Generalized Nash Equilibrium Seeking via Continuous-Time Coordination Dynamics Over Digraphs

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***Abstract*—This article studies a generalized Nash equilibrium problem with coupling equality constraints and local action sets, where the cost function of each player has a general form that depends on the actions of other players in this game. In the case that the players cannot directly use the others' actions, all players are allowed to estimate their opponents' actions by communicating with their neighbors over a digraph. In this regard, continuous-time coordination dynamics are proposed for two kinds of directed communication topologies including weight-balanced and weight-unbalanced digraphs. When the pseudogradient is strongly monotone and Lipschitz continuous as well as the extended pseudogradient is Lipschitz continuous, it is theoretically shown that the proposed dynamics could solve the generalized Nash equilibrium problem with and without local action sets, respectively. Finally, the obtained theoretical results are illustrated by numerical simulations.**

***Index Terms*—Continuous-time coordinated dynamics, coupling constraints, digraphs, generalized Nash equilibrium (NE) problem.**

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I. Introduction

NETWORKED games have gained considerable attention due to their wide applications in the context of multiagent networks such as sensor networks [1]; smart grids [2], [3]; and cloud systems [4]; just to name a few. The objective for a networked game is to find its Nash equilibrium (NE) or generalized Nash equilibrium (GNE) by designing distributed algorithms, where an NE (GNE) is referred to as a best response strategy at which any player does not benefit by deviating its local strategy unilaterally [5].

Recently, distributed computation on NE (GNE) has been extensively studied via a variety of NE (GNE)-seeking algorithms. Roughly speaking, the algorithm design is mainly based on two types of communication modes among players. One is that with a communication graph described by an interference graph, each player's neighbors in the interference graph are determined by the number of the other players whose actions influence that player's cost function. Typical results within this context can be found in [6]–[9]. Note that the above-mentioned case would lead to all-to-all communication interactions among players, especially when each player's local cost function depends on all the other opponents' actions. However, requiring a complete communication graph is restrictive and impractical in large-scale systems, as pointed out in [10]. To overcome this limitation, a general undirected and connected communication graph was considered in [10]–[14], [18], [20], and [21]. In this case, since players cannot fully obtain the other opponents' actions, each player is allowed to have an additional variable for estimating the others' actions by exchanging the estimate information with its neighbors. By using this mechanism, discrete-time algorithms based on gossip and inexact alternating direction method of multipliers-type were proposed for solving the NE (GNE) problems [10]–[14]. On the other hand, due to the effective analysis tool provided by continuous-time control techniques in multiagent systems [15]–[17], continuous-time distributed algorithms were constructed, such as a consensus approach [18], an integration of dynamic average consensus protocol and gradient-play [19], and a passivity control approach [20], [21]. As a specific form of a general game, aggregative games were investigated in [22]–[27].

Except for the previous results on undirected graphs, distributed NE (GNE) problems over directed communication graphs were studied in [28]–[33]. The two-player zero game was addressed in [28] and [29] by continuous-time and discrete-time algorithms, respectively. In [30], an unconstrained multiagent game was investigated via a continuous-time distributed seeking algorithm. In [31], asynchronous gossip-based algorithms were presented for tackling networked games with local action sets. Moreover, the aggregative games over weight-balanced digraphs were solved in [32] and [33].

It can be found that the study on the GNE problems over digraphs focuses mainly on the aggregative games with coupling constraints, where each player uses its estimate of the aggregate of players' actions to update its action [32], [33]. However, for a general game, it is hard to complete the updates to players' actions by the aggregate of players' actions, as shown in [10]. Although a general type of game model without and with local action sets over directed communication topologies was, respectively, studied in [30] and [31], the proposed algorithms cannot be directly applied to the case with affine constraints that couple the players' actions.

This article focuses on a GNE problem subject to coupling equality constraints and local action sets over digraphs, and the aim is to design continuous-time coordination dynamics to find a GNE of the game. In this regard, continuous-time coordination dynamics with projection operation are designed over a weight-balanced digraph. When the communication topology is weight-unbalanced, the estimation protocol of the left eigenvector [34] is incorporated into the proposed dynamics to solve the case without local action sets, where the estimation protocol is used to neutralize the weight-unbalanced information. In comparison to the existing works, the main contributions of this article are listed as follows.

1) Similar to the estimation mechanism given in [20] and [21], each player estimates all the other players' actions and exchanges its own estimation information with its neighbors in the present coordination dynamics. This protects the players' actions from leaking to other players, unlike the literature [7]–[9] in which each player can access to all players' actions related to its cost function.
2) The present coordination dynamics are able to solve a general type of game model over digraphs compared to the aggregative game models studied in [24]–[27], [32], and [33]. Also, they can deal with the case with coupling equality constraints over digraphs, while this is not directly solved by those techniques given in [30] and [31]. It should be pointed out that the proposed projection algorithm is an upper semicontinuous system in comparison to the algorithms reported in [20], [21], and [33] that are related to the tangent cones of local action sets.
3) The convergence of the proposed coordination dynamics is analyzed in detail by using the properties associated with the projection operation and the Lyapunov stability theory. In particular, it proved the exponential convergence of the proposed coordination dynamics without projection operation.

The remainder of this article is described as follows. Some preliminaries and game formulation are given in Section II. Continuous-time coordination dynamics for solving a GNE problem over digraphs are presented and analyzed in Section III. Theoretical results are illustrated by numerical simulations in Section IV. Section V concludes this article and suggests the future work.

II. Preliminaries and Game Formulation

Notations: Let R^n, $R^{p\times q}$, and $\|\cdot\|$ denote the set of n-dimensional real vectors, the set of $p\times q$ dimensional real matrixes, and Euclidean norm, respectively. Let I_n and $1_n \in R^n$ represent the identity matrix and the vector with each element equal to 1, respectively. Let $\mathbf{0}$ denote a column vector with all entries being zeros, whose dimension depends on the context in which it is used. Let $\mathcal{V} = \{1, 2, \ldots, N\}$ be an index set. $\mathrm{col}(x_1, x_2, \ldots, x_N)$ represents a stacked column vector in the form of $(x_1^T, x_2^T, \ldots, x_N^T)^T$. Let $\mathrm{diag}(a_1, a_2, \ldots, a_n)$ denote a diagonal matrix, where a_i is its diagonal element. For a square matrix A, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the largest and the smallest eigenvalues of A, respectively. $B \otimes C$ denotes the Kronecker product of matrices B and C. Given a nonempty closed convex set K, the normal cone to K at $x \in K$ is $N_K(x) = \{u \in R^n : u^T(y - x) \le 0 \ \forall y \in K\}$ and $P_K(x) = \arg\min_{y\in K} \|x - y\|$ denotes the projection of a vector $x \in R^n$ on K. It is well known that P_K is nonexpansive. That is, it holds that $\|P_K(x) - P_K(y)\| \le \|x - y\|$ for any $x, y \in R^n$.

A. Monotone Operators and Projection Properties

The following concepts can be referred to [35]. Let $F : D \subset R^n \to R^n$ be a vector-valued function. F is monotone if $(x - y)^T(F(x) - F(y)) \ge 0$ for all $x, y \in D$, and strictly monotone if the strict inequality holds whenever $x \ne y$. F is $m-$strongly monotone if $(x - y)^T(F(x) - F(y)) \ge m\|x - y\|^2$ for all $x, y \in D$. F is $M-$Lipschitz continuous if $\|F(x) - F(y)\| \le M\|x - y\|$ for all $x, y \in D$.

Lemma 1: (see [36] and [37]) Let $K \subset R^n$ be a nonempty closed convex set. For any $x, y \in R^n$, define $V : R^n \to R$ as follows:

$$V(x, y) = \frac{1}{2}(\|x - P_K(y)\|^2 - \|x - P_K(x)\|^2).$$

Then, $V(x, y)$ satisfy the following statements.

- $(x - z)^T(P_K(x) - P_K(z)) \ge \|P_K(x) - P_K(z)\|^2$ $\forall x, y \in R^n$.
- $V(x, y) \ge \frac{1}{2}\|P_K(x) - P_K(y)\|^2$.
- $V(x, y)$ is continuously differentiable with respect to x and its gradient is $\nabla_x V(x, y) = P_K(x) - P_K(y)$.

B. Grapy Theory

Let a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denote the information exchange among players, where $\mathcal{V}$ is the player set and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the edge set. An edge $e_{ij} \in \mathcal{E}$ indicates that player i can receive information from its neighbor player j. The weighted adjacency matrix $A = (a_{ij})_{N\times N}$ associated with graph $\mathcal{G}$ is defined as $a_{ij} > 0$ if $e_{ij} \in \mathcal{E}$, and $a_{ij} = 0$, otherwise. Moreover, $a_{ii} = 0$, $i \in \mathcal{V}$. A directed path from player i to player j is composed of a sequence of edges in the form $(i, i_1), (i_1, i_2), \ldots, (i_k, j)$.

A digraph is strongly connected if there exists a directed path between any pair of distinct players. The in-neighbors and out-neighbors of player i are denoted as $N_i^{\text{in}} = \{j|a_{ij} > 0\}$ and $N_i^{\text{out}} = \{j|a_{ji} > 0\}$, respectively. Correspondingly, the in-degree and out-degree are $d_i^{\text{in}} = \sum_{j\in N_i^{\text{in}}} a_{ij}$ and $d_i^{\text{out}} = \sum_{j\in N_i^{\text{out}}} a_{ji}$. The Laplacian matrix $L_N = (l_{ij})_{N\times N}$ corresponding to graph $\mathcal{G}$ is defined by $L_N = D^{\text{in}} - A$, where $D^{\text{in}} = \text{diag}(d_1^{\text{in}}, d_2^{\text{in}}, \ldots, d_N^{\text{in}})$. A digraph is weight-balanced if and only if $d_i^{\text{in}} = d_i^{\text{out}}$ for all $i \in \mathcal{V}$. Equivalently, $1_N^T L_N = \mathbf{0}$.

Lemma 2: (see [15] and [16]) Assume that graph $\mathcal{G}$ is strongly connected with the Laplacian matrix L_N. Then:

- there is a positive left eigenvector $\xi = (\xi_1, \xi_2, \ldots, \xi_N)^T$ associated with the zero eigenvalue such that $\xi^T L_N = \mathbf{0}^T$ and $\sum_{i=1}^N \xi_i = 1$.
- $\min_{1_N^T x=0} x^T \bar{L} x \geq \lambda_2(\bar{L})\|x\|^2$, where $\bar{L} = \frac{\Xi L_N + L_N^T \Xi}{2}$ with $\Xi = \text{diag}(\xi_1, \xi_2, \ldots, \xi_N)$ and $\lambda_2(\bar{L})$ is of the second smallest eigenvalue of $\bar{L}$.
- $\lim_{t\to\infty} e^{-L_N t} = 1_N \xi^T$.

