Evolution of non-compact hypersurfaces by inverse mean curvature

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Abstract

We study the evolution of complete non-compact convex hypersurfaces in $\mathbb{R}^{n+1}$ by the inverse mean curvature flow. We establish the long time existence of solutions and provide the characterization of the maximal time of existence in terms of the tangent cone at infinity of the initial hypersurface. Our proof is based on an a'priori pointwise estimate on the mean curvature of the solution from below in terms of the aperture of a supporting cone at infinity. The strict convexity of convex solutions is shown by means of viscosity solutions. Our methods also give an alternative proof of the result by Huisken and Ilmanen in [30] on compact star-shaped solutions, based on maximum principle argument.

1 Introduction

A one-parameter family of immersions $F : M^n \times [0,T] \to \mathbb{R}^{n+1}$ is a smooth complete solution to the *inverse mean curvature flow* (IMCF) in $\mathbb{R}^{n+1}$ if each $M_t := F(\cdot,t)(M^n)$ is a smooth strictly mean convex complete hypersurface satisfying

$$\frac{\partial}{\partial t}F(p,t) = H^{-1}(p,t)\,\nu(p,t) \tag{1.1}$$

where $H(p,t) > 0$ and $\nu(p,t)$ denote the mean curvature and outward unit normal of M_t, pointing opposite to the mean curvature vector.

This flow has been extensively studied for compact hypersurfaces. Gerhardt [20] and Urbas [42] showed compact smooth star-shaped strictly mean convex hypersurface admits a unique smooth solution for all times $t \geq 0$. Moreover, the solution approaches to a homothetically expanding sphere as $t \to \infty$.

For non-starshaped initial data it is well known that singularities may develop (See [27] [39]). This happens when the mean curvature vanishes in some regions which makes the classical flow undefined. However, in [27, 28] Huisken and Ilmanen developed a level set approach to *weak variational solutions* of the flow which allows the solutions to *jump outwards* in possible regions where $H = 0$. Using the weak formulation, they gave the first proof of the *Riemannian Penrose inequality* in General Relativity. One key observation in [28] was the fact the Hawking mass of surface in 3-manifold of nonnegative scalar curvature is monotone under the weak flow, which was first discovered for classical solutions by Geroch [23]. Note that the Riemannian Penrose inequality was shown in more general settings by Bray [3] and Bray-Lee [4] by different methods. Using

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similar techniques, the IMCF has been used to show geometric inequalities in various settings. For instance, see [24, 6] for Minkowski type inequalities, [32] for Penrose inequalities and [34, 14, 19] for Alexandrov-Fenchel type inequalities among other results. Note another important application of the flow by Bray and Neves in [2].

In [30] Huisken and Ilmanen studied the IMCF running from compact star-shaped weakly mean convex initial data. Using star-shapedness and the ultra-fast diffusion character of the flow, they derive a bound from above on H^{-1} for $t > 0$ *which is independent of the initial curvature assumption*. This follows by a *Stampacchia iteration* argument and utilizes the *Michael-Simon Sobolev inequality*. The C^∞ regularity of solutions for $t > 0$ easily follows from the bound on H^{-1}. The estimate in [30] is local *in time*, but necessarily *global in space* as it depends on the area of the initial hypersurface M_0 and uses global integration on M_t. As a consequence, the techniques in [30] cannot be applied directly to the non-compact setting. Note that [33] and [45] provide similar estimates for the IMCF in some negatively curved ambient spaces.

This work addresses *the long time existence* of *non-compact* smooth convex solutions to the IMCF embedded in *Euclidean space* $\mathbb{R}^{n+1}$. While extrinsic geometric flows have been extensively studied in the case of compact hypersurfaces, much remains to be investigated for non-compact cases. The important works by K. Ecker and G. Huisken [16, 17] address the evolution of entire graphs by *mean curvature flow* and establish a surprising result: existence for all times with the only assumption that the initial data M_0 is a *locally Lipschitz* entire graph and *no assumption of the growth at infinity of* M_0. This result is based on priori estimates which are *localized in space*. In addition, the main local bound on the second fundamental form $|A|^2$ of M_t is achieved without any bound assumption on $|A|^2$ on M_0. An open question between experts in the field has been whether the techniques of Ecker and Huisken in [16, 17] can be extended to the fully-nonlinear setting, in particular on entire convex graphs evolving by the *α-Gauss curvature flow* (powers K^α of the Gaussian curvature) and the *inverse mean curvature flow*.

In [10] the second author, jointly with Kyeongsu Choi, Lami Kim and Kiahm Lee, established the long time existence of the *α-Gauss curvature flow* on any strictly convex complete non-compact hypersurface and for any $\alpha > 0$. They showed similar estimates as in [16, 17] which are *localized in space* can be obtained for this flow, however the method is more involved due to the *degenerate* and *fully-nonlinear character* of the Monge-Ampére type of equation involved. However, such localized results are not expected to hold for the *inverse mean curvature flow* where the *ultra-fast diffusion* tends to cause instant propagation from spatial infinity. In fact, one sees certain similarities between the latter two flows and the well known quasilinear models of diffusion on $\mathbb{R}^n$

$$u_t = \operatorname{div}\big(u^{m-1}\,\nabla u\big). \tag{1.2}$$

Exponents $m > 1$ correspond to degenerate diffusion while exponents $m < 0$ to ultra-fast diffusion. We will see in the sequel that under the IMCF the mean curvature H satisfies an equation which is similar to (1.2) with $m = -1$. Our goal is to study this phenomenon and establish the long time existence, an analogue of the results in [16, 17] and [10].

Let us remark existing results on the IMCF of hypersurfaces other than closed ones. In [1], B. Allen investigated non-compact solutions in the hyperbolic space which are graphs on the horosphere. One key estimate in [1] was to show that uniform upper and lower bounds on the mean curvature persist under some initial assumptions. The second author and Huisken [13] studied non-compact solutions in $\mathbb{R}^{n+1}$ under some initial conditions and we will discuss this result later this section. The flow with free boundary, i.e. solutions with Neumann-type boundary condition,

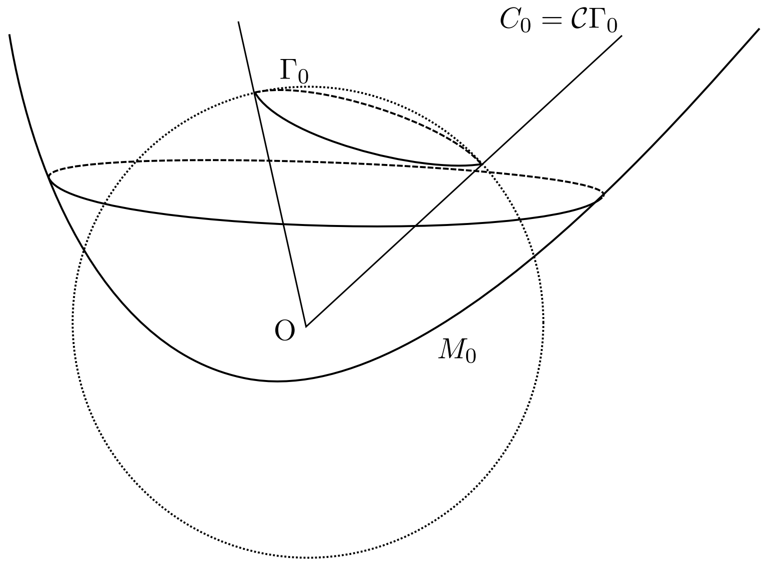

Figure 1: Definition 1.1

has quite extensive literature. We refer the reader to [40, 41],[18] and citations there-in for the mean curvature flow and [35, 36],[31] for the IMCF.

We will next state our main results. The following observation motivates the formulation of our theorem.

Example 1.1 (Conical solutions to IMCF). For a solution Γ_t to the IMCF in $\mathbb{S}^n$, the *family of cones generated by* Γ_t

$$\mathcal{C}\Gamma_t := \{rx \in \mathbb{R}^{n+1} \,:\, r \geq 0,\, x \in \Gamma_t\}$$

is a solution to the IMCF in $\mathbb{R}^{n+1}$ which is smooth except from the origin. When Γ_0 is a compact smooth strictly convex, Gerhardt [22] and Makowski-Scheuer [34] showed the unique existence of solution for time $t \in [0, T)$ with $T < \infty$ and the convergence of solution to an equator as $t \to T$. Moreover, we have explicit formula $T = \ln |\mathbb{S}^{n-1}| - \ln |\Gamma_0|$ by the exponential growth of area in time, (2) in Lemma 2.5. Note also that $\mathcal{C}\Gamma_t$, restricted to the unit ball in $\mathbb{R}^{n+1}$, moves by the IMCF with free boundary on $\mathbb{S}^n$ in the sense of [31].

From Example 1.1 and the ultra-fast diffusive character of the equation, it is reasonable to guess that the behavior of non-compact convex solution and its maximal time of existence is governed by the asymptotics at infinity. For a non-compact convex set $\hat{M}_0$ and the associated hypersurface $M_0 = \partial \hat{M}_0$, we recall the definition of the *blow-down*, so called *the tangent cone at infinity*.

Definition 1.1 (Tangent cone at infinity)**.** *Let $\hat{M}_0 \subset \mathbb{R}^{n+1}$ be a non-compact closed convex set. For a point $p \in \hat{M}_0$, we denote the tangent cone of $\hat{M}_0$ at infinity by*

$$\hat{C}_0 := \cap_{\lambda > 0} \lambda(\hat{M}_0 - p).$$

The definition is independent of $p \in \hat{M}_0$. $C_0 := \partial \hat{C}_0$ is called the tangent cone of $M_0 = \partial \hat{M}_0$ at infinity. $\hat{\Gamma}_0 := \hat{C}_0 \cap \mathbb{S}^n$ and $\Gamma_0 := C_0 \cap \mathbb{S}^n$ are called the links of tangent cones $\hat{C}_0$ and C_0, respectively.

In this work, we say M_0 is convex hypersurface if it is the boundary of a closed convex set with non-empty interior. See Definition 2.1 and subsequent discussion for more details. For convex hypersurface M_0 in $\mathbb{R}^{n+1}$, Lemma 2.3 shows $M_0 = N_0 \times \mathbb{R}^k$ for some convex hypersurface N_0 in $\mathbb{R}^{n+1-k}$ which is homeomorphic to either $\mathbb{S}^{n-k}$ or $\mathbb{R}^{n-k}$. In the first case, the existence of compact IMCF, say N_t, running from N_0 is known in [30] and thus $N_t \times \mathbb{R}^k$ becomes a solution with initial data M_0. Therefore, the essential remaining case is when M_0 is homeomorphic to $\mathbb{R}^n$. We state our existence result.

Theorem 1.2. *Let $\hat{M}_0$ in $\mathbb{R}^{n+1}$ with $n \geq 2$ be a non-compact convex set with interior whose boundary M_0 is $C^{1,1}_{loc}$ and homeomorphic to $\mathbb{R}^n$, and $T = T(M_0)$ be a number defined by*

$$T = \ln |\mathbb{S}^{n-1}| - \ln P(\hat{\Gamma}_0) \in [0, \infty]. \tag{1.3}$$

Here, $\hat{\Gamma}_0$ is the link of tangent cone of $\hat{M}_0$ at infinity, $|\cdot| := \mathcal{H}^{n-1}(\cdot)$, and $P(\hat{\Gamma})$ is the perimeter of $\hat{\Gamma}$ in $\mathbb{S}^n$ defined by

$$P(\hat{\Gamma}_0) = \begin{cases} |\Gamma_0| & \text{if } \hat{\Gamma}_0 \text{ has non-empty interior in } \mathbb{S}^n \\ 2|\Gamma_0| & \text{if } \hat{\Gamma}_0 \text{ has empty interior in } \mathbb{S}^n. \end{cases} \tag{1.4}$$

If $T > 0$, a smooth convex solution to the IMCF, say $M_t = \partial \hat{M}_t$, exists for $0 < t < T$. M_0 is the initial data in the sense that M_t converges to M_0 locally uniformly as $t \to 0$. M_t is strictly convex if and only if $\hat{M}_0$ contains no infinite straight line inside.

Remark 1.2.

(i) $\hat{\Gamma}_0$ can be an arbitrary convex set in $\mathbb{S}^n$ which may possibly have empty interior. (See Definition 2.1 and Lemma 2.2 regarding the definition of convexity in $\mathbb{S}^n$.) In that case the perimeter $P(\hat{\Gamma}_0)$ is the limit of outside areas of decreasing sequence of convex sets with interior in $\mathbb{S}^n$ which approximate $\hat{\Gamma}$. (See Lemma A.10 and Lemma A.11.) According to (1.3), $T = \infty$ when $P(\hat{\Gamma}_0) = 0$ and this happens if only if $\hat{\Gamma}$ has Hausdorff dimension less than $n-1$. Note that the definition of $P(\cdot)$ is not related with the notion of perimeter used in geometric measure theory.

(ii) The tangent cone of M_t at infinity, say Γ_t, also evolves by IMCF in $\mathbb{S}^n$ in some generalized sense (Lemma 4.5), and becomes flat as $t \to T^-$ when $T < \infty$. In Remark 4.2 we further discuss this in connection with the *asymptotic behavior* of M_t as $t \to T$.

(iii) According to (1.3), $T = 0$ when $P(\hat{\Gamma}_0) = |\mathbb{S}^{n-1}|$. In [9], it was shown that $P(\hat{\Gamma}_0) = |\mathbb{S}^{n-1}|$ if and only if $\hat{\Gamma}_0$ is either a hemisphere or a wedge

$$\hat{W}_{\theta_0} = \mathbb{S}^n \cap \big(\{(r\sin\theta, r\cos\theta) : \theta \in [0, \theta_0], \text{ and } r > 0\} \times \mathbb{R}^{n-1}\big) \text{ for some } \theta_0 \in [0, \pi) \tag{1.5}$$

up to an isometry of $\mathbb{S}^n$. We show in Theorem 1.3 no solution exists from such a M_0.

Remark 1.3. Let us emphasize Theorem 1.2 allows $H = 0$ on a possibly non-compact region of M_0. Even in that case, H becomes strictly positive for $t > 0$ and this is due to Theorem 1.4. A similar phenomenon was observed for solutions to the Cauchy problem of the ultra-fast diffusion equation (1.2) with $m < 0$ on $\mathbb{R}^n$. See Remark 4.3 for more details.

Next result asserts that $T = T(M_0)$ in Theorem 1.2 is the maximal time of existence. The result holds not only for the solutions constructed in Theorem 1.2, but applies to arbitrary solutions.

Theorem 1.3. *Let $M_0 = \partial \hat{M}_0$ satisfy the same assumptions as in Theorem 1.2 and $T = T(M_0)$ be given by* (1.3)*. If $T < \infty$, there is no smooth solution N_t which is the boundary of $\hat{N}_t$, $\cap_{t>0} \hat{N}_t = \hat{M}_0$, and existing $0 < t < T + \tau$ some $\tau > 0$. In particular, no solution exists if $T = 0$.*

Non-compact IMCF in $\mathbb{R}^{n+1}$ was first considered by the second author and G. Huisken in [13], where they established the existence and uniqueness of smooth solution to the IMCF, under the assumption that the initial hypersurface M_0 is an entire C^2 graph, $x_{n+1} = u_0(x')$ with $H > 0$, in the following two cases:

(i) M_0 has *super linear growth* at infinity and it is *strictly star-shaped*, that is $H\langle F - x_0, \nu\rangle \geq \delta > 0$ holds, for some $x_0 \in \mathbb{R}^{n+1}$;

(ii) M_0 a *convex graph* satisfying $0 < c_0 \leq H \langle F - x_0, e_{n+1}\rangle \leq C_0 < +\infty$, for some $x_0 \in \mathbb{R}^{n+1}$ and lies between *two round cones of the same aperture*, that is

$$\alpha_0 |x'| \leq u_0(x') \leq \alpha_0 |x'| + k, \qquad \alpha_0 > 0,\ k > 0. \tag{1.6}$$

In the first case, a unique smooth solution exists up to time $T = \infty$, while in the second case a unique smooth convex solution M_t exists for $t \in [0, T)$ where $T > 0$ is the time when the round conical solution from $\{x_{n+1} = \alpha_0 |x'|\}$ becomes flat. In the latter case, the solution M_t lies between two evolving round cones and becomes flat as $t \to T$. To derive a local lower bound of H, a parabolic Moser's iteration argument was used along with a variant of Hardy's inequality, which plays a similar role as the Micheal-Simon Sobolev inequality used in [30].

Theorem 1.2 and the results in [13] show that convex surfaces with *linear growth at infinity* have *critical behavior* in the sense that in this case the *maximal time of existence is finite* and it depends on the *behavior at infinity* of the initial data. However, while the techniques in [13] only treat this critical linear case under the condition (1.6), Theorem 1.2 allows any behavior at infinity. Moreover, the techniques in [13] require to assume that H is globally controlled from below at initial time, namely that $H\langle F - x_0, \nu\rangle \geq \delta > 0$ in the case of *super-linear growth* and $H \langle F - x_0, e_{n+1}\rangle \geq c > 0$ in the case of *linear growth.*

In this work we depart from the techniques in [13],[30], and establish a priori bound on H^{-1} which is *local in time.* For this, we develop a new method based on the maximum principle rather than the integrations used in [13],[30]. Our key estimate roughly says that a convex solution has a global bound on $(H|F|)^{-1}$ as long as a nontrivial convex cone is supporting the solution from outside.

Theorem 1.4. *Let $F : M^n \times [0, T] \to \mathbb{R}^{n+1}$, $n \geq 2$ and $T > 0$, be a compact smooth convex solution to the IMCF and suppose there is $\theta_1 \in (0, \pi/2)$ for which*

$$\langle F, e_{n+1}\rangle \geq \sin\theta_1 \, |F| \quad \text{on } M^n \times [0, T]. \tag{1.7}$$

Then

$$\frac{1}{H|F|} \leq C \left(1 + \frac{1}{t^{1/2}}\right) \quad \text{on} \quad M^n \times [0, T] \tag{1.8}$$

for a constant $C = C(\theta_1) > 0$.

The compactness assumption on M_t above *will only be used to apply maximum principle* and will not affect the application of the estimate in proving of our non-compact result, Theorem 1.2, as we will approximate non-compact solutions by compact ones. Note the estimate does not depend on initial bound on $(H|F|)^{-1}$, which will allow initial data with flat regions as described in Remark 1.3. Moreover, the new method developed while showing Theorem 1.4 leads us to a new proof of Theorem 1.1 in [30], the H^{-1} estimate for compact, star-shaped solutions. This is included in Theorem A.5 in the appendix. In fact, one expects that similar estimates as in Theorem A.5 can be possibly derived for the IMCF in other ambient spaces, including some positively curved spaces or asymptotically flat spaces, using this new method and this generalizes the results of [30, 33, 45]. See in [29] for a consequence of such an estimate when this is shown in asymptotically flat ambient spaces.

Remark 1.4. Recently, the first author and P.-K. Hung in [9] addressed the IMCF on convex solutions allowing singularities on M_0. Using Theorem 1.4 as a key ingredient, [9] shows the tangent cone obtained after blowing-up at a singularity point evolves by the IMCF. As a corollary, one can consider an arbitrary non-compact convex hypersurface M_0 in Theorem 1.2 and obtain the following necessary and sufficient condition for the existence of a smooth solution: *for an arbitrary non-compact convex M_0 with $T(M_0) > 0$, there is a smooth solution if and only if M_0 has density one everywhere. i.e.* $\Theta_0(p) = \lim_{r\to 0} \frac{|B_r(p)\cap M_0|}{\omega_n r^n} = 1$ *for all $p \in M_0$.* See [9] for more details.

A brief outline of this paper is as follows: In Section 2, we introduce basic notation, evolution equations of basic geometric quantities, and prove some useful identities. Section 3 is devoted to the proof of main a priori estimate Theorem 1.4. In Section 4, we prove the long time existence of solution (Theorem 1.2 and Theorem 1.3) via an approximation argument. Here, the passage to a limit relies on the estimate Theorem 1.4. In Appendix A.1, we prove the convexity of solution is preserved and show the solution becomes strictly convex immediately unless the lowest principle curvature λ_1 is zero everywhere. This will be shown for the solutions to the IMCF in space forms as this adds no difficulty in the proof but could be useful in other application. In Appendix A.2, we give an alternative proof of H^{-1} estimate shown in [30] using a maximum principle argument, showing how the star-shapedness condition can be incorporated in our method. Finally, in Appendix A.3 we show the approximation theorems of convex hypersurfaces in $\mathbb{R}^{n+1}$ and $\mathbb{S}^n$ that are used throughout the paper.

