Robust Analysis of Linear Systems with Uncertain Delays using PIEs ⋆

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Abstract: This paper establishes a PIE (Partial Integral Equation)-based technique for the robust stability and H_∞ performance analysis of linear systems with interval delays. The delays considered are time-invariant but uncertain, residing within a bounded interval excluding zero. We first propose a structured class of PIE systems with parametric uncertainty, then propose a Linear PI Inequality (LPI) for robust stability and H_∞ performance of PIEs with polytopic uncertainty. Next, we consider the problem of robust stability and H_∞ performance of multi-delay systems with interval uncertainty in the delay parameters and show this problem is equivalent to robust stability and performance of a given PIE with parametric uncertainty. The robust stability and H_∞ performance of the uncertain time-delay system are then solved using the LPI solver in the MATLAB PIETOOLS toolbox. Numerical examples are given to prove the effectiveness and accuracy of the method. This paper adds to the expanding field of PIE approach and can be extended to linear partial differential equations.

Keywords: PIEs, linear delay system, polytopic uncertainty, robust stability, H_∞ performance

1. INTRODUCTION

The uncertain time delay phenomenon appears frequently and can severely affect the stability and performance of control systems causing, for example, thermoacoustic instability in combustion systems and chatter instability in machining (Guo et al., 2019; Fazelinia, 2007). Numerous results have been published to deal with the robust stability of delay systems in recent decades. Despite this extensive work, the problem of finding the maximum/exact time-delay range for which the system is stable at a minimum cost of computation remains unresolved.

The most commonly used method for estimating the maximum stable delay interval is to pose the problem as a linear matrix inequality (LMI) - see Park et al. (2015); Seuret and Gouaisbaut (2013); Zeng et al. (2015); Li et al. (2017). Fundamentally, the LMI approach to robust stability and control is a feasibility problem over parameter-dependent set of LMIs, with feasibility required to hold for the entire set of uncertain parameters - making the problem infinite-dimensional (Oliveira and Peres, 2007). One approach to solve this problem is to use bounding techniques including the well known Jensen inequality, free weighting matrices, Bessel inequality (Gouaisbaut and Seuret, 2015), etc. However, the bounds used in such techniques are often conservative. By scaling the number of inequalities, techniques such as the Bessel inequality have the potential to approach necessity in the limit, but at the cost of high computational complexity. Alternatively, the use of Integral Quadratic Constraints (IQCs) has been used to address the analysis problem of uncertain time-delay systems (Jun and Safonov, 2010), but this approach generally can not always be extended to the interval dependent problem. For example, Matthieu et al. (2020) tested the robust stability of a linear delay system, but only for the single delay case.

While there exist very effective methods for robust analysis and control of finite-dimensional linear state-space systems (ODEs), linear time delay systems (TDSs) are infinite dimensional - making generalization of the methods developed for ODEs to robust analysis of TDSs challenging. Recently, efforts have been made to represent linear time-delay systems in a manner which makes extension of ODE-based methods more straightforward. Specifically, the Partial Integral Equation (PIE) representation for linear infinite dimensional systems was recently proposed in Peet (2020). The PIE representation has a form similar to linear state-space ODEs - $\dot{x}(t) = Ax(t) + Bu(t)$ and is parameterized by the algebra of Partial Integral (PI) operators, which are a generalization of matrices to infinite-dimensions. The PIE framework, then, is intended to provide a way to generalize the mature theory for ODEs to infinite dimensional systems such as time-delay systems. This approach has already been studied for the estimation and control issue of linear delay systems Peet and Gu

⋆ This work was supported by the National Science Foundation under grants No. 1739990 and 1935453.

(2019); Wu et al. (2019) - an approach which yielded less conservative numerical results (almost identical to available analytical results). It is worth noting that the PIE-based approach builds upon the Sum of Squares method (it uses the same parameterization of positive Lyapunov functionals), and is typically more computationally efficient compared with LMI methods for compareable accuracy.

The objective of this paper is to use PIE-based methods to analyze stability and H_∞ performance problem of linear TDSs with uncertain delays. Firstly, a class of PIE system with parametric uncertainties is provided, which can be used to equivalently represent the solutions of a set linear systems with uncertain delays in a compact and linear time-invariant system (LTI) form. Then based on the corresponding uncertain PIE system, we obtain robust stability and H_∞ performance conditions for a set of linear delay systems with interval uncertainty in the delay. These conditions are defined by Linear PI Inequalities (LPIs) which can be solved efficiently using the MATLAB toolbox PIETOOLS Shivakumar et al. (2020a). Numerical examples are given to illustrate the proposed method.

1.1 Notations

I denotes the identity matrix with dimension clear from context. A block-diagonal matrix is denoted by diag$\{\cdots\}$. We use $L_2^n[T]$ to denote the vector-valued Lesbesque square integrable functions which map $T \to \mathbb{R}^n$. The space $Z_{m,n}$:=$\mathbb{R}^m\times$ $L_2^n[-1,0]$ is an inner-product space with the inner product defined as

$$\left\langle \begin{bmatrix} y \\ \psi \end{bmatrix}, \begin{bmatrix} x \\ \phi \end{bmatrix} \right\rangle = y^T x + \int_{-1}^{0} \psi(s)^T \phi(s) ds,$$

where $x, y \in \mathbb{R}^m$ and $\psi, \phi \in L_2^n[-1,0]$. The inner product $\langle \cdot, \cdot \rangle$ is in $Z_{m,n}$ space without any special notation.

