Achievable rate-region for 3−User Classical-Quantum Interference Channel using Structured Codes

Touheed Anwar Atif
University of Michigan, USA

Arun Padakandla
University of Tennessee, USA

S. Sandeep Pradhan
University of Michigan, USA

***Abstract*—We consider the problem of characterizing an inner bound to the capacity region of a $3-$user classical-quantum interference channel ($3-$CQIC). The best known coding scheme for communicating over CQICs is based on unstructured random codes and employs the techniques of message splitting and superposition coding. For classical $3-$user interference channels (ICs), it has been proven that coding techniques based on coset codes - codes possessing algebraic closure properties - strictly outperform all coding techniques based on unstructured codes. In this work, we develop analogous techniques based on coset codes for $3$to$1-$CQICs - a subclass of $3-$user CQICs. We analyze its performance and derive a new inner bound to the capacity region of $3$to$1-$CQICs that subsume the current known largest and strictly enlarges the same for identified examples.**

I. Introduction

We consider the scenario of communicating over a $3-$user classical-quantum interference channel ($3-$CQIC) (Fig. 1). We undertake a Shannon-theoretic study for characterizing an inner bound to its capacity region. The current known coding schemes for CQICs [1]–[4] are based on unstructured codes. In this work, we propose a new coding scheme for a $3-$CQIC based on *nested coset codes* (NCCs) - codes possessing algebraic structure. Analyzing its performance, we derive a new inner bound (Sec. III) to the capacity region of 3to$1-$CQIC - a sub-class of $3-$CQICs. The inner bound is proven to subsume any current known inner bounds based on unstructured codes. Furthermore, we identify examples of 3to$1-$CQICs for which the derived inner bound is strictly larger. These findings are a first step towards characterizing a new inner bound to the capacity region of a general $3-$CQIC.

The current approach of characterizing the performance limits of CQ channels is based on unstructured codes, which remained for several decades the de facto ensemble of codes for information-theoretic study of any classical channels. Inspired by the work in [5] and followed by findings in a multitude of network communication scenarios [6]–[11], it has been analytically proven that coding schemes designed using codes endowed with algebraic closure properties can strictly outperform all known unstructured coding schemes.

The goal of this work is to build on this and enhance the current known coding schemes in the context of CQ channels. Our experience with classical channels suggests that a first step toward this is to design and analyze coding schemes for basic building block channels. Indeed, the ensemble of NCCs studied in the simple context of point-to-point (PTP) channels form an important element of this work [12]. On the other hand, the mathematical complexity of analyzing CQ channels makes it challenging to generalize even well known coding schemes to the CQ setting. In the light of this, our work maybe

This work was supported by NSF grant CCF-1717523.

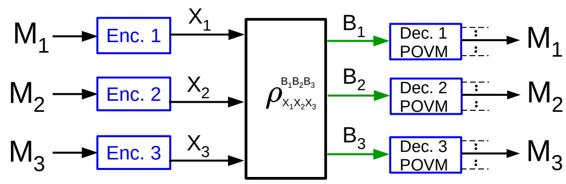

Fig. 1. Communication over $3-$CQIC.

viewed as a first step in designing new coding schemes for network CQ channels based on coset codes.

In the context of CQICs, the focus of current research is on $2-$user. There has been considerable effort in [1]–[4], [13], [14] at proving the achievability of the Han-Kobayashi rate-region (CHK) [15] for $2-$user ICs. Analogous to these, one can leverage all known coding techniques - message splitting, superposition coding, Marton's binning - and derive an achievable rate region for a $3-$CQIC. See discussion in [10, Sec. III]. This rate region, henceforth referred to as the $\mathscr{U}\mathscr{S}\mathscr{B}-$region contains the largest current known inner bound for any $3-$CQIC. In this work, we focus on 3to$1-$CQICs (Defn. 3) - a subclass of $3-$IC in which only one receiver (Rx) experiences interference. We propose a coding scheme based on NCCs and derive an inner bound for this sub-class that subsumes the $\mathscr{U}\mathscr{S}\mathscr{B}-$region in general, and strictly larger for identified examples (see Ex. 2).

To study coset code based coding schemes for basic building block channels, and for pedagogical reasons, we present our findings in two steps. In the first step (Thm. 1), we demonstrate a construction of a n-letter POVM that can simultaneously decode (i) the correct message and (ii) a bivariate interference component. This first step enables us study performance of NCCs for CQ-PTP channels (Sec. IV) and simultaneous decoding of unstructured and NCC codes (Sec. III). Our analysis of this simultaneous decoder builds on the technique proposed in [16]. In the next step, we leverage these building blocks and employ a multi-terminal simultaneous decoder [14] to derive a new achievable rate region for 3to$1-$CQICs.

II. Preliminaries and Problem Statement

For $n \in \mathbb{N}$, $[n] \triangleq \{1, \cdots, n\}$. We let an underline denote an appropriate aggregation of objects. For example, $\underline{\mathcal{X}} \triangleq \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$, $\underline{x} \triangleq (x_1, x_2, x_3) \in \underline{\mathcal{X}}$ and in regards to Hilbert spaces $\mathcal{H}_{Y_i} : i \in [3]$, we let $\mathcal{H}_{\underline{Y}} \triangleq \otimes_{i=1}^{3} \mathcal{H}_{Y_i}$.

Consider a (generic) $3-$*CQIC* $(\rho_{\underline{x}} \in \mathcal{D}(\mathcal{H}_{\underline{Y}}) : \underline{x} \in \underline{\mathcal{X}}, \kappa_j : j \in [3])$ specified through (i) three finite sets $\mathcal{X}_j : j \in [3]$, (ii) three Hilbert spaces $\mathcal{H}_{Y_j} : j \in [3]$, (iii) a collection $(\rho_{\underline{x}} \in \mathcal{D}(\mathcal{H}_{\underline{Y}}) : \underline{x} \in \underline{\mathcal{X}})$ and (iv) three cost functions $\kappa_j : \mathcal{X}_j \rightarrow [0, \infty) : j \in [3]$. The cost function is assumed to be additive, i.e., cost expended by encoder j in preparing

2021 IEEE International Symposium on Information Theory (ISIT) | 978-1-5386-8209-8/21/$31.00 ©2021 IEEE | DOI: 10.1109/ISIT45174.2021.9518095

the state $\otimes_{t=1}^{n}\rho_{x_{1t}x_{2t}x_{3t}}$ is $\overline{\kappa}_j^n \triangleq \frac{1}{n}\sum_{t=1}^{n}\kappa_j(x_{jt})$. Reliable communication on a $3-$CQIC entails identifying a code.