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2 Preliminaries

In this section we present some basic preliminary results. Let us begin by clarifying some notions and simple facts from convex geometry. Convex hypersurfaces are studied in both convex geometry and differential geometry of submanifolds. As a result there are different notions of convexity preferred in different subjects. In this paper, we use the following definition.

Definition 2.1 (Convex hypersurfaces in $\mathbb{R}^{n+1}$ and $\mathbb{S}^n$)**.**

(i) A hypersurface $M_0 \subset \mathbb{R}^{n+1}$ is called convex if it is the boundary of a convex set $\hat{M}_0$ of non-empty interior and $\hat{M}_0 \neq \mathbb{R}^{n+1}$.

(ii) A set $\hat{\Gamma}_0 \subset \mathbb{S}^n$ is convex if for all p and q in $\hat{\Gamma}_0$ at least one minimal geodesic connecting the two points is contained in $\hat{\Gamma}_0$. Γ_0 is a convex hypersurface in $\mathbb{S}^n$ if it is the boundary of a convex set $\hat{\Gamma}_0$ with non-empty interior and $\hat{\Gamma}_0 \neq \mathbb{S}^n$.

(iii) A C^2 convex hypersurface is strictly convex at a given point if the second fundamental form with respect to the inner the normal is positive definite.

There is a useful characterization of convexity in $\mathbb{S}^n$ which is immediate from Definition 2.1.

Lemma 2.2. *A set $\hat{\Gamma}_0 \subset \mathbb{S}^n$ is convex if and only if it is connected and $\mathcal{C}\hat{\Gamma}_0 := \{rx \in \mathbb{R}^{n+1} : r \geq 0,\, x \in \hat{\Gamma}_0\}$ is convex in $\mathbb{R}^{n+1}$.*

We prefer Definition 2.1 to the other one that defines convexity through certain properties of the embedding or immersion since we will deal with convex hypersurfaces of low regularity. These two notions are, however, equivalent under suitable assumptions. For example, if $M_0 \subset \mathbb{R}^{n+1}$ (or $\Gamma_0 \subset \mathbb{S}^n$) is a convex hypersurface, then it is a complete connected embedded submanifold. Furthermore, if M_0 (or Γ_0) is C^2, then the second fundamental form with respect to the inner normal is nonnegative definite. The converse question, namely under what conditions an immersed or embedded hypersurface of nonnegative sectional curvature (or semi-definite second fundamental form) bounds a convex set, has a long history and has been answered, for instance, by Hadamard [25], Sacksteder-van Heijenoort [38, 26], H. Wu [44], do Carmo-Warner [8], Makowski-Scheuer [34] under different assumptions. We refer the reader to the results and references cited in these papers.

The following simple observation will be used throughout this paper.

Lemma 2.3. *Let $M_0 = \partial \hat{M}_0$ be the boundary of a closed convex set with interior $\hat{M}_0 \subset \mathbb{R}^{n+1}$, that is M_0 is a convex hypersurface in $\mathbb{R}^{n+1}$. Then either $M_0 = \mathbb{R}^n$ or $M_0 = \mathbb{R}^k \times N_0$, for some $0 \leq k < n$ and $N_0 = \partial \hat{N}_0$ where $\hat{N}_0 \subset \mathbb{R}^{n+1-k}$ is a closed convex set with interior which contains no infinite line. Moreover, such N_0 is either homeomorphic to $\mathbb{S}^{n-k}$ or $\mathbb{R}^{n-k}$.*

Proof. If $\hat{M}_0$ contains an infinite straight line, then $\hat{M}_0$ splits off in the direction of the line by the following elementary argument: suppose a line $\{te_{n+1} \in \mathbb{R}^{n+1} : t \in \mathbb{R}\}$ is contained in $\hat{M}_0$ and let us denote its cross-sections $\hat{\Omega}_l \times \{l\} := \hat{M}_0 \cap \{x_{n+1} = l\}$ for $l \in \mathbb{R}$. By convexity, for any $0 \leq t_1 < t_2$ the set $\frac{t_2 - t_1}{t_2}\hat{\Omega}_0 \times \{t_1\}$ is contained in $\hat{M}_0 \cap \{x_{n+1} = t_1\}$. By taking $t_2 \to \infty$ while fixing t_1, we have $\hat{\Omega}_0 \times \{t_1\} \subset \hat{M}_0 \cap \{x_{n+1} = t_1\}$. We can do a similar argument for $t_2 < t_1 \leq 0$ and thus $\hat{\Omega}_0 \times \mathbb{R} \subset \hat{M}_0$. Similarly, we can do the same argument for all other sections $\hat{\Omega}_l \times \{l\}$ and obtain that $\Omega_l \times \mathbb{R} \subset \hat{M}_0$. Therefore $\hat{\Omega}_{l_1} = \hat{\Omega}_{l_2}$ for all $l_1 \neq l_2$ and $\hat{M}_0 = \hat{\Omega}_0 \times \mathbb{R}$. By repeating this splitting, we conclude that $\hat{M}_0 = \hat{N}_0 \times \mathbb{R}^k$, for some $k \geq 0$ where $\hat{N}_0$ does not contain any infinite lines. In this case, a classical simple result in convex geometry (see for instance Lemma 1 in [44]), implies that $\partial \hat{N}_0$ is homeomorphic to either $\mathbb{S}^{n-k}$ or $\mathbb{R}^{n-k}$.

□

We next derive some properties of smooth solutions to IMCF. Let $\nabla := \nabla^{g(t)}$ and $\Delta := \Delta_{g(t)}$ denote the connection and Laplacian on M^n with respect to the induced metric $g_{ij}(t) = \langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j} \rangle$. Recall that on a local system of coordinates $\{x^i\}$ on M^n,

$$\frac{\partial^2 F}{\partial x^i \partial x^j} = -h_{ij}\nu + \Gamma^k_{ij}\frac{\partial F}{\partial x^k} \qquad \text{and} \qquad \left\langle \frac{\partial F}{\partial x^j}, \frac{\partial \nu}{\partial x^i} \right\rangle = h_{ij} \tag{2.1}$$

where ν denotes the outer unit normal. We also define the operator

$$\Box := \left(\partial_t - \frac{1}{H^2}\Delta\right)$$

and use it frequently as this is the linearized operator of the IMCF. The IMCF or generally curvature flows of homogeneous degree -1, have the following scaling property which we will frequently use:

Lemma 2.4 (Scaling of IMCF)**.** *If $M_t^n \subset \mathbb{R}^{n+1}$ is a solution to the IMCF, then $\tilde{M}_t^n = \lambda\, M_t^n$ is again a solution for $\lambda > 0$.*

Lemma 2.5 (Huisken, Ilmanen [30])**.** *Any smooth solution of the IMCF* (1.1) *in $\mathbb{R}^{n+1}$ satisfies*

(1) $\partial_t g_{ij} = \frac{2}{H}\, h_{ij}$

(2) $\partial_t d\mu = d\mu$, *where $d\mu$ is the volume form induced from g_{ij}*

(3) $\partial_t \nu = -\nabla H^{-1} = \frac{1}{H^2}\,\nabla H$

(4) $\big(\partial_t - \frac{1}{H^2}\Delta\big)h_{ij} = -\frac{2}{H^3}\nabla_i H \nabla_j H + \frac{|A|^2}{H^2}\, h_{ij}$

(5) $\partial_t H = \nabla_i\big(\frac{1}{H^2}\,\nabla_i H\big) - \frac{|A|^2}{H} = \frac{1}{H^2}\Delta H - \frac{2}{H^3}|\nabla H|^2 - \frac{|A|^2}{H}$

(6) $\big(\partial_t - \frac{1}{H^2}\Delta\big)H^{-1} = \frac{|A|^2}{H^2}\, H^{-1}$

(7) $\big(\partial_t - \frac{1}{H^2}\Delta\big)\langle F - x_0, \nu\rangle = \frac{|A|^2}{H^2}\, \langle F - x_0, \nu\rangle$.

Remark 2.1. If the ambient space is not $\mathbb{R}^{n+1}$, then the evolution equations of g_{ij}, $d\mu$, and ν remain the same as in $\mathbb{R}^{n+1}$, but the evolution of curvature h_{ij} is different and complicated. On a space form of sectional curvature K, the formula hugely simplifies becoming

$$\partial_t h_{ij} = \frac{1}{H^2}\Delta h_{ij} + \frac{|A|^2}{H^2} h_{ij} - \frac{2}{H^3}\nabla_i H \nabla_j H - \frac{nKh_{ij}}{H^2} \tag{2.2}$$

(See Chapter 2 in [21].) In this paper we will mostly focus on the flow in Euclidean space and we will only use (2.2) in Appendix A.1.

Using Lemma 2.5 one can easily deduce the following formulas.

Lemma 2.6. *For a fixed vector ω in $\mathbb{R}^{n+1}$, the smooth solutions to the IMCF* (1.1) *in $\mathbb{R}^{n+1}$ satisfy*

(1) $\big(\partial_t - \frac{1}{H^2}\Delta\big)|F - x_0|^2 = -\frac{2n}{H^2} + \frac{4}{H}\langle F - x_0, \nu\rangle$

(2) $\big(\partial_t - \frac{1}{H^2}\Delta\big)\langle \omega, \nu\rangle = \frac{|A|^2}{H^2}\langle \omega, \nu\rangle$

(3) $\big(\partial_t - \frac{1}{H^2}\Delta\big)\langle \omega, F - x_0\rangle = \frac{2}{H}\langle \omega, \nu\rangle$.

Proof. By (2.1) we have

$$\Delta F = g^{ij}(\partial^2_{ij} F - \Gamma^k_{ij} F_k) = g^{ij}(-h_{ij}\nu + \Gamma^k_{ij} F_k - \Gamma^k_{ij} F_k) = -H\,\nu.$$

which combined with $\partial_t F = H^{-1}\nu$ implies (3). Next,

$$\Delta|F - x_0|^2 = 2\langle \Delta F, F - x_0\rangle + 2\langle \nabla F, \nabla F\rangle = -2H\langle \nu, F - x_0\rangle + 2n$$

implies (1). Finally,

$$\begin{aligned}\Delta\nu &= g^{ij}(\partial^2_{ij}\nu - \Gamma^k_{ij}\partial_k\nu) = g^{ij}(\partial_j(h^k_i F_k) - \Gamma^k_{ij}h^l_k F_l)\\ &= g^{ij}((\partial_j h^k_i)F_k - h^k_i h_{jk}\nu + \Gamma^l_{jk}h^k_i F_l - \Gamma^k_{ij}h^l_k F_l)\\ &= -|A|^2\nu + g^{ij}\nabla_j h^k_i F_k = -|A|^2\nu + \nabla H\end{aligned}$$

where we used the Codazzi identity in the last equation. This implies (2). □

The following simple lemma, which commonly appears in Pogorelov type computations, will be useful in the sequel when we compute the evolution of products.

Lemma 2.7. *For any C^2 functions $f_i(p,t)$, $i = 1, \ldots, m$, denote*

$$w := f_1^{\alpha_1} f_2^{\alpha_2} \ldots f_m^{\alpha_m}.$$

Then on the region where $w \neq 0$, we have

$$\Big(\partial_t - \frac{1}{H^2}\Delta\Big)\ln|w| = \frac{(\partial_t - H^{-2}\Delta)w}{w} + \frac{1}{H^2}\frac{|\nabla w|^2}{w^2} = \sum_{i=1}^m \alpha_i\left(\frac{(\partial_t - H^{-2}\Delta)f_i}{f_i} + \frac{1}{H^2}\frac{|\nabla f_i|^2}{f_i^2}\right). \quad (2.3)$$

Proof. The lemma simply follows from

$$\Big(\partial_t - \frac{1}{H^2}\Delta\Big)\ln|f| = \frac{(\partial_t - H^{-2}\Delta)f}{f} + \frac{1}{H^2}\frac{|\nabla f|^2}{f^2}. \quad (2.4)$$

□

Next two lemmas are straightforward computations and we leave their proofs for readers.

Lemma 2.8. *For any two C^2 functions f, g defined on $M^n \times (0,T)$ and any C^2 function $\psi : \mathbb{R} \to \mathbb{R}$,*

$$\Box(fg) = (\Box f)g + f(\Box g) - \frac{2}{H^2}\langle\nabla f, \nabla g\rangle$$

and

$$\Box\psi(f) = \psi'(f)\Box f - \frac{\psi''(f)}{H^2}|\nabla f|^2$$

where $\Box := (\partial_t - H^{-2}\Delta)$.

Lemma 2.9. *If a C^2 function f is defined on a solution M_t of the IMCF and satisfies*

$$\left(\frac{\partial}{\partial t} - \frac{1}{H^2}\Delta\right) f = \frac{|A|^2}{H^2} f$$

then for any fixed $\beta \neq 0$ we have

$$\left(\frac{\partial}{\partial t} - \frac{1}{H^2}\Delta\right) f^\beta = \beta\frac{|A|^2}{H^2} f^\beta - \frac{\beta(\beta-1)}{\beta^2}\frac{|\nabla f^\beta|^2}{H^2 f^\beta}.$$

For instance, H^{-1}, $\langle\omega,\nu\rangle$ and $\langle F - x_0, \nu\rangle$ are examples of such a function f.

We finish with the following local estimate which is an easy consequence of Proposition 2.11 in [13]. Here B_r denotes an extrinsic ball of radius $r > 0$ in $\mathbb{R}^{n+1}$.

Proposition 2.10 (Proposition 2.11 [13])**.** *For a solution M_t, $t \in [0,T]$, to the IMCF, there is a constant $C_n > 0$ such that*

$$\sup_{M_t \cap B_{r/2}} H \le C_n \max\big(\sup_{M_0\cap B_r} H,\, r^{-1}\big).$$

Proof. Although this proposition is proven in [13] we include below its proof for completeness. For fixed $r > 0$, let $\eta := (r^2 - |F|^2)_+^2$ be a cut-off function defined in the ambient space. Using Lemma 2.6 and 2.8,

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \frac{1}{H^2}\Delta\right)\eta &= -2(r^2 - |F|^2)_+\left[\left(\frac{\partial}{\partial t} - \frac{1}{H^2}\Delta\right)|F|^2\right] - \frac{2}{H^2}|\nabla(r^2-|F|^2)_+|^2\\ &= -2\eta^{1/2}\left(-\frac{2n}{H^2} + \frac{4}{H}\langle F,\nu\rangle\right) - \frac{8}{H^2}|F^T|^2\end{aligned}$$

and

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \frac{1}{H^2}\Delta\right)\eta H &= \frac{4n\eta^{1/2}}{H} - 8\eta^{1/2}\langle F,\nu\rangle - \frac{8}{H}|F^T|^2 + \eta\left(-2\frac{|\nabla H|^2}{H^3} - \frac{|A|^2}{H}\right) - \frac{2}{H^2}\langle\nabla\eta,\nabla H\rangle\\ &\le \frac{4n\eta^{1/2}}{H} + 8\eta^{1/2}r - \frac{2}{H^3}\langle\nabla(\eta H),\nabla H\rangle - \frac{\eta H}{n}.\end{aligned}$$

In the last inequality, we used $|A|^2 \ge n^{-1}H^2$ and $|\langle F,\nu\rangle| \le |F| \le r$. Let $m(t)$ be the maximum of ηH on M_t. Then the above inequality implies

$$\partial_t m(t) \le 4n\frac{\|\eta\|_\infty^{3/2}}{m(t)} + 8\|\eta\|_\infty^{1/2}r - \frac{m(t)}{n} \le 4n\frac{r^6}{m(t)} + 8r^3 - \frac{1}{n}m(t).$$

Thus $m(t)$ will decrease if

$$r^6\frac{4n}{m(t)} - \frac{m(t)}{n} + 8r^3 \le 0 \iff m^2(t) - 8nr^3m(t) - 4n^2r^6 \ge 0 \iff m(t) \ge (4+2\sqrt{5})n\,r^3.$$

Therefore, $m(t) \le \max\big(m(0), (4+2\sqrt{5})n\,r^3\big)$. The proposition is implied since

$$\begin{aligned}\sup_{M_t\cap B_{r/2}}(r^2 - (r/2)^2)^2 H &\le \sup \eta H \le \max\big(m(0), (4+2\sqrt{5})n\,r^3\big)\\ &\le \max\big(\sup_{M_0\cap B_r} r^4 H, (4+2\sqrt{5})n\,r^3\big).\end{aligned}$$

□

3 L^∞ bound of $(H|F|)^{-1}$

Before giving the proof of Theorem 1.4, lets us introduce some notations. We consider spherical coordinates with respect to the origin in $\mathbb{R}^{n+1}$, namely

$$x = (x_1,\dots,x_{n+1}) = (r\omega\sin\theta, r\cos\theta) \quad \text{with } r \ge 0,\ \omega \in \mathbb{S}^{n-1},\ \text{and } \theta\in[0,\pi]$$

which are smoothly well-defined away from x_{n+1}-axis. We will also denote by $\bar\nabla$ and ∇ metric-induced connections on $(\mathbb{R}^{n+1}, g_{\text{euc}})$ and (M^n, F^*g_{euc}), respectively. Before the proof, we need the evolution equation of θ, defined in the ambient space as follows:

Definition 3.1. We define

$$\theta : \mathbb{R}^{n+1} \setminus \{0\} \to [0, \pi] \quad \text{by} \quad \theta(x) := \arccos\left(\frac{\langle x, e_{n+1}\rangle}{|x|}\right) \tag{3.1}$$

and

$$r : \mathbb{R}^{n+1} \to [0, \infty) \quad \text{by } r(x) := |x|.$$

Moreover, we define smooth unit orthogonal vector fields

$$e_\theta(x) = e_\theta(x', x_{n+1}) := \frac{1}{|x|}\frac{\partial}{\partial \theta} = \left(\frac{x'}{|x|}\frac{\cos\theta}{\sin\theta}, -\sin\theta\right) \quad \text{on } \mathbb{R}^{n+1} \setminus \{x_{n+1} - \text{axis}\}$$

and

$$e_r(x) := \frac{\partial}{\partial r} = \frac{x}{|x|} \quad \text{on } \mathbb{R}^{n+1} \setminus \{0\}.$$

Though θ is not smooth at the points on the x_{n+1}-axis, note that θ^2, $\cos\theta$, and $\sec\theta$ are all smooth on $\{x_{n+1} > 0\}$.

Note θ represents a scaled distance from the north pole measured in the sphere. The first negative term on the right hand side of (3.2) will justify the use of θ in the estimate. Compare (3.2) with (1) in Lemma 2.6.

