Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME XX, NUMBER Y (201Z)
(Print) ISSN 1311-0454
(Electronic) ISSN 1314-2224

RESEARCH PAPER

ANALYSIS AND FAST APPROXIMATION OF A STEADY-STATE SPATIALLY-DEPENDENT DISTRIBUTED-ORDER SPACE-FRACTIONAL DIFFUSION EQUATION

Jinhong Jia [1], Xiangcheng Zheng [2], Hong Wang [3]

Abstract

We prove the wellposedness of a distributed-order space-fractional diffusion equation with variably distribution and its support, which could adequately model the challenging phenomena such as the anomalous diffusion in multiscale heterogeneous porous media, and smoothing properties of its solutions. We develop and analyze a collocation scheme for the proposed model based on the proved smoothing properties of the solutions. Furthermore, we approximately expand the stiffness matrix by a sum of Toeplitz matrices multiplied by diagonal matrices, which can be employed to develop the fast solver for the approximated system. We prove that it suffices to apply $O(\log N)$ terms of expansion to retain the accuracy of the numerical discretization of degree N, which reduces the storage of the stiffness matrix from $O(N^2)$ to $O(N \log N)$, and the computational cost of matrix-vector multiplication from $O(N^2)$ to $O(N \log^2 N)$. Numerical results are presented to verify the effectiveness and the efficiency of the fast method.

MSC 2010: 65F05, 65M70, 65R20, 26A33

Key Words and Phrases: distributed-order space-fractional diffusion equation; variably distribution; collocation method; fast method; Toeplitz matrix

pp. xxxx–xxxx , DOI: 10.1515/fca-2021-yyyy

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1. **Introduction**

Space-fractional diffusion equations (sFDEs) accurately describe the superdiffusive transport characterized by highly skewed power-law decays, observed in solute transport in heterogeneous porous materials and other applications [3, 6, 29, 30]. However, sFDEs admit solutions with boundary weak singularity [9, 10, 19, 37, 39, 41], because they were derived in the free space as the diffusion limit of the continuous time random walk in the phase plane [29, 30] and so do not properly model the transport near the boundary. A two-scale sFDE, which consists of a fractional derivative term and a second-order derivative term, was proposed in [6] to improve the modeling near the boundary where the superdiffusive transport should behave more like a Fickian diffusion due to the impact of the boundary condition, while retaining the accurate description of the superdiffusive transport away from the boundary.

As the order of sFDEs is determined by the fractal dimension of the surrounding porous medium via the Hurst index [29], a constant-order sFDE can hardly model the superdiffusive transport in highly heterogeneous porous media. Instead, the distributed-order fractional operators, in which the constant-order fractional operators are integrated over a range of the fractional order with respect to some density function $\nu(\alpha)$ are proposed in [2, 4, 5, 7, 9, 12, 13, 23, 24, 32],

$$\begin{aligned} D_x^\nu g &:= \int_0^1 \nu(\alpha)\big[\gamma I_x^\alpha g''(x) + (1-\gamma)\hat{I}_x^\alpha g''(x)\big]d\alpha, \quad 0 \le \gamma \le 1, \\ I_x^\alpha g &:= \frac{1}{\Gamma(\alpha)}\int_0^x \frac{g(s)}{(x-s)^{1-\alpha}}ds, \quad \hat{I}_x^\alpha g := \frac{1}{\Gamma(\alpha)}\int_x^1 \frac{g(s)}{(s-x)^{1-\alpha}}ds, \end{aligned} \tag{1.1}$$

to account for the integrate impact of a family of fractional differential operators and of uncertainties, e.g., due to the limited information and noise in the data [8, 11, 15, 20, 21, 26, 27, 28, 31]. Furthermore, due to the strong heterogeneity of the surrounding medium, the density function ν and its support may be spatially dependent [33, 40, 42]. To date, there is no rigorous mathematical and numerical analysis on variably distributed-order sFDEs reported in the literature.

In this paper we consider a spatially-dependent distributed-order sFDE

$$\begin{aligned} -u''(x) - d(x)D_x^\omega u(x) = f(x),\ x \in (0,1),\ u(0) = u(1) = 0, \\ D_x^\omega g(x) := \int_0^1 \omega(\alpha, x)\big[\gamma I_x^\alpha g''(x) + (1-\gamma)\hat{I}_x^\alpha g''(x)\big]d\alpha. \end{aligned} \tag{1.2}$$

Here $d \ge 0$ is a fractional diffusivity, γ and $1-\gamma$ with $0 \le \gamma \le 1$ indicate the relative weights of forward versus backward transition probability [29], f is the source or sink term. The space-dependence of ω and its support further complicates the mathematical analysis of the problem. Computationally, the distributed-order derivative (1.1) is often discretized as a finite sum of constant-order fractional derivatives via a numerical quadrature on the distributed-order integral [8, 31]. Then each of the fractional derivatives is further discretized, yielding Toeplitz stiffness matrices [36]. Consequently, discrete fast Fourier transform (FFT) based fast numerical methods with linear storage and almost linear computational complexity were developed [16, 25, 35]. However, due to the space-dependence of ω and its support, the corresponding discretization of problem (1.2) loses the Toeplitz structure, and so the fast numerical methods developed for the distributed-order sFDEs of form (1.1) no longer applies [15].

We analyze the well-posedness and smoothing properties of problem (1.2). Base on the proved regularity of its solution, we develop an indirect collocation method for the problem and prove its error estimates without any artificial regularity assumption on the exact solution. Finally, we combine the ideas in [16, 22, 36, 38] to utilize the power-law decaying property of the matrix entries to develop a low-rank approximation, yielding a fast solution method with almost linear memory requirement and computational complexity. The rest of the paper is organized as follows. Section 2 goes over preliminaries. In Section 3 we prove the well-posedness and smoothing properties of problem (1.2). In Section 4 we develop a collocation scheme and prove its error estimates. In Section 5 we develop and analyze a fast approximation to the problem. In Section 6 we investigate the effectiveness and efficiency of the fast approximation. Section 7 is Appendix.

