

THE NILPOTENT CONE FOR CLASSICAL LIE SUPERALGEBRAS

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(Communicated by Sarah Witherspoon)

ABSTRACT. In this paper the authors introduce an analogue of the nilpotent cone, $\mathcal{N}$, for a classical Lie superalgebra, $\mathfrak{g}$, that generalizes the definition for the nilpotent cone for semisimple Lie algebras. For a classical simple Lie superalgebra, $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with $\text{Lie } G_{\bar{0}} = \mathfrak{g}_{\bar{0}}$, it is shown that there are finitely many $G_{\bar{0}}$ -orbits on $\mathcal{N}$. Later the authors prove that the Duflo-Serganova commuting variety, $\mathcal{X}$, is contained in $\mathcal{N}$ for any classical simple Lie superalgebra. Consequently, our finiteness result generalizes and extends the work of Duflo-Serganova on the commuting variety. Further applications are given at the end of the paper.

1. INTRODUCTION

1.1. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over the complex numbers and $\mathcal{N}$ be the set of nilpotent elements, often referred to as the *nilpotent cone*. In the case when $\mathfrak{g}$ is semisimple with $\mathfrak{g} = \text{Lie } G$, it is well-known that $\mathcal{N}$ has finitely many G -orbits. For the classical families of simple Lie algebras (root systems of types A - D) a parametrization of orbits is given by partitions under suitable conditions, and for the exceptional Lie algebras one can either use the Bala-Carter labelling or weighted Dynkin diagrams. For a semisimple Lie algebra, $\mathfrak{g}$, fundamental results in geometric representation theory have involved investigating the geometry of the nilpotent cone $\mathcal{N}$ (also its Springer resolution $\tilde{\mathcal{N}}$) and its relationship to the representation theory for $\mathfrak{g}$ (cf. [HTT]).

The nilpotent cone $\mathcal{N}$ can be realized as the zero set of the G -invariant polynomials with constant term zero on $\mathfrak{g}$. The G -invariant polynomials also have a direct connection with the semisimple elements. Under the Chevalley isomorphism theorem the restriction map induces an isomorphism $\text{res} : S(\mathfrak{g}^*)^G \rightarrow S(\mathfrak{t}^*)^W$ where $\mathfrak{t}$ is a maximal torus of $\mathfrak{g}$ and W is the Weyl group. The semisimple elements are those elements in $\mathfrak{g}$ that are G -conjugate to an element in $\mathfrak{t}$ [Hum, Section 0.1].

1.2. A similar picture arises in the study of classical simple Lie superalgebras, $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$. Boe, Kujawa and Nakano [BKN1] used invariant theory for reductive groups to show that there are natural classes of “subalgebras” that detect the cohomology. These subalgebras arise from considering “semisimple” elements of

Received by the editors November 17, 2020, and, in revised form, March 12, 2021.

2020 *Mathematics Subject Classification*. Primary 17B05, 17B10.

Key words and phrases. Lie superalgebras, representation theory.

Research of the first author was supported in part by NSF (RTG) grant DMS-1344994.

Research of the second author was supported in part by NSF grants DMS-1701768 and DMS-2101941.

the $G_{\bar{0}}$ -action on $\mathfrak{g}_{\bar{1}}$, and fall into two families: $\mathfrak{f}$ (when $\mathfrak{g}$ is stable) and $\mathfrak{e}$ (when $\mathfrak{g}$ is polar). If $\mathfrak{g}$ is a classical simple Lie superalgebra, then $\mathfrak{g}$ admits a stable action and in most cases $\mathfrak{g}$ admits a polar action (cf. [BKN1, Table 5]).

When stable and polar actions exist (cf. [LR, DK]), the restriction maps induce isomorphisms:

$$\begin{array}{ccccc} H^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}, \mathbb{C}) & \longrightarrow & H^\bullet(\mathfrak{f}, \mathfrak{f}_{\bar{0}}, \mathbb{C})^N & \longrightarrow & H^\bullet(\mathfrak{e}, \mathfrak{e}_{\bar{0}}, \mathbb{C})^{W_{\mathfrak{e}}} \\ \downarrow & & \downarrow & & \downarrow \\ S^\bullet(\mathfrak{g}_{\bar{1}}^*)^{G_{\bar{0}}} & \longrightarrow & S^\bullet(\mathfrak{f}_{\bar{1}}^*)^N & \longrightarrow & S^\bullet(\mathfrak{e}_{\bar{1}}^*)^{W_{\mathfrak{e}}} \end{array}$$

where N is a reductive group and $W_{\mathfrak{e}}$ is a finite pseudoreflection group. The finite generation of these cohomology rings was used in [BKN1] to define support varieties for $\mathfrak{g}$ -modules. In [GGNW] it was shown that the support varieties (appropriately) defined over $\mathfrak{g}$, $\mathfrak{f}$ and $\mathfrak{e}$ are isomorphic.

In order to have a complete picture, it is natural to ask whether there exists an algebraic variety consisting of “nilpotent elements” for classical Lie superalgebras that fits into this framework. Since $\mathfrak{g}$ is classical, $\mathfrak{g}_{\bar{0}} = \text{Lie } G_{\bar{0}}$ where $G_{\bar{0}}$ is a reductive algebraic group. In this paper we study a generalization of the nilpotent cone

$$\mathcal{N} = \mathcal{N}_{\mathfrak{g}} = Z(S^\bullet(\mathfrak{g}_{\bar{1}}^*)_+^{G_{\bar{0}}})$$

where $Z(S^\bullet(\mathfrak{g}_{\bar{1}}^*)_+^{G_{\bar{0}}})$ is the zero-set of $G_{\bar{0}}$ -invariant polynomials with constant term zero on $\mathfrak{g}_{\bar{1}}$. When $\mathfrak{g} = \mathfrak{q}(n)$, one obtains the nilpotent cone for the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$.

The construction in our paper is inspired by work of Kac [K] in 1980. He defined the nilvariety as the zero locus of the constant term zero $G_{\bar{0}}$ -invariant polynomials on V where V is a rational $G_{\bar{0}}$ -module. Kac’s results are in a more general context than our paper and there is some overlap in our results. We anticipate that the varieties $\mathcal{N}$ will play an important role in the representation theory for Lie superalgebras. This is evident in Section 5 with the strong connections with the Duflo-Serganova commuting varieties that were introduced in the mid 2000’s. In the late 2000’s, Gruson and Leidwanger [GL] investigated the case of the nilpotent cone in the orthosymplectic case and constructed a resolution of singularities. The aim of our paper is to present a unified self-contained treatment of $\mathcal{N}$ for classical simple Lie superalgebras that can be easily referenced by those working in super representation theory.

1.3. The paper is organized as follows. In Section 2, the nilpotent cone for classical Lie superalgebras is defined. We also indicate how this definition generalizes the definition of the nilpotent cone for complex semisimple Lie algebras. Our first main result (Theorem 3.1.1) in Section 3 demonstrates that the nilpotent cone for $\mathfrak{gl}(m|n)$ has finitely many $G_{\bar{0}}$ -orbits. Explicit orbit representatives are determined. In Section 4, we prove a theorem that allows us to extend the finiteness result to the nilpotent cone for other classical simple Lie superalgebras. The ideas of the theorem are originally due to Richardson and can be applied in cases when there is a suitable embedding of the classical simple Lie superalgebras into a general linear Lie superalgebra. With these tools, we show that for classical simple Lie superalgebras, $\mathcal{N}$ has finitely many $G_{\bar{0}}$ -orbits.

Duflo and Serganova [DS] introduced the commuting variety $\mathcal{X}$ for any finite-dimensional Lie superalgebra. They proved that for basic classical Lie superalgebras, $\mathcal{X}$ has finitely many $G_{\bar{0}}$ -orbits. In Section 5, we prove that for all classical simple Lie superalgebras, one has an inclusion $\mathcal{X} \subseteq \mathcal{N}$. In this way, one should consider the nilpotent cone as a larger algebraic variety whose geometric properties should encompass that of $\mathcal{X}$. We show that our results on the finiteness of orbits for $\mathcal{N}$ allow us to extend the finiteness results in [DS] to a wider class of Lie superalgebras (cf. Corollary 5.3.1).

