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Calculus of Variations

Symmetry of positive solutions of elliptic equations with mixed boundary conditions in a super-spherical sector

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Abstract
In this paper we establish some symmetry results for positive solutions of semilinear elliptic equations with mixed boundary conditions. In particular, we show that the positive solution in a super-spherical sector must be symmetric. The monotonicity property is also proved. Our proof is based on the well-known moving plane methods.

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1 Introduction

In this paper we investigate qualitative properties of the classical solutions of the equation

$$\Delta u + f(u) = 0 \text{ in } \Omega \tag{1.1}$$

with mixed boundary conditions.

Symmetry properties of partial differential equations are interesting since it is natural to ask whether or not solutions inherit the same symmetry from the differential operator and from the domain and boundary conditions. There is a large literature on this topic. Alexandrov [1] introduced the reflection principle and showed that a closed embedded hypersurface in the Euclidean space must be a sphere. The reflection principle was also used by Serrin [37] for a symmetry result of overdetermined problems and by Gidas, Ni and Nirenberg [17,18]

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to derive radial symmetry results for positive solutions of semilinear elliptic equations. The reflection principle is usually called the method of moving planes after these seminal papers. This method was revisited in the influential paper [4] of Berestycki and Nirenberg. In [4], they generalized the methods of moving planes and introduced the sliding method to prove the monotonicity by a version of the maximum principle in domains of small volume which allows one to handle symmetry results for rough domains. Later, many other authors have devoted attention to these questions, without being exhaustive we mention the papers by Li [27], Li and Ni [28], Chen and Li [10,11], Caffarelli, Gidas and Spruck [7], and Berestycki, Caffarelli and Nirenberg [2,3] and the references therein. In [12,21,30,34,35], the axial symmetry of solutions with lower Morse index is also studied.

The zero Dirichlet condition is a very forceful condition and implies the monotonicity of positive solutions near the boundary. From the results of Gidas, Ni and Nirenberg [17], the zero Dirichlet condition will "force" all positive solutions of (1.1) to posses the same radial symmetry when Ω is a ball, while the zero Neumann condition allows many other possibilities. In fact, for problem (1.1) with boundary condition of Neumann type, the least energy solutions are often not radially symmetric and many researchers have focused on the least energy solution, single peak solution and multi-peak solution; See [12,15,20,22–26,29,31–33,39,40]. Naturally, it is interesting to see how different domains and boundary value conditions may influence the symmetry properties of positive solutions.

When the domain is a spherical sector, Berestycki and Pacella [6] proved the radial symmetry properties of positive solutions of (1.1) with mixed boundary conditions, provided the amplitude of spherical sector is less or equal to π. Zhu [42] proved a similar result for singular solutions when the amplitude may be greater than π and f satisfies some supercritical growth conditions. The first two authors [9,41] proved the symmetry and monotonicity properties of positive elliptic solutions with mixed boundary conditions in a standard spherical cone and in a super-spherical cone. Researchers are also concerned about the symmetry results for mixed boundary problems, see [8,13,14,38].

In this paper, we will use a variety of the moving plane method and bootstrap method to prove some symmetry and monotonicity results for positive solutions of a semilinear elliptic equation under mixed boundary conditions in a super-spherical sector.

First let us introduce some terminology. For $\beta \in (0, \pi]$ and $a \in (-\infty, 1)$, we let $\Sigma = \Sigma_{\beta,a}$ be the intersection of the open cone $\mathcal{C}$ and the unit ball $B = \{x \in \mathbb{R}^n : |x| < 1\}$ where $\mathcal{C}$ is an open sector such that

$$\mathcal{C} = \{(x_1, x_2, x') \in \mathbb{R}^n : x_1 - a > |x_2| \cot \frac{\beta}{2}\}.$$

We call $\Sigma_{\beta,a}$ a super-spherical sector if $a \in (0, 1)$, $\Sigma_{\beta,a}$ a (standard) spherical sector if $a = 0$, $\Sigma_{\beta,a}$ a sub-spherical sector if $a < 0$.

We consider the following equation with mixed boundary conditions

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Sigma, \\ u > 0 & \text{in } \Sigma, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_N, \\ u = 0 & \text{on } \Gamma_D \end{cases} \tag{1.2}$$

where ν is the unit outer normal to Σ, $\Gamma_D = \partial\Sigma \cap \partial B$ is the sphere boundary and $\Gamma_N = \partial\Sigma \backslash \partial B$ is the sector boundary.

Through this paper, we always assume that f is a local Lipschitz continuous function on $[0, \infty)$, $a \in (0, 1)$ and the solution u is classical, say $u \in C^2(\Sigma) \cap C^1(\Sigma \cup \Gamma_N) \cap C(\overline{\Sigma})$.

The main result is as follows:

Theorem 1 *Assume that f is local Lipschitz in $[0, \infty)$, $0 < \beta \le \pi$ and $a \in (0, 1)$. Let u be a classical solution of (1.2). Then we have the following properties:*

(i) *u is symmetric with respect to the hyperplane $\{x_2 = 0\}$;*

(ii) *u is monotone in x_2, that is, $x_2 u_{x_2} < 0$ for $x \in \Sigma$ satisfying $x_2 \neq 0$;*

(iii) *u is monotone in x_1, that is, $u_{x_1} < 0$ in Σ;*

(iv) *u is radially symmetric with respect to $x' \in \mathbb{R}^{n-2}$.*

Since the moving plane method strongly relies on the equation and geometric structure of the domain and boundary conditions, so the condition $0 < \beta \le \pi$ in Theorem 1 is needed. We expect it to be optimal due to a similar phenomenon in [6] and the condition is geometrical in terms of related isoperimetric inequalities.

Comparing with the result in Berestycki and Pacella [6], our domain is the super-spherical sector, which is different than the spherical symmetry in [6]. The boundary conditions force the expected symmetry to be evenly symmetric (instead of spherical symmetry) since the center of ball does not lie in the vertices set of the sector $\mathcal{C}$ nor is the domain spherical symmetric.

The proof of this theorem is based on the methods of moving planes [17] and the ideas developed in [6], and there are some new difficulties due to the Neumann boundary. To overcome the difficulties, the main idea is to compare u with its reflection $u^{\lambda,\vartheta}$. First, the maximum principle for mixed boundary problems in narrow domains will be used to show the negativity of $w^{\lambda,\vartheta} = u - u^{\lambda,\vartheta}$. Second, the parallel hyperplanes will move along both the lower and upper Neumann boundary. Third, one can obtain a priori information of the Neumann or Dirichlet boundary conditions for $w^{\lambda,\vartheta}$ on the boundary caused by Γ_N. In another word, the proof depends deeply on the understanding and obtaining a priori information about the signs of some directional derivatives of u.

Since the solution u is assumed to be classical, the standard elliptic theory implies that u is also C^2 up to the smooth boundary point. The solution may not be C^2 at non-smooth boundary point including the mixed boundary $\partial\mathcal{C} \cap \partial B$ and the vertex set $\partial\mathcal{C} \cap \{x_1 = a, x_2 = 0\}$ of the sector $\mathcal{C}$.

We point out that Theorem 1 is valid for $\beta = \pi$ and $\beta = \pi/2$. In fact, using the mirror reflection along the Neumann boundary, one can obtain a positive Dirichlet solution in the even extension of the domain, and then the results of Theorem 1 follows from [4,17].

The structure of this paper is as follows. In Sect. 2, we introduce a version of the maximum principle for the mixed boundary conditions in a narrow domain. Section 3 illustrates the symmetry for some special case that β is a right angle or a straight angle. The proof of (i)-(iii) in Theorem 1 will be divided into different range of β, this is given through Sects. 4–7. In Sect. 8, we prove the radial symmetry of u in $x' \in \mathbb{R}^{n-2}$.

2 Maximum principle for mixed boundary conditions

In this section, we consider the maximum principle of the following linear equation

$$\begin{cases} \mathcal{L}[u] = \Delta u + c(x)u = f & \text{in } \Omega, \\ \mathcal{B}[u] = u = g & \text{on } \Gamma_0, \\ \mathcal{B}[u] = \nabla u \cdot \gamma + \beta u = h & \text{on } \Gamma_1. \end{cases} \tag{2.1}$$

We usually assume that $\Omega \subset \mathbb{R}^n$, $n \geq 2$ is an open bounded subset, Γ_0 and Γ_1 are two disjoint subsets of $\partial\Omega$ such that Γ_0 is relatively closed, Γ_1 is open C^1 manifold and that the closure of $\Gamma_0 \cup \Gamma_1$ is $\partial\Omega$. Here γ is a vector valued function on Γ_1 such that $|\gamma| = 1$ and $\gamma \cdot \nu > 0$ on Γ_1 where ν is the outer unit normal vector to $\partial\Omega$. We always assume that β is a nonnegative function on Γ_1 and c is a bounded function in Ω,

$$|c(x)| < c_0 \text{ for } x \in \Omega \tag{2.2}$$

where c_0 is a positive constant.

It is well-known [19,36] that under the condition $c \leq 0$ in Ω we have

$$\text{if } f \leq 0 \text{ in } \Omega,\ g \geq 0 \text{ on } \Gamma_0,\ h \geq 0 \text{ on } \Gamma_1, \text{ then } u \geq 0 \text{ in } \Omega.$$

This property is called the maximum principle for $(\mathcal{L}, \mathcal{B})$ in Ω. When Γ_1 is an empty set, this is the usual maximum principle, and there are some well-known sufficient conditions (see [4,5,19,36]):

(1) $c \leq 0$ in Ω;
(2) There exists a positive continuous function g over $\overline{\Omega}$ such that $g \in W^{2,\infty}_{loc}(\Omega)$ and $\mathcal{L}[g] \leq 0$ in Ω;
(3) Ω lies in a narrow band, $\Omega \subset \{0 < (x - x_0) \cdot e < \eta\}$ for some $x_0 \in \mathbb{R}^n$, $|e| = 1$ where $\eta > 0$ is some constant depending only on c_0;
(4) The measure satisfies $|\Omega| < \delta$, provided $\operatorname{diam}(\Omega) < d$ and δ depends only on c_0 and d;
(5) $\lambda_1(\mathcal{L}, \Omega) > 0$, this is a sufficient and necessary condition. Here the first principal eigenvalue $\lambda_1(\mathcal{L}, \Omega)$ is defined by

$$\lambda_1(\mathcal{L}, \Omega) = \sup\{\lambda \in \mathbb{R} : \exists \phi > 0 \text{ in } \Omega \text{ satisfying } (\mathcal{L} + \lambda)\phi \leq 0 \text{ in } \Omega\} \tag{2.3}$$

where $\phi \in W^{2,n}_{loc}(\Omega) \cap C(\overline{\Omega})$.

The third and fourth sufficient conditions above are very important and play a key role in the process of the method of moving planes.

For the mixed boundary cases, we have a modified sufficient condition as in (2).

Lemma 1 *([41]) Assume that there exists a function $g \in C^2(\Omega) \cap C(\overline{\Omega}) \cap C^1(\Omega \cup \Gamma_1)$ such that*

$$\begin{cases} \mathcal{L}[g] = \Delta g + c(x) g \leq 0 & \text{in } \Omega, \\ g > 0 & \text{on } \overline{\Omega}, \\ \nabla g \cdot \gamma + \beta g \geq 0 & \text{on } \Gamma_1. \end{cases}$$

Then the maximum principle holds for $(\mathcal{L}, \mathcal{B})$ in Ω.

Using this lemma, we can prove the maximum principle holds in some sectorial domain.

Lemma 2 *Let (2.2) hold. Assume that $(\mathcal{L}, \mathcal{B}, \Omega)$ satisfy*

(1) Ω is contained in a sector $\mathcal{C} = \{x \in \mathbb{R}^n : x_1 > \beta x_2, x_2 > 0\}$ for some $\beta \in \mathbb{R}$;
(2) $\Gamma_1 \subset \partial\mathcal{C}$ and $\mathcal{B} = \nu \cdot \nabla$ is the outward norm derivative operator on Γ_1;
(3) $\Omega \subset \{x \in \mathbb{R}^n : x_1^2 + x_2^2 < \eta^2\}$ where

$$\eta = \frac{j_{0,1}}{\sqrt{c_0}}$$

and $j_{0,1}$ is the first zero of the first kind bessel function J_0.

Then the maximum principle holds for $(\mathcal{L}, \mathcal{B})$ *in* Ω.

Proof Let

$$g(x) = J_0(\frac{j_{0,1}}{d}\varrho), \quad \varrho = \sqrt{x_1^2 + x_2^2}.$$

Then

$$\begin{cases} \Delta g + c(x)g = (c - (\frac{j_{0,1}}{d})^2)g < 0 & \text{in } \overline{\Omega}, \\ g > 0 & \text{in } \overline{\Omega}, \\ \partial_\nu g = 0 & \text{in } \Gamma_1. \end{cases}$$

Thus, we finish the proof by applying Lemma 1. □

Now we show the positivity of solution at the nonsmooth point of Neumann boundary.