2. Preliminaries

Let $0 < \mu < 1$, $m, n \in \mathbb{N}$ and X, Y be Banach spaces. Let $C^m[0,1]$ be the space of continuous functions with continuous derivatives up to order m on $[0,1]$. Let $C^{m,\mu}[0,1] \subset C^m[0,1]$ consist of functions with the mth order derivative being Hölder continuous with index μ. All the spaces are equipped with the standard norms [1]. Furthermore, $\mathcal{L}(X,Y)$ and $\mathcal{K}(X,Y)$ refer to the spaces of bounded and compact linear mappings from X to Y, respectively. Throughout this paper we make the following assumption:

Assumption (A): For any $x \in [0,1]$, (i) $\omega(\alpha,x) \geq 0$, $\int_0^1 \omega(\alpha,x)d\alpha = 1$; (ii) there exist $0 < \check{\alpha} < \hat{\alpha} \leq 1$ such that $\operatorname{supp}\omega(\alpha,x) \subset [\underline{\alpha}(x), \overline{\alpha}(x)] \subset [\check{\alpha}, \hat{\alpha}]$; (iii) $\underline{\alpha}, \overline{\alpha} \in C^1[0,1]$, $\omega(\alpha,x) \in C([\check{\alpha},\hat{\alpha}],[0,1])$, and $\omega_x(\alpha,x) \in L^1(\check{\alpha},\hat{\alpha})$ for $x \in [0,1]$ a.e.

Let $v := u''$ and rewrite problem (1.2) as the Volterra integral equation

$$v(x) + d(x) \int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \omega(\alpha,x)\big(\gamma\, I_x^\alpha + (1-\gamma)\, \hat{I}_x^\alpha\big) v\, d\alpha = -f(x). \tag{2.1}$$

Interchange the order of the iterated integrals to get

$$d(x)\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \omega(\alpha,x)\big(\gamma\, I_x^{\alpha} + (1-\gamma)\,\hat{I}_x^{\alpha}\big)v(x)\,d\alpha = \int_0^1 k(x,s)v(s)ds.$$

Here $k(x,s) := l(x,s)/|x-s|^{1-\check{\alpha}}$, 1_A is the indicator function of set A, and

$$l(x,s) = d(x)\big(\gamma 1_{\{s<x\}}(s) + (1-\gamma)1_{\{s>x\}}(s)\big)\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \frac{\omega(\alpha,x)}{\Gamma(\alpha)|x-s|^{\check{\alpha}-\alpha}}d\alpha.$$

We use the identity operator I to formulate (2.1) and recover u from v by

$$\begin{aligned} (I+K)v = -f, \qquad Kv &:= \int_0^1 k(x,s)v(s)ds, \quad x\in[0,1],\\ u(x) = \int_0^x (x-s)v(s)ds &- x\int_0^1 (1-s)v(s)ds. \end{aligned} \tag{2.2}$$

In the rest of the paper, we may omit the interval $[0,1]$ in spaces and norms if no confusion occurs.

3. **Wellposedness and smoothing properties**

THEOREM 3.1. *Suppose Assumption (A) holds and $d, f \in C^{\lambda}[0,1]$ with $\lambda := \min\{\underline{\alpha}(0), \underline{\alpha}(1)\}$ ($\lambda < 1$ by the assumptions on ω). If (2.1) with $f \equiv 0$ has only the trivial solution, then equation (2.1) has a unique solution $v \in C^{\lambda}[0,1]$ and there is a constant $Q > 0$ such that*

$$\|v\|_{C^{\kappa}[0,1]} \le Q\|f\|_{C^{\kappa}[0,1]}, \quad 0 \le \kappa \le \lambda. \tag{3.1}$$

P r o o f. $l(x,s)$ is clearly bounded. To verify $|l_x(x,s)| \le Q/|x-s|$ for all $x \neq s$, only consider $s < x$ by symmetry,

$$\begin{aligned} l_x(x,s) &= \gamma d(x)\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \frac{(x-s)^{\alpha-\check{\alpha}}\big(\omega_x(\alpha,x)(x-s) + \omega(\alpha,x)(\alpha-\check{\alpha})\big)}{\Gamma(\alpha)(x-s)}d\alpha \\ &\quad + \frac{\gamma d(x)\overline{\alpha}'(x)\omega(\overline{\alpha}(x),x)}{\Gamma(\overline{\alpha}(x))(x-s)^{\check{\alpha}-\overline{\alpha}(x)}} - \frac{\gamma d(x)\underline{\alpha}'(x)\omega(\underline{\alpha}(x),x)}{\Gamma(\underline{\alpha}(x))(x-s)^{\check{\alpha}-\underline{\alpha}(x)}} \\ &\quad + \int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \frac{\gamma d'(x)\omega(\alpha,x)d\alpha}{\Gamma(\alpha)(x-s)^{\check{\alpha}-\alpha}} = I_1 + I_2 + I_3 + I_4. \end{aligned}$$

We use Assumption (A) to conclude $I_2 + I_3 + I_4$ is bounded and

$$|I_1| \le \frac{Q}{x-s}\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \left|\frac{\omega_x(\alpha,x)(x-s) + \omega(\alpha,x)(\alpha-\check{\alpha})}{\Gamma(\alpha)}\right| d\alpha \le \frac{Q}{x-s}.$$

