Distributed Average Tracking in Weight-Unbalanced Directed Networks

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***Abstract*—This article studies a distributed average tracking (DAT) problem, in which a collection of agents work collaboratively, subject to local communication, to track the average of a set of reference signals, each of which is available to a single agent. Our primary objective is to seek a design methodology for DAT under possibly weight-unbalanced directed networks—the most general and thus most challenging case from the network topology perspective, which has few results in the literature. For this purpose, we propose a distributed algorithm based on a chain of two integrators that are coupled with a distributed estimator. It is found that the convergence depends on not only the network topology but also the deviations among the reference signal accelerations. Another primary interest of this article stems from the dynamics perspective—a point perceived as a main source of control design difficulty for multiagent systems. Indeed, we devise a nonlinear algorithm that is capable of achieving DAT under weight-unbalanced directed networks for agents subject to high-order integrator dynamics. The results show that the convergence to the vicinity of the average of the reference signals is guaranteed as long as the signals' states and control inputs are all bounded. Both algorithms are robust to initialization errors, i.e., DAT is insured even if the agents are not correctly initialized, enabling the potential applications in a wider spectrum of application domains.**

***Index Terms*—Distributed average tracking, multiagent system, weight-unbalanced directed graphs.**

I. Introduction

In distributed average tracking (DAT), the agents are coupled through the common task that they try to track the average of a set of reference signals, each of which is available to a single agent and is generally time varying; the task should be completed on the basis of local information and local communication among the agents. Recent years have witnessed a growing interest in the study of DAT, partially due to its broad applications. DAT has found applications in distributed sensor fusion [1] and distributed Kalman filtering [2], where the technique has mainly been applied from an estimation perspective. There are also various applications, where DAT is employed to design control laws for physical agents. Examples include dynamic region-following formation control [3] and distributed convex optimization [4]. It has been recognized that DAT has its own unique difficulties and faces not only theoretical but also practical challenges, since the tracking objective of DAT is time varying and unavailable to any agent.

From the estimation perspective, the goal of the DAT problem is to, in a distributed manner, fuse information or compute common estimates of certain time-varying quantities of interest. A typical example is to estimate and track the averaged position of a moving target by multiple cameras. In this case, the local reference signal is the position data, sensed by each camera, of the moving target. Some DAT results from the estimation perspective have been presented in [5]–[11]. For example, in [5], the authors propose a linear algorithm to achieve DAT for reference signals with steady states. A proportional algorithm and a proportional-integral (PI) algorithm are proposed in [6] to achieve DAT with a bounded tracking error under constant or slowly-varying inputs. Based on the nonsmooth sliding mode control theory for nonlinear systems, [8] presents a distributed nonlinear algorithm to achieve accurate DAT for time-varying reference signals with bounded derivatives. In order to remove the chattering effect caused by the discontinuous signum function, the authors in [9] propose a class of distributed continuous nonlinear algorithms with, respectively, static and adaptive coupling strengths for signals generated by linear dynamics. Different from [9], our article focuses on DAT over a weight-unbalanced directed graph, which introduces more challenges than its undirected counterpart. Furthermore, in [10], considering the robustness to initial errors, the authors develop a nonlinear DAT algorithm for arbitrary reference signals with known bounded derivatives.

From the control perspective, some physical agents cooperatively track a desired trajectory generated by multiple reference signals. For example, the desired trajectory might be the geometric center of multiple leader robots. In this case, the local reference signal is the state of each leader robot. In practice, the physical agents might have more complicated dynamics than single-integrator dynamics. Some researchers have solved the DAT problem via linear distributed algorithms [12], [13], and some researchers have employed nonlinear distributed algorithms [3], [14]–[17]. Both the linear algorithms and the nonlinear algorithms have their features and advantages while with tradeoff. For weight-balanced directed graphs, considering single-integrator dynamics, the authors in [12] investigate a continuous algorithm to make agents track the average of the dynamic inputs with a bounded steady-state error. Recently, the authors in [13] propose a linear distributed algorithm with a chain of two integrators for single-integrator dynamics, which can deal with a class of reference signals with steady deviations among the reference signal velocities. However, in the linear algorithms, a common assumption is that the multiple reference signals tend to constant values, and most of the results cannot guarantee accurate tracking. Therefore, to achieve accurate DAT, the nonlinear algorithm in [8] is further extended in [14] to double-integrator systems for reference signals with bounded accelerations. To address the DAT problem for physical agents with nonlinear systems, in [15], the authors introduce an exact DAT algorithm for systems with heterogeneous unknown nonlinear dynamics, where no constraints are imposed on the input reference signals. Furthermore, a distributed algorithm is developed in [16] for agents with nonlinear dynamics to achieve DAT in finite time. DAT algorithms are proposed for agents with general linear dynamics in [3] and for agents with additional Lipschitz-type nonlinear dynamics in [17], where exact DAT is achieved. However, the tradeoff is that the signum function used in some of the above nonlinear algorithms may cause chattering phenomena.

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It should be recognized, nevertheless, that the DAT works alluded to above are all built upon the assumption that the network topology is either undirected or directed but weight-balanced; both cases are highly idealistic and seldom seen in practice. For example, if a camera is used to get the relative positions between agents, due to the limited field of view, it is possible that one agent can sense another agent but not vice versa. In addition, if the agents use communication devices to exchange information with others, the agents might broadcast at different power levels. As a result, the above situations might result in weight-unbalanced directed graphs. Moreover, if the convergence of an algorithm remains unchanged even after removing a few slow communication links or package loss, which might in turn result in a weight-unbalanced directed graph, the algorithm will be more robust and reliable. In order to solve DAT for generic directed networks, the authors in [18] propose a distributed algorithm to drive the states of all agents to a neighborhood of the average of the reference signals. A prerequisite for the algorithm to work is that the left eigenvector, corresponding to the zero eigenvalue, of the Laplacian matrix should be available to the agents, which is seldom possible in practice, particularly for large networks.

This article is devoted to establishing DAT algorithms for generic directed networks, which are possibly weight-unbalanced. To the best of our knowledge, the DAT problem has not yet been addressed in the literature for weight-unbalanced directed graphs without knowing the left eigenvector of the Laplacian matrix. Specifically, we introduce two algorithms for different application scenarios, each of which has its own relative benefits.

In the first algorithm, we consider single-integrator dynamics and avoid the use of the left eigenvector of the Laplacian matrix. The proposed algorithm accounts for a generic directed network and a wide class of time-varying reference signals of which the accelerations have bounded deviations; hence, it is practically more relevant and meaningful. Particularly, we introduce a distributed linear algorithm with a chain of two integrators coupled with a distributed estimator for the left eigenvector of the Laplacian matrix associated with the zero eigenvalue. The algorithm is inspired by [13], [19], and [20]. Specifically, the distributed estimator in the emerging algorithm is motivated by [19] and [20]. However, the problem studied here is on DAT, which aims at tracking the average of multiple time-varying reference signals. In contrast, [19] and [20] focus on a distributed optimization problem, where the team optimal value is a constant. The time-varying nature of the DAT problem makes the analysis and design in the current article significantly different from those in [19] and [20]. Moreover, the usage of the chain of two integrators is inspired by [13] but the algorithms therein are limited to undirected graphs. In contrast, the current article deals with weight-unbalanced directed graphs. The asymmetric nature of the weight-unbalanced directed graphs and the coupling of the distributed estimator with the chain of two integrators make the analysis and design in this article much more challenging than those in [13]. We prove that if the deviations among the reference signal accelerations tend to zero (respectively, bounded), the algorithm can achieve DAT with zero (respectively, bounded) tracking error.

In the second algorithm, we consider agents with high-order integrator dynamics. We propose a distributed nonlinear algorithm coupled with a distributed estimator for the left eigenvector of the Laplacian matrix associated with the zero eigenvalue. The algorithm is motivated by [3]. Specifically, our proposed algorithm and [3] both can achieve DAT for agents subject to certain linear dynamics. However, our proposed algorithm can solve the DAT problem under general weight-unbalanced directed graphs. In contrast, [3] poses an assumption that the graph is undirected. The relaxation of such an assumption makes our proposed algorithm amenable to more applications but in turn, poses more technical challenges. In addition, we replace the signum function in [3] with a continuous approximation in order to remove the chattering effect caused by the discontinuous signum function. The approximate function is widely adopted in the sliding mode control field [21]. The results show that if the reference signals and signal control inputs are bounded, the algorithm can achieve DAT with arbitrarily small tracking errors. The convergence of the algorithm to the vicinity of the average of the reference signals is established via Lyapunov stability theory and input-to-state stability theory.

Some preliminary results of this article (Section III) are presented in [22]. This article extends [22] by introducing new results on the nonlinear algorithm for high-order integrator systems in weight-unbalanced directed graphs.

II. Preliminaries

A. Notation

Let $\mathbb{R}, \mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the sets of real numbers, real vectors of dimension n, and real matrices of size $n \times m$, respectively. Let $\mathbb{R}_{>0}$ represent the set of positive real numbers. Let $\mathbf{1}_n$ (resp. $\mathbf{0}_n$) be the vector of n ones (resp. n zeros), I_n denote the $n \times n$ identity matrix, and $\bar{\mathbf{0}}_n$ (resp. $\bar{\mathbf{0}}_{m\times n}$) denote the $n \times n$ (resp. $m \times n$) matrix of all zeros. For a matrix $A \in \mathbb{R}^{m\times n}$, $\sigma_{\max}(A)$ denotes the maximal singular value of matrix A, A^T is the transpose of A, and vec $(A) = [\mathrm{col}_1(A)^T, \ldots, \mathrm{col}_n(A)^T]^T \in \mathbb{R}^{nm}$ is the column vector of size $nm \times 1$ obtained by stacking the columns of A, where $\mathrm{col}_i(A) \in \mathbb{R}^m$ represents the i th column of matrix A. For a square matrix $A \in \mathbb{R}^{m\times m}$, A^{-1} denotes the inverse of A. For a vector $x \in \mathbb{R}^{n\times 1}$, $\mathrm{diag}(x) \in \mathbb{R}^{n\times n}$ represents the diagonal matrix with the elements in the main diagonal being the elements of x, $\|x\|_p$ denotes the p-norm of the vector x. Let $\otimes$ be the Kronecker product. Let $n!$ be the product of n consecutive natural numbers from 1 to n. Let $f_2 \circ f_1(\cdot)$ be the composition of two functions $f_1(\cdot)$ and $f_2(\cdot)$. Let $f^{-1}(\cdot)$ denote the inverse of a function $f(\cdot)$. Let $\lfloor a \rfloor$ be the largest integer that is smaller than or equal to a.