Defn. 1. *A $3-$CQIC code $c=(n,\underline{\mathcal{M}},\underline{e},\underline{\lambda})$ of B-L n consists of three (i) message index sets $[\mathcal{M}_j]: j\in[3]$, (ii) encoder maps $e_j:[\mathcal{M}_j]\rightarrow\mathcal{X}_j^n: j\in[3]$ and (iii) POVMs $\lambda_j\triangleq\{\lambda_{j,m}:\mathcal{H}_j^{\otimes n}\rightarrow\mathcal{H}_j^{\otimes n}: m\in[\mathcal{M}_j]\}: j\in[3]$. The average probability of error of the $3-$CQIC code $(n,\underline{\mathcal{M}},\underline{e},\lambda^{[3]})$ is*

$$\overline{\xi}(\underline{e},\underline{\lambda})\triangleq 1-\frac{1}{\mathcal{M}_1\mathcal{M}_2\mathcal{M}_3}\sum_{\underline{m}\in\underline{\mathcal{M}}}\operatorname{tr}\Big(\lambda_{\underline{m}}\rho_{c,\underline{m}}^{\otimes n}\Big).$$

where $\lambda_{\underline{m}}\triangleq\otimes_{j=1}^{3}\lambda_{j,m_j}$, $\rho_{c,\underline{m}}^{\otimes n}\triangleq\otimes_{t=1}^{n}\rho_{x_{1t}x_{2t}x_{3t}}$ where $(x_{jt}: 1\leq t\leq n)=x_j^n(m_j)\triangleq e_j(m_j)$ for $j\in[3]$. Average cost per symbol of transmitting message $\underline{m}\in\underline{\mathcal{M}}\in\underline{\tau}(\underline{e}|\underline{m})\triangleq\big(\overline{\kappa}_j^n(e_j(m_j)):j\in[3]\big)$ and the average cost per symbol of $3-$CQIC code is $\underline{\tau}(\underline{e})\triangleq\frac{1}{|\underline{\mathcal{M}}|}\sum_{\underline{m}\in\underline{\mathcal{M}}}\underline{\tau}(\underline{e}|\underline{m})$.

Defn. 2. *A rate-cost vector $(R_1,R_2,R_3,\tau_1,\tau_2,\tau_3)\in[0,\infty)^6$ is achievable if there exists a sequence of $3-$CQIC code $(n,\underline{\mathcal{M}}^{(n)},\underline{e}^{(n)},\underline{\lambda}^{(n)})$ for which $\lim_{n\rightarrow\infty}\overline{\xi}(\underline{e}^{(n)},\underline{\lambda}^{(n)})=0$,*
$\lim_{n\rightarrow\infty}n^{-1}\log\mathcal{M}_j^{(n)}=R_j$, and $\lim_{n\rightarrow\infty}\underline{\tau}(\underline{e})_j\leq\tau_j: j\in[3]$.

The capacity region $\mathcal{C}(\rho_{\underline{x}}:\underline{x}\in\mathcal{X})$ of the $3-$CQIC $(\rho_{\underline{x}}\in\mathcal{D}(\mathcal{H}_{\underline{Y}}):\underline{x}\in\underline{\mathcal{X}})$ is the set of all achievable rate-cost vectors. Now, we define the sub-class of 3to1$-$CQICs.

Defn. 3. *A $3-$CQIC $(\rho_{\underline{x}}\in\mathcal{D}(\mathcal{H}_{\underline{Y}}):\underline{x}\in\underline{\mathcal{X}})$ is a 3to1$-$CQIC if (i) for every $\Lambda\in\mathcal{P}(\mathcal{H}_{Y_2})$, $\Gamma\in\mathcal{P}(\mathcal{H}_{Y_3})$, $\operatorname{tr}((I\otimes\Lambda\otimes I)\rho_{x_1x_2x_3})=\operatorname{tr}((I\otimes\Lambda\otimes I)\rho_{\hat{x}_1\hat{x}_2\hat{x}_3})$ for every $\underline{x},\hat{\underline{x}}\in\underline{\mathcal{X}}$ satisfying $x_2=\hat{x}_2$, and (ii) $\operatorname{tr}((I\otimes I\otimes\Gamma)\rho_{x_1x_2x_3})=\operatorname{tr}((I\otimes I\otimes\Gamma)\rho_{\hat{x}_1\hat{x}_2\hat{x}_3})$ for every $\underline{x},\hat{\underline{x}}\in\underline{\mathcal{X}}$ satisfying $x_3=\hat{x}_3$.*

A. Illustration of the Central Idea

The goal here is to demonstrate the utility of algebraic closure in coding schemes for $3-$ICs. While, we state Ex. 1 in the context of 3to1$-$CQICs, we discus in the context of a classical 3to1$-$IC. The latter provides an exposition on the utility of algebraic closure in network scenarios.

Ex. 1. *Let $\mathcal{X}_j=\mathcal{X}=\{0,1\}$, $\mathcal{H}_j=\mathcal{H}$, $\sigma_b^{(j)}\in\mathcal{D}(\mathcal{H})$ for $j\in[3]$ and $b\in\mathcal{X}$. For $\underline{x}\in\underline{\mathcal{X}}$, let $\rho_{\underline{x}}\triangleq\sigma_{x_1\oplus x_2\oplus x_3}^{(1)}\otimes\sigma_{x_2}^{(2)}\otimes\sigma_{x_3}^{(3)}$. For $x\in\{0,1\}$, we let $\kappa_1(x)=x$ and $\kappa_k(x)=0$ for $k=2,3$.*

Let $\mathcal{H}=\mathbb{C}^2$, $\sigma_b(\eta)\triangleq(1-\eta)\left|b\right\rangle\left\langle b\right|+\eta\left|1-b\right\rangle\left\langle 1-b\right|$ for $b\in\mathcal{X}$, $\eta\in[0,1]$. Let $\sigma_b^{(1)}\triangleq\sigma_b(\delta_1)$ and $\sigma_b^{(2)}\triangleq\sigma_b^{(3)}\triangleq\sigma_b(\delta)$ for $b\in\mathcal{X}$. In addition, let $\tau\in(0,\frac{1}{2})$ specify a Hamming cost constraint on Tx 1's input. With this choice, one identifies the above example with a 3to1$-$IC $Y_1=X_1\oplus X_2\oplus X_3\oplus N_1$, $Y_k=X_k\oplus N_k: k=2,3$ with $N_1\sim\text{Ber}(\delta_1)$, $N_k\sim\text{Ber}(\delta)$ $k=2,3$ being independent. Tx $k\in\{2,3\}$ splits its information into U_k,X_k. Rx 1 decodes U_2,U_3,X_1, while Rx $k\in\{2,3\}$ decodes U_k,X_k. So long as $H(U_k|X_k)>0$ for either $k\in\{2,3\}$, it can be shown that $H(X_2\oplus X_3|U_2,U_3)>0$ implying Tx-Rx 1 cannot achieve $h_b(\delta_1*\tau)-h_b(\delta_1)$ - its interference free cost constrained capacity. If $h_b(\delta_1*\tau)-h_b(\delta_1)+2(1-h_b(\delta))>1-h_b(\delta_1)$, it can be shown that $H(U_k|X_k)>0$ for either $k\in\{2,3\}$ precluding Tx-Rx 1 achieving a rate $h_b(\delta_1*\tau)-h_b(\delta_1)$ using unstructured coding. Suppose users $2,3$ employ codes of rate $1-h_b(\delta)$ that are cosets of the *same linear code*, then the above condition does not preclude Tx-Rx 1 from achieving a rate $h_b(\delta_1*\tau)-h_b(\delta_1)$, so long as $\tau*\delta<\delta$, even if $1+h_b(\tau*\delta_1)>2h_b(\delta)$. Hence, for this 3to1$-$IC, if $h_b(\delta_1*\tau)-h_b(\delta_1)+2(1-h_b(\delta))>1-h_b(\delta_1)$ and $\tau*\delta<\delta<\frac{1}{2}$ hold, then coset codes are strictly more efficient than unstructured codes.

III. Rate region using Coset Codes for 3to1-CQIC

In this section we consider the above described 3to1-CQIC and provide an achievable rate-region.