We use Lemma 7.1 to conclude that $K \in \mathcal{K}(C[0,1], C[0,1])$ and hence (2.1) has a unique solution $v \in C^{\check{\alpha}}[0,1]$ from Lemma 7.2. Therefore, estimate (3.1) holds for $\kappa = \check{\alpha}$. Decompose the iterated integrals in (2.1) as follows:

$$\begin{aligned}
\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \omega(\alpha,x) I_x^\alpha v d\alpha &= \phi_v^{l_1}(x) + \phi_v^{l_2}(x),\\
\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \omega(\alpha,x) \hat{I}_x^\alpha v d\alpha &= \phi_v^{r_1}(x) + \phi_v^{r_2}(x),\\
\phi_v^{l_2}(x) &:= v(0)\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \frac{\omega(\alpha,x)x^\alpha d\alpha}{\Gamma(\alpha+1)},\\
\phi_v^{r_2}(x) &:= v(1)\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \frac{\omega(\alpha,x)(1-x)^\alpha d\alpha}{\Gamma(\alpha+1)},\\
\phi_v^{l_1}(x) &:= \int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \frac{\omega(\alpha,x)}{\Gamma(\alpha)} \int_0^x \frac{v(s)-v(0)}{(x-s)^{1-\alpha}} ds d\alpha,\\
\phi_v^{r_1}(x) &:= \int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \frac{\omega(\alpha,x)}{\Gamma(\alpha)} \int_x^1 \frac{v(s)-v(1)}{(x-s)^{1-\alpha}} ds d\alpha.
\end{aligned}$$

We reformulate equation (2.1) as

$$v(x) = -d(x)\big(\gamma\phi_v^{l_1} + (1-\gamma)\phi_v^{r_1} + \gamma\phi_v^{l_2} + (1-\gamma)\phi_v^{r_2}\big) - f(x). \tag{3.2}$$

By Lemma 7.5, $\gamma\phi_v^{l_2} + (1-\gamma)\phi_v^{r_2} \in C^\lambda[0,1]$. Let $M \in \mathbb{N}^+$ be such that $M\breve{\alpha} < \lambda < (M+1)\breve{\alpha}$ (if λ is exactly a multiple of $\breve{\alpha}$, we could just slightly reduce the value of $\breve{\alpha}$ to ensure the existence of such M). Use $v \in C^{\breve{\alpha}}[0,1]$ to conclude $\phi_v^{l_1}, \phi_v^{r_1} \in C^{2\breve{\alpha}}[0,1]$ by (7.1) with the estimate

$$\|\phi_v^{l_1}\|_{C^{2\breve{\alpha}}[0,1]} + \|\phi_v^{r_1}\|_{C^{2\breve{\alpha}}[0,1]} \le Q\|v\|_{C^{\breve{\alpha}}[0,1]} \le Q\|f\|_{C^{\breve{\alpha}}[0,1]}.$$

The terms on the right-hand side of (3.2) are in $C^{2\breve{\alpha}}[0,1]$, $v \in C^{2\breve{\alpha}}[0,1]$ and

$$\begin{aligned}
\|v\|_{C^{2\breve{\alpha}}[0,1]} &\le Q\big(\|\phi_v^{l_1}\|_{C^{2\breve{\alpha}}[0,1]} + \|\phi_v^{r_1}\|_{C^{2\breve{\alpha}}[0,1]} + |v(0)| + |v(1)|\big) + \|f\|_{C^{2\breve{\alpha}}[0,1]}\\
&\le Q\|f\|_{C^{2\breve{\alpha}}[0,1]}.
\end{aligned}$$

Repeat this process $M-1$ times to conclude $v \in C^{M\breve{\alpha}}[0,1]$ with $\|v\|_{C^{M\breve{\alpha}}[0,1]} \le Q\|f\|_{C^{M\breve{\alpha}}[0,1]}$. As $\lambda < 1$, we apply $v \in C^{M\breve{\alpha}}[0,1] \subset C^{\lambda-\breve{\alpha}}[0,1]$ and repeat the procedure once more to deduce $v \in C^\lambda[0,1]$ with estimate (3.1). □

THEOREM 3.2. *Under assumptions of Theorem 3.1, equation (1.2) has a unique solution $u \in C^{2,\lambda}[0,1]$ and there exists a constant $Q > 0$ such that*

$$\|u\|_{C^{2,\lambda}[0,1]} \le Q\|f\|_{C^\lambda[0,1]}. \tag{3.3}$$

Proof. By Theorem 3.1, the equation (2.1) has a unique solution $v \in C^\lambda[0,1]$, and then the function u defined in (2.2) belongs to $C^{2,\lambda}[0,1]$ and solves (1.2) with the estimate (3.3). If there exists another solution $\tilde{u} \in C^{2,\lambda}[0,1]$, then $e := u - \tilde{u}$ satisfies the homogeneous analogue of (1.2), and $v = e''$ satisfies the homogeneous analogue of (2.1). Then an application of Theorem 3.1 yields $v \equiv 0$, which indicates e in a linear function. Then the homogeneous boundary conditions in (1.2) ensures that $e \equiv 0$, which shows the uniqueness of the solution to (1.2). □

THEOREM 3.3. *Suppose Assumption (A) holds and $d, f \in C^1[0,1]$. If for some constant $0 \le \mu < 1$, d satisfies*

$$d(x) \le \mu\Big(\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \frac{\omega(\alpha,x)}{\Gamma(\alpha+1)} d\alpha\Big)^{-1}, \quad x \in [0,1], \tag{3.4}$$

then problem (2.2) has a unique solution $v \in C^1(0,1)$ and

$$\|\rho v'\|_{C[0,1]} \le Q\|f\|_{C^1[0,1]}, \qquad \rho(x) := x(1-x). \tag{3.5}$$