Lemma 3.2. *On the region* $\{\theta \neq 0, \pi\} \cap \{|x| \neq 0\}$,

$$(\partial_t - \frac{1}{H^2}\Delta)\theta = -\frac{1}{H^2 r^2}\left(\frac{n - |\nabla r|^2}{\tan\theta}\right) + \frac{1}{H^2}\frac{|\nabla\theta|^2}{\tan\theta} + \frac{2}{H^2}\langle\frac{\nabla r}{r}, \nabla\theta\rangle + \frac{2}{H}\langle\nu, \bar\nabla\theta\rangle. \tag{3.2}$$

Proof. Consider a spherical coordinate chart

$$(r, \theta, (w^\alpha)_{\alpha=1\dots n-1}) \quad \text{with } r > 0,\ \theta \in (0, \pi),\ (w^\alpha) \in \mathbb{S}^{n-1}$$

around a point $\{\theta \neq 0, \pi\} \cap \{|x| \neq 0\}$ in $\mathbb{R}^{n+1}$, where w^α is a coordinate chart of $\mathbb{S}^{n-1}$. On this chart,

$$g_{\text{euc}} = dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta \sigma_{\alpha\beta}\, dw^\alpha dw^\beta. \tag{3.3}$$

Also note that

$$\operatorname{grad}\theta = \frac{1}{r^2}\frac{\partial}{\partial\theta} = \frac{1}{r}e_\theta \quad \text{and} \quad \operatorname{grad} r = \frac{\partial}{\partial r} = e_r \quad \text{on } (\mathbb{R}^{n+1}, g_{\text{euc}}). \tag{3.4}$$

At a given $p \in M^n$ with $\{\theta \neq 0, \pi\} \cap \{|x| \neq 0\}$, let us choose a geodesic normal coordinate of M^n, say $\{y^i\}_{i=1}^n$. In this coordinate at this point,

$$\begin{aligned}\Delta\theta &= \sum_i \partial_i\partial_i\theta = \sum_i \frac{\partial}{\partial y^i} d\theta\left(\frac{\partial}{\partial y^i}\right) = \sum_i \frac{\partial}{\partial y^i}\left(\frac{1}{r^2}\langle\frac{\partial}{\partial\theta}, \frac{\partial}{\partial y^i}\rangle\right) \\ &= \sum_i -\frac{2}{r^2}\langle\frac{\partial}{\partial y^i}, e_\theta\rangle\langle\frac{\partial}{\partial y^i}, e_r\rangle + \frac{1}{r^2}\langle\bar\nabla_{\partial_i}\frac{\partial}{\partial\theta}, \frac{\partial}{\partial y^i}\rangle + \frac{1}{r^2}\langle\frac{\partial}{\partial\theta}, -h_{ii}\nu\rangle.\end{aligned}$$

Since $\left(\left(\frac{\partial}{\partial y^i}\right)_{i=1}^n, \nu\right)$ constitutes an orthonormal basis of $T_{F(p)}\mathbb{R}^{n+1}$,

$$\sum_i \langle\frac{\partial}{\partial y^i}, e_\theta\rangle\langle\frac{\partial}{\partial y^i}, e_r\rangle + \langle\nu, e_r\rangle\langle\nu, e_\theta\rangle = \langle e_r, e_\theta\rangle = 0. \tag{3.5}$$

Therefore,

$$\Delta\theta = -\frac{H}{r}\langle\nu, e_\theta\rangle + \frac{2}{r^2}\langle\nu, e_r\rangle\langle\nu.e_\theta\rangle + \frac{1}{r^2}\sum_i\langle\bar\nabla_{\partial_i}\frac{\partial}{\partial\theta}, \frac{\partial}{\partial y^i}\rangle. \tag{3.6}$$

Claim 3.1.

$$\sum_{i=1}^{n}\langle\bar\nabla_{\partial_i}\frac{\partial}{\partial\theta},\frac{\partial}{\partial y^i}\rangle=\frac{\cos\theta}{\sin\theta}\left(n-(1-\langle\nu,e_r\rangle^2)-(1-\langle\nu,e_\theta\rangle^2)\right). \tag{3.7}$$

Proof of Claim 3.1 . By computing the Christoffel symbols from the metric (3.3), we get:

$$\bar\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\theta}=-r\frac{\partial}{\partial r},\quad \bar\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial\theta}=\frac{1}{r}\frac{\partial}{\partial\theta},\quad \bar\nabla_{\frac{\partial}{\partial w^\alpha}}\frac{\partial}{\partial\theta}=\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial w^\alpha}. \tag{3.8}$$

Suppose $\partial_i=\partial_{y^i}=a_\theta\partial_\theta+a_r\partial_r+\sum_\alpha a_\alpha\partial_{w^\alpha}$. Then $\bar\nabla_{\partial_i}\frac{\partial}{\partial\theta}=-ra_\theta\partial_r+\frac{a_r}{r}\partial_\theta+\sum_\alpha a_\alpha\frac{\cos\theta}{\sin\theta}\partial_{w^\alpha}$ and hence

$$\begin{aligned}\langle\bar\nabla_{\partial_i}\frac{\partial}{\partial\theta},\frac{\partial}{\partial y^i}\rangle&=-ra_\theta a_r+ra_\theta a_r+r^2\sin^2\theta\frac{\cos\theta}{\sin\theta}a_\alpha a_\beta\sigma^{\alpha\beta}\\&=\frac{\cos\theta}{\sin\theta}\left[\left|\frac{\partial}{\partial y^i}\right|^2-\left\langle\frac{\partial}{\partial y^i},e_r\right\rangle^2-\left\langle\frac{\partial}{\partial y^i},e_\theta\right\rangle^2\right].\end{aligned}$$

The claim follows by summing this over i. ☐

Now $\partial_t\theta=d\theta(\partial_tF)=\dfrac{1}{H}\langle\nu,\operatorname{grad}\theta\rangle=\dfrac{\langle\nu,e_\theta\rangle}{rH}$, (3.6) and (3.7) imply

$$(\partial_t-\frac{1}{H^2}\Delta)\theta=\frac{2\langle\nu,e_\theta\rangle}{rH}-\frac{1}{(rH)^2}\left[\frac{\cos\theta}{\sin\theta}[n-(1-\langle\nu,e_r\rangle^2)-(1-\langle\nu,e_\theta\rangle^2)]+2\langle\nu,e_r\rangle\langle\nu,e_\theta\rangle\right].$$

Hence, the lemma follows by using (3.4) and the orthonormality of $\left(\left(\frac{\partial}{\partial y^i}\right)_{i=1}^n,\nu\right)$ in the equation above. ☐

Proof of Theorem 1.4. Using the definition (3.1), our condition (1.7) can be written as $\theta(p,t)\le\pi/2-\theta_1$. Setting $c:=\dfrac{\pi-\theta_1}{\pi-2\theta_1}>1$, we have $c\,\theta\le\frac{\pi}{2}-\frac{\theta_1}{2}<\frac{\pi}{2}$ and $\sec(c\theta)\le2\sec\theta$ for $\theta=\theta(p,t)$ on $t\in[0,T]$.

By lemma 2.8,

$$\begin{aligned}\Box\sec(c\theta)&=c\sec(c\theta)\tan(c\theta)\Box\theta-\frac{1}{H^2}c^2[\sec(c\theta)\tan^2(c\theta)+\sec^3(c\theta)]|\nabla\theta|^2]\\&=\sec(c\theta)\left[c\tan(c\theta)\Box\theta-\frac{c^2}{H^2}(2\tan^2(c\theta)+1)|\nabla\theta|^2\right].\end{aligned}$$

After defining $\varphi:=\sec(c\theta)$, Lemma 3.2 and $\frac{\nabla\varphi}{\varphi}=c\tan(c\theta)\nabla\theta$ imply

$$\begin{aligned}\frac{\Box\varphi}{\varphi}&=-\frac{c}{H^2r^2}\frac{\tan(c\theta)}{\tan\theta}(n-|\nabla r|^2-r^2|\nabla\theta|^2)+\frac{2}{H^2}\langle\frac{\nabla r}{r},\frac{\nabla\varphi}{\varphi}\rangle+\frac{2}{H}\langle\nu,\frac{\bar\nabla\varphi}{\varphi}\rangle\\&\quad-\frac{2}{H^2}\frac{|\nabla\varphi|^2}{\varphi^2}-\frac{1}{H^2}c^2|\nabla\theta|^2\\&\quad\left(\text{since } n-|\nabla r|^2-r^2|\nabla\theta|^2=n-2\ge0\quad\text{and}\quad\frac{\tan(c\theta)}{\tan\theta}\ge c\right)\\&\le-\frac{c^2}{H^2r^2}(n-|\nabla r|^2)-\frac{2}{H^2}\frac{|\nabla\varphi|^2}{\varphi^2}+\frac{2}{H^2}\langle\frac{\nabla r}{r},\frac{\nabla\varphi}{\varphi}\rangle+\frac{2}{H}\langle\nu,\frac{\bar\nabla\varphi}{\varphi}\rangle.\end{aligned}$$

Note that this inequality holds on $\{x_{n+1} > 0\}$, where our solution M_t is located. Let $w := \dfrac{\sec(c\theta)t}{Hr} = \varphi\psi r^{-1}t$ where $\psi := H^{-1}$. Then by Lemma 2.7 and the previous inequality

$$\begin{aligned}
&\frac{\Box w}{w} + \frac{1}{H^2}\frac{|\nabla w|^2}{w^2} \\
=&\Big(\frac{\Box \varphi}{\varphi} + \frac{1}{H^2}\frac{|\nabla \varphi|^2}{\varphi^2}\Big) + \Big(\frac{|A|^2}{H^2} + \frac{1}{H^2}\frac{|\nabla \psi|^2}{\psi^2}\Big) - \frac{1}{2}\Big(\frac{\Box r^2}{r^2} + \frac{1}{H^2}\frac{|\nabla r^2|^2}{r^4}\Big) + \frac{1}{t} \\
=&\Big(\frac{\Box \varphi}{\varphi} + \frac{1}{H^2}\frac{|\nabla \varphi|^2}{\varphi^2}\Big) + \Big(\frac{|A|^2}{H^2} + \frac{1}{H^2}\frac{|\nabla \psi|^2}{\psi^2}\Big) + \Big(\frac{n}{H^2r^2} - \frac{2}{H}\langle \nu, \frac{\bar\nabla r}{r}\rangle - \frac{2}{H^2}\frac{|\nabla r|^2}{r^2}\Big) + \frac{1}{t} \\
\leq& \left[\frac{|A|^2}{H^2} + \frac{1}{t} + \frac{2}{H}\langle \nu, \frac{\bar\nabla \varphi}{\varphi}\rangle - \frac{2}{H}\langle \nu, \frac{\bar\nabla r}{r}\rangle\right] \\
&\quad + \frac{1}{H^2}\left[\frac{|\nabla\psi|^2}{\psi^2} + \frac{n}{r^2} - 2\frac{|\nabla r|^2}{r^2} - c^2\frac{n-|\nabla r|^2}{r^2} - \frac{|\nabla \varphi|^2}{\varphi^2} + 2\langle\frac{\nabla r}{r}, \frac{\nabla\varphi}{\varphi}\rangle\right] \\
=&:(1) + (2).
\end{aligned} \tag{3.9}$$

Suppose a nonzero maximum of $w(p,t)$ on $M^n \times [0,t_1]$ is achieved at (p_0,t_0) with $t_0 \in (0,t_1]$. At this point,

$$0 = \frac{\nabla w}{w} = \frac{\nabla\psi}{\psi} + \frac{\nabla\varphi}{\varphi} - \frac{\nabla r}{r}$$

and therefore

$$\frac{|\nabla\psi|^2}{\psi^2} = \left|\frac{\nabla r}{r} - \frac{\nabla\varphi}{\varphi}\right|^2 = \frac{|\nabla r|^2}{r^2} + \frac{|\nabla\varphi|^2}{\varphi^2} - 2\langle\frac{\nabla r}{r}, \frac{\nabla\varphi}{\varphi}\rangle.$$

At the maximum point, by plugging this into (2) in (3.9), $(2) = -(c^2-1)\dfrac{n-|\nabla r|^2}{H^2r^2}$. Therefore at the maximum point,

$$0 \leq (1) - (c^2-1)\frac{n-|\nabla r|^2}{H^2r^2}.$$

Let us estimate terms in (1). Note that by our choice of $c > 1$,

$$\left|\frac{\bar\nabla\varphi}{\varphi}\right| = |c\tan(c\theta)\bar\nabla\theta| \leq \frac{c}{r}\tan(c\theta) = \frac{1}{r}\sin(c\theta)\sec(c\theta) \leq \frac{2}{r\cos\theta} \leq \frac{2}{r}\frac{1}{\sin\theta_1} = \frac{C}{r}$$

for some $C = C(\theta_1)$. Next, $\frac{|A|^2}{H^2} \leq 1$ from convexity and $|\bar\nabla r| \leq 1$ imply at (p_0,t_0),

$$\begin{aligned}
0 &\leq -(n-|\nabla r|^2)\frac{c-1}{H^2r^2} + \frac{C}{Hr} + 1 + \frac{1}{t_0} \\
&\leq -\frac{c-1}{H^2r^2} + \frac{C}{Hr} + 1 + \frac{1}{t_0} \quad (\text{since } |\nabla r|^2 \leq 1,\ n \geq 2) \\
&\leq -\frac{c-1}{2H^2r^2} + \frac{C}{2(c-1)} + \frac{1}{t_0}.
\end{aligned}$$

Note that $0 < t_0 \leq t_1$ and $1 \leq \varphi \leq C$ on $M \times [0,t_1]$. Multiplication of $(\varphi(p_0,t_0)t_0)^2$ implies

$$w^2(p_0,t_0) = \left(\frac{\varphi t_0}{Hr}\right)^2 \leq Ct_1\,(t_1+1).$$

On any other point $p \in M$ at $t = t_1$,

$$\frac{1}{H^2r^2}(p,t_1) = \left(\frac{w(p,t_1)}{t_1\varphi(p,t_1)}\right)^2 \leq \frac{w^2(p_0,t_0)}{t_1^2\varphi^2(p,t_1)} \leq C\left(1 + \frac{1}{t_1}\right).$$

We used $\varphi \geq 1$ in the last inequality. This finishes the proof of Theorem 1.4. $\Box$

Remark 3.1. If we define $\bar{w} := \varphi\psi r^{-1}$ and follow the rest similarly, we get an estimate which includes the initial bound

$$\frac{1}{H|F|} \leq C \max\left(\sup_{M_0} \frac{1}{H|F|}, 1\right).$$

After the preprint of this work has been posted, M.N. Ivaki pointed that $\varphi(\sec\theta)$ with φ of form (A.12) could replace the use of $\sec(c\theta)$ in the previous proof. If $\varphi(\sec\theta)$ is used, one may avoid computing the evolution of θ as the evolution of $\sec\theta = \langle F, e_{n+1}\rangle/|F|$ can be derived from Lemma 2.6.

4 Long time existence of non-compact solutions

Let us provide a sketch on how we prove Theorem 1.2. Since Theorem 1.4 was shown for compact solutions, we first construct a family of *compact convex* approximating solutions $M_{i,t} = \partial\hat{M}_{i,t}$ which is monotone increasing in i. Each compact expanding solution $M_{i,t}$ exists for all time by [20][42], thus we may define the limit $\hat{M}_t = \lim_{i\to+\infty}\hat{M}_{i,t}$ as a set. We will see, however, that the limit $\hat{M}_t$ is non-trivial only up to time $t < T$ and the proof of Theorem 1.3 will show $\hat{M}_t = \mathbb{R}^{n+1}$ for $t > T$, meaning $M_t = \partial\hat{M}_t$ is empty.

Let us briefly explain where the connection between our solution M_t in Euclidean space and solutions on the sphere is revealed. Recall $\hat{\Gamma}_0$ is the link of the tangent cone of $\hat{M}_0$. For each $\delta > 0$, there is a smooth strictly convex $\Gamma_0^\delta \subset \mathbb{S}^n$ such that $\hat{\Gamma}_0 \subset\subset \hat{\Gamma}_0^\delta$ and $T_\delta := \ln|\mathbb{S}^n| - \ln|\Gamma_0^\delta| > T - \delta$. Here and later, we use $\hat{A} \subset\subset \hat{B}$ to denote $\overline{\hat{A}} \subset \mathrm{int}(B)$. As explained in Example 1.1, Γ_0^δ admits a smooth Γ_t^δ in $\mathbb{S}^n$ which exists up to time T_δ and we will make use of $\mathcal{C}\Gamma_t^\delta$ as a barrier which contains $M_{i,t}$. Indeed, by moving its vertex far away from M_0 initially, we can make $\mathcal{C}\Gamma_t^\delta$ (after the initial translation) contain $M_{i,t}$ up to time T_δ, implying that each $M_{i,t}$ satisfies condition (1.7) in Theorem 1.4 up to time $T - \delta$ for all $\delta > 0$. Theorem 1.4 then leads to an upper bound on $(|F|\,H)^{-1}$ showing that the IMCF on $M_{i,t}$ is locally uniformly parabolic and the rest is straightforward.

The proof of Theorem 1.2 consists of four steps. First, we define our solution $M_t = \partial\hat{M}_t$ as a limit of compact approximating solutions. Second, we show that M_t is a smooth non-trivial solution for $t \in (0, T)$ using the idea mentioned above. Third, we show that M_t locally converges to M_0, as $t \to 0$. Finally, we show the strict convexity assertion of Theorem 1.2. Since the proof is long, we will address some proofs of technical lemmas in Appendix.

Proof of Theorem 1.2. Suppose $T = T(M_0)$ satisfies $0 < T \leq \infty$ following the assumption of theorem. We may also assume without loss of generality that $\hat{M}_0$ *does not contain any infinite straight lines.* Let us justify this claim. By Lemma 2.3, $M_0 = \mathbb{R}^k \times N_0$ for some non-compact convex hypersurface $N_0 \subset \mathbb{R}^{n+1-k}$ and it is homeomorphic to $\mathbb{R}^{n-k}$ as M_0 is homeomorphic to $\mathbb{R}^n$. Note $0 \leq k \leq n-2$ since $k = n-1$ or n would imply $\hat{\Gamma}_{0,\hat{M}_0}$ is a wedge in (1.5) (when $k = n-1$) or a hemisphere (when $k = n$), and $T = 0$ in both cases. If $\hat{\Gamma}_{0,\hat{M}_0} \subset \mathbb{S}^n$ and $\hat{\Gamma}_{0,\hat{N}_0} \subset \mathbb{S}^{n-k}$ are the links of the tangent cones of $\hat{M}_0$ and $\hat{N}_0$, then we have

$$T(M_0) = \ln|\mathbb{S}^{n-1}| - \ln P(\hat{\Gamma}_{0,\hat{M}_0}) = \ln|\mathbb{S}^{n-k-1}| - \ln P(\hat{\Gamma}_{0,\hat{N}_0}) = T(N_0). \tag{4.1}$$

In conclusion, $N_0^{n-k} = \partial\hat{N}_0$ is a non-compact convex hypersurface in $\mathbb{R}^{(n-k)+1}$ (with $n-k \geq 2$) homeomorphic to $\mathbb{R}^{n-k}$, i.e. N_0 satisfies the assumption of Theorem 1.2 and $\hat{N}_0$ has no infinite straight line inside. Since the existence of a solution N_t of IMCF in $\mathbb{R}^{n-k+1}$ with initial data N_0 implies that $M_t := \mathbb{R}^k \times N_t$ is a solution of IMCF in $\mathbb{R}^{n+1}$ with initial data M_0, we conclude that it

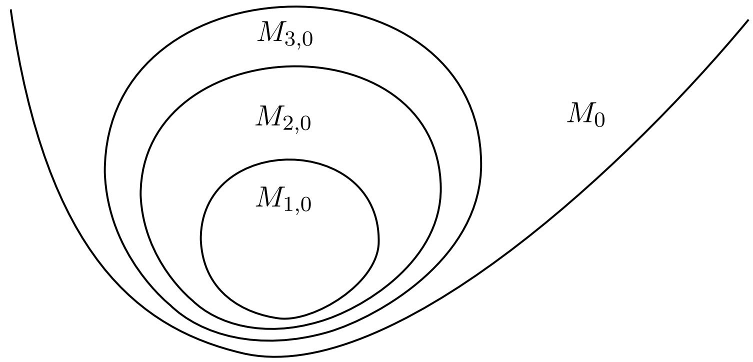

Figure 2: Approximation of M_0

would be sufficient, as we claimed, to assume that $\hat{M}_0$ does not contain any infinite straight lines. In this case, the *link* $\hat{\Gamma}_0$ of the tangent cone of $\hat{M}_0$ *does not contain any antipodal points* and is *compactly contained in some open hemisphere* $H(v_0) := \{p \in \mathbb{S}^n : \langle p, v_0 \rangle > 0\}$ for some $v_0 \in \mathbb{S}^n$ (see Lemma 3.8 in [34]).

Next, we create hypersurfaces $M_{i,0} = \partial \hat{M}_{i,0}$ with certain properties which approximate M_0 from inside. A sequence of sets $\hat{A}_i$ is called strictly increasing if $\overline{\hat{A}_i} \subset \text{int}(\hat{A}_{i+1})$ (and we write $A_i \subset\subset A_{i+1}$). Let us denote the ball of radius r centered at p by $B_r(p)$. Setting $\hat{\Sigma}_{i,0} := B_i(0) \cap \hat{M}_0$, $\hat{\Sigma}_{i,0}$ is a (weakly) increasing sequence of convex sets. By Lemma A.6, each $\Sigma_{i,0}$ admits a strictly increasing approximation by compact smooth strictly convex hypersurfaces $\Sigma_{i,j} = \partial \hat{\Sigma}_{i,j}$. Furthermore, we may assume $d_H(\hat{\Sigma}_{i,0}, \hat{\Sigma}_{i,j}) \leq j^{-1}$, where d_H is the Hausdorff metric (A.17). Then a diagonal argument gives a sequence $n_i \to \infty$ so that $\hat{M}_{i,0} := \hat{\Sigma}_{i,n_i}$ strictly increases to $\hat{M}_0$. By [42] and [22], each $M_{i,0}$ admits a unique smooth IMCF $M_{i,t} = \partial \hat{M}_{i,t}$, which exists for all $t \in [0, \infty)$. Note $M_{i,t}$ is smooth strictly convex hypersurface (see Remark A.1) which is strictly monotone increasing in i by the comparison principle. We use the sequence of solutions $M_{i,t}$ to define next the notion of *innermost candidate* solution from $\hat{M}_0$.