Remark 1 If the assumption in Lemma 2 holds, and if $Lu \le 0$ in Ω, $\mathcal{B}u \ge 0$ on $\partial\Omega$, then either u vanishes completely in Ω or u is positive in $\overline{\Omega}\backslash\Gamma_0$.

Proof By the maximum principle in Lemma 2, u is nonnegative in Ω. Suppose that u is positive somewhere in Ω. By the strong maximum and Hopf lemma, u is positive in all $\overline{\Omega}$ except at the Dirichlet boundary and the corner formed by two smooth Neumann boundary $(x_1 = x_2 = 0)$. Thus, $\Delta u - c_0 u \le -(c + c_0)u < 0$ in Ω.

In order to prove the positivity along the corner, we choose an arbitrary fixed point $P \in \{x_1 = x_2 = 0\} \cap \Gamma_1$ and suppose $P = O$ is the origin for simplicity. There exists a small $\delta > 0$ such that $Q_{2\delta} \cap \Gamma_0 = \emptyset$ where

$$Q_\delta = \{x \in \mathbb{R}^n : x_1^2 + x_2^2 \le \delta^2, |x_i| \le \delta, \forall\, 3 \le i \le n\}.$$

We construct an auxiliary function v over Q_δ as follows

$$v(x) = I_0(\gamma\varrho)\prod_{i=3}^{n}\cos\frac{\pi x_i}{2\delta} \text{ with } \gamma = \sqrt{c_0 + \frac{(n-2)\pi^2}{4\delta^2}}.$$

Then $\Delta v - c_0 v = 0$ and

$$u - \varepsilon v \ge 0 \text{ on } \partial Q_\delta \cap \overline{\Omega}$$

for fixed ε satisfying $0 < \varepsilon < \inf\{u(x)/v(x) : x \in \partial Q_\delta \cap \overline{\Omega}, \varrho = \delta\}$. Therefore,

$$\begin{cases} \Delta(u - \varepsilon v) - c_0(u - \varepsilon v) < 0 & \text{in } \Omega \cap Q_\delta, \\ (u - \varepsilon v) \ge 0 & \text{on } \overline{\Omega} \cap \partial Q_\delta, \\ \partial_\nu(u - \varepsilon v) \ge 0 & \text{on } \partial\Omega \cap Q_\delta. \end{cases}$$

It follows by the maximum principle that $u - \varepsilon v \ge 0$ in $\Omega \cap Q_\delta$. Therefore, $u(P) \ge \varepsilon v(P) > 0$. The proof is complete. □

We point out that another proof of Remark 1 is given by reflection and the Hopf's lemma, see [13, Lemma 2.4].

Lastly, we state two useful lemmas about the monotonicity near the Dirichlet boundary and Neumann boundary. We consider a solution $u(x)$ of the equation

$$\Delta u + f(u) = 0 \text{ in } \Omega, \tag{2.4}$$

where f is a local Lipschitz continuous function and Ω is a bounded domain.

Lemma 3 *Let $\bar{x} \in \partial\Omega$ and let $\nu(\bar{x})$ be the outer unit normal vector at the point $\bar{x} \in \partial\Omega$. Let γ be a unit vector in $\mathbb{R}^n$ satisfying $\nu(\bar{x}) \cdot \gamma > 0$. For some $\varepsilon > 0$ assume that u is a C^2 solution in $\overline{\Omega}_\varepsilon$ where $\Omega_\varepsilon = \Omega \cap \{|x - \bar{x}| < \varepsilon\}$, $u \geq 0, u \not\equiv 0$ in Ω_ε and $u = 0$ on Γ_ε. Moreover, suppose that the boundary Γ_ε is C^2. Then there exists a $\delta > 0$ such that*

$$\frac{\partial u}{\partial \gamma} < 0 \text{ in } \Omega_\delta = \Omega \cap \{|x - \bar{x}| < \delta\}.$$

Proof See the proof in Lemma 2.1 of [17]. □

Lemma 4 *Assume that $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in \partial\Omega$ and $\Omega_\varepsilon(\bar{x}) = B_\varepsilon^+(\bar{x})$ for some $\varepsilon > 0$ where $\Omega_\varepsilon(\bar{x}) = \Omega \cap B_\varepsilon(\bar{x})$ and $B_\varepsilon^+(\bar{x}) = B_\varepsilon(\bar{x}) \cap \{x_1 > \bar{x}_1\}$. Suppose that u is a C^2 solution in $\overline{\Omega}_\varepsilon(\bar{x})$ satisfying Neumann boundary condition $\partial_{x_1} u = 0$ on $B_\varepsilon(\bar{x}) \cap \{x_1 = \bar{x}_1\}$ and*

$$u(x', x_n) < u(x', 2\bar{x}_n - x_n) \text{ for } x_n > \bar{x}_n \text{ and } x \in \Omega_\varepsilon(\bar{x}).$$

Then

$$\frac{\partial u}{\partial x_n}(\bar{x}) < 0.$$

Proof This is proved by using Serrin's boundary lemma. The readers can find details in the proof in Theorem 2.4 of [6]. □

3 The special case β is a straight angle or a right angle

In this section we consider the special case that β is a straight angle or a right angle via reflection along flat Neumann boundary.

Lemma 5 *Assume that $\beta = \pi$. Then the solution u of (1.2) have the following properties:*

(1) u is symmetric with respect to the hyperplane $\{x_2 = 0\}$, that is,

$$u(x_1, -x_2, x_3, x_4, \ldots, x_n) = u(x);$$

(2) $x_2 u_{x_2} < 0$ for $x \in \Sigma$ satisfying $x_2 \neq 0$;
(3) $u_{x_1} < 0$ for $x \in \Sigma$.

Proof This can be proved by the origin moving plane method. Indeed, if one denotes u by reflection;

$$\tilde{u}(x) = \begin{cases} u(x), & x_1 \geq a, \\ u(2a - x_1, x_2, \ldots, x_n), & x_1 < a \end{cases}$$

Then $\tilde{u}$ is a positive Dirichlet solution of (1.1) in $\tilde{\Sigma} = \Sigma \cup \Sigma' \cup \Gamma_N$ where Σ' is the reflection domain of Σ with respect to the hyperplane $\{x_1 = a\}$. Applying the well-known result in [4,17], one deduces the symmetry and monotonicity properties of $\tilde{u}$ and u. □

Lemma 6 *Assume that $\beta = \pi/2$. Then the solution u of (1.2) has the following properties:*

(1) $u_{x_1} \pm u_{x_2} < 0$ in Σ;
(2) $x_2 u_{x_2} < 0$ for $x \in \Sigma$ satisfying $x_2 \neq 0$;
(3) $x_2 u_{x_1} - x_1 u_{x_2} > 0$ for $x \in \Sigma$ satisfying $x_2 \neq 0$.

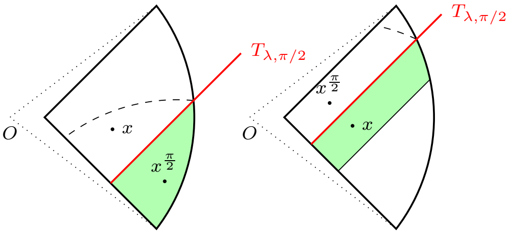

Fig. 1 The shape $D_{\lambda,\pi/2,0}$ for $\beta = \pi/2$

Proof (Method 1: By Reflection). We can reflect the solution u along the (flat) Neumann boundary Γ_N to obtain a solution with a Dirichlet condition in a larger domain. Indeed, if one denotes $\tilde{u}$ the reflection function of u with respect to the Neumann boundary Γ_N, that is

$$\tilde{u}(x) = \begin{cases} u(x_1, x_2, x'), & x \in \overline{\Sigma}, \\ u(x_2, x_1, x'), & x \in \overline{\Sigma_1} \\ u(-x_2, -x_1, x'), & x \in \overline{\Sigma_2} \end{cases}$$

where Σ_1 is the reflection of Σ w.r.t. Γ_N^+, Σ_2 the reflection of $\Sigma \cup \Sigma_1$ w.r.t. Γ_N^-, and $\hat{\Sigma}$ is the interior of the closure of $\Sigma \cup \Sigma_1 \cup \Sigma_2$. Here we assume the vertex set of the sector $\mathcal{C}$ passing though $V = (0, 0, 0, \ldots, 0)$ by translation. Then it is clear that $\tilde{u}$ is a classical positive solution of

$$\Delta\tilde{u} + f(\tilde{u}) = 0 \text{ in } \hat{\Sigma}, \quad \tilde{u} = 0 \text{ on } \partial\hat{\Sigma}$$

Note that $\hat{\Sigma}$ is symmetric and convex along the direction $e_1, e_2, e_1 \pm e_2$. Applying the well-known result in [4,17], one deduces the symmetry and monotonicity properties of $\tilde{u}$ and u.

(Method 2: By the method of moving plane directly). One can use the moving plane method and the maximum principle for mixed boundary to obtain $u_{x_1} \pm u_{x_2} < 0$ in Σ (see Figure 1). Then one can obtain that u is symmetric w.r.t. to x_2 and $u_{x_2} > 0$ for $x \in \Sigma$ satisfying $x_2 < 0$. The details will be illuminated later in Lemma 7. □

As proved by the reflection method, we see that the solution u is C^2 at the point of nonsmooth Neumann boundary $\{x_1 = a, x_2 = 0\} \cap \partial\mathcal{C}$ when $\beta = \pi/2$. By using this reflection, one can see that $u \in C^2(\overline{\Sigma}\backslash(\partial B \cap \partial\mathcal{C}))$ when $\pi/\beta \in \mathbb{N}$ where $\mathbb{N}$ is the collection of all nonnegative integers. We note that for general case of β, u may not be C^2 at the point of nonsmooth Neumann boundary.

4 The case $\beta \in (\pi/2, 2\pi/3]$

In this section we will prove the symmetry of u with variable x_2. Let us define the moving hyperplanes and moving domains to be used in the process of the methods of moving planes, and these notations will be also used for all range of $\beta \in (0, \pi]$.

For simplicity, we always assume that the vertex line of the sector $\mathcal{C}$ passing though $V = (0, 0, 0, \ldots, 0)$, and the center of the ball is $O = (-a, 0, 0, \ldots, 0)$. Denote the lower Neumann boundary and upper Neumann boundary by $\Gamma_N^- = \Gamma_N \cap \{x_2 < 0\}$ and $\Gamma_N^+ = \Gamma_N \cap \{x_2 > 0\}$. Set $P_\lambda = (\lambda\cos(\beta/2), -\lambda\sin(\beta/2), 0, \ldots, 0)$. We consider the moving hyperplane $T_{\lambda,\vartheta}$ passing through P_λ which forms with the lower boundary Γ_N^- an angle

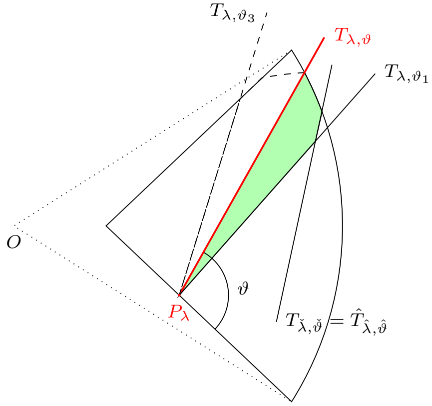

Fig. 2 The moving hyperplane $T_{\lambda,\vartheta}$ and the moving domain $D_{\lambda,\vartheta,\vartheta_1}$

$\vartheta \in (0, (\pi+\beta)/2]$, that is,

$$T_{\lambda,\vartheta} = \{x \in \mathbb{R}^n : [x_1 - \lambda\cos\frac{\beta}{2}]\sin(\vartheta - \frac{\beta}{2}) - [x_2 + \lambda\sin\frac{\beta}{2}]\cos(\vartheta - \frac{\beta}{2}) = 0\}. \quad (4.1)$$

See Figure 2. As usual Σ', B' will be the reflection of Σ, B with respect to $T_{\lambda,\vartheta}$ and $x^{\lambda,\vartheta}$ is the symmetry point of x. Similarly to $T_{\lambda,\vartheta}$, we denote $\hat{T}_{\lambda,\vartheta}$ by the reflection of $T_{\lambda,\vartheta}$ with respect to the hyperplane $\{x_2 = 0\}$,

$$\hat{T}_{\lambda,\vartheta} = \{x \in \mathbb{R}^n : [x_1 - \lambda\cos\frac{\beta}{2}]\sin(\vartheta - \frac{\beta}{2}) + [x_2 - \lambda\sin\frac{\beta}{2}]\cos(\vartheta - \frac{\beta}{2}) = 0\}.$$