If $d, f \in C^{1,\lambda}[0,1]$ with λ given in Theorem 3.1, then $\rho v' \in C^\lambda[0,1]$ and

$$\|\rho v'\|_{C^\lambda[0,1]} \le Q\|f\|_{C^{1,\lambda}[0,1]}. \tag{3.6}$$

P r o o f. Let v_0 be the solution to the homogeneous analogue of (2.1),

$$\begin{aligned}
|v_0(x)| &= \Big| d(x) \int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \frac{w(\alpha,x)}{\Gamma(\alpha)} \Big(\int_0^x \frac{\gamma v_0(s)ds}{(x-s)^{1-\alpha}} + \int_x^1 \frac{(1-\gamma)v_0(s)ds}{(s-x)^{1-\alpha}} \Big) d\alpha \Big| \\
&\le \|v_0\|_{C[0,1]} \Big(d(x) \int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \frac{\omega(\alpha,x)}{\Gamma(\alpha+1)} \Big(\gamma x^\alpha + (1-\gamma)(1-x)^\alpha \Big) d\alpha \Big) \\
&\le \|v_0\|_{C[0,1]} \Big(d(x) \int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \frac{\omega(\alpha,x)}{\Gamma(\alpha+1)} d\alpha \Big) = Z(x)\|v_0\|_{C[0,1]}.
\end{aligned}$$

Under condition (3.4), $Z(x) < 1$. Hence, $\|v_0\|_{C[0,1]} = 0$. Thus, the homogeneous analogue of problem (2.1) has only the trivial solution. By Theorem 3.1, problem (2.1) has a unique solution v with the estimate (3.1).

We use relation $\rho(x) = \rho(s) + (x-s)(1-x-s)$ to split $\rho(x)v(s)$ as

$$\rho(x)v(s) = \rho(s)v(s) + (x-s)(1-x-s)v(s). \tag{3.7}$$

We multiply (2.1) by $\rho(x)$, use (3.7) and $(\rho v)(0) = (\rho v)(1) = 0$ to get

$$\begin{aligned}
\rho(x)v(x) &= -\rho(x)f(x) \\
&\quad - \gamma d(x) \int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \frac{\omega(\alpha,x)}{\Gamma(\alpha)} \int_0^x \frac{\rho(s)v(s) + (x-s)(1-x-s)v(s)}{(x-s)^{1-\alpha}} ds d\alpha \\
&\quad - (1-\gamma)d(x) \int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \frac{\omega(\alpha,x)}{\Gamma(\alpha)} \int_x^1 \frac{\rho(s)v(s) + (x-s)(1-x-s)v(s)}{(s-x)^{1-\alpha}} ds d\alpha \\
&= -\rho(x)f(x) - d(x)\big(\gamma\phi_{\rho v}^{l_1} + (1-\gamma)\phi_{\rho v}^{r_1}\big) - d(x)\big(\gamma G_1 - (1-\gamma)G_2\big),
\end{aligned} \tag{3.8}$$

where

$$G_1 := \int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \frac{\omega(\alpha,x)}{\Gamma(\alpha)} \int_0^x (1-x-s)(x-s)^\alpha v(s)dsd\alpha$$
$$G_2 := \int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \frac{\omega(\alpha,x)}{\Gamma(\alpha)} \int_x^1 (1-x-s)(s-x)^\alpha v(s)dsd\alpha.$$

Since G_1 and G_2 have continuous kernels and so in $C^1[0,1]$,

$$\begin{aligned} G_1'(x) = &\frac{\overline{\alpha}'(x)\omega(\overline{\alpha}(x),x)}{\Gamma(\bar{\alpha}(x))} \int_0^x (1-x-s)(x-s)^{\overline{\alpha}(x)} v(s)ds \\ &- \frac{\underline{\alpha}'(x)\omega(\underline{\alpha}(x),x)}{\Gamma(\underline{\alpha}(x))} \int_0^x (1-x-s)(x-s)^{\underline{\alpha}(x)} v(s)ds \\ &+ \int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \frac{\omega_x(\alpha,x)}{\Gamma(\alpha)} \int_0^x (1-x-s)(x-s)^\alpha v(s)dsd\alpha \\ &+ \int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \frac{\omega(\alpha,x)}{\Gamma(\alpha)} \int_0^x \frac{\alpha-(1+\alpha)x+(1-\alpha)s}{(x-s)^{1-\alpha}} v(s)dsd\alpha. \end{aligned} \tag{3.9}$$

Thus, we bound G_1 (and similarly G_2) by

$$\begin{aligned} \|G_1\|_{C^1[0,1]} \le Q\|v\|_{C[0,1]} \Big\| &\int_{\underline{\alpha}(x)}^{\bar{\alpha}(x)} \frac{|\omega(\alpha,x)| + |\omega_x(\alpha,x)|}{\Gamma(\alpha)} \\ &\times \int_0^x \frac{ds}{(x-s)^{1-\alpha}} d\alpha \Big\|_{C[0,1]} \le Q\|v\|_{C[0,1]}. \end{aligned}$$

Since $\rho v \in C^\lambda[0,1]$, a similar lifting argument to Theorem 3.1 concludes $\rho v \in C^1[0,1]$ (if $M\breve{\alpha} > 1$, we apply estimate (7.2) in Lemma 7.6 instead). As $\rho^{-1} \in C(0,1)$, we deduce that $v \in C^1(0,1)$ and

$$\|\rho v\|_{C^1[0,1]} \le Q\|f\|_{C^1[0,1]}.$$

We apply $\rho v' = (\rho v)' - \rho' v = (\rho v)' - (1-2x)v$ and Theorem 3.1 to get the estimate of $\rho v'$ as

$$\|\rho v'\| \le \|\rho v\|_{C^1[0,1]} + \|v\|_{C[0,1]} \le Q\|f\|_{C^1[0,1]}.$$

We integrate $\phi_{\rho v}^{l_1}$ by parts to get

$$\begin{aligned} \phi_{\rho v}^{l_1} &= -\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \frac{\omega(\alpha,x)}{\Gamma(\alpha+1)} \Big(\int_0^x \rho(s)v(s)d(x-s)^\alpha \Big) d\alpha \\ &= \int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \frac{\omega(\alpha,x)}{\Gamma(\alpha+1)} \Big(\int_0^x (x-s)^\alpha (\rho v)'(s)ds \Big) d\alpha. \end{aligned}$$

