Robust Distributed Average Tracking for Double-Integrator Agents Without Velocity Measurements Under Event-Triggered Communication

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***Abstract*—This article focuses on an event-triggered mechanism to solve the distributed average tracking problem for double-integrator agents without velocity measurements. In some practical applications, velocity measurements may be unavailable due to technology and space limitations, and it is also usually less accurate and more expensive to implement. Before deriving the event-triggered approach, we first present a base algorithm without using velocity measurements, which sets the stage for the development of the event-triggered algorithm. The base algorithm has an advantage over the existing related works in the sense that there is no global information requirement for parameter design. Building on the base algorithm, we present an event-triggered algorithm that further removes the continuous communication requirement and is free of Zeno behavior. It is suitable for practical implementation, since in reality, the bandwidth of the communication network and power capacity are usually constrained. The event-triggered algorithm overcomes some practical limitations, such as the unbounded growth of the adaptive gain and requirement of additional internal dynamics, by constructing a new triggering strategy. In addition, a continuous nonlinear function is used to approximate the signum function to reduce the chattering phenomenon in reality. Numerical simulations are provided to illustrate the obtained results.**

***Index Terms*—Distributed average tracking, double-integrator agents, event-triggered communication, velocity measurements.**

I. Introduction

DISTRIBUTED cooperative control of multiagent systems has drawn increasing attention from various scientific communities due to its wide range of applications, such as vehicle formation, sensor networks, and cooperative surveillance. Consensus is an important research subject in distributed cooperative control of multiagent systems, where all the agents reach an agreement on a state of interest. When the desired consensus state for a group of agents follows a certain trajectory, the distributed tracking problem is investigated. During the recent decade, a more general problem, the distributed average tracking problem, which includes consensus and distributed tracking as special cases, is formulated and addressed in the literature. In the distributed average tracking problem, each agent has a time-varying reference signal, and the goal is to design controllers for the agents based on local information such that all the agents are able to track the average of these reference signals. Because of the time-varying tracking objective and the lack of access to error signals, the distributed average tracking problem is theoretically more challenging compared with consensus and distributed tracking problems.

In the literature, there are cases where each agent aims to only estimate the average of these reference signals, which is often called dynamic average consensus. Some applications, such as feature-based map merging [1] and distributed Kalman filtering [2], have been reported in the literature. Several linear distributed algorithms are established to deal with the dynamic average consensus problem for certain types of reference signals. For instance, the dynamic average consensus problem is solved in [3]–[5] for reference signals with steady-state values, with a common denominator in their Laplace transforms, and slowly varying reference signals, respectively. The dynamic average consensus problem is solved with bounded steady-state error for a strongly connected weight-balanced interaction topology in [6], where the discrete-time counterparts are addressed as well. A class of nonlinear algorithms is proposed in [7] for reference signals with bounded deviations, and the dynamic average consensus error is bounded. A nonsmooth algorithm is proposed in [8], which enables each agent to keep track of the average of a class of reference signals with bounded derivatives. More recently, combined with an adaptive scheme, two dynamic average consensus algorithms without correct initialization are proposed in [9] such that each agent is able to estimate the average of the reference signals. Also, a robust dynamic average consensus algorithm is proposed for directed networks,

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which guarantees an arbitrary prescribed small steady-state error bound.

The aforementioned algorithms focus on estimator design, and in reality, some tasks, such as region following formation control [10] and coordinated path planning [11], require each agent to have a certain dynamics, and the goal is to design controllers for each agent such that its physical states track the average of multiple time-varying reference signals. In this context, the term distributed average tracking is often used. A nonsmooth algorithm is presented in [12] for double-integrator agents. It requires that the accelerations of the individual reference signals be bounded. For general linear systems, the distributed average tracking problem is addressed in [13]. The distributed average tracking algorithms mentioned above need full state information (e.g., both positions and velocities for double-integrator agents) to update the controllers.

However, in some practical applications, partial states may be unavailable due to technology and space limitations. Moreover, it is usually less accurate and more expensive to implement velocity measurements compared with position measurements. Hence, it is worth investigating the distributed average tracking problem for double-integrator agents without using velocity measurements. In [14], the authors investigate the problem described above. However, in [14], the lower bounds of the design parameters depend on the bounds related to the reference signals and the graph information including the largest and smallest nonzero eigenvalues of the Laplacian matrix, which are global information and may be inaccessible to the agents. Also, the algorithm in [14] is sensitive to parameter selection as a certain parameter is required to be exactly equal to a certain value.

All these aforementioned continuous-time distributed average tracking algorithms require each agent to continuously interact with its neighbors. However, in reality, it may not be practical due to the constrained bandwidth of the communication network and power source. In contrast, discrete-time distributed average tracking algorithms require agents to interact with each other periodically. It may result in a waste of network resources. Furthermore, with regard to general bounded reference signals, there usually exist tracking errors by using the discrete-time algorithms. Thus, it makes sense to employ event-triggered control strategies to address the distributed average tracking problem. They take advantage of opportunistic aperiodic sampling to improve efficiency. In [15], the authors extend the algorithm in [6] by incorporating an event-triggered communication strategy, but specific initialization is needed for a certain variable, and there exist nonzero tracking errors for general bounded reference signals. A robust dynamic average consensus algorithm under dynamic event-triggered communication is proposed in [16] for agents to estimate the average of the reference signals. These two works focus on the estimation aspect of the distributed average tracking problem, where the agents' dynamics are essentially single integrators.

The focus of this article is on an event-triggered mechanism to solve the distributed average tracking problem for double-integrator agents without using velocity measurements. Before deriving the event-triggered approach, we first present a base algorithm (see Section III) to solve the distributed average tracking under continuous communication. Then, we present an event-triggered distributed average tracking algorithm that further removes the continuous communication requirement. In contrast, Ghapani *et al.* [14] consider the problem of distributed average tracking of double-integrator agents without using velocity measurements under continuous communication, which does not enjoy the benefit of the event-triggered algorithm proposed in this article. While the base algorithm in this article has some connection with [14], it is worth mentioning that even this base algorithm has an advantage over [14] in the sense that no global information is needed for parameter design. We would also like to point out that the base algorithm has a different structure from the one in [14]. Such a structure and its independence on global information lay a solid base for the development of the event-triggered algorithm. The proposed event-triggered algorithm is able to achieve distributed average tracking with zero tracking errors, does not require correct initialization, and is free of Zeno behavior. In contrast to [16], which is limited to only single-integrator agents, double-integrator agents without using velocity measurements are considered in this article, which is a more complicated and challenging problem. It is also noted that there are some practical limitations for the algorithm in [16]. First, the time-varying gain may grow unbounded due to persistent disturbance, which would affect the convergence and the success of the event triggering scheme. Second, an extra internal dynamics is needed to ensure the exclusion of Zeno behavior, which may cost extra computational power and storage space. In addition, the use of the signum function may cause the chattering phenomenon in real applications. The proposed event-triggered algorithm overcomes the aforementioned limitations in [16]. In this algorithm, a new adaptive law and a new event-triggering strategy are constructed, and a continuous nonlinear function is used to approximate the signum function.

Some preliminary results of this article (see Section III) are presented in [18]. The current article improves on [18] by introducing a new event-triggered distributed average tracking algorithm to overcome some practical limitations. In addition, this article contains more detailed proofs and additional simulation results.

The remainder of this article is arranged as follows. In Section II, some preliminaries are presented, and the distributed average tracking problem is introduced. Section III provides a base algorithm to achieve relaxed parameter conditions without velocity measurements under continuous communication. In Section IV, an event-triggered distributed average tracking algorithm is proposed to remove the continuous communication requirement. Numerical examples are provided in Section V to explain the main results. Section VI concludes this article.