Proposition 1 [23]: Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times l}$ and $D \in \mathbb{R}^{l\times k}$. Then, vec $(ABD) = (D^T \otimes A)$ vec(B).

B. Graph Theory

A weighted directed graph, is denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, \ldots, n\}$ is the node set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ is the weighted adjacency matrix with $a_{ij} \in (0, \bar{a})$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise for some $\bar{a} \in \mathbb{R}_{>0}$. For a directed graph, an edge (j, i) implies that node i can receive information from j. Let $\mathcal{N}_i = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$ denote the set of in-neighbors of node i. A directed path is a sequence of nodes connected by edges. A directed graph is strongly connected if for every pair of nodes there is a directed path connecting them. The Laplacian matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{n\times n}$ associated with $\mathcal{A}$ is defined as $l_{ii} = \sum_{j=1,j\neq i}^{n} a_{ij}$ and $l_{ij} = -a_{ij}$, where $i \neq j$. Note that $\mathcal{L}\mathbf{1}_n = \mathbf{0}_n$. A directed graph is weight-balanced if and only if $\mathbf{1}_n^T\mathcal{L} = \mathbf{0}_n^T$.

Lemma 1 [24], [25]: Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a directed graph with the Laplacian matrix $\mathcal{L} \in \mathbb{R}^{n\times n}$. If $\mathcal{G}$ is strongly connected, then the following statements hold.

1) There exists a positive left eigenvector $p = [p_1, \ldots, p_n]^T$ of $\mathcal{L}$ associated with the zero eigenvalue, such that $p_i > 0, i = 1, \ldots, n$, $p^T\mathcal{L} = \mathbf{0}_n^T$, and $\sum_{i=1}^{n} p_i = 1$.
2) The Laplacian matrix $\mathcal{L}$ has a simple zero eigenvalue corresponding to the right eigenvector $\mathbf{1}_n$, and all the nonzero eigenvalues have positive real parts.
3) $\min_{a^T x=0, x\neq \mathbf{0}_n} x^T\bar{\mathcal{L}}x > \lambda_2(\bar{\mathcal{L}})x^Tx/n$, where x is any vector, $\bar{\mathcal{L}} = \mathcal{L}^TP + P\mathcal{L}$ and $P = \mathrm{diag}(p)$, a is any vector with positive entries, and $\lambda_2(\bar{\mathcal{L}})$ is the smallest nonzero eigenvalue of matrix $\bar{\mathcal{L}}$.
4) $\lim_{t\to\infty} \exp(-\mathcal{L}t) = \mathbf{1}_n p^T$.

III. Linear DAT Algorithm for Single-Integrator Dynamics

In this section, the DAT problem for multiagent systems with single-integrator dynamics over weight-unbalanced directed graphs is studied. Consider a multiagent system consisting of n agents with an interaction topology described by a weighted directed graph $\mathcal{G}$.

Assumption 1: The directed graph $\mathcal{G}$ is time invariant and strongly connected.[1]

Suppose that the agents follow the single-integrator dynamics

$$\dot{x}_i(t) = u_i(t), \qquad i = 1, \ldots, n \tag{1}$$

where $x_i(t) \in \mathbb{R}^m$ and $u_i(t) \in \mathbb{R}^m$ are, respectively, the ith agent's state and control input. Each agent has a time-varying reference signal $r_i(t) \in \mathbb{R}^m, i = 1, \ldots, n$, satisfying

$$\dot{r}_i(t) = v_i^r(t), \qquad \dot{v}_i^r(t) = a_i^r(t) \tag{2}$$

where $v_i^r(t) \in \mathbb{R}^m$ and $a_i^r(t) \in \mathbb{R}^m$ are, respectively, the velocity and acceleration of the ith reference signal. For example, the reference signal r_i might be the position, sensed by the ith camera, of a mobile target of interest.

Our main objective is to design a distributed algorithm for agent $i \in \mathcal{V}$ based on $r_i(t)$, $v_i^r(t)$, $a_i^r(t)$, $x_i(t)$ and $x_j(t), j \in \mathcal{N}_i$, such that it tracks the average of all the time-varying reference signals, i.e.,

$$\lim_{t\to\infty} \left\| x_i(t) - (1/n)\sum_{j=1}^{n} r_j(t) \right\|_2 = 0. \tag{3}$$

We call a DAT algorithm robust to initialization errors if the objective (3) can be achieved regardless of the agents' initial states. For notational simplicity, we will remove the time index t from variables in the remainder of the section.

A. DAT Algorithm Design

We propose the following algorithm:

$$\begin{aligned}
u_i &= -\kappa(x_i - r_i) - \kappa z_{ii}\sum_{j\in\mathcal{N}_i} a_{ij}(x_i - x_j) + v_i^r - w_{1i}\\
\dot{w}_{1i} &= \kappa^2 z_{ii}\sum_{j\in\mathcal{N}_i} a_{ij}(x_i - x_j) - z_{ii}\sum_{j\in\mathcal{N}_i} a_{ij}(w_{2i} - w_{2j})\\
\dot{w}_{2i} &= w_{1i} - \kappa r_i - v_i^r, \quad \dot{z}_i = -\sum_{j\in\mathcal{N}_i} a_{ij}(z_i - z_j)
\end{aligned} \tag{4}$$

where $\kappa \in \mathbb{R}_{>0}$ is a positive control gain, $z_i \in \mathbb{R}^n$ is agent i's estimate of the left eigenvector corresponding to the zero eigenvalue of the Laplacian matrix, z_{ii} is the ith component of z_i, $w_{1i} \in \mathbb{R}^m$ and $w_{2i} \in \mathbb{R}^m$ are the internal states of a chain of two integrators, and a_{ij} is the (i,j)th entry of the adjacency matrix. We initialize the internal states w_{1i}, w_{2i}, and the estimators z_i to satisfy the following conditions:

$$\sum_{i=1}^{n} w_{1i}(0) = \mathbf{0}_m, \; z_{ij}(0) = 0, \; \forall i \neq j, \; z_{ii}(0) = 1, \; \forall i \in \mathcal{V}. \tag{5}$$

We note that each component of x_i is decoupled in (4). Therefore, in the following, we will only tackle the one-dimensional case, i.e., $m = 1$. The same conclusion holds for any $m \geq 2$ by using the Kronecker product. Substituting (4) into (1) leads to a vector form as

$$\begin{aligned}
\dot{x} &= -\kappa(x - r) - \kappa Z_n\mathcal{L}x + v^r - w_1\\
\dot{w}_1 &= \kappa^2 Z_n\mathcal{L}x - Z_n\mathcal{L}w_2\\
\dot{w}_2 &= w_1 - \kappa r - v^r, \quad \dot{z} = -(\mathcal{L}\otimes I_n)z
\end{aligned} \tag{6}$$

where $r = [r_1, \ldots, r_n]^T \in \mathbb{R}^n$, $v^r = [v_1^r, \ldots, v_n^r]^T \in \mathbb{R}^n$, $x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$, $w_1 = [w_{11}, \ldots, w_{1n}]^T \in \mathbb{R}^n$, $w_2 = [w_{21}, \ldots, w_{2n}]^T \in \mathbb{R}^n$, $z = [z_1^T, \ldots, z_n^T]^T \in \mathbb{R}^{n^2}$ and $Z_n = \mathrm{diag}([z_{11}, z_{22}, \ldots, z_{nn}]) \in \mathbb{R}^{n\times n}$.

Lemma 2: If Assumption 1 holds and $z(0)$ satisfies (5), then $\lim_{t\to\infty} Z_n \to P$, where P is defined in Lemma 1.

[1]Note that there is no requirement that $\mathcal{G}$ be weight-balanced.

Proof: We know that $z = \exp((-\mathcal{L}\otimes I_n)t)z(0)$. By Lemma 1, it can be obtained that $\lim_{t\to\infty} z = \exp(\mathbf{1}_n p^T \otimes I_n)z(0) = \mathbf{1}_n \otimes p$ if $z(0)$ satisfies (5), yielding $\lim_{t\to\infty} Z_n \to P$. ■

Remark 1: Compared with [13] which requires the network be undirected, the algorithm (4) can work for generic directed networks. Due to Lemma 2, we know that Z_n is utilized to estimate the matrix P. It follows from $\mathbf{1}_n^T P\mathcal{L} = \mathbf{0}_n^T$ that $P\mathcal{L}$ is equivalent to the Laplacian matrix of a balanced directed graph [26].

Remark 2: In the proposed algorithm (4), a chain of two integrators with the internal states w_{1i} and w_{2i} are introduced to make (4) work for more general reference signals, the term $-\kappa(x_i - r_i)$ is introduced to achieve sum tracking, i.e., $\lim_{t\to\infty} \|\sum_{i=1}^n x_i - \sum_{i=1}^n r_i\|_2 = 0$, and the term $-\kappa z_{ii}\sum_{j\in\mathcal{N}_i} a_{ij}(x_i - x_j)$ is introduced to achieve consensus with the aid of the chain of two integrators w_{1i} and w_{2i}. The distributed estimator given by the last equation in (4) is used by agent i to estimate the left eigenvector, corresponding to the zero eigenvalue, of the Laplacian matrix.