Differentiate the equation to get

$$(\phi_{\rho v}^{l_1})' = \frac{\overline{\alpha}'(x)\omega(\overline{\alpha}(x),x)}{\Gamma(\overline{\alpha}(x)+1)}\int_0^x (x-s)^{\overline{\alpha}(x)}(\rho v)'(s)ds$$
$$-\frac{\underline{\alpha}'(x)\omega(\underline{\alpha}(x),x)}{\Gamma(\underline{\alpha}(x)+1)}\int_0^x (x-s)^{\underline{\alpha}(x)}(\rho v)'(s)ds$$
$$+\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)}\Big[\frac{\omega_x(\alpha,x)}{\Gamma(\alpha+1)}\int_0^x (x-s)^{\alpha}(\rho v)'(s)ds+\frac{\omega(\alpha,x)}{\Gamma(\alpha)}\int_0^x\frac{(\rho v)'(s)ds}{(x-s)^{1-\alpha}}\Big]d\alpha.$$

The last term on the right-hand side can be decomposed as

$$\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)}\frac{\omega(\alpha,x)}{\Gamma(\alpha)}\int_0^x\frac{(\rho v)'(s)dsd\alpha}{(x-s)^{1-\alpha}} = \phi_{(\rho v)'}^{l_1} + (\rho v)'(0)\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)}\frac{\omega(\alpha,x)x^{\alpha}d\alpha}{\Gamma(\alpha+1)}.$$

The same holds for $\phi_{\rho v}^{r_1}$ by symmetry. Differentiate (3.8) to obtain

$$\begin{aligned}(\rho v)' &= -d'\big(\gamma\phi_{\rho v}^{l_1} + (1-\gamma)\phi_{\rho v}^{r_1} + \gamma G_1 - (1-\gamma)G_2\big) - (\rho f)'\\ &\quad -d\big(\gamma(\phi_{\rho v}^{l_1})' + (1-\gamma)(\phi_{\rho v}^{r_1})'\big) - d\big(\gamma G_1' - (1-\gamma)G_2'\big)\\ &= -\gamma d(x)(\phi_{\rho v}^{l_1})' - (1-\gamma)d(x)(\phi_{\rho v}^{r_1})' - d\big(\gamma G_1' - (1-\gamma)G_2'\big) + P_1,\end{aligned}$$

where $P_1 = -d'\big(\gamma\phi_{\rho v}^{l_1} + (1-\gamma)\phi_{\rho v}^{r_1} + \gamma G_1 - (1-\gamma)G_2\big) - (\rho f)' \in C^{\lambda}[0,1]$.

We combine the preceding three equations to get

$$\begin{aligned}&(\rho v)' + \gamma d(x)\phi_{(\rho v)'}^{l_1} + (1-\gamma)d(x)\phi_{(\rho v)'}^{r_1}\\ &= -d\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)}\frac{\omega(\alpha,x)\big(\gamma(\rho v)'(0)x^{\alpha}+(1-\gamma)(\rho v)'(1)(1-x)^{\alpha}\big)}{\Gamma(\alpha+1)}d\alpha+P_2,\end{aligned} \tag{3.10}$$

$$\begin{aligned}P_2 &= -d\big(\gamma G_1' - (1-\gamma)G_2'\big) + P_1\\ &-d\Big[\frac{\overline{\alpha}'\omega(\overline{\alpha},x)}{\Gamma(\bar{\alpha}(x)+1)}\Big(\gamma\int_0^x (x-s)^{\overline{\alpha}(x)}(\rho v)'ds - (1-\gamma)\int_x^1 (s-x)^{\overline{\alpha}(x)}(\rho v)'ds\Big)\\ &-\frac{\underline{\alpha}'\omega(\underline{\alpha},x)}{\Gamma(\underline{\alpha}+1)}\Big(\gamma\int_0^x (x-s)^{\underline{\alpha}(x)}(\rho v)'ds - (1-\gamma)\int_x^1 (s-x)^{\underline{\alpha}(x)}(\rho v)'ds\Big)\\ &+\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)}\frac{\omega_x(\alpha,x)}{\Gamma(\alpha+1)}\Big(\gamma\int_0^x (x-s)^{\alpha}(\rho v)'ds - (1-\gamma)\int_x^1 (s-x)^{\alpha}(\rho v)'ds\Big)d\alpha\Big].\end{aligned}$$

As is proved in Lemma 7.5, the first term on the right-hand side of (3.10) is in $C^{\lambda}[0,1]$. The last three terms of P_2 have continuous kernels, and so belong to $C^1[0,1]$. We bound the most singular term of G_1' in (3.9) by

$$\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)}\frac{\omega(\alpha,x)}{\Gamma(\alpha)}\int_0^x\frac{v(s)}{(x-s)^{1-\alpha}}dsd\alpha = \phi_v^{l_1}(x) + v(0)\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)}\frac{\omega(\alpha,x)x^{\alpha}}{\Gamma(\alpha+1)}d\alpha.$$

By Lemmas 7.4-7.5, we conclude $G_1' \in C^\lambda[0,1]$ with the bound

$$\|G_1'\|_{C^\lambda[0,1]} \le Q\big(\|\phi_v^{l_1}\|_{C^\lambda[0,1]} + |v(0)|\big) \le Q\|v\|_{C^\lambda[0,1]} \le Q\|f\|_{C^\lambda[0,1]}.$$

The same holds for G_2'. Thus, all the terms on the right-hand side of (3.10) belong to $C^\lambda[0,1]$ with its C^λ norm bounded by $\|f\|_{C^{1,\lambda}[0,1]}$. A similar lifting argument to Theorem 3.1 applied to (3.10) concludes $\rho v' \in C^\lambda[0,1]$ with estimate (3.6). □

THEOREM 3.4. *If Assumption (A) and (3.4) hold, $d, f \in C^2[0,1]$, then*

$$\|\rho^2 v''\|_{C[0,1]} \le Q\|f\|_{C^2[0,1]}. \tag{3.11}$$

In addition, if $d, f \in C^{2,\lambda}[0,1]$ with λ given in Theorem 3.1, then

$$\|\rho^2 v''\|_{C^\lambda[0,1]} \quad \le Q\|f\|_{C^{2,\lambda}[0,1]}. \tag{3.12}$$