II. Preliminaries and Problem Statement

Denote by $\mathbb{R}_{\geq 0}$ the set of non-negative real numbers. For a given vector $x \in \mathbb{R}^p$, $\|x\|_2$, $\|x\|_1$, and $\|x\|_\infty$ denote the two norm, one norm, and infinity norm of x, respectively. For a set $\mathcal{S}$, $|\mathcal{S}|$ denotes the cardinality of $\mathcal{S}$. The transpose of matrix A is denoted by A^T. The Kronecker product of matrices A and B is denoted by $A \otimes B$. We use $\text{sgn}(\cdot)$ to denote the signum function defined componentwise. Let $\mathbf{0}_{m\times n} \in \mathbb{R}^{m\times n}$ denote the $m \times n$ dimensional zero matrix, and for simplicity, let $\mathbf{0}_m = \mathbf{0}_{m\times 1}$. $I_n \in \mathbb{R}^{n\times n}$ denotes the identity matrix. For any symmetric matrix M, the notation $M \succ 0$ is used to say that M is positive definite.

A. Graph Theory

For a multiagent system consisting of N agents, the interaction topology can be modeled by an undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{1, \ldots, N\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denote the node set and edge set, respectively. An edge denoted by $(i, j) \in \mathcal{E}$, means that agent i and j can obtain information from each other. In an undirected graph, the edges (i, j) and (j, i) are equivalent. It is assumed that $(i, i) \notin \mathcal{E}$. The neighbor set of node i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} \mid (j, i) \in \mathcal{E}\}$. The adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{N\times N}$ of the graph G is defined such that $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. For an undirected graph, $a_{ij} = a_{ji}$. By arbitrarily assigning an orientation for every edge in $\mathcal{G}$, let $B = [B_{ij}] \in \mathbb{R}^{N\times|\mathcal{E}|}$ denote the incidence matrix associated with graph $\mathcal{G}$, where $B_{ij} = -1$ if edge e_j leaves node i, $B_{ij} = 1$ if it enters node i, and $B_{ij} = 0$ otherwise. An undirected path between node i_1 and i_k is a sequence of edges of the form $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$, where $i_k \in \mathcal{V}$. A connected graph means that there exists an undirected path between any pair of nodes in $\mathcal{V}$.

B. Problem Formulation

In this article, we consider N physical agents, and the interaction topology among these agents is characterized as the undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Unless otherwise stated, throughout this article, we assume a time-invariant graph. Each agent i is modeled by double-integrator dynamics

$$\dot{x}_i(t) = v_i(t), \quad \dot{v}_i(t) = u_i(t), \quad i \in \mathcal{V} \tag{1}$$

where $x_i(t) \in \mathbb{R}^p$ and $v_i(t) \in \mathbb{R}^p$ are the ith agent's position and velocity, respectively, and u_i is its control input.

Each agent has a time-varying reference signal $x_i^r \in \mathbb{R}^p$, $i \in \mathcal{V}$ satisfying

$$\dot{x}_i^r(t) = v_i^r(t), \quad \dot{v}_i^r(t) = u_i^r(t), \quad i \in \mathcal{V} \tag{2}$$

where $v_i^r(t) \in \mathbb{R}^p$ and $u_i^r(t) \in \mathbb{R}^p$ are the velocity and acceleration of the ith agent's reference signal, respectively. We assume that the reference signals are generated internally by the agents, and that each agent has access to its own reference signal, and the velocity and acceleration of the reference signal. In this article, we make the following assumption on the reference signals and the velocities and accelerations of the reference signals.

Assumption 1: For any two connected agents, the local difference in reference signals $x_i^r(t)$, their velocities $v_i^r(t)$, and their accelerations $a_i^r(t)$ are bounded, i.e.,

$$\sup_{\substack{t\in[0,\infty)\\ \forall(i,j)\in\mathcal{E}}} \|x_i^r(t) - x_j^r(t)\|_\infty \le \bar{x}^r,$$

$$\sup_{\substack{t\in[0,\infty)\\ \forall(i,j)\in\mathcal{E}}} \|v_i^r(t) - v_j^r(t)\|_\infty \le \bar{v}^r,$$

and

$$\sup_{\substack{t\in[0,\infty)\\ \forall(i,j)\in\mathcal{E}}} \|u_i^r(t) - u_j^r(t)\|_\infty \le \bar{a}^r.$$

In the distributed average tracking for a group of double-integrator agents, the objective is to design controller u_i for agent $i \in \mathcal{V}$ such that each agent's position (velocity) is capable of tracking the group average of their reference signals (their reference signals' velocity). That is, for any $i \in \mathcal{V}$, it is achieved that $\lim_{t\to\infty} \|x_i(t) - \frac{1}{N}\sum_{j=1}^N x_j^r(t)\|_2 = 0$ and $\lim_{t\to\infty} \|v_i(t) - \frac{1}{N}\sum_{j=1}^N v_j^r(t)\|_2 = 0$. In this article, we are particularly interested in developing a controller for each agent without velocity measurement and in the absence of any correct initialization. The motivation behind this is that employing a velocity measuring device is usually costly in the aspect of finance and energy. Also, the velocity measurements are less accurate compared with position measurements. In contrast, perfect initialization is hard to achieve in reality.

Before moving onto the main results, a lemma is presented in the following.

Lemma 1 (see [19]): For any symmetric real matrix, M, of the form $M = \begin{bmatrix} D_{11} & D_{12} \\ D_{12}^T & D_{22} \end{bmatrix}$, it holds that $M \succ 0$ if and only if one of the following conditions holds: 1) $D_{11} \succ 0$ and $D_{22} - D_{12}^T D_{11}^{-1} D_{12} \succ 0$; and 2) $D_{22} \succ 0$ and $D_{11} - D_{12} D_{11}^{-1} D_{12}^T \succ 0$.

III. Distributed Average Tracking Without Velocity Measurements

In this section, we introduce a distributed average tracking algorithm for double-integrator agents without using the velocity measurements and in the absence of any correct initialization. In the rest of this article, we omit the argument t for brevity.

We design a filter for each agent i as

$$\begin{aligned}\dot{\phi}_i = & -\kappa(x_i - x_i^r) - 2\kappa(w_i - v_i^r) + u_i^r \\ & - \sum_{j=1}^N a_{ij}\pi_{ij}\mathrm{sgn}(x_i - x_j + w_i - w_j) \\ w_i = & \ \phi_i + \kappa(x_i - x_i^r), \quad i \in \mathcal{V}\end{aligned} \tag{3}$$

where $\kappa \in \mathbb{R}$ is a positive constant to be determined, $\phi_i \in \mathbb{R}^p$ is the internal state of the filter, $w_i \in \mathbb{R}^p$ is the output of the filter, and π_{ij} is a time-varying gain for the edge $(i, j) \in \mathcal{E}$, satisfying the following adaptation law:

$$\dot{\pi}_{ij} = a_{ij}\left\|x_i - x_j + w_i - w_j\right\|_1, \quad i \in \mathcal{V} \tag{4}$$

with $\pi_{ij}(0) > 0$ if $(i, j) \in \mathcal{E}$. In addition, each agent i needs to coordinate with its neighbor $j \in \mathcal{N}_i$ to ensure $\pi_{ij}(0) = \pi_{ji}(0)$. In this way, the gains π_{ij} and π_{ji} remain equal to each other. We design the controller for agent i as

$$\begin{aligned}u_i = & -\kappa(x_i - x_i^r) - \kappa(w_i - v_i^r) + u_i^r \\ & - \sum_{j=1}^N a_{ij}\pi_{ij}\mathrm{sgn}(x_i - x_j + w_i - w_j), \quad i \in \mathcal{V}.\end{aligned} \tag{5}$$

Essentially, the filter is designed such that its output is capable of tracking the average of the reference signals' velocities, and the controller is applied to drive each agent's position to the average of the reference signals and velocity to the output of the filter. Note that the designs of the filter (3) and the controller (5) for each agent i depend on only local information and the positions and filter's outputs from its neighbors. Therefore, it is implementable in reality.