C. Game Formulation

Consider a noncooperative game with N players. In this game, each player $i \in \mathcal{V}$ has its own local action set $\Omega_i \subset R^{n_i}$. Besides, all players share a coupling constraint $\sum_{i=1}^N B_i x_i = \sum_{i=1}^N b_i$, where $x_i \in \Omega_i$, $B_i \in R^{m\times n_i}$, and $b_i \in R^m$. The aim of player i is to choose its action x_i from its feasible action set $X_i(x_{-i}) = \{x_i \in \Omega_i | (x_i, x_{-i}) \in \Omega \cap X\}$ to minimize its cost function $f_i(x_i, x_{-i})$, given the other players' action $x_{-i} = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$, where $\Omega = \prod_{i=1}^N \Omega_i$ and $X = \{x \in R^n | Bx = b\}$. The considered noncooperative game problem can be mathematically described as follows:

$$\min f_i(x_i, x_{-i}), \quad \text{s.t.} \quad x_i \in X_i(x_{-i}). \tag{1}$$

Let $n = \sum_{i=1}^N n_i$, $B = [B_1, B_2, \ldots, B_N]$, $b = \sum_{i=1}^N b_i$, and $x = \text{col}(x_1, x_2, \ldots, x_N) \in R^n$. The strategy set of all players is denoted as $\Omega \cap X$. A strategy profile $x^* = \text{col}(x_1^*, x_2^*, \ldots, x_N^*)$ is called a GNE of the game (1) if for any $i \in \mathcal{V}$,

$$f_i(x_i, x_{-i}^*) \geq f_i(x_i^*, x_{-i}^*) \quad \forall x_i \in X_i(x_{-i}^*). \tag{2}$$

Assumption 1: For each $i \in \mathcal{V}$, the set Ω_i is convex, compact, and for each fixed x_{-i}, the local cost function $f_i(x_i, x_{-i})$ is continuously differentiable and convex with regard to x_i.

Assumption 2: There exists an interior point $\bar{x} \in \Omega$ such that X is nonempty.

Remark 1: Assumptions 1 and 2 ensure the existence of a GNE of game (1).

For each $i \in \mathcal{V}$, let $\nabla_i f_i(x_i, x_{-i})$ denote the gradient of the cost function $f_i(x_i, x_{-i})$ with respect to the action x_i. Then, $F(x) = \text{col}(\nabla_1 f_1(x_1, x_{-1}), \nabla_2 f_2(x_2, x_{-2}), \ldots, \nabla_N f_N(x_N, x_{-N}))$ is called the pseudogradient [38]. From [38, Th. 3.9], it follows that if $x^* \in \Omega \cap X$ is a solution of the following variational inequality:

$$(x - x^*)^T F(x^*) \geq 0 \quad \forall x \in \Omega, Bx = b \tag{3}$$

then x^* is a GNE of game (1). Actually, the variational inequality (3) is equivalent to the following optimization problem:

$$\min_x x^T F(x^*), \quad \text{s.t.} \quad x \in \Omega, Bx = b. \tag{4}$$

By [39, Th. 3.3], it is derived that the optimal conditions of (4) satisfy

$$0 \in F(x^*) + B^T \lambda^* + N_\Omega(x^*), \quad Bx^* = b. \tag{5}$$

The above-mentioned statements are formally characterized as follows.

Lemma 3: [38] With Assumptions 1 and 2, if x^* is a solution of the variational inequality (3), then x^* is a GNE of game (1) and there exists a $\lambda^* \in R^m$ such that x^* and λ^* satisfy (5).

Next, a mild assumption is made for the pseudogradient $F(x)$ [7], [8], [18], [20], [21], which ensures the uniqueness of the GNE problem (1).

Assumption 3: The pseudogradient $F : \Omega \to R^n$ is $m-$strong monotone and $M-$Lipschitz continuous.

Throughout this article, (1) is thought of as a multiagent game, where each player (agent) i only knows its own cost function f_i and could not fully observe the others' actions. Since each f_i depends on all players' actions, to be able to find the GNE of game (1), it is necessary to assume that players are willing to collaborate with their neighbors. The collaboration means that the players exchange their estimates of all others' actions with their individual neighbors, where the information exchange among players is described by a digraph, as shown in the following assumption.

Assumption 4: The information exchange among players is a strongly connected digraph.

III. GNE Seeking via Continuous-Time Coordination Dynamics

A. GNE Seeking Over a Weight-Balanced Digraph

This section first presents a class of continuous-time coordination dynamics for solving game (1) over a weight-balanced digraph. In this case, for each $i \in \mathcal{V}$, player i carries out the following GNE seeking strategy:

$$\begin{aligned}
\dot{w}_i^i &= -w_i^i + y_i^i - \nabla f_i(y_i^i, y_{-i}^i) - \varepsilon \sum_{j\in N_i^{\text{in}}} a_{ij}(y_i^i - y_i^j) \\
&\quad - B_i^T \lambda_i, \quad y_i^i = P_{\Omega_i}(w_i^i) \\
\dot{y}_{-i}^i &= -\varepsilon \sum_{j\in N_i^{\text{in}}} a_{ij}(y_{-i}^i - y_{-i}^j) \\
\dot{\lambda}_i &= B_i y_i^i - b_i - a \sum_{j\in N_i^{\text{in}}} a_{ij}(\lambda_i - \lambda_j) - z_i \\
\dot{z}_i &= a \sum_{j\in N_i^{\text{in}}} a_{ij}(\lambda_i - \lambda_j)
\end{aligned} \tag{6}$$

where $y_i^i = x_i$ denotes player i's own action that is viewed as the output of (6) and is equal to the projection of the auxiliary variable w_i^i on Ω_i, y_j^i represents player i's estimate of player j's action, y_{-i}^i denotes player i's estimate of all the others' actions,

λ_i is player i's estimate of the global multiplier associated with $Bx = b$, z_i is an auxiliary variable of player i, and $\varepsilon > 0$ and $a > 0$ are tuning parameters.

Remark 2: In (6), the design of $\dot{w}_i^i$ is based on player is pseudogradient dynamics that is integrated into a projection output feedback mechanism. Meanwhile, player i updates y_{-i}^i by using the received information from its neighbors so that all estimates reach consensus (see, $\dot{y}_{-i}^i$). The updates of λ_i and z_i are designed via a decomposition of $Bx = b$, where λ_i is used to estimate the global multiplier in a distributed way with the help of the auxiliary variable z_i^i.

Remark 3: Actually, the projection operation $P_\Omega(\cdot)$ [39] and the differential projection operation $P_{T_\Omega}(\cdot)$ [40], [41], in theory, can deal with the case with convex constraint set Ω. Note that the algorithms based on differential projection operation are not upper-semicontinuous at the boundary of Ω [20], [21], [33]. To avoid the discontinuity of $P_{T_\Omega}(\cdot)$ at the boundary of Ω, it is preferred to choose $P_\Omega(\cdot)$ to deal with the boundary of Ω.

For notational brevity, let $\boldsymbol{\lambda} = \mathrm{col}(\lambda_1, \lambda_2, \ldots, \lambda_N) \in R^{Nm}$, $\boldsymbol{z} = \mathrm{col}(z_1, z_2, \ldots, z_N) \in R^{Nm}$, $\boldsymbol{y} = \mathrm{col}(y^1, y^2, \ldots, y^N) \in R^{Nn}$ with $y^i \triangleq (y_i^i, y_{-i}^i) = \mathrm{col}(y_1^i, y_2^i, \ldots, y_N^i) \in R^n$, $\mathcal{R} = \mathrm{diag}(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_N) \in R^{n\times Nn}$ with $\mathcal{R}_i = (\, 0_{n_i\times\sum_{j<i} n_j} \; I_{n_i} \; 0_{n_i\times\sum_{j>i} n_j} \,) \in R^{n_i\times n}$, $\mathcal{S} = \mathrm{diag}(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_N) \in R^{(Nn-n)\times Nn}$ with $\mathcal{S}_i = \begin{pmatrix} I_{\sum_{j<i} n_j\times\sum_{j<i} n_j} & 0_{\sum_{j<i} n_j\times n_i} & 0_{\sum_{j<i} n_j\times\sum_{j>i} n_j} \\ 0_{\sum_{j>i} n_j\times\sum_{j<i} n_j} & 0_{\sum_{j>i} n_j\times n_i} & I_{\sum_{j>i} n_j\times\sum_{j>i} n_j} \end{pmatrix} \in R^{(n-n_i)\times n}$, $\boldsymbol{B} = \mathrm{diag}(B_1, B_2, \ldots, B_N) \in R^{Nm\times n}$, $\boldsymbol{b} = \mathrm{col}(b_1, b_2, \ldots, b_N) \in R^{Nm}$, $\boldsymbol{F}(\boldsymbol{y}) = \mathrm{col}(\nabla_1 f_1(y_1^1, y_{-1}^1), \nabla_2 f_2(y_2^2, y_{-2}^2), \ldots, \nabla_N f_N(y_N^N, y_{-N}^N)) \in R^n$, $\boldsymbol{\Omega} = \prod_{i=1}^N \boldsymbol{\Omega}_i \in R^{Nn}$ with $\boldsymbol{\Omega}_i = R^{n_1} \times R^{n_2} \times \cdots \times R^{n_{i-1}} \times \Omega_i \times R^{n_{i+1}} \times \cdots \times R^{n_N} \in R^n$, $\boldsymbol{w} = \mathrm{col}(w^1, w^2, \ldots, w^N) \in R^{Nn}$, where $w^i \triangleq (w_i^i, w_{-i}^i) = \mathrm{col}(w_1^i, w_2^i, \ldots, w_N^i) \in R^n$ with $w_{-i}^i = y_{-i}^i$.

In terms of the definitions of $\boldsymbol{w}$, $\boldsymbol{y}$, $\boldsymbol{\lambda}$, $\boldsymbol{z}$, $\boldsymbol{F}(\boldsymbol{y})$, $\boldsymbol{\Omega}$, $\boldsymbol{B}$, $\boldsymbol{b}$, and $\mathcal{R}$, (6) is equivalent to

$$\begin{aligned} \dot{\boldsymbol{w}} &= -\boldsymbol{w} + \boldsymbol{y} - \mathcal{R}^T \boldsymbol{F}(\boldsymbol{y}) - \varepsilon(L_N \otimes I_n)\boldsymbol{y} - \mathcal{R}^T \boldsymbol{B}^T \boldsymbol{\lambda} \\ \dot{\boldsymbol{\lambda}} &= \boldsymbol{B}\mathcal{R}\boldsymbol{y} - \boldsymbol{b} - a(L_N \otimes I_m)\boldsymbol{\lambda} - \boldsymbol{z} \\ \dot{\boldsymbol{z}} &= a(L_N \otimes I_m)\boldsymbol{\lambda}, \quad \boldsymbol{y} = P_{\boldsymbol{\Omega}}(\boldsymbol{w}) \end{aligned} \tag{7}$$

where L_N is defined in Lemma 2.

Lemma 4: Suppose that Assumptions 1–4 hold, $1_N^T L_N = \mathbf{0}^T$ and the initial value $\boldsymbol{z}(0)$ satisfies $\sum_{i=1}^N \boldsymbol{z}_i(0) = \mathbf{0}$. Let $\mathrm{col}(\boldsymbol{w}^*, \boldsymbol{\lambda}^*, \boldsymbol{z}^*)$ be an equilibrium point of (7) with $\boldsymbol{y}^* = P_{\boldsymbol{\Omega}}(\boldsymbol{w}^*)$. Then, $\boldsymbol{y}^* = 1_N \otimes x^*$, where x^* is the GNE of game (1).