We want to prove

$$u_{x_1}\sin(\vartheta - \beta/2) - u_{x_2}\cos(\vartheta - \beta/2) < 0 \text{ on } T_{\lambda,\vartheta} \cap \Sigma, \quad (4.2a)$$

$$u_{x_1}\cos(\beta/2) - u_{x_2}\sin(\beta/2) < 0 \text{ on } T_{\lambda,\pi/2} \cap \Gamma_N^-, \quad (4.2b)$$

$$u_{x_1}\sin(\vartheta - \beta/2) + u_{x_2}\cos(\vartheta - \beta/2) < 0 \text{ on } \hat{T}_{\lambda,\vartheta} \cap \Sigma, \quad (4.2c)$$

$$u_{x_1}\cos(\beta/2) - u_{x_2}\sin(\beta/2) < 0 \text{ on } \hat{T}_{\lambda,\pi/2} \cap \Gamma_N^+ \quad (4.2d)$$

for $\lambda > 0$ and $\vartheta \in [\beta/2, (\pi+\beta)/2]$. The proofs of (4.2c) and (4.2d) are similar to (4.2a) and (4.2b), and the proof of (4.2a) and (4.2b) depends on (4.2c) and (4.2d) for larger λ. This is the reason why we discuss (4.2c) and (4.2d) here. Set

$$\begin{aligned} \lambda_M(\vartheta) &= \sup\{\lambda > 0 : T_{\lambda,\vartheta} \cap \Sigma \neq \emptyset\}, \quad \vartheta \in (0, (\pi+\beta)/2], \\ \lambda_{max} &= \sup\{\lambda_M(\vartheta) : \vartheta \in (0, (\pi+\beta)/2]\} = (1-a)\sec(\beta/2). \end{aligned} \quad (4.3)$$

Let $\sigma_\lambda(x) \in [0, 2\pi)$ be the polar angle as follows

$$\begin{aligned} x_1 - \lambda\cos(\beta/2) &= \sqrt{[x_1 - \lambda\cos(\beta/2)]^2 + [x_2 + \lambda\sin(\beta/2)]^2}\cos[\sigma_\lambda(x) - \beta/2], \\ x_2 + \lambda\sin(\beta/2) &= \sqrt{[x_1 - \lambda\cos(\beta/2)]^2 + [x_2 + \lambda\sin(\beta/2)]^2}\sin[\sigma_\lambda(x) - \beta/2]. \end{aligned} \quad (4.4)$$

For $\lambda \in \mathbb{R}^+$, $0 \le \vartheta_1 < \vartheta = (\vartheta_1 + \vartheta_3)/2 < \vartheta_3 \le \pi$ with $\vartheta \in [\beta/2, (\pi+\beta)/2]$, we define the moving domain as follows

$$D_{\lambda,\vartheta,\vartheta_1} = \{x \in \Sigma \cap \Sigma' : \vartheta_1 < \sigma_\lambda(x) < \vartheta\},$$

see Figure 2.

The boundary $\partial D_{\lambda,\vartheta,\vartheta_1}$ consists of three parts:

(1) $\Gamma^0_{\lambda,\vartheta,\vartheta_1} = T_{\lambda,\vartheta} \cap \partial D_{\lambda,\vartheta,\vartheta_1}$, this is always non-empty as $0 < \lambda < \lambda_M(\vartheta)$;
(1) $\Gamma^1_{\lambda,\vartheta,\vartheta_1} = (\partial B \cup \partial B') \cap (\partial D_{\lambda,\vartheta,\vartheta_1} \backslash T_{\lambda,\vartheta})$, this is the boundary caused by the sphere ∂B and its reflection $\partial B'$. $\Gamma^1_{\lambda,\vartheta,\vartheta_1}$ belongs ∂B if $\vartheta \geq \beta/2$, $\lambda > 0$;
(1) $\Gamma^2_{\lambda,\vartheta,\vartheta_1} = \partial D_{\lambda,\vartheta,\vartheta_1} \backslash (\Gamma^0_{\lambda,\vartheta,\vartheta_1} \cup \Gamma^1_{\lambda,\vartheta,\vartheta_1})$, and $\Gamma^2_{\lambda,\vartheta,\vartheta_1}$ contains two parts: $\Gamma^{2A}_\lambda = \Gamma^2_{\lambda,\vartheta,\vartheta_1} \cap T_{\lambda,\vartheta_1}$ and $\Gamma^{2B}_\lambda = \Gamma^2_{\lambda,\vartheta,\vartheta_1} \cap T_{\check{\lambda},2\vartheta-\beta}$.

For simplicity, we omit the subscripts ϑ, ϑ_1 and denote these notations by Γ^0_λ, Γ^1_λ, Γ^2_λ, Γ^{2A}_λ and Γ^{2B}_λ. Here $T_{\check{\lambda},\check{\vartheta}} = \hat{T}_{\hat{\lambda},\hat{\vartheta}}$ stands for the hyperplane related to the reflection of the upper boundary Γ^+_N w.r.t. $T_{\lambda,\vartheta}$ where $\check{\vartheta} = 2\vartheta - \beta$, $\hat{\vartheta} = \pi - 2\vartheta + 2\beta$, $\hat{\lambda}$ and $\check{\lambda}$ are given as follows

$$\hat{\lambda} = \frac{\lambda \sin\vartheta}{\sin(\vartheta - \beta)}, \quad \check{\lambda} = \lambda + \frac{\lambda \sin\beta}{\sin(2\vartheta - \beta)}, \tag{4.5}$$

and $\hat{\lambda} > \check{\lambda}$ if and only if $\pi - 2\vartheta + 2\beta < 2\vartheta - \beta$, i.e., $\vartheta > (\pi + 3\beta)/4$. We note that $\Gamma^{2A}_\lambda \cap \Gamma^{2B}_\lambda$ (which is nonempty or not) is a non-smooth part of Γ^2_λ, both $\Gamma^{2A}_\lambda \backslash \Gamma^{2B}_\lambda$ and $\Gamma^{2B}_\lambda \backslash \Gamma^{2A}_\lambda$ are relatively open and are smooth subsets of $\partial D_{\lambda,\vartheta,\vartheta_1}$. Observe that $\Gamma^1_{\lambda,\vartheta,\vartheta_1} \cup \Gamma^{2B}_{\lambda,\vartheta,\vartheta_1}$ is always non-empty for $0 < \lambda < \lambda_M(\vartheta)$.

Set

$$w^{\lambda,\vartheta}(x) = u(x) - u^{\lambda,\vartheta}(x) \tag{4.6}$$

where $u^{\lambda,\vartheta}(x) = u(x^{\lambda,\vartheta})$. Clearly, $w^{\lambda,\vartheta}$ satisfies

$$\Delta w^{\lambda,\vartheta} + c^{\lambda,\vartheta}(x) w^{\lambda,\vartheta} = 0 \text{ in } D_{\lambda,\vartheta,\vartheta_1}, \tag{4.7a}$$

$$w^{\lambda,\vartheta} = 0 \text{ on } \Gamma^0_\lambda, \tag{4.7b}$$

$$w^{\lambda,\vartheta} < 0 \text{ on } \Gamma^1_\lambda \tag{4.7c}$$

for $\vartheta \in [\beta/2, (\pi + \beta)/2]$ where

$$c^{\lambda,\vartheta} = \frac{f(u^{\lambda,\vartheta}(x)) - f(u(x))}{u^{\lambda,\vartheta}(x) - u(x)}$$

is a uniformly (w.r.t. λ, ϑ) bounded function, say, $|c^{\lambda,\vartheta}| < c_0$ for some constant $c_0 > 0$.

In order to prove (4.2a), it suffices to prove that

$$w^{\lambda,\vartheta}(x) = u(x) - u^{\lambda,\vartheta}(x) < 0 \text{ for } x \in D_{\lambda,\vartheta,\vartheta_1}. \tag{4.8}$$

Using Hopf boundary lemma, we see (4.2a) is a direct consequence of (4.8).

Lemma 7 *($T_{\lambda,\vartheta}$ is orthogonal to the boundary of sector) Let $\beta \in (0, 2\pi/3]$. Assume that our conclusion (4.2) holds for all $\lambda > \Lambda$. Then (4.2) holds for $\vartheta = \pi/2$ and $\lambda > \Lambda_1 = \eta_1 \Lambda$ where*

$$\eta_1 = \max\{\frac{1}{2}, \cos\beta\}. \tag{4.9}$$

Remark 1 (4.2) holds for $\vartheta = \pi/2$ and $\lambda \geq (1 - a)\csc(\beta/2)\max\{\frac{1}{2}, \cos\beta\}$.

Proof Let $\vartheta = \pi/2$ and $\vartheta_1 = 0$ be fixed. For $\lambda > \Lambda_1$, we see that
(i) if $\pi/3 \leq \beta \leq 2\pi/3$, then $\Gamma^{2B}_\lambda \subset T_{2\lambda,\pi-\beta}$, $\beta/2 \leq \pi - \beta \leq (\pi + \beta)/2$ and $2\lambda \geq \Lambda$;
(ii) if $0 < \beta < \pi/3$, then $\Gamma^{2B}_\lambda \subset T_{\lambda\sec\beta,2\beta}$, $\beta/2 \leq 2\beta \leq (\pi + \beta)/2$ and $\lambda \sec\beta \geq \Lambda$.

By the assumption, we see that $w = w^{\lambda,\vartheta}$ satisfies

$$\begin{cases} \Delta w^{\lambda,\vartheta} + c^{\lambda,\vartheta}(x) w^{\lambda,\vartheta} = 0 & \text{in } D_{\lambda,\vartheta,\vartheta_1} \\ w^{\lambda,\vartheta} = 0 & \text{on } \Gamma_\lambda^0, \\ w^{\lambda,\vartheta} < 0 & \text{on } \Gamma_\lambda^1, \\ \nabla w^{\lambda,\vartheta} \cdot \nu \le 0 & \text{on } \Gamma_\lambda^{2A}, \\ \nabla w^{\lambda,\vartheta} \cdot \nu < 0 & \text{on } \Gamma_\lambda^{2B}. \end{cases} \tag{4.10}$$

Step 1: Starting the moving plane. Let η be the small constant that the maximum principle holds for Dirichlet boundary condition or mixed boundary condition when the domain width is less than η. From the definition of $D_{\lambda,\vartheta,\vartheta_1}$, we see that there exists a $\delta_0 > 0$ such that for every $\lambda \in (\lambda_M(\vartheta) - \delta_0, \lambda_M(\vartheta))$,

$$\operatorname{diam}(D_{\lambda,\vartheta,\vartheta_1}) < \eta$$

and

$$\Gamma^1_{\lambda,\vartheta,\vartheta_1} \neq \emptyset, \quad \Gamma^2_{\lambda,\vartheta,\vartheta_1} \subset T_{\lambda,\vartheta_1}.$$

Applying the maximum principle with mixed boundary condition in Lemma 2, we deduce the negativity of $w^{\lambda,\vartheta,\vartheta_1}$ in $D_{\lambda,\vartheta,\vartheta_1}$. Moreover, the Hopf boundary lemma implies the negativity of $w^{\lambda,\vartheta,\vartheta_1}$ on $\Gamma^2_{\lambda,\vartheta,\vartheta_1} \subset T_{\lambda,\vartheta_1}$. Hence,

$$w^{\lambda,\vartheta} < 0 \text{ in } \overline{D_{\lambda,\vartheta,\vartheta_1}} \backslash T_{\lambda,\vartheta}$$

and (4.2a) hold for $\lambda \in (\lambda_M(\vartheta) - \delta_0, \lambda_M(\vartheta))$.