Remark 3: In the proposed algorithm (4), only correct initializations of internal states $w_{1i}(0)$ and $z_i(0)$ are needed, and correct initializations of agents' states $x_i(0)$ and $w_{2i}(0)$ are not required, which makes the algorithm robust to the state initialization errors. Note that the initialization condition (5) can be easily satisfied, e.g., to satisfy $\sum_{i=1}^n w_{1i}(0) = 0$, we can choose $w_{1i} = 0, \quad \forall i = 1, \ldots, n$.

B. Convergence Analysis

The main assumption and result of this section are stated in the following theorem.

Assumption 2: The deviations among the accelerations of the references all tend to zero, i.e., $\lim_{t\to\infty}(a_i^r - a_j^r) = 0, i \neq j$.

Theorem 1: Using (4) for (1), if Assumptions 1 and 2 and the initial condition (5) hold, and $\kappa \gg 1$, then $\lim_{t\to\infty} \|x_i - \frac{1}{n}\sum_{j=1}^n r_j\|_2 = 0$ for all $i = 1, \ldots, n$.

Proof: Define $\tilde{x} = x - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T r$, $\tilde{w}_1 = w_1 - \kappa\Omega r - \Omega v^r$, $\tilde{w}_2 = Z_n\mathcal{L}w_2 + \kappa\Omega v^r + \Omega a^r$, and $Y = [\tilde{x}^T, \tilde{w}_1^T, \tilde{w}_2^T]^T$ where $\Omega = I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T$. It can be verified that $\mathcal{L}\tilde{x} = \mathcal{L}x$ and $\mathcal{L}\Omega = \mathcal{L}$. The first three equations in (6) can be rewritten in terms of Y as

$$\dot{Y} = f(Y) + g(Y) + h \tag{7}$$

where

$$f(Y) = \begin{bmatrix} -\kappa\tilde{x} - \kappa P\mathcal{L}\tilde{x} - \tilde{w}_1 \\ \kappa^2 P\mathcal{L}\tilde{x} - \tilde{w}_2 \\ P\mathcal{L}\tilde{w}_1 \end{bmatrix}$$

$$g(Y) = \begin{bmatrix} \kappa(P - Z_n)\mathcal{L}\tilde{x} \\ \kappa^2(Z_n - P)\mathcal{L}\tilde{x} \\ (Z_n - P)\mathcal{L}\tilde{w}_1 \end{bmatrix}, h = \begin{bmatrix} 0 \\ 0 \\ \kappa\Omega a^r + \Omega\dot{a}^r \end{bmatrix}.$$

Based on Assumption 2, we know that h will approach zero as time goes to infinity. Therefore, by taking h in (7) as the system input, we first analyze the stability and convergence properties of the unforced system, i.e.,

$$\dot{Y} = f(Y) + g(Y). \tag{8}$$

Due to Lemma 2, we know that as $t \to \infty$, $Z_n - P$ tends to zero, so by Corollary 9.1 and Lemma 9.5 in [28] the convergence of (8) can be analyzed via $\dot{Y} = f(Y)$ only, i.e.,

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{w}}_1 \\ \dot{\tilde{w}}_2 \end{bmatrix} = \tilde{A}\begin{bmatrix} \tilde{x} \\ \tilde{w}_1 \\ \tilde{w}_2 \end{bmatrix} \tag{9}$$

where

$$\tilde{A} = \begin{bmatrix} -\kappa - \kappa P\mathcal{L} & -I_n & \bar{\mathbf{0}}_n \\ \kappa^2 P\mathcal{L} & \bar{\mathbf{0}}_n & -I_n \\ \bar{\mathbf{0}}_n & P\mathcal{L} & \bar{\mathbf{0}}_n \end{bmatrix}.$$

In the following, we show that the dynamical system (9) is stable and convergent by studying the dynamics of two related systems. Define $T_1 = [\, q_1 \; Q^T \,]^T$, where $q_1 = \frac{1}{\sqrt{n}}\mathbf{1}_n$, $Qq_1 = \mathbf{0}_{n-1}$ and $QQ^T = I_{n-1}$. It follows that $T_1 P\mathcal{L} T_1^T = \begin{bmatrix} 0 & \mathbf{0}_{n-1}^T \\ \mathbf{0}_{n-1} & \Lambda \end{bmatrix}$, where Λ is an upper triangular matrix whose diagonal entries are the nonzero eigenvalues of $P\mathcal{L}$. Thus $\hat{x} = [\hat{x}_1 \;\; \hat{x}_{2:n}^T]^T = T_1\tilde{x}$, $\hat{w}_1 = [\hat{w}_{11} \;\; \hat{w}_{12:1n}^T]^T = T_1\tilde{w}_1$ and $\hat{w}_2 = [\hat{w}_{21} \;\; \hat{w}_{22:2n}^T]^T = T_1\tilde{w}_2$, where $\hat{x}_1, \hat{w}_{11}$ and $\hat{w}_{21} \in \mathbb{R}$. We can rewrite (9) as

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{w}}_{11} \\ \dot{\hat{w}}_{21} \end{bmatrix} = A \begin{bmatrix} \hat{x}_1 \\ \hat{w}_{11} \\ \hat{w}_{21} \end{bmatrix}, \quad \begin{bmatrix} \dot{\hat{x}}_{2:n} \\ \dot{\hat{w}}_{12:1n} \\ \dot{\hat{w}}_{22:2n} \end{bmatrix} = \hat{A} \begin{bmatrix} \hat{x}_{2:n} \\ \hat{w}_{12:1n} \\ \hat{w}_{22:2n} \end{bmatrix} \tag{10}$$

where

$$A = -\begin{bmatrix} \kappa & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \hat{A} = \begin{bmatrix} -\kappa - \kappa\Lambda & -I_{n-1} & \bar{\mathbf{0}}_{n-1} \\ \kappa^2\Lambda & \bar{\mathbf{0}}_{n-1} & -I_{n-1} \\ \bar{\mathbf{0}}_{n-1} & \Lambda & \bar{\mathbf{0}}_{n-1} \end{bmatrix}.$$

The matrix A $****$ has two eigenvalues 0 (with multiplicity 2) and $-\kappa$. Define

$$T_2 = \begin{bmatrix} I_{n-1} & \bar{\mathbf{0}}_{n-1} & \bar{\mathbf{0}}_{n-1} \\ \kappa I_{n-1} & I_{n-1} & \bar{\mathbf{0}}_{n-1} \\ \bar{\mathbf{0}}_{n-1} & \bar{\mathbf{0}}_{n-1} & I_{n-1} \end{bmatrix}$$

such that $[\bar{x}^T, \bar{w}_1^T, \bar{w}_2^T]^T = T_2[\hat{x}_{2:n}^T, \hat{w}_{12:1n}^T, \hat{w}_{22:2n}^T]^T$. It follows from (10) that

$$\begin{bmatrix} \dot{\bar{x}} \\ \dot{\bar{w}}_1 \\ \dot{\bar{w}}_2 \end{bmatrix} = \bar{A} \begin{bmatrix} \bar{x} \\ \bar{w}_1 \\ \bar{w}_2 \end{bmatrix} \tag{11}$$

where

$$\bar{A} = \begin{bmatrix} -\kappa\Lambda & -I_{n-1} & \bar{\mathbf{0}}_{n-1} \\ \bar{\mathbf{0}}_{n-1} & -\kappa I_{n-1} & -I_{n-1} \\ -\kappa\Lambda & \Lambda & \bar{\mathbf{0}}_{n-1} \end{bmatrix}.$$

The determinant $\det(\lambda I_{n-1} - \bar{A})$ is given by

$$\det \begin{bmatrix} \lambda I_{n-1} + \kappa\Lambda & I_{n-1} & \bar{\mathbf{0}}_{n-1} \\ \bar{\mathbf{0}}_{n-1} & \lambda I_{n-1} + \kappa I_{n-1} & I_{n-1} \\ \kappa\Lambda & -\Lambda & \lambda I_{n-1} \end{bmatrix}$$
$$= \det \begin{bmatrix} \lambda I_{n-1} + \kappa\Lambda & I_{n-1} & \bar{\mathbf{0}}_{n-1} \\ \bar{\mathbf{0}}_{n-1} & \lambda I_{n-1} + \kappa I_{n-1} & I_{n-1} \\ \bar{\mathbf{0}}_{n-1} & \bar{\mathbf{0}}_{n-1} & \Gamma \end{bmatrix} \tag{12}$$

where $\Gamma = \lambda I_{n-1} + (\lambda I_{n-1} + \kappa I_{n-1})^{-1}[\Lambda + \kappa(\lambda I_{n-1} + \kappa\Lambda)^{-1}\Lambda]$. Noting that the inverse of an upper triangular matrix is also an upper triangular matrix, and the multiplication of two upper triangular matrices is also an upper triangular matrix, it follows that Γ is an upper triangular matrix. Define $\Xi \in \mathbb{R}^{n-1\times n-1}$ where $\Xi_{i,i} = \Lambda_{i,i}$ for all $i = 1, 2, \ldots, n-1$ and $\Xi_{i,j} = 0$ for all $i \neq j$. We have

$$\det \begin{bmatrix} \lambda I_{n-1} + \kappa\Lambda & I_{n-1} & \bar{\mathbf{0}}_{n-1} \\ \bar{\mathbf{0}}_{n-1} & \lambda I_{n-1} + \kappa I_{n-1} & I_{n-1} \\ \bar{\mathbf{0}}_{n-1} & \bar{\mathbf{0}}_{n-1} & \Gamma \end{bmatrix}$$
$$= \det \begin{bmatrix} \lambda I_{n-1} + \kappa\Xi & \bar{\mathbf{0}}_{n-1} & \bar{\mathbf{0}}_{n-1} \\ \bar{\mathbf{0}}_{n-1} & \lambda I_{n-1} + \kappa I_{n-1} & \bar{\mathbf{0}}_{n-1} \\ \bar{\mathbf{0}}_{n-1} & \bar{\mathbf{0}}_{n-1} & \tilde{\Gamma} \end{bmatrix} \tag{13}$$

where $\tilde{\Gamma} = \lambda I_{n-1} + (\lambda I_{n-1} + \kappa I_{n-1})^{-1}[\Xi + \kappa(\lambda I_{n-1} + \kappa\Xi)^{-1}\Xi]$. The eigenvalues λ of $\bar{A}$ can be obtained by calculating the following equations

$$\lambda^3 + \kappa\lambda^2 + \kappa\Xi_{ii}\lambda^2 + \kappa^2\Xi_{ii}\lambda + \Xi_{ii}\lambda + \kappa\Xi_{ii}^2 + \kappa\Xi_{ii} = 0,$$
$$i = 1, .., n-1. \tag{14}$$