Proof: According to the definition of $\mathrm{col}(\boldsymbol{w}^*, \boldsymbol{\lambda}^*, \boldsymbol{z}^*)$, one gets

$$\boldsymbol{y}^* - \mathcal{R}^T \boldsymbol{F}(\boldsymbol{y}^*) - \varepsilon(L_N \otimes I_n)\boldsymbol{y}^* - \mathcal{R}^T \boldsymbol{B}^T \boldsymbol{\lambda}^* = \boldsymbol{w}^* \tag{8}$$

$$\boldsymbol{B}\mathcal{R}\boldsymbol{y}^* - \boldsymbol{b} - a(L_N \otimes I_m)\boldsymbol{\lambda}^* - \boldsymbol{z}^* = \mathbf{0} \tag{9}$$

$$(L_N \otimes I_m)\boldsymbol{\lambda}^* = \mathbf{0}. \tag{10}$$

The above-mentioned equation means that $\boldsymbol{\lambda}^* = 1_N \otimes \lambda^*$ with $\lambda^* \in R^m$. With $\sum_{i=1}^N \boldsymbol{z}_i(0) = \mathbf{0}$ and $\mathcal{R}\boldsymbol{y}^* = x^*$, we can obtain that $Bx^* = b$, by left multiplying (9) by $1_N^T \otimes I_m$. Note that $\mathcal{S}\mathcal{R}^T = 0$, $\mathcal{R}\mathcal{R}^T = I_n$, $\mathcal{S}\boldsymbol{y} = \mathrm{col}(y_{-1}^1, y_{-2}^2, \ldots, y_{-N}^N)$, $\mathcal{S}\boldsymbol{w} = \mathrm{col}(w_{-1}^1, w_{-2}^2, \ldots, w_{-N}^N)$, and $w_{-i}^i = y_{-i}^i, \forall i \in \mathcal{V}$. Thus, the following facts hold:

$$\mathcal{S}(L_N \otimes I_n)\boldsymbol{y}^* = \mathbf{0} \tag{11}$$

$$\mathcal{R}\boldsymbol{y}^* - \boldsymbol{F}(\boldsymbol{y}^*) - \varepsilon\mathcal{R}(L_N \otimes I_n)\boldsymbol{y}^* - \boldsymbol{B}^T\boldsymbol{\lambda}^* = \mathcal{R}\boldsymbol{w}^*. \tag{12}$$

Applying the similar analysis given in [20] and [21] yields $(L_N \otimes I_n)\boldsymbol{y}^* = \mathbf{0}$, which indicates that all $(y^i)^*$, $i \in \mathcal{V}$ reach consensus. Together with $(y_i^i)^* = x_i^*$, we have $\boldsymbol{y}^* = 1_N \otimes x^*$. By substituting $\mathcal{R}\boldsymbol{y}^* = x^*$, $\mathcal{R}\boldsymbol{w}^* = \mathrm{col}((w_1^1)^*, (w_2^2)^*, \ldots, (w_N^N)^*)$, $\boldsymbol{F}(\boldsymbol{y}^*) = \boldsymbol{F}(1_N \otimes x^*) = F(x^*)$, $P_\Omega(\mathcal{R}\boldsymbol{w}^*) = x^*$, and $\boldsymbol{\lambda}^* = 1_N \otimes \lambda^*$ into (12), we get $P_\Omega(x^* - F(x^*) - B^T\lambda^*) = x^*$, which is equivalent to $\mathbf{0} \in F(x^*) + B^T\lambda^* + N_\Omega(x^*)$ by [40]. In other words, x^* and λ^* satisfy (5). This shows that x^* is the GNE of game (1) by Lemma 3. □

An additional assumption for $\boldsymbol{F}(\boldsymbol{y})$ used in [20] and [21] is given below.

Assumption 5: The extended pseudogradient $\boldsymbol{F}$ is c-Lipschitz continuous on $\boldsymbol{\Omega}$.

Remark 4: As shown in [20] and [21], $\boldsymbol{F}$ can be viewed an extension of the pseudogradient F. Correspondingly, Assumption 5 extends the Lipschitz continuity of F to $\boldsymbol{F}$. In fact, Assumption 5 plays a key role in ensuring the existence and uniqueness of the solutions to (7). Specifically, from $\boldsymbol{y} = P_{\boldsymbol{\Omega}}(\boldsymbol{w})$, it follows that $\boldsymbol{y}' = P_{\boldsymbol{\Omega}}(\boldsymbol{w}')$ and $\boldsymbol{y}'' = P_{\boldsymbol{\Omega}}(\boldsymbol{w}'')$ for any $\boldsymbol{w}' \in R^{Nn}$ and $\boldsymbol{w}'' \in R^{Nn}$. Using Assumption 5 and the nonexpansive property of $P_{\boldsymbol{\Omega}}$ to yield $\|\boldsymbol{F}((P_{\boldsymbol{\Omega}}(\boldsymbol{w}')) - \boldsymbol{F}((P_{\boldsymbol{\Omega}}(\boldsymbol{w}''))\| = \|\boldsymbol{F}(\boldsymbol{y}') - \boldsymbol{F}(\boldsymbol{y}'')\| \le c\|\boldsymbol{y}' - \boldsymbol{y}''\| = c\|P_{\boldsymbol{\Omega}}(\boldsymbol{w}') - P_{\boldsymbol{\Omega}}(\boldsymbol{w}'')\| \le c\|\boldsymbol{w}' - \boldsymbol{w}''\|$. Thus, one can obtain that the composite mapping $\boldsymbol{F}(P_{\boldsymbol{\Omega}})$ is c-Lipschitz continuous on R^{Nn}. Denote the left-hand and the right-hand sides of (7) as $\mathrm{col}(\dot{\boldsymbol{w}}, \dot{\boldsymbol{\lambda}}, \dot{\boldsymbol{z}})$ and $\Psi(\boldsymbol{w}, \boldsymbol{\lambda}, \boldsymbol{z})$, respectively. Then, (7) can be read as $\mathrm{col}(\dot{\boldsymbol{w}}, \dot{\boldsymbol{\lambda}}, \dot{\boldsymbol{z}}) = \boldsymbol{\Psi}(\boldsymbol{w}, \boldsymbol{\lambda}, \boldsymbol{z})$. It is not difficult to find that $\boldsymbol{\Psi}$ is Lipschitz continuous on $R^{Nn} \times R^{Nm} \times R^{Nm}$. As a result, it follows that the existence and the uniqueness of the solutions to (7) can be guaranteed by [42, Th. 3.2]. Also, Assumptions 3 and 5 are two key conditions in ensuring the convergence of the proposed GNE seeking strategies as shall be seen in Appendix A.

Theorem 1: Suppose that Assumptions 1–5 hold, $1_N^T L_N = \mathbf{0}^T$ and the initial value $\boldsymbol{z}(0)$ satisfies $\sum_{i=1}^N z_i(0) = \mathbf{0}$. Let a and ε satisfy the following inequalities:

$$\varepsilon > \frac{4mc + (M+c)^2}{4m\lambda_2(\tilde{L})}, \quad a > \frac{(\kappa+1)^2}{\kappa\lambda_2(\tilde{L})} \tag{13}$$

where $\tilde{L} = (L_N + L_N^T)/2$, $\kappa > \{\frac{\|\boldsymbol{B}\|^2}{\lambda_{\min}(Q)} - 1, 0\}$, $\lambda_2(\tilde{L})$ is the second smallest eigenvalue of $\tilde{L}$ and

$$Q = \begin{pmatrix} \lambda_2(\tilde{L})\varepsilon - c & -\dfrac{M+c}{2\sqrt{N}} \\ -\dfrac{M+c}{2\sqrt{N}} & \dfrac{m}{N} \end{pmatrix}$$

is a positive definite matrix. Then, the trajectory $\mathrm{col}(\boldsymbol{w}(t), \boldsymbol{\lambda}(t), \boldsymbol{z}(t))$ of system (7) is bounded and $\boldsymbol{y}(t)$

converges asymptotically to $1_N \otimes x^*$, where x^* is the GNE of game (1).

Proof: The proof is given in Appendix A. □

B. GNE Seeking Over a Weight-Unbalanced Digraph

It can be seen that the GNE seeking strategy (6) is based on a weight-balanced digraph among players. As shown in [31], the weight-unbalanced communication is ubiquitous in reality such as sensor networks. To solve the GNE problem (1) under a weight-unbalanced digraph and facilitate the subsequent analysis, it is assumed that $\Omega_i = R^{n_i}$ for any $i \in \mathcal{N}$. In this case, for each $i \in \mathcal{N}$, player i runs the following GNE seeking strategy:

$$\begin{aligned}
\dot{y}_i^i &= -(\mu_i^i)^{-1}\nabla f_i(y_i^i, y_{-i}^i) - \varepsilon \sum_{j\in N_i^{\text{in}}} a_{ij}(y_i^i - y_i^j)\\
&\quad - (\mu_i^i)^{-1} B_i^T \lambda_i\\
\dot{y}_{-i}^i &= -\varepsilon \sum_{j\in N_i^{\text{in}}} a_{ij}(y_{-i}^i - y_{-i}^j)\\
\dot{\lambda}_i &= (\mu_i^i)^{-1}(B_i y_i^i - b_i) - a \sum_{j\in N_i^{\text{in}}} a_{ij}(\lambda_i - \lambda_j) - z_i\\
\dot{z}_i &= a \sum_{j\in N_i^{\text{in}}} a_{ij}(\lambda_i - \lambda_j)\\
\dot{\mu}_i &= -\sum_{j\in N_i^{\text{in}}} a_{ij}(\mu_i - \mu_j)
\end{aligned} \tag{14}$$

where $\mu_i = (\mu_i^1, \mu_i^2, \ldots, \mu_i^N) \in R^N$.

Remark 5: The design of the dynamics (14) is based on the integration of the consensus protocol $\dot{\mu}_i$ [34] and the dynamics (6) with $\Omega_i = R^{n_i}$. In comparison to the push-sum mechanism [43], [44], the present distributed setting based on the Laplacian matrix with a zero row sum is more practical in broadcast-based communications, as shown in [34] and [45]. Instead, there still needs a matrix with a zero column sum like the Laplacian matrix with a zero column sum given in the push-sum mechanism [44]. To be specific, in (14), μ_i is used to estimate the left eigenvector ξ, and its ith component μ_i^i is used in $\dot{\lambda}_i$ and $\dot{y}_i^i$. As seen in the subsequent proof given in Lemma 5, the use of μ_i^i is to ensure a balance matrix ΞL_N such that its balance is utilized to obtain the optimal conditions of (4) with $\Omega = R^n$ at the steady state of system (15), where a key property is that the matrix ΞL_N satisfies a zero column sum. Here, it should be emphasized that an estimation way is used to obtain such a matrix with a zero column sum instead of requiring that the Laplacian matrix satisfies a zero column sum by using the out-degrees or controlling the outgoing weights.