Remark 1: Note that there is no requirement on the initialization of each agents' position and velocity, as well as the internal state of the filter. Thus, the proposed algorithm (3)–(5) is called robust distributed average tracking algorithm.

Let $x = [x_1^T, \ldots, x_N^T]^T$, $v = [v_1^T, \ldots, v_N^T]^T$, and $w = [w_1^T, \ldots, w_N^T]^T$. Define $\tilde{x} = (M \otimes I_p)x$, $\tilde{v} = (M \otimes I_p)v$, and $\tilde{w} = (M \otimes I_p)w$, where $M = I_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T$. For brevity, define $\alpha = [\alpha_1^T, \ldots, \alpha_N^T]^T$ with $\alpha_i = \kappa x_i^r + \kappa v_i^r + u_i^r$. Then, we have

$$\begin{aligned}\dot{\tilde{x}} &= \tilde{v}\\ \dot{\tilde{v}} &= -\kappa\tilde{x} - \kappa\tilde{w} + (M \otimes I_p)\alpha\\ &\quad - (B\Pi \otimes I_p)\text{sgn}\left[\left(B^T \otimes I_p\right)(\tilde{x} + \tilde{w})\right] \end{aligned} \tag{6}$$

and

$$\begin{aligned}\dot{\tilde{w}} &= -\kappa\tilde{x} - 2\kappa\tilde{w} + \kappa\tilde{v} + (M \otimes I_p)\alpha\\ &\quad - (B\Pi \otimes I_p)\text{sgn}\left[\left(B^T \otimes I_p\right)(\tilde{x} + \tilde{w})\right] \end{aligned} \tag{7}$$

where $\Pi \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{E}|}$ is a time-varying diagonal matrix, and the sth diagonal entry, denoted by Π_{ss}, represents the weight on the sth edge. That is, if the sth edge is between agent i and agent j, then $\Pi_{ss} = \pi_{ij}$.

Theorem 1: Suppose that the undirected graph $\mathcal{G}$ is connected, and Assumption 1 holds. Using the algorithm (3)–(5) for (1), distributed average tracking is achieved asymptotically if $\kappa > \frac{3+2\sqrt{3}}{3}$.

Proof: We prove this statement in two steps. In the first step, we prove that for any $i \in \mathcal{V}$, $x_i \to \frac{1}{N}\sum_{j=1}^N x_j$ and $v_i \to \frac{1}{N}\sum_{j=1}^N v_j$ as $t \to \infty$. In the second step, we prove that for any $i \in \mathcal{V}$, $\sum_{j=1}^N x_j \to \sum_{j=1}^N x_j^r$ and $\sum_{j=1}^N v_j \to \sum_{j=1}^N v_j^r$ as $t \to \infty$. Combining these two steps, it can be concluded that $\lim_{t\to\infty} \|x_i - \frac{1}{N}\sum_{j=1}^N x_j^r\|_2 = 0$ and $\lim_{t\to\infty} \|v_i - \frac{1}{N}\sum_{j=1}^N v_j^r\|_2 = 0$ hold for all $i \in \mathcal{V}$. For simplicity, we denote these two steps by consensus and sum-tracking steps, respectively. Define $X = [\tilde{x}^T, \tilde{v}^T, \tilde{w}^T]^T$. Consider a Lyapunov function candidate as

$$V = \frac{1}{2}X^T P X + \sum_{i=1}^N\sum_{j=1}^N \frac{(\pi_{ij} - \pi_m)^2}{4} \tag{8}$$

where

$$P = \begin{bmatrix} \mu I_{Np} & \mathbf{0}_{Np\times Np} & I_{Np}\\ \mathbf{0}_{Np\times Np} & I_{Np} & -I_{Np}\\ I_{Np} & -I_{Np} & 2I_{Np}\end{bmatrix} \tag{9}$$

and π_m is a positive constant to be determined. By Lemma 1 and the properties of the Kronecker product, it holds that P is positive definite if and only if $\mu > 1$. Therefore, V is positive definite.

Taking the derivative of V along (6) and (7) yields

$$\begin{aligned}\dot{V} =& -X^T Q X + (\tilde{x} + \tilde{w})^T (M \otimes I_p)\alpha\\ &- \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N a_{ij}\pi_{ij}\left\|x_i - x_j + w_i - w_j\right\|_1\\ &+ \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N \pi_{ij}\dot{\pi}_{ij} - \frac{\pi_m}{2}\sum_{i=1}^N\sum_{j=1}^N \dot{\pi}_{ij}\end{aligned}$$

where

$$Q = \begin{bmatrix} \kappa I_{Np} & -\frac{\mu+\kappa}{2}I_{Np} & \frac{3\kappa}{2}I_{Np}\\ -\frac{\mu+\kappa}{2}I_{Np} & \kappa I_{Np} & -\frac{1+3\kappa}{2}I_{Np}\\ \frac{3\kappa}{2}I_{Np} & -\frac{1+3\kappa}{2}I_{Np} & 3\kappa I_{Np}\end{bmatrix}. \tag{10}$$

Note that $\|\alpha_i - \alpha_j\|_\infty \le \bar{\alpha}$ by Assumption 1, and let $N_{\max} = \max_{i\in\mathcal{V}} |\mathcal{N}_i|$. Then, it holds that

$$\begin{aligned}\|(M \otimes I_p)\alpha\|_\infty &\le \frac{1}{N}\max_{i\in\mathcal{V}}\left\{\sum_{j=1, j\ne i}^N \|\alpha_i - \alpha_j\|_\infty\right\}\\ &\le \frac{N-1}{2N}\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\|\alpha_i - \alpha_j\|_\infty \le \frac{\bar{\alpha}N_{\max}(N-1)}{2}\end{aligned} \tag{11}$$

where $\bar{\alpha} = \kappa\bar{x}^r + \kappa\bar{v}^r + \bar{a}^r$. For brevity, define

$$\beta = \frac{\bar{\alpha}N_{\max}(N-1)}{2}. \tag{12}$$

Note that

$$\begin{aligned}\|\tilde{x} + \tilde{w}\|_1 &\le \frac{1}{N}\sum_{i=1}^N\sum_{j=1, j\ne i}\|x_i - x_j + w_i - w_j\|_1\\ &\le \max_{i\in\mathcal{V}}\left\{\sum_{j=1, j\ne i}^N \|x_i - x_j + w_i - w_j\|_1\right\}\\ &\le \frac{N-1}{2}\sum_{i=1}^N\sum_{j=1}^N a_{ij}\left\|x_i - x_j + w_i - w_j\right\|_1.\end{aligned}$$

It then holds that

$$\begin{aligned}&(\tilde{x} + \tilde{w})^T (M \otimes I_p)\alpha\\ &\le \frac{(N-1)\beta}{2}\sum_{i=1}^N\sum_{j=1}^N a_{ij}\|x_i - x_j + w_i - w_j\|_1.\end{aligned}$$