Definition 4.1. For a convex set $\hat{M}_0$ with non-empty interior, let $\hat{M}_{i,0}$ be a sequence of compact sets with smooth strictly convex boundary which strictly increases to $\hat{M}_0$. We define the *innermost candidate* solution from $\hat{M}_0$ as

$$\hat{M}_t := \overline{\cup_{i=1}^{\infty} \hat{M}_{i,t}}, \quad \text{for } t \in [0, \infty) \tag{4.2}$$

where $M_{i,t} = \partial \hat{M}_{i,t}$ is a sequence of compact smooth strictly convex solutions with initial data $M_{i,0}$. $\hat{M}_t$ is convex by definition and the definition does not depend on $\hat{M}_{i,t}$ (See Remark 4.1.)

It remains to show that $M_t := \partial \hat{M}_t$ *defines a non-empty strictly convex smooth solution to the flow, for* $t \in (0, T(M_0))$*, and converges to* M_0 *locally uniformly as* $t \to 0^+$. We need an approximation lemma for $\hat{\Gamma}_0$, the link of the tangent cone of $\hat{M}_0$ at infinity.

Claim 4.1. *Let* $\hat{\Gamma}_0 \subset \mathbb{S}^n$ *be a closed convex set contained in an open hemisphere. Then there is a sequence of smooth, strictly convex hypersurfaces* $\Gamma_0^j = \partial \hat{\Gamma}_0^j$ *in the open hemisphere which strictly decreases and* $\cap_j \hat{\Gamma}_0^j = \hat{\Gamma}_0$*. For every such sequence,* $|\Gamma_0^j| = P(\hat{\Gamma}_0^j) \to P(\hat{\Gamma}_0)$*.*

Proof of Claim 4.1. This is a direct consequence of Lemma A.9 and Lemma A.10. Since their proofs require some other results from convex geometry, we prove them separately in the appendix. ☐

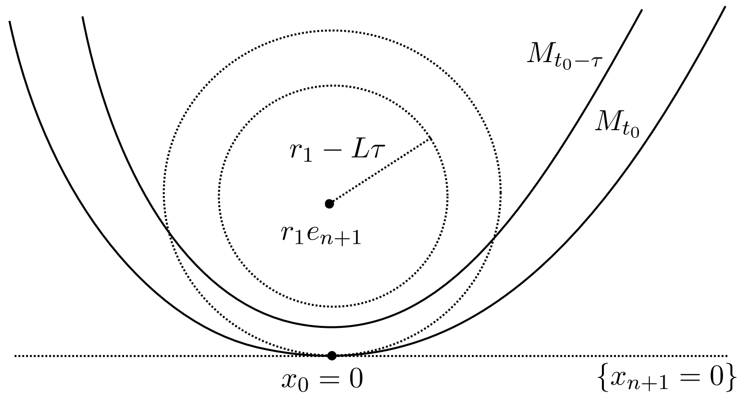

Figure 3: Outside hyperplane and inside sphere barriers around $x_0 \in M_{t_0}$

Now fix an arbitrary time $t_0 \in (0, T)$. By the claim, we may find a j such that $T^j := \ln|\mathbb{S}^{n-1}| - \ln|\Gamma_0^j| > t_0$. Since $\hat{\Gamma}_0$ is contained in the interior of $\hat{\Gamma}_0^j$, we may find a vector $v_j' \in \mathbb{R}^{n+1}$ so that $\hat{M}_0 \subset \mathcal{C}\hat{\Gamma}_0^j + v_j'$. By the definition of $\hat{M}_0$, it then follows that $\hat{M}_{i,0} \subset \hat{M}_0 \subset \mathcal{C}\hat{\Gamma}_0^j + v_j'$, for all i. Theorem 1.4 [34] guarantees the existence of a smooth strictly convex IMCF solution Γ_t^j in $\mathbb{S}^n$ with initial data Γ_0^j, for $t \in [0, T^j)$. Note $\mathcal{C}\Gamma_t^j$ is an IMCF which is smooth away from the origin. Unless the convex cone $\mathcal{C}\Gamma_t^j$ is flat, the origin can not be touched from inside by a C^2 hypersurface. Therefore a version of comparison principle can be justified and we obtain $\hat{M}_{i,t} \subset \mathcal{C}\hat{\Gamma}_t^j + v_j'$ for $t \in [0, T^j)$. Since Γ_t^j is a strictly convex solution which converges to an equator, we may find a direction $\omega_0 \in \mathbb{S}^n$ and small $\delta_0 > 0$ such that

$$\langle F - v_j', \omega_0 \rangle \geq (\sin \delta_0)\, |F - v_j'|, \qquad \text{for } t \in [0, t_0] \text{ on } M_{i,t}.$$

By Theorem 1.4, we have a uniform bound

$$(H|F - v_j'|)^{-1} \leq C(1 + t^{-1/2}) \quad \text{for } M_{i,t} \text{ on } t \in (0, t_0]. \tag{4.3}$$

The barrier $\mathcal{C}\hat{\Gamma}_t^j + v_j'$ also shows $\hat{M}_t \neq \mathbb{R}^{n+1}$ and hence M_t *is non-empty for* $t \in [0, t_0]$.

Let us next prove that M_t, for $t \in (0, T(M_0))$, is a smooth solution of IMCF. First, note that $\hat{M}_0 \subset \operatorname{int}(\hat{M}_t)$ for $t > 0$: indeed, since M_0 is locally in $C^{1,1}$, for every point $p \in M_0$, there is an inscribed sphere at p whose largest radius depends on the local $C^{1,1}$ norm of M_0. By the comparison principle between the sphere solution running from this inscribed sphere and $M_{i,t}$, we conclude that $p \in \operatorname{int}(\hat{M}_t)$ for all $t > 0$. (In practice, if $N_t = \partial \hat{N}_t$ is a smooth compact solution containing the origin and $\hat{N}_0 \subset \hat{M}_\tau$ some $\tau \geq 0$, then the comparison and Lemma 2.4 imply $(1-\epsilon)\hat{N}_t \subset \hat{M}_{i,\tau+t}$ for $i \geq i_\epsilon$, showing $(1-\epsilon)\hat{N}_t \subset \hat{M}_{\tau+t}$. We then take $\epsilon \to 0$ to conclude $\hat{N}_t \subset \hat{M}_{\tau+t}$.)

Next goal is to show, for each $(x_0, t_0) \in \mathbb{R}^{n+1} \times (0, T(M_0))$ with $x_0 \in M_{t_0}$, there is a spacetime neighborhood, say $U \times [t_0 - \tau, t_0]$, around (x_0, t_0) such that the portions of $M_{i,t}$ in this neighborhood can be represented as graphs over a fixed hyperplane with uniformly bounded C^1 norm. We may assume that $x_0 = 0$ and that $\{x_{n+1} = 0\}$ is a supporting hyperplane of $\hat{M}_{t_0}$ satisfying $\hat{M}_{i,t} \subset \{x_{n+1} \geq 0\}$ for $t \leq t_0$. For the discussion below we refer the reader to Figure 3. The observation in the previous paragraph says $2r_0 := \operatorname{dist}(\hat{M}_0, 0) > 0$. Note that the estimate on H shown in Proposition 2.10 holds even when $M_0 \cap B_r(x_0)$ is empty. Thus, applying this proposition gives that H is bounded by $c_n r_0^{-1}$ on $M_{i,t} \cap B_{r_0}(0)$ for all i and $t > 0$. Recall that for smooth convex hypersurfaces, one has $|A|^2 \leq H^2$. Since a (local) uniform limit of smooth functions with bounded C^2 norm is in $C^{1,1}$, M_{t_0} has to be $C^{1,1}$ around 0 and hence there is some r_1 such that $B_{r_1}(r_1 e_{n+1}) \subset \hat{M}_{t_0}$. Let us choose r_1 sufficiently small so that $B_{r_1}(r_1 e_{n+1}) \subset B_{r_0}(0)$. The uniform bound on $(H|F - v_j'|)^{-1}$ in (4.3) implies that there is L such that $H^{-1} \le L$ on $B_{r_0}(0) \cap M_{i,t}$, for $t \in [t_0/2, t_0]$. Using this speed bound, we obtain that

$$B_{r_1 - L\tau}(r_1 e_{n+1}) \subset \hat{M}_{t_0 - \tau} \quad \text{for all } \tau \in [0, \min(\tfrac{t_0}{2}, \tfrac{r_1}{L})].$$

To prove this, let us define, for $-\frac{t_0}{2} \le s \le 0$,

$$d(s) := \operatorname{dist}(r_1 e_{n+1}, M_{t_0+s}) \text{ and } d_i(s) := \operatorname{dist}(r_1 e_{n+1}, M_{i,t_0+s}).$$

Note that $d_i(s)$ is a Lipschitz function and the bound on H^{-1} gives

$$0 \le \dot{d}_i(s) \le L \quad \text{if } s \in [-\tfrac{t_0}{2}, 0] \quad \text{and} \quad r_1 e_{n+1} \in \hat{M}_{i,t_0+s}. \tag{4.4}$$

Since $d(s) = \lim_{i\to\infty} d_i(s)$, $d(s)$ is Lipschitz and (4.4) holds for $d(s)$ as well. Since $d(0) = r_1$ and $r_1 e_{n+1} \in \hat{M}_{t_0+0}$, we may integrate (4.4) to conclude

$$d(s) \ge r_1 + Ls \quad \text{and} \quad r_1 e_{n+1} \in \hat{M}_{t_0+s} \quad \text{for all } s \in [-\min(\tfrac{t_0}{2}, \tfrac{r_1}{L}), 0].$$

In summary, we have shown that there are positive r', h', and τ' such that for $i > i'$ large, we have

$$B_{r'}(h' e_{n+1}) \subset \hat{M}_{i,t} \subset \{x_{n+1} \ge 0\} \text{ for all } t \in [t_0 - \tau', t_0].$$

Therefore: *if $D_{r'} := \{x' \in \mathbb{R}^n : |x'| \le r'\}$, we can write $M_{i,t} \cap (D_{r'} \times [-h', h'])$ as a graph $x_{n+1} = u^{(i)}(x', t)$ on $D_{r'} \times [t_0 - \tau', t_0]$.* Note we have a uniform bound of $|D_x u|$ on $D_{r'/2}$ by the ball and hyperplane barriers above and below. Since $M_{i,t}$ are solutions to IMCF, the graphs $x_{n+1} = u^{(i)}(x', t)$ evolve by the fully nonlinear parabolic equation

$$\partial_t u = -\frac{(1 + |Du|^2)^{1/2}}{H} = -(1 + |Du|^2)^{1/2} \left[\operatorname{div} \left(\frac{Du}{(1 + |Du|^2)^{1/2}} \right) \right]^{-1} \tag{4.5}$$

and the equation is uniformly parabolic if $|Du|$, H, H^{-1} are bounded. Therefore, our estimates above show that $u^{(i)}$ are solutions to a uniformly parabolic equation on $D_{r'/2} \times [t_0 - \tau', t_0]$ and moreover they are uniformly bounded, since $|u^{(i)}| \le h'$. Standard parabolic regularity theory implies a smooth subsequential convergence $u^i \to u$ on $D_{r'/4} \times [t_0 - \tau'/2, t_0]$. Since the sequence of surfaces $M_{i,t}$ is monotone in i, this proves that $x_{n+1} = u(x', t)$ is a smooth graphical parametrization of M_t which is a solution to (4.5). Our argument holds in a neighborhood around any point $x_0 \in M_{t_0}$ and for any $t_0 \in (0, T(M_0))$, therefore showing that M_t is a smooth solution to the IMCF for $t \in (0, T)$.

We will next obtain the local uniform convergence of $M_t \to M_0$, as $t \to 0$, by showing that $\cap_{t>0} \hat{M}_t = \hat{M}_0$. Arguing by contradiction, suppose that $\hat{M}_0 \subsetneq \cap_{t>0} \hat{M}_t$, that is there exists a point $p \in \operatorname{int}(\cap_{t>0} \hat{M}_t) \setminus \hat{M}_0$. This means for each $t > 0$, there is i_t such that $\hat{M}_{i,t}$ contains p if $i > i_t$. Let us define $d_i(t) := \operatorname{dist}(p, \hat{M}_{i,t})$. Note that $d_i(0) > 0$ and $d_i(t)$ is nonnegative decreasing function. In view of the bound (4.3), by choosing $t_0 = T/2$, we conclude that there is $C = C(p, M_0) > 0$ such that if $0 < t < T/2$, then

$$\dot{d}_i(t) \ge -C\, t^{-1/2}. \tag{4.6}$$

Furtermore, the function $d_i(t)$ is Lipschitz continuous and hence the above inequality holds a.e. One obtains this inequality by considering those points attaining the distance at each fixed time and estimate the rate of changes in the distances between those points and p using the bound on H^{-1}. Integrating (4.6) from 0 to t, we get $d_i(t) - d_i(0) \geq -2C\,t^{1/2}$, for $0 < t < T/2$. Note that $d_i(0) \geq \mathrm{dist}\,(p, \hat{M}_0) > 0$. There is $t_1 > 0$ such that $d_i(t) > \mathrm{dist}\,(p, \hat{M}_0)/2$ for all i and $t \in (0, t_1)$. This is a contradiction to the assumption which says $d_i(t) = 0$ for large $i > i_t$.

Finally, we prove the strict convexity assertion in Theorem 1.2 using Appendix A.1. If $\hat{M}_0$ contains an infinite line, a solution at later time $\hat{M}_t$ also contains the same line and hence M_t it is not strictly convex by Lemma 2.3. Now suppose $\hat{M}_0$ has no infinite straight. In view of Corollary A.4, it suffices to show $\mathcal{H}^n(\nu[M_t]) > 0$ for all $t \in (0, T)$. Let us fix an arbitrary $t_0 \in (0, T)$. In the construction of $\hat{M}_t$, we have shown that M_{i,t_0} (and hence M_{t_0}) are contained in some round cone $\hat{C} := \{x \in \mathbb{R}^{n+1} : \langle x - v, \omega\rangle \geq (\sin\delta)|x - v|\}$. Observe that the outward normal of each supporting hyperplane of $\hat{C}$ should belong to $\nu[M_{t_0}]$ as we may translate the hyperplane to make it support $\hat{M}_{t_0}$ somewhere. We directly compute the set of outward normals of supporting hyperplanes of $\hat{C}$ as $\{v \in \mathbb{S}^n : \langle v, -\omega\rangle \geq \cos\delta\} =: \hat{\Gamma}'$. This shows $\mathcal{H}^n(\nu[M_{t_0}]) \geq \mathcal{H}^n(\hat{\Gamma}') > 0$, finishing the proof.

□

Remark 4.1. The definition of *innermost candidate* in (4.2) does not depend on the choice of approximation $\hat{M}_{i,0}$: if $\hat{M}_{i,0}$ and $\hat{M}'_{i,0}$ are two approximations of $\hat{M}_0$, we have $\hat{M}_{i,0} \subset\subset \hat{M}'_{n_i,0}$ for large n_i, showing that $\hat{M}_{i,t} \subset \cup_j \hat{M}'_{j,t}$ and vice versa. By the same argument, the comparison principle holds between two innermost candidates if one contains the other at initial time. The solution is innermost in the sense described in Lemma 4.2. This fact will be used in the remaining of the section.

Lemma 4.2. *Let $N_t = \partial\hat{N}_t$ for $t > 0$ be a smooth solution to the IMCF with initial data $\hat{N}_0 := \cap_{t>0}\hat{N}_t$ and $\hat{M}_t$ be the innermost candidate from $\hat{M}_0$ by Definition 4.1. If $\hat{M}_0 \subset \hat{N}_0$, then $\hat{M}_t \subset \hat{N}_t$ as long as $N_t = \partial\hat{N}_t$ exists.*

Proof. $\hat{M}_{i,0} \subset\subset \hat{N}_0$ implies $\hat{M}_{i,t} \subset \hat{N}_t$ by the comparison principle, showing $\hat{M}_t \subset \hat{N}_t$. □

Next lemma shows conical solutions can be used as barriers from inside.

Lemma 4.3. *Let $\Gamma_0 = \partial\hat{\Gamma}_0 \subset \mathbb{S}^n$ be a smooth strictly convex hypersurface in $\mathbb{S}^n$ and Γ_t be the unique solution to the IMCF obtained by [34] and [22]. Let $\hat{N}_t$ the innermost candidate from $\hat{N}_0$ by Definition 4.1. If $\mathcal{C}\hat{\Gamma}_0 \subset \hat{N}_0$, then $\mathcal{C}\hat{\Gamma}_t \subset \hat{N}_t$ for $t \in [0, \ln|\mathbb{S}^{n-1}| - \ln|\Gamma_0|)$.*

Proof. After a rotation, we may assume that e_{n+1} is in the interior of $\hat{\Gamma}_0$ in $\mathbb{S}^n$. Then $\mathcal{C}\Gamma_0$ can be written as a graph of an entire homogeneous function $x_{n+1} = f(x)$, $x \in \mathbb{R}^n$, which is uniformly Lipschitz. Since the graph is a cone, we have $f(\lambda x) = \lambda f(x)$. Let η be a usual smooth radially symmetric mollifier supported on $B_1(0)$, and define

$$f_\epsilon(x) := f * \eta_\epsilon(x) = \int_{\mathbb{R}^n} f(x + w)\eta(\epsilon^{-1}w)\epsilon^{-n}\,dw.$$

The convexity of this mollified function f_ϵ can be easily checked:

$$\begin{aligned} f_\epsilon(\lambda x + (1-\lambda)y) &= \int f(\lambda x + (1-\lambda)y + w)\eta(\epsilon^{-1}w)\epsilon^{-n}dw \\ &\geq \int [\lambda f(x + w) + (1-\lambda)f(y + w)]\eta(\epsilon^{-1}w)\epsilon^{-n}\,dw \\ &= \lambda f_\epsilon(x) + (1-\lambda)f_\epsilon(y). \end{aligned} \tag{4.7}$$

Moreover, $f_\epsilon \geq f$ since

$$\begin{aligned} f_\epsilon(x) &= \int f(x+w)\,\frac{\eta(\epsilon^{-1}w)+\eta(-\epsilon^{-1}w)}{2}\epsilon^{-n}dw \\ &= \int \frac{f(x+w)+f(x-w)}{2}\eta(\epsilon^{-1}w)\epsilon^{-n}dw \geq \int f(x)\eta(\epsilon^{-1}w)\epsilon^{-n}dw = f(x), \end{aligned} \tag{4.8}$$

the uniform Lipschitz condition of f implies that $\|f_\epsilon - f\|_\infty < \infty$ for all $\epsilon > 0$, and that $\|f_\epsilon - f\|_\infty \to 0$, as $\epsilon \to 0$. Next, observe that

$$\frac{f_1(\lambda x)}{\lambda} = \int \frac{f(\lambda x + w)}{\lambda}\eta(w)dw = \int f(x+\frac{w}{\lambda})\eta(w)dw = \int f(x+y)\eta(\lambda y)\lambda^n dy = f_{\lambda^{-1}}(x). \tag{4.9}$$

Let $M_0 = \partial \hat{M}_0$ be the convex hypersurface $\{(x, x_{n+1}) : x_{n+1} = f_1(x)\}$. Then (4.9) implies that the tangent cone of M_0 at infinity is $\mathcal{C}\Gamma_0$. Theorem 1.2 shows the existence of a smooth solution M_t, for $t \in [0, \ln|\mathbb{S}^{n-1}| - \ln|\Gamma_0|)$ with initial data M_0.

We will show next that: *for $t \in [0, \ln|\mathbb{S}^{n-1}| - \ln|\Gamma_0|)$, ϵM_t converges to $\mathcal{C}\Gamma_t$ in L^∞_{loc}.* Assuming this, let us first finish the proof of the lemma: for each $\epsilon > 0$, (4.9) implies $\epsilon M_0 = \{x_{n+1} = f_\epsilon(x)\}$ and thus (4.8) implies $\epsilon \hat{M}_0$ is contained in $\hat{N}_0$. $\epsilon \hat{M}_t$ is an innermost candidate as $\hat{M}_t$ is. Thus $\epsilon \hat{M}_t \subset \hat{N}_t$ by the comparison in Remark 4.1 and, by taking $\epsilon \to 0$, we conclude $\mathcal{C}\hat{\Gamma}_t \subset \hat{N}_t$.