Let $\boldsymbol{\mu} = \text{col}(\mu_1, \mu_2, \ldots, \mu_N)$ and $E = \text{diag}(\mu_1^1, \mu_2^2, \ldots, \mu_N^N)$. With $\boldsymbol{y}$, $\boldsymbol{\lambda}$, $\boldsymbol{z}$, $\boldsymbol{F}(\boldsymbol{y})$, $\boldsymbol{\mu}$, E, $\boldsymbol{B}$, $\boldsymbol{b}$, and $\mathcal{R}$, (14) becomes

$$\begin{aligned}
\dot{\boldsymbol{y}} &= -(E^{-1}\otimes I_n)\mathcal{R}^T\boldsymbol{F}(\boldsymbol{y}) - \varepsilon(L_N\otimes I_n)\boldsymbol{y}\\
&\quad - (E^{-1}\otimes I_n)\mathcal{R}^T\boldsymbol{B}^T\boldsymbol{\lambda}\\
\dot{\boldsymbol{\lambda}} &= (E^{-1}\otimes I_m)(\boldsymbol{B}\mathcal{R}\boldsymbol{y} - \boldsymbol{b}) - a(L_N\otimes I_m)\boldsymbol{\lambda} - \boldsymbol{z}\\
\dot{\boldsymbol{z}} &= a(L_N\otimes I_m)\boldsymbol{\lambda}\\
\dot{\boldsymbol{\mu}} &= -(L_N\otimes I_N)\boldsymbol{\mu}.
\end{aligned} \tag{15}$$

Remark 6: With Assumption 4 and the initial value $\boldsymbol{\mu}(0)$ satisfying $\mu_i^i(0) = 1$ and $\mu_i^j(0) = 0, j \neq i, i, j \in \mathcal{V}$, the existence of E^{-1} is guaranteed as shown in [34].

Lemma 5: Suppose that Assumptions 1–4 hold, the initial values $\boldsymbol{z}(0)$ and $\boldsymbol{\mu}(0)$ satisfy $\sum_{i=1}^N \xi_i z_i(0) = \mathbf{0}$, and $\mu_i^i(0) = 1$ and $\mu_i^j(0) = 0, j \neq i, i, j \in \mathcal{V}$, respectively. Let $\text{col}(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{z}}, \tilde{\boldsymbol{\mu}})$ be an equilibrium point of system (15). Then, $\tilde{\boldsymbol{y}} = 1_N \otimes \tilde{x}$, where $\tilde{x}$ is the GNE of the relaxed game (1) with $\Omega = R^n$.

Proof: Note that $\boldsymbol{\mu}(t) = e^{-(L_N\otimes I_N)t}\boldsymbol{\mu}(0)$ and $\lim_{t\to\infty} e^{-L_N t} = 1_N \xi^T$. With the definition of the initial value $\boldsymbol{\mu}(0)$, it holds that $\lim_{t\to\infty}\boldsymbol{\mu}(t) = 1_N \otimes \xi$ and $\lim_{t\to\infty} E(t) = \Xi$, where Ξ is defined in Lemma 2. Then, $\tilde{\boldsymbol{y}}$, $\tilde{\boldsymbol{\lambda}}$, and $\tilde{\boldsymbol{z}}$ satisfy

$$\mathcal{R}^T\boldsymbol{F}(\tilde{\boldsymbol{y}}) + \varepsilon(\Xi L_N\otimes I_n)\tilde{\boldsymbol{y}} + \mathcal{R}^T\boldsymbol{B}^T\tilde{\boldsymbol{\lambda}} = \mathbf{0} \tag{16}$$

$$\boldsymbol{B}\mathcal{R}\tilde{\boldsymbol{y}} - \boldsymbol{b} - a(\Xi L_N\otimes I_m)\tilde{\boldsymbol{\lambda}} - (\Xi\otimes I_n)\tilde{\boldsymbol{z}} = \mathbf{0} \tag{17}$$

$$(L_N\otimes I_m)\tilde{\boldsymbol{\lambda}} = \mathbf{0}. \tag{18}$$

Following the analysis similar to the proof of Lemma 4, it is concluded that $\tilde{\boldsymbol{y}} = 1_N\otimes\tilde{x}$, $\tilde{\boldsymbol{\lambda}} = 1_N\otimes\tilde{\lambda}$, and $F(\tilde{x}) + B^T\tilde{\lambda} = \mathbf{0}$, $B\tilde{x} = b$. This implies that $\tilde{x}$ is the GNE point of the relaxed game (1) with $\Omega = R^n$. □

Theorem 2: Suppose that Assumptions 1–5 hold and all B_i, $i \in \mathcal{V}$ are row full rank. Let the initial values $\boldsymbol{z}(0)$ and $\boldsymbol{\mu}(0)$ satisfy $\sum_{i=1}^N \xi_i z_i(0) = \mathbf{0}$, and $\mu_i^i(0) = 1$ and $\mu_i^j(0) = 0, j \neq i$, $i, j \in \mathcal{V}$, respectively. Let a and ε satisfy the following inequalities:

$$\varepsilon > \frac{4mc + (M+c)^2}{4m\lambda_2(\bar{L})}, \quad a > \frac{(\theta+1)^2}{\theta\lambda_2(\bar{L})} \tag{19}$$

where $\theta > \{0, \frac{\|\boldsymbol{B}\|^2}{\xi_{\min}\lambda_{\min}(\bar{Q})} - 1\}$, $\xi_{\min} = \min\{\xi_1, \xi_2, \ldots, \xi_N\}$, $\lambda_2(\bar{L})$ is defined in Lemma 2, and

$$\bar{Q} = \begin{pmatrix} \lambda_2(\bar{L})\varepsilon - c & -\frac{M+c}{2\sqrt{N}} \\ -\frac{M+c}{2\sqrt{N}} & \frac{m}{N} \end{pmatrix}$$

is a positive definite matrix. Then, the trajectory $\text{col}(\boldsymbol{y}(t), \boldsymbol{\lambda}(t), \boldsymbol{z}(t))$ generated by system (15) converges exponentially to the point $\text{col}(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{z}})$ defined in Lemma 5.

Remark 7: The proposed coordination dynamics (7) and (15) adopt the same estimation mechanism in dealing with the actions of other players as in [20] and [21]. In comparison to [20] and [21], the present dynamics provide a new insight in addressing directed communication graphs and affine equality constraints.

Remark 8: In Theorems 1 and 2, the parameters ε and a are used to guarantee the convergence of the proposed dynamics. It can be found that the lower bounds of ε and a depend on the global information of the concerned communication graph, and the global constants associated with the strong monotonicity and the Lipschitz continuity. Some distributed schemes [17], [46] for estimating those global information can be adopted before running (7) and (15).

IV. Simulation Results

In this section, the proposed dynamics are executed to solve two GNE seeking problems.

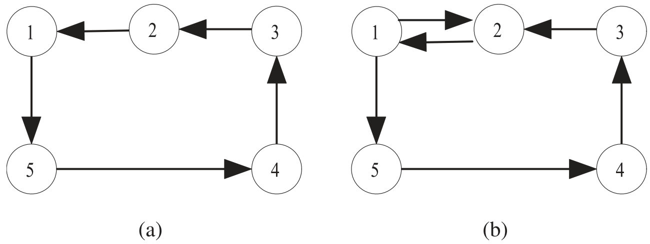


Fig. 1. Underlying communication topologies. (a) Balanced digraph. (b) Unbalanced digraph.

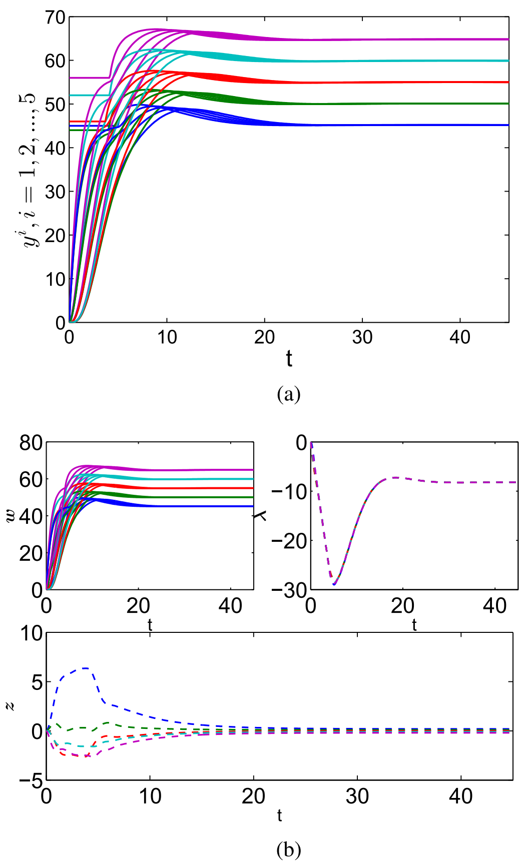


Fig. 2. (a) GNE of Example 1 by executing system (7). (b) Profiles of the trajectory col$(\boldsymbol{w}, \boldsymbol{\lambda}, \boldsymbol{z})$ with respect to time t.

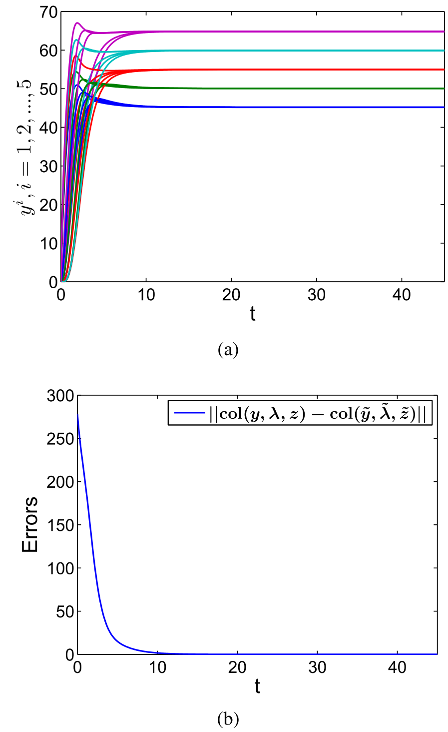


Fig. 3. (a) GNE of the relaxed Example 1 by executing system (15). (b) Evolution of $\mathrm{col}(\boldsymbol{y}, \|\boldsymbol{\lambda}, \boldsymbol{z}) - \mathrm{col}(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{z}})\|$ with respect to time t.