Furthermore, by the same argument in Theorem 2.4 of [6] (see Lemma 4), one can prove that $|Du| \neq 0$ along the lower boundary of the sector,

$$u_{x_1} \cos(\beta/2) - u_{x_2} \sin(\beta/2) < 0 \text{ on } \Gamma_N^- \cap T_{\lambda,\pi/2}. \tag{4.11}$$

Let $\bar\lambda$ be

$$\bar\lambda = \inf\{\lambda' > 0 : (4.8) \text{ and } (4.2a) \text{ hold for every } \lambda \ge \lambda'\}.$$

Step 2: Proving that $\bar\lambda \le \Lambda_1$. Otherwise, we assume $\bar\lambda > \Lambda_1$. By continuity, we have

$$w^{\bar\lambda,\vartheta}(x) \le 0 \text{ for } x \in D_{\bar\lambda,\vartheta,\vartheta_1}.$$

Recalling that $\Gamma^1_{\lambda,\vartheta,\vartheta_1} \cup \Gamma^{2B}_{\lambda,\vartheta,\vartheta_1}$ is always non-empty, we obtain from the strong maximum principle and Hopf boundary lemma that

$$w^{\bar\lambda,\vartheta}(x) < 0 \text{ for } x \in \overline{D_{\bar\lambda,\vartheta,\vartheta_1}} \backslash T_{\bar\lambda,\vartheta} \tag{4.12}$$

where the negative at $\Gamma^{2A}_{\lambda,\vartheta,\vartheta_1} \cap \Gamma^{2B}_{\lambda,\vartheta,\vartheta_1}$ is obtained by Figure 1. Therefore, we have strictly inequalities

$$u_{x_1} \sin(\vartheta - \beta/2) - u_{x_2} \cos(\vartheta - \beta/2) < 0 \text{ on } T_{\bar\lambda,\vartheta} \cap \Sigma, \tag{4.13a}$$

$$u_{x_1} \cos(\beta/2) - u_{x_2} \sin(\beta/2) < 0 \text{ on } T_{\bar\lambda,\pi/2} \cap \Gamma_N^-. \tag{4.13b}$$

Since u has strictly monotonicity properties near the smooth boundary, we denote by Ξ the intersection of mixed boundary $\partial\mathcal{C} \cap \partial B$ and $T_{\bar\lambda,\pi/2}$. Then Ξ has two parts $\Xi^- = \Xi \cap \{x_2 < 0\}$

and $\Xi^+ = \Xi \cap \{x_2 > 0\}$. Clearly $\Xi^- \subset \{x_1 = \bar{x}_1, x_2 = \bar{x}_2\}$ and $\Xi^+ \subset \{x_1 = \bar{\bar{x}}_1, x_2 = \bar{\bar{x}}_2\}$ where

$$\bar{x}_1 = \bar{\lambda}\cos(\beta/2), \quad \bar{x}_2 = -\bar{\lambda}\sin(\beta/2),$$
$$\bar{\bar{x}}_1 = \bar{\bar{\lambda}}\cos(\beta/2), \quad \bar{\bar{x}}_2 = \bar{\bar{\lambda}}\sin(\beta/2), \quad \bar{\bar{\lambda}} = \frac{\bar{\lambda}\sin(\vartheta)}{\sin(\vartheta-\beta)}.$$

We suppose that both Ξ^- and Ξ^+ are non-empty set. One can choose a neighborhood $\mathcal{N}_-$ of Ξ^- and a neighborhood $\mathcal{N}_+$ of Ξ^+ such that (i) $\overline{\mathcal{N}_-} \cap \overline{\mathcal{N}_+} = \emptyset$; (ii) $\mathcal{N}_- \subset \{|x_1 - \bar{x}_1|^2 + |x_2 - \bar{x}_2|^2 < \eta^2\}$, $\mathcal{N}_+ \subset \{|x_1 - \bar{\bar{x}}_1|^2 + |x_2 - \bar{\bar{x}}_2|^2 < \eta^2\}$ where η is the small constant for the maximum principle in narrow domain to hold; (iii) $\mathcal{N}_- \cap \Gamma_\lambda^{2B} = \emptyset$, $\mathcal{N}_+ \cap \Gamma_\lambda^{2A} = \emptyset$ for every $\lambda \in (\bar{\lambda} - \delta_1, \bar{\lambda})$ for some $\delta_1 > 0$. Recall that u has strictly monotone property near the Dirichlet boundary (see Lemma 3), along the Neumann boundary (see (4.13b) and (4.2d)), and in the interior (see (4.13a)). Combining this with the negativity of $w^{\bar{\lambda},\vartheta}$ in (4.12), we derive from continuity that

$$w^{\lambda,\vartheta} < 0 \text{ in } \overline{D_{\lambda,\vartheta}} \backslash (T_{\lambda,\vartheta} \cup \mathcal{N}_- \cup \mathcal{N}_+)$$

for $|\lambda - \bar{\lambda}| < \delta_2$ for some $\delta_2 > 0$ (assuming $\delta_2 < \delta_1$).

In the rest of the domain $\tilde{D} = D_{\lambda,\vartheta} \cap (\mathcal{N}_- \cup \mathcal{N}_+)$, we have

$$\begin{cases} \Delta w^{\lambda,\vartheta} + c^{\lambda,\vartheta} w^{\lambda,\vartheta} = 0, \\ \partial_\nu w^{\lambda,\vartheta} = 0 \text{ on } \partial(D_{\lambda,\vartheta} \cap \mathcal{N}_-) \cap \Gamma_\lambda^{2A}, \\ w^{\lambda,\vartheta} \le, \not\equiv 0 \text{ on } \partial(D_{\lambda,\vartheta} \cap \mathcal{N}_-) \backslash \Gamma_\lambda^{2A}, \\ \partial_\nu w^{\lambda,\vartheta} = 0 \text{ on } \partial(D_{\lambda,\vartheta} \cap \mathcal{N}_+) \cap \Gamma_\lambda^{2B}, \\ w^{\lambda,\vartheta} \le, \not\equiv 0 \text{ on } \partial(D_{\lambda,\vartheta} \cap \mathcal{N}_+) \backslash \Gamma_\lambda^{2B}. \end{cases}$$

It follows by the maximum principle that $w^{\lambda,\vartheta} < 0$ in $\tilde{D}$ and then (4.8) holds. Therefore, (4.2a) holds for $\vartheta = \pi/2$ and $0 < \bar{\lambda} - \lambda \ll 1$. This contradicts the definition of $\bar{\lambda}$. Hence $\bar{\lambda} = \Lambda_1$, (4.8) and (4.2a) hold for $\vartheta = \pi/2$ and $\lambda > \Lambda$. Similarly, (4.2c) holds and the proof is finished. □

Lemma 8 *Let $\beta \in (\pi/2, 2\pi/3]$. Assume that (4.2) holds for all $\lambda > \Lambda$, then (4.2) holds for $\vartheta = 3\pi/4$ and $\lambda > \Lambda_2 = \eta_2\Lambda$ where*

$$\eta_2 = \max\{\frac{1}{2}, \sin\beta + \cos\beta\}.$$

Proof Let $\vartheta = 3\pi/4$ and $\vartheta_1 = \pi/2, \vartheta_3 = \pi$ be fixed. According to Lemma 7, $w^{\lambda,\vartheta,\vartheta_1}$ satisfies the boundary condition on Γ_λ^{2A} for $\lambda > \Lambda_1$. Now for $\lambda > \Lambda_2$, we see that $\Gamma_\lambda^2 \subset T_{\hat{\lambda},2\beta-\pi/2}$, $\beta/2 \le 2\beta - \pi/2 \le (\pi+\beta)/2$, $\lambda > \Lambda_1$ and

$$\hat{\lambda} = \frac{\lambda\sin(3\pi/4)}{\sin(3\pi/4-\beta)} > \Lambda.$$

Thus, (4.10) holds for $\lambda > \Lambda_2$. The remain of the proof is similar to Lemma 7. □

Lemma 9 *Let $\beta \in (\pi/2, 2\pi/3]$. Assume that (4.2) holds for all $\lambda > \Lambda$, then (4.2) holds for $\vartheta \in [\beta/2, 3\pi/4]$ and $\lambda > \Lambda_3 = \eta_3\Lambda$ where*

$$\eta_3 = \max\{\sin\beta + \cos\beta, \frac{1}{1+\sin\beta}\}. \tag{4.14}$$

Proof The proof will be divided into several steps.

Step 1. We check the boundary condition on Γ_λ^{2B} for every $\lambda > \Lambda_3$.

(C1) $\vartheta \in [(\pi + 3\beta)/4, (\pi + \beta)/2]$. In this case, $\Gamma_\lambda^{2B} \subset T_{\hat{\lambda},\hat{\vartheta}}$, $\hat{\vartheta} \in [\beta/2, (\pi + \beta)/2]$, and

$$\hat{\lambda} = \frac{\lambda \sin\vartheta}{\sin(\vartheta - \beta)} \geq \frac{\lambda \sin(3\pi/4)}{\sin(3\pi/4 - \beta)} > \Lambda.$$

(C2) $\vartheta \in [3\beta/4, (\pi + 3\beta)/4]$. In this case, $\Gamma_\lambda^{2B} \subset T_{\check{\lambda},\check{\vartheta}}$, $\check{\vartheta} \in [\beta/2, (\pi + \beta)/2]$ and

$$\check{\lambda} = \lambda + \frac{\lambda \sin\beta}{\sin(2\vartheta - \beta)} \geq \lambda(1 + \sin\beta) > \Lambda.$$

(C3) $\vartheta \in (0, \beta)$. In this case, we will choose $\vartheta_3 \leq \beta$ and hence $\Gamma_\lambda^{2B} = \emptyset$.

Therefore, $w^{\lambda,\vartheta}$ satisfies strictly the boundary condition of (4.10) on Γ_λ^{2B}. The boundary condition of Γ_λ^{2A} will be given in the process of mathematical induction below.

Step 2. We shall show that (4.2a) holds for $\vartheta \in [\pi/2, 3\pi/4]$. We claim that for every $m \in \mathbb{N}^+$ (the set of nonnegative integers), (4.2a) holds for every $\vartheta \in J_m$ and for every $\lambda > \Lambda_2$ where

$$J_m = \{\frac{k\pi}{2^{m+1}} \in [\frac{\pi}{2}, \frac{3\pi}{4}] : k \in \mathbb{N}^+\}.$$

By Lemma 7 and Lemma 8, we see the assertion holds for $m = 0, 1$. Assume the assertion holds for some $m \geq 1$, that is, (4.2a) holds for $\vartheta \in J_m$ and for $\lambda > \Lambda_2$. Now let $\vartheta \in J_{m+1} \backslash J_m$. Then

$$\vartheta = \frac{(2k+1)\pi}{2^{m+2}} \in (\frac{\pi}{2}, \frac{3\pi}{4}) \tag{4.15}$$

with $m, k \in \mathbb{N}^+$. We define ϑ_1 and ϑ_3 as follows

$$\vartheta_1 = \frac{2k\pi}{2^{m+2}}, \quad \vartheta_3 = \frac{(2k+2)\pi}{2^{m+2}}$$

and then ϑ_1 and ϑ_3 belong to J_m. Thus, (4.10) is satisfied for all $\lambda > \Lambda_3$. Therefore, as in the proof in Lemma 7, one can prove (4.2a) holds for $\lambda > \Lambda_3$. Thus, (4.2a) holds for $\vartheta \in J_{m+1}$ and for $\lambda > \Lambda_3$. Thus, the assertion follows by mathematical induction.

Since $J_\infty = \cup_{m=0}^\infty J_m$ is dense in $[\pi/2, 3\pi/4]$, we deduce by continuity that

$$u_{x_1} \sin(\vartheta - \beta/2) - u_{x_2} \cos(\vartheta - \beta/2) \leq 0 \text{ on } T_{\lambda,\vartheta} \cap \Sigma \tag{4.16}$$

holds for $\vartheta \in [\pi/2, 3\pi/4]$ and $\lambda > \Lambda_3$. For every fixed $\vartheta \in (\pi/2, 3\pi/4)$, one can find ϑ_1, ϑ_3 in $[\pi/2, 3\pi/4]$ such that $\vartheta - \vartheta_1 = \vartheta_3 - \vartheta > 0$. Noting that $\vartheta > \pi/2 \geq 3\beta/4$, we see $w^{\lambda,\vartheta}$ satisfies (4.10) for all $\lambda > \Lambda_3$. Therefore, as in the proof in Lemma 7, one can prove that the strict inequality (4.2a) holds for $\lambda > \Lambda_3$. Hence Step 2 is completed.

Now we know that (4.2a) holds for $\vartheta = \beta$. In the next step we will consider $\theta \in [\beta/2, \beta)$ and choose $\vartheta_1 \in \{0\} \cup [\beta/2, \beta]$ and $\vartheta_3 \in (0, \beta]$ so that Γ_λ^{2B} is an empty set.

Step 3. We shall show that (4.2a) holds for $\vartheta \in [\beta/2, \beta]$ and $\lambda > \Lambda_3$. We claim that (4.2a) holds $\vartheta \in \tilde{J}_m$ and $\lambda > \Lambda_3$ where

$$\tilde{J}_m = \{\frac{k\beta}{2^m} \in [\frac{\beta}{2}, \beta] : k \in \mathbb{N}^+\}$$

From step 2, this claim is valid for $m = 0$. Assume that this claim holds for some $m \geq 0$. Now let $\vartheta \in \tilde{J}_{m+1} \setminus \tilde{J}_m$. Then

$$\vartheta = \frac{(2k+1)\beta}{2^{m+1}} \in [\frac{\beta}{2}, \beta) \tag{4.17}$$

with $m, k \in \mathbb{N}$. We define ϑ_1 and ϑ_3 as follows

$$\vartheta_1 = \frac{k\beta}{2^m}, \quad \vartheta_3 = \frac{(k+1)\pi}{2^m}$$

and then ϑ_1 and ϑ_3 belong to $\tilde{J}_m \cup \{0\}$. Thus, $w^{\lambda,\vartheta}$ satisfies (4.10) for $\lambda > \Lambda_3$. Therefore, as in the proof in Lemma 7, one can prove (4.2a) holds for $\lambda > \Lambda_3$. Thus, (4.2a) holds for $\vartheta \in J_{m+1}$ and for $\lambda > \Lambda_3$. Thus, the assertion follows by mathematical induction. Observing that $J_\infty = \cup_{m=0}^{\infty} J_m$ is dense in $[\beta/2, \beta]$, we deduce by continuity that (4.16) holds for $\vartheta \in [\beta/2, \beta]$ and $\lambda > \Lambda_3$. For every fixed $\vartheta \in (\beta/2, \beta)$, one can find ϑ_1, ϑ_3 is contained in $[\beta/2, \beta]$ such that $\vartheta - \vartheta_1 = \vartheta_3 - \vartheta > 0$. Noting that $\vartheta_3 \leq \beta$ and Γ_λ^{2B} is empty, we see $w^{\lambda,\vartheta}$ satisfies (4.10) for all $\lambda > \Lambda_3$. Therefore, as in the proof in Lemma 7, one can prove that the strict inequality (4.2a) holds for $\lambda > \Lambda_3$. Step 3 is finished.