We are left to prove the convergence of ϵM_t to $\mathcal{C}\Gamma_t$. Following the construction in Theorem 1.2, let $M_{i,t}$ be compact convex solutions which approximate M_t from inside. Since $\hat{M}_{i,0}$ is contained in $\mathcal{C}\hat{\Gamma}_0$, the comparison principle implies $\hat{M}_{i,t} \subset \mathcal{C}\hat{\Gamma}_t$, showing $\hat{M}_t \subset \mathcal{C}\hat{\Gamma}_t$. Let us express M_t by the entire graphs $x_{n+1} = f_1(x,t)$ and $\mathcal{C}\Gamma_t$ by $x_{n+1} = f(x,t)$. Observe that the gradients $|Df_1|$ and $|Df|$ are uniformly bounded on $(x,t) \in \mathbb{R}^n \times [0, |\mathbb{S}^{n-1}| - \ln|\Gamma_0|)$. This is because e_{n+1} is an interior point of $\hat{\Gamma}_0$ and hence $\hat{M}_t$ and $\mathcal{C}\hat{\Gamma}_t$ contain a fixed round convex cone whose axis in positive e_{n+1} direction. Next, note $f(\cdot,0) \in C^\infty_{loc}(\mathbb{R}^n \setminus \{0\})$ and $f_\epsilon(\cdot,0) \to f(\cdot,0)$ in $C^\infty_{loc}(\mathbb{R}^n \setminus \{0\})$ as $\epsilon \to 0$. This convergence and (4.9) imply that there is $C > 0$ such that $H(|x|+1) \leq C$ for M_0. Proposition 2.10 then implies that the mean curvature $H(x,t)$ of M_t at $(x, f_1(x,t))$ satisfies the bound $H(x,t)(|x|+1) \leq C'$. Next, since $\mathcal{C}\Gamma_t$ works as a conical barrier outside, Theorem 1.4 (see also Remark 3.1) can be applied to the approximating compact solutions $M_{i,t}$ to conclude that $(H|F|)^{-1} \leq C_\delta$ on M_t for $t \in [0, T(M_0) - \delta]$.

All the bounds above imply that the solutions ϵM_t when viewed as entire graphs, have uniform gradient bounds, locally uniform height bounds, and locally uniform bounds of H and H^{-1} on compact domains which do not contain the origin. In the previous statement, the uniformity of estimates holds both in $\epsilon > 0$ and $t \in [0, T(M_0) - \delta]$ for all fixed $\delta > 0$. By the regularity estimates of uniformly parabolic equations, we may pass to a sub-sequential limit and obtain, as $\epsilon \to 0$, $\epsilon f(\epsilon^{-1}x, t)$ converges to some $f_0(x,t)$ smoothly on $(B_{\delta^{-1}} \setminus B_\delta) \times [0, T-\delta)$ for all $\delta > 0$. It follows that $\{x_{n+1} = f_0(x,t)\}$ is a smooth solution to IMCF on $\mathbb{R}^n \setminus \{0\}$. Meanwhile, $\{x_{n+1} = f_0(x,t)\}$ represents a convex cone as it is the blow-down of M_t. Combining these together, $\{x_{n+1} = f_0(x,t)\} = \mathcal{C}\Gamma'_t$ for some smooth convex hypersurface $\Gamma'_t \subset \mathbb{S}^n$ and Γ'_t evolves by the IMCF. Since Γ_t is the unique solution to IMCF with initial data $\Gamma_0 = \Gamma'_0$, we conclude that $\Gamma_t = \Gamma'_t$. This proves that $f_0(x,t) = f(x,t)$ and the convergence of ϵM_t to $\mathcal{C}\Gamma_t$.

□

Proposition 4.4. *Let $M_t = \partial \hat{M}_t$ be a convex non-compact solution obtained from Theorem 1.2 and $\hat{\Gamma}_t$ be the link of the tangent cone of $\hat{M}_t$. Suppose $\hat{\Gamma}_0$ has no interior in $\mathbb{S}^n$ but $\mathcal{H}^{n-1}(\hat{\Gamma}_0) > 0$. Then $\hat{\Gamma}_t$ has interior for $t > 0$.*

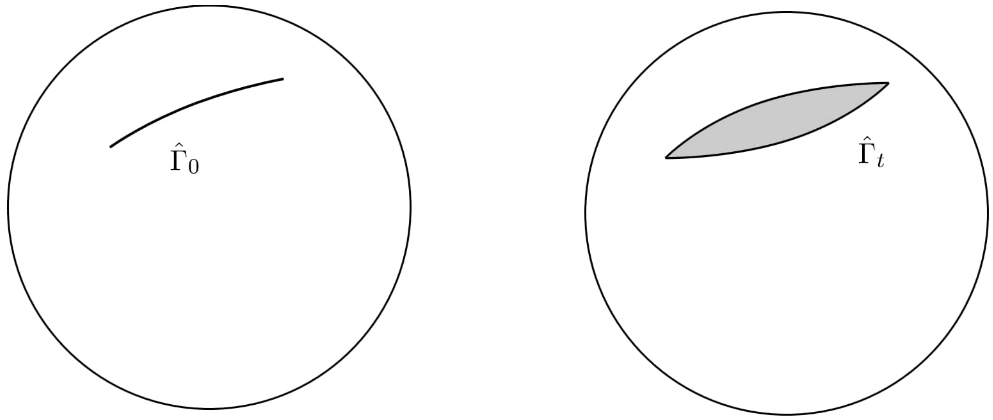

Figure 4: Proposition 4.4

Proof. We may assume that $\mathcal{H}^{n-1}(\hat{\Gamma}_0 \cap \{x_{n+1} > 0\}) > 0$ and $0 \in \operatorname{int}(\hat{M}_0)$, by rotating and translating the coordinates respectively. Since $\mathcal{C}\hat{\Gamma}_0$ is convex in $\mathbb{R}^{n+1}$, $\mathcal{C}\hat{\Gamma}_0 \cap \{x_{n+1} = 1\} =: \hat{\Omega}$ is convex in $\mathbb{R}^n$. Moreover, $\hat{\Omega}$ has no interior in $\mathbb{R}^n$ and $\mathcal{H}^{n-1}(\hat{\Omega}) > 0$ since the gnomonic projection (A.18) is (locally) bi-Lipschitz map between $\mathbb{R}^n$ and $\mathbb{S}^n \cap \{x_{n+1} > 0\}$. If a convex set in Euclidean space has no interior, then it should be contained in some hyperplane. Therefore, $\hat{\Omega}$ is contained in a $(n-1)$-plane and $\hat{\Omega}$ has interior in that $(n-1)$-plane as otherwise $\hat{\Omega}$ would be contained in a $(n-2)$-plane, contradicting $\mathcal{H}^{n-1}(\hat{\Omega}) > 0$. As a result, $\hat{\Omega}$ in $\mathbb{R}^n \times \{1\}$ contains a $(n-1)$-dimensional disk of some radius r_0. The cone generated by this $(n-1)$-disk is contained in $\mathcal{C}\hat{\Gamma}_0$ and thus $\mathcal{C}\hat{\Gamma}_0$ should contain a rotated image of the following n-dimensional cone for some $0 < r \leq r_0$:

$$\mathcal{C} := \{(x_1, x_2, \dots, x_n, 0) \in \mathbb{R}^{n+1} : \sqrt{x_1^2 + \cdots + x_{n-1}^2} \leq r x_n\}.$$

By letting $c := r/\sqrt{1+r^2}$, observe $B^n_{ca}(ae_{n+1}) \times \{0\}$ is contained in $\mathcal{C}$ for all $a > 0$. Since we assumed $0 \in \operatorname{int}(\hat{M}_0)$, there are $\vec{v} \in \hat{\Gamma}_0 \subset \mathbb{S}^n$, $\epsilon > 0$, and a rotation operator J such that the family of expanding thin disks $a\vec{v} + J(B^n_{ca}(0) \times (-\epsilon, \epsilon))$ are contained in $\hat{M}_0$ for all $a > 0$.

Claim 4.2. *Let $\hat{D}_{R,\epsilon} := B^n_R(0) \times (-\epsilon, \epsilon) \subset \mathbb{R}^{n+1}$ be a thin disk $\epsilon \in (0, R/100)$ and $N_t = \partial \hat{N}_t$ be a smooth IMCF. If $\hat{D}_{R,\epsilon} \subset \hat{N}_0$, then there is $c_n > 0$ such that $B^{n+1}_{c_n Rt}(0) \subset \hat{N}_t$ for $t \in [0, c_n]$.*

Proof of Claim. Let us smoothen out the edges of $D_{R,\epsilon} := \partial \hat{D}_{R,\epsilon}$ to obtain a smooth pancake like convex hypersurface $\Sigma_{R,\epsilon}$ by a similar method of Lemma 4.3: consider the convex conical hypersurface in $\mathbb{R}^{n+2}$ generated by $D_{R,\epsilon} \times \{1\} \subset \mathbb{R}^{n+2}$ from the origin. We can represent this cone by an entire graph $x_{n+2} = f(x)$ of a 1-homogeneous function $f(x)$. Then $f^{-1}(\{1\}) = D_{R,\epsilon}$. If we consider the regularization of f, say f_δ, as constructed in Lemma 4.3, then $f_\delta \geq f$ and it is smooth convex function. For sufficiently small δ, the level set $f_\delta^{-1}(\{1\}) =: \Sigma_{R,\epsilon}$ is a smooth convex hypersurface in $\mathbb{R}^{n+1}$ which is contained in $\hat{D}_{R,\epsilon}$. Since the regularization of a linear function is the same as itself, $D_{R,\epsilon}$ and $\Sigma_{R,\epsilon}$ coincide on $B^{n+1}_{R/2}(0)$ for small $\delta > 0$.

Observe that $\Sigma_{R,\epsilon}$ has the same symmetry as $D_{R,\epsilon}$. i.e. $O(n)$-rotational symmetry and reflection symmetry with respect to $\{x_{n+1} = 0\}$. Thus the IMCF $\Sigma_{R,\epsilon}(t)$ starting at $\Sigma_{R,\epsilon}$ must contain two points $(0, \epsilon + c(t))$ and $(0, -\epsilon - c(t))$, for each $t > 0$, at which the normal vectors to $\Sigma_{R,\epsilon}(t)$ are e_{n+1} and $-e_{n+1}$, respectively. In view of Lemma 2.10, $c'(t) > cR$ as long as $\epsilon + c(t) < R/2$. Since $\Sigma_{R,\epsilon}(t)$ contains these two points and the disk $B^n_{R/2} \times \{0\}$, convexity implies that $\hat{\Sigma}_{R,\epsilon}(t)$ includes our desired ball. This finishes the proof as $\hat{\Sigma}_{R,\epsilon}(t) \subset \hat{N}_t$ by the comparison principle. □

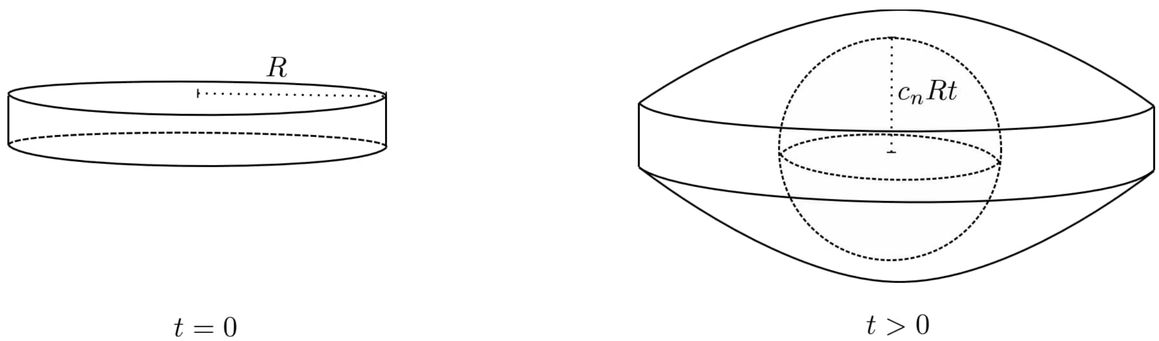

Figure 5: Claim 4.2

By the claim, $a\vec{v} + B^{n+1}_{c_n cat}(0) \subset \hat{M}_t$ for all $0 \le t \le c_n$ and $a \ge a_0$ some $a_0 > 0$. $\hat{\Gamma}_t$ has interior since $\hat{M}_t$ contains a round cone.

□

We now prove Theorem 1.3. Note that the same proof works for both $T = 0$ and $T \in (0, \infty)$.

Proof of Theorem 1.3. Let $\hat{M}_t$, for $t \ge 0$, be the innermost candidate solution from $\hat{M}_0$ (see Definition 4.1). Since $\hat{M}_t \subset \hat{N}_t$ (by Lemma 4.2), it suffices to show $\hat{M}_{T+\tau} = \mathbb{R}^{n+1}$ for $\tau > 0$. One useful observation is that if $\hat{M}_{t_0}$ contains a half space at $t_0 \ge 0$ then $\hat{M}_t = \mathbb{R}^{n+1}$ for $t > t_0$: suppose $\{x_{n+1} \ge 0\} \subset \hat{M}_{t_0}$. By the comparison with spherical solutions, $\partial B_{re^{\tau'/n}}(re_{n+1}) \subset \hat{M}_{t_0+\tau'}$. Since $B_{r(e^{\tau'/n}-1)}(0) \subset B_{re^{\tau'/n}}(re_{n+1}) \subset \hat{M}_{t_0+\tau'}$ for all $r > 0$, we get $\hat{M}_{t_0+\tau'} = \mathbb{R}^{n+1}$.

We may assume $0 \in \mathrm{int}(\hat{M}_0)$. Suppose $\hat{M}_{t_0}$ contains a cone $\mathcal{C}\hat{\Gamma}'_0$ with a smooth strictly convex link Γ'_0 in an open hemisphere. Since the IMCF running from Γ'_0 converges to an equator as $t \to \ln|\mathbb{S}^{n-1}| - \ln|\Gamma'_0|$ (by [22][34]), Lemma 4.3 and the observation above imply that $\hat{M}_t = \mathbb{R}^{n+1}$ for $t > t_0 + \ln|\mathbb{S}^{n-1}| - \ln|\Gamma'_0|$. In view of the approximation in Lemma A.9, the same assertion holds when $\hat{\Gamma}'_0$ is a convex set with interior and contained in an open hemisphere.

In general, the link of initial tangent cone $\hat{\Gamma}_0$ is a convex set in a closed hemisphere. After a rotation, we may assume that $e_{n+1} \in \hat{\Gamma}_0$ and represent $\hat{M}_0$ using a convex function f on a convex domain $\Omega_0 \subset \mathbb{R}^n$ by $\hat{M}_0 = \{(x, x_{n+1}) : x_{n+1} \ge f(x), x \in \Omega_0\}$. Let us define $\hat{M}_{\epsilon,0} := \{(x, x_{n+1}) : x_{n+1} \ge f(x) + \epsilon\sqrt{|x|^2+1}, x \in \Omega_0\}$ and observe that it is still convex with $C^{1,1}_{loc}$ boundary. The links of the tangent cones of $\hat{M}_{\epsilon,0}$, say $\hat{\Gamma}_{\epsilon,0}$, are contained in a fixed open hemisphere and $\hat{\Gamma}_{\epsilon,0}$ increases to $\hat{\Gamma}_0$ as $\epsilon \to 0^+$. $P(\hat{\Gamma}_{\epsilon,0})$ is monotone in ϵ by Lemma A.11. Let us assume the following claim for the moment.

Claim 4.3. *$P(\hat{\Gamma}_{\epsilon,0})$ increases to $P(\hat{\Gamma}_0)$ as $\epsilon \to 0^+$.*

Choose $\hat{M}_{\epsilon',0}$ such that $T' := \ln|\mathbb{S}^{n-1}| - P(\hat{\Gamma}_{\epsilon',0}) < T + \tau/2$ and note $T' > 0$ by Lemma A.11. By Theorem 1.2, there is a smooth strictly convex solution $M_{\epsilon',t}$ for $0 < t < T'$. At $t_0 := \min(\tau/4, T'/2)$, the link of the tangent cone of $\hat{M}_{\epsilon',t_0}$, say $\hat{\Gamma}_{\epsilon',t_0}$, has interior by Proposition 4.4. $\hat{\Gamma}_{\epsilon',t_0}$ is contained in an open hemisphere due to strict convexity of M_{ϵ',t_0}. By Lemma A.11, $T'' := \ln|\mathbb{S}^{n-1}| - \ln|\Gamma_{\epsilon',t_0}| \le \ln|\mathbb{S}^{n-1}| - P(\hat{\Gamma}_{\epsilon',0}) < T + \tau/2$. $\hat{M}_{\epsilon,t_0} \subset \hat{M}_{t_0}$ by Lemma 4.2. Because $\mathcal{C}\hat{\Gamma}_{\epsilon',t_0} \subset \hat{M}_{t_0}$, we apply the assertion in the second paragraph to conclude that $\hat{M}_t = \mathbb{R}^{n+1}$ for $t > t_0 + T''$. Note that $t_0 + T'' < T + 3\tau/4$ finishing the proof.

Proof of Claim. Let us define a locally Lipschitz map

$$\psi_\epsilon : \mathbb{R}^{n+1} - \{0\} \longrightarrow \mathbb{R}^{n+1} \quad \text{by} \quad \psi_\epsilon(x, x_{n+1}) = \frac{(x, x_{n+1} + \epsilon|x|)}{|(x, x_{n+1} + \epsilon|x|)|},$$

and observe $\hat{\Gamma}_{\epsilon,0} = \psi_\epsilon(\hat{\Gamma}_0)$. ψ_ϵ induces a bijection between $\{x_{n+1} = \alpha|x|\} \cap \mathbb{S}^n$ and $\{x_{n+1} = (\alpha + \epsilon)|x|\} \cap \mathbb{S}^n$ for all $\alpha \in \mathbb{R}$, hence ψ_ϵ induces a bijection from and onto $\mathbb{S}^n$. Let us define $\phi(x, x_{n+1}) := |x|e_{n+1}$ so that $\psi_\epsilon(p) = \frac{p+\epsilon\phi(p)}{|p+\epsilon\phi(p)|}$. At each differentiable point of ψ_ϵ, the differential of ψ_ϵ, $D\psi_\epsilon$, represented by a $(n+1) \times (n+1)$ matrix is

$$D\psi_\epsilon(p) = \frac{1}{|p+\epsilon\phi|}(I_{n+1} + \epsilon D\phi) - \frac{p+\epsilon\phi}{|p+\epsilon\phi|^3}(p+\epsilon\phi)^T(I_{n+1} + \epsilon D\phi).$$

Using $|p| = 1$, $|\phi| \leq 1$ and $|D\phi| \leq 1$ on $p \in \mathbb{S}^n$, we may find positive C and ϵ_0 such that for $0 < \epsilon \leq \epsilon_0$ at each $p \in \mathbb{S}^n - \{\pm e_{n+1}\}$ and $V \in T_p\mathbb{S}^n = \{W \in \mathbb{R}^{n+1} : \langle W, p\rangle = 0\}$

$$(1 - C\epsilon)|V| \leq |(d\psi_\epsilon)_p(V)| = |D\psi_\epsilon(p)V| \leq (1 + C\epsilon)|V|.$$

The shows that $(1 - C\epsilon)^{n-1}|\Gamma_0| \leq |\Gamma_{\epsilon,0}| \leq (1 + C\epsilon)^{n-1}|\Gamma_0|$.

□

□

In the remaining part of this section, we assume $0 < T(M_0) < \infty$ and discuss the behavior of innermost solution $\hat{M}_t$ as $t \to T^-$. By Theorem 1.2, M_t is a smooth IMCF for $0 < t < T$ which is innermost by Lemma 4.2. Observe $\hat{\Gamma}_t$, the link of the tangent cone of $\hat{M}_t$, is a kind of weak solution to the IMCF on $\mathbb{S}^n$.

Lemma 4.5. *Suppose $\hat{\Gamma}'_0 \subset \hat{\Gamma}_0$ for some compact convex set $\hat{\Gamma}'_0$ with smooth strictly convex boundary and let Γ'_t be the IMCF from Γ'_0. Then, $\hat{\Gamma}'_t \subset \hat{\Gamma}_t$. Similarly, if $\hat{\Gamma}_0 \subset \hat{\Gamma}'_0$ for some $\hat{\Gamma}'_0 \neq \mathbb{S}^n$ with smooth strictly convex boundary, then $\hat{\Gamma}_t \subset \hat{\Gamma}'_t$. The inequalities hold as long as Γ'_t exists.*

Proof. For such a $\hat{\Gamma}'_0 \subset \hat{\Gamma}_0$, we may find a vector $v \in \mathbb{R}^{n+1}$ such that $\mathcal{C}\hat{\Gamma}'_0 + v \subset \hat{M}_0$. Then Lemma 4.3 implies that $\mathcal{C}\hat{\Gamma}'_t + v \subset \hat{M}_t$, hence $\hat{\Gamma}'_t \subset \hat{\Gamma}_t$.