Then, it follows that

$$\begin{aligned}\dot{V} \le& -X^T Q X - \frac{\pi_m - (N-1)\beta}{2}\\ &\times \sum_{i=1}^N\sum_{j=1}^N a_{ij}\|x_i - x_j + w_i - w_j\|_1,\end{aligned}$$

where the fact that

$$(\tilde{x}+\tilde{w})^T(B\Pi\otimes I_p)\text{sgn}[(B^T\otimes I_p)(\tilde{x}+\tilde{w})]$$
$$=\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\pi_{ij}\|(x_i-x_j)+(w_i-w_j)\|_1$$

is used. Selecting an π_m such that $\pi_m \geq \beta$, one has

$$\dot{V}\leq -X^TQX := -W[X].$$

By Lemma 1, the matrix Q is positive definite if and only if $\kappa > \mu + \frac{1}{3(\mu-1)} = f(\mu)$, which implies that Q is positive definite if $\kappa > \min_{\mu>1} f(\mu) = \frac{3+2\sqrt{3}}{3}$. Thus, $\dot{V}\leq 0$, which implies that V is nonincreasing. Then, it follows that X and π_{ij} are bounded. Note that V is bounded from below by zero. Thus, $\lim_{t\to\infty} V$ exists and is finite. Note that

$$\int_0^t W[X(\tau)]d\tau \leq -\int_0^t \dot{V}[X(\tau),\{\pi_{ij}(\tau)\}_{i,j\in\mathcal{V}}]d\tau$$
$$= V[X(0),\{\pi_{ij}(0)\}_{i,j\in\mathcal{V}}] - V[X,\{\pi_{ij}\}_{i,j\in\mathcal{V}}].$$

Therefore, $\lim_{t\to\infty}\int_0^t W[X(\tau)]d\tau$ exists and is finite. It follows from (6), (7), and Assumption 1 that $\dot{\tilde{x}}$, $\dot{\tilde{v}}$, and $\dot{\tilde{w}}$ are bounded. Hence, $\tilde{x}$, $\tilde{v}$, and $\tilde{w}$ are uniformly continuous. Consequently, $W[X]$ is uniformly continuous by the definition of $W[X]$ and X. By Barbalat's Lemma, it can be concluded that $W[X]\to 0$ as $t\to\infty$, which implies that $\lim_{t\to\infty} X = \mathbf{0}_{np}$. This completes the consensus step. Second, define $S_x = \sum_{j=1}^N x_j - \sum_{j=1}^N x_j^r$, $S_v = \sum_{j=1}^N v_j - \sum_{j=1}^N v_j^r$, and $S_w = \sum_{j=1}^N w_j - \sum_{j=1}^N v_j^r$. Then, we have that $\dot{S} = \left(\begin{bmatrix} 0 & 1 & 0 \\ -\kappa & 0 & -\kappa \\ -\kappa & \kappa & -2\kappa \end{bmatrix}\otimes I_p\right) S = (A\otimes I_p)S$, where $S = [S_x^T, S_v^T, S_w^T]^T$. The characteristic polynomial of A is

$$p_A(s) = s^3 + 2\kappa s^2 + (\kappa+\kappa^2)s + \kappa^2.$$

According to the Routh–Hurwitz stability criterion, it is easy to verify that if $\kappa > 0$, all the zeros of $p_A(s) = 0$ have negative real parts, which means that A is Hurwitz. Note that $\kappa > \frac{3+2\sqrt{3}}{3} > 0$. Then, the matrix A is Hurwitz, which indicates $\lim_{t\to\infty} S = \mathbf{0}_{3p}$. This completes the sum-tracking step. ■

Note that the dynamics (6) is discontinuous due to the introduction of the signum function in the controller and filter design (3)–(5). Then, the solutions should be understood in terms of differential inclusion by using nonsmooth analysis [20], [21]. However, since the signum function is measurable and locally essentially bounded, the Filippov solutions for the closed-loop dynamics always exist. The Lyapunov function used in the proof is continuously differentiable. Then, its set-valued Lie derivative is a singleton at the discontinuous points. Therefore, the proof is valid as in the case without discontinuities.

Remark 2: Note that the algorithm (3)–(5) has some connection with the first algorithm in [14]. In the first algorithm in [14], there are multiple design parameters, the design of which depends on the largest and smallest nonzero eigenvalues of the Laplacian matrix, the bounds on the reference signals, and the total number of the agents in the network. Also, the first algorithm in [14] is sensitive to parameter selection as a certain parameter is required to be exactly equal to a certain value. However, the algorithm (3)–(5) overcomes these limitations in [14] and solves the distributed average tracking problem if κ is greater than a constant. It is easy to select a suitable value for κ and implement the algorithm. It is also worth noting that the structures of the controller (5) and the filter (3) are different from the ones in [14]. Such newly designed structures and their independence of global information lay a solid base for the development of event-triggered approaches.

Remark 3: The algorithm (3)–(5) is implementable since κ is constant, which can be chosen offline before running the algorithm and embedded to each agent. Once the algorithm starts to run, the agents communicate with only local neighbors, and there is no need to have access to any global information. If each agent chooses its own $\kappa_i(0)$ offline such that $\kappa_i(0) > \frac{3+2\sqrt{3}}{3}$, then each agent can run the max consensus algorithm in [22]: $\kappa_i(k+1) = \max_{j\in\mathcal{N}_i\cup\{i\}}\{\kappa_j(k)\}$, where k is discrete time instance, to drive each agent to reach consensus on $\max_{j\in\mathcal{V}}\kappa_j(0)$. It is proved that the max consensus algorithm converges in finite time. To determine when to stop the max consensus algorithm, each agent needs to know the diameter of the graph. However, one can always be more conservative to run the max consensus algorithm long enough, which guarantees the convergence.

Remark 4: Theorem 1 shows that the agents are capable of achieving distributed average tracking under any fixed connected undirected communication network. It is actually able to extend to the case of arbitrarily switching connected communication networks with positive dwelling time. The function defined in (8) can be used as a common Lyapunov function during the proof process.

IV. Event-Triggered Distributed Average Tracking Without Velocity Measurements

The algorithm (3)–(5) in Section III requires each agent i to continuously exchange the position, x_i, and the output of the filter, w_i, with its neighbors. However, continuous communication may not be practical due to the constrained bandwidth of the communication network in reality. To this end, we investigate the event-triggered distributed average tracking, which removes the requirement of continuous communication. It is worth mentioning that no velocity measurements and no initialization requirements are needed as well.

It is noted that there are several practical limitations for the event-triggered algorithm in [16]. First, due to the nature of the adaptation law, the adaptive gains can only increase. It is normally the case that there exist measurement/communication noise and/or persistent disturbances in practical systems. In such a case, perfect consensus cannot be achieved, and consequently, the adaptive gains and the control inputs will grow unbounded, which would affect the convergence and the success of the event-triggered scheme. Second, implementing the algorithm in [16] requires each agent to maintain an additional internal dynamics to ensure the exclusion of Zeno behavior. Such additional dynamics may cost extra computational power and storage space. Finally, the use of the signum function in the algorithm design will cause chattering phenomenon in real applications. To overcome these limitations, we propose a novel event-triggered distributed average tracking algorithm without using velocity measurements and requiring correct initialization.

