 4 = -G_5^{(1)} 4 + 2 G_6 6; \\
&- G_6 5 = G_5^{(1)} 3 + (G_6 3 + G_6 6); \qquad -G_6 6 = \frac{1}{2} G_5^{(1)} 5 + \frac{1}{2} G_6 7;
\end{aligned}
\tag{B.4}
$$

$$
\begin{aligned}
&-G_5^{(j)}1 = G_4^{(j)}1 + 2G_5^{(j)}2; \qquad -G_5^{(j)}2 = G_4^{(j)}2 + (G_5^{(j)}3 + G_5^{(j)}4);\\
&-G_5^{(j)}3 = \frac{1}{2}G_4^{(j)}3 + \frac{1}{2}G_5^{(j)}5; \qquad -G_5^{(j)}4 = G_4^{(j)}4 + 2G_5^{(j)}5;
\end{aligned}
\tag{B.5}
$$

$$
-G_4^{(j)}1 = G_3^{(j)}1 + 2G_4^{(j)}2; \quad -G_4^{(j)}2 - G_4^{(j)}3 = G_3^{(j)}2 + G_4^{(j)}4; \quad -G_4^{(j)}4 = \frac{1}{3}G_3^{(j)}3.
\tag{B.6}
$$

By these relations, we calculate

$$
\begin{aligned}
& I_1 + I_2 + a_1G_7 + a_2G_6\\
&= (1 + 3a_1)G_71 + 42a_1G_72 + 126a_1(G_73 + G_74) + 210a_1G_75 + 630a_1(G_76 + G_77)\\
&+ 420a_1G_78\\
&+ (3a_2 - 42)G_61 + 36a_2G_62 + 90a_2(G_63 + G_64) + 60a_2G_65 + 360a_2G_66 + 90a_2G_67\\
&= (36a_1 - 2)G_72 + ... + (36a_1 + 3a_2 - 30)G_61 + ...\\
&= (90a_1 + 2)(G_73 + G_74) + ...(360a_1 + 30a_2 + 36)G_62 + ... - (36a_1 + 3a_2 - 30)G_5^{(1)}1\\
&= (120a_1 - 2)(G_75 + 3G_76) + 630a_1G_77 + 420a_1G_78\\
&+ (720a_1 + 60a_2 - 12)(G_63 + G_64)\\
&+ 60a_2G_65 + 360a_2G_66 + 90a_2G_67 - (36a_1 + 3a_2 - 30)G_5^{(1)}1\\
&- (360a_1 + 30a_2 + 36)G_5^{(1)}2\\
&= (270a_1 + 6)G_77 + 7G_78 + 3(720a_1 + 60a_2 - 12)G_66 + 90a_2G_67\\
&- (36a_1 + 3a_2 - 30)G_5^{(1)}1\\
&- (360a_1 + 30a_2 + 36)G_5^{(1)}2 - (720a_1 + 60a_2 - 12)(G_5^{(1)}3 + G_5^{(1)}4)\\
&= 7G_78 + 42G_67 - (36a_1 + 3a_2 - 30)G_5^{(1)}1 - ... - (1080a_1 + 90a_2 - 18)G_5^{(1)}5.
\end{aligned}
$$

Let

$$
II = I_1 + I_2 + a_1G_7 + a_2G_6 - 7G_78 - 42G_67 \text{ and } 12a_1 + a_2 = t.
$$

Then it follows from (B.5) that

$$
\begin{aligned}
II &= -(3t - 30)G_5^{(1)}1 - (30t + 36)G_5^{(1)}2 - (60t - 12)(G_5^{(1)}3 + G_5^{(1)}4) - (90t - 18)G_5^{(1)}5\\
&= (3t - 30)G_4^{(1)}1 - (24t + 96)G_5^{(1)}2 - ...\\
&= (3t - 30)G_4^{(1)}1 + (24t + 96)G_4^{(1)}2 - (36t - 108)(G_5^{(1)}3 + G_5^{(1)}4) - (90t - 18)G_5^{(1)}5\\
&= (3t - 30)G_4^{(1)}1 + (24t + 96)G_4^{(1)}2 + 18(t - 3)(G_4^{(1)}3 + G_4^{(1)}4) - 252G_5^{(1)}5.
\end{aligned}
$$