2. PRELIMINARIES

2.1. Notation. Throughout this paper we will use the conventions in [BKN1, BKN2, BKN3, BKN4, LNZ, GGNW, Jan1]. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a *classical Lie superalgebra* over $k = \mathbb{C}$. This means there exists $G_{\bar{0}}$, a corresponding connected reductive algebraic group, such that $\mathrm{Lie} G_{\bar{0}} = \mathfrak{g}_{\bar{0}}$ where $\mathfrak{g}_{\bar{1}}$ is a $G_{\bar{0}}$ -module via the adjoint action. The Lie superalgebra $\mathfrak{g}$ is a *basic classical* if it is a classical Lie superalgebra with a nondegenerate invariant supersymmetric even bilinear form.

Let $\mathfrak{g}$ be a classical Lie superalgebra and $S^{\bullet}(\mathfrak{g}_{\bar{1}}^*)$ be the symmetric algebra on the dual of $\mathfrak{g}_{\bar{1}}$. We will often regard $S^{\bullet}(\mathfrak{g}_{\bar{1}}^*)$ as the polynomial functions on $\mathfrak{g}_{\bar{1}}$. Let $S^{\bullet}(\mathfrak{g}_{\bar{1}}^*)_+$ be the polynomials with constant term equal to zero. The algebraic group $G_{\bar{0}}$ acts on $\mathfrak{g}_{\bar{1}}$, so we can consider the $G_{\bar{0}}$ -invariant polynomials with zero constant term on $\mathfrak{g}_{\bar{1}}$ denoted by $S^{\bullet}(\mathfrak{g}_{\bar{1}}^*)_+^{G_{\bar{0}}}$. The *nilpotent cone*, $\mathcal{N}$, for $\mathfrak{g}$ is the zero set of these polynomials:

$$\mathcal{N} = Z(S^{\bullet}(\mathfrak{g}_{\bar{1}}^*)_+^{G_{\bar{0}}}).$$

Observe that $\mathcal{N} \subseteq \mathfrak{g}_{\bar{1}}$. The algebraic variety $\mathcal{N}$ is a $G_{\bar{0}}$ -invariant closed cone in $\mathfrak{g}_{\bar{1}}$.

2.2. Simple classical Lie superalgebra. The main results of the paper will be stated for classical “simple” Lie superalgebras. We will use the term simple Lie superalgebra to refer to the Lie superalgebras of general interest that are not simple in the true sense, but close enough to being simple (cf. [GGNW]). The Lie superalgebras that will be considered “simple” include:

- $\mathfrak{gl}(m|n)$, $\mathfrak{sl}(m|n)$, $\mathfrak{psl}(n|n)$
- $\mathfrak{osp}(m, n)$
- $D(2, 1, \alpha)$
- $G(3)$
- $F(4)$
- $\mathfrak{q}(n)$, $\mathfrak{psq}(n)$
- $\mathfrak{p}(n)$, $\tilde{\mathfrak{p}}(n)$

For the Lie superalgebras of Type Q, $\mathfrak{q}(n)$ is the Lie superalgebra with even and odd parts $\mathfrak{gl}(n)$, while $\mathfrak{psq}(n)$ is the corresponding simple subquotient of $\mathfrak{q}(n)$ (cf. [PS]). The Lie superalgebras that fall into the family of Type P include $\mathfrak{p}(n)$ and its enlargement $\tilde{\mathfrak{p}}(n)$.

2.3. Generalization of the ordinary nilpotent cone. We now indicate how our results generalize known results for the nilpotent cone for complex semisimple Lie algebras. Let $\mathfrak{a}$ be a complex semisimple Lie algebra, and set $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with $\mathfrak{g}_{\bar{0}} = \mathfrak{a} = \mathfrak{g}_{\bar{1}}$ as vector spaces. We can make $\mathfrak{g}$ into a Lie superalgebra by defining the bracket on $\mathfrak{g}_{\bar{0}}$ to be the ordinary Lie bracket on $\mathfrak{a}$. The bracket of an element

in $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is given by the adjoint action, and the bracket of any two elements in $\mathfrak{g}_{\bar{1}}$ is zero.

Let $G_{\bar{0}}$ be the semisimple simply connected group such that $\text{Lie } G_{\bar{0}} = \mathfrak{g}_{\bar{0}}$. Then $\mathcal{N}$ is the ordinary nilpotent cone for $\mathfrak{g}_{\bar{0}} = \mathfrak{a}$. One can also set this up for fields of prime characteristic, if one considers Lie algebras that arise as the Lie algebra of a semisimple algebraic group.

3. $G_{\bar{0}}$ -ORBITS ON $\mathcal{N}$: $\mathfrak{gl}(m|n)$ CASE

3.1. The adjoint action of $G_{\bar{0}} = GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$ on $\mathfrak{g}_{\bar{1}}$ is given by conjugation. Explicitly,

$$\begin{aligned} \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right] \cdot \left[\begin{array}{c|c} 0 & X^+ \\ \hline X^- & 0 \end{array} \right] &= \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right]^{-1} \left[\begin{array}{c|c} 0 & X^+ \\ \hline X^- & 0 \end{array} \right] \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right] \\ &= \left[\begin{array}{c|c} 0 & A^{-1}X^+B \\ \hline B^{-1}X^-A & 0 \end{array} \right]. \end{aligned}$$

In this case, [Fuks, Section 2.6] has determined the generators of the invariants, $S(\mathfrak{g}_{\bar{1}}^*)_{+}^{G_{\bar{0}}}$, to be

$$\text{Tr}((X^+X^-)^k), \quad k = 1, \dots, l$$

where $l = \min\{m, n\}$.

For Theorem 3.1.1 we recall the notion of a matrix in column echelon form. A matrix is in (reduced) column echelon form if it satisfies the following conditions:

- all columns that consist entirely of zero entries appear as the right-most columns of the matrix
- the first nonzero entry of each column is called the pivot, and the pivot is the only nonzero entry in its row
- if $j > i$, then the pivot of the nonzero column c_j lies in a row strictly below the row of the pivot of column c_i .

We also use the convention that each pivot element is 1. By transposing the matrix, the notion of row echelon form can be defined in a similar way.

In this section we will show that $\mathfrak{gl}(m|n)$ has finitely many $G_{\bar{0}}$ -orbits. Furthermore, explicit orbit representatives for this action will be exhibited. The results are summarized in Theorem 3.1.1 and the proof will be given in the next subsection.

Theorem 3.1.1. *Let $\mathfrak{g} = \mathfrak{gl}(m|n)$.*

- The number of $G_{\bar{0}}$ -orbits of the adjoint action on $\mathcal{N}$ is finite.*
- A complete set of orbit representatives is given by matrices*

$$Y = \left[\begin{array}{c|c} 0 & Y^+ \\ \hline Y^- & 0 \end{array} \right],$$

where

$$Y^+ = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array} \right] \quad \text{and} \quad Y^- = \left[\begin{array}{ccc|ccc} J_1 & & & | & 0 & 0 \\ & \ddots & & | & C_{r_1} & 0 \\ & & J_t & | & 0 & 0 \\ \hline - & R_{r_2} & - & | & 0 & 0 \\ 0 & 0 & 0 & | & 0 & I_s \\ 0 & 0 & 0 & | & 0 & 0 \end{array} \right].$$

Here I_r (resp. I_s) is the $r \times r$ (resp. $s \times s$) identity matrix, $J_1, \dots, J_t$ are Jordan blocks with zero eigenvalues, where the Jordan block J_i is of size $k_i \times k_i$, $i = 1, 2, \dots, t$ with $k_1 \geq k_2 \geq \dots \geq k_t$ with $r = \sum_{i=1}^t k_i$. Furthermore, the matrix C_{r_1} (resp. R_{r_2}) is of size $r \times r_1$ (resp. $r_2 \times r$) and of the form

$$C_{r_1} = (e_{i_1} \ e_{i_2} \ \cdots \ e_{i_{r_1}}), \quad R_{r_2} = (e_{j_1} \ e_{j_2} \ \cdots \ e_{j_{r_2}})^T$$

where e_k is the column vector with a single 1 in the k -th row and zeroes elsewhere. Each index i_p (resp. j_q) belongs to the set $\{k_1, k_1 + k_2, \dots, k_1 + k_2 + \dots + k_t\}$ (resp. $\{1, 1 + k_1, \dots, 1 + k_1 + \dots + k_{t-1}\}$).