Consequently, $v \in C^m(0,1)$ can be bounded by

$$\big|v^{(m)}(x)\big| \le Q \max\{x^{\lambda-m}, (1-x)^{\lambda-m}\}\|f\|_{C^{m,\lambda}[0,1]}, \quad m = 1, 2. \tag{3.13}$$

P r o o f. We multiply (3.10) by $\rho(x)$ to get

$$\begin{aligned}
&\rho(\rho v)'(x) + \gamma d(x)\rho(x)\phi^{l_1}_{(\rho v)'} + (1-\gamma)d(x)\rho(x)\phi^{r_1}_{(\rho v)'}\\
&= -\gamma(\rho v)'(0)d(x)\rho(x)\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \frac{\omega(\alpha,x)x^\alpha}{\Gamma(\alpha+1)}d\alpha\\
&\quad -(1-\gamma)(\rho v)'(1)d(x)\rho(x)\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \frac{\omega(\alpha,x)(1-x)^\alpha}{\Gamma(\alpha+1)}d\alpha + \rho(x)P_2.
\end{aligned} \tag{3.14}$$

Set $z = \rho(\rho v)'$, use the fact that $\rho(0) = 0$, and apply the splitting (3.7) to rewrite the second term on the left-hand side of (3.14) as

$$\begin{aligned}
\rho(x)\phi^{l_1}_{(\rho v)'}(x) = \phi_z^{l_1} + \int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \frac{\omega(\alpha,x)}{\Gamma(\alpha)} \int_0^x (x-s)^\alpha(1-x-s)(\rho v)'(s)ds d\alpha\\
-(\rho v)'(0)\rho(x)\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \frac{\omega(\alpha,x)x^\alpha}{\Gamma(\alpha+1)}d\alpha.
\end{aligned}$$

We similarly decompose $\rho\phi^{r_1}_{(\rho v)'}$ and reformulate (3.14) as

$$\begin{aligned}
&z + \gamma d(x)\phi_z^{l_1} + (1-\gamma)d(x)\phi_z^{r_1}\\
&= \rho(x)P_2 - \gamma d(x)\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \frac{\omega(\alpha,x)}{\Gamma(\alpha)} \int_0^x (x-s)^\alpha(1-x-s)(\rho v)'(s)ds d\alpha\\
&\quad + (1-\gamma)d(x)\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \frac{\omega(\alpha,x)}{\Gamma(\alpha)} \int_x^1 (s-x)^\alpha(1-s-x)(\rho v)'(s)ds d\alpha.
\end{aligned} \tag{3.15}$$

We use (3.9) to bound the most singular terms $\rho G_1'$ and $\rho G_2'$ in ρP_2 by

$$
\begin{aligned}
&\rho(x)\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)}\frac{\omega(\alpha,x)}{\Gamma(\alpha)}\int_0^x\frac{v(s)ds}{(x-s)^{1-\alpha}}d\alpha\\
&=\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)}\frac{\omega(\alpha,x)}{\Gamma(\alpha)}\Big[\int_0^x\frac{(\rho v)(s)ds}{(x-s)^{1-\alpha}}+\int_0^x(x-s)^\alpha(1-x-s)v(s)ds\Big]d\alpha\\
&=\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)}\frac{\omega(\alpha,x)}{\Gamma(\alpha)}\int_0^x\Big(\frac{1}{\alpha}(x-s)^\alpha(\rho v)'+(x-s)^\alpha(1-x-s)v(s)\Big)dsd\alpha.
\end{aligned}
$$

The right-hand side has continuous integrand, so $\rho G_1'\in C^1[0,1]$. We drop the smooth coefficient d and decompose the most singular term in $(\rho G_1')'$ by

$$
\begin{aligned}
&\Big[\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)}\frac{\omega(\alpha,x)}{\Gamma(\alpha)}\int_0^x(x-s)^\alpha(\rho v)'dsd\alpha\Big]'\\
&=\int_0^x\Big(\frac{\overline{\alpha}'\omega(\overline{\alpha}(x),x)}{\Gamma(\overline{\alpha}(x))}\int_0^x(x-s)^{\overline{\alpha}(x)}-\frac{\underline{\alpha}'\omega(\underline{\alpha}(x),x)}{\Gamma(\underline{\alpha}(x))}(x-s)^{\underline{\alpha}(x)}\Big)(\rho v)'ds\\
&\quad+\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)}\int_0^x\Big(\frac{\omega_x(\alpha,x)}{\Gamma(\alpha)}(x-s)^\alpha+\frac{\alpha\omega(\alpha,x)}{\Gamma(\alpha)(x-s)^{1-\alpha}}\Big)(\rho v)(s)'dsd\alpha.
\end{aligned}
$$

We split the (more singular) last term on the right-hand side by

$$
\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)}\frac{\omega(\alpha,x)}{\Gamma(\alpha)}\int_0^x\frac{(\rho v)'(s)dsd\alpha}{(x-s)^{1-\alpha}}=\phi^{l_1}_{(\rho v)'}+(\rho v)'(0)\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)}\frac{\omega(\alpha,x)x^\alpha d\alpha}{\Gamma(\alpha+1)}. \quad (3.16)
$$

By Theorem 3.3, $(\rho v)'\in C^\lambda[0,1]$ and the left-hand side belongs to $C^\lambda[0,1]$. We conclude $\rho G_1'\in C^{1,\lambda}[0,1]$ with estimate $\|\rho G_1'\|_{C^{1,\lambda}[0,1]}\le Q\|f\|_{C^{1,\lambda}[0,1]}$. Similarly, the remaining terms in P_2 are in $C^{1,\lambda}[0,1]$ with the same bound.