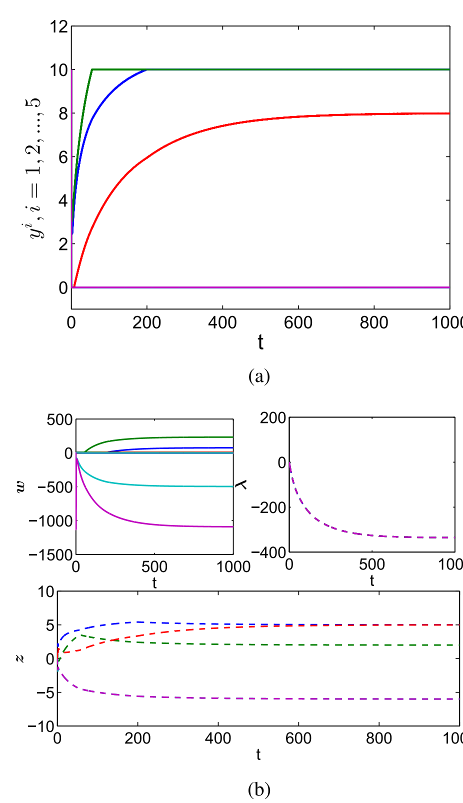
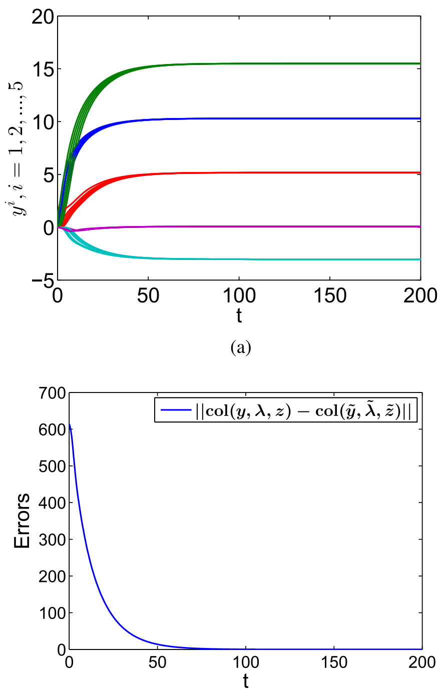


Fig. 4. (a) GNE of Example 2 by executing system (7). (b) Profiles of the trajectory col$(\boldsymbol{w}, \boldsymbol{\lambda}, \boldsymbol{z})$ with respect to time t.

Example 1: Consider a five-player aggregative game problem [27] over a strongly connected digraph. In this game, the cost function of player $i \in \{1, 2, \ldots, 5\}$ is given by $f_i(x_i, x_{-i}) = (x_i - d_i)^2 + p(x)x_i + 5x_i$, where $d_1 = 50$, $d_2 = 55$, $d_3 = 60$, $d_4 = 65$, $d_5 = 70$, and $p(x) = 0.04\sum_{i=1}^{5} x_i$. The local action sets and the shared equality constraint are $\Omega_1 = [45, 55]$, $\Omega_2 = [44, 66]$, $\Omega_3 = [46, 72]$, $\Omega_4 = [52, 78]$, $\Omega_5 = [56, 84]$, and $\sum_{i=1}^{5} x_i = \sum_{i=1}^{5} b_i$ with $b_1 = 45$, $b_2 = 50$, $b_3 = 55$, $b_4 = 60$, and $b_5 = 65$.

Under the balanced digraph Fig. 1(a), by executing (7), it can be found that for all $i \in 1, 2, \ldots, 5$, y^i converge to the GNE of Example 1, as shown in Fig. 2(a), which is consistent with



(a)

(b)

Fig. 5. (a) GNE of the relaxed Example 2 by executing system (15). (b) Evolution of $\|\mathrm{col}(\boldsymbol{y},\boldsymbol{\lambda},\boldsymbol{z})-\mathrm{col}(\tilde{\boldsymbol{y}},\tilde{\boldsymbol{\lambda}},\tilde{\boldsymbol{z}})\|$ with respect to time t.

the result given in [27]. Correspondingly, Fig. 2(b) shows the boundedness of the trajectory $\mathrm{col}(\boldsymbol{w},\boldsymbol{\lambda},\boldsymbol{z})$. By implementing (15) under the unbalanced digraph Fig. 1(b), the GNE of the relaxed Example 1 without Ω_i is illustrated in Fig. 3(a). Meanwhile, Fig. 3(b) shows the fact that the trajectory $\mathrm{col}(\boldsymbol{y},\boldsymbol{\lambda},\boldsymbol{z})$ tends to its convergent point exponentially fast.

Example 2: Consider a five-player nonaggregative game problem over a directed strongly connected graph. The cost functions of these five players are, respectively,

$$
\begin{aligned}
f_1(x_1,x_{-1}) &= x_1^3-3x_1x_2,\\
f_2(x_2,x_{-2}) &= (-2x_1+4x_2+\frac{1}{2}x_4+x_5)^2-48x_2,\\
f_3(x_3,x_{-3}) &= (x_1+4x_3-x_4-x_5)^2,\\
f_4(x_4,x_{-4}) &= (2x_1+4x_3+8x_4-x_5)^2, \text{and}\\
f_5(x_5,x_{-5}) &= (x_1+4x_3+8x_4+17x_5)^2.
\end{aligned}
$$

Additionally, the local action sets $\Omega_i=[0,10]$, $i\in\{1,2,\ldots,5\}$, and the shared equality constraint $\sum_{i=1}^N x_i=\sum_{i=1}^N b_i$ for this game are imposed, where $\mathrm{col}(b_1,b_2,\ldots,b_5)^T=(5,8,3,6,6)^T$.

From Fig. 4, it can be seen that all trajectories y^i, $i\in 1,2,\ldots,5$, converge to the GNE of Example 2 and the trajectory $\mathrm{col}(\boldsymbol{w},\boldsymbol{\lambda},\boldsymbol{z})$ is bounded by running (7) under the balanced digraph Fig. 1(a). Fig. 5 gives the simulation results on the relaxed Example 2 with $\Omega_i=R$ by implementing (15) under the unbalanced digraph Fig. 1(b), where the GNE of the relaxed Example 2 and the exponential convergence of the trajectory $\mathrm{col}(\boldsymbol{y},\boldsymbol{\lambda},\boldsymbol{z})$ are shown. These simulations verify the effectiveness of the proposed GNE seeking dynamics.

V. Conclusion

This article has investigated the GNE seeking problem subject to linear equality constraints and local action sets. By allowing each player to estimate its opponents' actions on a directed communication topology, we established the corresponding coordinated GNE seeking dynamics for the game problem and its relaxed case without local action sets, respectively, for weight-balanced and weight-unbalanced digraphs. By virtue of stability theory from nonlinear systems, we showed the convergence of the proposed dynamics with the strong monotonicity and Lipschitz continuity of the pseudogradient as well as the Lipschitz continuity of the extended pseudogradient. The simulations illustrated the validity of the proposed dynamics.

Note that when the communication topology is weight-unbalanced, this article is mainly devoted to studying the GNE problem with linear equality constraints but without local action sets. At present, there is still a certain challenge in terms of convergence analysis for the case with local action sets over weight-unbalanced digraphs, which is worthy of further study.

Appendix A
Proof of Theorem 1

Proof. To analyze the convergence of system (7), construct the following Lyapunov function:

$$
V=\frac{\kappa+1}{2}\left(\|\boldsymbol{w}-P_{\boldsymbol{\Omega}}(\boldsymbol{w}^*)\|^2-\|\boldsymbol{w}-P_{\boldsymbol{\Omega}}(\boldsymbol{w})\|^2\right)
+\frac{\kappa}{2}\|\boldsymbol{\lambda}-\boldsymbol{\lambda}^*\|^2+\frac{1}{2}\|\boldsymbol{\lambda}-\boldsymbol{\lambda}^*+\boldsymbol{z}-\boldsymbol{z}^*\|^2
$$

where $\mathrm{col}(\boldsymbol{w}^*,\boldsymbol{\lambda}^*,\boldsymbol{z}^*)$ and κ are defined in Lemma 4 and Theorem 1, respectively. From Lemma 1, it follows that

$$
V\geq\frac{\kappa+1}{2}\|\boldsymbol{y}-\boldsymbol{y}^*\|^2+\frac{\kappa}{2}\|\boldsymbol{\lambda}-\boldsymbol{\lambda}^*\|^2+\frac{1}{2}\|\boldsymbol{\lambda}-\boldsymbol{\lambda}^*+\boldsymbol{z}-\boldsymbol{z}^*\|^2\geq 0
$$

and $\nabla_{\boldsymbol{w}}V=(\kappa+1)(\boldsymbol{y}-\boldsymbol{y}^*)$. It is clear that $\nabla_{\boldsymbol{\lambda}}V=\kappa(\boldsymbol{\lambda}-\boldsymbol{\lambda}^*)+(\boldsymbol{\lambda}-\boldsymbol{\lambda}^*+\boldsymbol{z}-\boldsymbol{z}^*)$ and

$$
\nabla_{\boldsymbol{z}}V=\boldsymbol{\lambda}-\boldsymbol{\lambda}^*+\boldsymbol{z}-\boldsymbol{z}^*.
$$

Then, the derivative of V along with system (7) is

$$
\begin{aligned}
\dot{V} &= (\kappa+1)(\boldsymbol{y}-\boldsymbol{y}^*)^T\left(-\boldsymbol{w}+\boldsymbol{y}-\mathcal{R}^T\boldsymbol{F}(\boldsymbol{y})-\mathcal{R}^T\boldsymbol{B}^T\boldsymbol{\lambda}-\varepsilon\right.\\
&\quad \left.(L_N\otimes I_n)\boldsymbol{y}\right)+\kappa(\boldsymbol{\lambda}-\boldsymbol{\lambda}^*)^T\left(\boldsymbol{B}\mathcal{R}\boldsymbol{y}-\boldsymbol{b}-\boldsymbol{z}-a(L_N\otimes I_m)\boldsymbol{\lambda}\right)\\
&\quad +(\boldsymbol{\lambda}-\boldsymbol{\lambda}^*+\boldsymbol{z}-\boldsymbol{z}^*)^T\left(\boldsymbol{B}\mathcal{R}\boldsymbol{y}-\boldsymbol{b}-\boldsymbol{z}\right)\\
&= (\kappa+1)v_1+(\kappa+1)v_2+\varepsilon(\kappa+1)v_3+\kappa a v_4\\
&\quad +(\kappa+1)v_5+v_6-\|\boldsymbol{z}-\boldsymbol{z}^*\|^2
\end{aligned}
$$

where

$$
\begin{aligned}
v_1 &= -(\boldsymbol{y}-\boldsymbol{y}^*)^T(\boldsymbol{w}-\boldsymbol{w}^*)+\|\boldsymbol{y}-\boldsymbol{y}^*\|^2\\
v_2 &= -(\boldsymbol{y}-\boldsymbol{y}^*)^T\mathcal{R}^T\left(\boldsymbol{F}(\boldsymbol{y})-\boldsymbol{F}(\boldsymbol{y}^*)\right)\\
v_3 &= -(\boldsymbol{y}-\boldsymbol{y}^*)^T(L_N\otimes I_n)(\boldsymbol{y}-\boldsymbol{y}^*)
\end{aligned}
$$

$$v_4 = -(\boldsymbol{\lambda} - \boldsymbol{\lambda}^*)^T (L_N \otimes I_m)(\boldsymbol{\lambda} - \boldsymbol{\lambda}^*)$$
$$v_5 = -(\boldsymbol{\lambda} - \boldsymbol{\lambda}^*)^T (\boldsymbol{z} - \boldsymbol{z}^*)$$
$$v_6 = (\boldsymbol{z} - \boldsymbol{z}^*)^T \boldsymbol{B}\mathcal{R}(\boldsymbol{y} - \boldsymbol{y}^*).$$