Theorem 1. *Given a 3to1-CQIC $(\rho_{\underline{x}}\in\mathcal{D}(\mathcal{H}_{\underline{Y}}):\underline{x}\in\underline{\mathcal{X}},\kappa_j: j\in[3])$ and a PMF $p_{V_2V_3X_1X_2X_3}=p_{X_1}p_{V_2X_2}p_{V_3X_3}$ on $\mathcal{V}_2\times\mathcal{V}_3\times\mathcal{X}_2\times\mathcal{X}_3$ where $\mathcal{V}_2=\mathcal{V}_3=\mathcal{F}_q$, a rate-cost triple $(R_1,R_2,R_2,\tau_1,\tau_2,\tau_3)$ is achievable if it satisfies the following*

$$R_1\leq I(Y_1;X_1|U)_{\sigma_1},\quad R_j\leq I(Y_j;V_j)_{\sigma_2},$$
$$R_j\leq\min\{H(V_2),H(V_3)\}-H(U)+I(Y_1;U|X_1)_{\sigma_1},$$
$$R_1+R_j\leq\min\{H(V_2),H(V_3)\}-H(U)+I(Y_1;V_1U)_{\sigma_1},$$

for $j=2,3$, and $\mathbb{E}[\kappa_j(X_j)]\leq\tau_j: j\in[3]$, where

$$\sigma_1^{\underline{Y}}\triangleq\sum_{x_1\in\mathcal{X}_1,u\in\mathcal{F}_q}p_{X_1}(x_1)p_U(u)\rho_{x_1,u}^{\underline{Y}}\otimes|x_1\rangle\langle x_1|\otimes|u\rangle\langle u|,$$
$$\rho_{x_1,u}^{\underline{Y}}\triangleq\sum_{v_2,v_3}\sum_{x_2,x_3}p_{V_2,V_3,X_2,X_3|U}(v_2,v_3,x_2,x_3|u)\rho_{\underline{x}}^{\underline{Y}}$$
$$\sigma_2=\sum_{v_1,v_2,v_3}p_{\underline{X}V_2V_3}(\underline{x},v_2,v_3)\rho_{\underline{x}}^{\underline{Y}}\otimes|v_2\rangle\langle v_2|\otimes|v_3\rangle\langle v_3|,$$

for $U\triangleq V_2\oplus V_3$, and $\{|v_2\rangle\}$, $\{|v_3\rangle\}$ as some basis on $\mathcal{H}_Y$.

Ex. 2. *Let $\mathcal{X}_j=\mathcal{X}=\{0,1\}$, $\mathcal{H}_j=\mathbb{C}^2$ and*

$$\sigma_0=\begin{bmatrix}2/3 & 0\\ 0 & 1/3\end{bmatrix},\ \text{and}\ \sigma_1=\begin{bmatrix}1/2 & 1/6\\ 1/6 & 1/2\end{bmatrix}.$$

Let $$\rho_{\underline{x}}\triangleq[(1-\delta_1)\sigma_{x_1\oplus x_2\oplus x_3}+\delta_1\sigma_{x_1\oplus x_2\oplus x_3\oplus 1}]\otimes[(1-\delta)\sigma_{x_2}+\delta\sigma_{x_2\oplus 1}]\otimes[(1-\delta)\sigma_{x_3}+\delta\sigma_{x_3\oplus 1}],$$

for $\underline{x}\in\underline{\mathcal{X}}$, where N_1, N_2 and N_3 are mutually independent Bernoulli random variables with biases δ_1,δ and δ, respectively. We let $\delta_1,\delta\in(0,0.5)$. For $x\in\{0,1\}$, we let $\kappa_1(x)=x$ and $\kappa_k(x)=0$ for $k=2,3$. Let $\rho(p):=p\sigma_0+(1-p)\sigma_1$. Note that $\rho(p)$ and $\rho(1-p)$ do not commute except for $p=0.5$. It can be checked that $S(\rho(p))$ is a symmetric concave function of $p\in(0,1)$. Consider the case when $\tau*\delta_1\leq\delta$. Using NCC, the three users can achieve their PTP capacities simultaneously: $S(\rho(\tau*\delta_1))-S(\rho(\delta_1))$, $S(\rho(0.5))-S(\rho(\delta))$, and $S(\rho(0.5))-S(\rho(\delta))$, respectively. These correspond to the rates given by $I(X_1;B_1|X_2\oplus X_3)$, $I(X_2;B_2)$, and $I(X_3;B_3)$. One can show that if $S(\rho(\tau*\delta_1))-S(\rho(\delta_1))+2(S(\rho(0.5))-S(\rho(\delta)))>S(\rho(0.5))-S(\rho(\delta_1))$, then using unstructured codes, all three users cannot achieve their respective capacities simultaneously. This condition is equivalent to the condition: $S(\rho(\tau*\delta_1))+S(\rho(0.5))>2S(\rho(\delta))$. Hence by choosing $\tau*\delta_1=\delta$, and $\delta<0.5$, we see that NCC-based coding scheme enables all users achieve their respective capacities simultaneously, while this is not possible in unstructured coding scheme.

Proof. We divide the proof into three parts entailing the encoding, decoding and error analysis techniques.

A. Encoding Technique

Consider a PMF $p_{V_2V_3\underline{X}}$ on $\mathcal{V}_2\times\mathcal{V}_3\times\underline{\mathcal{X}}$ with $\mathcal{V}_2=\mathcal{V}_3=\mathcal{F}_q$, and choose n and $R_j : j=[3]$ as non-negative integers. For encoder 1, we use the conventional random coding strategy and construct a codebook $\mathcal{C}_1 \triangleq \{x_1(m_1) : m_1 \in [2^{nR_1}]\}$ on $\mathcal{X}_1$ using the marginal PMF $p^n_{X_1}$. Let $e_1(m_1) \triangleq x_1(m_1) : m_1 \in [2^{nR_1}]$ denote this encoding map. However, to construct the codebooks for encoders 2 and 3, we employ the nested coset codes based technique. Since, the structure and encoding rule are identical for these two encoders, we describe it using a generic index $j \in \{2,3\}$. Let $e_j : \mathcal{F}_q \rightarrow \mathcal{X}^n_j : j=1,2$ denote the encoding maps. We define an NCC as follows.

Defn. 4. *An $(n,k,l,g_I,g_{O/I},b^n)$ NCC built over a finite field $\mathcal{V}=\mathcal{F}_q$ comprises of (i) generator matrices $g_I \in \mathcal{V}^{k\times n}$, $g_{O/I} \in \mathcal{V}^{l\times n}$ (ii) a dither/bias vector b^n, an encoder map $e:\mathcal{V}^l \rightarrow \mathcal{V}^k$. We let $v^n(a,m) = ag_I \oplus_q mg_{O/I} \oplus_q b^n : (a,m) \in \mathcal{V}^k \times \mathcal{V}^l$ denote elements in its range space.*

Consider two NCCs with parameters $(n,k,l,g_I,g_{O/I},b^n_j) : j \in \{2,3\}$ defined using the above definition, with their range spaces denoted by $v^n_j(a_j,m_j) : j \in \{2,3\}$, respectively. Note that the choice of g_I and $g_{O/I}$ are identical for the two NCCs. Further, let $\theta_j(m_j) \triangleq \sum_{a_j\in\mathcal{F}^k_q} \mathbb{1}_{\{v^n_j(a_j,m_j)\in\mathcal{T}^{(n)}_\delta(p_{V_j})\}}$. For every message m_j the encoder j looks for a codeword in the coset $v^n_j(a_j,m_j) : a_j \in \mathcal{F}^k_q$ that is typical according to p_{V_j}. If it finds at least one such codeword, one of them, say $v^n_j(\alpha_j(m_j),m_j)$, is chosen randomly and uniformly. Using this codeword, an $e_j(m_j)$ is generated according to $p^n_{X_j|V_j}(\cdot|v^n_j(\alpha_j(m_j),m_j))$ and is transmitted on the CQIC. Otherwise, if it finds none in the coset that is typical according to p_{V_j}, and error is declared. This specifies the encoding rule for the three encoders. Now we describe the decoding rule.