2. PRELIMINARIES

Before we proceed to the main results, we define the class of linear time-invariant PIE systems without uncertainty and an associated notion of stability. The definition and properties of Partial Integral operators used in this paper are also introduced.

2.1 LMI-based robust stability condition of LTI systems

Lemma 1. (Horisberger and Belanger, 1976) Suppose there exists a positive symmetric matrix P satisfying

$$A_i^T P + P A_i < 0, \forall i \in \{1, \cdots, N\}. \tag{1}$$

Then for any initial condition, the system

$$\dot{x}(t) = \sum_{i=1}^{N} \alpha_i A_i x(t), \alpha \in \Delta$$

is robustly stable over $\Delta = \{\alpha \in \mathbb{R}^N : \alpha_i \in [0,1], \sum_{i=1}^{N} \alpha_i = 1\}$.

This LMI-based robust stability lemma can be obtained for LTI systems through a common quadratic Lyapunov function $V(x(t)) = x(t)^T P x(t)$. Note that this lemma actually proves the stronger notion of quadratic stability, which ensures stability with respect to time-varying uncertainty.

2.2 4-PI operators

Partial Integral (PI) operators are an extension of matrices to infinite-dimensional spaces. Specifically, the class of 4-PI operators form an algebra of bounded linear multiplier and integral operators defined jointly on $\mathbb{R}^n$ and L_2. We say $\mathcal{P}$ is a 4-PI operator if it has the form

$$\left(\mathcal{P}\begin{bmatrix} P, & Q_1 \\ Q_2, & \{R_i\}_{i=0}^2 \end{bmatrix} \begin{bmatrix} x \\ \Phi \end{bmatrix}\right)(s) := \begin{bmatrix} Px + \int_{-1}^{0} Q_1(s)\Phi(s)ds \\ Q_2(s)x + \left(\mathcal{P}_{\{R_i\}_{i=0}^2}\right)\Phi(s) \end{bmatrix}$$

where

$$\mathcal{P}_{\{R_i\}_{i=0}^2}\phi(s) := R_0(s)\phi(s) + \int_{-1}^{s} R_1(s,\theta)\phi(\theta)d\theta + \int_{s}^{0} R_2(s,\theta)\phi(\theta)d\theta$$

For any two 4-PI operators, $\mathcal{P}_1$ and $\mathcal{P}_2$, we have

a. $\mathcal{P}_1 + \mathcal{P}_2$ is also a 4-PI operator.
b. $\mathcal{P}_1^*$ stands for the adjoint of $\mathcal{P}_1$ and is also a 4-PI operator.
c. $\mathcal{P}_1\mathcal{P}_2$ represents the composition of $\mathcal{P}_1$ and $\mathcal{P}_2$ and is also a 4-PI operator.
d. $\mathcal{P}_1(: X \to Z) \succ 0$ if $\forall \sigma \in X$, $\langle \sigma, \mathcal{P}_1\sigma \rangle \succ 0$.

Remark 1. More details on the PI operator such as the definition of the adjoint and verification of the positivity using MATLAB package PIETOOLS are omitted here for the space reasons and can be found in Shivakumar et al. (2020a).

2.3 Defintion of PIE systems

A PIE system is a class of system described by a set of differential equations that are parameterized by PI operators. Specifically, we say $\mathbf{z} \in Z_{m,n}$ solves the PIE for initial condition $z_0 \in Z_{m,n}$ if

$$\begin{aligned} \mathcal{T}\dot{\mathbf{z}}(t) &= \mathcal{A}\mathbf{z}(t) + \mathcal{B}w(t) \\ z(t) &= \mathcal{C}\mathbf{z}(t) + Dw(t) \\ \mathbf{z}(0) &= \mathbf{z}_0 \in Z_{m,n} \end{aligned} \tag{2}$$

where signals $w \in \mathbb{R}^p$ is the external input and $z \in \mathbb{R}^q$ is the regulated output. Here $\mathcal{T}, \mathcal{A} : Z_{m,n} \to Z_{m,n}$, $\mathcal{B} : \mathbb{R}^p \to Z_{m,n}$, $\mathcal{C} : Z_{m,n} \to \mathbb{R}^q$, and $D : \mathbb{R}^p \to \mathbb{R}^q$ are PI operators. The PIE formulation provides a new alternative representation to a large class of linear infinite dimensional systems including delay differential (DDF) formulation and PDE systems (Peet, 2020; Shivakumar et al., 2020b). The stability of a PIE system is defined as follows.

Definition 2. The PIE system (2) defined by $\{\mathcal{T}, \mathcal{A}\}$ with $w(t) \equiv 0$ is said to be stable if any solution to the PIE system (2) satisfies $\lim_{t\to\infty} \|\mathcal{T}\mathbf{z}\| \to 0$.