We bound the rest on the right-hand side of (3.15), which belong to $C^1[0,1]$ since they have continuous integrands. For the sake of clarity, we just bound the derivative of the second term that is representative

$$
\begin{aligned}
&\Big(\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)}\frac{\omega(\alpha,x)}{\Gamma(\alpha)}\int_0^x(x-s)^\alpha(1-x-s)(\rho v)'(s)dsd\alpha\Big)'\\
&=\frac{\overline{\alpha}'(x)\omega(\overline{\alpha}(x),x)}{\Gamma(\overline{\alpha}(x))}\int_0^x(x-s)^{\overline{\alpha}(x)}(1-x-s)(\rho v)'(s)ds\\
&\quad-\frac{\underline{\alpha}'(x)\omega(\underline{\alpha}(x),x)}{\Gamma(\underline{\alpha}(x))}\int_0^x(x-s)^{\underline{\alpha}(x)}(1-x-s)(\rho v)'(s)ds \quad (3.17)\\
&\quad+\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)}\frac{\omega_x(\alpha,x)}{\Gamma(\alpha)}\int_0^x(x-s)^\alpha(1-x-s)(\rho v)'(s)dsd\alpha\\
&\quad+\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)}\frac{\omega(\alpha,x)}{\Gamma(\alpha)}\int_0^x\frac{\alpha-(1+\alpha)x+(1-\alpha)s}{(x-s)^{1-\alpha}}(\rho v)'(s)dsd\alpha.
\end{aligned}
$$

The (most singular) last term in (3.17) is bounded similar to (3.16). Thus the second term on the right-hand side of (3.15) is bounded by

$$\Big\|\gamma d\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)}\frac{\omega(\alpha,x)}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha}(1-s-x)(\rho v)'(s)dsd\alpha\Big\|_{C^1[0,1]}\le Q\|f\|_{C^{1,\lambda}[0,1]}.$$

We similarly prove that the third term, and so all the terms on the right-hand side of (3.15) belong to $C^{1,\lambda}[0,1]$. We then apply a similar lifting as in Theorem 3.1 to (3.15) to deduce that $z\in C^1[0,1]$ with the estimate

$$\|z\|_{C^1[0,1]}\le Q\|(\rho v)'\|_{C^1[0,1]}\le Q\|f\|_{C^2[0,1]}.$$

We finally lift the regularity of z from $C^1[0,1]$ to $C^{1,\lambda}[0,1]$. Using $z(s)-z(0)=\int_0^s z'(\theta)d\theta$ and interchanging the order of the integrations to get

$$\phi_z^{l_1}=\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)}\frac{\omega(\alpha,x)}{\Gamma(\alpha+1)}\int_0^x(x-s)^{\alpha}z'(s)dsd\alpha.$$

We consequently split the derivative $(\phi_z^{l_1})'$ by

$$\begin{aligned}(\phi_z^{l_1})'=&\int_0^x\Big(\frac{\overline{\alpha}'\omega(\overline{\alpha}(x),x)^x}{\Gamma(\overline{\alpha}(x))}(x-s)^{\overline{\alpha}(x)}-\frac{\underline{\alpha}'\omega(\underline{\alpha}(x),x)}{\Gamma(\underline{\alpha}(x))}(x-s)^{\underline{\alpha}(x)}\Big)z'(s)ds\\&+\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)}\frac{\omega_x(\alpha,x)}{\Gamma(\alpha+1)}\int_0^x(x-s)^{\alpha}z'dsd\alpha+\phi_{z'}^{l_1}+z'(0)\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)}\frac{\omega(\alpha,x)x^{\alpha}d\alpha}{\Gamma(\alpha+1)}.\end{aligned}$$

We differentiate (3.15) to get

$$\begin{aligned}&z'+\gamma d(x)\phi_{z'}^{l_1}+(1-\gamma)d(x)\phi_{z'}^{r_1}\\&=-d\Big[\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)}\frac{\omega(\alpha,x)\big(\gamma z'(0)x^{\alpha}+(1-\gamma)z'(1)(1-x)^{\alpha}\big)}{\Gamma(\alpha+1)}d\alpha\\&\quad+\frac{\overline{\alpha}'\omega(\overline{\alpha}(x),x)}{\Gamma(\overline{\alpha}(x))}\Big(\gamma\int_0^x(x-s)^{\overline{\alpha}(x)}z'(s)ds-(1-\gamma)\int_x^1(s-x)^{\overline{\alpha}(x)}z'(s)ds\Big)\\&\quad-\frac{\underline{\alpha}'\omega(\underline{\alpha}(x),x)}{\Gamma(\underline{\alpha}(x))}\Big(\gamma\int_0^x(x-s)^{\underline{\alpha}(x)}z'(s)ds-(1-\gamma)\int_x^1(s-x)^{\underline{\alpha}(x)}z'(s)ds\Big)\\&\quad+\int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)}\frac{\omega_x(\alpha,x)}{\Gamma(\alpha+1)}\Big(\gamma\int_0^x(x-s)^{\alpha}z'(s)ds\\&\quad-(1-\gamma)\int_x^1(s-x)^{\alpha}z'(s)ds\Big)d\alpha\Big]-d'(x)\big(\gamma\phi_z^{l_1}+(1-\gamma)\phi_z^{r_1}\big)+P_3'.\end{aligned}$$