3.2. Proof of Theorem 3.1.1. The proof of theorem in the prior subsection will entail several steps. We start with a general element $X \in \mathcal{N}$ where $\mathfrak{g} = \mathfrak{gl}(m|n)$. Through a series of conjugations (i.e., applications of elements in $G_{\bar{0}}$), the element X will be transformed into a matrix Y of the form in Theorem 3.1.1(b). In the process of this transformation, we will use Y (including Y^+ and Y^-) to denote the current matrix under the series of transformations.

3.2.1. Let $X \in \mathcal{N}$. Using standard results in linear algebra (involving equivalence of matrices) and the action of $(A, B) \in G_{\bar{0}}$ on X , there exists $(A, B) \in G_{\bar{0}}$ such that $A^{-1}X^+B = Y^+$ where

$$Y^+ = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array} \right]$$

and where r is the rank of X^+ .

3.2.2. The next step is to identify and later work with $(A, B) \in G_{\bar{0}}$ that centralize Y^+ , which is equivalent to the condition: $A^{-1}Y^+B = Y^+$. In order to elaborate further, it will be useful to consider A and B in block matrix form:

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \right] \quad B = \left[\begin{array}{c|c} B_{11} & B_{21} \\ \hline B_{21} & B_{22} \end{array} \right].$$

The centralizing condition is equivalent to

$$\left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline 0 & 0 \end{array} \right] = \left[\begin{array}{c|c} A_{11} & 0 \\ \hline A_{21} & 0 \end{array} \right].$$

Therefore, one has

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array} \right] \quad B = \left[\begin{array}{c|c} A_{11} & 0 \\ \hline B_{21} & B_{22} \end{array} \right].$$

Note we have a formula for B^{-1} in terms of blocks:

$$B^{-1} = \left[\begin{array}{c|c} A_{11}^{-1} & 0 \\ \hline -B_{22}^{-1}B_{21}A_{11}^{-1} & B_{22}^{-1} \end{array} \right].$$

We can now provide a formula for the action of (A, B) in the centralizer of Y^+ on Y^- in block form:

$$\begin{aligned} B^{-1}Y^-A &= \left[\begin{array}{c|c} A_{11}^{-1} & 0 \\ \hline -B_{22}^{-1}B_{21}A_{11}^{-1} & B_{22}^{-1} \end{array} \right] \left[\begin{array}{c|c} Y_{11}^- & Y_{12}^- \\ \hline Y_{21}^- & Y_{22}^- \end{array} \right] \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array} \right] \\ &= \left[\begin{array}{c|c} A_{11}^{-1}Y_{11}^-A_{11} & A_{11}^{-1}(Y_{11}^-A_{12} + Y_{12}^-A_{22}) \\ \hline B_{22}^{-1}(Y_{21}^-A_{11} - B_{21}A_{11}^{-1}Y_{11}^-A_{11}) & B_{22}^{-1}(Y_{21}^-A_{12} + Y_{22}^-A_{22}) - B_{22}^{-1}B_{21}A_{11}^{-1}(Y_{11}^-A_{12} + Y_{12}^-A_{22}) \end{array} \right]. \end{aligned}$$

As long as we work with $(A, B) \in G_{\bar{0}}$ that centralize Y^+ (i.e., $(A, B) \in c_{G_{\bar{0}}}(Y^+)$), we can focus on transforming Y^- into the desired form.

3.2.3. Observe that Y_{11}^- can be put into its Jordan form J (upper triangular) via A_{11} . If one chooses A_{11} to be in the centralizer of J , one can replace both Y_{11}^- and $A_{11}^{-1}Y_{11}^-A_{11}$ with J in the expression above. By operating with elements $(A, B) \in c_{G_0}(Y^+)$ (i.e., elements in the centralizer of Y^+ in G_0) with A_{11} that centralizes J , our new expression for the action on Y^- is

$$B^{-1}Y^-A = \left[\begin{array}{c|c} J & A_{11}^{-1}(JA_{12} + Y_{12}^-A_{22}) \\ \hline B_{22}^{-1}(Y_{21}^-A_{11} - B_{21}J) & B_{22}^{-1}(Y_{21}^-A_{12} + Y_{22}^-A_{22}) - B_{22}^{-1}B_{21}A_{11}^{-1}(JA_{12} + Y_{12}^-A_{22}) \end{array} \right].$$

3.2.4. We also observe that all the eigenvalues of J are zero since

$$Y^+Y^- = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array} \right] \left[\begin{array}{c|c} J & X_{12}^- \\ \hline Y_{21}^- & Y_{22}^- \end{array} \right] = \left[\begin{array}{c|c} J & Y_{12}^- \\ \hline 0 & 0 \end{array} \right].$$

The condition that $\text{Tr}((Y^+Y^-)^k) = 0$ for $k = 1, \dots, l$ where $l = \min\{m, n\}$ implies that if $\lambda_1, \lambda_2, \dots, \lambda_r$ are the eigenvalues of J then $\lambda_1^k + \lambda_2^k + \dots + \lambda_r^k = 0$ for $k = 1, 2, \dots, l$. Since $r \leq l$ this implies that $\lambda_j = 0$ for all j .

3.2.5. Let J be a Jordan canonical form with Jordan blocks $J_1, J_2, \dots, J_t$ of sizes $k_1 \geq k_2 \geq \dots \geq k_t$ with all Jordan blocks having eigenvalue zero:

$$J = \begin{pmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & J_t \end{pmatrix}.$$

Consider the action of $(A, B) \in c_{G_0}(Y^+)$ with A_{11} , A_{22} , and B_{22} to be (appropriately sized) identity matrices. Then

$$B^{-1}Y^-A = \left[\begin{array}{c|c} J & JA_{12} + Y_{12}^- \\ \hline Y_{21}^- - B_{21}J & * \end{array} \right].$$

The matrix JA_{12} (resp. $B_{21}J$) consists of matrix entries with zeros in rows (resp. columns) $k_1, k_1 + k_2, \dots, k_1 + k_2 + \dots + k_t$ (resp. $1, 1 + k_1, 1 + k_1 + k_2, \dots, 1 + k_1 + k_2 + \dots + k_{t-1}$) and arbitrary entries in the other rows (resp. columns). Therefore, one can choose entries in A_{12} (resp. B_{21}) to make $JA_{12} + Y_{12}^-$ (resp. $Y_{21}^- - B_{21}J$) into a matrix with possibly non-zero entries in rows (resp. columns) $k_1, k_1 + k_2, \dots, k_1 + k_2 + \dots + k_t$ (resp. $1, 1 + k_1, 1 + k_1 + k_2, \dots, 1 + k_1 + k_2 + \dots + k_{t-1}$) and zeros in the other rows (resp. columns).