Note that under this definition, LPI-based stability, H_∞ performance, estimation and stabilization conditions for PIE system (2) are studied in Shivakumar et al. (2020b), Wu et al. (2019), and Peet (2020). However such results have not been extended to uncertain PIEs.

3. LPI-BASED ROBUST ANALYSIS OF UNCERTAIN PIE SYSTEMS

This section proposes the structure of a class of uncertain PIE system with parametric uncertainty. Then, for uncertain PIEs with polytopic uncertainty, we propose LPI conditions for robust stability and input-out properties (H_∞ performance) of these uncertain PIEs.

3.1 PIE systems with parametric uncertainty

While uncertainties may enter the system in multiple ways, for simplicity and clarity we only consider the case where the uncertainty only appears in the generator $\mathcal{A}$. Stability and performance conditions when the parametric uncertainties appear in $\mathcal{B}$, $\mathcal{C}, \mathcal{D}$ can be obtained in the similar manner. The uncertain PIE system is defined as follows.

$$\begin{aligned} \mathcal{T}\dot{\mathbf{z}}(t) &= \mathcal{A}(\alpha)\mathbf{z}(t) + \mathcal{B}w(t), \alpha \in \Delta \\ z(t) &= \mathcal{C}\mathbf{z}(t) + Dw(t), \\ \mathbf{z}(0) &= \mathbf{z}_0 \in Z_{m,n} \end{aligned} \tag{3}$$

where $w \in \mathbb{R}^p$, $z \in \mathbb{R}^q$, $\alpha \in \mathbb{R}^u$, $\mathcal{T}, \mathcal{A}(\alpha) : Z_{m,n} \to Z_{m,n}$, $\mathcal{B} : \mathbb{R}^p \to Z_{m,n}$, $\mathcal{C} : Z_{m,n} \to \mathbb{R}^q$, and $D : \mathbb{R}^p \to \mathbb{R}^q$ are PI operators. We define robust stability of an uncertain PIE system (3) as follows.

Definition 3. The PIE system (3) defined by $\{\mathcal{T}, \mathcal{A}(\alpha)\}$ ($w(t) \equiv 0$) is said to be robustly stable over Δ if the PIE system (3) defined by $\{\mathcal{T}, \mathcal{A}(\alpha)\}$ is stable for any given $\alpha \in \Delta$.

3.2 Robust stability of PIEs with polytopic uncertainty

In this subsection, we consider the case where $\mathcal{A}(\alpha)$ is linear in the uncertain parameters and the parameters lie in a polytope. In this case, the uncertain PIE is parameterized by the vertex values $\mathcal{A}_i$, so that $\mathcal{A}(\alpha) := \sum_{i=1}^N \alpha_i \mathcal{A}_i, \Delta = \{\alpha \in \mathbb{R}^N : \alpha_i \in [0,1], \sum_{i=1}^N \alpha_i = 1\}$. Inspired by Lemma 1, a sufficient LPI-based robust stability condition for (3) is obtained as follows.

Theorem 4. Suppose there exist a PI operator $\mathcal{P}$ satisfying $\mathcal{P}^* = \mathcal{P} \succ 0$ and

$$\mathcal{A}_i^* \mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A}_i \prec 0, i = 1, 2, \cdots, N. \tag{4}$$

Then the PIE system (3) defined by $\{\mathcal{T}, \sum_{i=1}^N \alpha_i \mathcal{A}_i\}$ with $w(t) \equiv 0$ is robustly stable over $\Delta = \{\alpha \in \mathbb{R}^N : \alpha_i \in [0,1], \sum_{i=1}^N \alpha_i = 1, i = 1, 2, \cdots, N\}$.

Proof. Consider the Lyapunov candidate function

$$V(\mathbf{z}) = \langle \mathcal{T}\mathbf{z}, \mathcal{P}\mathcal{T}\mathbf{z} \rangle .$$

Since $\mathcal{P}$ is bounded and positive, we get there exist positive scalars λ_1, λ_2, $\lambda_1 \|\mathcal{T}\mathbf{z}\|^2 \le V(\mathbf{z}) \le \lambda_2 \|\mathcal{T}\mathbf{z}\|^2$. Then differentiating $V(\mathbf{z})$ along the solutions of the PIE (3) defined by $\{\mathcal{T}, \sum_{i=1}^N \alpha_i \mathcal{A}_i\}$ we obtain

$$\begin{aligned} \dot{V}(\mathbf{z}(t)) &= \langle \mathcal{T}\dot{\mathbf{z}}(t), \mathcal{P}\mathcal{T}\mathbf{z}(t) \rangle + \langle \mathcal{T}\mathbf{z}(t), \mathcal{P}\mathcal{T}\dot{\mathbf{z}}(t) \rangle \\ &= \left\langle \left(\sum_{i=1}^N \alpha_i \mathcal{A}_i \right) \mathbf{z}(t), \mathcal{P}\mathcal{T}\mathbf{z}(t) \right\rangle \\ &\quad + \left\langle \mathcal{T}\mathbf{z}(t), \mathcal{P} \left(\sum_{i=1}^N \alpha_i \mathcal{A}_i \right) \mathbf{z}(t) \right\rangle \\ &= \sum_i \alpha_i \left\langle \mathbf{z}(t), \left(\mathcal{A}_i^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A}_i \right) \mathbf{z}(t) \right\rangle < 0. \end{aligned}$$

Thus for any $\alpha \in \Delta$, $\dot{V}(\mathbf{z}(t)) < 0$ holds, and we have the PIE system (3) defined by $\{\mathcal{T}, \sum_{i=1}^N \alpha_i \mathcal{A}_i\}$ is stable for any given $\alpha \in \Delta$. We conclude from Definition 3, that the PIE system (3) defined by $\{\mathcal{T}, \sum_{i=1}^N \alpha_i \mathcal{A}_i\}$ with $w(t) \equiv 0$ is robustly stable over Δ.