Following these steps above we conclude (4.2a) holds for $\vartheta \in [\beta/2, 3\pi/4]$, $\lambda \geq \Lambda_3$. Similarly, (4.2c) holds for $\vartheta \in [\beta/2, 3\pi/4]$, $\lambda \geq \Lambda_3$. The proof is finished. □

Lemma 10 *Let $\beta \in (0, \pi]$. Suppose (4.2) holds for all $\lambda > \Lambda_3$ and $\vartheta \in [\beta/2, a_0]$ for some a_0 satisfying*

$$\max\{\frac{\pi}{2}, \frac{\pi + 4\beta}{5}\} \leq a_0 < \frac{\pi + \beta}{2}. \tag{4.18}$$

Then (4.2) holds for all $\lambda > \Lambda_3$ and $\vartheta \in [\beta/2, (\pi + \beta)/2]$.

Proof We divide the proof into three steps. For simplicity we set

$$a_{k+1} = \frac{\pi + 2\beta - a_k}{2}, k = 0, 1, 2, \dots. \tag{4.19}$$

Step 1: (4.2) **holds for** $\lambda > \Lambda_3$ **and**

$$\max\{a_0, a_1, \pi/2 + \beta/4\} \leq \vartheta \leq (\pi + \beta)/2. \tag{4.20}$$

Note that

$$\begin{aligned} &\vartheta \in [a_0, (\pi + \beta)/2], \quad \hat{\vartheta} \in [\beta/2, a_0], \quad 2\vartheta - \pi \in [\beta/2, a_0] \\ \Longleftrightarrow & a_0 \leq \vartheta \leq (\pi + \beta)/2, a_1 \leq \vartheta \leq \pi/2 + 3\beta/4, \pi/2 + \beta/4 \leq \vartheta \leq (\pi + a_0)/2 \\ \Longleftrightarrow & \max\{a_0, a_1, \pi/2 + \beta/4\} \leq \vartheta \leq (\pi + \beta)/2. \end{aligned}$$

Thus, when ϑ satisfies (4.20), one can choose $\vartheta_1 = 2\vartheta - \pi \in [\beta/2, a_0]$, $\vartheta_3 = \pi$, so that $w^{\lambda,\vartheta}$ satisfies (4.10) for $\lambda > \Lambda_3$. Following the same process in the proof of Lemma 7, we deduce that (4.8) and (4.2a) hold for every $\lambda > \Lambda_3$. Similarly, (4.2c) is also valid for $\lambda > \Lambda_3$. In conclusion, (4.2) holds for $\lambda > \Lambda_3$ and ϑ satisfying (4.20) and in particular it holds for $\vartheta = (\pi + \beta)/2$.

Step 2: (4.2) **holds for** $\lambda > \Lambda_3$ **and**

$$\max\{a_0, a_1\} \leq \vartheta \leq (\pi + \beta)/2. \tag{4.21}$$

Assume $a_1 < \pi/2 + \beta/4$ (otherwise, this step is valid). Note that $\vartheta \in [a_1, \pi/2 + \beta/4]$ implies $\Gamma_\lambda^{2B} \in T_{\hat{\lambda},\hat{\vartheta}}$ and

$$\hat{\vartheta} \in [3\beta/2, a_0] \subset [\beta/2, a_0].$$

Thus, $w^{\lambda,\vartheta}$ satisfies strictly the boundary condition in (4.10) on Γ_λ^{2B} for $\lambda > \Lambda_3$. Following the same proof as in Lemma 7, we deduce that (4.2a) (and similarly (4.2c)) holds for $\vartheta \in [a_1, \pi/2 + \beta/4]$ and $\lambda > \Lambda_3$. Therefore, we finish this step.

In particular, Lemma 10 finishes if $a_0 \geq a_1$ (i.e., $a_0 \geq (\pi + 2\beta)/3$). So we assume $a_0 < (\pi + 2\beta)/3$ in the rest of the argument.

Step 3: (4.2) holds for $\lambda > \Lambda_3$ and $\beta/2 \leq \vartheta \leq (\pi + \beta)/2$.

We first claim that (4.2) holds for $\lambda > \Lambda_3$ and $\vartheta \in \cup_{k=0}^{\infty} J_k$ where

$$J_k = [\frac{\beta}{2}, a_{2k}] \cup [a_{2k+1}, \frac{\pi + \beta}{2}]$$

and $\{a_k\}$ is defined in (4.19). Clearly, the series $\{a_k\}$ has the following properties:

$$\lim_{k\to\infty} a_k = \frac{\pi + 2\beta}{3}, \quad a_{2k} < a_{2k+2} < \frac{\pi + 2\beta}{3} < a_{2k+3} < a_{2k+1}, k = 0, 1, 2, \ldots.$$

Now suppose that this claim holds for $k = \kappa \geq 0$. Let $\vartheta \in [a_{2\kappa}, a_{2\kappa+2}]$ be fixed. For the choice of ϑ_1 and ϑ_3 and the boundary conditions for $w^{\lambda,\vartheta}$ on Γ_λ^{2A}, we observe that

$$\begin{aligned} I =& \{\vartheta = (\vartheta_1 + \vartheta_3)/2 : \vartheta_1 \in [\beta/2, a_{2\kappa}], \vartheta_3 \in [a_{2\kappa+1}, (\pi + \beta)/2]\} \\ =& [(2a_{2\kappa+1} + \beta)/4, 2a_{2\kappa} + \pi + \beta)/4] \\ \supset& [a_{2\kappa}, a_{2\kappa+2}] \end{aligned} \tag{4.22}$$

since

$$\begin{aligned} (2a_{2\kappa+1} + \beta) - 4a_{2\kappa} &= (\pi + 3\beta) - 5a_{2\kappa} < 0, \\ (2a_{2\kappa} + \pi + \beta) - 4a_{2\kappa+2} &= a_{2\kappa} - \beta > 0. \end{aligned}$$

For the boundary condition on Γ_λ^{2B}, we see $\Gamma_\lambda^{2B} \subset T_{\check{\lambda},\hat{\vartheta}} = T_{\hat{\lambda},\bar{\vartheta}}$ with

$$\hat{\vartheta} \in [a_{2\kappa+1}, (\pi + \beta)/2], \hat{\lambda} > \lambda > \Lambda_3 \text{ when } \vartheta \geq (\pi + 3\beta)/4, \tag{4.23a}$$

$$\check{\vartheta} \in [a_{2\kappa+1}, (\pi + \beta)/2], \check{\lambda} > \lambda > \Lambda_3 \text{ when } \vartheta < (\pi + 3\beta)/4 \tag{4.23b}$$

where we have used $2a_0 - \beta \geq a_1$, i.e., $a_0 \geq (\pi + 4\beta)/5$. It immediately follows that $w^{\lambda,\vartheta}$ satisfies (4.10) for $\lambda > \Lambda_3$. As in the proof of Lemma 7, we deduce that (4.8) and (4.2a) hold for $\lambda > \Lambda_3$. Similarly, (4.2c) holds for $\lambda > \Lambda_3$.

Let $\vartheta \in [a_{2\kappa+3}, (\pi + \beta)/2]$. Note that

$$\begin{aligned} I =& \{\vartheta = (\vartheta_1 + \vartheta_3)/2 : \vartheta_1 \in [\beta/2, a_{2\kappa+2}], \vartheta_3 \in [a_{2\kappa+1}, (\pi + \beta)/2]\} \\ =& [(2a_{2\kappa+1} + \beta)/4, (2a_{2\kappa+2} + \pi + \beta)/4] \\ \supset& [a_{2\kappa+3}, a_{2\kappa+1}] \end{aligned} \tag{4.24}$$

since

$$\begin{aligned} (2a_{2\kappa+1} + \beta) - 4a_{2\kappa+3} &= a_{2\kappa+1} - (\pi + \beta) < 0, \\ (2a_{2\kappa+2} + \pi + \beta) - 4a_{2\kappa+1} &= 6a_{2\kappa+2} - (\pi + 3\beta) > 0. \end{aligned}$$

We also have $\Gamma_\lambda^{2B} \subset T_{\hat{\lambda},\hat{\vartheta}}$ and

$$\hat{\vartheta} \in [\beta, a_{2\kappa+2}] \subset [\beta/2, a_{2\kappa+2}].$$

It immediately follows that $w^{\lambda,\vartheta}$ satisfies (4.10) for $\lambda > \Lambda_3$. As in the proof of Lemma 7, we deduce that (4.2) holds for $\lambda > \Lambda_3$ and $\vartheta \in [a_{2\kappa+3}, (\pi + \beta)/2]$. Therefore, the assertion holds for $k = \kappa + 1$. By mathematical induction we finish the proof of the claim for $\vartheta \in \cup_{k=0}^{\infty} J_k = [\beta/2, (\pi + \beta)/2]\backslash\{(\pi + 2\beta)/3\}$.

For $\theta = (\pi + 2\beta)/3$, one can prove that

$$w^{\lambda,(\pi+2\beta)/3} < 0 \text{ in } D_{\lambda,(\pi+2\beta)/3,(\pi+5\beta)/6}.$$

and (4.2a) hold for $\lambda > \Lambda_3$. Similarly, (4.2c) holds for $\lambda > \Lambda_3$. Therefore, the proof is complete. □

Combining these lemmas above, we conclude the symmetry of u.

Theorem 2 *Let $\beta \in (\pi/2, 2\pi/3]$. Then (4.2) holds for $\vartheta \in [\beta/2, (\pi + \beta)/2]$ for $\lambda > 0$. In particular we have*

(i) $u_{x_1} < 0$ in Σ;
(ii) $x_2 u_{x_2} < 0$ in $\Sigma \cap \{x_2 \neq 0\}$;
(iii) u is symmetric with respect to the hyperplane $\{x_2 = 0\}$.

Proof Step 1: The monotonicity properties of u.

Set $\Lambda_0 = \lambda_{max}$ where

$$\lambda_{max} = \sup\{\lambda : T_{\lambda,(\pi+\beta)/2} \cap \Sigma \neq \emptyset\} = (1 - a)\sec(\beta/2).$$

Clearly, $T_{\lambda,\vartheta/2} \cap \Sigma = \emptyset$ for all $\lambda > \Lambda_0$ and $\vartheta \in [\beta/2, (\pi + \beta)/2]$. Denote $\bar{\eta} = \eta_3$ where η_3 is given in (4.14). Using Lemma 7-10, we deduce that (4.2) holds for $\vartheta \in [\beta/2, (\pi + \beta)/2]$ for $\lambda > \bar{\eta}\Lambda_0$. Repeating these lemmas serval times we conclude that (4.2) holds for $\vartheta \in [\beta/2, (\pi + \beta)/2]$ for $\lambda > \bar{\eta}^k \Lambda_0$ for every $k \geq 1$. This leads to the first part of the conclusion.

Conclusion (i) is a direct consequence of (4.2a) with $\vartheta = (\pi + \beta)/2$. Conclusion (ii) is a direct consequence of (4.2b), (4.2d) with $\vartheta = \beta/2$. By continuity, we deduce that $u_{x_2} = 0$ for $x \in \Sigma$ satisfying $x_2 = 0$.

Step 2: The even symmetry of u with respect to variable x_2. Indeed, we shall give two methods to prove the even symmetry.

Methods 1: The moving plane method. From the proof of Lemma 9, we deduce $w^{\lambda,\beta/2} < 0$ in $D_{\lambda,\beta/2,0}$ for all $\lambda > 0$. By letting $\lambda \to 0^+$, we see $u(x) \leq u(x_1, -x_2, x_3, \ldots, x_n)$ for $x_2 < 0$. Similarly, by moving $\hat{T}_{\lambda,\beta/2}$ from large λ to $\lambda = 0$, one can conclude that $u(x) \leq u(x_1, -x_2, x_3, \ldots, x_n)$ for $x_2 > 0$. This leads to the even symmetry of u with respect to x_2, i.e., $u(x) = u(x_1, -x_2, x_3, \ldots, x_n)$ for $x \in \Sigma$.