B. Decoding Description

Since we have a $3to1$ CQIC, the decoder employed by the user 1 is naturally different from the other two, so we begin the describing first decoder. Unlike a generic 3 CCIC decoding technique of recovering the three messages, the first decoder constructs his POVM to recover his own message and only a bi-variate function of the two interfering messages. Since, the POVMs here require joint typicality of two messages, we employ the POVM construction similar to [13], but additionally incorporate the capability of decoding a bi-variate function. For this, we equip the decoder 1 with the NCC $(n,k,l,g_I,g_{O/I},b^n)$, with $b^n=b^n_1\oplus b^n_2$, and define $u^n(a,l)$ as its range space. We let

$$\pi_{m_1} \triangleq \pi_{x^n_1(m_1)}, \quad \pi_{a,l} \triangleq \pi_{u^n(a,l)}\mathbb{1}_{\{u^n(a,l)\in\mathcal{T}^{(n)}_\delta(p_U)\}},$$
$$\pi^{a,l}_{m_1} \triangleq \pi_{x^n_1(m_1),u^n(a,l)}\mathbb{1}_{\{(x^n_1(m_1),u^n(a,l))\in\mathcal{T}^{(n)}_\delta(p_{X_1U})\}},$$

denote the conditional typical projectors (as defined in [17, Def. 15.2.4]) with respect to the states $\rho^{Y_1}_{x_1} \triangleq \sum_u p_U(u)\rho^{Y_1}_{x_1,u}$, $\rho^{Y_1}_u \triangleq \sum_{x_1} p_{X_1}(x_1)\rho^{Y_1}_{x_1,u}$ and $\rho^{Y_1}_{x_1,u}$, respectively, where $\rho^{Y_1}_{x_1,u}$ is as defined in the theorem statement. In addition, let $\pi^{Y_1}_\rho$ denote the typical projector with respect to the state $\rho \triangleq \sum_{x_1,u} p_{X_1}(x_1)p_U(u)\rho^{Y_1}_{x_1,u}$. Using these projectors, we define the POVM $\lambda^{Y_1}_{\mathcal{I}_1} \triangleq \{\lambda^{Y_1}_{m_1,a,l}\}$, as

$$\lambda^{Y_1}_{m_1,a,l} \triangleq \Bigg(\sum_{\substack{\hat{m}_1\in\\ [2^{nR_1}]}}\sum_{\substack{\hat{a}\in\mathcal{F}^k_q\\ \hat{l}\in\mathcal{F}^l_q}}\gamma^{\hat{a},\hat{l}}_{\hat{m}_1}\Bigg)^{-1/2}\gamma^{a,l}_{m_1}\Bigg(\sum_{\substack{\hat{m}_1\in\\ [2^{nR_1}]}}\sum_{\substack{\hat{a}\in\mathcal{F}^k_q\\ \hat{l}\in\mathcal{F}^l_q}}\gamma^{\hat{a},\hat{l}}_{\hat{m}_1}\Bigg)^{-1/2},$$

$\lambda_{-1} \triangleq I - \sum_{m_1\in[2^{nR_1}]}\sum_{a\in\mathcal{F}^k_q}\sum_{l\in\mathcal{F}^l_q}\lambda^{Y_1}_{m_1,a,l}$ and $\gamma^{a,l}_{m_1} \triangleq \pi_\rho\pi_m\pi^{a,l}_{m_1}\pi_{m_1}\pi_\rho$. Having described the first decoder, we move on to describing the other two. Since these two decoders are identical, we use a generic variable j to refer to each of these. We define π^j_ρ and $\pi^j_{a_j,m_j}$ as the typical and the conditional typical projectors [17, Def. 15.2.4] with respect to the states $\rho^{Y_j} \triangleq \sum_{v_j} p_{V_j}(v_j)\rho^{Y_j}_{v_j}$ and $\rho^{Y_j}_{v_j}$, respectively. Using this, we construct the POVM $\lambda^{Y_j}_{\mathcal{I}_j} \triangleq \{\lambda^{Y_j}_{m_j,a_j}\}$, for encoder j as

$$\lambda^{Y_j}_{a_j,m_j} \triangleq \Bigg(\sum_{\hat{a_j}\in\mathcal{F}^k_q}\sum_{\hat{m}_j\in\mathcal{F}^l_q}\zeta_{\hat{a_j},\hat{m}_j}\Bigg)^{-1/2}\zeta_{a_j,m_j}\Bigg(\sum_{\hat{a_j}\in\mathcal{F}^k_q}\sum_{\hat{m}_j\in\mathcal{F}^l_q}\zeta_{\hat{a_j},\hat{m}_j}\Bigg)^{-1/2},$$

$\lambda^{Y_j}_{-1} \triangleq I - \sum_{m\in\mathcal{V}^l}\sum_{a\in\mathcal{V}^k}\lambda^{Y_j}_{a,m}$ and $\zeta_{a_j,m_j} \triangleq \pi^j_\rho\pi^j_{a_j,m_j}\pi^j_\rho$. Lastly, we provide the distribution of the random NCC.

Distribution of the Random Coset Code : The objects $g_I \in \mathcal{V}^{k\times n}, g_{O/I} \in \mathcal{V}^{l\times n}, b^n \in \mathcal{V}^n$ and the collection $(a_m \in s(m) : m \in \mathcal{V}^l)$ specify a NCC CQ-PTP code unambiguously. A distribution for a random code is therefore specified through a distribution of these objects. We let upper case letters denote the associated random objects, and obtain

$$\mathcal{P}\left(\begin{matrix} G_I=g_I, G_{O/I}=g_{O/I} \\ B^n_j=b^n_j, \alpha_j(m_j)=a_j : m_j\in\mathcal{F}^l_q\end{matrix}\right) = \prod_{m\in\mathcal{F}^l_q}\frac{q^{-(k+l+1)n}}{\Theta_j(m_j)}.$$

C. Error Analysis

As in a general information theoretic setting, we derive upper bounds on probability of error $\overline{\xi}(\underline{e},\underline{\lambda})$ by averaging over the random code of the first user and the ensemble of nested coset codes used by the other two users. The error probability of this code is given by $\overline{\xi}(\underline{e},\underline{\lambda}) \triangleq 1 - \frac{1}{\mathcal{M}_1\mathcal{M}_2\mathcal{M}_3}\sum_{\underline{m}\in\underline{\mathcal{M}}}\mathrm{tr}\Big(\lambda^{\underline{Y}}_{\underline{m}}\rho^{\otimes n}_{c,\underline{m}}\Big)$. Using the inequality

$$(I-\lambda^{\underline{Y}}_{\underline{m}}) \leq \sum_{i=1}^{3}(I-\lambda^{Y_i}_{m_i})\otimes I^{\underline{Y}\setminus Y_i},$$

from [18], we get $\overline{\xi}(\underline{e},\underline{\lambda}) \leq S_1+S_2+S_3$, where

$$S_j \triangleq \frac{1}{\underline{\mathcal{M}}}\sum_{\underline{m}\in\underline{\mathcal{M}}}\mathrm{tr}\Big(\Big((I-\lambda^{Y_j}_{m_j})\otimes I^{\underline{Y}\setminus Y_i}\Big)\rho^{\otimes n}_{c,\underline{m}}\Big) : j\in[3].$$

Using the definition of $3to1$-CQIC, we can further simplify S_2 and S_3 as $S_j = \frac{1}{\mathcal{M}_j}\sum_{m_j}\mathrm{tr}\Big((I-\lambda^{Y_j}_{m_j})\rho_{e(m_j)}\Big) : j\in\{2,3\}$.