3.3 H_∞ performance condition for PIEs

We now consider the H_∞ performance of the uncertain PIE system (3). The aim is to find a smallest γ for which any solution of the PIE satisfies $\|z\|_{L_2[0,\infty]} \le \gamma \|w\|_{L_2[0,\infty]}$ for any $\alpha \in \Delta$. By generalizing of the bounded real lemma for uncertain PIE systems, we get the following theorem.

Theorem 5. Suppose there exist a positive scalar γ and a bounded PI operator $\mathcal{P}$ satisfying $\mathcal{P}^* = \mathcal{P} \succ 0$ and

$$\begin{bmatrix} \mathcal{T}^*\mathcal{P}\mathcal{A}_i + \mathcal{A}_i^*\mathcal{P}\mathcal{T} & \mathcal{T}^*\mathcal{P}\mathcal{B} & \mathcal{C}^* \\ \mathcal{B}^*\mathcal{P}\mathcal{T} & -\gamma I & D^T \\ \mathcal{C} & D & -\gamma I \end{bmatrix} \prec 0, \quad i = 1, 2, \cdots, N. \tag{5}$$

Then if $\mathbf{z}_0 \equiv 0$, for any $w \in L_2$, any solution of the PIE system (3) defined by $\{\mathcal{T}, \sum_{i=1}^N \alpha_i \mathcal{A}_i, \mathcal{B}, \mathcal{C}, D\}$ satisfies $\|z\|_{L_2[0,\infty]} \le \gamma \|w\|_{L_2[0,\infty]}$ for any $\Delta = \{\alpha \in \mathbb{R}^N : \alpha_i \in [0,1], \sum_{i=1}^N \alpha_i = 1, i = 1, 2, \cdots, N\}$.

Proof. Define the Lyapunov candidate function (storage function) as

$$V(\mathbf{z}) = \langle \mathcal{T}\mathbf{z}, \mathcal{P}\mathcal{T}\mathbf{z} \rangle .$$

Since $\mathcal{P}$ is bounded and positive, there exist positive scalars λ_1, λ_2, $\lambda_1 \|\mathcal{T}\mathbf{z}\|^2 \le V(\mathbf{z}) \le \lambda_2 \|\mathcal{T}\mathbf{z}\|^2$. Set $v(t) = \frac{1}{\gamma} z(t)$ for the positive scalar γ. Then differentiating $V(\mathbf{z})$ along the solutions of PIE (3) we find

$$\begin{aligned} &\dot{V}(\mathbf{z}(t)) - \gamma \|w(t)\|^2 - \gamma \|v(t)\|^2 + 2 \langle v(t), z(t) \rangle \\ &= \langle \mathcal{T}\mathbf{z}(t), \mathcal{P}\mathcal{B}w(t) \rangle + \langle \mathcal{B}w(t), \mathcal{P}\mathcal{T}\mathbf{z}(t) \rangle - \gamma \|w(t)\|^2 \\ &\quad - \gamma \|v(t)\|^2 + \langle v(t), \mathcal{C}\mathbf{z}(t) \rangle + \langle \mathcal{C}\mathbf{z}(t), v(t) \rangle + \langle v(t), \mathcal{D}w(t) \rangle \\ &\quad + \langle \mathcal{D}w(t), v(t) \rangle + \left\langle \mathcal{T}\mathbf{z}(t), \mathcal{P} \left(\sum_{i=1}^N \alpha_i \mathcal{A}_i \right) \mathbf{z}(t) \right\rangle \\ &\quad + \left\langle \left(\sum_{i=1}^N \alpha_i \mathcal{A}_i \right) \mathbf{z}(t), \mathcal{P}\mathcal{T}\mathbf{z}(t) \right\rangle \\ &= \sum_{i=1}^N \alpha_i \begin{bmatrix} \mathbf{z}(t) \\ w(t) \\ v(t) \end{bmatrix}^T \begin{bmatrix} \mathcal{T}^*\mathcal{P}\mathcal{A}_i + (\cdot)^* & \mathcal{T}^*\mathcal{P}\mathcal{B} & \mathcal{C}^* \\ \mathcal{B}^*\mathcal{P}\mathcal{T} & -\gamma I & D^T \\ \mathcal{C} & D & -\gamma I \end{bmatrix} \begin{bmatrix} \mathbf{z}(t) \\ w(t) \\ v(t) \end{bmatrix}. \end{aligned}$$

Therefore, if Eqn (5) is satisfied, we have

$$\dot{V}(\mathbf{z}(t)) - \gamma \|w(t)\|^2 + \frac{1}{\gamma} \|z(t)\|^2 < 0$$

for any $\alpha \in \Delta$. Integration of this inequality with respect to t yields

$$V(\mathbf{z}(t)) - V(\mathbf{z}(0)) - \gamma \int_0^t \|w(s)\|^2 ds + \frac{1}{\gamma} \int_0^t \|z(s)\|^2 ds < 0.$$

Since $V(\mathbf{z}(0)) = 0$ and $V(\mathbf{z}(t)) \ge 0$ for any $t \ge 0$, then as $t \to \infty$, any solution of the PIE system (3) satisfies $\|z\|_{L_2[0,\infty]} \le \gamma \|w\|_{L_2[0,\infty]}$ for any $\alpha \in \Delta$.