To solve (14), we consider the corresponding perturbed cubic equation

$$\lambda^3 + \left(\kappa + \kappa\Xi_{ii} + \frac{1}{\kappa} - \varepsilon\right)\lambda^2 + \left(\kappa^2\Xi_{ii} + \Xi_{ii} + 1 - \kappa\varepsilon\right)\lambda$$
$$+ \kappa\Xi_{ii}^2 + \kappa^2\Xi_{ii}\varepsilon = 0, \quad i = 1, .., n-1. \tag{15}$$

With ε beging the perturbation parameter, it is worth noting that when $\varepsilon = \frac{1}{\kappa}$, the perturbed cubic (15) reduces to (14). It follows from $\kappa \gg 1$ that $(1/\kappa) \ll 1$. Based on perturbation theory [27], the eigenvalues λ are given by $\lambda = \lambda_0 + \varepsilon\lambda_1 + O(\varepsilon) = \lambda_0 + \frac{1}{\kappa}\lambda_1 + O(\varepsilon)$, where λ_0 is the solution of the following equations:

$$\left(\lambda + \frac{1}{\kappa}\right)(\lambda + \kappa)(\lambda + \kappa\Xi_{ii}) = 0, \quad i = 1, .., n-1 \tag{16}$$

and $\lambda_1 = \frac{\lambda_0^2 + \kappa\lambda_0 - \kappa^2\Xi_{ii}^2}{\lambda_0^2 + 2\lambda_0 + 2\Xi_{ii}\kappa + 2\kappa + 2/\kappa + \Xi_{ii}\kappa^2 + \Xi_{ii} + 1}$, $i = 1, \ldots, n-1$, with $O(\varepsilon)$ being the higher order term.

It follows from (16) that for each $i = 1, \ldots, n-1$, $\lambda_0 = -1/\kappa, -\kappa, -\kappa\Xi_{ii}$, along with

$$\lambda_1 = \begin{cases} \frac{(1/\kappa)^2 - 1 - \kappa^2\Xi_{ii}^2}{(1/\kappa)^2 + 2\Xi_{ii}\kappa + 2\kappa + \Xi_{ii}\kappa^2 + \Xi_{ii} + 1} & \text{if } \lambda_0 = -\frac{1}{\kappa} \\ \frac{-\kappa^2\Xi_{ii}^2}{(\kappa)^2 + 2\Xi_{ii}\kappa + 2/\kappa + \Xi_{ii}\kappa^2 + \Xi_{ii} + 1} & \text{if } \lambda_0 = -\kappa \\ \frac{-\kappa^2\Xi_{ii}}{(\kappa\Xi_{ii})^2 + 2\kappa + 2/\kappa + \Xi_{ii}\kappa^2 + \Xi_{ii} + 1} & \text{if } \lambda_0 = -\kappa\Xi_{ii}. \end{cases} \tag{17}$$

Recall that the nonzero eigenvalues of $P\mathcal{L}$ all have positive real parts, therefore λ_0 and λ_1 all have negative real parts, which indicates that the eigenvalues λ of (14) all have negative real parts. Hence the dynamical system (11) is exponentially stable. Noting that T_1 and T_2 are all nonsingular matrices, the dynamical system (9) must be exponentially stable.

The null space of the system matrix $\tilde{A}$ of (9) is spanned by $[\mathbf{1}_n^T, -\kappa\mathbf{1}_n^T, \mathbf{0}_n^T]^T$, the eigenvector associated with the zero eigenvalue. Therefore, (9) converges exponentially fast to the set $\{(\tilde{x}, \tilde{w}_1, \tilde{w}_2) \mid \tilde{x} = a\mathbf{1}_n, \tilde{w}_1 = -a\kappa\mathbf{1}_n, \tilde{w}_2 = \mathbf{0}_n, a \in \mathbb{R}\}$. According to the definition of $\tilde{w}_1$, we know that $\mathbf{1}_n^T\tilde{w}_1 = \mathbf{1}_n^T w_1$. It follows from $\mathbf{1}_n^T(\kappa^2 P\mathcal{L}x - P\mathcal{L}w_2) = 0$ that $\mathbf{1}_n^T\dot{w}_1 = 0$, leading to $\mathbf{1}_n^T\tilde{w}_1 = \mathbf{1}_n^T w_1 = \mathbf{1}_n^T w_1(0) = 0$. Therefore, if (5) holds, system (9) converges exponentially to the set $\{(\tilde{x}, \tilde{w}_1, \tilde{w}_2) \mid \tilde{x} = \mathbf{0}_n, \tilde{w}_1 = \mathbf{0}_n, \tilde{w}_2 = \mathbf{0}_n\}$. Based on Corollary 9.1 and Lemma 9.5 in [28], the perturbed system (8) is exponentially stable with respect to the equilibrium point. Therefore, under Assumption 2, Y in (7) asymptotically converges to $\mathbf{0}_{3n}$. ■

In practice, the reference signals may not always satisfy Assumption 2. A more realistic assumption is to require the deviations among the reference signal accelerations be bounded, which is formally stated in the following.

Assumption 3: The deviations among the accelerations of the reference signals are bounded, i.e., there exists $\bar{a}^r \in \mathbb{R}_{>0}$ such that $\sup_{t\in[0,\infty)}(a_i^r - a_j^r) = \bar{a}^r, \forall i \in \mathcal{V}, j \in \mathcal{V}, i \neq j$.

Theorem 2: Using (4) for (1), if Assumption 3 and the internal state's initial conditions (5) hold, and $\kappa \gg 1$, then $\sup_{t\in[0,\infty)} \left\| x_i - \frac{1}{n}\sum_{j=1}^n r_j \right\|_2$ is bounded for all $i = 1, \ldots, n$.

Proof: Because of the boundedness of h, system (7) is input-to-state stable by Lemma 4.6 given in [28]. If Assumption 3 holds, it follows from Definition 4.7 in [28] that the whole tracking error is upper bounded by $\lim_{t\to\infty}\sup\|\tilde{x}(t)\|_2 \leq \lim_{t\to\infty}\sup\|Y(t)\|_2 \leq \epsilon\bar{a}^r$, where ϵ is a positive constant. ■

IV. Nonlinear DAT Algorithm for High-Order Integrator Dynamics

In this section, the DAT problem for multiagent systems with high-order integrator dynamics over weight-unbalanced directed graphs is studied. In some applications, it might be more realistic to model the dynamics of the agents with high-order integrators. Unlike single-integrator dynamics, in the case of high-order integrator dynamics, the agents' system inputs might have a different dimension from that of the agents' state. Consider a network of n agents whose states are governed by

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) \tag{18}$$

where $x_i(t) \in \mathbb{R}^m$, $u_i(t) \in \mathbb{R}$ are the system state and control input of the ith agent,[2] $A \in \mathbb{R}^{m\times m}$ is the state matrix, and $B \in \mathbb{R}^{m\times 1}$ is the input matrix. Here A and B are defined as

$$A = \begin{bmatrix} \mathbf{0}_{m-1} & I_{m-1} \\ 0 & \mathbf{0}_{m-1}^T \end{bmatrix}, \quad B = \begin{bmatrix} \mathbf{0}_{m-1} \\ 1 \end{bmatrix}.$$

Each agent has a time-varying reference signal $r_i(t) \in \mathbb{R}^m$ given by

$$\dot{r}_i(t) = Ar_i(t) + Bu_i^r(t) \tag{19}$$

where $r_i(t) \in \mathbb{R}^m$, $u_i^r(t) \in \mathbb{R}$ is the state and control input of the ith time-varying reference signal. The input $u_i^r(t)$ can be properly designed such that (19) can generate a general time-varying reference signal $r_i(t)$. The following standard assumption is made:

Assumption 4: The reference signals are bounded, and their control inputs are bounded, i.e., there exist $\bar{r} > 0$ and $\bar{u}^r > 0$ such that $\sup_{t\in[0,\infty)} \|r_i(t)\|_2 \le \bar{r}$ and $\sup_{t\in[0,\infty)} \|u_i^r(t)\|_2 \le \bar{u}^r$, for all $i \in \mathcal{V}$.

Again, for notational simplicity, we will remove the time index t from variables in the remainder of this section and only keep it in some places when necessary.