Consider the terms S_2, S_3. Due to the nature of the 3to1−CQIC problem, the terms S_2 and S_3 are identical to a point-to-point (PTP) setup. Therefore, to bound these terms we construct a CQ-PTP problem setup in the following section and employ that as a module in bounding S_2, S_3. The following proposition formalizes this.

Prop. 1. *There exists $\epsilon_S(\delta), \delta_S(\delta)$, such that for all δ and sufficiently large n, we have $\mathbb{E}[S_2 + S_3] \le \epsilon_S(\delta)$, if $R_j \le I(Y_j;V_j)_{\sigma_2} + \delta_S : j = 2, 3$, where $\epsilon_S, \delta_S \searrow 0$ as $\delta \searrow 0$.*

Proof. The proof is provided in Section IV. □

Now, we move on to bounding the term S_1. Let $\mathscr{E} \triangleq \{\theta_1(m_1) = 0 \text{ or } \theta_2(m_2) = 0\}$. By noting that $S_1 \le 1$, we obtain $S_1 \le S_1' + \mathbb{1}_{\mathscr{E}}$, where $S_1' \triangleq S_1 \cdot \mathbb{1}_{\mathscr{E}^c}$. As a first step, we bound the indicator $\mathbb{1}_{\mathscr{E}}$ using the following proposition.

Prop. 2. *There exist $\epsilon_E(\delta), \delta_E(\delta)$, such that for all δ and sufficiently large n, we have $\mathbb{E}_{\mathcal{P}}[\mathscr{E}] \le \epsilon_E(\delta)$, if $\frac{k}{n} \ge \log q - \min\{H(V_1), H(V_2)\} + \delta_E$, where $\epsilon_E, \delta_E \searrow 0$ as $\delta \searrow 0$.*

Proof. The proof follows from [19, App. B]. □

Now considering the term S_1', and using the linearity of trace while ignoring negative terms, we get

$$S_1' \le \frac{1}{\underline{\mathcal{M}}} \sum_{\underline{m} \in \underline{\mathcal{M}}} \operatorname{tr}\Big((I - \lambda^{Y_1}_{m_1,a,l})\pi_{a,l}\rho^{Y_1}_{c,\underline{m}}\pi_{a,l}\Big)\mathbb{1}_{\mathscr{E}^c} + S_{11},$$

where $S_{11} \triangleq \left\|\pi_{a,l}\rho^{Y_1}_{c,\underline{m}}\pi_{a,l} - \rho^{Y_1}_{c,\underline{m}}\right\|_1, \rho^{Y_1}_{c,\underline{m}} \triangleq \operatorname{tr}_{Y_2Y_3}(\rho^{\otimes n}_{c,\underline{m}})$, $a \triangleq \alpha_1(m_1) \oplus \alpha_2(m_2)$, and $l \triangleq m_1 \oplus m_2$ and the inequality also uses $\operatorname{tr}(\lambda\rho) \le \operatorname{tr}(\lambda\sigma) + \|\rho - \sigma\|_1$ which holds for all $0 \le \rho, \sigma, \lambda \le 1$. Let T be any generic term within the summation of the first term in the right hand side of the above equation. This term T can be bounded using the Hayashi-Nagaoka inequality [17] as $T \le 2(1 - T_1) + 4T_2$, where

$$T_1 \triangleq \operatorname{tr}\left(\gamma^{a,l}_{m_1}\pi_{a,l}\rho^{Y_1}_{c,\underline{m}}\pi_{a,l}\right),\ T_2 \triangleq \sum_{\substack{(m_1',a',l') \\ \ne (m_1,a,l)}} \operatorname{tr}\left(\gamma^{a',l'}_{m_1'}\pi_{a,l}\rho^{Y_1}_{c,\underline{m}}\pi_{a,l}\right).$$

The objective now is to proof T_1 is close to one and T_2 is close to zero. As for T_1, consider the following proposition.

Prop. 3. *There exist $\epsilon_{T_1}(\delta), \delta_{T_1}(\delta)$, such that for all sufficiently small δ and sufficiently large n, we have $\mathbb{E}[T_1] \ge 1 - \epsilon_{T_1}(\delta)$, where $\epsilon_{T_1}, \delta_{T_1} \searrow 0$ as $\delta \searrow 0$.*

Proof. Using $\operatorname{tr}(\lambda\rho) \ge \operatorname{tr}(\lambda\sigma) - \|\rho - \sigma\|_1$, we have

$$T_1 \ge \operatorname{tr}\Big(\pi_{a,l}\rho^{Y_1}_{c,\underline{m}}\Big) - \left\|\pi_\rho\rho^{Y_1}_{c,\underline{m}}\pi_\rho - \rho^{Y_1}_{c,\underline{m}}\right\| - \left\|\pi^a_l\rho^{Y_1}_{c,\underline{m}}\pi^a_l - \rho^{Y_1}_{c,\underline{m}}\right\| - \left\|\pi_{m_1}\rho^{Y_1}_{c,\underline{m}}\pi_{m_1} - \rho^{Y_1}_{c,\underline{m}}\right\|.$$

Further, using pinching for non-commutating operators [17], [20] the following is true for a sufficiently large n: $\operatorname{tr}(\pi_{a,l}\rho^{Y_1}_{c,\underline{m}}), \operatorname{tr}(\pi_{m_1}\rho^{Y_1}_{c,\underline{m}}), \operatorname{tr}(\pi^a_l\rho^{Y_1}_{c,\underline{m}}), \operatorname{tr}(\pi_\rho\rho^{Y_1}_{c,\underline{m}}) \ge 1 - \epsilon_p(\delta)$, where $\epsilon_p(\delta) \searrow 0$ as $\delta \searrow 0$ (see [21] for a detailed set of pinching arguments). Using these bounds and the Gentle Measurement Lemma [17], the result follows. □

Now, we move on to bounding the term T_2. Firstly, note that the summation in T_2 can be split into seven different summations based on how many within the triple (m_1', a', l') are equal to (m_1, a, l). However, only three of these seven provide binding constraints on the rate triple (R_1, R_2, R_3). Building on this and by denoting $\kappa_{\underline{m}} \triangleq \pi_{a,l}\rho^{Y_1}_{c,\underline{m}}\pi_{a,l}$, we perform the split $T_2 = T_{21} + T_{22} + T_{23} + T_3$, where

$$T_{22} \triangleq \sum_{m_1' \ne m_1} \operatorname{tr}\left(\gamma^{a,l}_{m_1'}\kappa_{\underline{m}}\right), \quad T_{22} \triangleq \sum_{a' \ne a, l' \ne l} \operatorname{tr}\left(\gamma^{a',l'}_{m_1}\kappa_{\underline{m}}\right),$$
$$T_{23} \triangleq \sum_{\substack{m_1' \ne m_1, \\ a' \ne a, l' \ne l}} \operatorname{tr}\left(\gamma^{a',l'}_{m_1'}\kappa_{\underline{m}}\right).$$

represents the rate constraining (binding) terms, and $T_3 \triangleq T_2 - \sum_{i=1}^3 T_{2i}$. We provide the following set of propositions bounding each of these terms $T_{2i} : i \in [3]$.