Remark 2. Since a large class of uncertain DDFs and neutral-type systems fit the PIE structure, Theorems 4 and 5 have a wide application. However, the equivalence between the PIE representation and the original system formulation should be been proved firstly. Luckily this issue can be addressed with the help of Peet (2020) which has shown the existence and equivalence of solutions for PIEs, DDFs and neutral-type systems. Next section will show how the proposed framework works.

4. APPLICATION TO LINEAR SYSTEMS WITH UNCERTAIN DELAYS

In this section, we apply Theorem 4 and Theorem 5 to linear systems with uncertain delays. Specifically, we consider linear systems with multiple delays and interval delay uncertainty. These systems have the following form

$$\begin{aligned}\dot{x}(t) &= A_0x(t) + B_0w(t) + \sum_{i=1}^{k} A_ix(t-\tau_i)\\ z(t) &= C_{10}x(t) + D_{10}w(t) + \sum_{i=1}^{k} C_{1i}x(t-\tau_i)\\ x(s) &= x_0, s\in[-\tau,0], \tau = \max\{\tau_1,\cdots,\tau_k\}. \end{aligned} \quad (6)$$

where $x(t)\in\mathbb{R}^m$ is the system state with the initial function $x_0\in L_2[-\tau,0]$. $w(t)\in\mathbb{R}^p$ is the disturbance input. $z(t)\in\mathbb{R}^q$ is the regulated output. The delay parameters $\tau_i, i=1,2,\cdots,k$ are time-invariant but uncertain and

$$\tau\in\Delta_\tau := \{\tau\in\mathbb{R}_+^k : \tau_i\in\left[\tau_i^{[0]},\tau_i^{[1]}\right], i=1,2,\cdots,k\}. \quad (7)$$

where τ_i^0 and τ_i^1 are known positive constants defining the lower and upper bound of the τ_i respectively.

4.1 The equivalent PIE representation of the linear TDS

To apply the LPI-condition for uncertain PIE system to the uncertain TDS case, we first convert each instance of the uncertain linear delay system (6) to a corresponding PIE representation (3). First define

$$\begin{aligned}&\mathcal{T} := \mathcal{P}\begin{bmatrix} I, & 0\\ I, & \{0,0,-I\}\end{bmatrix}, \mathcal{B} := \mathcal{P}\begin{bmatrix} B_0, & 0\\ 0, & \{0\}\end{bmatrix},\\ &\mathcal{C} := \mathcal{P}\begin{bmatrix} C_{10}+\sum_{j=1}^{k} C_{1j}, & -\begin{bmatrix} C_{11} & \cdots & C_{1k}\end{bmatrix}\\ 0, & \{0,0,0\}\end{bmatrix}, D := D_{10}.\end{aligned} \quad (8)$$

Note that none of these PI operators depend on the τ_i. The effect of delay parameter is felt only in the generator $\hat{\mathcal{A}}(\hat{\tau})$ where $\hat{\tau}\in\mathbb{R}_+^k$ represents the vector of uncertain delay parameters in the uncertain TDS. Specifically, define $\hat{\mathcal{A}}(\hat{\tau})$ as

$$\begin{aligned}\hat{\mathcal{A}}(\hat{\tau}) &:= \mathcal{P}\begin{bmatrix} A_0+\sum_{j=1}^{k} A_j, & -\begin{bmatrix} A_1 & \cdots & A_k\end{bmatrix}\\ 0, & \left\{\text{diag}(\hat{\tau})^{-1},0,0\right\}\end{bmatrix} \\ \text{diag}(\hat{\tau}) &= \text{diag}\{\hat{\tau}_1 I_m,\cdots,\hat{\tau}_k I_m\}.\end{aligned} \quad (9)$$

In Peet (2020), it was shown that, using these definitions, for any choice of $\hat{\tau}$, any solution to the linear TDS corresponds to a solution of the PIE defined by $\{\mathcal{T},\hat{\mathcal{A}}(\tau),\mathcal{B},D\}$. This is stated in the following lemma.