We propose the following distributed average tracking algorithm with the filter:

$$\begin{aligned}\dot{\phi}_i =& -\kappa(x_i - x_i^r) - 2\kappa(w_i - v_i^r) + u_i^r \\ & - \sum_{j=1}^{N} a_{ij}\pi_{ij} h(\hat{x}_i - \hat{x}_j + \hat{w}_i - \hat{w}_j, t) \\ w_i =& \phi_i + \kappa(x_i - x_i^r), \quad i \in \mathcal{V}\end{aligned} \tag{13}$$

and the controller

$$\begin{aligned}u_i =& -\kappa(x_i - x_i^r) - \kappa(w_i - v_i^r) + u_i^r \\ & - \sum_{j=1}^{N} a_{ij}\pi_{ij} h(\hat{x}_i - \hat{x}_j + \hat{w}_i - \hat{w}_j, t), \quad i \in \mathcal{V}\end{aligned} \tag{14}$$

and π_{ij} is governed by the following adaptation law:

$$\begin{aligned}\dot{\pi}_{ij} = a_{ij}\big[&-\rho_{ij}\pi_{ij} + R_i + \left(\hat{x}_i - \hat{x}_j + \hat{w}_i - \hat{w}_j\right)^T \\ & \times h\left(\hat{x}_i - \hat{x}_j + \hat{w}_i - \hat{w}_j, t\right)\big], \quad i \in \mathcal{V}\end{aligned} \tag{15}$$

where $\hat{x}_j(t) = x_j(t_{k_j}^j)$ and $\hat{w}_j(t) = w_j(t_{k_j}^j)$, $t \in [t_{k_j}^j, t_{k_j+1}^j)$, denote the last broadcast position and filter output of agent j, respectively, and $t_{k_j}^j = \max\{t_k^j \mid t_k^j \le t\}$ is the latest triggering time instant of agent j, ρ_{ij} and R_i are positive constants to be determined, and $h : \mathbb{R}^p \times \mathbb{R}_{\ge 0} \to \mathbb{R}^p$ is a nonlinear function [24] defined as

$$h(z,t) = \frac{z}{\|z\|_2 + \eta e^{-ct}}$$

where η and c are positive constants. The boundary layer ηe^{-ct} is time varying, and as $t \to \infty$, the continuous function $h(z,t)$ approaches the discontinuous function $\mathrm{sgn}(z)$.

For each agent $i \in \mathcal{V}$, define

$$e_{x_i} = \hat{x}_i - x_i, \quad e_{w_i} = \hat{w}_i - w_i \tag{16}$$

the triggering time instant is determined by $t_1^i = 0$ and

$$t_{k+1}^i = \min\left\{t \mid f_i\left(t, x_i, w_i, \{\hat{x}_j, \hat{w}_j\}_{j\in\mathcal{N}_i\cup\{i\}}\right) > 0\right\} \tag{17}$$

where $f_i(t, x_i, w_i, \{\hat{x}_j, \hat{w}_j\}_{j\in\mathcal{N}_i\cup\{i\}})$ is agent i's triggering function, which is given by

$$\begin{aligned}& f_i\left(t, x_i, w_i, \{\hat{x}_j, \hat{w}_j\}_{j\in\mathcal{N}_i\cup\{i\}}\right) \\ & = \left|\|e_{x_i} + e_{w_i}\|_1 R_i + (e_{x_i} + e_{w_i})^T \hat{\zeta}_i\right| - \epsilon_i e^{-\varphi_i t}\end{aligned} \tag{18}$$

where

$$\hat{\zeta}_i = \sum_{j=1}^{N} a_{ij}\pi_{ij} h(\hat{x}_i - \hat{x}_j + \hat{w}_i - \hat{w}_j, t),$$

and R_i, ϵ_i, and φ_i are positive constants to be determined. Note that the triggering function in (18) takes values in $\mathbb{R}$ and depends on time t, its current position x_i and current filter's output w_i, and its own and neighbors' last broadcast positions $\{\hat{x}_j\}_{j\in\mathcal{N}_i\cup\{i\}}$ and filter's outputs $\{\hat{w}_j\}_{j\in\mathcal{N}_i\cup\{i\}}$. For agent i, at the triggering time instant, it updates its filter's input and controller by using its current position and filter's output and broadcasts its current position and filter's output to its neighbors. In the meantime, e_{x_i} and e_{w_i} are reset to zero. When an event is triggered at its neighboring agent j, it receives newly broadcast position and filter's output and update its filter's input and controller immediately.

Theorem 2: Suppose that the undirected graph $\mathcal{G}$ is connected, and Assumption 1 holds. Apply the algorithm (13)–(15) to (1) with $\kappa > \frac{3+2\sqrt{3}}{3}$, and the triggering time instant is determined by (17) with the triggering function defined in (18), where $\rho_{ij} > \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}$, $\epsilon_i > 0$, $\varphi_i > 0$, $\eta > 0$, $c > 0$, and the matrices P and Q are given in (9) and (10) with $\mu = \frac{3+\sqrt{3}}{3}$, respectively. Then, we have the following.

i) If $\beta \le R_i < \beta \max_{j\in\mathcal{N}_i}\{1, \rho_{ij}\sqrt{p}(N-1)\}$, distributed average tracking is achieved with bounded error.

ii) If $R_i \ge \beta \max_{j\in\mathcal{N}_i}\{1, \rho_{ij}\sqrt{p}(N-1)\}$, distributed average tracking is achieved with zero error.

In addition, the triggering law (17) excludes Zeno behavior while running the algorithm (13)–(15)

Proof: We first prove statement (i). The proof follows the same two steps described in that of Theorem 1. Use the same definitions of $\tilde{x}$, $\tilde{v}$, and $\tilde{w}$ as in Section III. For notational simplicity, let $\chi = \tilde{x} + \tilde{w}$ and $\hat{\chi} = (M \otimes I_p)(\hat{x} + \hat{w})$ with

$$\chi_i = \tilde{x}_i + \tilde{w}_i = x_i - \frac{1}{N}\sum_{j=1}^{N} x_j + w_i - \frac{1}{N}\sum_{j=1}^{N} w_j$$

and

$$\hat{\chi}_i = \hat{x}_i - \frac{1}{N}\sum_{j=1}^{N} \hat{x}_j + \hat{w}_i - \frac{1}{N}\sum_{j=1}^{N} \hat{w}_j.$$

Then, we have

$$\begin{aligned}\dot{\tilde{x}} =& \tilde{v} \\ \dot{\tilde{v}} =& -\kappa\tilde{x} - \kappa\tilde{w} + (M \otimes I_p)\alpha \\ & - \begin{bmatrix} \sum_{j=1}^{N} a_{1j}\pi_{1j} h(\hat{\chi}_1 - \hat{\chi}_j, t) \\ \vdots \\ \sum_{j=1}^{N} a_{Nj}\pi_{Nj} h(\hat{\chi}_N - \hat{\chi}_j, t) \end{bmatrix}.\end{aligned} \tag{19}$$

Consider the function V defined in (8). Taking the derivative of V along (19) yields

$$\begin{aligned}\dot{V} =& -X^T Q X + \chi^T (M \otimes I_p)\alpha \\ & - \chi^T \begin{bmatrix} \sum_{j=1}^{N} a_{1j}\pi_{1j} h(\hat{\chi}_1 - \hat{\chi}_j, t) \\ \vdots \\ \sum_{j=1}^{N} a_{Nj}\pi_{Nj} h(\hat{\chi}_N - \hat{\chi}_j, t) \end{bmatrix} \\ & + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \dot{\pi}_{ij}(\pi_{ij} - \pi_m).\end{aligned}$$