In the other case, if we first assume strict inclusion $\hat{\Gamma}_0 \subset\subset \hat{\Gamma}'_0$, then the proof goes similarly except that we don't use Lemma 4.3 and use the usual comparison principle between the approximating compact solutions $M_{i,t}$ and the conical barrier outside (and then take $i \to \infty$). For general $\hat{\Gamma}_0 \subset \hat{\Gamma}'_0$, consider a strictly decreasing approximating sequence $\{\hat{\Gamma}'_{i,0}\}$ of $\hat{\Gamma}'_0$, obtained from Lemma A.9, and apply the result which assumes strict inclusion. This shows that $\hat{\Gamma}_t \subset \cap_i \hat{\Gamma}'_{i,t}$ and $\Gamma'_t \subset \cap_i \hat{\Gamma}'_{i,t}$. In view of Lemma A.9, we have the following equalities

$$|\partial(\cap_i \hat{\Gamma}'_{i,t})| = \lim_{i\to\infty} |\Gamma'_{i,t}| = \lim_{i\to\infty} e^t|\Gamma'_{i,0}| = e^t|\Gamma'_0| = |\Gamma'_t|$$

and the strict outer area minimizing property (the last assertion in Lemma A.9) implies $\hat{\Gamma}'_t = \cap_i \hat{\Gamma}'_{i,t}$, and this shows $\hat{\Gamma}_t \subset \hat{\Gamma}'_t$.

□

Remark 4.2 (Asymptotic behavior of M_t, as $t \to T^-$). In the case when Γ_0 admits a smooth strictly convex IMCF, the comparison in Lemma 4.5 implies that Γ_t must be this solution and, by the result of [34] or [22], Γ_t converges to an equator in $C^{1,\alpha}$, as $t \to T^- = \ln|\mathbb{S}^{n-1}| - \ln P(\hat{\Gamma}_0)$. In other words, the solution M_t becomes flat, as $t \to T^-$, in the sense that the tangent cone $\mathcal{C}\Gamma_t$ converges to a hyperplane in $C^{1,\alpha}_{loc}$. Next, since $\cup_{t<T}\mathcal{C}\hat{\Gamma}_t \subset \mathcal{C}\hat{\Gamma}_T \subset \hat{M}_T + v$, for some $v \in \mathbb{R}^{n+1}$, $\hat{M}_T + v$ must contain the closure of $\cup_{t<T}\mathcal{C}\hat{\Gamma}_t$, a closed half space. Thus $\hat{M}_T$ is either a closed half space or $\mathbb{R}^{n+1}$. In the first case, M_t converges to a hyperplane as $t \to T^-$, and in the second case, M_t disappears to infinity as $t \to T^-$. Furthermore, in the first case the convergence to the hyperplane is in $C^{1,\alpha}_{loc}$ as we have a locally uniform C^2 bound (see Proposition 2.10). Note this result resembles $C^{1,\alpha}$ convergence of flow with boundary to a flat disk shown in [31]. One additional condition on $\hat{M}_0$ which leads to the first case is to assume there is $v \in \mathbb{R}^{n+1}$ such that $\hat{M}_0 \subset \mathcal{C}\hat{\Gamma}_0 + v$. Then $\mathcal{C}\hat{\Gamma}_t + v$ becomes a barrier which contains $\hat{M}_t$ and thus $\hat{M}_T$ has to be a half space.

In general Γ_0 may not admit a smooth IMCF solution. This gave the motivation for further research by the first author and Pei-Ken Hung [9] and one of the results in [9] shows the following: there is $t_0 < T := \ln|\mathbb{S}^{n-1}| - \ln P(\hat{\Gamma}_0)$ and $0 \le k \le n-2$ such that $\mathcal{C}\Gamma_t = \mathbb{R}^k \times \mathcal{C}\Gamma'_t$ and Γ'_t is a IMCF in $\mathbb{S}^{n-k}$ which becomes smooth strictly convex for $t \in (t_0, T)$. Therefore, in this general case one obtains the same asymptotic behavior as in the case of previous paragraph.

If we do not assume the result of [9], we still have some partial results on the asymptotic behavior of the solution. $\{\Gamma_t\}_{\in(0,T)}$ is a monotone family of convex hypersurfaces which are all contained in some hemisphere and that $|\Gamma_t| = e^t P(\hat{\Gamma}_0)$. Therefore, the closure of $\cup_{t<T}\hat{\Gamma}_t$ is a convex set in a (closed) hemisphere whose outer area is the same as the area of an equator $|\mathbb{S}^{n-1}|$. Such a convex set is either a hemisphere or a wedge discussed in (1.5). Since $\cup_{t<T}\hat{\Gamma}_t$ is contained in $\hat{\Gamma}_T$, $\mathcal{C}\hat{\Gamma}_T$ is either a wedge, a half space, or $\mathbb{R}^{n+1}$. In the first case (although there is no such case if the result of [9] is assumed), $M_T = \Sigma \times \mathbb{R}^{n-1}$ for some non-compact $C^{1,1}_{loc}$ convex curve $\Sigma = \partial\hat{\Sigma}$ in $\mathbb{R}^2$ and M_t converges, as $t \to T^-$, to $\Sigma \times \mathbb{R}^{n-1}$ in $C^{1,\alpha}_{loc}$. The cases when $\mathcal{C}\hat{\Gamma}_T$ is a half space or $\mathbb{R}^{n+1}$ were described the above.

Remark 4.3 (The connection with *ultra-fast diffusion* on $\mathbb{R}^n$). In [11, 12], the second author and M. del Pino studied the Cauchy problem of ultra-fast diffusion equations $u_t = \nabla\cdot(u^{m-1}\nabla u)$ on $\mathbb{R}^n$ for $m \le -1$. In an attempt to find the fastest possible decay of initial data u_0 which guarantees a solvability of the equation on $t \in (0,T)$, some partial necessary or sufficient conditions had been found. As pointed earlier, the evolution of H in the IMCF is similar to the ultra-fast diffusion equation of $m = -1$ and it shares similar features. Let us first summarize some of results when $m = -1$ from [11, 12]. First, there is $C(n)$ so that if the Cauchy problem $u_t = \nabla\cdot(u^{-2}\nabla u)$ with $u(x,0) = u_0(x) \ge 0$ has a solution for $t \in (0,T)$, then

$$\limsup_{R\to\infty} \frac{1}{R^{n-1}} \int_{B_R} u_0 dx \ge C\, T^{1/2}.$$

There exist, however, some $u_0(x) \ge 0$ such that

$$\lim_{R\to\infty} \frac{1}{R^{n-1}} \int_{B_R} u_0 = C > 0$$

but for which *no solution exist* with initial data u_0, for any $T > 0$. Such solutions are characterized by *a non-radial structure at spatial infinity*. Indeed, for initial data which is bounded from below near infinity by *positive radial functions* there is a necessary and sufficient for existence as follows:

there is an explicit constant $E^* > 0$ such that if the problem has a solution for $t \in (0, T)$, then

$$\limsup_{R\to\infty} \left[\frac{1}{R}\left(\int_0^R \frac{ds}{w_n s^{n-1}} \int_{B_s} u_0\, dx\right)\right] \geq E^* T^{1/2}.$$

Moreover, if u_0 is radially symmetric and locally bounded,

$$\liminf_{R\to\infty} \left[\frac{1}{R}\left(\int_0^R \frac{ds}{w_n s^{n-1}} \int_{B_s} u_0\, dx\right)\right] \geq E^* T^{1/2}$$

guarantees an existence of a solution on $\mathbb{R}^n \times (0, T)$. For non-radial u_0, there is a similar condition in Theorem 1.3. [12]. Every result mentioned here is in some sense sharp when explicit solutions

$$v^T(x,t) = \frac{\sqrt{2(n-1)(T-t)_+}}{|x|}$$

are considered. These results explain partial conditions for non-existence and existence of solutions, but a complete description was missing. For the convex IMCF, however, Theorem 1.2 and 1.3 depict a fairly complete picture. This was possible by the geometric estimate Theorem 1.4. Note that this lower bound has the same decay of $v^T(x,t)$ above. Instead of the integral operators used in [12], the asymptotic geometry of M_0 is used to provide the lower bound on H in Theorem 1.4. It would be interesting to see if a similar idea could be implemented in the theory of ultra-fast diffusion equation (1.2), with $m < 0$.

A Appendix

A.1 Strict convexity of solutions in space forms

Throughout this subsection, unless otherwise stated, we assume the solutions are smooth immersed n-dimensional possibly incomplete submanifods in $(N^{n+1}, \bar{g})$ which is a *space form* of sectional curvature $K \in \mathbb{R}$. This ambient space, in particular, includes Euclidean space, the sphere, or hyperbolic space. Since smooth solutions are strictly mean convex ($H > 0$), this necessarily implies that the solutions are orientable. As before, we denote the outward unit normal which is opposite to the mean curvature vector by ν, the norm of mean curvature by H, and the second fundamental form with respect to $-\nu$ by h_{ij}. Moreover, we say a solution is convex if h_{ij} is nonnegative definite everywhere. Note that the convex solution in this subsection is weaker notion than the convex solution in other sections which uses Definition 2.1. For instance, a C^2 hypersurface convex in the sense of Definition 2.1 should necessarily be complete and embedded.

Our aim is to prove Theorem A.3, a strong minimum principle on λ_1. However by looking at the evolution of the second fundamental form h_{ij} given in (2.2), it is not clear that the convexity is preserved. To do so we need to use a viscosity solution argument and we need the following lemma shown from [5].

Lemma A.1 (Lemma 5 in Section 4 [5]). *Suppose that ϕ is a smooth function such that $\lambda_1 \geq \phi$ everywhere and $\lambda_1 = \phi$ at $x = \bar{p} \in \Omega$. Let us choose an orthonormal frame so that*

$$h_{ij} = \lambda_i \delta_{ij} \text{ at } \bar{p} \in \Omega \quad \text{with } \lambda_1 = \lambda_2 = \ldots = \lambda_\mu < \lambda_{\mu+1} \leq \ldots \leq \lambda_n.$$

We denote $\mu \geq 1$ by the multiplicity of λ_1. Then at $\bar{p}$, $\nabla_i h_{kl} = \delta_{kl} \nabla_i \phi$ for $1 \leq k, l \leq \mu$. Moreover,

$$\nabla_i \nabla_i \phi \leq \nabla_i \nabla_i h_{11} - 2\sum_{j>\mu} (\lambda_j - \lambda_1)^{-1} (\nabla_i h_{1j})^2.$$

Proposition A.2. *For $n \geq 1$, let $F : \Omega \times (0,T) \to (N^{n+1}, \bar{g})$ be a smooth convex solution to the IMCF where $(N, \bar{g})$ is a space form. Let λ_1 denote the lowest eigenvalue of h^i_j. Then $u := \lambda_1/H$ is a viscosity supersolution to the equation*

$$\frac{\partial}{\partial t}u - \frac{1}{H^2}\Delta u + \frac{1}{H^3}\langle V, \nabla u\rangle + \left(\frac{W}{H^4}\right)u \geq 0 \tag{A.1}$$

where V is a vector field, and W is a scalar function such that

$$|W|, |V| \leq C(|\nabla H|, n) \quad \text{at each point.}$$

Proof. Using equation (2.2) in Remark 2.1, we compute the evolution of h^i_j/H:

$$(\partial_t - \frac{1}{H^2}\Delta)\frac{h^i_j}{H} = 2\frac{|A|^2}{H^2}\frac{h^i_j}{H} - 2\frac{h^{ik}h_{kj}}{H^2} + \frac{2}{H^4}(\nabla_m H\nabla^m h^i_j - \nabla^i H\nabla_j H). \tag{A.2}$$

Suppose a smooth function of space time, namely ϕ/H, touches λ_1/H from below at $(\bar{p}, \bar{t})$. At time $\bar{t}$ around $\bar{p}$, let us fix a time independent frame $\{e_i\}$ using the metric $g(\bar{t})$ as in Lemma A.1.

Since $\phi \leq \lambda_1 \leq h^1_1$ and they coincide at $(\bar{p}, \bar{t})$, $\partial_t\phi \geq \partial_t h^1_1$ at $(\bar{p}, \bar{t})$. At this point $(\bar{p}, \bar{t})$ with the frame $\{e_i\}$, we use Lemma A.1, equation (A.2), and the Codazzi identity $\nabla_i h_{jk} = \nabla_j h_{ik}$ to obtain

$$\begin{aligned}
\Box\frac{\phi}{H} &\geq \frac{\partial}{\partial t}\frac{h^1_1}{H} - \frac{1}{H^2}\Delta\frac{h^1_1}{H} + \frac{2}{H^3}\sum_i\sum_{j>\mu}(\lambda_j - \lambda_1)^{-1}|\nabla_i h_{1j}|^2 \\
&= \Box\frac{h^1_1}{H} + \frac{2}{H^3}\sum_{i\geq 1, j>\mu}(\lambda_j - \lambda_1)^{-1}|\nabla_1 h_{ij}|^2 \\
&= \frac{2\sum_j \lambda_j^2 - 2\lambda_1\sum_j\lambda_j}{H^2}\frac{\lambda_1}{H} + \frac{2}{H^4}\left[\nabla_m H\nabla_m h_{11} - |\nabla_1 H|^2 + H\sum_{i\geq 1, j>\mu}\frac{|\nabla_1 h_{ij}|^2}{\lambda_j - \lambda_1}\right] \\
&\geq \frac{2}{H^4}\left[\nabla_m H\nabla_m\phi - |\nabla_1 H|^2 + H\sum_{i\geq 1, j>\mu}\frac{|\nabla_1 h_{ij}|^2}{\lambda_j - \lambda_1}\right].
\end{aligned} \tag{A.3}$$

The last inequality uses $\lambda_1\sum_j\lambda_j \leq \sum_j\lambda_j^2 = |A|^2$, which holds on convex (or more generally on mean convex) hypersurfaces.

Next, note that

$$\nabla_1 H\nabla_1 H = \sum_{i,j}\nabla_1 h_{ii}\nabla_1 h_{jj} = 2\mu\nabla_1 H\nabla_1\phi - \mu^2|\nabla_1\phi|^2 + \sum_{i>\mu, j>\mu}\nabla_1 h_{ii}\nabla_1 h_{jj}. \tag{A.4}$$

Since $H\,\nabla\frac{\phi}{H} = \nabla\phi - \frac{\phi}{H}\nabla H$, we have the following for each fixed unit direction e_m

$$\nabla_m H\nabla_m\phi = H\nabla_m H\nabla_m\frac{\phi}{H} + \frac{\phi}{H}|\nabla_m H|^2. \tag{A.5}$$

We first plug (A.5) with $m = 1$ into (A.4) and then plug that into the last line of (A.3) to obtain

$$\begin{aligned}
\Box\frac{\phi}{H} \geq\ & \frac{2}{H^4}\sum_{m>1}|\nabla_m H|^2\frac{\phi}{H} + \frac{2(1-2\mu)}{H^4}|\nabla_1 H|^2\frac{\phi}{H} \\
& + \frac{2}{H^3}\sum_{m>1}\nabla_m H\nabla_m\frac{\phi}{H} + \frac{2(1-\mu)}{H^3}\nabla_1 H\nabla_1\frac{\phi}{H} + \frac{2\mu^2}{H^3}\frac{|\nabla_1\phi|^2}{H} \\
& + \frac{2}{H^4}\left[H\sum_{i\geq 1, j>\mu}(\lambda_j - \lambda_1)^{-1}|\nabla_1 h_{ij}|^2 - \sum_{i>\mu, j>\mu}\nabla_1 h_{ii}\nabla_1 h_{jj}\right].
\end{aligned} \tag{A.6}$$

We now use the convexity, $\lambda_1 \geq 0$, in the proof of the following claim.

Claim A.1. $\Big[H\sum_{i\geq 1, j>\mu}(\lambda_j-\lambda_1)^{-1}|\nabla_1 h_{ij}|^2 - \sum_{i>\mu,j>\mu}\nabla_1 h_{ii}\nabla_1 h_{jj}\Big] \geq 0$ *on* $\{\lambda_1 \geq 0\}$.

Assuming that the claim is true, then by taking away the good term $\dfrac{2\mu^2}{H^3}\dfrac{|\nabla_1\phi|^2}{H}$ in (A.6), we easily conclude that (A.1) holds by choosing a vector filed V and a scalar function W as a function of ∇H accordingly. Thus it remains to show the claim.

Proof of Claim A.1. Since $\lambda_1 \geq 0$, $H = \sum_{l\geq 1}\lambda_l \geq \sum_{l>\mu}\lambda_l$, the claim follows by:

$$\begin{aligned} H\sum_{i\geq 1,j>\mu}(\lambda_j-\lambda_1)^{-1}|\nabla_1 h_{ij}|^2 &\geq \sum_{l>\mu}\lambda_l\sum_{i>\mu}\lambda_i^{-1}|\nabla_1 h_{ii}|^2 = \sum_{i>\mu,j>\mu}\lambda_j\lambda_i^{-1}|\nabla_1 h_{ii}|^2 \\ &= \sum_{i>\mu,j>\mu}\frac{\lambda_j\lambda_i^{-1}|\nabla_1 h_{ii}|^2+\lambda_i\lambda_j^{-1}|\nabla_1 h_{jj}|^2}{2} \\ &\geq \sum_{i>\mu,j>\mu}\nabla_1 h_{ii}\nabla_1 h_{jj}. \end{aligned} \tag{A.7}$$

□

□

Now, let $M_t \subset N^{n+1}$ be a smooth complete convex solution for $t>0$, which could be either compact or non-compact. One expects M_t to be strictly convex, that is to have $\lambda_1 > 0$ for $t > 0$. Indeed, this follows easily by Proposition A.2 and the strong minimum principle for nonnegative supersolutions which is a consequence of the weak Harnack inequality for nonnegative viscosity super solutions to (locally) uniformly parabolic equations. (See Chapter 4 in [43]).

Theorem A.3. *Suppose* $F : M^n \times (0,T) \to (N^{n+1},\bar{g})$ *is a smooth complete convex solution to the IMCF with* $H>0$ *where* $(N^{n+1},\bar{g})$ *is a space form. If* $\lambda_1(p_0,t_0)=0$ *at some* (p_0,t_0) *with* $0<t_0<T$*, then* $\lambda_1 = 0$ *on* $M^n \times (0,t_0]$.

Proof. Since solution is smooth, $|H|$, $|\nabla H|$, and $|H^{-1}| = |\partial_t F|$ are locally bounded. Therefore, λ_1 is a nonnegative supersolution to equation (A.1) which is locally uniformly parabolic with bounded coefficients. We can apply strong minimum principle on a sequence $\{\Omega_k\}$ of expanding domains containing (p_0,t_0) such that $M^n = \cup_k\Omega_k$ and conclude that the theorem holds. □

Corollary A.4. *Let* $F : M^n \times (0,T) \to \mathbb{R}^{n+1}$ *be a smooth complete convex solution to the IMCF. If* $\mathcal{H}^n(\nu[M_{t_0}])>0$ *at* $t_0 \in (0,T)$*, then* M_{t_0} *is strictly convex.*

Proof. If it is not, Theorem A.3 implies $\lambda_1 \equiv 0$ for all $M^n \times (0,t_0]$. The Gauss map $\nu : M_{t_0} \to \mathbb{S}^n$ is Lipschitz. Thus the area formula implies

$$\int_{M_{t_0}} K\,d\mu = \int_{\mathbb{S}^n}\mathcal{H}^0(\nu^{-1}(\{z\}))d\mathcal{H}^n(z) \geq \mathcal{H}^n(\nu[M_{t_0}]). \tag{A.8}$$

Since $K\equiv 0$, this is a contradiction and proves the assertion. □

Remark A.1 (Strict convexity of compact solutions). We may use Theorem A.3 to show the initial strict convexity of smooth compact solution is preserved. Let M_t be the smooth IMCF running from smooth compact immersed hypersurface M_0 with positive h_{ij}. By considering the first time λ_1 becomes zero at some point, Theorem A.3 implies that there is no such time as long as the smooth solution exists and this proves h_{ij} is positive for M_t. Furthermore, if M_t is a flow in $\mathbb{R}^{n+1}$, then it is the boundary of a compact convex set with interior by Hadamard [25], showing M_t is convex in the sense of Definition 2.1.

A.2 Speed estimate for closed star-shaped solutions

The goal is this section is to give an alternative proof of Theorem 1.1 in [30] which will be based on the maximum principle. The theorem holds in any dimension $n \geq 1$.