From Lemma 1, we arrive at $v_1 \le 0$. In fact, we can make the following orthogonal decomposition for $\boldsymbol{y}$ and $\boldsymbol{\lambda}$, namely, $\boldsymbol{y} = \bar{\boldsymbol{y}} + \boldsymbol{y}^\perp$ and $\boldsymbol{\lambda} = \bar{\boldsymbol{\lambda}} + \boldsymbol{\lambda}^\perp$, where $\bar{\boldsymbol{y}} = 1_N \otimes \bar{y}$ with $\bar{y} \in R^n$, $\bar{\boldsymbol{y}}^T \boldsymbol{y}^\perp = 0$, $\bar{\boldsymbol{\lambda}} = 1_N \otimes \bar{\lambda}$ with $\bar{\lambda} \in R^m$ and $\bar{\boldsymbol{\lambda}}^T \boldsymbol{\lambda}^\perp = 0$. Note that $\boldsymbol{F}(\bar{\boldsymbol{y}}) - \boldsymbol{F}(\boldsymbol{y}^*) = F(\bar{y}) - F(x^*)$ and $\mathcal{R}(\bar{\boldsymbol{y}} - \boldsymbol{y}^*) = \bar{y} - x^*$. Then,

$$\begin{aligned}
v_2 &= -(\bar{\boldsymbol{y}}-\boldsymbol{y}^*)^T \mathcal{R}^T \left(\boldsymbol{F}(\boldsymbol{y})-\boldsymbol{F}(\boldsymbol{y}^*)\right) - \boldsymbol{y}^\perp \mathcal{R}^T \left(\boldsymbol{F}(\boldsymbol{y})-\boldsymbol{F}(\boldsymbol{y}^*)\right) \\
&= -(\bar{\boldsymbol{y}}-\boldsymbol{y}^*)^T \mathcal{R}^T \left(\boldsymbol{F}(\boldsymbol{y})-\boldsymbol{F}(\bar{\boldsymbol{y}})\right) - (\bar{y}-x^*)^T (F(\bar{y})-F(x^*)) \\
&\quad -(\boldsymbol{y}^\perp)^T \mathcal{R}^T (\boldsymbol{F}(\boldsymbol{y})-\boldsymbol{F}(\bar{\boldsymbol{y}})) - (\boldsymbol{y}^\perp)^T \mathcal{R}^T (F(\bar{y})-F(x^*)) \\
&\le c\|\bar{y}-x^*\|\|\boldsymbol{y}^\perp\| - m\|\bar{y}-x^*\|^2 + c\|\boldsymbol{y}^\perp\|^2 + M\|\boldsymbol{y}^\perp\|\|\bar{y}-x^*\| \\
&= (M+c)\|\bar{y}-x^*\|\|\boldsymbol{y}^\perp\| - m\|\bar{y}-x^*\|^2 + c\|\boldsymbol{y}^\perp\|^2
\end{aligned}$$

where the inequalities $\|F(\bar{y}) - F(x^*)\| \le M\|\bar{y} - x^*\|$, $-(\bar{y} - x^*)^T (F(\bar{y}) - F(x^*)) \le -m\|\bar{y} - x^*\|^2$ and $\|\boldsymbol{F}(\boldsymbol{y}) - \boldsymbol{F}(\bar{\boldsymbol{y}})\| \le c\|\boldsymbol{y} - \bar{\boldsymbol{y}}\| = c\|\boldsymbol{y}^\perp\|$ that are derived from Assumptions 3 and 5 are used. According to Lemma 2, the orthogonal decomposition of $\boldsymbol{\lambda}$ and $1_N^T L_N = \boldsymbol{0}$, we can compute that

$$v_3 = -(\boldsymbol{y}^\perp)^T (L_N \otimes I_n)\boldsymbol{y}^\perp \le \lambda_2(\tilde{L})\|\boldsymbol{y}^\perp\|^2$$
$$v_4 = -(\boldsymbol{\lambda}^\perp)^T (L_N \otimes I_m)\boldsymbol{\lambda}^\perp \le \lambda_2(\tilde{L})\|\boldsymbol{\lambda}^\perp\|^2.$$

Note that $\mathcal{W} = \{\mathrm{col}(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{z}) | \sum_{i=1}^N z_i(t) = \sum_{i=1}^N z_i(0) = \boldsymbol{0}\}$ is a strongly positive invariant set under system (7). Then, for any $\mathrm{col}(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{z}) \in \mathcal{W}$, $v_5 = -(\boldsymbol{\lambda}^\perp)^T (\boldsymbol{z} - \boldsymbol{z}^*)$. Furthermore, $(\kappa+1)v_5 \le (\kappa+1)^2 \|\boldsymbol{\lambda}^\perp\|^2 + \frac{1}{4}\|\boldsymbol{z} - \boldsymbol{z}^*\|^2$, where the inequality $x^T y \le \|x\|^2 + \frac{1}{4}\|y\|^2$ is adopted. Similarly,

$$v_6 \le \|\boldsymbol{B}\|^2 \|\boldsymbol{y} - \boldsymbol{y}^*\|^2 + \frac{1}{4}\|\boldsymbol{z} - \boldsymbol{z}^*\|^2.$$

Hence, $\dot{V}$ is bounded by

$$\begin{aligned}
\dot{V} &\le -(\kappa+1)u^T Q u + \|\boldsymbol{B}\|^2 \|\boldsymbol{y} - \boldsymbol{y}^*\|^2 \\
&\quad - \left(\kappa a \lambda_2(\tilde{L}) - (\kappa+1)^2\right) \|\boldsymbol{\lambda}^\perp\|^2 - \frac{1}{2}\|\boldsymbol{z} - \boldsymbol{z}^*\|^2
\end{aligned}$$

where $u = (\|\boldsymbol{y}^\perp\|, \|\bar{\boldsymbol{y}} - \boldsymbol{y}^*\|)^T$ and Q is defined in Theorem 1. It is obvious that Q is positive definite by the definition of ε given in (13). Thus, $\lambda_{\min}(Q) > 0$ and $u^T Q u \ge \lambda_{\min}(Q)\|u\|^2 = \lambda_{\min}(Q)\|\boldsymbol{y} - \boldsymbol{y}^*\|^2$. This further simplifies $\dot{V}$ as follows:

$$\begin{aligned}
\dot{V} &\le -\left((\kappa+1)\lambda_{\min}(Q) - \|\boldsymbol{B}\|^2\right)\|\boldsymbol{y} - \boldsymbol{y}^*\|^2 \\
&\quad - \left(\kappa a \lambda_2(\tilde{L}) - (\kappa+1)^2\right)\|\boldsymbol{\lambda}^\perp\|^2 - \frac{1}{2}\|\boldsymbol{z} - \boldsymbol{z}^*\|^2.
\end{aligned}$$

By invoking (13) and the definition of κ, we have that $\dot{V} \le 0$. Since V is radially unbounded with regard to $\mathrm{col}(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{z})$, we conclude that $\boldsymbol{y}$, $\boldsymbol{\lambda}$, and $\boldsymbol{z}$ are bounded. It is easy to verify that $\boldsymbol{w}$ is also bounded. Let $\mathcal{M}$ be the largest invariant set contained in $\mathcal{W} \cap \{\dot{V} = 0\}$. According to the LaSalle invariance principle [42], each solution starting in $\mathcal{W}$ approaches $\mathcal{M}$ as $t \to \infty$. Clearly, $\boldsymbol{y} = \boldsymbol{y}^* = 1_N \otimes x^*$ in $\mathcal{M}$. Therefore, $\boldsymbol{y} \to 1_N \otimes x^*$. ■

Appendix B
Proof of Theorem 2

Proof: Since each B_i, $i \in \mathcal{V}$, is full row rank, $(B_i B_i^T)^{-1}$ is well-defined and positive definite. Naturally, $(\boldsymbol{B}\boldsymbol{B}^T)^{-\frac{1}{2}}$ is also positive definite. To facilitate the analysis, $\mathrm{col}(\dot{\boldsymbol{y}}, \dot{\boldsymbol{\lambda}}, \dot{\boldsymbol{z}})$ in system (15) is rewritten as follows:

$$\mathrm{col}(\dot{\boldsymbol{y}}, \dot{\boldsymbol{\lambda}}, \dot{\boldsymbol{z}}) = \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{z}) + \boldsymbol{g}(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{z}, t) \tag{20}$$

where

$$\boldsymbol{f}(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{z}) = \begin{pmatrix} -(\Xi^{-1}\otimes I_n)\mathcal{R}^T \boldsymbol{F}(\boldsymbol{y}) - \varepsilon(L_N\otimes I_n)\boldsymbol{y} - (\Xi^{-1}\otimes I_n)\mathcal{R}^T \boldsymbol{B}^T\boldsymbol{\lambda}, \\ (\Xi^{-1}\otimes I_m)(\boldsymbol{B}\mathcal{R}\boldsymbol{y} - \boldsymbol{b}) - a(L_N\otimes I_m)\boldsymbol{\lambda} - \boldsymbol{z}, \\ a(L_N\otimes I_m)\boldsymbol{\lambda}, \end{pmatrix}$$

and

$$\boldsymbol{g}(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{z}, t) = \begin{pmatrix} ((\Xi^{-1} - E^{-1})\otimes I_n)\mathcal{R}^T \boldsymbol{F}(\boldsymbol{y}) + ((\Xi^{-1} - E^{-1})\otimes I_n)\mathcal{R}^T \boldsymbol{B}^T \boldsymbol{\lambda}, \\ ((E^{-1} - \Xi^{-1})\otimes I_m)\boldsymbol{B}\mathcal{R}(\boldsymbol{y} - \boldsymbol{b}), \\ \boldsymbol{0}, \end{pmatrix}.$$

To show the exponential stability of (20) at the point $\mathrm{col}(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{z}})$ defined in Lemma 5, consider the following Lyapunov function $\bar{V}$:

$$\bar{V} = V_1 + V_2$$

where

$$\begin{aligned}
V_1 &= \frac{\theta+1}{2}(\boldsymbol{y} - \tilde{\boldsymbol{y}})^T (\Xi\otimes I_n)(\boldsymbol{y} - \tilde{\boldsymbol{y}}) + \frac{\theta}{2}(\boldsymbol{\lambda} - \tilde{\boldsymbol{\lambda}})^T (\Xi\otimes I_m)(\boldsymbol{\lambda} - \tilde{\boldsymbol{\lambda}}) \\
&\quad + \frac{1}{2}(\boldsymbol{\lambda} - \tilde{\boldsymbol{\lambda}} + \boldsymbol{z} - \tilde{\boldsymbol{z}})^T (\Xi\otimes I_m)(\boldsymbol{\lambda} - \tilde{\boldsymbol{\lambda}} + \boldsymbol{z} - \tilde{\boldsymbol{z}}) \\
V_2 &= \frac{\sigma}{2}\left(\boldsymbol{y} - \tilde{\boldsymbol{y}} + \mathcal{R}^T \boldsymbol{B}^T (\boldsymbol{B}\boldsymbol{B}^T)^{-\frac{1}{2}}(\boldsymbol{\lambda} - \tilde{\boldsymbol{\lambda}})\right)^T \\
&\quad (\Xi\otimes I_n)\left(\boldsymbol{y} - \tilde{\boldsymbol{y}} + \mathcal{R}^T \boldsymbol{B}^T (\boldsymbol{B}\boldsymbol{B}^T)^{-\frac{1}{2}}(\boldsymbol{\lambda} - \tilde{\boldsymbol{\lambda}})\right)
\end{aligned}$$

with $\theta > 0$ and $\sigma > 0$. Then, the derivative of $\bar{V}$ along system (20) is

$$\dot{\bar{V}} = w_1 + w_2 + w_3$$

where $w_1 = (\frac{\partial V_1}{\partial \mathrm{col}(\boldsymbol{y},\boldsymbol{\lambda},\boldsymbol{z})})^T \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{z})$, $w_2 = (\frac{\partial V_2}{\partial \mathrm{col}(\boldsymbol{y},\boldsymbol{\lambda},\boldsymbol{z})})^T \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{z})$, and $w_3 = (\frac{\partial \bar{V}}{\partial \mathrm{col}(\boldsymbol{y},\boldsymbol{\lambda},\boldsymbol{z})})^T \boldsymbol{g}(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{z}, t)$.