Then, by using the facts that $\pi_{ij} = \pi_{ji}$ and $h(-z,t) = -h(z,t)$, it holds that

$$\dot{V} \leq -X^T QX - (e_x + e_w)^T (M \otimes I_p)\alpha + \sum_{i=1}^N \sum_{j=1}^N \frac{a_{ij}\bar{\beta}}{2} \|\hat{\chi}_i - \hat{\chi}_j\|_1 + \sum_{i=1}^N (e_{x_i} + e_{w_i})^T \sum_{j=1}^N a_{ij}\pi_{ij} h(\hat{x}_i - \hat{x}_j + \hat{w}_i - \hat{w}_j, t) + \sum_{i=1}^N \sum_{j=1}^N \frac{a_{ij}\rho_{ij}}{2} \left(-\pi_{ij} + \frac{R_i}{\rho_{ij}}\right)(\pi_{ij} - \pi_m) - \sum_{i=1}^N \sum_{j=1}^N \frac{a_{ij}\pi_m}{2} (\hat{\chi}_i - \hat{\chi}_j)^T h(\hat{\chi}_i - \hat{\chi}_j, t)$$

where $\bar{\beta} = (N-1)\beta$. Note that

$$\sum_{i=1}^N \sum_{j=1}^N a_{ij} \left[\frac{\bar{\beta}}{2} \|\hat{\chi}_i - \hat{\chi}_j\|_1 - \frac{\pi_m}{2} (\hat{\chi}_i - \hat{\chi}_j)^T h(\hat{\chi}_i - \hat{\chi}_j, t)\right] = \sum_{i=1}^N \sum_{j=1}^N a_{ij} \left(\frac{\bar{\beta}}{2} \|\hat{\chi}_i - \hat{\chi}_j\|_1 - \frac{\pi_m}{2} \frac{a_{ij}\|\hat{\chi}_i - \hat{\chi}_j\|_2^2}{\|\hat{\chi}_i - \hat{\chi}_j\|_2 + \eta e^{-ct}}\right) \leq \sum_{i=1}^N \sum_{j=1}^N a_{ij} \left(\frac{\bar{\beta}\sqrt{p} - \pi_m}{2} \|\hat{\chi}_i - \hat{\chi}_j\|_2 + \frac{\pi_m}{2} \eta e^{-ct}\right)$$

where the fact that $\|\hat{\chi}_i - \hat{\chi}_j\|_1 \leq \sqrt{p}\|\hat{\chi}_i - \hat{\chi}_j\|_2$ is used. Since $ab \leq \frac{\epsilon a^2}{2} + \frac{b^2}{2\epsilon}$ $\forall a, b \in \mathbb{R}$ holds for any $\epsilon > 0$, it then follows that:

$$\sum_{i=1}^N \sum_{j=1}^N \frac{a_{ij}\rho_{ij}}{2} \left(-\pi_{ij} + \frac{R_i}{\rho_{ij}}\right)(\pi_{ij} - \pi_m) = \sum_{i=1}^N \sum_{j=1}^N \frac{a_{ij}\rho_{ij}}{2} \left[-(\pi_{ij} - \pi_m)^2 + \left(\frac{R_i}{\rho_{ij}} - \pi_m\right)(\pi_{ij} - \pi_m)\right] \leq \sum_{i=1}^N \sum_{j=1}^N \frac{a_{ij}\rho_{ij}}{2} \left[-\frac{1}{2}(\pi_{ij} - \pi_m)^2 + \frac{1}{2}\left(\frac{R_i}{\rho_{ij}} - \pi_m\right)^2\right].$$

Thus, selecting a π_m such that $\pi_m \geq \bar{\beta}\sqrt{p}$ yields that

$$\dot{V} \leq -X^T QX + \sum_{i=1}^N \sum_{j=1}^N \frac{a_{ij}\rho_{ij}}{4} \left(\frac{R_i}{\rho_{ij}} - \pi_m\right)^2 + \frac{\pi_m}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij}\eta e^{-ct} - \sum_{i=1}^N \sum_{j=1}^N \frac{a_{ij}\rho_{ij}}{4} (\pi_{ij} - \pi_m)^2 + \frac{\bar{\alpha} N_{\max}(N-1)}{2} \sum_{i=1}^N \|e_{x_i} + e_{w_i}\|_1 + \sum_{i=1}^N (e_{x_i} + e_{w_i})^T \sum_{j=1}^N a_{ij}\pi_{ij} h(\hat{x}_i - \hat{x}_j + \hat{w}_i - \hat{w}_j, t)$$

where we have used the Hölder's inequality. Then, implementing the triggering condition (17), (18) yields

$$\dot{V} \leq -X^T QX + \sum_{i=1}^N \sum_{j=1}^N \frac{a_{ij}\rho_{ij}}{4} \left(\frac{R_i}{\rho_{ij}} - \pi_m\right)^2 + \sum_{i=1}^N \epsilon_i e^{-\varphi_i t} + \frac{\pi_m}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij}\eta e^{-ct} - \sum_{i=1}^N \sum_{j=1}^N \frac{a_{ij}\rho_{ij}}{4} (\pi_{ij} - \pi_m)^2$$
$$\leq -\lambda_{Q/P} V + \sum_{i=1}^N \epsilon_i e^{-\varphi_i t} + \frac{\pi_m}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij}\eta e^{-ct} - \frac{1}{4} \sum_{i=1}^N \sum_{j=1}^N a_{ij} \left(\rho_{ij} - \lambda_{Q/P}\right)(\pi_{ij} - \pi_m)^2 + \sum_{i=1}^N \sum_{j=1}^N \frac{a_{ij}\rho_{ij}}{4} \left(\frac{R_{2i}}{\rho_{ij}} - \pi_m\right)^2$$
$$\leq -\lambda_{Q/P} V + \sum_{i=1}^N \epsilon_i e^{-\varphi_i t} + \frac{\pi_m}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij}\eta e^{-ct} + \sum_{i=1}^N \sum_{j=1}^N \frac{a_{ij}\rho_{ij}}{4} \left(\frac{R_i}{\rho_{ij}} - \pi_m\right)^2$$

where $\lambda_{Q/P} = \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}$, and the last inequality holds because $\rho_{ij} > \lambda_{Q/P}$. According to the comparison lemma in [23], it holds that

$$V \leq e^{-\lambda_{Q/P} t} \left[V(0) + \lambda_{Q/P} \sum_{i=1}^N \sum_{j=1}^N \frac{a_{ij}\rho_{ij}}{4} \left(\frac{R_i}{\rho_{ij}} - \pi_m\right)^2\right] + \lambda_{Q/P} \sum_{i=1}^N \sum_{j=1}^N \frac{a_{ij}\rho_{ij}}{4} \left(\frac{R_i}{\rho_{ij}} - \pi_m\right)^2 + e^{-\lambda_{Q/P} t} \sum_{i=1}^N \int_0^t \left(\epsilon_i e^{-(\varphi_i - \lambda_{Q/P})\tau} + \frac{\pi_m}{2} |\mathcal{N}_i| \eta e^{-(c - \lambda_{Q/P})\tau}\right) d\tau.$$

Therefore,

$$\lim_{t\to\infty} V \leq \lambda_{Q/P} \sum_{i=1}^N \sum_{j=1}^N \frac{a_{ij}\rho_{ij}}{4} \left(\frac{R_i}{\rho_{ij}} - \pi_m\right)^2,$$

which implies that $\|x_i - \frac{1}{N}\sum_{j=1}^N x_j\|_2$, $\|v_i - \frac{1}{N}\sum_{j=1}^N v_j\|_2$, and $\|w_i - \frac{1}{N}\sum_{j=1}^N w_j\|_2$ are all bounded.