Methods 2: The methods for uniqueness of overdetermined problem. Set $v(x) = u(x_1, -x_2, x_3, \ldots, x_n)$. Then one can see that v and u satisfy the same equation on half domain $\Sigma^+ = \Sigma \cap \{x_2 > 0\}$, and same Dirichlet and Neumann datum on $\{x_2 = 0\}$

$$v = u, \ v_{x_2} = u_{x_2} = 0 \text{ on } \Sigma \cap \{x_2 = 0\}.$$

Applying Theorem 1 of [16], we derive that $v - u$ vanishes completely in Σ^+. The symmetry property follows. □

5 The case $\beta \in (2\pi/3, \pi)$

In this section we will consider the case $\beta > 2\pi/3$. In this case w^λ may not satisfy the Neumann type condition on Γ_λ^{2B}. The main task is to overcome this difficult.

For starting the moving plane process, we observe that $\Gamma^{2B}_{\lambda,\vartheta}$ is a empty set for large λ and suitable range of ϑ. Let κ be a fixed integer such that

$$\beta_\kappa < \beta \le \beta_{\kappa+1}$$

where $\beta_j = (1-2^{-j})\pi$, $j \in \mathbb{N}$. Now we set $\beta_* = \max\{\beta_{\kappa+1}, (\pi+4\beta)/5\}$. Then there exists a positive constant Λ_* such that $T_{\Lambda_*,\beta_*} \cap \Sigma \neq \emptyset$ and

$$\Gamma^{2B}_{\Lambda_*,\beta_*,0} = \emptyset. \tag{5.1}$$

We remark that (1) The choice of β_* is between β and $(\pi+\beta)/2$, and β_* will be used to apply Lemma 10. (2) The condition $T_{\Lambda_*,\beta_*} \cap \Sigma \neq \emptyset$ is used to guarantee that $\Lambda_* \in (0, \lambda_{max})$. (3) (5.1) implies that $\Gamma^{2B}_{\lambda,\vartheta,\vartheta_1}$ is empty for all $\lambda \ge \Lambda_*$ and $0 \le \vartheta_1 < \vartheta \le \beta_*$.

Lemma 11 *Let $\beta \in (0,\pi)$ and let Λ_* be chosen so that (5.1) holds. Then (4.2) holds for $\lambda \ge \Lambda_*$ and $\vartheta \in [\beta/2, (\beta+\pi)/2]$.*

Proof Since $\lambda \ge \Lambda_*$ is valid in this lemma, we know Γ^{2B}_λ is empty, so the results in Sect. 4 are valid without the hypothesis $\beta \le 2\pi/3$. As in the same process of Lemma 7, we conclude that (4.2) holds for $\vartheta = \beta_j$, $1 \le j \le \kappa+1$. Using the same process of Lemma 9, (4.2) holds for $\vartheta \in [\beta/2, \beta_{\kappa+1}]$. By setting $\vartheta = (\beta+\pi)/2$, $\vartheta_1 = \beta$ and $\hat{\vartheta} = \pi - 2\theta + 2\beta = \beta$, we see that $w^{\lambda,\vartheta}$ satisfies (4.10), and hence (4.8) and (4.2) hold for $\lambda \ge \lambda_*$. By the definition of β_*, one can check that $2\beta_* - (\beta+\pi)/2 \in [\beta/2, \beta_{\kappa+1}]$. Hence one can prove that (4.2) holds for $\vartheta = \beta_*$ and for $\vartheta \in [\beta/2, \beta_*]$. Finally, it follows from Lemma 10 that (4.2) holds for $\vartheta \in [\beta_*, (\pi+\beta)/2]$. □

Lemma 12 *Let $\beta \in (2\pi/3, \pi)$. Suppose that there exists some $\Lambda \in (0, \lambda_{max})$ such that (4.2) holds for $\lambda > \Lambda$ and $\vartheta \in [\beta/2, (\pi+\beta)/2]$. Then (4.2a) and (4.2c) holds for $\vartheta \in [\beta/2, (\pi+\beta)/2]$ and $\lambda = \Lambda$.*

Proof By continuity, we see the following inequalities

$$u_{x_1}\sin(\vartheta - \beta/2) - u_{x_2}\cos(\vartheta - \beta/2) \le 0 \text{ on } T_{\lambda,\vartheta} \cap \Sigma, \tag{5.2a}$$

$$u_{x_1}\sin(\vartheta - \beta/2) + u_{x_2}\cos(\vartheta - \beta/2) \le 0 \text{ on } \hat{T}_{\lambda,\vartheta} \cap \Sigma \tag{5.2b}$$

hold for $\lambda = \Lambda$ and $\vartheta \in [\beta/2, (\pi+\beta)/2]$.

Let $\vartheta \in [\beta/2, \beta)$ and then let $\vartheta_1 \in \{0\} \cup [\beta/2, \beta]$, $\vartheta_3 \in [\beta/2, \beta]$ be fixed so $\vartheta - \vartheta_1 = \vartheta_3 - \vartheta > 0$. Due to the negativity of $w^{\lambda,\vartheta}$ on a nonempty set Γ^1_λ, one see that (4.8), (4.2a) and (4.2c) hold for $\lambda \ge \Lambda$.

Let $\vartheta \in [\beta, (\pi+\beta)/2)$ and $\vartheta_1, \vartheta_3 \in [\beta/2, (\pi+\beta)/2]$ be fixed so $\vartheta - \vartheta_1 = \vartheta_3 - \vartheta > 0$. Then either $\check{\vartheta} \in [\beta, (\pi+\beta)/2]$ or $\hat{\vartheta} \in [\beta, (\pi+\beta)/2]$ and hence $w^{\lambda,\vartheta}$ satisfies a strict boundary condition on a non-empty set $\Gamma^{2B}_\lambda \cup \Gamma^1_\lambda$, so one can prove strict inequalities (4.8), (4.2a) and (4.2c) for $\lambda \ge \Lambda$.

Similarly, let $\vartheta = (\pi+\beta)/2$ and $\vartheta_1 = \beta$, $\vartheta_3 = \pi$ be fixed. Then $\hat{\vartheta} = \beta$ and hence $w^{\lambda,\vartheta}$ has a strict boundary condition on a non-empty set $\Gamma^{2B}_\lambda \cup \Gamma^1_\lambda$, so one can prove strict inequalities (4.8), (4.2a) and (4.2c) for $\lambda \ge \Lambda$. □

In order to overcome the boundary condition of Neumann type $w^{\lambda,\vartheta}$ on a non-empty set $\Gamma^{2B}_{\lambda,\vartheta}$, we observe and trace back to find a Dirichlet type boundary boundary condition as in the original moving plane method. Indeed, we will obtain the negativity of $w^{\lambda,\vartheta}$ by comparing the values of function u.

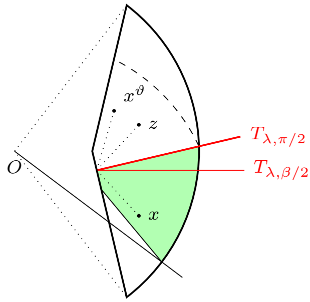

Fig. 3 The shape $D_{\lambda,\pi/2,0}$ for $\beta \in (2\pi/3, \pi)$

Lemma 13 *Let $\beta \in (2\pi/3, \pi)$. Suppose that there exists some $\Lambda \in (0, \lambda_{max})$ such that (4.2) holds for $\lambda > \Lambda$ and $\vartheta \in [\beta/2, (\pi + \beta)/2]$. Then (4.8) and (4.2) hold for $\vartheta = \pi/2$ and $\lambda > \Lambda_1 = \Lambda - \varepsilon$ for some $\varepsilon > 0$.*

Proof The proof will be divided into three steps.

Step 1: We prove that

$$w^{\Lambda,\pi/2}(x) = u(x) - u(x^{\Lambda,\pi/2}) < 0 \text{ for } x \in D_{\Lambda,\pi/2,0}. \tag{5.3}$$

Let $\check{\Lambda} = 2\Lambda$, $\check{\vartheta} = \pi - \beta$. Then $\Gamma_\Lambda^{2B} \subset T_{\check{\Lambda},\check{\vartheta}}$. When Γ_Λ^{2B} is a empty subset (this case occurs when Λ is large), (5.3) is valid as in Lemma 7. Hence we assume that Γ_Λ^{2B} is not a empty subset, i.e., $\Gamma_D \cap T_{\check{\Lambda},\check{\vartheta}}$ is not a empty subset. Lemma 3 gives the monotonicity properties near the Dirichlet boundary and hence $\partial_\nu w^{\Lambda,\pi/2} > 0$ for all points x satisfying $x \in \Gamma_\Lambda^{2B}$ and x is close to $\Gamma_D \cap T_{\check{\Lambda},\check{\vartheta}}$; see Figure 3. We will use a new observation to show the negativity of $w^{\Lambda,\pi/2}$.

Let $x \in T_{\Lambda,\psi_1}$ and $x^{\Lambda,\pi/2} \in T_{\Lambda,\psi_2}$ be fixed with $\psi_1 < \psi_2$, $\psi_1 + \psi_2 = \pi$. In order to prove

$$u(x) < u(x^{\Lambda,\pi/2}),$$

we will introduce another point z and prove that

$$u(x) < u(z), u(z) < u(x^{\Lambda,\pi/2}).$$

For simplicity, we shall use the polar coordinate (ϱ, ψ) as in (4.4) with $\lambda = \Lambda$ for the first two variables. The fixed point x and $x^{\Lambda,\pi/2}$ are denoted by $(\bar{\varrho}, \psi_1, x')$ $(\bar{\varrho}, \psi_2, x')$, or $(\bar{\varrho}, \psi_1)$ $(\bar{\varrho}, \psi_2)$ for simplicity (we omit $x' \in \mathbb{R}^{n-2}$). By the above argument we have proved that

$$w^{\Lambda,\beta/2} < 0 \text{ in } \overline{D_{\Lambda,\beta/2,0}} \backslash T_{\Lambda,\beta/2}, \tag{5.4}$$

$$w^{\Lambda,(\pi+\beta)/2} < 0 \text{ in } \overline{D_{\Lambda,(\pi+\beta)/2,\beta}} \backslash T_{\Lambda,(\pi+\beta)/2} \tag{5.5}$$

and

$$\vartheta \mapsto u(r, \vartheta) \text{ is strictly increasing in } \vartheta \in J \tag{5.6}$$

where $J = [\beta/2, (\pi + \beta)/2]$. (5.6) holds due to the fact that Σ is convex in $\vartheta \in J$, that is, (ϱ, ϑ_i), $\vartheta_i \in J$, $i = 1, 2$ implies $(\varrho, \vartheta) \in \Sigma$ for ϑ is between ϑ_1 and ϑ_2.

Case 1: $\psi_1 \in J$, $\psi_2 \in J$. By (5.6)

$$u(\bar{\varrho}, \psi_1) < u(\bar{\varrho}, \psi_2).$$

Case 2: $\psi_1 \notin J$, $\psi_2 \in J$. From (5.4),

$$u(\bar{\varrho}, \psi_1) < u(\bar{\varrho}, \psi_1')$$

where $\psi_1' = \beta - \psi_1 \in J$. Noting that $\psi_2 - \psi_1' = \pi - \beta > 0$, we deduce by (5.6) that

$$u(\bar{\varrho}, \psi_1') < u(\bar{\varrho}, \psi_2).$$

Case 3: $\psi_1 \in J$, $\psi_2 \notin J$. From (5.5),

$$u(\bar{\varrho}, \psi_2') < u(\bar{\varrho}, \psi_2)$$

where $\psi_2' = (\pi + \beta) - \psi_2 \in J$. Noting that $\psi_2' - \psi_1 = \beta > 0$, we deduce by (5.6) that

$$u(\bar{\varrho}, \psi_1) < u(\bar{\varrho}, \psi_2').$$

Case 4: $\psi_1 \notin J$, $\psi_2 \notin J$. From (5.4) and (5.5),

$$u(\bar{\varrho}, \psi_1) < u(\bar{\varrho}, \psi_1'), \quad u(\bar{\varrho}, \psi_2') < u(\bar{\varrho}, \psi_2)$$

where $\psi_1' = \beta - \psi_1 \in J$, $\psi_2' = (\pi + \beta) - \psi_2 \in J$. Noting that

$$\psi_2' - \psi_1' = [(\pi + \beta) - \psi_2] - [\beta - \psi_1] = 2\psi_1 > 0,$$

we deduce by (5.6) that

$$u(\bar{\varrho}, \psi_1') < u(\bar{\varrho}, \psi_2').$$

In all of these four cases we have proved $u(\bar{\varrho}, \psi_1) < u(\bar{\varrho}, \psi_2)$. Step 1 is finished.

Step 2: We claim that there holds

$$u_{x_1} \cos(\beta/2) - u_{x_2} \sin(\beta/2) < 0 \text{ on } T_{\Lambda,\pi/2} \cap \Gamma_N^-. \tag{5.7}$$

This can be done by using an argument of contradiction and applying Serrin's boundary lemma to $w^{\Lambda,\pi/2}$ in $D_{\Lambda,\pi/2,0}$, see the details in Theorem 2.4 of [6].