It is readily to compute that

$$\begin{aligned}
w_1 &\le -(\theta+1)(\boldsymbol{y} - \tilde{\boldsymbol{y}})^T \mathcal{R}^T (\boldsymbol{F}(\boldsymbol{y}) - \boldsymbol{F}(\tilde{\boldsymbol{y}})) - \varepsilon(\theta+1)(\boldsymbol{y} - \tilde{\boldsymbol{y}})^T \\
&\quad (\Xi L_N \otimes I_n)(\boldsymbol{y} - \tilde{\boldsymbol{y}}) - (\boldsymbol{z} - \tilde{\boldsymbol{z}})^T (\Xi \otimes I_m)(\boldsymbol{z} - \tilde{\boldsymbol{z}}) \\
&\quad - \theta a(\boldsymbol{\lambda} - \tilde{\boldsymbol{\lambda}})^T (\Xi L_N \otimes I_m)(\boldsymbol{\lambda} - \tilde{\boldsymbol{\lambda}}) + (\boldsymbol{z} - \tilde{\boldsymbol{z}})^T \boldsymbol{B}\mathcal{R}(\boldsymbol{y} - \tilde{\boldsymbol{y}}) \\
&\quad - (\theta+1)(\boldsymbol{\lambda} - \tilde{\boldsymbol{\lambda}})^T (\Xi \otimes I_m)(\boldsymbol{z} - \tilde{\boldsymbol{z}}).
\end{aligned}$$

Following similar procedures to those given in the proof of Theorem 1, w_1 is bounded by

$$w_1 \leq -\left((\theta+1)\lambda_{\min}(\bar{Q})-\frac{\|\boldsymbol{B}\|^2}{\xi_{\min}}\right)\|\boldsymbol{y}-\tilde{\boldsymbol{y}}\|^2-\frac{1}{2}(\boldsymbol{z}-\tilde{\boldsymbol{z}})^T (\Xi\otimes I_m)(\boldsymbol{z}-\tilde{\boldsymbol{z}})-\left(\theta a\lambda_2(\bar{L})-(\theta+1)^2\right)\|\boldsymbol{\lambda}^{\perp}\|^2.$$

On the other hand, we can derive that w_2 satisfies

$$w_2 \leq -\sigma\lambda_{\min}(\bar{Q})\|\boldsymbol{y}-\tilde{\boldsymbol{y}}\|^2+\sigma(\varpi_1+\varpi_2+\varpi_3+\varpi_4+\varpi_5+\varpi_6) -\sigma a\lambda_2(\bar{L})\|\boldsymbol{\lambda}^{\perp}\|^2-\sigma(\boldsymbol{\lambda}-\boldsymbol{\lambda}^*)^T(\boldsymbol{B}\boldsymbol{B}^T)^{\frac{1}{2}}(\boldsymbol{\lambda}-\boldsymbol{\lambda}^*)$$

where

$$\begin{aligned}
\varpi_1 &= -a(\boldsymbol{y}-\tilde{\boldsymbol{y}})^T(\Xi\otimes I_n)\mathcal{R}^T\boldsymbol{B}^T(\boldsymbol{B}\boldsymbol{B}^T)^{-\frac{1}{2}}(L_N\otimes I_m)(\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}})\\
&= -a(\boldsymbol{y}-\tilde{\boldsymbol{y}})^T(\Xi\otimes I_n)\mathcal{R}^T\boldsymbol{B}^T(\boldsymbol{B}\boldsymbol{B}^T)^{-\frac{1}{2}}(L_N\otimes I_m)\boldsymbol{\lambda}^{\perp}\\
\varpi_2 &= (\boldsymbol{y}-\tilde{\boldsymbol{y}})^T\mathcal{R}^T\boldsymbol{B}^T(\boldsymbol{B}\boldsymbol{B}^T)^{-\frac{1}{2}}\boldsymbol{B}\mathcal{R}(\boldsymbol{y}-\tilde{\boldsymbol{y}})\\
\varpi_3 &= -(\boldsymbol{y}-\tilde{\boldsymbol{y}})^T(\Xi\otimes I_n)\mathcal{R}^T\boldsymbol{B}^T(\boldsymbol{B}\boldsymbol{B}^T)^{-\frac{1}{2}}(\boldsymbol{z}-\tilde{\boldsymbol{z}})\\
\varpi_4 &= -(\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}})^T(\boldsymbol{B}\boldsymbol{B}^T)^{-\frac{1}{2}}\boldsymbol{B}(\boldsymbol{F}(\boldsymbol{y})-\boldsymbol{F}(\tilde{\boldsymbol{y}}))\\
\varpi_5 &= -\varepsilon(\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}})^T(\boldsymbol{B}\boldsymbol{B}^T)^{-\frac{1}{2}}\boldsymbol{B}\mathcal{R}(L_N\otimes I_n)(\boldsymbol{y}-\tilde{\boldsymbol{y}})\\
\varpi_6 &= -(\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}})^T(\Xi\otimes I_m)(\boldsymbol{z}-\tilde{\boldsymbol{z}}).
\end{aligned}$$

Applying the orthogonal decomposition of $\boldsymbol{\lambda}$ given in the proof of Theorem 1 and $L_N 1_N = \mathbf{0}$ to ϖ_1 yields $\varpi_1 = -a(\boldsymbol{y}-\tilde{\boldsymbol{y}})^T(\Xi\otimes I_n)\mathcal{R}^T\boldsymbol{B}^T(\boldsymbol{B}\boldsymbol{B}^T)^{-\frac{1}{2}}(L_N\otimes I_m)\boldsymbol{\lambda}^{\perp}$. For any two matrixes $A\in R^{p\times q}$ and $B\in R^{q\times p}$, [47, Th. 1.3.22] indicates that if the eigenvalues of AB (or BA) are non-negative, then $\lambda_{\max}(AB)=\lambda_{\max}(BA)$. Note that the positive semidefinite matrix $J\triangleq \boldsymbol{B}^T(\boldsymbol{B}\boldsymbol{B}^T)^{-\frac{1}{2}}(L_N L_N^T\otimes I_m)(\boldsymbol{B}\boldsymbol{B}^T)^{-\frac{1}{2}}\boldsymbol{B}$ implies that the eigenvalues of J are non-negative. Thus,

$$\begin{aligned}
\lambda_{\max}(J) &= \lambda_{\max}((L_N L_N^T\otimes I_m)(\boldsymbol{B}\boldsymbol{B}^T)^{-\frac{1}{2}}\boldsymbol{B}\boldsymbol{B}^T(\boldsymbol{B}\boldsymbol{B}^T)^{-\frac{1}{2}})\\
&= \lambda_{\max}(L_N L_N^T\otimes I_m) = \|L_N\|^2.
\end{aligned}$$

Then, ϖ_1 satisfies

$$\begin{aligned}
\varpi_1 &\leq \frac{a^2\lambda_{\max}(J)}{2}\|\boldsymbol{y}-\tilde{\boldsymbol{y}}\|^2+\frac{1}{2}\|\boldsymbol{\lambda}^{\perp}\|^2\\
&= \frac{a^2\|L_N\|^2}{2}\|\boldsymbol{y}-\tilde{\boldsymbol{y}}\|^2+\frac{1}{2}\|\boldsymbol{\lambda}^{\perp}\|^2.
\end{aligned}$$

Similarly, we can derive that

$$\begin{aligned}
\varpi_2 &\leq \lambda_{\max}(\boldsymbol{B}^T(\boldsymbol{B}\boldsymbol{B}^T)^{-\frac{1}{2}}\boldsymbol{B})\|\boldsymbol{y}-\tilde{\boldsymbol{y}}\|^2\\
&= \sqrt{\lambda_{\max}(\boldsymbol{B}\boldsymbol{B}^T)}\|\boldsymbol{y}-\tilde{\boldsymbol{y}}\|^2 = \|\boldsymbol{B}\|\|\boldsymbol{y}-\tilde{\boldsymbol{y}}\|^2\\
\varpi_3 &\leq \frac{1}{2\xi_{\min}}\|\boldsymbol{y}-\tilde{\boldsymbol{y}}\|^2+\frac{1}{2}(\boldsymbol{z}-\tilde{\boldsymbol{z}})^T(\Xi\otimes I_m)(\boldsymbol{z}-\tilde{\boldsymbol{z}})\\
\varpi_4 &\leq \frac{1}{4}(\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}})^T(\boldsymbol{B}\boldsymbol{B}^T)^{\frac{1}{2}}(\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}})+\frac{c^2}{\sqrt{\lambda_{\min}(\boldsymbol{B}\boldsymbol{B}^T)}}\|\boldsymbol{y}-\tilde{\boldsymbol{y}}\|^2\\
\varpi_5 &\leq \frac{1}{4}(\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}})^T(\boldsymbol{B}\boldsymbol{B}^T)^{\frac{1}{2}}(\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}})+\frac{\varepsilon^2\|L_N\|^2}{\sqrt{\lambda_{\min}(\boldsymbol{B}\boldsymbol{B}^T)}}\|\boldsymbol{y}-\tilde{\boldsymbol{y}}\|^2\\
\varpi_6 &\leq \frac{1}{2}\|\boldsymbol{\lambda}^{\perp}\|^2+\frac{1}{2}(\boldsymbol{z}-\tilde{\boldsymbol{z}})^T(\Xi\otimes I_m)(\boldsymbol{z}-\tilde{\boldsymbol{z}})
\end{aligned}$$

where $\xi_{\min}$ is defined in Theorem 2. Substituting the abovementioned inequalities into w_2 results in

$$\begin{aligned}
w_2 \leq{}& \sigma\left(\frac{a^2\|L_N\|^2\xi_{\min}+1}{2\xi_{\min}}+\|\boldsymbol{B}\|+\frac{\varepsilon^2\|L_N\|^2+c^2}{\sqrt{\lambda_{\min}(\boldsymbol{B}\boldsymbol{B}^T)}}-\lambda_{\min}(\bar{Q})\right)\\
&\times\|\boldsymbol{y}-\tilde{\boldsymbol{y}}\|^2-\sigma(a\lambda_2(\bar{L})-1)\|\boldsymbol{\lambda}^{\perp}\|^2-\frac{\sigma}{2}(\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}})^T\\
&\times(\boldsymbol{B}\boldsymbol{B}^T)^{\frac{1}{2}}(\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}})+\sigma(\boldsymbol{z}-\tilde{\boldsymbol{z}})^T(\Xi\otimes I_m)(\boldsymbol{z}-\tilde{\boldsymbol{z}}).
\end{aligned}$$