Lemma 6. Given w, x_0, positive constants $\tau_i^{[0]},\tau_i^{[1]}, i=1,2,\cdots,k$, and for any given $\tau\in\Delta_\tau$ defined in Eqn (7), the function x and z satisfy the linear TDS (6) defined by $\{A_0,A_i,B_0,C_{10},C_{1i},D_{10},\tau\}$ if and only if z and $\mathbf{z}$ satisfy the linear PIE (3) defined by $\{\mathcal{T},\hat{\mathcal{A}}(\tau),\mathcal{B},D\}$ where $\hat{\mathcal{A}}(\tau)$ is as in Eqn (9), $\mathcal{B},\mathcal{T},D$ are as in Eqn (8), and $\mathbf{z}(t) = \begin{bmatrix} x(t)\\ \partial_s\phi(t,s)\end{bmatrix}, \mathbf{z}_0 = \begin{bmatrix} x_0\\ \partial_s\phi_0\end{bmatrix}$ where $\phi(t,s) = \left[x(t+\tau_1 s)^T,\cdots,x(t+\tau_k s)^T\right]^T$ and $\phi_0(s) = \left[x_0^T,\cdots,x_0^T\right]^T$ for $s\in[-1,0]$.

Proof. This Lemma can be derived by a combination of Lemma 1 and Lemma 4 in Peet (2020).

4.2 The equivalence between two uncertain PIE systems

The interval delay uncertainty set $\tau\in\Delta_\tau$ in the linear TDS can be equivalently represented as the convex box formed by the set of vertices $\hat{\tau}\in T$ where the set of vertices is defined as

$$\begin{aligned}T := \Big\{&\hat{\tau}\in\mathbb{R}^k : \hat{\tau} = \left[\tau_1^{[\gamma_1]},\cdots,\tau_k^{[\gamma_k]}\right],\\ &\gamma = [\gamma_1,\cdots,\gamma_k]\in\{0,1\}^k\Big\}.\end{aligned} \quad (10)$$

For convenience, we define and order the corresponding vertices of $\hat{\mathcal{A}}(\tau)$ as

$$\{\hat{\mathcal{A}}_1,\cdots,\hat{\mathcal{A}}_{2^k}\} := \left\{\hat{\mathcal{A}}(\hat{\tau}) : \hat{\mathcal{A}}(\hat{\tau}) \text{ satisfies Eqn (9)}, \hat{\tau}\in T\right\}. \quad (11)$$

Using this notation, we have the following lemma which establishes equivalence between robust stability of the linear TDS and robust stability of an uncertain PIE.

Lemma 7. Given positive constants $\tau_i^{[0]},\tau_i^{[1]}, i=1,\cdots,k$, suppose that $\mathcal{T}$ satisfies Eqn (8), $\hat{\mathcal{A}}(\tau)$ satisfies Eqn (9), the $\hat{\mathcal{A}}_i$ are as defined in Eqn (11), and Δ_τ is as defined in Eqn (7). Then the PIE system (3) defined by $\{\mathcal{T},\hat{\mathcal{A}}(\tau)\}$ is robustly stable over Δ_τ if and only if the PIE system (3) defined by $\{\mathcal{T},\sum_{i=1}^{2^k}\beta_i\hat{\mathcal{A}}_i\}$ is robustly stable over $\Delta_\beta = \{\beta\in\mathbb{R}^{2^k} : \beta_i\in[0,1], \sum_{i=1}^{2^k}\beta_i = 1\}$.

Proof. Suppose the PIE system (3) defined by $\{\mathcal{T},\hat{\mathcal{A}}(\tau)\}$ is robustly stable over Δ_τ. Then the PIE system (3) defined by $\{\mathcal{T},\hat{\mathcal{A}}(\tau)\}$ is stable for any given $\tau\in\Delta_\tau$. Since for any given $\tau\in\Delta_\tau$, there exist a unique $\beta\in\Delta_\beta$ such that $\tau = \sum_{i=1}^{2^k}\beta_i\hat{\tau}_i$ and $\hat{\mathcal{A}}(\tau) = \sum_i^{2^k}\beta_i\hat{\mathcal{A}}_i$ and the PIE system (3) defined by $\{\mathcal{T},\sum_{i=1}^{2^k}\beta_i\hat{\mathcal{A}}_i\}$ is stable. Further, one gets the PIE system (3) defined by $\{\mathcal{T},\sum_{i=1}^{2^k}\beta_i\hat{\mathcal{A}}_i\}$ is robustly stable over Δ_β. This establishes necessity. Sufficiency follows by the same argument.

4.3 Robust stability synthesis of the uncertain TDS

We now propose LPI conditions for robust stability of the uncertain linear TDS. Before proceeding to the main result in Theorem 9, we define robust stability of the linear uncertain delay system in Eqn (6).

Definition 8. Given $w=0$, positive constants $\tau_i^{[0]},\tau_i^{[1]}, i=1,2,\cdots,k$, the linear TDS (6) defined by $\{A_0,A_i,B_0,C_{10},C_{1i},D_{10},\tau\}$ is robustly stable over Δ_τ defined in Eqn (7) if the linear TDS (6) defined by $\{A_0,A_i,B_0,C_{10},C_{1i},D_{10},\tau\}$ is stable for any given $\tau\in\Delta_\tau$.

We now give the main result.