Theorem A.5 (Theorem 1.1 in [30])**.** *Let $F : M^n \times [0,T] \to \mathbb{R}^{n+1}$ be a smooth closed star-shaped solution to* (1.1) *such that $M_0 := F_0(M^n)$ satisfies*

$$0 < R_1 \leq \langle F, \nu \rangle \leq R_2. \tag{A.9}$$

Then, there is a constant $C_n > 0$ depending only on n such that

$$\frac{1}{H} \leq C_n \left(\frac{R_2}{R_1}\right) \left(1 + \frac{1}{t^{1/2}}\right) R_2\, e^{\frac{t}{n}} \tag{A.10}$$

holds everywhere on $M^n \times [0,T]$.

Proof. Since M_0 satisfies (A.9), by Proposition 1.3 in [30], we have

$$R_1 \leq R_1\, e^{\frac{t}{n}} \leq \langle F, \nu \rangle \leq |F| \leq R_2\, e^{\frac{t}{n}} \tag{A.11}$$

for all $0 < t < +\infty$. Let us denote $w := \langle F, \nu \rangle^{-1}$ and we will consider a function

$$Q := \frac{\varphi^{1-\epsilon}(w)\, e^{\gamma |F|^2}}{H}$$

for some function $\varphi := \varphi(w)$, constants $\gamma > 0$ and $\epsilon \in (0,1)$ which will be chosen shortly.

By (i) in Lemma 2.6,

$$\big(\partial_t - \frac{1}{H^2}\Delta\big) \ln e^{|F|^2} = \big(\partial_t - \frac{1}{H^2}\Delta\big)|F|^2 = -\frac{2n}{H^2} + \frac{4}{Hw}.$$

Moreover, by (7) in Lemma 2.5 and Lemma 2.9 with $\beta = -1$,

$$\big(\partial_t - \frac{1}{H^2}\Delta\big) w = -\frac{|A|^2}{H^2} w - \frac{2}{wH^2}|\nabla w|^2$$

and hence, on $\{\varphi \neq 0\}$,

$$\big(\partial_t - \frac{1}{H^2}\Delta\big) \ln \varphi = \frac{(\partial_t - H^{-2}\Delta)\varphi}{\varphi} + \frac{1}{H^2}\frac{|\nabla \varphi|^2}{\varphi^2} = -\frac{|A|^2}{H^2}\frac{\varphi' w}{\varphi} - \frac{|\nabla w|^2}{H^2}\left(2\frac{\varphi'}{w\varphi} + \frac{\varphi''}{\varphi} - \frac{\varphi'^2}{\varphi^2}\right).$$

Inspired by the choice of φ in the well known interior curvature estimate by Ecker and Huisken in [17] (see also [7]), we define

$$\varphi(s) := \left(\frac{s}{2R_1^{-1} - s}\right). \tag{A.12}$$

For this $\varphi := \varphi(w)$, under the notation $\varphi' = \varphi'(w)$ and $\varphi'' = \varphi''(w)$, a direct computation yields

$$\frac{\varphi' w}{\varphi} = -\Big(\frac{2}{2 - wR_1}\Big) \quad \text{and} \quad 2\,\frac{\varphi'}{w\varphi} + \frac{\varphi''}{\varphi} - \frac{\varphi'^2}{\varphi^2} = \frac{\varphi'^2}{\varphi^2}.$$

Lemma 2.7 and the computations above imply

$$\begin{aligned}\big(\partial_t - \frac{1}{H^2}\Delta\big)\ln Q &= \left[\frac{|A|^2}{H^2} + \frac{1}{H^2}\frac{|\nabla H^{-1}|^2}{H^{-2}}\right] + \gamma\left[\frac{4}{Hw} - \frac{2n}{H^2}\right] - (1-\epsilon)\left[\frac{|A|^2}{H^2}\frac{\varphi' w}{\varphi} + \frac{1}{H^2}\frac{|\nabla\varphi|^2}{\varphi^2}\right]\\ &= -\Big(\frac{wR_1 - 2\epsilon}{2 - wR_1}\Big)\frac{|A|^2}{H^2} + \Big(\gamma\frac{4}{Hw} - \gamma\frac{2n - 4\epsilon^{-1}\gamma\,|F|^2\left|\nabla|F|\right|^2}{H^2}\Big)\\ &\quad - \frac{1}{H^2}\left[(1-\epsilon)\frac{|\nabla\varphi|^2}{\varphi^2} - \frac{|\nabla H^{-1}|^2}{H^{-2}} + \epsilon^{-1}\gamma^2\frac{|\nabla e^{|F|^2}|^2}{|e^{|F|^2}|^2}\right].\end{aligned} \tag{A.13}$$

Note that we have added and subtracted the term $\frac{\epsilon^{-1}\gamma^2\left|\nabla|F|^2\right|^2}{H^2}$ in the last equality. At a nonzero critical point of Q,

$$0 = \frac{\nabla Q}{Q} = (1-\epsilon)\frac{\nabla\varphi}{\varphi} + \gamma\frac{\nabla e^{|F|^2}}{e^{|F|^2}} + \frac{\nabla H^{-1}}{H^{-1}},$$

and thus

$$\begin{aligned}\left|\frac{\nabla H^{-1}}{H^{-1}}\right|^2 = \left|(1-\epsilon)\frac{\nabla\varphi}{\varphi} + \gamma\frac{\nabla e^{|F|^2}}{e^{|F|^2}}\right|^2 &= (1-\epsilon)^2\left|\frac{\nabla\varphi}{\varphi}\right|^2 + 2(1-\epsilon)\gamma\left\langle\frac{\nabla\varphi}{\varphi}, \frac{\nabla e^{|F|^2}}{e^{|F|^2}}\right\rangle + \gamma^2\left|\frac{\nabla e^{|F|^2}}{e^{|F|^2}}\right|^2\\ &\le ((1-\epsilon)^2 + \epsilon(1-\epsilon))\left|\frac{\nabla\varphi}{\varphi}\right|^2 + (1 + \frac{1-\epsilon}{\epsilon})\gamma^2\left|\frac{\nabla e^{|F|^2}}{e^{|F|^2}}\right|^2\\ &= (1-\epsilon)\frac{|\nabla\varphi|^2}{\varphi^2} + \epsilon^{-1}\gamma^2\frac{|\nabla e^{|F|^2}|^2}{e^{2|F|^2}}.\end{aligned}$$

For a given $T > 0$, note that $\frac{R_1}{R_2 e^{\frac{T}{n}}} \le wR_1 \le 1$. It remains to choose ϵ and γ. The choice $\epsilon := \frac{R_1}{2R_2 e^{\frac{T}{n}}}$ makes the first term on RHS of the second equality in (A.13) nonpositive. Next, choose $\gamma := \frac{\epsilon}{4n}\frac{1}{(R_2 e^{\frac{T}{n}})^2} > 0$ so that $4\epsilon^{-1}\gamma|F|^2 \le n$ on M_t for $t \in [0,T]$. Combining the choices and estimates, at a nonzero spatial critical point of Q,

$$\big(\partial_t - \frac{1}{H^2}\Delta\big)\ln Q = \frac{(\partial_t - H^{-2}\Delta)Q}{Q} + \frac{|\nabla Q|^2}{Q^2} \le \gamma\left(-\frac{n}{H^2} + \frac{4}{H\,w}\right). \tag{A.14}$$

We will now apply the maximum principle on $\hat{Q} := tQ$. Suppose that nonzero maximum of $\hat{Q}$ on $M^n \times [0,T]$ occurs at the point (p_0, t_0), which necessarily implies $t_0 > 0$. At this point, (A.14) implies

$$0 \le \big(\partial_t - \frac{1}{H^2}\Delta\big)\ln\hat{Q} \le \gamma\left(-\frac{n}{H^2} + \frac{4}{H\,w}\right) + \frac{1}{t_0} \le \gamma\left(-\frac{n}{2H^2} + \frac{8}{n}R_2^2 e^{2\frac{T}{n}}\right) + \frac{1}{t_0} \tag{A.15}$$

where the second inequality comes from

$$\frac{4}{H\,w} \le \frac{8}{nw^2} + \frac{n}{2H^2} \le \frac{8}{n}R_2^2 e^{2\frac{T}{n}} + \frac{n}{2H^2}.$$

The rest is a standard argument shown in the proof of Theorem 3.1 [17]. By the choices of ϵ, γ, bounds (A.11) and

$$\frac{R_1}{2R_2e^{T/n}} \le \varphi((R_2e^{T/n})^{-1}) \le \varphi(w) \le \varphi(R_1^{-1}) = 1,$$

we proceed and obtain, for every $(p,t) \in M^n \times (0,T]$,

$$\frac{1}{H^2}(p,t) \le C_n \left(\frac{R_2}{R_1}e^{\frac{T}{n}}\right)^{2-\epsilon} (R_2e^{\frac{T}{n}})^2 \left(1+\frac{1}{t}\right). \tag{A.16}$$

Now for time $t > 1$, we can alway apply this estimate starting at time $t-1$. Inequality (A.11) implies that the ratio between star-shapedness bounds from above and below remains unchanged over time. This way we can replace $(R_2e^{\frac{T}{n}}/R_1)^{2-\epsilon}$ in the above estimate by $(R_2/R_1)^{2-\epsilon}$ after possibly enlarging the constant C_n. Since $(R_2/R_1)^{2-\epsilon} \le (R_2/R_1)^2$, the theorem follows. □

A.3 Smooth approximation of convex hypersurfaces

In this appendix, unless it is stated otherwise, *convergence* of compact convex sets (or their boundary hypersurfaces) means *the convergence in the Hausdorff metric*, defined as

$$d_H(A,B) := \max(\sup_{x\in A}\inf_{y\in B}\|x-y\|, \sup_{x\in A}\inf_{y\in B}\|x-y\|). \tag{A.17}$$

In case where A, B are compact convex sets, it is known that $d_H(A,B) = d_H(\partial A, \partial B)$ (Lemma 1.8.1 [37]). For a compact convex set $\hat{M} \subset \mathbb{R}^{n+1}$, $\hat{M}^\delta$ denotes the δ-envelope

$$\hat{M}^\delta := \{x \in \mathbb{R}^{n+1} : \text{dist}\,(x,\hat{M}) \le \delta\} = \hat{M} + \delta\overline{B_0(1)}.$$

Here the dist $(x,\hat{M})$ is measured in Euclidean distance.

Lemma A.6. *For $n \ge 1$, let $\hat{M} \subset \mathbb{R}^{n+1}$ be a compact convex set with non-empty interior. Suppose $0 \in int(\hat{M})$. Then there is a sequence compact convex sets $\hat{M}_k$ with smooth strictly convex boundaries such that*

$$\hat{M}_k^{de} := (1+k^{-1})\hat{M}_k \text{ is strictly decreasing,} \quad \hat{M}_k^{in} := (1-k^{-1})\hat{M}_k \text{ is strictly increasing,}$$

and both converge to $\hat{M}$ as $k \to \infty$. Here, we say that $\hat{\Sigma}_k^{de}$ ($\hat{\Sigma}_k^{in}$) is strictly decreasing (strictly increasing) if $\hat{\Sigma}_{k+1}^{de} \subset int\,(\hat{\Sigma}_k^{de})$ ($\hat{\Sigma}_k^{in} \subset int\,(\hat{\Sigma}_{k+1}^{in})$, respectively).

Proof. By Theorem 3.4.1 in [37] and its immediate following discussion, there is a sequence of compact convex sets $\hat{M}_k$ with non-empty interior and smooth strictly convex boundaries such that $d_H(\hat{M},\hat{M}_k) \to 0$. Note that $(1+k^{-1})\hat{M}$ and $(1-k^{-1})\hat{M}$ are strictly monotone sets. Hence, using a diagonal argument, we may choose a subsequence of $\hat{M}_k$ so that $(1+k^{-1})\hat{M}_k$ and $(1-k^{-1})\hat{M}_k$ are strictly monotone as well. □

Let $\hat{\Gamma}$ be a compact convex set in the upper open hemisphere $\mathbb{S}^n \cap \{x_{n+1} > 0\} =: \mathbb{S}^n_+$. Since $\mathcal{C}\hat{\Gamma}$ is convex in $\mathbb{R}^{n+1}$, if we define $\hat{\Omega} \times \{1\} := \mathcal{C}\hat{\Gamma} \cap \{x_{n+1} = 1\}$, then $\hat{\Omega}$ is a compact convex set in $\mathbb{R}^n$. $\hat{\Gamma}$ and $\hat{\Omega}$ are related by $\hat{\Gamma} = \varphi(\hat{\Omega})$ using the gnomonic projection $(x,1) \in \mathbb{R}^{n+1} \mapsto \varphi(x) \in \mathbb{S}^n_+$

$$\varphi(x) = \frac{1}{(1+|x|^2)^{1/2}}(x,1) \quad \text{for } x \in \mathbb{R}^n. \tag{A.18}$$

Since $(\mathbb{R}^n, \varphi^* g_{\mathbb{S}^n})$ and $(\mathbb{S}^n_+, g_{\mathbb{S}^n})$ are isometric, we have $|\partial\hat{\Gamma}| = |\partial\hat{\Omega}|_{\varphi^* g_{\mathbb{S}^n}}$. Here, the surface area measure $|\cdot|_{\varphi^* g_{\mathbb{S}^n}}$ is $(n-1)$-Hausdorff measure on metric space $(\mathbb{R}^n, d_{\varphi^* g_{\mathbb{S}^n}})$, induced from Riemannian structure $(\mathbb{R}^n, \varphi^* g_{\mathbb{S}^n})$. Since $\partial\hat{\Omega}$ is $(n-1)$-rectifiable, we will use the area formula and avoid using the metric $d_{\varphi^* g_{\mathbb{S}^n}}$ in actual computation of $|\partial\hat{\Omega}|_{\varphi^* g_{\mathbb{S}^n}}$.

Lemma A.7. *For $n \geq 1$, assume that a sequence of compact convex sets $\hat{M}_k \subset \mathbb{R}^{n+1}$ converges to a compact convex set $\hat{M}$ with non-empty interior. Then $\lim_{k\to\infty} |M_k| = |M|$. Similarly, if a sequence of compact convex sets $\hat{\Gamma}_k \subset \mathbb{S}^n_+$ converges to a compact convex set with non-empty interior $\hat{\Gamma} \subset \mathbb{S}^n_+$, then $\lim_{k\to\infty} |\Gamma_k| = |\Gamma|$.*

Proof. Our proof is a modification of the proof of Theorem 4.2.3 in [37]. For the first part, we may assume $0 \in \operatorname{int}(\hat{M}_k)$ for all k. Let $\rho(\hat{M}_k, \cdot)$ be a spherical parametrization of M_k around 0, meaning that $\rho(\hat{M}_k, y)\, y$ for $y \in \mathbb{S}^n$ is a point in M_k. Let $\nu(\hat{M}_k, y)$ denote an arbitrary outer unit normal (in the sense of supporting normal) of $\hat{M}_k$ at $\rho(\hat{M}_k, y)y$. The normal $\nu(\hat{M}_k, y)$ is unique for $\mathcal{H}^n$-almost all $y \in \mathbb{S}^n$ (Theorem 2.2.5 in [37]). If $\hat{M}_k$ converges to $\hat{M}$ as $k \to \infty$, then $\rho(\hat{M}_k, \cdot) \to \rho(\hat{M}, \cdot)$ everywhere. Moreover, $\nu(\hat{M}_k, \cdot) \to \nu(\hat{M}, \cdot)$ $\mathcal{H}^n$-almost everywhere, otherwise if $\nu(\hat{M}_{i_k}, y') \to \nu'$ for some ν' as $i_k \to \infty$, this would implies that M has a supporting hyperplane $\{\langle x - \rho(\hat{M}, y')y', \nu' \rangle = 0\}$ at $\rho(\hat{M}, y')y'$, but on the other hand the outer normal of M uniquely exists $\mathcal{H}^n$-almost everywhere. Finally, since $\hat{M}_k$ contains the origin in its interior, there is a uniform $\delta > 0$ such that $\langle y, \nu(\hat{M}_k, y)\rangle \geq \delta$ for all k and y.

Let $\psi : U \subset \mathbb{R}^n \to \mathbb{S}^n$, $z = (z^1, \dots, z^n) \mapsto \psi(z) = y$, be a smooth local coordinate chart of $\mathbb{S}^n$ and g_{ij} be the metric $g_{\mathbb{S}^n}$ on U. Note that $\rho(\hat{M}_k, \psi(\cdot))$ is a Lipschitz function. At each point y where the function $\rho_k(y) := \rho(\hat{M}_k, y)$ is differentiable, one can directly compute that

$$\nu(\hat{M}_k, y) = \frac{\rho_k y - g^{ij} \frac{\partial \rho_k}{\partial z^i} \frac{\partial y}{\partial z^j}}{\sqrt{\rho_k^2 + \|d\rho_k\|^2_{g_{\mathbb{S}^n}}}}.$$

Thus from the convergence of $\nu(\hat{M}_k, \cdot)$ and the lower bound $\langle y, \nu(\hat{M}_k, y)\rangle \geq \delta$, if $\rho_\infty(y) := \rho(\hat{M}, y)$, then we have $\left|\frac{\partial \rho_k}{\partial z^i}\right| \leq C_\delta$ and $\frac{\partial \rho_k}{\partial z^i} \to \frac{\partial \rho_\infty}{\partial z^i}$ almost everywhere for all $i = 1, \dots, n$.

Denote by f_k and $f_\infty : U \to \mathbb{R}^{n+1}$ the functions $f_k(z) := \rho(\hat{M}_k, \psi(z))\psi(z)$ and $f_\infty(z) := \rho(\hat{M}, \psi(z))\psi(z)$. By the area formula, we have $|f_k(U)| = \int_U J_{f_k}(z) dz^n$, where

$$\begin{aligned} J_{f_k}(z) &= \sqrt{\det\left[\left\langle \frac{\partial f_k}{\partial z^i}(z), \frac{\partial f_k}{\partial z^j}(z)\right\rangle\right]} = \sqrt{\det\left[\frac{\partial \rho_k}{\partial z^i}\frac{\partial \rho_k}{\partial z^j} + \rho_k^2 g_{ij}\right]} \\ &= \sqrt{\rho_k^{2n-2}(\rho_k^2 + \|d\rho_k\|^2_{g_{\mathbb{S}^n}})}\sqrt{\det g_{ij}} = \rho_k^{n-1}\sqrt{\rho_k^2 + \|d\rho_k\|^2_{g_{\mathbb{S}^n}}}\sqrt{\det g_{ij}} \end{aligned}$$

holds almost everywhere. The Lebesgue dominated convergence theorem, then yields that

$$|f_k(U)| = \int_U J_{f_k}(z) dz^n \to \int_U \rho_\infty^{n-1}\sqrt{\rho_\infty^2 + \|d\rho_\infty\|^2_{g_{\mathbb{S}^n}}}\sqrt{\det g_{ij}} = |f_\infty(U)|, \quad \text{as } k \to \infty.$$

Using a standard partition of unity argument, we conclude that $|M_k| \to |M|$ as $k \to \infty$, which concludes the proof of the first assertion of the lemma.

We will now prove the second assertion of the lemma. Note that the preimages of $\hat{\Gamma}_k$ and $\hat{\Gamma}$ under φ are compact convex sets in $\mathbb{R}^n$. On a given compact set, the metrics induced by $(\mathbb{R}^n, \varphi^* g_{\mathbb{S}^n})$ and $(\mathbb{R}^n, g_{\mathbb{R}^n})$ are equivalent. i.e. one is less than a constant multiple of the other and the constant depends on the compact set. We may assume that there is some $p \in \mathbb{R}^n$ such that $\hat{M}_k := \varphi^{-1}(\hat{\Gamma}_k) - p$ and $\hat{M} := \varphi^{-1}(\hat{\Gamma}) - p$ contain the origin in their interiors and $d_H(\hat{M}_k, M) \to 0$ as $k \to \infty$.