Joint with (B.6), we further have

$$
\begin{aligned}
II + 252G_5^{(1)}5 &= (3t-30)G_4^{(1)}1 + (24t+96)G_4^{(1)}2 + 18(t-3)(G_4^{(1)}3 + G_4^{(1)}4) \\
&= -3(t-10)G_3^{(1)}1 + 6(3t+26)G_4^{(1)}2 + 18(t-3)(G_4^{(1)}3 + G_4^{(1)}4) \\
&= -3(t-10)G_3^{(1)}1 - 6(3t+26)G_3^{(1)}2 - 210G_4^{(1)}3 + 6(3t-44)G_4^{(1)}4 \\
&= -3(t-10)G_3^{(1)}1 - 6(3t+26)G_3^{(1)}2 - 2(3t-44)G_3^{(1)}3 - 210G_4^{(1)}3.
\end{aligned}
$$

Repeating the above arguments, we find

$$
\begin{aligned}
-\frac{1}{126}I_3 &= G_5^{(2)}1 = G_3^{(2)}1 - 4G_3^{(2)}2 + 2G_3^{(2)}3 - 5G_4^{(2)}3 - 5G_5^{(2)}5; \\
\frac{1}{840}I_4 &= G_4^{(3)}1 = -G_3^{(3)}1 + 2G_3^{(3)}2 - \frac{2}{3}G_3^{(3)}3 + 2G_4^{(3)}3.
\end{aligned}
$$

Then we compute

$$
\begin{aligned}
& II + I_3 + I_4 + I_5 + tG_3^{(1)} \\
&= II - 126G_5^{(2)}1 + 840G_4^{(3)}1 + 2520G_3^{(4)}1 + tG_3^{(1)} \\
&= -252G_5^{(1)}5 + 630G_5^{(2)}5 - 210G_4^{(1)}3 + 630G_4^{(2)}3 + 1680G_4^{(3)}3 \\
&+ \left(30G_3^{(1)}1 - 126G_3^{(2)}1 - 840G_3^{(3)}1 + 2520G_3^{(4)}1\right) \\
&+ \left(1680G_3^{(3)}2 - 156G_3^{(1)}2 + 504G_3^{(2)}2\right) \\
&+ \left(88G_3^{(1)}3 - 252G_3^{(2)}3 - 1260G_3^{(3)}3\right) \\
&:= III_1 + III_2 + \sum_{i=1}^{3} III_3^{(i)}.
\end{aligned}
$$

By (B.2), we derive

$$
\begin{aligned}
III_1 &= -252G_5^{(1)}5 + 630G_5^{(2)}5 \\
&= \int_{-1}^{1} \left[630\left(1-5x^2\right)\left(1-x^2\right)^4 - 252\left(1-11x^2\right)\left(1-x^2\right)^4\right] G'(G'')^2 \\
&= 378\int_{-1}^{1} \left(1-x^2\right)^5 G'(G'')^2;
\end{aligned}
$$

$$
III_2 = -210G_4^{(1)}3 + 630G_4^{(2)}3 + 1680G_4^{(3)}3 = 0;
$$

and

$$
\begin{aligned}
III_3^{(1)} &= 30G_3^{(1)}1 - 126G_3^{(2)}1 - 840G_3^{(3)}1 + 2520G_3^{(4)}1 \\
&= 72\int_{-1}^{1}(25x^4 - 48x^2 + 19)(1-x^2)^2G^2G^{(3)} \\
&:= 72\int_{-1}^{1} X_3^{(5)}G^2G^{(3)};
\end{aligned}
$$

$$
\begin{aligned}
III_3^{(2)} &= 1680G_3^{(3)}2 - 156G_3^{(1)}2 + 504G_3^{(2)}2 = 216\int_{-1}^{1}(15x^4 - 26x^2 + 3)(1-x^2)^2GG'G'' \\
&:= 216\int_{-1}^{1} X_3^{(6)}(1-x^2)^2GG'G'';
\end{aligned}
$$

$$
\begin{aligned}
III_3^{(3)} &= 88G_3^{(1)}3 - 252G_3^{(2)}3 - 1260G_3^{(3)}3 = 72\int_{-1}^{1}(15x^4 - 26x^2 + 3)(1-x^2)^2(G')^3 \\
&:= 72\int_{-1}^{1} X_3^{(6)}(1-x^2)^2(G')^3.
\end{aligned}
$$