A. DAT Algorithm Design

We study the following control algorithm:

$$u_i = u_i^r + K_1(x_i - r_i) - \beta\tilde{h}\left(z_{ii}\sum_{j\in\mathcal{N}_i} a_{ij}K_2(x_i - x_j)\right)$$
$$\dot{z}_i = -\sum_{j\in\mathcal{N}_i} a_{ij}(z_i - z_j) \tag{20}$$

where $z_i \in \mathbb{R}^n$ is agent i's estimate of the left eigenvector corresponding to the zero eigenvalue of the Laplacian matrix, z_{ii} is the ith element in vector z_i, a_{ij} is the (i,j)th element of the adjacency matrix $\mathcal{A}$, and $\tilde{h}(\cdot)$ is a function defined component-wise as $\tilde{h}(s) = \begin{cases} \frac{s}{|s|} & \text{if } |s|\ge\epsilon, \\ \frac{s}{\epsilon} & \text{otherwise,} \end{cases}$ where $\epsilon \in \mathbb{R}_{>0}$ is a small positive constant. In addition

$$K_1 \triangleq -(K_2A + K_2) \in \mathbb{R}^{1\times m}$$
$$K_2 \triangleq [C_{m-1}^0, C_{m-1}^1, \ldots, C_{m-1}^{m-1}] \in \mathbb{R}^{1\times m} \tag{21}$$

where $C_{m-1}^k = \frac{(m-1)!}{(m-1-k)!k!}$, $k \in [0, m-1]$, and β is a control gain satisfying

$$\beta > \frac{2n^{\frac{5}{2}}\sigma_{\max}(P\mathcal{L})\left(\bar{u}^r + \sigma_{\max}(K_1)\bar{r}\right)}{\lambda_2(\bar{\mathcal{L}})} \tag{22}$$

where $\bar{\mathcal{L}}$ and P are defined in Lemma 1. We initialize the estimators $z_i, i \in \mathcal{V}$, to satisfy the last two equations in (5). Define $z \in \mathbb{R}^{n^2} = [z_1^T, \ldots, z_n^T]^T$, $Z_n = \text{diag}([z_{11}, z_{22}, \ldots, z_{nn}]) \in \mathbb{R}^{n\times n}$, $x \triangleq [x_1^T, \ldots, x_n^T]^T \in \mathbb{R}^{nm}$, $u \triangleq [u_1, \ldots, u_n]^T \in \mathbb{R}^n$, $r \triangleq [r_1^T, \ldots, r_n^T]^T \in \mathbb{R}^{nm}$ and $u^r \triangleq [u_1^r, \ldots, u_n^r]^T \in \mathbb{R}^n$. Then the closed-loop system (18) can be written in a vector form as

$$\dot{x} = (I_n \otimes A)x + (I_n \otimes B)$$
$$\times \left(u^r + (I_n \otimes K_1)(x - r) - \beta\tilde{h}\left((Z_n\mathcal{L} \otimes K_2)x\right)\right)$$
$$\dot{z} = -(\mathcal{L} \otimes I_n)z. \tag{23}$$

[2]Note that the dimensions of the control input u_i in Secs. III and IV are different.

Remark 4: For the high-order integrator case in Section IV, we only tackle the one-dimension case (i.e., $x_i \in \mathbb{R}^m$ and $u_i \in \mathbb{R}$ for m-order integrators), since each dimension of the high-order integrators is decoupled. Please note that our algorithm can be easily extended to multidimension high-order integrator cases (i.e., $x_i \in \mathbb{R}^{mq}$ and $u_i \in \mathbb{R}^q$ for m-order integrators) by using the Kronecker product.

Remark 5: In the proposed algorithm (20), the term $K_1(x_i - r_i)$ is introduced to achieve sum tracking (i.e., $\lim_{t\to\infty} \|\sum_{i=1}^n x_i(t) - \sum_{i=1}^n r_i(t)\|_2 = 0$) with the help of the distributed estimator z_i, and the term $-\beta\tilde{h}(z_{ii}\sum_{j\in\mathcal{N}_i} a_{ij}K_2(x_i - x_j))$ is introduced to guarantee consensus.

B. Convergence Analysis

This section establishes the convergence properties of the system (18) under the controller (20).

Lemma 3: For any strongly connected directed graph $\mathcal{G}$ of order n, let $(P\mathcal{L})^+ \in \mathbb{R}^{n\times n}$ be the generalized inverse of $P\mathcal{L}$ with P and $\mathcal{L}$ being defined in Section II.B, we have $(P\mathcal{L})^+(P\mathcal{L}) = I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T$.

Proof: Note that $\mathbf{1}_n^T P\mathcal{L} = \mathbf{0}_n^T$ and $P\mathcal{L}\mathbf{1}_n = \mathbf{0}_n$, i.e., $P\mathcal{L}$ can be viewed as the Laplacian matrix of a weight-balanced directed graph. Consequently, the proof follows directly from the proof of Lemma 3 in [10]. ■

Lemma 4: Let Assumption 1 hold. Using (20) for (18), if $\|(P\mathcal{L} \otimes K_2)x(t)\|_1$ is bounded for all $t\ge 0$ and

$$\lim_{t\to\infty} \sup \|(P\mathcal{L} \otimes K_2)x(t)\|_1 \le \bar{b} \tag{24}$$

where $\bar{b}$ is an arbitrary positive constant, then $\lim_{t\to\infty} \sup \|(\Omega \otimes I_m)x(t)\|_1 \le \|(P\mathcal{L})^+ \otimes I_m\|_1 2^{m-1}\bar{b}\sum_{i=1}^m i!(\prod_{j=0}^{i-1} C_j^{\lfloor j/2\rfloor})$, where $\Omega = I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T$, and $(P\mathcal{L})^+$ is the generalized inverse of $P\mathcal{L}$.

Proof: Define $X = [x_1, \ldots, x_n] \in \mathbb{R}^{m\times n}$, where $x_i \in \mathbb{R}^m$, $i \in [1, n]$, is defined in (18). Define $X_i \in \mathbb{R}^{1\times n}$ as the ith row of X. It follows from Proposition 1 and (21) that

$$\begin{aligned}(P\mathcal{L} \otimes K_2)x &= (P\mathcal{L} \otimes K_2)\text{vec}(X) = \text{vec}\left(K_2X(P\mathcal{L})^T\right) \\ &= \text{vec}\left((C_{m-1}^0X_1 + \cdots + C_{m-1}^{m-1}X_m)(P\mathcal{L})^T\right) \\ &= C_{m-1}^0\text{vec}\left(X_1(P\mathcal{L})^T\right) + \cdots + C_{m-1}^{m-1}\text{vec}\left(X_m(P\mathcal{L})^T\right) \\ &= C_{m-1}^0\tilde{X}_1 + \cdots + C_{m-1}^{m-1}\tilde{X}_m \end{aligned} \tag{25}$$

where $\tilde{X}_i = \text{vec}(X_i(P\mathcal{L})^T)$. For $\ell \in [0, m-1]$, define

$$s_\ell = C_{m-\ell-1}^0\tilde{X}_1 + C_{m-\ell-1}^1\tilde{X}_2 + \cdots + C_{m-\ell-1}^{m-\ell-1}\tilde{X}_{m-\ell} \tag{26}$$

and thus $\dot{s}_\ell = C_{m-\ell-1}^0\dot{\tilde{X}}_1 + C_{m-\ell-1}^1\dot{\tilde{X}}_2 + \cdots + C_{m-\ell-1}^{m-\ell-1}\dot{\tilde{X}}_{m-\ell}$. By (18), we have $\dot{\tilde{X}}_k = \tilde{X}_{k+1}$ for $k \in [1, m-1]$. It follows that for $\ell \in [1, m-1]$, $s_\ell + \dot{s}_\ell = C_{m-\ell-1}^0\tilde{X}_1 + C_{m-\ell-1}^1\tilde{X}_2 + \cdots + C_{m-\ell-1}^{m-\ell-1}\tilde{X}_{m-\ell} + C_{m-\ell-1}^0\dot{\tilde{X}}_1 + C_{m-\ell-1}^1\dot{\tilde{X}}_2 + \cdots + C_{m-\ell-1}^{m-\ell-1}\dot{\tilde{X}}_{m-\ell} = C_{m-\ell-1}^0\tilde{X}_1 + C_{m-\ell-1}^1\tilde{X}_2 + \cdots + C_{m-\ell-1}^{m-\ell-1}\tilde{X}_{m-\ell} + C_{m-\ell-1}^0\tilde{X}_2 + C_{m-\ell-1}^1\tilde{X}_3 + \cdots + C_{m-\ell-1}^{m-\ell-1}\tilde{X}_{m-\ell+1}$. Because $C_n^k = C_{n-1}^{k-1} + C_{n-1}^k$ and $C_n^0 = C_n^n = 1$, we have for $\ell \in [1, m-1]$,

$$\begin{aligned} s_\ell + \dot{s}_\ell &= C_{m-\ell-1}^0\tilde{X}_1 + \left(C_{m-\ell-1}^1 + C_{m-\ell-1}^0\right)\tilde{X}_2 + \cdots \\ &\quad + \left(C_{m-\ell-1}^{m-\ell-1} + C_{m-\ell-1}^{m-\ell-2}\right)\tilde{X}_{m-\ell} + C_{m-\ell-1}^{m-\ell-1}\tilde{X}_{m-\ell+1} \\ &= C_{m-\ell}^0\tilde{X}_1 + C_{m-\ell}^1\tilde{X}_2 + \cdots \\ &\quad + C_{m-\ell}^{m-\ell-1}\tilde{X}_{m-\ell} + C_{m-\ell}^{m-\ell}\tilde{X}_{m-\ell+1} = s_{\ell-1}. \end{aligned} \tag{27}$$