Theorem 9. Given positive constants $\tau_i^{[0]},\tau_i^{[1]}, i=1,\cdots,k$, Suppose there exist a constant real-valued matrix $P\in\mathbb{R}^{m\times m}$ and matrix-valued polynomials $Q:[a,b]\to\mathbb{R}^{m\times n}, R_0:[a,b]\to,\mathbb{R}^{n\times n}$, and $R_1,R_2:[a,b]\times[a,b]\to\mathbb{R}^{n\times n}$, such that $\mathcal{P} := \mathcal{P}\begin{bmatrix} P, & Q\\ Q^T, & \{R_0,R_1,R_2\}\end{bmatrix}$ satisfies $\mathcal{P}^* = \mathcal{P}\succ 0$ and

$$\hat{\mathcal{A}}_i^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\hat{\mathcal{A}}_i \prec 0, i=1,2,\cdots,2^k \quad (12)$$

where $n = m \cdot k$, $\hat{\mathcal{A}}_i$ is as defined in Eqn (11) and $\mathcal{T}$ is as defined in Eqn (8). Then the linear TDS (6) with $w \equiv 0$ is robustly stable over Δ_τ as defined in Eqn (7).

Proof. For any solution, $x(t)$ of the linear TDS, define $\mathbf{z}(t) = \begin{bmatrix} x(t) \\ \partial_s \phi(t,s) \end{bmatrix}, \mathbf{z}_0 = \begin{bmatrix} x_0 \\ \partial_s \phi_0 \end{bmatrix}$ where $\phi(t,s) = \left[x(t+\tau_1 s)^T, \cdots, x(t+\tau_k s)^T \right]^T, s \in [-1,0]$, and $\phi_0(0) = \left[x_0^T, \cdots, x_0^T \right]^T$. From Lemma 6, $\mathbf{z}(t)$ satisfies the PIE system (3) defined by $\{\mathcal{T}, \hat{\mathcal{A}}(\tau)\}$ ($w \equiv 0$ and $z \equiv 0$). Suppose the LPI (12) is satisfied. From Theorem 4, the PIE system (3) defined by $\{\mathcal{T}, \sum_{i=1}^{2^k} \beta_i \hat{\mathcal{A}}_i\}$ with $w \equiv 0$ is robustly stable over $\Delta_\beta = \{\beta \in \mathbb{R}^{2^k} : \beta_i \in [0,1], \sum_{i=1}^{2^k} \beta_i = 1\}$. Then from Lemma 7, the PIE system (3) defined by $\{\mathcal{T}, \hat{\mathcal{A}}(\tau)\}$ is robustly stable over Δ_τ. This implies that for any given $\tau \in \Delta_\tau$, the PIE system (3) defined by $\{\mathcal{T}, \hat{\mathcal{A}}(\tau)\}$ is stable and any solution to the the PIE system (3) satisfies $\lim_{t\to\infty} \|\mathcal{T}\mathbf{z}(t)\| \to 0$. Since $\mathcal{T}\mathbf{z}(t) = \mathcal{T}\begin{bmatrix} x(t) \\ \partial_s \phi(t,s) \end{bmatrix} = \begin{bmatrix} x(t) \\ \phi(t,s) \end{bmatrix}$ and $\|x(t)\| \le \left\| \begin{bmatrix} x(t) \\ \phi(t,s) \end{bmatrix} \right\|$, one gets $\lim_{t\to\infty} \|x(t)\| \to 0$ for any given $\tau \in \Delta_\tau$. Thus and the linear TDS (6) is stable for any given $\tau \in \Delta_\tau$. This implies the linear TDS (6) is robustly stable over Δ_τ.

4.4 H_∞ performance

To determine input-output performance of linear TDSs with uncertain delays, we extend the methodology proposed for robust stability to Theorem 5.

Theorem 10. Given positive constants $\tau_i^{[0]}, \tau_i^{[1]}, i = 1, \cdots, k$, suppose there exist a positive scalar γ, a constant real-valued matrix $P \in \mathbb{R}^{m\times m}$, matrix-valued polynomials $Q : [a,b] \to \mathbb{R}^{m\times n}$, $R_0 : [a,b] \to, \mathbb{R}^{n\times n}$, and $R_1, R_2 : [a,b] \times [a,b] \to \mathbb{R}^{n\times n}$, such that $\mathcal{P} := \mathcal{P}\begin{bmatrix} P, & Q \\ Q^T, & \{R_0, R_1, R_2\} \end{bmatrix}$ satisfying $\mathcal{P}^* = \mathcal{P} \succ 0$ and

$$\begin{bmatrix} \mathcal{T}^*\mathcal{P}\hat{\mathcal{A}}_i + \hat{\mathcal{A}}_i^*\mathcal{P}\mathcal{T} & \mathcal{T}^*\mathcal{P}\mathcal{B} & \mathcal{C}^* \\ \mathcal{B}^*\mathcal{P}\mathcal{T} & -\gamma I & D^T \\ \mathcal{C} & D & -\gamma I \end{bmatrix} \prec 0, \quad i = 1, 2, \cdots, 2^k \tag{13}$$

where $n = m \cdot k$, $\hat{\mathcal{A}}_i$ is defined by Eqn (11) and $\mathcal{T}$, $\mathcal{B}, \mathcal{C}, D$ are as defined in Eqn (8). Then for $x_0 \equiv 0$, for any $w \in L_2$, the solution of the linear TDS (6) satisfies $\|z\|_{L_2[0,\infty]} \le \gamma \|w\|_{L_2[0,\infty]}$ for any $\tau \in \Delta_\tau$ where Δ_τ is as defined in Eqn (7).