It follows from (A.4) that

$$
\begin{aligned}
III_3 := \sum_{i=1}^{3} III_3^{(i)} &= -72\int_{-1}^{1}\left(X_3^{(5)}\right)' G^2G'' + \int_{-1}^{1}\left(216X_3^{(5)} - 144X_3^{(6)}\right)GG'G'' \\
&\quad + 72\int_{-1}^{1} X_3^{(6)}(1-x^2)^2(G')^3 \\
&= -72\int_{-1}^{1}\left(X_3^{(5)}\right)' G^2G'' - 288\int_{-1}^{1} X_2G(G')^2 + 1260\int_{-1}^{1}(1-x^2)^4(G')^3,
\end{aligned}
$$

where $X_2 = x(1-x^2)(155x^4 - 16x^2 + 19)$. Then we consider

$$
\begin{aligned}
III_3 + I_6 + I_7 = -144\int_{-1}^{1} X_2\left[G^2G'' + 2G(G')^2\right] - 5040\int_{-1}^{1}(1-x^2)^3G^2G' \\
+ 1260\int_{-1}^{1}(1-x^2)^4(G')^3
\end{aligned}
$$

$$= -48\int_{-1}^{1} X_2(G^3)' - 1680\int_{-1}^{1}(1-x^2)^3(G^3)' + 1260\int_{-1}^{1}(1-x^2)^4(G')^3.$$

Put these results together, we conclude that

$$\begin{aligned}\sum_{i=1}^{7} I_i &= -a_1\int_{-1}^{1}(1-x^2)^6(G^3)^{(7)} - a_2\int_{-1}^{1} x(1-x^2)^5(G^3)^{(6)} - t\int_{-1}^{1} X_3^{(1)}(G^3)^{(3)}\\ &- 48\int_{-1}^{1} X_2(G^3)'' - 1680\int_{-1}^{1}(1-x^2)^3(G^3)' + 7\int_{-1}^{1}(1-x^2)^6 G'(G^{(3)})^2\\ &+ 42\int_{-1}^{1} x(1-x^2)^5(G'')^3\\ &+ 378\int_{-1}^{1}\left(1-x^2\right)^5 G'(G'')^2 + 1260\int_{-1}^{1}(1-x^2)^4(G')^3\\ &= 7\int_{-1}^{1}(1-x^2)^6 G'(G^{(3)})^2 + 42\int_{-1}^{1} x(1-x^2)^5(G'')^3 + 378\int_{-1}^{1}\left(1-x^2\right)^5 G'(G'')^2\\ &+ 1260\int_{-1}^{1}(1-x^2)^4(G')^3.\end{aligned}$$

Note that

$$\begin{aligned}\int_{-1}^{1}[(1-x^2)G]^{(3)}(1-x^2)^5(G'')^2 &= \int_{-1}^{1}(1-x^2)^6(G'')^2G^{(3)}\\ - 6\int_{-1}^{1} x(1-x^2)^5(G'')^3 &- 6\int_{-1}^{1}(1-x^2)^5G'(G'')^2\\ = -2\int_{-1}^{1} x(1-x^2)^5(G'')^3 &- 6\int_{-1}^{1}(1-x^2)^5G'(G'')^2\end{aligned}$$

Therefore,

$$\int_{-1}^{1}(1-x^2)^3G^2[(1-x^2)^3G]^{(7)} = 1260\int_{-1}^{1}(1-x^2)^4(G')^3 + 252\int_{-1}^{1}(1-x^2)^5G'(G'')^2$$

$$+7\int_{-1}^{1}(1-x^2)^6G'(G^{(3)})^2 - 21\int_{-1}^{1}[(1-x^2)G]^{(3)}(1-x^2)^5(G'')^2.$$

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