Prop. 4. *There exists $\epsilon_{T_{21}}(\delta), \delta_{T_{21}}(\delta)$, such that for all sufficiently small δ and sufficiently large n, we have $\mathbb{E}[T_{21}] \le \epsilon_{T_{21}}(\delta)$ if $R_1 + \frac{2k}{n}\log q \le 2\log q - H(V_1, V_2) + I(Y_1; X_1|U)_{\sigma_1} + \delta_{T_{21}}$, where $\epsilon_{T_{21}}, \delta_{T_{21}} \searrow 0$ as $\delta \searrow 0$.*

Prop. 5. *There exists $\epsilon_{T_{22}}(\delta), \delta_{T_{22}}(\delta)$, such that for all sufficiently small δ and sufficiently large n, we have $\mathbb{E}[T_{22}] \le \epsilon_{T_{22}}(\delta)$ if $\frac{3k+l}{n}\log q \le 3\log q - H(V_1, V_2) - H(U) + I(Y_1; U|X_1)_{\sigma_1} + \delta_{T_{22}}$, where $\epsilon_{T_{22}}, \delta_{T_{22}} \searrow 0$ as $\delta \searrow 0$.*

Prop. 6. *There exists $\epsilon_{T_{23}}(\delta), \delta_{T_{23}}(\delta)$, such that for all sufficiently small δ and sufficiently large n, we have $\mathbb{E}[T_{23}] \le \epsilon_{T_{23}}(\delta)$ if $R_1 + \frac{3k+l}{n}\log q \le 3\log q - H(V_1, V_2) - H(U) + I(Y_1; X_1, U)_{\sigma_1} + \delta_{T_{23}}$, where $\epsilon_{T_{23}}, \delta_{T_{23}} \searrow 0$ as $\delta \searrow 0$.*

Proof. Proof of Props. 4-6 are provided in [21]. □

For the terms in the expression T_3, we do not obtain any new rate constraints, so we bound them in [21]. Now, we provide the result stating NCC codes achieve capacity of a CQ-PTP channel (as discussed in the proof of Proposition 1).

IV. Coset Codes for communicating over CQ-PTP

As discussed in Sec. III, here we shall build and analyze a NCC for a point-to-point CQ channel [17] and employ it as a module for the $3to1$ CQ-IC result. Towards that, we begin by formalizing the definition of a CQ-PTP code.

Defn. 5. *A CQ-PTP code $c_m = (n, \mathcal{I}, e, \lambda_{\mathcal{I}})$ for a CQ-PTP $(\rho_x \in \mathcal{D}(\mathcal{H}_Y) : x \in \mathcal{X})$ consists of (i) an index set $\mathcal{I}$, (ii) and encoder map $e : \mathcal{I} \to \mathcal{X}^n$ and a decoding POVM $\lambda_{\mathcal{I}} = \{\lambda_m \in \mathcal{P}(\mathcal{H}_Y) : m \in \mathcal{I}\}$. For $m \in \mathcal{I}$, we let $\rho^{\otimes n}_{c,m} = \otimes_{i=1}^n \rho_{x_i}$ where $e(m) = x_1 \cdots x_n$.*

Defn. 6. *A CQ-PTP code $(n, \mathcal{I} = \mathcal{F}_q^l, e, \lambda_{\mathcal{I}})$ is an NCC CQ-PTP if there exists an $(n, k, g_I, g_{O/I}, b^n)$ NCC such that $e(m) \in \{u^n(a, m) : a \in \mathcal{F}_q^k\}$ for all $m \in \mathcal{F}_q^l$.*

Theorem 2. *Given a CQ-PTP $(\rho_v \in \mathcal{D}(\mathcal{H}_Y) : v \in \mathcal{F}_q)$ and a PMF p_V on $\mathcal{V}$, $\epsilon > 0$ there exists a CQ-PTP code $c = (n, \mathcal{I} = \mathcal{F}_q^l, e, \lambda_{\mathcal{I}})$ such that (i) $q^{-l}\sum_{\hat{m} \ne [\mathcal{I}] \setminus \{m\}} \operatorname{tr}\big(\lambda_{\hat{m}}\rho^{\otimes n}_{c,m}\big) \le \epsilon$, (ii) $c = (n, \mathcal{I} = \mathcal{F}_q^l, e, \lambda_{\mathcal{I}})$ is a NCC CQ-PTP, (iii) $\frac{k\log_2 q}{n} > \log_2 q - H(V)$ and $\frac{(k+l)\log_2 q}{n} < \log_2 q - H(V) + \chi(\{p_v, \rho_v\})$ for all n sufficiently large.*

Proof. The proof has two parts: (i) error probability analysis for a generic fixed code and (ii) an upper bound on the latter via code randomization.

Upper bound on Error Prob. for a generic fixed code : Consider a generic NCC $(n,k,l,g_I,g_{O/I},b^n)$ with its range space $v^n(a,m)=ag_I\oplus_q mg_{O/I}\oplus_q b^n:(a,m)\in\mathcal{V}^k\times\mathcal{V}^l$ and define a CQ-PTP code $(n,\mathcal{I}=\mathcal{F}_q^l,e,\lambda_{\mathcal{I}})$ that is an NCC CQ-PTP. Towards that, let $\theta(m)\triangleq\sum_{a\in\mathcal{V}^k}\mathbb{1}_{\{v^n(a,m)\in T_\delta^n(p_V)\}}$ and

$$s(m)\triangleq\begin{cases}\{a\in\mathcal{V}^K:v^n(a,m)\in T_\delta^n(p_V)\} & \text{if } \theta(m)\geq 1\\ \{0^k\} & \text{if } \theta(m)=0,\end{cases}$$

for each $m\in\mathcal{V}^l$. For $m\in\mathcal{V}^l$, a predetermined element $a_m\in s(m)$ is chosen. On receiving message $m\in\mathcal{V}^l$, the encoder prepares the state $\rho_m^{\otimes n}\triangleq\rho_{v^n(a_m,m)}^{\otimes n}\triangleq\otimes_{i=1}^n\rho_{v_i(a_m,m)}$ and communicates it. The encoding map e is therefore determined via the collection $(a_m\in s(m):m\in\mathcal{V}^l)$.