3.2.6. Now consider the action of $(A, B) \in c_{G_0}(Y^+)$ with A_{11} being the identity matrix, $A_{12} = 0$ and $B_{21} = 0$. Then

$$B^{-1}Y^-A = \left[\begin{array}{c|c} J & Y_{12}^-A_{22} \\ \hline B_{22}^{-1}Y_{21}^- & * \end{array} \right].$$

Now one can make A_{22} (resp. B_{22}^{-1}) into a product of a permutation matrix and elementary matrices to transform $Y_{12}^-A_{22}$ (resp. $B_{22}^{-1}Y_{21}^-$) into a column (resp. row) echelon form with possibly non-zero entries in rows (resp. columns) $k_1, k_1 + k_2, \dots, k_1 + k_2 + \dots + k_t$ (resp. $1, 1 + k_1, 1 + k_1 + k_2, \dots, 1 + k_1 + k_2 + \dots + k_{t-1}$) and zeros in the other rows (resp. columns).

3.2.7. The next step is to transform Y_{12}^- into a matrix C_{r_1} of the form stated in the theorem. Let $(A, B) \in c_{G_0}(Y^+)$ with $A_{12} = 0$, $B_{21} = 0$ and A_{22} , B_{22} being identity matrices. Then

$$B^{-1}Y^-A = \left[\begin{array}{c|c} \frac{A_{11}^{-1}JA_{11}}{Y_{21}^-A_{11}} & \frac{A_{11}^{-1}Y_{12}^-}{*} \end{array} \right].$$

By using the action of $A_{11} \in c(J)$ (centralizer of J), we can assume without a loss of generality that the pivots in Y_{12}^- are all 1's. Normally we can transform Y_{12}^- into an elementary row echelon form by making A_{11}^{-1} into a product of elementary matrices that performs the row operations used in Gaussian elimination, and since Y_{12}^- is already in column echelon form from the previous step, this is equivalent to the form of C_{r_1} . The issue is that this product of elementary matrices need not centralize J . This problem can be remedied for the cases of the elementary row operations of row addition and row scaling as follows.¹

The matrix Y_{12}^- is in column echelon form and has non-zero entries in rows $k_1, k_1+k_2, \dots, k_1+k_2+\dots+k_t$ with zeros in the other rows. Set $f(j) = k_1+k_2+\dots+k_j$, $j = 1, 2, \dots, t$. Furthermore, the non-zero pivots can lie in matrix positions $(f(j), j)$ for $j = 1, 2, \dots, t$. We want to use A_{11}^{-1} to clear the matrix entries in the columns directly below the pivots. If there was no restriction on A_{11}^{-1} one can accomplish this with a product of elementary matrices. For example, if one wants to eliminate a non-zero entry α in position $(f(i), j)$, then one can apply the elementary matrix $E_{f(j), f(i)}(-\alpha)$ on the left which replaces Row $f(j)$ with $-\alpha \times \text{Row } f(j) + \text{Row } f(i)$.

Consider the matrix $L_{f(j), f(i)}(-\alpha)$ which is in the centralizer of J where

$$L_{f(j), f(i)}(-\alpha) = \begin{pmatrix} I_{k_1} & 0 & 0 & \cdots & 0 \\ Z_{2,1} & I_{k_2} & 0 & \cdots & 0 \\ Z_{3,1} & Z_{3,2} & I_{k_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ Z_{t,1} & Z_{t,2} & Z_{t,3} & \cdots & I_{k_t} \end{pmatrix}.$$

Here I_{k_j} are $k_j \times k_j$ identity matrices and $Z_{j,i}$ are $k_j \times k_i$ matrices where $i = 1, 2, \dots, t-1$ and $j = 2, 3, \dots, t$. Moreover, set $Z_{j',i'} = 0$ for $(j, i) \neq (j', i')$, and the block matrix

$$Z_{j,i} = \begin{pmatrix} 0 & -\alpha I_{k_j} \end{pmatrix}.$$

It can be directly verified that $L_{f(j), f(i)}(-\alpha)$ will have the same effect as $E_{f(j), f(i)}(-\alpha)$ on the Gaussian elimination process on Y_{12}^- .

Similarly, it can be directly verified that the matrix

$$M_{k_j}(\alpha) = \begin{pmatrix} I_{k_1} & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \cdots & 0 \\ 0 & 0 & \alpha I_{k_j} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & I_{k_t} \end{pmatrix}$$

will scale row $k_1 + \dots + k_j$ by α and also centralizes J .

¹The permutation matrices necessary to perform the row swapping operation of Gaussian elimination are also not in the centralizer of J ; however there are only finitely many possible forms of C_{r_1} up to permutation.

Therefore, if the rank of the current iteration of Y_{12}^- is r_1 with pivots in rows $i_1, \dots, i_{r_1}$, then it is transformed to

$$\begin{pmatrix} C_{r_1} & 0 & \cdots & 0 \end{pmatrix}$$

where

$$C_{r_1} = (e_{i_1} \quad e_{i_2} \quad \cdots \quad e_{i_{r_1}})$$

and where e_{i_p} is the column vector with a 1 in the i_p -th row and zeroes elsewhere. Furthermore, each index i_p is an element of $\{k_1, k_1 + k_2, \dots, k_1 + k_2 + \dots + k_t\}$. That is, the nonzero entry of each column e_{i_p} occurs in one of the aforementioned rows.

3.2.8. We now perform the same procedure as in the last step to transform Y_{21}^- into a matrix R_{r_2} with form analogous to the transpose of C_{r_1} . Recall from Section 3.2.6 that we have already transformed Y_{21}^- into row echelon form. Let $(A, B) \in c_{G_0}(Y^+)$ with $A_{12} = 0$, $B_{21} = 0$ and A_{22} , B_{22} being identity matrices. Here one makes A_{11} into a product of matrices that are upper triangular, are in the centralizer of J , and perform column operations to clear out the entries in the pivot rows of all non-zero entries (except for the pivot). In this case each index j_q is an element of $\{1 + k_1, 1 + k_1 + k_2, \dots, 1 + k_1 + k_2 + \dots + k_{t-1}\}$ and so the nonzero entry of each row occurs in one of these columns.

Note that in the process we have changed Y_{12}^- out of the form of C_{r_1} . Consider A_{11} . Then A_{11}^{-1} will still be upper triangular and $A_{11}^{-1}Y_{12}^-$ will be a matrix with 1's in the same pivot entries as Y_{12}^- with possible non-zero entries above and to the right of the position of the pivot entries in Y_{12}^- . The next two steps will correct this issue.

3.2.9. Now let $(A, B) \in c_{G_0}(Y^+)$ with A_{11} , A_{22} , and B_{22} be identity matrices, and $B_{21} = 0$. Then

$$B^{-1}Y^-A = \left[\begin{array}{c|c} J & JA_{12} + Y_{12}^- \\ \hline Y_{21}^- & * \end{array} \right].$$

We can now transform Y_{12}^- into a matrix in row echelon form by choosing A_{12} to kill the non-zero entries in the “non-pivot” rows. Note that Y_{21}^- is unchanged.

3.2.10. Let $(A, B) \in c_{G_0}(Y^+)$ with A_{11} , B_{22} being identity matrices, and $A_{12} = 0$, $B_{21} = 0$. Then

$$B^{-1}Y^-A = \left[\begin{array}{c|c} J & Y_{12}^-A_{22} \\ \hline Y_{21}^- & * \end{array} \right].$$

The matrix A_{22} can be chosen to clear out the non-zero entries that are not pivots to make Y_{12}^- in column echelon form. Again Y_{21}^- is unchanged in the process.

3.2.11. The final step is to transform Y_{22}^- into a matrix of the desired form. We will work with (A, B) in the centralizer of Y^+ with A_{11} centralizing J . Let $A_{11} = I$ and $A_{12}, B_{21} = 0$. Then Y_{22}^- is transformed to $B_{22}^{-1}Y_{22}^-A_{22}$. Let $\text{rank}(C_{r_1}) = r_1$ and $\text{rank}(R_{r_2}) = r_2$.