Defining $\rho_k,\ \rho_\infty : \mathbb{S}^{n-1} \to \mathbb{R}$, a smooth local chart $\psi : U \subset \mathbb{R}^{n-1} \to \mathbb{S}^{n-1}$, and the functions $f_k,\ f_\infty : U \to \mathbb{R}^n$ similarly as in the previous case (note that the dimension is 1 less than the dimension in the previous case), we get the convergence of ρ_k to ρ_∞ with positive uniform upper and lower bounds and the convergence of $\frac{\partial \rho_k}{\partial z^i}$ to $\frac{\partial \rho_\infty}{\partial z^i}$ with uniform bound on their absolute value. With $\tilde{\varphi}(x) := \varphi(x+p)$, the area formula says that $|\varphi(f_k(U)+p)| = \int_U J_{\tilde{\varphi}\circ f_k}(z)dz^n$, where

$$J_{\tilde{\varphi}\circ f_k}(z) = \sqrt{\det\left[\left\langle \frac{\partial \tilde{\varphi}}{\partial x^\alpha}(f_k(z))\frac{\partial f_k^\alpha}{\partial z^i}(z), \frac{\partial \tilde{\varphi}}{\partial x^\alpha}(f_k(z))\frac{\partial f_k^\alpha}{\partial z^j}(z)\right\rangle\right]}.$$

Note that φ is a smooth function, which in particular has bounded higher order derivatives on each compact domain. Therefore the Lebesgue dominated convergence theorem yields that

$$|\varphi(f_k(U)+p)| \to |\varphi(f_\infty(U)+p)|, \quad \text{as } k \to \infty$$

and a partition of unity can be used to show

$$|\varphi(M_k+p)| = |\Gamma_k| \to |\varphi(M+p)| = |\Gamma|.$$

□

The following is a well known lemma and a stronger statement than this also holds, but we provide a proof of this simple version for the completeness of our work.

Lemma A.8. *For $n \geq 1$, a convex hypersurface $M = \partial\hat{M}$ in $\mathbb{R}^{n+1}$ has the outer area minimizing property among convex hypersurfaces, i.e. if $\hat{M} \subset \hat{M}'$ then $|M| \leq |M'|$. Moreover, when M is smooth strictly convex, then the property is strict in the sense that equality holds if and only if $M = M'$.*

Proof. The proof uses a standard calibration argument. Assume $0 \in \mathrm{int}(\hat{M})$. Assume M is smooth and strictly convex. Then λM, $\lambda \geq 1$, gives a foliation of smooth strictly convex hypersurfaces. Let $\hat{M}'$ be a set containing $\hat{M}$. The foliation gives a smooth vector field consisting of the outer normal vectors ν of $\{\lambda M\}_{\lambda\geq 1}$. If we denote the unit normal on $\partial(\hat{M}' \setminus \hat{M})$ pointing from $\hat{M}' \setminus \hat{M}$ by ν', then the divergence theorem implies

$$0 \leq \int_{\hat{M}'\setminus\hat{M}} H = \int_{\hat{M}'\setminus\hat{M}} \mathrm{div}\ \nu = \int_{M'} \langle \nu, \nu' \rangle dA + \int_M \langle \nu, \nu' \rangle dA = \int_{M'} \langle \nu, \nu' \rangle dA - \int_M dA$$

and hence

$$|M| = \int_M dA \leq \int_{M'} \langle \nu, \nu' \rangle dA \leq \int_{M'} dA = |M'|.$$

(The divergence theorem can be applied for a set with rough boundary when the boundary consists of convex hypersurfaces. One could also avoid doing this by approximating $\hat{M}'$ from outside using Lemma A.6 and Lemma A.7). The strict outer area minimizing is a consequence from the fact $H > 0$ on $\hat{M}' \setminus \hat{M}$.

For a general convex $M = \partial\hat{M}$, we consider smooth approximation from inside, say $\hat{M}_k^{in}$ which was shown to exist in Lemma A.6. By the first case, $|M_k^{in}| \leq |M'|$. Lemma A.7 implies that $|M_k^{in}| \to |M|$ as $k \to \infty$ and this finishes the proof. □

We do have a similar result for convex hypersurfaces in $\mathbb{S}^n$.

Lemma A.9. *For $n \geq 2$, let $\hat{\Gamma} \subset \mathbb{S}^n \cap \{x_{n+1} > 0\} = \mathbb{S}^n_+$ be a compact convex set with non-empty interior. Then $\hat{\Gamma}$ can be approximated from inside (and outside) by a strictly monotone sequence of compact sets in $\mathbb{S}^n_+$ with smooth strictly convex boundaries. For any sequence of compact convex sets $\hat{\Gamma}_k$ converging to $\hat{\Gamma}$, we have $|\Gamma_k| \to |\Gamma|$ as $k \to \infty$. Moreover, $\hat{\Gamma}$ satisfies the outer area minimizing property on $\mathbb{S}^n_+$. That is, if $\hat{\Gamma} \subset \hat{\Gamma}' \subset \mathbb{S}^n_+$ and $\hat{\Gamma}'$ is convex, then $|\Gamma| \leq |\Gamma'|$, and if Γ is smooth strictly convex, then $|\Gamma| = |\Gamma'|$ if and only if $\Gamma = \Gamma'$.*

Proof. Let $\hat{\Omega} := \varphi^{-1}(\hat{\Gamma}) \subset \mathbb{R}^n$. Then by Lemma A.6, $\hat{\Omega}$ could be approximated from inside (and outside) by strictly monotone sequence of compact sets with smooth strictly convex boundaries. The images of these sequences of sets under φ give the desired approximating sequences. The convergence of area is shown in Lemma A.7.

The proof of the second part is similar to the proof of Lemma A.8. Suppose first $\Gamma = \partial\hat{\Gamma}$ is smooth and $\hat{\Gamma} \subset \hat{\Gamma}' \subset \mathbb{S}^n \cap \{x_{n+1} > 0\}$. Fix $p \in \hat{\Omega} = \varphi^{-1}(\hat{\Gamma})$ and consider the foliation $\{\lambda(\Omega - p) + p\}_{\lambda \geq 1}$. Then the image of this foliation under φ, that is

$$\varphi(\lambda(\Omega - p) + p) \subset \mathbb{S}^n, \text{ for } \lambda \geq 1,$$

gives a foliation of the region $\mathbb{S}^n_+ - \mathrm{int}(\hat{\Gamma})$ by smooth convex hypersurfaces in $\mathbb{S}^n$. By the same calibration argument, we obtain $|\Gamma'| \geq |\Gamma|$. In the non-smooth case we approximate Γ from inside by smooth sets and apply Lemma A.7.

□

The next approximation lemma concerns with the case where $\hat{\Gamma} \subset \mathbb{S}^n_+$ has empty interior.

Lemma A.10. *For $n \geq 2$, suppose $\hat{\Gamma}$ in $\mathbb{S}^n_+$ is a compact convex set which has empty interior in $\mathbb{S}^n$. Then there is $\{\hat{\Gamma}_k\}$ a sequence of compact convex sets with non-empty interior and smooth strictly convex boundaries which strictly decreases to $\hat{\Gamma}$. For any such sequence $\hat{\Gamma}_k$, $|\Gamma_k| = |\partial\hat{\Gamma}_k|$ decreases to $P(\hat{\Gamma})$ as $k \to \infty$. Here $P(\hat{\Gamma})$ is defined by (1.4).*

Proof. $\hat{\Omega} = \varphi^{-1}(\hat{\Gamma})$ has empty interior in $\mathbb{R}^n$. Consider the set $\hat{\Omega}^\delta := \delta$-envelope of $\hat{\Omega}$ in $(\mathbb{R}^n, g_{\mathbb{R}^n})$. Then $\hat{\Omega}^\delta$ is a compact convex set with non-empty interior, implying that $\varphi(\hat{\Omega}^\delta)$ is a closed convex set with non-empty interior in $\mathbb{S}^n$. By a diagonal argument applied to the approximations of $\varphi(\hat{\Omega}^{1/k})$, which is similar to the proof of Lemma A.6, we obtain the existence of a strictly decreasing approximation.

Let us now prove the convergence $|\Gamma_k| \to P(\hat{\Gamma})$, as $k \to \infty$. If a convex set $\hat{\Omega} \subset \mathbb{R}^n$ has empty interior, it is contained in a hyperplane of $\mathbb{R}^n$. Since a rotation is an isometry of $(\mathbb{R}^n, \varphi^* g_{\mathbb{S}^n})$, we may assume that $\hat{\Omega} \subset \{x_n = l\}$, for some $l > 0$. Let $\hat{\Sigma} \subset \mathbb{R}^{n-1}$ denote the projection of $\hat{\Omega}$ to $\{x_n = 0\} = \mathbb{R}^{n-1}$ and $\hat{\Sigma}^\delta$ denote the δ-envelope of $\hat{\Sigma}$ in $\mathbb{R}^{n-1}$ with respect to the standard Euclidean metric. Observe that $\hat{\Sigma}^\delta \times \{l\} = \hat{\Omega}^\delta \cap \{x_n = l\}$. Moreover, $\hat{\Omega}^\delta \subset \hat{\Sigma}^\delta \times [l - \delta, l + \delta]$. The outer area minimizing property (Lemma A.9) implies that

$$\begin{aligned} |\partial\hat{\Omega}^\delta|_{\varphi^* g_{\mathbb{S}^n}} &\leq |\partial(\hat{\Sigma}^\delta \times [-\delta, \delta]))|_{\varphi^* g_{\mathbb{S}^n}} \\ &= |\partial\hat{\Sigma}^\delta \times [l - \delta, l + \delta]|_{\varphi^* g_{\mathbb{S}^n}} + |\hat{\Sigma}^\delta \times \{l - \delta\}|_{\varphi^* g_{\mathbb{S}^n}} + |\hat{\Sigma}^\delta \times \{l + \delta\}|_{\varphi^* g_{\mathbb{S}^n}}. \end{aligned}$$

It is clear that as $\delta \to 0$, the first term in the last line is of order $O(\delta)$. Since $\hat{\Sigma}^\delta$ decreases to $\hat{\Sigma}$, together with the smoothness of φ, we conclude that each of remaining two terms converges to $|\hat{\Sigma} \times \{l\}|_{\varphi^* g_{\mathbb{S}^n}}$. This shows $\limsup_{\delta \to 0} |\partial \hat{\Omega}^\delta|_{\varphi^* g_{\mathbb{S}^n}} \leq P(\hat{\Gamma})$. Next, $\partial \hat{\Omega}^\delta$ contains $\hat{\Sigma} \times \{l - \delta\}$ and $\hat{\Sigma} \times \{l + \delta\}$, implying that

$$|\hat{\Sigma} \times \{l - \delta\}|_{\varphi^* g_{\mathbb{S}^n}} + |\hat{\Sigma} \times \{l + \delta\}|_{\varphi^* g_{\mathbb{S}^n}} \leq |\partial \hat{\Omega}^\delta|_{\varphi^* g_{\mathbb{S}^n}}$$

and hence

$$2|\hat{\Sigma} \times \{l\}|_{\varphi^* g_{\mathbb{S}^n}} \leq \liminf_{\delta \to 0} |\hat{\Omega}^\delta|_{\varphi^* g_{\mathbb{S}^n}}.$$

This proves that $\lim_{\delta \to 0} |\hat{\Omega}^\delta|_{\varphi^* g_{\mathbb{S}^n}} = P(\hat{\Gamma})$. Now, for any decreasing approximation by convex sets with non-empty interior $\hat{\Gamma}_k$, $\lim_{k \to \infty} |\Gamma_k|$ exists as it is a decreasing sequence. For each k, we may find $\delta_1(k)$ and $\delta_2(k)$ which converge to 0 as $k \to \infty$ such that $\varphi(\hat{\Omega}^{\delta_1(k)}) \subset \hat{\Gamma}_k \subset \varphi(\hat{\Omega}^{\delta_2(k)})$ and this, in particular, implies that $|\hat{\Omega}^{\delta_1(k)}|_{\varphi^* g_{\mathbb{S}^n}} \leq |\Gamma_k| \leq |\hat{\Omega}^{\delta_2(k)}|_{\varphi^* g_{\mathbb{S}^n}}$. We conclude that $|\Gamma_k| \to P(\hat{\Gamma})$, as $k \to \infty$.

□

We conclude this appendix by the following generalized outer area minimizing property.

Lemma A.11. *For $n \geq 2$, if $\hat{\Gamma}_1, \hat{\Gamma}_2$, are compact convex sets such that $\hat{\Gamma}_1 \subset \hat{\Gamma}_2 \subset \mathbb{S}^n_+$. Then,* $P(\hat{\Gamma}_1) \leq P(\hat{\Gamma}_2) < |\mathbb{S}^{n-1}|$.

Proof. By Lemma A.9 and A.10, there are approximating sequences of strictly decreasing compact convex sets $\hat{\Gamma}_{1,k}$ and $\hat{\Gamma}_{2,k}$ in $\mathbb{S}^n_+$ with smooth strictly convex boundaries. $|\Gamma_{1,k}| \to P(\hat{\Gamma}_1)$ and $|\Gamma_{2,k}| \to P(\hat{\Gamma}_2)$ by the two lemmas say. By the strict monotonicity of the sequences, for fixed k there is l_0 such that $\hat{\Gamma}_{1,l} \subset \hat{\Gamma}_{2,k}$ for $l > l_0$. Taking $l \to \infty$, the outer area minimizing property in Lemma A.9 implies $P(\hat{\Gamma}_1) \leq |\Gamma_{2,k}|$. The first inequality now follows by letting $k \to \infty$. The second inequality is implied by $\hat{\Gamma}_2 \subset \mathbb{S}^n \cap \{x_{n+1} \leq \epsilon\}$ for small $\epsilon > 0$, the first inequality, and $|\mathbb{S}^n \cap \{x_{n+1} = \epsilon\}| < |\mathbb{S}^{n-1}|$.

□

References

[1] B. Allen. Non-compact solutions to inverse mean curvature flow in hyperbolic space. *Ph.D. thesis*, University of Tennessee, Knoxville, 2016.

[2] H. Bray and A. Neves. Classification of prime 3-manifolds with Yamabe invariant greater than RP^3. *Ann. of Math.* , (2) 159 (2004), 407–424.

[3] H.L. Bray. Proof of the Riemannian Penrose inequality using the positive mass theorem. *J. of Differential Geometry*, 59.2 (2001): 177-267.

[4] H.L. Bray and D.A. Lee. On the Riemannian Penrose inequality in dimensions less than eight. *Duke Mathematical Journal*, 148.1 (2009): 81-106.

[5] S. Brendle, K. Choi, and P. Daskalopoulos. Asymptotic behavior of flows by powers of the gaussian curvature. *Acta Mathematica*, 219(1):1–16, 2017.

[6] S. Brendle, P.-K. Hung and M.-T. Wang. A minkowski inequality for hypersurfaces in the anti-de sitter-schwarzschild manifold. *Communications on Pure and Applied Mathematics*, 69(1):124–144, 2016.

[7] L. Caffarelli, L. Nirenberg and J. Spruck. Nonlinear second-order elliptic equations v. the dirichlet problem for weingarten hypersurfaces. *Communications on pure and applied mathematics*, 41(1):47–70, 1988.

[8] M. do Carmo and F.W. Warner. Rigidity and convexity of hypersurfaces in spheres. *Journal of Differential Geometry*, 4.2 (1970): 133-144.

[9] B. Choi and P.-K. Hung. Inverse mean curvature flow with singularities. *arXiv preprint arXiv:1811.04497*

[10] K. Choi, P. Daskalopoulos, L. Kim and K.-A. Lee. The evolution of complete non-compact graphs by powers of Gauss curvature. *J. Reine Angenw. Math.*, to appear (2018).

[11] P. Daskalopoulos and M. del Pino. On fast diffusion nonlinear heat equations and a related singular elliptic problem. *Indiana University Mathematics Journal*, pages 703–728, 1994.

[12] P. Daskalopoulos and M. del Pino. On nonlinear parabolic equations of very fast diffusion. *Archive for Rational Mechanics and Analysis*, 137(4):363–380, 1997.

[13] P. Daskalopoulos and G. Huisken. Inverse mean curvature evolution of entire graphs. *arXiv preprint arXiv:* 1709.06665, 2017.

[14] L.L. De Lima and F. Girão. An Alexandrov-Fenchel-type inequality in hyperbolic space with an application to a Penrose inequality. In *Annales Henri Poincaré*, volume 17, pages 979–1002. Springer, 2016.

[15] L.C. Evans and F. Gariepy. Measure theory and fine properties of functions. Chapman and Hall/CRC, 2015.

[16] K. Ecker and G. Huisken. Mean curvature evolution of entire graphs. *Annals of Mathematics*, 130.3 (1989): 453-471.

[17] K. Ecker and G. Huisken. Interior estimates for hypersurfaces moving by mean curvature. *Inventiones mathematicae*, 105(1):547–569, 1991.

[18] N. Edelen, R. Haslhofer, M.N. Ivaki and J.J. Zhu Mean convex mean curvature flow with free boundary. arXiv preprint arXiv:1911.01186 (2019).

[19] Y. Ge, G. Wang and J. Wu. Hyperbolic Alexandrov-Fenchel quermassintegral inequalities II. *Journal of Differential Geometry*, 98(2):237–260, 2014.

[20] C. Gerhardt. Flow of nonconvex hypersurfaces into spheres. *Journal of Differential Geometry*, 32(1):299–314, 1990.

[21] C. Gerhardt. Curvature problems, series in geometry and topology, vol. 39, 2006.

[22] C. Gerhardt. Curvature flows in the sphere. *Journal of Differential Geometry*, 100(2):301–347, 2015.

[23] R. Geroch Energy extraction. *Annals of the New York Academy of Sciences*, 224.1 (1973): 108-117.

[24] P. Guan and J. Li. The quermassintegral inequalities for k-convex starshaped domains. *Advances in Mathematics*, 221.5 (2009): 1725-1732.

[25] J. Hadamard. Sur certaines propriétés des trajectoires en dynamique. *J. Math. Pures. Appl.*, 3 (1897) 331-387.

[26] J. van Heijenoort. On locally convex manifolds *Comm. Pure Appl. Math.*, 5 (1952) 223-242.

[27] G. Huisken and T. Ilmanen. The Riemannian Penrose inequality. *International Mathematics Research Notices*, 1997(20):1045–1058, 1997.

[28] G. Huisken and T. Ilmanen. The inverse mean curvature flow and the Riemannian Penrose inequality. *Journal of Differential Geometry*, 59(3):353–437, 2001.

[29] G. Huisken and T. Ilmanen. Energy inequalities for isolated systems and hypersurfaces moving by their curvature. *General Relativity and Gravitation*, 2002. 162-173.

[30] G. Huisken and T. Ilmanen. Higher regularity of the inverse mean curvature flow. *Journal of differential geometry*, 80(3):433–451, 2008.

[31] B. Lambert and J. Scheuer. The inverse mean curvature flow perpendicular to the sphere. *Mathematische Annalen*, 364.3-4 (2016): 1069-1093.

[32] D.A. Lee and A. Neves. The Penrose inequality for asymptotically locally hyperbolic spaces with nonpositive mass. *Communications in Mathematical Physics*, 339(2):327–352, 2015.

[33] H. Li and Y. Wei. On inverse mean curvature flow in Schwarzschild space and Kottler space. *Calculus of Variations and Partial Differential Equations*, 56(3):62, 2017.

[34] M. Makowski and J. Scheuer. Rigidity results, inverse curvature flows and Alexandrov-Fenchel type inequalities in the sphere. *Asian Journal of Mathematics*, 20(5):869–892, 2016.

[35] T. Marquardt Inverse mean curvature flow for star-shaped hypersurfaces evolving in a cone. *Journal of Geometric Analysis*, 23.3 (2013): 1303-1313.

[36] T. Marquardt Weak solutions of inverse mean curvature flow for hypersurfaces with boundary. *Journal für die reine und angewandte Mathematik*, 2017.728 (2017): 237-261.

[37] S. Rolf. Convex bodies: the Brunn–Minkowski theory. No. 151. Cambridge university press, 2014.

[38] R. Sacksteder. On hypersurfaces with nonnegative sectional curvatures. *Amer. J. Math.*, 82 (1960) 609-630.

[39] K. Smoczyk. Remarks on the inverse mean curvature flow. *Asian Journal of Mathematics*, 4(2):331–336, 2000.

[40] A. Stahl Regularity estimates for solutions to the mean curvature flow with a Neumann boundary condition. *Calculus of Variations and Partial Differential Equations*, 4.4 (1996): 385-407.

[41] A. Stahl Convergence of solutions to the mean curvature flow with a Neumann boundary condition. *Calculus of Variations and Partial Differential Equations*, 4.5 (1996): 421-441.

[42] J. Urbas. On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures. *Mathematische Zeitschrift*, 205(1):355–372, 1990.

[43] L. Wang. On the regularity theory of fully nonlinear parabolic equations: I. *Communications on pure and applied mathematics*, 45.1 (1992): 27-76.

[44] H. Wu. The spherical images of convex hypersurfaces. *Journal of Differential Geometry*, 9.2 (1974): 279-290.

[45] H. Zhou. Inverse mean curvature flows in warped product manifolds. *The Journal of Geometric Analysis*, pages 1–24, 2017.