Step 3: We conclude the proof. Indeed, as in same process of step 2 in the proof of Lemma 7, we see that (4.8), (4.2a) and (4.2b) hold for $\vartheta = \pi/2$, $\vartheta_1 = 0$ and $\lambda \in (\Lambda - \varepsilon_1, \Lambda]$ for some small $\varepsilon_1 > 0$. Similarly, one can prove (4.2c) and (4.2d) hold for $\vartheta = \pi/2$ and $\lambda \in (\Lambda - \varepsilon_2, \Lambda]$ for some small $\varepsilon_2 > 0$. Lemma 13 follows by taking $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. □

Lemma 14 *Let $\beta \in (2\pi/3, \pi)$. Suppose that there exists some $\Lambda \in (0, \lambda_{max})$ and $\Lambda_1 \in (0, \Lambda)$ such that (4.2) holds for $\lambda > \Lambda$, $\vartheta \in [\beta/2, (\pi + \beta)/2]$ and for $\lambda > \Lambda_1$, $\vartheta = \pi/2$. Then (4.2) holds for $\vartheta \in [\beta/2, a_0]$ and $\lambda > \Lambda_3$ where $a_0 = (\pi + 3\beta)/4$, $\Lambda_3 = \max\{\Lambda_1, \eta_3\Lambda\}$ and*

$$\eta_3 = \frac{1}{\sin\beta + 1}.$$

Proof We first deal with the boundary condition on Γ_λ^{2B}. For $\vartheta \in [\beta/2, a_0]$ and $\lambda > \Lambda_3$, we have

(C1) $\vartheta \in [3\beta/4, (\pi + 3\beta)/4]$. In this case, $\Gamma_\lambda^{2B} \subset T_{\check{\lambda},\check{\vartheta}}$, $\check{\vartheta} \in [\beta/2, (\pi + \beta)/2]$ and

$$\check{\lambda} = \lambda + \frac{\lambda \sin\beta}{\sin(2\vartheta - \beta)} \geq \lambda(1 + \sin\beta) > \Lambda.$$

(C2) $\vartheta \in (0, \beta)$. In this case, we will choose $\vartheta_3 \leq \beta$ and hence $\Gamma_\lambda^{2B} = \emptyset$.

Therefore, $w^{\lambda,\vartheta}$ satisfies the strict boundary condition on Γ_λ^{2B}. Choosing $\vartheta_1 = \pi/2$, $\vartheta_3 = \pi$ and following by the same process of Lemma 7, we see (4.2) holds for $\lambda > \Lambda_3$ and $\vartheta = 3\pi/4$.

Now there are two possibilities: $\beta \in [3\pi/4, \pi)$ and $\beta \in (2\pi/3, 3\pi/4)$.

Assume $\beta \in [3\pi/4, \pi)$. Then $(3\pi/4+\pi)/2 = 7\pi/8 \le (\pi+\beta)/2$. We have the following cases:

(1) (4.2) holds for $\lambda > \Lambda_3$ and $\vartheta \in [3\pi/4, 7\pi/8]$. In fact, following by the same process of Lemma 7, this can be done for $\vartheta = 7\pi/8$ by choosing $\vartheta_1 = 3\pi/4$, $\vartheta_3 = \pi$. As in step 2 of Lemma 9, this can be done for $\vartheta \in (3\pi/4, 7\pi/8)$.
(2) (4.2) holds for $\lambda > \Lambda_3$ and $\vartheta \in [3\pi/4, a_0]$ (in particular for $\vartheta = \beta$). Indeed, following the proof of Lemma 7, one can prove (4.2) holds for $\vartheta \in [\beta_{j+2}, \beta_{j+3}] \cap [\pi/2, a_0]$ for every $j \in \mathbb{N}$ where $\beta_j = (1 - 2^{-j})\pi$.
(3) (4.2) holds for $\lambda > \Lambda_3$ and $\vartheta \in [\beta/2, \beta]$. This can be done similarly as in step 3 of Lemma 9.

Assume $\beta \in (2\pi/3, 3\pi/4)$. Then $(3\pi/4 + \pi/2)/2 = 5\pi/8 \in (3\beta/4, (\pi + \beta)/2)$. We have the following cases:

(1) (4.2) holds for $\lambda > \Lambda_3$ and $\vartheta \in [5\pi/8, 3\pi/4]$. In fact, following by the same process of Lemma 7, this can be done for $\vartheta = 5\pi/8$ by choosing $\vartheta_1 = \pi/2$, $\vartheta_3 = 3\pi/4$. As in step 2 of Lemma 9, this can be done for $\vartheta \in (5\pi/8, 3\pi/4)$.
(2) (4.2) holds for $\lambda > \Lambda_3$ and $\vartheta \in [3\beta/4, 3\pi/4]$. Indeed, by induction and the proof of Lemma 7, one can prove that (4.2) holds for $\vartheta \in [b_{j+2}, b_{j+1}] \cap [3\beta/4, 3\pi/4]$ for every $j \in \mathbb{N}$ where $b_j = (1 + 2^{-j})\pi/2$.
(3) (4.2) holds for $\lambda > \Lambda_3$ and $\vartheta = c_2$, $\vartheta \in [c_1, c_2]$ and then for $\vartheta \in [c_1, a_0]$ where $c_1 = 3\beta/4$ and $c_{j+1} = (\pi + c_j)/2$. Indeed, noting that $c_2 < (\pi + 3\beta)/4$, one can prove (4.2) holds for $\vartheta \in [c_{j+1}, c_{j+2}] \cap [c_1, (\pi + 3\beta)/4]$ for every $j \in \mathbb{N}$.
(4) (4.2) holds for $\lambda > \Lambda_3$ and $\vartheta \in [\beta/2, \beta]$. This can be done similarly as in step 3 of Lemma 9.

The lemma is proven. □

Combining these lemmas above, we conclude the symmetry of u.

Theorem 3 *Let $\beta \in (2\pi/3, \pi)$. Then (4.2) holds for $\vartheta \in [\beta/2, (\pi + \beta)/2]$ for $\lambda > 0$. In particular we have*

(i) $u_{x_1} < 0$ in Σ;
(ii) $x_2 u_{x_2} < 0$ in $\Sigma \cap \{x_2 \neq 0\}$;
(iii) u is symmetric with respect to the hyperplane $\{x_2 = 0\}$.

Proof Set $\Lambda_0 = \lambda_{max}$ where

$$\lambda_{max} = \sup\{\lambda : T_{\lambda,(\pi+\beta)/2} \cap \Sigma \neq \emptyset\} = (1 - a)\sec(\beta/2).$$

Let us define

$$\mathcal{S} = \{\lambda' \in (0, \lambda_{max}) : (4.2) \text{ holds for every } \vartheta \in [\beta/2, (\pi + \beta)/2],\ \lambda \in [\lambda', \lambda_{max})\}.$$

From Lemma 11, $[\Lambda_*, \lambda_{max}] \subset \mathcal{S}$ and $\mathcal{S}$ is not a empty set. Lemma 12 implies $\mathcal{S}$ is relatively closed in $(0, \lambda_{max})$. Lemma 13, Lemma 14 and Lemma 10 tell us that the set $\mathcal{S}$ is open. In conclusion, S is relatively closed, open, nonempty set of $(0, \lambda_{max})$. Therefore, $S = (0, \lambda_{max})$. This gives the first two conclusions. The symmetry property follows from step 2 of Theorem 2. □

6 The case $\beta \in (0, \pi/3]$

In this section we consider the case $\beta \le \pi/3$.

Lemma 15 *Let $\beta \in (0, \pi/3]$. Assume that there exists a constant $\Lambda > 0$ such that (4.2) holds for every $\lambda > \Lambda$. Then (4.2) holds for $\vartheta \in [\beta/2, \pi/2]$ and $\lambda > \Lambda_3$ where $\Lambda_3 = \eta_3\Lambda$ and*

$$\eta_3 = \max\{\cos\beta, \frac{1}{1+\sin\beta}\}. \tag{6.1}$$

Proof Step 1. The function $w^{\lambda,\vartheta}$ satisfies a suitable boundary condition on Γ^{2B} for $\lambda \ge \Lambda_3$. In fact, for $\vartheta \in [(\pi+3\beta)/4, \pi/2]$, we see $\Gamma^{2B}_\lambda \subset T_{\hat\lambda,\hat\vartheta}$ with

$$\hat\vartheta = \pi - 2\vartheta + 2\beta \in [\beta/2, (\pi+\beta)/2] \text{ and } \hat\lambda = \frac{\lambda\sin\vartheta}{\sin(\vartheta-\beta)} \ge \frac{\lambda}{\cos\beta} \ge \Lambda.$$

While for $\vartheta \in [\pi/4, (\pi+3\beta)/4]$, we see $\Gamma^{2B}_\lambda \subset T_{\check\lambda,\check\vartheta}$ with

$$\check\vartheta = 2\vartheta - \beta \in [\beta/2, (\pi+\beta)/2] \text{ and } \check\lambda = \lambda + \frac{\lambda\sin\beta}{\sin(2\vartheta-\beta)} = \lambda(1+\sin\beta) \ge \Lambda$$

where the assumption $\beta \le \pi/3$ is used.

Step 2. The conclusion holds for $\vartheta \in [\pi/4, \pi/2]$. In fact, by choosing $\vartheta = \pi/2$ and $\vartheta_1 = 0$, one deduces from Lemma 7 that (4.8) and (4.2) hold for all $\lambda > \Lambda_2$. Note that the hypothesis $\beta \le \pi/3$ is used to guarantee that $\pi/4 \in (3\beta/4, (\pi+\beta)/2]$. By choosing $\vartheta = \pi/4$ and $\vartheta_1 = 0$, one deduces from Lemma 7 that (4.8) and (4.2) hold for all $\lambda > \Lambda_2$. By mathematical induction as in step 2 of Lemma 9, we can get that (4.2) holds for all $\lambda > \Lambda_2$ and $\vartheta \in (\pi/4, \pi/2)$.

Step 3. The conclusion holds for $\vartheta \in [\beta/2, \pi/2]$. Indeed, taking $\vartheta_1 = 0, \vartheta_3 = 2\vartheta$ in the proof of Lemma 7, one can get by induction that (4.2) holds for all $\lambda > \Lambda_2$, $\vartheta \in [2^{-j-2}\pi, 2^{-j-1}\pi] \cap [3\beta/4, \pi/2]$, for every $j \in \mathbb{N}$ and then for $\vartheta \in [3\beta/4, \pi/2]$. Following step 3 of Lemma 9, we deduce that (4.2) holds for all $\lambda > \Lambda_2$ and $\vartheta \in \cap[\beta/2, \beta/2]$. □

Theorem 4 *Under the condition $\beta \in (0, \pi/3]$, we have the same conclusion as stated in Theorem 2.*

Proof Note that $\beta \in (0, \pi/3]$ implies $\max\{\pi/2, (\pi+4\beta)/5\} \le \pi/2$. Using Lemma 15 and Lemma 10, we can conclude this result by the same proof of Theorem 2. □

7 The case $\beta \in (\pi/3, \pi/2)$

In this subsection we focus on the case $\beta \in (\pi/3, \pi/2)$. In this case $\Gamma^{2B}_{\lambda,\pi/4} \subset T_{\check\lambda,\check\vartheta}$ with $\check\vartheta = \pi/2 - \beta < \beta/2$. Hence $w^{\lambda,\pi/4}$ does not satisfy the Neumann boundary condition on $\Gamma^{2B}_{\lambda,\pi/4}$ when λ is small. Thus, we will prove the negativity of $w^{\lambda,\pi/4}$ as in step 1 in Lemma 13.

Lemma 16 *Let $\beta \in (\pi/3, \pi/2)$. Assume that (4.2) holds for $\vartheta \in [\beta/2, (\pi+\beta)/2]$ and $\lambda > \Lambda$. Then we have that*

(i) (4.2) holds for $\vartheta = \pi/2$ and $\lambda > \Lambda_1$ where $\Lambda_1 = \Lambda/2$;

(ii) there exists a small constant $\varepsilon > 0$ such that (4.2) holds for $\vartheta = \pi/4$ and $\lambda > \Lambda_2$ where $\Lambda_2 = \max\{\Lambda_1, \Lambda - \varepsilon\}$.

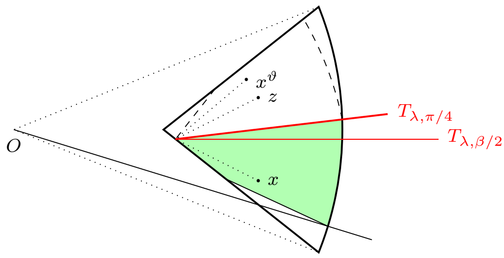

Fig. 4 The shape $D_{\lambda,\pi/4,0}$ for $\beta \in (\pi/3, \pi/2)$

Proof From Lemma 7, (4.2) holds for $\vartheta = \pi/2$ and $\lambda > \Lambda_1 = \Lambda/2$. This gives part (i).