Write

$$Y_{22}^- = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix}$$

where ξ_{11} is an $r_2 \times r_1$ matrix. Moreover, ξ_{12} is $r_2 \times (n - r - r_1)$, ξ_{21} is $(m - r - r_2) \times r_1$, and ξ_{22} is $(m - r - r_2) \times (n - r - r_1)$.

Now to centralize C_{r_1} and R_{r_2} we require the first r_1 rows of A_{22} to be the first r_1 rows of the identity matrix and similarly the first r_2 columns of B_{22}^{-1} to be the first r_2 columns of the identity matrix. Therefore, we can write A_{22} and B_{22}^{-1} in block form (in a similar way with Y_{22}^-) as

$$A_{22} = \begin{pmatrix} I_{r_1} & 0 \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \text{ and } B_{22}^{-1} = \begin{pmatrix} I_{r_2} & \beta_{12} \\ 0 & \beta_{22} \end{pmatrix}.$$

Then

$$B_{22}^{-1} Y_{22}^- A_{22} = \begin{pmatrix} \xi_{11} + \beta_{12} \xi_{21} + \xi_{12} \alpha_{21} + \beta_{12} \xi_{22} \alpha_{21} & \xi_{12} \alpha_{22} + \beta_{12} \xi_{22} \alpha_{22} \\ \beta_{22} \xi_{21} + \beta_{22} \xi_{22} \alpha_{21} & \beta_{22} \xi_{22} \alpha_{22} \end{pmatrix}.$$

This shows that we can send ξ_{22} to

$$\begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}$$

where $s = \text{rank}(\xi_{22})$.

3.2.12. Now choose $(A, B) \in c_{G_0}(Y^+)$ with A_{11}, A_{22} and B_{22}^{-1} to be the identity, while allowing A_{12} and B_{21} to be free. This action will fix ξ_{22} and image of Y_{22}^- is

$$(3.2.1) \quad -B_{21} J A_{12} - B_{21} Y_{12}^- + Y_{21}^- A_{12} + Y_{22}^-.$$

Set B_{21} to be zero. Let the pivots of Y_{21}^- be in columns $j_1, \dots, j_{r_2}$. Then set the j_1 -th row of A_{12} to be the negative of the 1st row of Y_{22}^- , the j_2 -th row of A_{12} to be the negative of the 2nd row of Y_{22}^- and so forth. Then by (3.2.1), (A, B) sends ξ_{11} and ξ_{12} to 0.

3.2.13. Finally, we need to choose $(A, B) \in c_{G_0}(Y^+)$ that stabilizes the current $Y_{11}^-, Y_{12}^-, Y_{21}^-, \xi_{11}, \xi_{12}, \xi_{22}$ and sends ξ_{21} to 0. This can be accomplished by setting A_{11}, A_{22} and B_{22}^{-1} to be the identity, and A_{12} to be zero. If the pivots of Y_{12}^- are in rows $i_1, \dots, i_{r_1}$, then set the i_1 -th column of B_{21} to be the 1st column of Y_{22}^- , the i_2 -th column of B_{21} to be the 2nd column of Y_{22}^- and so forth. This concludes the proof of Theorem 3.1.1.

Note that it can be checked using our proof that none of the orbit representatives constructed are in the same $G_{\bar{0}}$ -orbit.

4. $G_{\bar{0}}$ -ORBITS ON $\mathcal{N}$: GENERAL CASE

4.1. In the ordinary Lie algebra case, the adjoint action of the algebraic group G on $\mathfrak{g}$ is known to have finitely many nilpotent orbits via Richardson's Theorem (see [Hum, Theorem 3.8]). In this section, we prove an appropriate generalization for Lie superalgebras. We begin by stating Lemma 4.1.1 whose proof can be found in [Jan2, Section 2.4].

Lemma 4.1.1. *Let G be an algebraic group and let H be a closed subgroup of G . Let X be a G -variety and let Y be a closed and H -invariant subvariety of X . Suppose that for all $y \in Y$,*

$$(4.1.1) \quad T_y(G \cdot y) \cap T_y(Y) \subseteq (d\pi_y)_{id}(\text{Lie } H),$$

where T_y denotes the tangent space at y and $\pi_y : G \rightarrow G \cdot y$ sends g to gy . Then the intersection with Y of each G -orbit in X is a finite union of H -orbits.

4.2. Generalization of Richardson's Theorem. We can now state a generalization of Richardson's Theorem in the context of Lie superalgebras.

Theorem 4.2.1. *Let $G_{\bar{0}}$ be a closed subgroup of some $GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra with $\text{Lie } G_{\bar{0}} = \mathfrak{g}_{\bar{0}}$. Suppose there exists a supersubspace $M \subseteq \mathfrak{gl}(m|n)$ such that $\mathfrak{gl}(m|n) = M \oplus \mathfrak{g}$ and $[\mathfrak{g}, M] \subseteq M$. Then the intersection with $\mathfrak{g}$ of each $GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$ -orbit in $\mathfrak{gl}(m|n)$ is a union of finitely many $G_{\bar{0}}$ -orbits.*

Proof. Assume such a complement M exists. We show that condition (4.1.1) of Lemma 4.1.1 is satisfied where $X = \mathfrak{gl}(m|n)$, $Y = \mathfrak{g}$, $G = GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$, and $H = G_{\bar{0}}$. This means one must show for every $y \in \mathfrak{g}$

$$T_y(G \cdot y) \cap T_y(\mathfrak{g}) \subseteq (d\pi_y)_{id}(\mathfrak{g}_{\bar{0}}).$$

We observe that $T_y(\mathfrak{g}) = \mathfrak{g}$ and $(d\pi_y)_{id}(\mathfrak{g}_{\bar{0}}) = [\mathfrak{g}_{\bar{0}}, y]$.

Now showing (4.1.1) is equivalent in our case to showing

$$(4.2.1) \quad T_y(G \cdot y) \cap \mathfrak{g} \subseteq [\mathfrak{g}_{\bar{0}}, y],$$

for every $y \in \mathfrak{g}$.

Note that $\text{Lie } G_x = (\mathfrak{gl}(m) \oplus \mathfrak{gl}(n))_x$, and by a standard fact in [Bor, Prop. 9.1] this is equivalent to $T_y(G \cdot y) = [\mathfrak{gl}(m) \oplus \mathfrak{gl}(n), y]$.

Now apply the complement condition to obtain that

$$[\mathfrak{gl}(m) \oplus \mathfrak{gl}(n), y] = [M_{\bar{0}} \oplus \mathfrak{g}_{\bar{0}}, y] = [M_{\bar{0}}, y] + [\mathfrak{g}_{\bar{0}}, y].$$

By assumption $[M_{\bar{0}}, y] \subseteq M$, so (4.2.1) becomes

$$T_y(G \cdot y) \cap \mathfrak{g} \subseteq (M + [\mathfrak{g}_{\bar{0}}, y]) \cap \mathfrak{g}$$

and since $M \cap \mathfrak{g} = \{0\}$ and $[\mathfrak{g}_{\bar{0}}, y] \subseteq \mathfrak{g}$, this reduces to

$$T_y(G \cdot y) \cap \mathfrak{g} \subseteq [\mathfrak{g}_{\bar{0}}, y].$$

So (4.2.1) is satisfied and the result follows by Lemma 4.1.1. $\square$

4.3. In the case when Richardson's Theorem is applied to show the finiteness of G -orbits for the nilpotent cone for complex semisimple Lie algebras, the existence of such an M is guaranteed via complete reducibility. More specifically, this follows from regarding $\mathfrak{gl}(n)$ as a G -module under the adjoint action. Then $\mathfrak{g} = \text{Lie } G$ is a submodule and therefore has a vector space complement M in $\mathfrak{gl}(n)$ that is invariant under the adjoint action of $\mathfrak{g}$.