The proof will proceed by the mathematical induction method. Recall the definition of s_ℓ given by (26). It follows from (25) that $s_0 = C_{m-1}^0\tilde{X}_1 + \cdots + C_{m-1}^{m-1}\tilde{X}_m = (P\mathcal{L} \otimes K_2)x$. Therefore, it follows from (24) that $s_0(t)$ is bounded for all $t\ge 0$ and $\lim_{t\to\infty} \sup \|s_0(t)\|_1 \le \bar{b}$. Next, we will prove that for $\ell \in [1, m-1]$, if $s_{\ell-1}(t)$ is bounded for all $t\geq 0$ and $\lim_{t\to\infty}\sup\|s_{\ell-1}(t)\|_1\leq\tilde{b}$ with $\tilde{b}$ being a positive constant, then $s_\ell(t)$ is bounded for all $t\geq 0$ and $\lim_{t\to\infty}\sup\|s_\ell(t)\|_1\leq 2\tilde{b}$, using the input-to-state stability concept. It follows from (27) that $\dot{s}_\ell=-s_\ell+s_{\ell-1}$, which is obviously an input-to-state stable system by viewing s_ℓ as the state and $s_{\ell-1}$ as the input. Consider the Lyapunov function $\bar{V}(s_\ell)=\frac{1}{2}\|s_\ell\|_2^2$. It follows that $\alpha_1(\|s_\ell\|_2)\leq\bar{V}(s_\ell)\leq\alpha_2(\|s_\ell\|_2)$, where $\alpha_1(y)=\frac{1}{2}y^2$ and $\alpha_2(y)=\frac{1}{2}y^2$ are both class $\mathcal{K}_\infty$ functions. The derivative of $\bar{V}(s_\ell)$ is given by $\dot{\bar{V}}(s_\ell)=s_\ell^T\dot{s}_\ell=s_\ell^T(-s_\ell+s_{\ell-1})\leq-\|s_\ell\|_2^2+\|s_\ell\|_2\|s_{\ell-1}\|_2$. It follows that for all $\|s_\ell\|_2\geq\rho(\|s_{\ell-1}\|_2)$, where $\rho(y)=2y$ is a class $\mathcal{K}$ function, $\dot{\bar{V}}(s_\ell)\leq-\frac{1}{2}\|s_\ell\|_2^2$. Based on Theorem 4.19 and Definition 4.7 in [28], the system $\dot{s}_\ell=-s_\ell+s_{\ell-1}$ is an input-to-state stable system by viewing s_ℓ as the state and $s_{\ell-1}$ as the input, and there must exist a class $\mathcal{KL}$ function $\alpha(\cdot,t)$ and a class $\mathcal{K}$ function $\gamma(y)=\alpha_1^{-1}\circ\alpha_2\circ\rho=2y$ such that for any initial state $s_\ell(t_0)$ and any bounded input $s_{\ell-1}(t)$, the solution $s_\ell(t)$ exists and satisfies

$$\|s_\ell(t)\|_1\leq\alpha(\|s_\ell(t_0)\|_1,t-t_0)+2\left(\sup_{t_0\leq\tau\leq t}\|s_{\ell-1}(\tau)\|_1\right).\quad(28)$$

Since $\|s_{\ell-1}(t)\|_1\leq\tilde{b}$ as $t\to\infty$, for an arbitrary number $\zeta>0$, there must exist a time $T_1>0$ such that $\|s_{\ell-1}(t)\|_1\leq\tilde{b}+\zeta$ for all $t\geq T_1$. Consequently, for all $t\geq T_1$, we have

$$\begin{aligned}\|s_\ell(t)\|_1&\leq\alpha(\|s_\ell(T_1)\|_1,t-T_1)+2\left(\sup_{T_1\leq\tau\leq t}\|s_{\ell-1}(\tau)\|_1\right)\\&\leq\alpha(\|s_\ell(T_1)\|_1,t-T_1)+2\left(\tilde{b}+\zeta\right).\end{aligned}\quad(29)$$

Since $\alpha(\|s_\ell(T_1)\|_1,t-T_1)$ is a class $\mathcal{KL}$ function, $\alpha(\|s_\ell(T_1)\|_1,t-T_1)\to 0$ as $t\to\infty$. Therefore, there must exist a time $T_2>T_1$ such that $\alpha(\|s_\ell(T_1)\|_1,t-T_1)<\zeta$ for all $t\geq T_2$. It follows that for all $t\geq T_2$, $\|s_\ell(t)\|_1<\zeta+2\tilde{b}+2\zeta=3\zeta+2\tilde{b}$. Since ζ is an arbitrary positive number, we have $\|s_\ell(t)\|_1\leq 2\tilde{b}$ as $t\to\infty$. Therefore, we obtain the conclusion that for $\ell\in[1,m-1]$, if $s_{\ell-1}(t)$ is bounded for all $t\geq 0$ and $\lim_{t\to\infty}\sup\|s_{\ell-1}(t)\|\leq\tilde{b}$, then $s_\ell(t)$ is bounded for all $t\geq 0$ and $\lim_{t\to\infty}\sup\|s_\ell(t)\|\leq 2\tilde{b}$. Since we have proved that $s_0(t)$ is bounded for all $t\geq 0$ and $\lim_{t\to\infty}\sup\|s_0(t)\|_1\leq\bar{b}$, we have that $s_\ell(t),\ell\in[0,m-1]$, is bounded for all $t\geq 0$ and

$$\lim_{t\to\infty}\sup\|s_\ell(t)\|_1\leq 2^\ell\bar{b}.\quad(30)$$

Then, we will derive the bound of $\|\tilde{X}_i\|_1,\ i\in[1,m]$, based on the bound of $\|s_\ell\|_1$, $\ell\in[0,m-1]$, using the mathematical induction method again. It can be verified that $s_{m-1}=\tilde{X}_1$ by (26). Then it follows from (30) that $\lim_{t\to\infty}\sup\|\tilde{X}_1(t)\|_1=\lim_{t\to\infty}\sup\|s_{m-1}(t)\|_1\leq 2^{m-1}\bar{b}$. Therefore there exists a positive constant $\bar{B}_2=2^{m-1}\bar{b}>2^{m-2}\bar{b}$ such that $\lim_{t\to\infty}\sup\|\tilde{X}_1(t)\|_1\leq\bar{B}_2$. Next, we will prove that for $i\in[2,m]$, if there exists a positive constant $\bar{B}_i\geq 2^{m-i}\bar{b}$ such that $\lim_{t\to\infty}\sup\|\tilde{X}_k(t)\|_1\leq\bar{B}_i,\ \forall k\in[1,i-1]$, then there exists a positive constant $\bar{B}_{i+1}=iC_{i-1}^{\lfloor\frac{i-1}{2}\rfloor}\bar{B}_i\geq 2^{m-i-1}\bar{b}$ such that $\lim_{t\to\infty}\sup\|\tilde{X}_k(t)\|_1\leq\bar{B}_{i+1}$, $\forall k\in[1,i]$. For $i\in[2,m]$, if there exists a positive constant $\bar{B}_i\geq 2^{m-i}\bar{b}$ such that $\lim_{t\to\infty}\sup\|\tilde{X}_k(t)\|_1\leq\bar{B}_i$, $\forall k\in[1,i-1]$, it follows from (26) and (30) that

$$\begin{aligned}&\lim_{t\to\infty}\sup\|\tilde{X}_i(t)\|_1\\&=\lim_{t\to\infty}\sup\|s_{m-i}(t)-C_{i-1}^0\tilde{X}_1(t)-\cdots-C_{i-1}^{i-2}\tilde{X}_{i-1}(t)\|_1\\&\leq\lim_{t\to\infty}\sup\|s_{m-i}(t)\|_1+C_{i-1}^0\lim_{t\to\infty}\sup\|\tilde{X}_1(t)\|_1+\cdots\\&\quad+C_{i-1}^{i-2}\lim_{t\to\infty}\sup\|\tilde{X}_{i-1}(t)\|_1\\&\leq C_{i-1}^{\lfloor\frac{i-1}{2}\rfloor}\lim_{t\to\infty}\sup\|s_{m-i}(t)\|_1\\&\quad+C_{i-1}^{\lfloor\frac{i-1}{2}\rfloor}\lim_{t\to\infty}\sup\|\tilde{X}_1(t)\|_1+\cdots\\&\quad+C_{i-1}^{\lfloor\frac{i-1}{2}\rfloor}\lim_{t\to\infty}\sup\|\tilde{X}_{i-1}(t)\|_1\\&\leq C_{i-1}^{\lfloor\frac{i-1}{2}\rfloor}2^{m-i}\bar{b}+C_{i-1}^{\lfloor\frac{i-1}{2}\rfloor}\bar{B}_i+\cdots+C_{i-1}^{\lfloor\frac{i-1}{2}\rfloor}\bar{B}_i\\&\leq C_{i-1}^{\lfloor\frac{i-1}{2}\rfloor}\bar{B}_i+C_{i-1}^{\lfloor\frac{i-1}{2}\rfloor}\bar{B}_i+\cdots+C_{i-1}^{\lfloor\frac{i-1}{2}\rfloor}\bar{B}_i=iC_{i-1}^{\lfloor\frac{i-1}{2}\rfloor}\bar{B}_i\end{aligned}$$

where the last third inequality holds since $C_{i-1}^{\lfloor\frac{i-1}{2}\rfloor}\geq 1$ is the biggest coefficient among all $C_{i-1}^l,l\in[0,i-1]$. Since $i>1$ and $C_{i-1}^{\lfloor\frac{i-1}{2}\rfloor}\geq 1$, we have $iC_{i-1}^{\lfloor\frac{i-1}{2}\rfloor}\bar{B}_i\geq\bar{B}_i$. If $\bar{B}_i\geq 2^{m-i}\bar{b}$, it follows that $iC_{i-1}^{\lfloor\frac{i-1}{2}\rfloor}\bar{B}_i\geq 2^{m-i}\bar{b}\geq 2^{m-i-1}\bar{b}$. Therefore, for $i\in[2,m]$, if there exists a positive constant $\bar{B}_i\geq 2^{m-i}\bar{b}$ such that $\lim_{t\to\infty}\sup\|\tilde{X}_k(t)\|_1\leq\bar{B}_i,\ \forall k\in[1,i-1]$, then there exists a positive constant $\bar{B}_{i+1}=iC_{i-1}^{\lfloor\frac{i-1}{2}\rfloor}\bar{B}_i\geq 2^{m-i-1}\bar{b}$ such that $\lim_{t\to\infty}\sup\|\tilde{X}_k(t)\|_1\leq\bar{B}_{i+1},\ \forall k\in[1,i]$. We have proved that there exists $\bar{B}_2=2^{m-1}\bar{b}>2^{m-2}\bar{b}$ such that $\lim_{t\to\infty}\sup\|\tilde{X}_k(t)\|_1\leq\bar{B}_2$, $\forall k\in[1]$. It thus follows that $\lim_{t\to\infty}\sup\|\tilde{X}_i(t)\|_1\leq i!(\prod_{j=0}^{i-1}C_j^{\lfloor j/2\rfloor})2^{m-1}\bar{b}$ for $i\in[1,m]$.