Second, define S_x, S_v, and S_w as in the proof of Theorem 1. Note that

$$\sum_{i=1}^N \sum_{j=1}^N a_{ij}\pi_{ij} h(\hat{\chi}_i - \hat{\chi}_j) = 0$$

holds for any $i \in \mathcal{V}$ because $\pi_{ij} = \pi_{ji}$ and $h(-z,t) = -h(z,t)$. As a result, by a similar proof of Theorem 1, it follows that

$$\lim_{t\to\infty} \sum_{j=1}^{N} x_j = \sum_{j=1}^{N} x_j^r$$

and

$$\lim_{t\to\infty} \sum_{j=1}^{N} v_j = \sum_{j=1}^{N} v_j^r.$$

Therefore, $\lim_{t\to\infty}(x_i - \frac{1}{N}\sum_{j=1}^{N} x_j^r)$ and $\lim_{t\to\infty}(v_i - \frac{1}{N}\sum_{j=1}^{N} v_j^r)$ are bounded.

For the proof of the statement (ii), we consider the following Lyapunov function candidate:

$$V_2 = \frac{1}{2}X^T P X + \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\pi_{ij} - \frac{R_i}{\rho_{ij}}\right)^2.$$

Taking the derivative yields that

$$\begin{aligned}\dot{V}_2 \leq\ & -X^T Q X - (e_x + e_w)^T (M \otimes I_p)\alpha \\ & + \sum_{i=1}^{N}(e_{x_i} + e_{w_i})^T \sum_{j=1}^{N} a_{ij}\pi_{ij} h(\hat{x}_i - \hat{x}_j + \hat{w}_i - \hat{w}_j, t) \\ & - \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{a_{ij}\rho_{ij}}{2}\left(\pi_{ij} - \frac{R_i}{\rho_{ij}}\right)^2 \\ & + \frac{\beta}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\left\|\hat{\chi}_i - \hat{\chi}_j\right\|_1 \\ & - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{R_i}{\rho_{ij}} a_{ij}(\hat{\chi}_i - \hat{\chi}_j)^T h(\hat{\chi}_i - \hat{\chi}_j, t).\end{aligned}$$

Notice that $-(e_x + e_w)^T (M \otimes I_p)\alpha \leq \beta\|e_x + e_w\|_1$ Implementing the triggering condition (17), (18) yields that

$$\begin{aligned}\dot{V}_2 &\leq -X^T Q X + \sum_{i=1}^{N}\epsilon_i e^{-\varphi_i t} + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{R_i}{\rho_{ij}} a_{ij}\eta e^{-ct} \\ &\quad - \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{a_{ij}\rho_{ij}}{2}\left(\pi_{ij} - \frac{R_i}{\rho_{ij}}\right)^2 \\ &\leq -\lambda_{Q/P} V + \sum_{i=1}^{N}\epsilon_i e^{-\varphi_i t} + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{R_i}{\rho_{ij}} a_{ij}\eta e^{-ct} \\ &\quad - \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\left(\frac{\rho_{ij}}{2} - \frac{\lambda_{Q/P}}{4}\right)\left(\pi_{ij} - \frac{R_i}{\rho_{ij}}\right)^2 \\ &\leq -\lambda_{Q/P} V + \sum_{i=1}^{N}\epsilon_i e^{-\varphi_i t} + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{R_i}{\rho_{ij}} a_{ij}\eta e^{-ct}\end{aligned}$$

where the last inequality holds by noting that $\rho_{ij} > \lambda_{Q/P}$ in the statement. Following the similar line of analysis as in the proof of statement (i), we have $\lim_{t\to\infty} V_2 = 0$, which implies that $x_i \to \frac{1}{N}\sum_{j=1}^{N} x_j$, $v_i \to \frac{1}{N}\sum_{j=1}^{N} v_j$, and $w_i \to \frac{1}{N}\sum_{j=1}^{N} w_j$, as $t \to \infty$. Hence, the consensus step is completed. The sum-tracking step can be completed by the same analysis to that in the proof of statement (i). Therefore, the distributed average tracking is achieved with zero tracking error.

Next, we prove that the proposed event-triggering mechanism (17), (18) is able to exclude Zeno behavior. Since V (or V_2) is bounded according to the analysis above, it is concluded that $\|x_i\|_1$, $\|w_i\|_1$ $\forall i \in \mathcal{V}$, and $|\pi_{ij}|$ $\forall (i,j) \in \mathcal{E}$ are all bounded. It then follows that $\|\dot{x}_i\|_1$ and $\|\dot{w}_i\|_1$ are bounded. Let $\dot{w}_i^{\max} = \sup_{t\in[0,\infty)}\|\dot{x}_i\|_1$, and $\dot{w}_i^{\max} = \sup_{t\in[0,\infty)}\|\dot{w}_i\|_1$. Note that

$$\begin{aligned}&\left|\left\|e_{x_i} + e_{w_i}\right\|_1 R_i + (e_{x_i} + e_{w_i})^T\hat{\zeta}_i\right| \\ &\leq \left\|e_{xi} + e_{wi}\right\|_1 R_i + \left|(e_{xi} + e_{wi})^T\hat{\zeta}_i\right| \\ &\leq \left\|e_{x_i} + e_{w_i}\right\|_1\left(R_i + \left\|\hat{\zeta}_i\right\|_\infty\right) \\ &\leq \left\|\hat{x}_i - x_i + \hat{w}_i - w_i\right\|_1\left(R_i + \left\|\hat{\zeta}_i\right\|_\infty\right) \\ &\leq (t - t^*)\left(\dot{x}_i^{\max} + \dot{w}_i^{\max}\right)\left(R_i + \left\|\hat{\zeta}_i\right\|_\infty\right)\end{aligned}$$

where $\hat{\zeta}_i$ is defined in (18). The next event will not be triggered before $\left|\|e_{x_i} + e_{w_i}\|_1 R_i + (e_{x_i} + e_{w_i})^T\hat{\zeta}_i\right| = \epsilon_i e^{-\varphi_i t}$. Thus, a lower bound is given by $\tau^* = t - t^*$ that solves the equation

$$(t - t^*)\left(\dot{x}_i^{\max} + \dot{w}_i^{\max}\right)\left(R_i + \left\|\hat{\zeta}_i\right\|_\infty\right)\tau^* = \epsilon_1 e^{-\varphi_i \tau^*} e^{-\varphi_i t^*}.$$

It is apparent that $\tau^* > 0$, which implies no Zeno behavior. This completes the proof. ■

From the triggering condition (17), (18) and the proof of Theorem 2, the function $\epsilon_i e^{-\phi_i t}$ serves as the time-varying threshold for the term $\left|\|e_{x_i} + e_{w_i}\|_1 R_i + (e_{x_i} + e_{w_i})^T\hat{\zeta}_i\right| := F(e_{x_i} + e_{w_i})$. Once $F(e_{x_i} + e_{w_i})$ reaches the threshold, the agent is triggered. Therefore, selecting proper ϵ_i and ϕ_i allows one to affect the rate of triggering times. To be exact, a larger value of ϵ_i and a smaller value of ϕ_i intuitively lead to a lower triggering rate.

The matrices P and Q are accessible to agents, since once κ is determined, the form of these two matrices are fixed. Then, the eigenvalues of P and Q can be easily computed by each agent.

Remark 5: As indicated in Theorem 2, the lower bound of the design parameter R_i depends on some global information such as the total number of agents in the network and the bounds related to the reference signals. However, the parameter is constant and can be determined offline before running the algorithms. One can always be more conservative to select a large enough number for R_i. Moreover, due to the challenging nature of the problem studied in this article, it might be inevitable to have certain piece of global information to determine the lower bound for the design parameter. This is also the case in the literature [15], [16], even when solving a simpler problem compared to the one studied in this article. In addition, to obtain a better estimate of the lower bound of the design parameter, one can use some existing algorithms in the literature [22], [25] to estimate the global information by interacting with local neighbors.