In the situation for Lie superalgebras, we can apply this same reasoning to $G_{\bar{0}}$ acting on $\mathfrak{g}_{\bar{0}}$ to produce an $M_{\bar{0}}$ satisfying $\mathfrak{gl}(m|n)_{\bar{0}} = \mathfrak{g}_{\bar{0}} \oplus M_{\bar{0}}$ and $G_{\bar{0}} \cdot M_{\bar{0}} \subseteq M_{\bar{0}}$. Then $M_{\bar{0}}$ will also be invariant under the derived $\mathfrak{g}_{\bar{0}}$ action, so that $[\mathfrak{g}_{\bar{0}}, M_{\bar{0}}] \subseteq M_{\bar{0}}$. However, an issue arises when considering $\mathfrak{g}_{\bar{1}}$. We can still regard $\mathfrak{g}_{\bar{1}}$ as a $G_{\bar{0}}$ -module and produce a complement $M_{\bar{1}}$, but since the derived action involves only $\mathfrak{g}_{\bar{0}}$, we know nothing about $[\mathfrak{g}_{\bar{1}}, M_{\bar{1}}]$.

In order to prove finiteness of orbits for $\mathcal{N}$ in the superalgebra case we construct M in a case-by-case manner and show directly that $[\mathfrak{g}_{\bar{i}}, M_{\bar{j}}] \subseteq M_{\overline{i+j}}$. Details can be found in Appendix A. It is worth noting that the methods we use to produce these complements are analogous to the methods used to produce complements in the characteristic p case of Richardson's Theorem. Therefore, even though all of the Lie superalgebras considered here are over $\mathbb{C}$, we still need to use ideas from the characteristic p case in order to produce compatible complements.

4.4. We can now verify the finiteness of $G_{\bar{0}}$ -orbits on $\mathcal{N}$.

Theorem 4.4.1. *Let $\mathfrak{g}$ be a classical simple Lie superalgebra over $\mathbb{C}$. Then $\mathcal{N}$ has finitely many $G_{\bar{0}}$ -orbits.*

Proof. For classical simple Lie superalgebras $\mathfrak{g}$ other than $D(2, 1, \alpha)$, $G(3)$ and $F(4)$, there exists an embedding $\mathfrak{g} \hookrightarrow \mathfrak{g}' \cong \mathfrak{gl}(m|n)$ and a supersubspace $M \subseteq \mathfrak{g}'$ such that $\mathfrak{g}' = M \oplus \mathfrak{g}$ and $[\mathfrak{g}, M] \subseteq M$. The embeddings and complements are described in Section 4.3 and Appendix A.

Next we need to show that $\mathcal{N}_{\mathfrak{g}} \subseteq \mathcal{N}_{\mathfrak{g}'}$. We will prove a stronger statement that $\mathcal{N}_{\mathfrak{g}'} \cap \mathfrak{g}_{\bar{1}} = \mathcal{N}_{\mathfrak{g}}$. First we prove that $\mathcal{N}_{\mathfrak{g}} \subseteq \mathcal{N}_{\mathfrak{g}'} \cap \mathfrak{g}_{\bar{1}}$. Let $z \in \mathcal{N}_{\mathfrak{g}}$; then $f(z) = 0$ for every $f(x) \in S^{\bullet}(\mathfrak{g}_{\bar{1}}^*)^{G_{\bar{0}}}$. We have the identifications:

$$(4.4.1) \quad S^{\bullet}((\mathfrak{g}'_{\bar{1}})^*)^{G'_{\bar{0}}} \subseteq S^{\bullet}((\mathfrak{g}'_{\bar{1}})^*)^{G_{\bar{0}}} = [S^{\bullet}(\mathfrak{g}_{\bar{1}}^*) \otimes S^{\bullet}(M_{\bar{1}}^*)]^{G_{\bar{0}}}.$$

Under this identification, one can regard $g(x) = g(p, q) \in S^{\bullet}((\mathfrak{g}'_{\bar{1}})^*)^{G'_{\bar{0}}}$. If $z \in \mathcal{N}_{\mathfrak{g}} \subseteq \mathfrak{g}_{\bar{1}}$ then $g(z) = g(z, 0) = 0$; thus $z \in \mathcal{N}_{\mathfrak{g}'}$.

The other inclusion, $\mathcal{N}_{\mathfrak{g}'} \cap \mathfrak{g}_{\bar{1}} \subseteq \mathcal{N}_{\mathfrak{g}}$, uses properties of the embedding described in the first paragraph. One has $S^{\bullet}(\mathfrak{g}_{\bar{1}}^*)^{G_{\bar{0}}} = S^{\bullet}(\mathfrak{g}_{\bar{1}}^*)^{\mathfrak{g}_{\bar{0}}}$, and it will be more convenient to use Lie algebra invariants. We have

$$(4.4.2) \quad S^{\bullet}((\mathfrak{g}'_{\bar{1}})^*)^{\mathfrak{g}'_{\bar{0}}} = [S^{\bullet}(\mathfrak{g}_{\bar{1}}^*) \otimes S^{\bullet}(M_{\bar{1}}^*)]^{\mathfrak{g}'_{\bar{0}}} \subseteq [S^{\bullet}(\mathfrak{g}_{\bar{1}}^*) \otimes S^{\bullet}(M_{\bar{1}}^*)]^{\mathfrak{g}_{\bar{0}}},$$

and

$$(4.4.3) \quad S^{\bullet}(\mathfrak{g}_{\bar{1}}^*)^{\mathfrak{g}_{\bar{0}}} \subseteq [S^{\bullet}(\mathfrak{g}_{\bar{1}}^*) \otimes S^{\bullet}(M_{\bar{1}}^*)]^{\mathfrak{g}_{\bar{0}}}.$$

The inclusion (4.4.3) is given by $f(x) \mapsto f(x) \otimes 1$. Let $h(x) \in S^{\bullet}(\mathfrak{g}_{\bar{1}}^*)^{\mathfrak{g}_{\bar{0}}}$. Let $p = g_{\bar{0}} + m_{\bar{0}} \in \mathfrak{g}'_{\bar{0}}$ where $g_{\bar{0}} \in \mathfrak{g}_{\bar{0}}$ and $m_{\bar{0}} \in M_{\bar{0}}$. Using the inclusion in (4.4.3) and the fact that $[M_{\bar{0}}, \mathfrak{g}_{\bar{1}}] \subseteq M_{\bar{1}}$, it follows that

$$p.h(x) = -h([g_{\bar{0}}, x] + [m_{\bar{0}}, x]) = -h([g_{\bar{0}}, x]) = g_{\bar{0}}.h(x) = 0.$$

From (4.4.2), if $z \in \mathcal{N}_{\mathfrak{g}'} \cap \mathfrak{g}_{\bar{1}}$ then $z \in \mathcal{N}_{\mathfrak{g}}$.

We can now prove the finiteness of $G_{\bar{0}}$ -orbits on $\mathcal{N} := \mathcal{N}_{\mathfrak{g}}$. Let $G_{\bar{0}} \cdot y \in \mathcal{N}$. Set $G'_{\bar{0}} = GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$. Then $G'_{\bar{0}} \cdot y$ contains $G_{\bar{0}} \cdot y$, and $y \in \mathcal{N}_{\mathfrak{g}'}$. Now the finiteness of $G_{\bar{0}}$ -orbits on $\mathcal{N}$ follows from the finiteness of orbits for the nilpotent cone of $\mathfrak{gl}(m|n)$ and the fact that the intersection of any orbit in $\mathcal{N}_{\mathfrak{g}'}$ with $\mathfrak{g}$ contains only finitely many $G_{\bar{0}}$ -orbits (see Theorem 4.2.1).