Towards specifying the decoding POVM, for any $v^n\in\mathcal{V}^n$, let π_{v^n} be the conditional typical projector as in [17, Defn. 15.2.4] with respect ρ_v and let π_ρ be the (unconditional) typical projector of the state $\rho\triangleq\sum_{v\in\mathcal{V}}p_V(v)\rho_v$ as in [17, Defn. 15.1.3]. For $(a,m)\in\mathcal{V}^k\times\mathcal{V}^l$, we let $\pi_{a,m}\triangleq\pi_{v^n(a,m)}\mathbb{1}_{\{v^n(a,m)\in T_\delta^n(p_V)\}}$. We let $\lambda_{\mathcal{I}}\triangleq\{\sum_{a\in\mathcal{V}^k}\lambda_{a,m}:m\in\mathcal{I}=\mathcal{V}^l,\lambda_{-1}\}$, where

$$\lambda_{a,m}\triangleq\Big(\sum_{\hat{a}\in\mathcal{V}^k}\sum_{\hat{m}\in\mathcal{V}^l}\gamma_{\hat{a},\hat{m}}\Big)^{-1/2}\gamma_{a,m}\Big(\sum_{\tilde{a}\in\mathcal{V}^k}\sum_{\tilde{m}\in\mathcal{V}^l}\gamma_{\tilde{a},\tilde{m}}\Big)^{-1/2},$$

$\lambda_{-1}\triangleq I-\sum_{m\in\mathcal{V}^l}\sum_{a\in\mathcal{V}^k}\lambda_{a,m}$ and $\gamma_{a,m}\triangleq\pi_\rho\pi_{a,m}\pi_\rho$. Since $0\leq\gamma_{a,m}\leq I$, we have $0\leq\lambda_{a,m}\leq I$. It can be verified that $\lambda_{\mathcal{I}}$ is a POVM. We have thus associated an NCC $(n,k,l,g_I,g_{O/I},b^n)$ and a collection $(a_m\in s(m):m\in\mathcal{V}^l)$ with a CQ-PTP code. The error probability of this code is

$$q^{-l}\sum_{m\in\mathcal{I}}\text{tr}((I-\sum_{a\in\mathcal{V}^k}\lambda_{a,m})\rho_m^{\otimes n})\leq q^{-l}\sum_{m\in\mathcal{I}}\text{tr}((I-\lambda_{a_m,m})\rho_m^{\otimes n}).$$

Denoting event $\mathscr{E}=\{\theta(m)<1\}$, a generic term in the RHS of the above sum satisfies

$$\begin{aligned}&\text{tr}((I-\lambda_{a_m,m})\rho_m^{\otimes n})\mathbb{1}_{\mathscr{E}^c}+\text{tr}((I-\lambda_{a_m,m})\rho_m^{\otimes n})\mathbb{1}_{\mathscr{E}}\\ &\leq\ \mathbb{1}_{\mathscr{E}^c}+\sum_{i=1}^3T_{2i},\ \text{where } T_{21}=2\text{tr}((I-\gamma_{a_m,m})\rho_m^{\otimes n})\mathbb{1}_{\mathscr{E}},\end{aligned}$$

$$T_{22}=4\sum_{\hat{a}\neq a_m}\text{tr}(\gamma_{\hat{a},m}\rho_m^{\otimes n})\mathbb{1}_{\mathscr{E}},\ T_{23}=4\sum_{\hat{m}\neq m}\sum_{\tilde{a}}\text{tr}(\gamma_{\tilde{a},\hat{m}}\rho_m^{\otimes n})\mathbb{1}_{\mathscr{E}}$$

and the inequality follows by Hayashi-Nagaoka inequality [22], for $0\leq S\leq I$, and $T\geq 0$, with S and T identified as $\gamma_{a_m,m}$ and $\sum_{\hat{a}\neq a_m}\gamma_{\hat{a},m}+\sum_{\hat{a}\in\mathcal{V}^k}\sum_{\hat{m}\neq m}\gamma_{\hat{a},\hat{m}}$, respectively. Note that S and T satisfy the required hypothesis which can be verified from earlier stated facts.

Distribution of the Random Code : The objects $g_I\in\mathcal{V}^{k\times n},g_{O/I}\in\mathcal{V}^{l\times n},b^n\in\mathcal{V}^n$ and the collection $(a_m\in s(m):m\in\mathcal{V}^l)$ specify a NCC CQ-PTP code unambiguously. Therefore we let upper case letters denote the associated random objects, and obtain

$$\mathcal{P}\begin{pmatrix}G_I=g_I,G_{O/I}=g_{O/I}\\ B^n=b^n,A_m=a_m:m\in S(m)\end{pmatrix}=q^{-(k+l+1)n}\prod_{m\in\mathcal{V}^l}\frac{1}{\Theta(m)}.$$

Using this we analyze the expectation of $\mathscr{E}$ and $T_{2i};i\in[1,3]$. We begin by $\mathbb{E}_{\mathcal{P}}[\mathscr{E}]=\mathcal{P}(\sum_{a\in\mathcal{V}^k}\mathbb{1}_{\{V^n(a,m)\in T_\delta^n(p_V)\}}<1)$. For this, we provide the following proposition.

Prop. 7. *There exist $\epsilon_{T_1}(\delta),\delta_{T_1}(\delta)$, such that for all δ and sufficiently large n, we have $\mathbb{E}_{\mathcal{P}}[\mathscr{E}]\leq\epsilon_{T_1}(\delta)$, if $\frac{k}{n}\geq\log q-H(V)+\delta_S$, where $\epsilon_S,\delta_S\searrow 0$ as $\delta\searrow 0$.*

Proof. The proof follows from Appendix B of [19]. □

We now consider T_{21}. Since this term can be bounded by a using straight-forward extension of the pinching technique described in [17, Def. 15.2.4], we provide its complete details in [21]. We now analyze $\mathbb{E}_{\mathcal{P}}[T_{22}]$. Denoting

$$\mathcal{J}\triangleq\begin{Bmatrix}\Theta(m)\geq 1,V^n(\hat{a},\hat{m})=\hat{x}^n\\ A_m=d,V^n(d,m)=x^n\end{Bmatrix}\subseteq\mathcal{K}\triangleq\begin{Bmatrix}V^n(\hat{a},\hat{m})=\hat{x}^n\\ V^n(d,m)=x^n\end{Bmatrix}\ (1)$$

we perform the following steps.

$$\mathbb{E}_{\mathcal{P}}[T_{22}]=\sum_{d\in\mathcal{V}^k}\sum_{\hat{a}\neq d}\sum_{x^n\in T_\delta^n(p_V)}\sum_{\hat{x}^n\in\mathcal{V}^n}\mathbb{E}\left[\text{tr}(\Gamma_{\hat{a},m}\rho_m^{\otimes n})\mathbb{1}_{\mathcal{J}}\right]$$

where the restriction of the summation x^n to $T_\delta^n(p_V)$ is valid since $S(m)\geq\tau_c>1$ forces the choice $A_m\in S(m)$ such that $V^n(A_m,m)\in T_\delta^n(p_V)$. Going further, we have

$$\mathbb{E}_{\mathcal{P}}[T_{22}]\leq 2^{-n\left[\chi(\{p_V;\rho_v\})+\epsilon_V-2H(p_V)-\frac{2k}{n}\log q+2\log q\right]}\quad(2)$$

We now derive an upper bound on $\mathbb{E}_{\mathcal{P}}[T_{23}]$. We have

$$\mathbb{E}_{\mathcal{P}}[T_{23}]\ \leq\ 2^{-n\left[\chi(\{p_V;\rho_v\})+2\log_2 q-2H(p_V)-\frac{2k+l}{n}\log_2 q+\epsilon_V\right]}.$$