Recall the definition of $\tilde{X}_i$. Based on Proposition 1, we have

$$\begin{aligned}\|(P\mathcal{L}\otimes I_m)\,\mathrm{vec}(X)\|_1&=\left\|\mathrm{vec}\left(X(P\mathcal{L})^T\right)\right\|_1\\&=\sum_{i=1}^m\left\|\mathrm{vec}\left(X_i(P\mathcal{L})^T\right)\right\|_1=\sum_{i=1}^m\left\|\tilde{X}_i\right\|_1.\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{t\to\infty}\sup\|(P\mathcal{L}\otimes I_m)\,x(t)\|_1&=\lim_{t\to\infty}\sup\|(P\mathcal{L}\otimes I_m)\,\mathrm{vec}(X(t))\|_1\\&=\lim_{t\to\infty}\sup\sum_{i=1}^m\left\|\tilde{X}_i(t)\right\|_1\\&\leq 2^{m-1}\bar{b}\sum_{i=1}^m i!\left(\prod_{j=0}^{i-1}C_j^{\lfloor j/2\rfloor}\right).\end{aligned}$$

It follows from Lemma 3 that if Assumption 1 holds, then $(P\mathcal{L})^+P\mathcal{L}=I_n-\frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T$, which further leads to

$$\begin{aligned}&\lim_{t\to\infty}\sup\|(\Omega\otimes I_m)\,x(t)\|_1\\&=\lim_{t\to\infty}\sup\|\left((P\mathcal{L})^+\otimes I_m\right)(P\mathcal{L}\otimes I_m)\\&\quad\times x(t)\|_1\leq\|(P\mathcal{L})^+\otimes I_m\|_1 2^{m-1}\bar{b}\sum_{i=1}^m i!\left(\prod_{j=0}^{i-1}C_j^{\lfloor j/2\rfloor}\right).\end{aligned}$$

■

The main result of this section is given as follows.

Theorem 3: Using (20) for (18), if Assumptions 1 and 4, the gain condition (22) and last two initial conditions in (5) hold, then $\lim_{t\to\infty}\sup\sum_{k=1}^n\|x_k(t)-(1/n)\sum_{j=1}^n r_j(t)\|_1\leq\|(P\mathcal{L})^+\otimes I_m\|_1 n\epsilon 2^{m-1}\sum_{i=1}^m i!(\prod_{j=0}^{i-1}C_j^{\lfloor j/2\rfloor})$, where $(P\mathcal{L})^+$ is the generalized inverse of matrix $P\mathcal{L}$.

Proof: Consider the Lyapunov function candidate

$$V(x)=\sum_{i=1}^n\tilde{g}\left(p_i\sum_{j\in\mathcal{N}_i}a_{ij}K_2(x_i-x_j)\right)\quad(31)$$

where p_i is defined in Lemma 1, $\tilde{g}(s) : \mathbb{R} \to \mathbb{R}$ is a function defined as $\tilde{g}(s) = \begin{cases} |s| & \text{if } |s| \geq \epsilon, \\ \frac{s^2}{2\epsilon} + \frac{\epsilon}{2} & \text{otherwise.} \end{cases}$ It can be shown that the derivative of $V(x)$ is

$$\dot{V}(x) = \sum_{i=1}^{n} \xi_i \left(p_i \sum_{j \in \mathcal{N}_i} a_{ij} K_2 (\dot{x}_i - \dot{x}_j) \right) \tag{32}$$

where $\xi_i \in \mathbb{R},\ i \in [1, \ldots, n]$, is defined as

$$\xi_i = \begin{cases} 1 & \text{if } p_i \sum_{j \in \mathcal{N}_i} a_{ij} K_2 (x_i - x_j) \geq \epsilon, \\ -1 & \text{if } p_i \sum_{j \in \mathcal{N}_i} a_{ij} K_2 (x_i - x_j) \leq -\epsilon, \\ \frac{p_i \sum_{j \in \mathcal{N}_i} a_{ij} K_2 (x_i - x_j)}{\epsilon} & \text{otherwise.} \end{cases} \tag{33}$$

Define $\xi = [\xi_1, \ldots, \xi_n]^T \in \mathbb{R}^n$. Then the derivative of $V(x)$ along the trajectory of (23) can be calculated as follows:

$$\begin{aligned} \dot{V}(x) &= \xi^T (P\mathcal{L} \otimes K_2)\, \dot{x} \\ &= \xi^T (P\mathcal{L} \otimes K_2) \Big((I_n \otimes A)x + (I_n \otimes B) \\ &\quad \times \Big(u^r + (I_n \otimes K_1)(x - r) - \beta \tilde{h}\left((Z_n \mathcal{L} \otimes K_2)x\right) \Big) \Big) \\ &= \xi^T (P\mathcal{L} \otimes K_2 B) u^r + \xi^T (P\mathcal{L} \otimes K_2 (A + BK_1))\, x \\ &\quad - \xi^T (P\mathcal{L} \otimes K_2 B K_1) r \\ &\quad - \beta \xi^T (P\mathcal{L} \otimes K_2 B) \tilde{h}\left((Z_n \mathcal{L} \otimes K_2)x\right). \end{aligned} \tag{34}$$

Recall the definition of K_2 and B, we have $K_2 B = C_{m-1}^{m-1} = 1$, it follows that

$$\begin{aligned} \dot{V}(x) &= \xi^T P\mathcal{L} u^r + \xi^T (P\mathcal{L} \otimes K_2 (A + BK_1))\, x \\ &\quad - \xi^T (P\mathcal{L} \otimes K_1) r - \beta \xi^T P\mathcal{L} \tilde{h}\left((Z_n \mathcal{L} \otimes K_2)x\right). \end{aligned} \tag{35}$$

Consider the first term in (35). It follows from the Schwartz inequality that

$$\xi^T P\mathcal{L} u^r \leq \|\xi\|_2 \|P\mathcal{L}\|_2 \|u^r\|_2 \leq n^{\frac{3}{2}} \sigma_{\max}(P\mathcal{L}) \bar{u}^r. \tag{36}$$

Similarly, we have $\xi^T (P\mathcal{L} \otimes K_1) r \leq n^{\frac{3}{2}} \sigma_{\max}\ (P\mathcal{L}) \sigma_{\max}\ (K_1) \bar{r}$. Next, consider the second term in (35). Due to the definitions of K_1 and K_2 in (21), we know that $K_2(A + BK_1) = -K_2$. Recall the definition of ξ in (33), it follows that $\xi = \tilde{h}((P\mathcal{L} \otimes K_2)x)$ and thus ξ_i and the ith element of $(P\mathcal{L} \otimes K_2)x$ have the same sign. Therefore $\xi^T (P\mathcal{L} \otimes K_2 (A + BK_1))x = -\xi^T (P\mathcal{L} \otimes K_2)x \leq 0$.

Consider the fourth term in (35). Define $\eta \in \mathbb{R}^n = \tilde{h}((Z_n \mathcal{L} \otimes K_2)x)$ and $\psi \in \mathbb{R}^n = \eta - \xi$, then we have

$$\begin{aligned} & -\beta \xi^T P\mathcal{L} \tilde{h}\left((Z_n \mathcal{L} \otimes K_2)x\right) = -\beta \xi^T P\mathcal{L} \eta \\ &= -\beta \xi^T P\mathcal{L} (\xi + \psi) = -\frac{\beta}{2} \xi^T P\mathcal{L} \xi - \frac{\beta}{2} \xi^T \mathcal{L}^T P \xi - \beta \xi^T P\mathcal{L} \psi \\ &= -\frac{\beta}{2} \xi^T \bar{\mathcal{L}} \xi - \beta \xi^T P\mathcal{L} \psi \leq -\frac{\beta}{2} \xi^T \bar{\mathcal{L}} \xi + \beta n^{\frac{1}{2}} \sigma_{\max}(P\mathcal{L}) \|\psi\|_2. \end{aligned}$$

Here $\bar{\mathcal{L}}$ is defined in Lemma 1. In addition, if there exists $i \in \mathcal{V}$ such that $\left| p_i \sum_{j \in \mathcal{N}_i} a_{ij} K_2 (x_i - x_j) \right| \geq \epsilon$, there is at least a 1 and a negative number or a -1 and a positive number in the elements of vector ξ. Therefore, there must exist a positive vector a such that $a^T \xi = 0$. It follows from Lemma 1 that if there exists $i \in \mathcal{V}$ such that $\left| p_i \sum_{j \in \mathcal{N}_i} a_{ij} K_2 (x_i - x_j) \right| \geq \epsilon$, then $\|\xi\|_2^2 \geq 1$ and

$$\tfrac{\beta}{2} \xi^T \bar{\mathcal{L}} \xi > \tfrac{\beta}{2n} \lambda_2(\bar{\mathcal{L}}) \|\xi\|_2^2 \geq \tfrac{\beta}{2n} \lambda_2(\bar{\mathcal{L}}). \tag{37}$$