Next we consider the remaining cases when $\mathfrak{g}$ is an exceptional Lie superalgebra. Let $\mathfrak{g} = D(2, 1, \alpha)$. Then $G_{\bar{0}} \cong SL_2 \times SL_2 \times SL_2$ with $\mathfrak{g}_{\bar{1}} = V \boxtimes V \boxtimes V$ where V is the two-dimensional natural representation. When $\alpha = 1$, one has $D(2, 1, \alpha) = \mathfrak{osp}(4, 2)$ [CW, 1.1.5], so in this case $\mathcal{N}$ has finitely many $G_{\bar{0}}$ -orbits from the argument in the preceding paragraph. Now the action of $G_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ for $D(2, 1, \alpha)$ does not depend on α . Hence, for arbitrary α , $\mathcal{N}$ has finitely many orbits.

For $\mathfrak{g} = G(3)$ or $F(4)$, one can argue the finiteness as follows. Let $\mathfrak{g} = G(3)$. In this case $\mathfrak{g}_{\bar{1}} = V \boxtimes Z$ where V is the 2-dimensional natural representation for $SL_2 := SL_2(\mathbb{C})$ and Z is the 7-dimensional irreducible representation for G_2 . Let $v_H = (1, 0)^T$ and $v_L = (0, 1)^T$ be vectors forming the standard basis for V and let z_H be a highest weight vector for Z . If $x \in \mathcal{N}$ then $x = v_H \otimes p_1 + v_L \otimes p_2$. If $p_1 = 0$ or $p_2 = 0$ then we can use the fact that SL_2 acts transitively on V and G_2 acts transitively on Z to show that x is $G_{\bar{0}}$ -conjugate to $v_H \otimes z_H$.

Now suppose that $p_1 \neq 0$ and $p_2 \neq 0$. First, we can conjugate x to $x_1 = v_H \otimes z_H + v_L \otimes p'_2$. Using the Bruhat decomposition for G_2 one can show that

if B is a Borel subgroup (corresponding to the positive roots) for G_2 then there are finitely many B -orbits on Z with orbit representatives given by weight vectors of the form z_γ where γ is a short root for G_2 . The group $B = T \ltimes U$ where U acts trivially on z_H and T acts by scaling z_H . Thus, x_1 is $G_{\bar{0}}$ -conjugate to $x_2 = \alpha(v_H \otimes z_H) + v_L \otimes z_\gamma \neq 0$. Moreover, since $x_2 \in \mathcal{N}$ and satisfies a 4th degree $T_{\bar{0}}$ -invariant polynomial ($T_{\bar{0}}$ a maximal torus for $G_{\bar{0}}$), it follows that z_γ is not a multiple of a lowest weight vector z_L . Now one can use $T_{\bar{0}}$ to conjugate x_2 to $x_3 = v_H \otimes z_H + v_L \otimes z_\gamma$. Consequently, there are only finitely many $G_{\bar{0}}$ -orbits on $\mathcal{N}$.

A similar argument can be used to prove the finiteness of $G_{\bar{0}}$ -orbits for $\mathfrak{g} = F(4)$. Our conclusions on the finiteness for $G(3)$ and $F(4)$ can also be found in [K, Table IV]. $\square$

5. CONNECTIONS WITH THE DUFLO-SERGANOVA SELF-COMMUTING VARIETY

5.1. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a finite-dimensional complex Lie superalgebra with $\text{Lie } G_{\bar{0}} = \mathfrak{g}_{\bar{0}}$. Duflo and Serganova defined the *self-commuting variety* as

$$\mathcal{X} = \{x \in \mathfrak{g}_{\bar{1}} : [x, x] = 0\}.$$

The variety $\mathcal{X}$ is a $G_{\bar{0}}$ -invariant conical variety of $\mathfrak{g}_{\bar{1}}$. In [DS], it was shown for a finite-dimensional $\mathfrak{g}$ -module, M , one can define a subvariety $\mathcal{X}_M$ of $\mathcal{X}$. The collection of these associated varieties governs the representation theory of $\mathfrak{g}$.

5.2. Theorem 5.2.1 shows that under suitable conditions on $\mathfrak{g}$, the self-commuting variety is contained in the nilpotent cone of $\mathfrak{g}$.

Theorem 5.2.1. *Let $\mathfrak{g}$ be a classical Lie superalgebra such that*

- (a) *there exists an embedding $\mathfrak{g} \hookrightarrow \mathfrak{g}' \cong \mathfrak{gl}(m|n)$,*
- (b) *there exists a supersubspace $M \subseteq \mathfrak{g}'$ such that $\mathfrak{g}' = M \oplus \mathfrak{g}$ and $[\mathfrak{g}, M] \subseteq M$.*

Then $\mathcal{X} \subseteq \mathcal{N}$.

Proof. Let $\mathcal{X} = \mathcal{X}_{\mathfrak{g}}$ (resp. $\mathcal{X}_{\mathfrak{g}'}$) be the self-commuting variety of $\mathfrak{g}$ (resp. $\mathfrak{g}'$). Similarly, denote the nilpotent cone of $\mathfrak{g}$ (resp. $\mathfrak{g}'$) by $\mathcal{N} = \mathcal{N}_{\mathfrak{g}}$ (resp. $\mathcal{N}_{\mathfrak{g}'}$).

Recall from Section 3.1 that $\mathcal{N}_{\mathfrak{g}'}$ is defined as the zero set of $\text{Tr}((X^+ X^-)^k)$, $k = 1, \dots, l$ where $l = \min\{m, n\}$. This characterization can be used to show that

$$(5.2.1) \quad \mathcal{X}_{\mathfrak{g}'} \subseteq \mathcal{N}_{\mathfrak{g}'}.$$

Moreover, using the definition of the self-commuting variety, one has

$$(5.2.2) \quad \mathcal{X}_{\mathfrak{g}} \subseteq \mathcal{X}_{\mathfrak{g}'}.$$

Now from the proof of Theorem 4.4.1, $\mathcal{N}_{\mathfrak{g}'} \cap \mathfrak{g}_{\bar{1}} \subseteq \mathcal{N}_{\mathfrak{g}}$. Consequently, $\mathcal{X}_{\mathfrak{g}} \subseteq \mathcal{N}_{\mathfrak{g}}$. $\square$

5.3. We can now state and prove generalizations of the finiteness of $G_{\bar{0}}$ -orbits on $\mathcal{X}$ due to Duflo and Serganova (cf. [DS, Theorem 4.2]). Note that their work is stated under the assumption that $\mathfrak{g}$ is a contragredient Lie superalgebra with indecomposable Cartan matrix.

Corollary 5.3.1. *Let $\mathfrak{g}$ be a classical simple Lie superalgebra over $\mathbb{C}$. Then*

- (a) $\mathcal{X} \subseteq \mathcal{N}$,
- (b) $\mathcal{X}$ *has finitely many $G_{\bar{0}}$ -orbits.*

Proof. We handle the case first when $\mathfrak{g}$ is not isomorphic to $D(2, 1, \alpha)$, $F(4)$ or $G(3)$. In this situation, Theorem 5.2.1 applies. Therefore, $\mathcal{X} \subseteq \mathcal{N}$ and $\mathcal{X}$ has finitely many $G_{\bar{0}}$ -orbits.

Now consider the case when $\mathfrak{g} = D(2, 1, \alpha)$, $F(4)$ or $G(3)$. One can obtain the inclusion $\mathcal{X} \subseteq \mathcal{N}$ because $\mathcal{X}$ is the closure of $G_{\bar{0}} \cdot v_H$ where v_H is the highest weight vector (cf. [DS, pf. of Theorem 4.2]). Since v_H satisfies the defining equation for $\mathcal{N}$, one obtains the inclusion. The finiteness result for $\mathcal{X}$ for the exceptional Lie superalgebras follows from the finiteness results in Theorem 4.4.1. $\square$

5.4. For $\mathfrak{gl}(m|n)$, we can use the parametrization of $G_{\bar{0}}$ -orbit representatives for $\mathcal{N}$ to recover the Duflo-Serganova parametrization of $G_{\bar{0}}$ -orbit representatives for $\mathcal{X}$ (cf. [DS, Theorem 4.2]).