Combining w_1 and w_2, it can be obtained that

$$\begin{aligned}
w_1+w_2 \leq{}& -\psi_1\|\boldsymbol{y}-\tilde{\boldsymbol{y}}\|^2-\psi_2(\boldsymbol{z}-\tilde{\boldsymbol{z}})^T(\Xi\otimes I_m)(\boldsymbol{z}-\tilde{\boldsymbol{z}})\\
&-\psi_3\|\boldsymbol{\lambda}^{\perp}\|^2-\frac{\sigma}{2}(\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}})^T(\boldsymbol{B}\boldsymbol{B}^T)^{\frac{1}{2}}(\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}})
\end{aligned}$$

where $\psi_1=(\theta+1)\lambda_{\min}(\bar{Q})-\frac{\|\boldsymbol{B}\|^2}{\xi_{\min}}-\sigma\Big(\frac{a^2\|L_N\|^2\xi_{\min}+1}{2\xi_{\min}}+\|\boldsymbol{B}\|+\frac{\varepsilon^2\|L_N\|^2+c^2}{\sqrt{\lambda_{\min}(\boldsymbol{B}\boldsymbol{B}^T)}}-\lambda_{\min}(\bar{Q})\Big)$, $\psi_2=\frac{1}{2}-\sigma$, and $\psi_3=\theta a\lambda_2(\bar{L})-(\theta+1)^2+\sigma(a\lambda_2(\bar{L})-1)$.

Let

$$\sigma<\min\left\{\frac{1}{2},\frac{(\theta+1)\lambda_{\min}(\bar{Q})-\frac{\|\boldsymbol{B}\|^2}{\xi_{\min}}}{\frac{a^2\|L_N\|^2}{2}+\frac{1}{2\xi_{\min}}+\|\boldsymbol{B}\|+\frac{\varepsilon^2\|L_N\|^2+c^2}{\sqrt{\lambda_{\min}(\boldsymbol{B}\boldsymbol{B}^T)}}-\lambda_{\min}(\bar{Q})}\right\}.$$

According to (19) and the definition of θ and σ, it follows that $\psi_1>0$, $\psi_2>0$, and $\psi_3>0$. With $\psi\triangleq\min\{\psi_1,\psi_2\xi_{\min},\frac{\sigma}{2}\lambda_{\min}(\boldsymbol{B}\boldsymbol{B}^T)\}$,

$$w_1+w_2\leq-\psi\|\mathrm{col}(\boldsymbol{y},\boldsymbol{\lambda},\boldsymbol{z})-\mathrm{col}(\tilde{\boldsymbol{y}},\tilde{\boldsymbol{\lambda}},\tilde{\boldsymbol{z}})\|^2. \tag{21}$$

Next, w_3 is analyzed. In fact, we can verify that there are some positive constants α_1, α_2, and α_3 such that $\bar{V}$ and $\frac{\partial\bar{V}}{\partial\mathrm{col}(\boldsymbol{y},\boldsymbol{\lambda},\boldsymbol{z})}$ satisfy

$$\begin{aligned}
&\alpha_1\|\mathrm{col}(\boldsymbol{y}-\tilde{\boldsymbol{y}},\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}},\boldsymbol{z}-\tilde{\boldsymbol{z}})\|^2\leq\bar{V}\leq\alpha_2\|\mathrm{col}(\boldsymbol{y}-\tilde{\boldsymbol{y}},\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}},\boldsymbol{z}-\tilde{\boldsymbol{z}})\|^2\\
&\left\|\frac{\partial\bar{V}}{\partial\mathrm{col}(\boldsymbol{y},\boldsymbol{\lambda},\boldsymbol{z})}\right\|\leq\alpha_3\|\mathrm{col}(\boldsymbol{y}-\tilde{\boldsymbol{y}},\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}},\boldsymbol{z}-\tilde{\boldsymbol{z}})\|.
\end{aligned}$$

Given $\boldsymbol{\mu}(0)$ in Theorem 2, we can find from the proof of Lemma 5 that $\boldsymbol{\mu}(t)$ converges exponentially to $1_N\otimes\xi$. Correspondingly, E^{-1} converges to Ξ exponentially fast. Meanwhile, we can verify that there exist $\beta_1>0$ and $\beta_2>0$ such that $\|E^{-1}-\Xi^{-1}\|\leq\beta_1 e^{-\beta_2 t}$. After some computations, we have that $\|((\Xi^{-1}-E^{-1})\otimes I_n)\mathcal{R}^T(\boldsymbol{F}(\boldsymbol{y})-\boldsymbol{F}(\tilde{\boldsymbol{y}}))+((\Xi^{-1}-E^{-1})\otimes I_n)\mathcal{R}^T\boldsymbol{B}^T(\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}})\|\leq\sqrt{2c^2+\|\boldsymbol{B}\|^2}\beta_1 e^{-\beta_2 t}\|\mathrm{col}(\boldsymbol{y}-\tilde{\boldsymbol{y}},\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}})\|$, $\|((E^{-1}-\Xi^{-1})\otimes I_m)\boldsymbol{B}\mathcal{R}(\boldsymbol{y}-\tilde{\boldsymbol{y}})\|\leq\|\boldsymbol{B}\|\beta_1 e^{-\beta_2 t}\|\boldsymbol{y}-\tilde{\boldsymbol{y}}\|$, $\|((\Xi^{-1}-E^{-1})\otimes I_n)\mathcal{R}^T(\boldsymbol{F}(\tilde{\boldsymbol{y}})+\boldsymbol{B}^T\tilde{\boldsymbol{\lambda}})\|\leq\beta_1 e^{-\beta_2 t}\|\boldsymbol{F}(\tilde{\boldsymbol{y}})+\boldsymbol{B}^T\tilde{\boldsymbol{\lambda}}\|$, and $\|((E^{-1}-\Xi^{-1})\otimes I_m)\boldsymbol{B}\mathcal{R}\tilde{\boldsymbol{y}}\|\leq\|\boldsymbol{B}\|\beta_1 e^{-\beta_2 t}\|\tilde{\boldsymbol{y}}\|$. Applying the above-mentioned inequalities to $\|\boldsymbol{g}(\boldsymbol{y},\boldsymbol{\lambda},\boldsymbol{z},t)\|$

leads to

$$\|\boldsymbol{g}(\boldsymbol{y},\boldsymbol{\lambda},\boldsymbol{z},t)\| \le d_1 e^{-\beta_2 t}\|\mathrm{col}(\boldsymbol{y}-\tilde{\boldsymbol{y}},\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}},\boldsymbol{z}-\tilde{\boldsymbol{z}})\| + d_2 e^{-\beta_2 t}$$

where $d_1 = \beta_1\sqrt{2c^2+\|\boldsymbol{B}\|^2}$ and $d_2 = \beta_1\|\boldsymbol{B}\|\|\tilde{\boldsymbol{y}}\|\|\boldsymbol{F}(\tilde{\boldsymbol{y}}) + \boldsymbol{B}^T\tilde{\boldsymbol{\lambda}}\|$. Hence, $w_3 \le \left\|\frac{\partial \bar{V}}{\partial \mathrm{col}(\boldsymbol{y},\boldsymbol{\lambda},\boldsymbol{z})}\right\| \|\boldsymbol{g}(\boldsymbol{y},\boldsymbol{\lambda},\boldsymbol{z},t)\| \le \alpha_3 d_1 e^{-\beta_2 t}\| \mathrm{col}(\boldsymbol{y}-\tilde{\boldsymbol{y}},\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}},\boldsymbol{z}-\tilde{\boldsymbol{z}})\|^2 + \alpha_3 d_2 e^{-\beta_2 t}\|\mathrm{col}(\boldsymbol{y}-\tilde{\boldsymbol{y}},\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}},\boldsymbol{z}-\tilde{\boldsymbol{z}})\|$. Combining with $w_1 + w_2$ results in

$$\begin{aligned}\dot{\bar{V}} &\le -(\psi - \alpha_3 d_1 e^{-\beta_2 t})\|\mathrm{col}(\boldsymbol{y}-\tilde{\boldsymbol{y}},\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}},\boldsymbol{z}-\tilde{\boldsymbol{z}})\|^2 \\ &\quad + \alpha_3 d_2 e^{-\beta_2 t}\|\mathrm{col}(\boldsymbol{y}-\tilde{\boldsymbol{y}},\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}},\boldsymbol{z}-\tilde{\boldsymbol{z}})\| \\ &\le -\left(\frac{\psi}{2} - \alpha_3 d_1 e^{-\beta_2 t}\right)\|\mathrm{col}(\boldsymbol{y}-\tilde{\boldsymbol{y}},\boldsymbol{\lambda}-\tilde{\boldsymbol{\lambda}},\boldsymbol{z}-\tilde{\boldsymbol{z}})\|^2 \\ &\quad + \frac{\alpha_3^2 d_2^2}{2\psi}e^{-2\beta_2 t} \le -\left(\frac{\psi}{2\alpha_2} - \frac{\alpha_3 d_1}{\alpha_1}e^{-\beta_2 t}\right)\bar{V} + \frac{\alpha_3^2 d_2^2}{2\psi}e^{-2\beta_2 t}.\end{aligned}$$

According to the comparison lemma given in [42], $\bar{V}(t)$ satisfies

$$\bar{V}(t) \le d_3 e^{-\frac{\psi t}{2\alpha_2}} + d_4 e^{-\frac{\psi t}{2\alpha_2}}\int_0^t e^{(\frac{\psi}{2\alpha_2} - 2\beta_2)\tau d\tau} \tag{22}$$

where $d_3 = e^{\frac{\alpha_3 d_1}{\alpha_1 \beta_2}}\bar{V}(0)$ and $d_4 = \frac{\alpha_3^2 d_2^2}{2\psi}e^{\frac{\alpha_3 d_1}{\alpha_1\beta_2}}$. From (22), it follows that:

$$\begin{cases} \bar{V}(t) \le d_3 e^{-\frac{\psi t}{2\alpha_2}} + d_4 t e^{-\frac{\psi t}{2\alpha_2}}, & \frac{\psi}{2\alpha_2} = 2\beta_2, \\ \bar{V}(t) \le d_3 e^{-\frac{\psi t}{2\alpha_2}} + \frac{2\alpha_2 d_4}{\psi - 4\alpha_2\beta_2}(e^{-2\beta_2 t} - e^{-\frac{\psi t}{2\alpha_2}}), & \frac{\psi}{2\alpha_2} \ne 2\beta_2. \end{cases}$$

In either case, we can conclude that $\bar{V}(t)$ converges exponentially to zero, which implies that the assertion of Theorem 2 holds. □

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