By the same proof of Lemma 12, one can obtain that (4.2) holds for $\vartheta \in [\beta/2, (\pi+\beta)/2]$ and $\lambda = \Lambda$. The case for $\vartheta = \pi/4$ is more complicated. One can prove that

$$w^{\Lambda,\pi/4} < 0 \text{ in } \overline{D_{\Lambda,\pi/4,0}} \backslash T_{\Lambda,\pi/4}. \tag{7.1}$$

Indeed, the proof is similar to step 1 in Lemma 13. We will do it directly by proving

$$u(x) < u(z),\ u(z) < u(x^{\Lambda,\pi/4})$$

for some point z; see Figure 4. Let $x \in \overline{D_{\Lambda,\pi/4,0}} \backslash T_{\Lambda,\pi/4}$ be fixed. Then $x \in T_{\Lambda,\psi_1}$ and $x^{\Lambda,\pi/2} \in T_{\Lambda,\psi_2}$ be fixed with $\psi_1 < \psi_2$, $\psi_1 + \psi_2 = \pi/2$. For simplicity, we use the polar coordinate (ϱ, ψ) as in (4.4) for the first two variables and omit the remain $n-2$ variables $x' \in \mathbb{R}^{n-2}$. The fixed point x and $x^{\Lambda,\pi/4}$ are denoted by $(\bar\varrho, \psi_1)$, $(\bar\varrho, \psi_2)$. By the above argument we have proved that

$$w^{\lambda,\beta/2,0} < 0 \text{ in } \overline{D_{\lambda,\beta/2,0}} \backslash T_{\lambda,\beta/2} \tag{7.2}$$

and

$$\vartheta \mapsto u(\varrho, \vartheta) \text{ is strictly increasing in } \vartheta \in J \tag{7.3}$$

where $J = [\beta/2, (\pi+\beta)/2]$. (7.3) holds due to the fact that Σ is convex in $\vartheta \in J$, that is, (ϱ, ϑ_i), $\vartheta_i \in J$, $i = 1, 2$ implies $(\varrho, \vartheta) \in \Sigma$ for ϑ is between ϑ_1 and ϑ_2.

Case 1: $\psi_1 \in [\beta/2, \pi/2]$, $\psi_2 \in [\beta/2, \pi/2]$. By (7.3)

$$u(\bar\varrho, \psi_1) < u(\bar\varrho, \psi_2).$$

Case 2: $\psi_1 \in [0, \beta/2)$, $\psi_2 \in [\beta/2, \pi/2]$. From (7.2),

$$u(\bar\varrho, \psi_1) < u(\bar\varrho, \psi_1')$$

where $\psi_1' = \beta - \psi_1 \in [\beta/2, \pi/2]$. Noting that $\psi_2 - \psi_1' = \pi/2 - \beta > 0$, we deduce by (7.3) that

$$u(\bar\varrho, \psi_1') < u(\bar\varrho, \psi_2).$$

In both cases we have proved $u(\bar\varrho, \psi_1) < u(\bar\varrho, \psi_2)$ and hence (7.1) is proven.

The remaining part is similar to the proof of Lemma 13. □

Lemma 17 *Let $\beta \in (\pi/3, \pi/2)$. Then (4.2) holds for $\vartheta \in [\beta/2, (\pi+3\beta)/4]$ and $\lambda > \Lambda_3$ where $\Lambda_3 = \max\{\Lambda_2, \Lambda/(1+\sin\beta)\}$.*

Proof If $\vartheta \in [3\beta/4, (\pi+3\beta)/4]$ and $\lambda > \Lambda_3$, we see $\Gamma_\lambda^{2B} \in T_{\check\lambda,\check\vartheta}$ with

$$\check\vartheta = 2\vartheta - \beta \in [\beta/2, (\pi+\beta)/2],$$

and

$$\check{\lambda} = \lambda(1 + \frac{\sin\beta}{\sin(2\vartheta - \beta)}) \geq \lambda(1 + \sin\beta) \geq \Lambda.$$

Step 1. Since $3\pi/8 = (\pi/4 + \pi/2)/2 \in [3\beta/4, \pi/2]$, we see that for $\lambda > \Lambda_3$, (4.2) holds for $\vartheta \in J_\infty = \cup_{m=1}^{\infty} J_m$ and then for $\vartheta \in [3\pi/8, \pi/2]$ where

$$J_m = \{\frac{j\pi}{2^m} \in [\frac{3\pi}{8}, \frac{\pi}{2}] : j = 1, 2, \ldots\}, m = 1, 2, \ldots.$$

This can be done as in step 2 of Lemma 9.

Step 2. For $\lambda > \Lambda_3$, (4.2) holds for $\vartheta \in [3\beta/4, \pi/2] = \cup_{m=1}^{\infty} \tilde{J}_m$ where

$$\tilde{J}_m = [\frac{(1 + 2^{1-m})\pi}{4}, \frac{\pi}{2}] \cap [\frac{3\beta}{4}, \frac{\pi}{2}], m = 1, 2, \ldots.$$

Step 3. For $\lambda > \Lambda_3$, (4.2) holds for $\vartheta \in [\beta/2, \beta]$. This can be done as in step 3 of Lemma 9. Here in this step we have used $\vartheta_3 \leq \beta$ and $\Gamma_\lambda^{2B} = \emptyset$.

Step 4. By choosing $\vartheta_1 = \beta$, $\vartheta_3 = \pi$, $\vartheta = (\pi + \beta)/2$ and then $\hat{\vartheta} = \beta$, we can prove that for $\lambda > \Lambda_3$, (4.2) holds for $\vartheta = (\pi + \beta)/2$

Step 5. For $\vartheta \in [\pi/2, (\pi + 3\beta)/4]$, we can take $\vartheta_3 = (\pi + \beta)/2$,

$$\vartheta_1 = 2\vartheta - \vartheta_3 \in [(\pi - \beta)/2, \beta] \subset [\beta/2, \pi/2].$$

Following the process in Lemma 7, one can show that (4.2) holds for and $\lambda > \Lambda_3$. □

Theorem 5 *Under the condition $\beta \in (\pi/3, \pi/2)$, we have the same conclusion as stated in Theorem 2.*

Proof Using Lemma 11, Lemma 16, Lemma 17 and Lemma 10, we can conclude this result by the same proof of Theorem 3. □

8 Radial symmetry in $x' \in \mathbb{R}^{n-2}$

In this section we gives the proof of the symmetry in the last $n - 2$ variables $x' \in \mathbb{R}^{n-2}$. The proof does not need the symmetry and monotonicity result with respect to x_1 and x_2.

Lemma 18 *The solution u is radially symmetric in the last $n - 2$ variables $x' \in \mathbb{R}^{n-2}$.*

Proof We prove that u is symmetric with respect to every hyperplane T_0 which is orthogonal to the upper Neumann boundary Γ_N^+ and the lower Neumann boundary Γ_N^-. For simplicity we assume that T_0 is the hyperplane $\{x_n = 0\}$ and denote by T_λ the hyperplane parallels to T_0, that is,

$$T_\lambda = \{x \in \mathbb{R}^n : x_n = \lambda\}.$$

Denote $\lambda_0 = \sup\{\lambda : T_\lambda \cap \Sigma \neq \emptyset\}$. For $\lambda \in (0, \lambda_0)$, the open cap above T_λ will be denoted by Σ_λ, $\Sigma_\lambda = \{x \in \Sigma : x_n > \lambda\}$. The symmetry point of x with respect to T_λ will denote by x^λ, $x^\lambda = (x_1, \ldots, x_{n-1}, 2\lambda - x_n)$. We set $u^\lambda(x) = u(x^\lambda)$ and

$$w^\lambda(x) = u(x) - u(x^\lambda) \text{ for } x \in \Sigma_\lambda.$$

Note that u^λ satisfies the same equation and boundary condition as u. We deduce that w^λ satisfies

$$\begin{cases} \Delta w^\lambda + c^\lambda(x) w^\lambda = 0 & \text{in } \Sigma_\lambda, \\ w^\lambda = 0 & \text{on } T_\lambda \cap \overline{\Sigma}, \\ w^\lambda < 0 & \text{on } \Gamma_D \cap \partial\Sigma_\lambda \\ \frac{\partial w^\lambda}{\partial \nu} = 0 & \text{on } \Gamma_N \cap \partial\Sigma_\lambda, \end{cases} \tag{8.1}$$

where

$$c^\lambda(x) = \frac{f(u(x)) - f(u(x^\lambda))}{u(x) - u(x^\lambda)}$$

is a uniform bounded function, $|c^\lambda(x)| < c_0$ for some $c_0 > 0$.

Our aim is to prove that

$$u(x) < u(x^\lambda) \text{ for } x \in \overline{\Sigma_\lambda} \backslash T_\lambda, \tag{8.2}$$

for every $\lambda \in (0, \lambda_0)$. We let $\bar{\lambda}$ be the supremum of λ such that (8.2) holds false, that is,

$$\bar{\lambda} = \inf\{\lambda' \in (0, \lambda_0) : (8.2) \text{ holds for every } \lambda \in (\lambda', \lambda_0)\}. \tag{8.3}$$

Step 1. We claim that (8.2) holds for all λ such that $\lambda_0 - \lambda$ is positive and sufficiently small. Indeed, by the definition of Σ_λ, the diameter of Σ_λ is very small when λ is close to λ_0. By the maximum principle in Lemma 2 and Remark 1, we get that (8.2) holds for λ satisfying $0 < \lambda_0 - \lambda \ll 1$. Thus, the constant $\bar{\lambda}$, given in (8.3), is well-defined and $\bar{\lambda} \in [0, \lambda_0)$.

Step 2. We claim that (8.2) holds for $\lambda \in (0, \lambda_0)$ and hence $w^0(x) \le 0$ for $x \in \Sigma_0$. Suppose that the assertion is false, then $\bar{\lambda} > 0$. Then, by continuity, $w^{\bar{\lambda}}(x) \le 0$ in $\Sigma_{\bar{\lambda}}$. By the strong maximum principle and Remark 1 we obtain

$$w^{\bar{\lambda}} < 0 \text{ in } \overline{\Sigma_{\bar{\lambda}}} \backslash T_{\bar{\lambda}}, \tag{8.4}$$

It follows from the Hopf boundary lemma that $u_{x_n} < 0$ on $T_{\bar{\lambda}} \cap \Sigma$. From Lemma 4, we can obtain the strict monotonicity along the Neumann boundary and hence

$$u_{x_n} < 0 \text{ on } T_{\bar{\lambda}} \cap (\Sigma \cup \Gamma_N) \cap \{x_1^2 + x_2^2 > 0\} \tag{8.5}$$

where we always assume that the vertex line of the sector $\mathcal{C}$ passes though $V = (0, 0, 0, \dots, 0)$.

Now let us fix a small subset $\mathcal{N} = \{x : \sum_{i=1}^{n-1} x_i^2 + (x_n - \bar{\lambda})^2 < \eta/2\}$ where η is a small constant for the maximum principle in a narrow domain to hold; see Lemma 2. By the monotonicity near Dirichlet boundary (see Lemma 3), strict monotonicity properties in (8.5), and the negativity of $w^{\bar{\lambda}}$ in (8.4), one sees that

$$w^\lambda < 0 \text{ in } \overline{\Sigma_\lambda} \backslash (T_\lambda \cup \mathcal{N})$$

for $|\lambda - \bar{\lambda}| < \delta_2$ for some $\delta > 0$ (assuming $\delta < \eta/2$).

In the rest of the domain $\tilde{D} = \Sigma_\lambda \cap \mathcal{N}$, we have

$$\begin{cases} \Delta w^{\lambda,\pi/2} + c^{\lambda,\pi/2} w^{\lambda,\pi/2} = 0, \\ \partial_\nu w^\lambda = 0 \text{ on } \partial\tilde{D} \cap \Gamma_N, \\ w^\lambda \le, \not\equiv 0 \text{ on } \partial\tilde{D} \backslash \Gamma_N. \end{cases}$$

It follows by the maximum principle that $w^\lambda < 0$ in $\tilde{D}$. Therefore, (8.2) holds for $0 < \bar{\lambda} - \lambda \ll 1$. This contradicts the definition of $\bar{\lambda}$. Hence $\bar{\lambda} = 0$, (8.2) holds for every $\lambda \in (0, \lambda_0)$ and then $w^0(x) \leq 0$ for $x_n > 0$.

Step 3. If the hyperplanes are moved in the opposite direction, then we conclude that u is symmetric with respect to T_0. Because of the fact that T_0 is any hyperplane orthogonal to the Neumann boundary Γ_N, we deduce that u is radially symmetric with respect to $x' \in \mathbb{R}^{n-2}$. □

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