Let Y be an orbit representative as described in Theorem 3.1.1(b). Then $Y \in \mathcal{X}$ if and only if $[Y, Y] = 2Y^2 = 0$. A direct calculation shows that $Y^2 = 0$ if and only if $Y^-Y^+ = 0$ and $Y^+Y^- = 0$. This is equivalent to the Jordan blocks $J_i = 0$ for $i = 1, 2, \dots, t$, $C_{r_1} = 0$, and $R_{r_2} = 0$. Hence, $Y \in \mathcal{X}$ if and only if

$$Y^+ = \left[\begin{array}{c|c|c} I_r & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right] \quad \text{and} \quad Y^- = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & I_s & 0 \\ \hline 0 & 0 & 0 \end{array} \right].$$

This corresponds to taking a representative of a subset of linearly independent set of mutually orthogonal isotropic odd roots under the action of the Weyl group for $G_{\bar{0}}$ (see the paragraph after [DS, Theorem 4.2]) which is precisely how Duflo and Serganova describe their orbit representatives for $\mathcal{X}$.

APPENDIX A. CONSTRUCTION OF COMPLEMENTS M

A.1. For each classical Lie superalgebra $\mathfrak{g}$, an explicit matrix realization of $\mathfrak{g}$ is well-known (for example, see [K]). We construct a matrix realization for the complement M in Table A.1.1 below.

A.2. We now check that each of the non-exceptional classical Lie superalgebras $\mathfrak{g}$ (except $\mathfrak{gl}(m|n)$) satisfies the hypotheses of Theorem 4.2.1 case-by-case. From the construction of $M = M_{\bar{0}} \oplus M_{\bar{1}}$ in each case below it follows that $\mathfrak{gl}(m|n)_{\bar{i}} = \mathfrak{g}_{\bar{i}} \oplus M_{\bar{i}}$ for $i = 0, 1$ since each generator E_{ij} of $\mathfrak{gl}(m|n)$ can be written as a sum of a (homogeneous) element of $\mathfrak{g}$ and an element of M in an obvious way. Direct calculation shows that $[\mathfrak{g}_{\bar{i}}, M_{\bar{j}}] \subseteq M_{\overline{i+j}}$ in each case. Sample calculations are given below for some of the classical Lie superalgebras when $i = j = 1$. In each case, let $X \in \mathfrak{g}_{\bar{1}}$ and $Y \in M_{\bar{1}}$ so that $[X, Y] = XY + YX$.

- $\mathfrak{sl}(m|n) : Y = 0$, so $[X, Y] = 0$.
- $\mathfrak{q}(n) : X = \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & d \\ -d & 0 \end{bmatrix}$. Then

$$[X, Y] = \begin{bmatrix} db - bd & 0 \\ 0 & -(db - bd) \end{bmatrix}.$$
- $\mathfrak{p}(n) : X = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & a \\ d & 0 \end{bmatrix}$ with b, d symmetric and a, c skew-symmetric. Then

$$[X, Y] = \begin{bmatrix} bd + ac & 0 \\ 0 & ca + db \end{bmatrix} = \begin{bmatrix} bd + ac & 0 \\ 0 & (bd + ac)^t \end{bmatrix}.$$

TABLE A.1.1. Block matrix realization of M for classical Lie superalgebras

$\mathfrak{g}$	M	
$\mathfrak{sl}(m n)$	$\begin{bmatrix} \lambda I_m & 0 \\ 0 & -\lambda I_n \end{bmatrix},$	I_m, I_n identity matrices, $\lambda \in \mathbb{C}$
$\mathfrak{psl}(n n)$	$\begin{bmatrix} (\mu + \lambda)I_n & 0 \\ 0 & (\mu - \lambda)I_n \end{bmatrix},$	I_n identity matrix, $\lambda, \mu \in \mathbb{C}$
$\mathfrak{osp}(2m+1 2n)$	$\left[\begin{array}{ccc cc} \delta & u^t & v^t & x & x_1 \\ v & a & b & y & y_1 \\ u & c & a^t & z & z_1 \\ \hline x_1^t & z_1^t & y_1^t & d & e \\ -x^t & -z^t & -y^t & f & d^t \end{array} \right],$	b, c symmetric, e, f skew-symmetric
$\mathfrak{osp}(2m 2n)$	$\left[\begin{array}{ccc cc} a & b & & y & y_1 \\ c & a^t & & z & z_1 \\ \hline z_1^t & y_1^t & d & e \\ -z^t & -y^t & f & d^t \end{array} \right],$	b, c symmetric, e, f skew-symmetric
$\mathfrak{q}(n)$	$\begin{bmatrix} a & b \\ -b & -a \end{bmatrix}$	
$\mathfrak{psq}(n)$	$\begin{bmatrix} a + \lambda I_n & b + \mu I_n \\ -b + \mu I_n & -a + \lambda I_n \end{bmatrix},$	I_n identity matrix, $\lambda, \mu \in \mathbb{C}$
$\tilde{\mathfrak{p}}(n)$	$\begin{bmatrix} a & b \\ c & a^t \end{bmatrix},$	b skew-symmetric, c symmetric
$\mathfrak{p}(n)$	$\begin{bmatrix} a + \lambda I_n & b \\ c & a^t + \lambda I_n \end{bmatrix},$	b skew-symmetric, c symmetric, I_n identity matrix, $\lambda \in \mathbb{C}$

$$\bullet \mathfrak{osp}(2m+1|2n) : X = \left[\begin{array}{ccc|cc} & & & x & x_1 \\ & & & y & y_1 \\ & & & z & z_1 \\ \hline x_1^t & z_1^t & y_1^t & & \\ -x^t & -z^t & -y^t & & \end{array} \right],$$

$$Y = \left[\begin{array}{ccc|cc} & a & a_1 & & \\ & b & b_1 & & \\ & c & c_1 & & \\ \hline a_1^t & c_1^t & b_1^t & & \\ -a^t & -c^t & -b^t & & \end{array} \right]. \text{ Then } [X, Y] = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \text{ where the matrices } A, B \text{ have block forms}$$

$$A = \begin{bmatrix} xa_1^t - x_1a^t - ax_1^t + a_1x^t & xc_1^t - x_1c^t - az_1^t + a_1z^t & xb_1^t - x_1b^t - ay_1^t + a_1y^t \\ ya_1^t - y_1a^t - bx_1^t + b_1x^t & yc_1^t - y_1c^t - bz_1^t + b_1z^t & yb_1^t - y_1b^t - by_1^t + b_1y^t \\ za_1^t - z_1a^t - cx_1^t + c_1x^t & zc_1^t - z_1c^t - cz_1^t + c_1z^t & zb_1^t - z_1b^t - cy_1^t + c_1y^t \end{bmatrix}$$

$$B = \begin{bmatrix} -x_1^ta - z_1^tb - y_1^tc + a_1^tx + c_1^ty + b_1^tz & -x_1^ta_1 - z_1^tb_1 - y_1^tc_1 + a_1^tx_1 + c_1^ty_1 + b_1^tz_1 \\ x^ta + z^tb + y^tc - a^tx - c^ty - b^tz & x^ta_1 + z^tb_1 + y^tc_1 - a^tx_1 - c^ty_1 - b^tz_1 \end{bmatrix}$$

with the blocks satisfying the relations

$$A_{12} = A_{31}^t, A_{13} = A_{21}^t, A_{33} = A_{22}^t, B_{22} = B_{11}^t$$

and with A_{23}, A_{32} symmetric and B_{12}, B_{21} skew-symmetric.

- $\mathfrak{osp}(2m|2n)$: Since this superalgebra is obtained by deleting the first row and first column of $\mathfrak{osp}(2m+1|2n)$, the calculations in this case are obtained in a similar manner.

ACKNOWLEDGMENT

We thank Vera Serganova for comments and suggestions on an earlier version of our manuscript.

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