Proof. The proof is similar to that for Theorem 9.

5. NUMERICAL IMPLEMENTATION

To demonstrate the accuracy and competitive performance of the method, we apply the LPI conditions to several numerical examples. In all cases, the LPI conditions are enforced using the PIETOOLS Matlab interface. We first test robust stability in Examples 1, 2, 3. In Example 3, the robust H_∞ performance is also analyzed. Moreover, it is worth noting that when we set the bounds of delay interval to the same value, Theorem 9 can also be used to compute the maximum upper bound of the delay which the linear TDS is stable. In this case, Example 4 is provided

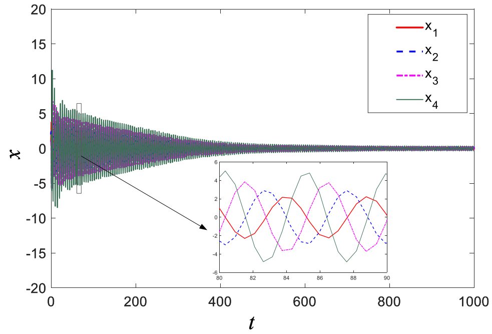

Fig. 1. State response for Example 2

to prove the superiority of our method. Noted that with the increase of number of independent delays, the number of LPIs to be solved increases as well as the computation cost, which is a common issue of the LMI-based method.

Example 1. Consider the following linear TDS.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-\tau)$$

Here τ is a constant delay satisfying $\tau \in [\tau^{[0]}, \tau^{[1]}]$. The robust stability region of this system has been well-studied and the analytical delay interval is known to be $[0.100169, 1.7178]$, as listed in Table 1. It is worth noting that using Theorem 9 we are able to prove robust stability for $\tau \in [0.100169, 1.7178]$ - precisely matching the analytical results.

Table 1. The maximum admissible range of τ

Methods	Delay interval
Seuret and Gouaisbaut (2013)	[0.1003, 1.5406]
Park et al. (2015)(Theorem 1)	[0.1002, 1.5954]
Zeng et al. (2015)	[0.100169, 1.7122]
Li et al. (2017)	[0.100169, 1.7146]
Theorem 9	[0.100169, 1.7178]
the analytical range of τ	[0.100169, 1.7178]

Example 2. Consider the linear system with commensurate delays as follows

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -3 & -5 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{bmatrix} x(t-\tau)$$
$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & -1 & -1 & 0 \end{bmatrix} x(t-2\tau) + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & -2 \end{bmatrix} x(t-3\tau).$$

From Chen (1995), this system is stable for $\tau \le 0.3783$. We get by Theorem 9 the maximum delay interval which can assure the robust stability is $\tau \in \left[1.0 \times 10^{-11}, 0.3786\right]$. Fig 1. plots the state response when $\tau = 0.3786$, which shows the system is stable.

Example 3. Consider the following linear TDS system

$$\dot{x}(t) = \begin{bmatrix} -3.09 & 2.67 \\ -9.80 & 2.83 \end{bmatrix} x(t) + \begin{bmatrix} 0.57 & 0.02 \\ 1.26 & 0.80 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(t)$$
$$z(t) = [-1 \ 0]\, x(t) + 0.5 w(t).$$

When $w(t) = 0$, the exact delay bound is found in Roozbehani and Knospe (2005) to be $\tau \in [0.2319, 0.8609]$. Using Theorem 9, a maximum delay interval is derived as $[0.2319, 0.8609]$ exactly matching the analytical result. When $w(t) \neq 0$, we compute the robust H_∞ performance via Theorem 10 to obtain an L_2 gain bound. When $\tau \in [0.28, 0.6]$, the analytical γ^*_{min} is 4.962. While the result in Roozbehani and Knospe (2005) obtains a bound of $\gamma_{min} = 5.200$, our results based on Theorem 10 yield $\gamma_{min} = 4.9692$, which is much closer to the analytical γ^*_{min}.

Example 4. Consider the linear TDS

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -100 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0.2 \end{bmatrix} x(t-\tau)$$

where τ is a constant delay. The upper bound of delay parameter which keep the system stability are derived by using Theorem 9 with $\tau^{[0]} = \tau^{[1]}$. Table 2 lists the computed upper bounds by different methods showing a larger delay bound using our method than previous results.

Table 2. The maximum admissible range of τ

Methods	Upper bound τ_M
Park et al. (2015)	0.126
Hien and Trinh (2015)	0.577
Zhao et al. (2017)	0.675
Tian et al. (2020)	0.728
Tao et al. (2018)	0.7495
Theorem 9	0.7519

6. CONCLUSION

This paper provides a new approach to robust analysis of a class of linear infinite dimensional systems with polytopic uncertainties. By making use of the recently proposed PIE representation of linear infinite dimensional systems and the PIETOOLs, a more convenient, more adaptable and less conservative method is provided. The effectiveness has been shown through application to the problem of robust analysis of linear systems with uncertain delays. Future work will address the problems of robust control and extension to the problem of time-varying delays.

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