Remark 6: The adaptation law (15) is partially inspired by Zhao *et al.* [13]. The difference is the adoption of R_i in (15) for each agent. From Theorem 2, we can see that the value of R_i has an effect on the tracking error. As stated in Theorem 2, distributed average tracking is achieved with zero tracking error when R_i is sufficient larger. It is also worth noting that in this article, an event-triggered communication mechanism is proposed to avoid continuous interactions and reduce the communication cost. In addition, only position measurements are used. These two points distinguish the present work from the one in [13].

Remark 7: The distributed average tracking problem is solved by the proposed event-triggered algorithm, and Zeno behavior is excluded, which removes the requirement of continuous interactions among agents. Compared with the existing works on event-triggered distributed average tracking algorithms [15], [16], the proposed one (13)–(15) contributes in the following two aspects: 1) the algorithm is able to be implemented for double-integrator agents without using velocity measurements, which is economical and energy efficient; and 2) several practical limitations have been overcome by the newly designed triggering strategy.

V. Illustrative Examples

In this section, we provide examples to illustrate the results obtained in this article.

We consider a group of 20 physical agents ($N = 20$) given in (1), which are labeled as $1, \ldots, 20$. The agents form a ring topology. In the simulation, we set

$$u_i^r = A_i \sin(\vartheta_i t + \varphi_i)$$

in (2) with

$$\begin{aligned} A_i =& -0.04(0.7i + 0.5)^2[2(i - 3.5) - 2(-1)^i], \\ \vartheta_i =& \, 0.2(0.7i + 0.5), \end{aligned}$$

and

$$\varphi_i = (2i\pi/N) - \pi.$$

Select $\kappa = 5$ and $\pi_{ij}(0) = 1000$ for any i and j that are connected. Implement the algorithm (3)–(5) for (1). The simulation results are shown in Fig. 1. It can be seen that all the agents' physical states, positions and velocities, are capable of tracking $\frac{1}{20}\sum_{j=1}^{20} x_j^r$ and $\frac{1}{20}\sum_{j=1}^{20} v_j^r$, respectively.

In the following, we use the algorithm (13)–(15) for (1) with the same set of reference signals. The triggering time instants are determined as in (17) with the triggering function defined in (18). For simplicity, we set $R_i = 2000$, $\epsilon_i = 1000$, $\rho_i = 5$, and $\varphi_i = 10^{-4}$ for any $i = 1, \ldots, 20$. Let $\eta = 10$ and $c = 1$. The position and velocity trajectories for those 20 agents are shown in Fig. 2. It can be seen that all the agents' physical states, positions and velocities, are capable of tracking $\frac{1}{20}\sum_{j=1}^{20} x_j^r$ and $\frac{1}{20}\sum_{j=1}^{20} v_j^r$, respectively. The number of triggering time instants for each agent is presented in Fig. 3. In this simulation, we use a fixed-step solver to solve the system, and the fixed-step size is 10^{-5}. In the 10-s simulation time, agents 1–20 are triggered 3.69%, 3.80%, 3.84%, 3.93%, 3.91%, 4.03%, 3.99%,

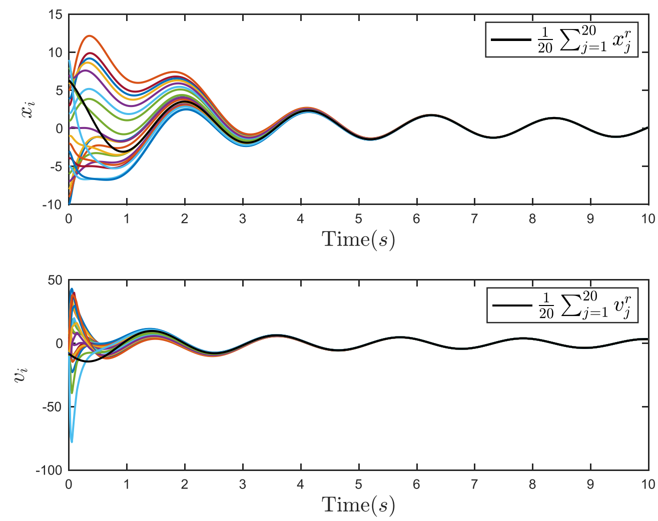

Fig. 1. Using algorithm (3)–(5) for (1), 20 agents' position and velocity trajectories. The black lines denote the average of the reference signals and their velocities. The rest are the position and velocity trajectories of these 20 agents.

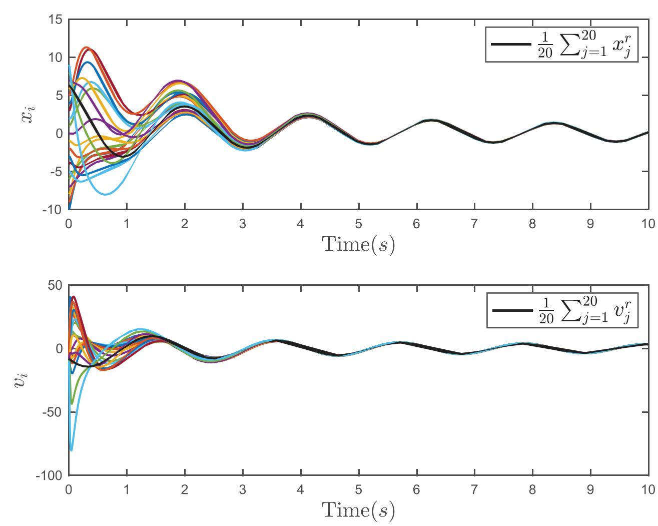

Fig. 2. Using algorithm (13)–(15) for (1), 20 agents' position and velocity trajectories.

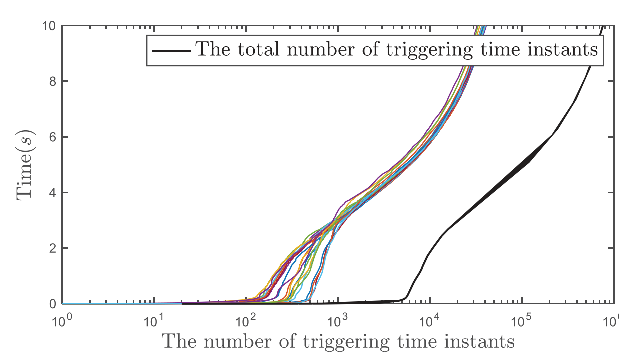

Fig. 3. Number of triggering time instants of the agents while using algorithm (13)–(15) for (1). The black line denotes the total number of triggering time instants. The rest are the number of triggering time instants for these 20 agents.

4.08%, 4.02%, 3.90%, 4.09%, 3.78%, 4.02%, 3.79%, 3.85%, 3.68%, 3.49%, 3.27%, 3.34%, and 3.72% of times. Therefore, the proposed distributed average tracking algorithm (13)–(15) avoids continuous communication.

VI. Conclusion

This article investigated the distributed average tracking problem for double-integrator agents without velocity measurements under event-triggered communication. First, a base algorithm was proposed, which removed the dependence of the design parameters' lower bounds on global information. Built on the base algorithm, an event-triggered distributed average tracking algorithm was designed to remove the continuous communication requirement. The event-triggered algorithm was developed with a new adaptation law and a new triggering condition, which overcame several practical limitations. In addition, a continuous nonlinear function was used approximate the signum function to reduce the chattering phenomenon in reality. Finally, several examples were provided to illustrate the results in this article.

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