Then we have if there exists $i \in \mathcal{V}$ such that $\left| p_i \sum_{j \in \mathcal{N}_i} a_{ij} K_2 (x_i - x_j) \right| \geq \epsilon$,

$$\begin{aligned} \dot{V}(x) &\leq -\xi^T (P\mathcal{L} \otimes K_2)\, x + n^{\frac{3}{2}} \sigma_{\max}(P\mathcal{L}) \left(\bar{u}^r + \sigma_{\max}(K_1) \bar{r} \right) \\ &\quad - \frac{\beta}{2n} \lambda_2(\bar{\mathcal{L}}) + \beta n^{\frac{1}{2}} \sigma_{\max}(P\mathcal{L}) \|\psi\|_2 \\ &< -\xi^T (P\mathcal{L} \otimes K_2)\, x + \beta n^{\frac{1}{2}} \sigma_{\max}(P\mathcal{L}) \|\psi\|_2. \end{aligned} \tag{38}$$

Here the last inequality is under the gain condition (22). Next, we show that all $\left| p_i \sum_{j \in \mathcal{N}_i} a_{ij} K_2 (x_i - x_j) \right|$ remains in a bounded region. According to the definition of $V(x)$, we know that the existence of $i \in \mathcal{V}$ such that $\left| p_i \sum_{j \in \mathcal{N}_i} a_{ij} K_2 (x_i - x_j) \right| \to \infty$ is a necessary and sufficient condition of $V(x) \to \infty$. If there exists $i \in \mathcal{V}$ such that $\left| p_i \sum_{j \in \mathcal{N}_i} a_{ij} K_2 (x_i - x_j) \right| \to \infty$, it will also hold that $\xi^T (P\mathcal{L} \otimes K_2)x \to \infty$ and $\dot{V}(x) < 0$, which will result in a bounded $V(x)$ and thus all bounded $\left| p_i \sum_{j \in \mathcal{N}_i} a_{ij} K_2 (x_i - x_j) \right|$. It follows that all $\left| p_i \sum_{j \in \mathcal{N}_i} a_{ij} K_2 (x_i - x_j) \right|$ and thus $V(x)$ remain bounded. According to the gain condition in (22), we know that there exists a positive number ς such that $\dot{V} \leq \varsigma - \xi^T (P\mathcal{L} \otimes K_2)x + \beta n^{\frac{1}{2}} \sigma_{\max}(P\mathcal{L}) \|\psi\|_2$ if there exists $i \in \mathcal{V}$ such that $\left| p_i \sum_{j \in \mathcal{N}_i} a_{ij} K_2 (x_i - x_j) \right| \geq \epsilon$. It follows from Lemma 2 that $\psi \to \mathbf{0}_n$ as $t \to \infty$, therefore there must exist a time T_3 such that $\beta n^{\frac{1}{2}} \sigma_{\max}(P\mathcal{L}) \|\psi\|_2 < \varsigma$ for all $t \geq T_3$. Then we have $\dot{V}(x) < 0$ for all $t \geq T_3$ if there exists $i \in \mathcal{V}$ such that $\left| p_i \sum_{j \in \mathcal{N}_i} a_{ij} K_2 (x_i - x_j) \right| \geq \epsilon$. Then we can get the conclusion that $\|(P\mathcal{L} \otimes K_2)x(t)\|_1$ is bounded for all $t \geq 0$, and all $\left| p_i \sum_{j \in \mathcal{N}_i} a_{ij} K_2 (x_i - x_j) \right| \leq \epsilon$ as $t \to \infty$ and thus

$$\limsup_{t \to \infty} \|(P\mathcal{L} \otimes K_2)x(t)\|_1 \leq n\epsilon. \tag{39}$$

Note that Assumption 1 holds. It then follows from Lemma 4 that

$$\begin{aligned} & \limsup_{t \to \infty} \|(\Omega \otimes I_m)x(t)\|_1 \\ &= \limsup_{t \to \infty} \sum_{k=1}^{n} \left\| x_k(t) - \frac{1}{n} \sum_{j=1}^{n} x_j(t) \right\|_1 \\ &\leq \|(P\mathcal{L})^+ \otimes I_m\|_1 n\epsilon 2^{m-1} \sum_{i=1}^{m} i! \left(\prod_{j=0}^{i-1} C_j^{\lfloor j/2 \rfloor} \right). \end{aligned} \tag{40}$$

In what follows, the term $(\mathbf{1}_n^T \otimes I_m)(x - r)$ is analyzed. The derivative of $(\mathbf{1}_n^T \otimes I_m)(x - r)$ can be calculated as follows:

$$\begin{aligned} & \frac{\mathrm{d}\left((\mathbf{1}_n^T \otimes I_m)(x - r)\right)}{\mathrm{d}t} = (\mathbf{1}_n^T \otimes I_m)(\dot{x} - \dot{r}) = (\mathbf{1}_n^T \otimes I_m) \\ &\quad \times \Big((I_n \otimes (A + BK_1))\, (x - r) - \beta (I_n \otimes B) \tilde{h}((Z_n \mathcal{L} \otimes K_2)x) \Big) \\ &= \left(\mathbf{1}_n^T \otimes (A + BK_1) \right) (x - r) - \beta (\mathbf{1}_n^T \otimes B) \tilde{h}\left((Z_n \mathcal{L} \otimes K_2)x\right). \end{aligned} \tag{41}$$

It follows from the definitions of K_1 and K_2 that $\det(\lambda I_m - (A + BK_1)) = \begin{bmatrix} \lambda & -1 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \lambda & -1 \\ C_m^0 & C_m^1 & C_m^2 & \cdots & \lambda + C_m^{m-1} \end{bmatrix} = (\lambda + 1)^m$, which indicates that the eigenvalues of $A + BK_1$ all have negative real parts. Here $\det(\cdot)$ denotes the determinant of a matrix. Define the variable $S = (\mathbf{1}_n^T \otimes I_m)(x - r)$, then we can rewrite (41) as

$$\dot{S} = (I_n \otimes (A + BK_1))\, S - \beta (\mathbf{1}_n^T \otimes B) \tilde{h}\left((Z_n \mathcal{L} \otimes K_2)x\right). \tag{42}$$

Then we can use the input-to-state stability to analyze the system (42) by treating the term $\beta (\mathbf{1}_n^T \otimes B) \tilde{h}((Z_n \mathcal{L} \otimes K_2)x)$ as the input and S as the state. From Lemma 2, we know that $Z_n(t) \to P$ as $t \to \infty$, leading to $\tilde{h}((Z_n(t)\mathcal{L} \otimes K_2)x(t)) \to \tilde{h}((P\mathcal{L} \otimes K_2)x(t))$. It has been proved that $|p_i \sum_{j \in \mathcal{N}_i} a_{ij} K_2 (x_i(t) - x_j(t))| \leq \epsilon, \forall i \in \mathcal{V}$, is reached as $t \to \infty$, therefore we have $\tilde{h}((Z_n(t)\mathcal{L} \otimes K_2)x(t)) \to \frac{((P\mathcal{L} \otimes K_2)x(t))}{\epsilon}$ as $t \to \infty$. It thus follows that $\beta (\mathbf{1}_n^T \otimes B) \tilde{h}((Z_n(t)\mathcal{L} \otimes K_2)x(t)) \to \beta \frac{(\mathbf{1}_n^T P\mathcal{L} \otimes BK_2)x(t)}{\epsilon} = \mathbf{0}_m$ as $t \to \infty$, which gives $S \to \mathbf{0}_m$ and thus $\lim_{t \to \infty} \sum_{i=1}^{n} (x_i(t) - r_i(t)) \to \mathbf{0}_m$ as $t \to \infty$. It follows from (40) that

$$\limsup_{t\to\infty} \sum_{k=1}^{n} \left\| x_k(t) - \frac{1}{n}\sum_{j=1}^{n} r_j(t) \right\|_1$$
$$\leq \|(P\mathcal{L})^+ \otimes I_m\|_1 2^{m-1} n\epsilon \sum_{i=1}^{m} i! \left(\prod_{j=0}^{i-1} C_j^{\lfloor j/2 \rfloor} \right). \quad (43)$$

■

Remark 6: Both the linear and nonlinear algorithms have their unique features and advantages while with tradeoffs. The advantage of the linear algorithm (4) is that it is smooth and linear and hence is easier to implement in practice. However, the tradeoff is that the tracking error is zero only for reference signals whose acceleration deviations approach zero and bounded for signals with bounded acceleration deviations. On the other hand, the advantages of the nonlinear algorithm (20) are that it can achieve DAT with relatively small tracking errors for reference signals whose states and velocities are both bounded (it follows from (43) that the tracking error can be arbitrarily small by adjusting ϵ) and it can deal with more general high-order integrator systems. But the tradeoff is that the nonlinear algorithm may be more "expensive" to implement than the linear one in practice.

V. Conclusion

In this article, we have studied distributed average tracking in weight-unbalanced directed graphs, which attempts to push a set of networked agents to track the average of the locally available time-varying reference signals, where each agent can only receive information from its neighbors. We first propose a linear algorithm for single-integrator dynamics. We have shown that the tracking error is upper bounded if the reference signals have bounded acceleration deviations. We also investigate a nonlinear algorithm for high-order integrator dynamics, which guarantees that distributed average tracking can be achieved with arbitrarily small tracking errors if the reference signals and their velocities are all bounded, and the control gain is properly chosen. Future works include the extension to nonlinear agent models, discrete-time multiagent systems and stochastic reference signals.

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