The reader is referred to [21] where detailed arguments are provided bounding each of the terms T_{22} and T_{23}. We have therefore obtained three bounds $\frac{k}{n}>1-\frac{H(p_V)}{\log_2 q}$, $\frac{2k}{n}<2+\frac{\chi(\{p_V;\rho_v\})-2H(p_V)}{\log_2 q}$, $\frac{2k+l}{n}<2+\frac{\chi(\{p_V;\rho_v\})-2H(p_V)}{\log_2 q}$. A rate of $\chi(\{p_V;\rho_v\})-\epsilon$ is achievable by choosing $\frac{k}{n}=1-\frac{H(p_V)}{\log_2 q}+\frac{\epsilon}{2}$, $\frac{l}{n}=\frac{\chi(\{p_V;\rho_v\})-\epsilon\log_2\sqrt{q}}{\log_2 q}$ thus completing the proof. ⊟

V. Rate-region using NCC and message splitting for 3to1− CQIC

Theorem 3. *Given a 3to1-CQIC $(\rho_{\underline{x}}\in\mathcal{D}(\mathcal{H}_{\underline{Y}}):\underline{x}\in\underline{\mathcal{X}})$ and a PMF $p_{U_2U_3V_2V_3X_2X_3}=p_{U_2V_2X_2}p_{U_3V_3X_3}$ on $\mathcal{U}_1\times\mathcal{V}_1\times\mathcal{X}_1\times\mathcal{U}_2\times\mathcal{V}_2\times\mathcal{X}_2$ where $\mathcal{V}_1=\mathcal{V}_2=\mathcal{F}_q$, a rate triple is achievable if it satisfies the following: $R_j\leq I(U_jX_j;Y_j)_{\sigma_j}$,*

$$R_1\leq\min_{j=2,3}\{0,H(U_j)-H(W|Y_1)_{\sigma_1}\}+I(X_1;WY_1)_{\sigma_1}$$

$$R_1+R_j\leq I(X_j;Y_j|U_j)_{\sigma_j}+I(X_1;W,Y_1)_{\sigma_1}+H(U_j)-H(W|Y_1)_{\sigma_1}$$

for $j=2,3$, where

$$\sigma_1^Y\triangleq\sum_{x_1\in\mathcal{X}_1,w\in\mathcal{F}_q}p_{X_1}(x_1)p_W(w)\rho_{x_1,w}^{\underline{Y}}\otimes|x_1\rangle\langle x_1|\otimes|w\rangle\langle w|,$$

$$\rho_{x_1,w}^{\underline{Y}}\triangleq\sum_{\substack{u_2,v_2,x_2\\u_3,v_3,x_3}}p_{V_2,V_3U_2U_3X_2X_3|W}(v_2,v_3,u_2,u_3,x_2,x_3|w)\rho_{\underline{x}}^{\underline{Y}}$$

$$\sigma_2\triangleq\sum_{v_1,v_2,v_3}p_{U_2U_3V_2V_3\underline{X}}(u_2,u_3,v_2,v_3,\underline{x})\rho_{\underline{x}}^{\underline{Y}}\bigotimes_{j=2}^3|u_j,x_j\rangle\langle u_j,x_j|,$$

for $W\triangleq U_2\oplus U_3$, and $\{|u_j\rangle\}$ and $\{|x_j\rangle\}$ as some orthonormal basis on $\mathcal{H}_Y$ for $j=2,3$.

Proof. Steps for the proof are provided in [21]. □

By choosing $W=\phi$, we can recover the $\mathscr{USB}$−rate region from the above inner bound.

References

[1] P. Sen, "Achieving the han-kobayashi inner bound for the quantum interference channel," in 2012 IEEE International Symposium on Information Theory Proceedings. IEEE, 2012, pp. 736–740.

[2] I. Savov, "Network information theory for classical-quantum channels," arXiv preprint arXiv:1208.4188, 2012.

[3] P. Sen, "Inner bounds via simultaneous decoding in quantum network information theory," arXiv preprint arXiv:1806.07276, 2018.

[4] C. Hirche, C. Morgan, and M. M. Wilde, "Polar codes in network quantum information theory," IEEE Transactions on Information Theory, vol. 62, no. 2, pp. 915–924, 2016.

[5] J. Korner and K. Marton, "How to encode the modulo-two sum of binary sources (corresp.)," IEEE Transactions on Information Theory, vol. 25, no. 2, pp. 219–221, 1979.

[6] D. Krithivasan and S. S. Pradhan, "Distributed source coding using abelian group codes: A new achievable rate-distortion region," IEEE Transactions on Information Theory, vol. 57, no. 3, pp. 1495–1519, 2011.

[7] B. Nazer and M. Gastpar, "Computation over multiple-access channels," IEEE Trans. on Info. Th., vol. 53, no. 10, pp. 3498 –3516, oct. 2007.

[8] T. Philosof and R. Zamir, "On the loss of single-letter characterization: The dirty multiple access channel," IEEE Trans. on Info. Th., vol. 55, pp. 2442–2454, June 2009.

[9] A. Jafarian and S. Vishwanath, "Achievable rates for k-user Gaussian interference channels," IEEE Transactions on information theory, vol. 58, no. 7, pp. 4367–4380, 2012.

[10] A. Padakandla, A. G. Sahebi, and S. S. Pradhan, "An achievable rate region for the three-user interference channel based on coset codes," IEEE Transactions on Information Theory, vol. 62, no. 3, pp. 1250–1279, 2016.

[11] A. Padakandla and S. S. Pradhan, "Computing sum of sources over an arbitrary multiple access channel," in 2013 IEEE International Symposium on Information Theory. IEEE, 2013, pp. 2144–2148.

[12] S. S. Pradhan, A. Padakandla, and F. Shirani, "An algebraic and probabilistic framework for network information theory," Foundations and Trends® in Communications and Information Theory, vol. 18, no. 2, pp. 173–379, 2020. [Online]. Available: http://dx.doi.org/10.1561/0100000083

[13] O. Fawzi, P. Hayden, I. Savov, P. Sen, and M. M. Wilde, "Classical communication over a quantum interference channel," IEEE Transactions on Information Theory, vol. 58, no. 6, pp. 3670–3691, 2012.

[14] P. Sen, "A one-shot quantum joint typicality lemma," arXiv preprint arXiv:1806.07278, 2018.

[15] T. Han and K. Kobayashi, "A new achievable rate region for the interference channel," IEEE Transactions on Information Theory, vol. 27, no. 1, pp. 49–60, January 1981.

[16] O. Fawzi, P. Hayden, I. Savov, P. Sen, and M. M. Wilde, "Classical communication over a quantum interference channel," IEEE Transactions on Information Theory, vol. 58, no. 6, pp. 3670–3691, 2012.

[17] M. M. Wilde, Quantum Information Theory, 1st ed. USA: Cambridge University Press, 2013.

[18] A. Abeyesinghe, I. Devetak, P. Hayden, and A. Winter, "The mother of all protocols: Restructuring quantum information's family tree," Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, vol. 465, no. 2108, pp. 2537–2563, 2009.

[19] A. Padakandla and S. S. Pradhan, "An achievable rate region based on coset codes for multiple access channel with states," IEEE Transactions on Information Theory, vol. 63, no. 10, pp. 6393–6415, 2017.

[20] D. Sutter, "Approximate quantum markov chains," in Approximate Quantum Markov Chains. Springer, 2018, pp. 75–100.

[21] T. A. Atif, A. Padakandla, and S. S. Pradhan, "Achievable rate-region for $3-$ user classical-quantum interference channel using structured codes," arXiv preprint arXiv:2103.03978, 2021.

[22] M. Hayashi and H. Nagaoka, "General formulas for capacity of classical-quantum channels," IEEE Transactions on Information Theory, vol. 49, no. 7, pp. 1753–1768, 2003.