Here $P_3\in C^{1,\lambda}[0,1]$ represent all the terms on the right-hand side of (3.15). Thus, all the right-hand side terms belong to $C^{\lambda}[0,1]$. Apply the similar lifting to Theorem 3.1 to conclude that $z\in C^{1,\lambda}[0,1]$ with the estimate

$$\|z\|_{C^{1,\lambda}[0,1]}\le Q\|f\|_{C^{2,\lambda}[0,1]}.$$

We use $\rho^2v''=(\rho(\rho v)')'-(\rho')^2v-\rho\rho''v-3\rho\rho'v'$ to arrive at estimate (3.12). We use estimates (3.5) and (3.11) for $m=1,2$ to obtain

$$|\rho(x)^m v(x)^m - 0| \le Q\|\rho^m v^m\|_{C^\lambda[0,1]} x^\lambda \le Q\|f\|_{C^{m,\lambda}[0,1]} x^\lambda,$$

which leads to estimate (3.13). □

THEOREM 3.5. *If Assumption (A) and (3.4) hold, $d, f \in C^2[0,1]$, then*

$$\|\rho^2 u^{(4)}\|_{C[0,1]} \le Q\|f\|_{C^2[0,1]}.$$

Furthermore, if $d, f \in C^{2,\lambda}[0,1]$, then

$$\|\rho^2 u^{(4)}\|_{C^\lambda[0,1]} \le Q\|f\|_{C^{2,\lambda}[0,1]}.$$

4. A collocation method and its error analysis

Let $x_n := (n-1)h$ for $n = 1, \dots, N+2$ with $h := 1/(N+1)$ be a partition of $[0,1]$. Let S_h be the space of continuous and piecewise linear functions with the partition. Let $\{\phi_n(x)\}_{n=1}^{N+2}$ be the piecewise linear basis functions with $\phi_n(x_n) = 1$ and $\phi_n(x_k) = 0$ for $k \ne n$. Then each function $v_h(x) \in S_h$ can be represented by $v_h(x) = \sum_{n=1}^{N+2} v_n \phi_n(x)$ with $v_n = v_h(x_n)$.

A collocation method is formulated as follows:

- Find $v_h \in S_h$ such that for $n = 1, 2, \dots, N+2$,

$$\begin{aligned} &\big[(I+K)v_h\big](x_n) = -f(x_n), \ n = 1, 2, \dots, N+2,\\ &(Kv_h)(x_n) = d(x_n)\int_{\underline{\alpha}(x_n)}^{\overline{\alpha}(x_n)} \frac{\omega(\alpha, x_n)}{\Gamma(\alpha)}\Big(\gamma\int_0^{x_n}\frac{v_h(s)ds}{(x_n-s)^{1-\alpha}} \\ &\qquad + (1-\gamma)\int_{x_n}^1 \frac{v_h(s)ds}{(s-x_n)^{1-\alpha}}\Big)d\alpha. \end{aligned} \tag{4.1}$$

- Postprocess u_h by

$$u_h(x) := \int_0^x (x-s)v_h(s)ds - x\int_0^1 (1-s)v_h(s)ds, \quad x \in [0,1]. \tag{4.2}$$

The diagonal, lower triangular, and upper triangular entries of the stiffness matrix $\mathbf{A} = \big(a_{n,i}\big)_{n,i=1}^{N+2}$ are given by (4.3)–(4.5), respectively

$$a_{n,n} = \begin{cases} 1 + (1-\gamma)d(x_1)\displaystyle\int_{\underline{\alpha}(x_1)}^{\overline{\alpha}(x_1)} \frac{\omega(\alpha, x_1)}{h}^{\alpha}\Gamma(\alpha+2)d\alpha, & n = 1,\\ 1 + d(x_n)\displaystyle\int_{\underline{\alpha}(x_n)}^{\overline{\alpha}(x_n)} \frac{\omega(\alpha, x_n)h^\alpha}{\Gamma(\alpha+2)}d\alpha, & 2 \le n \le N+1,\\ 1 + \gamma d(x_{N+2})\displaystyle\int_{\underline{\alpha}(x_n)}^{\overline{\alpha}(x_n)} \frac{\omega(\alpha, x_n)h^\alpha}{\Gamma(\alpha+2)}d\alpha, & n = N+2; \end{cases} \tag{4.3}$$

$$a_{n,i}=\begin{cases}\gamma d(x_n)\displaystyle\int_{\underline{\alpha}(x_n)}^{\overline{\alpha}(x_n)}\frac{\omega(\alpha,x_n)h^\alpha}{\Gamma(\alpha+2)}\big((n-i+1)^{\alpha+1}-2(n-i)^{\alpha+1}\\ \quad+(n-i-1)^{\alpha+1}\big)d\alpha,\ 3\le n\le N+2,\ 2\le i\le n-1,\\ \gamma d(x_n)\displaystyle\int_{\underline{\alpha}(x_n)}^{\overline{\alpha}(x_n)}\frac{\omega(\alpha,x_n)h^\alpha}{\Gamma(\alpha+1)}\Big[(n-i)^\alpha-\frac{(n-i)^{\alpha+1}}{\alpha+1}\\ \quad+\dfrac{(n-i-1)^{\alpha+1}}{\alpha+1}\Big]d\alpha,\ 2\le n\le N+2,\ i=1;\end{cases}\tag{4.4}$$

$$a_{n,i}=\begin{cases}(1-\gamma)d(x_n)\displaystyle\int_{\underline{\alpha}(x_n)}^{\overline{\alpha}(x_n)}\frac{\omega(\alpha,x_n)h^\alpha}{\Gamma(\alpha+2)}\big((i-n-1)^{\alpha+1}-2(i-n)^{\alpha+1}\\ \quad+(i-n+1)^{\alpha+1}\big)d\alpha,\ 1\le n\le N,\ n+1\le i\le N+1,\\ (1-\gamma)d(x_n)\displaystyle\int_{\underline{\alpha}(x_n)}^{\overline{\alpha}(x_n)}\frac{\omega(\alpha,x_n)h^\alpha}{\Gamma(\alpha+1)}\Big[(i-n)^\alpha-\frac{(i-n)^{\alpha+1}}{\alpha+1}\\ \quad+\dfrac{(i-n-1)^{\alpha+1}}{\alpha+1}\Big]d\alpha,\ 1\le n\le N+1,\ i=N+2.\end{cases}\tag{4.5}$$

Let $\mathbf{v}:=[v_h(x_1),v_h(x_2),\cdots,v_h(x_{N+2})]^\top$, $\mathbf{f}:=[f(x_1),f(x_2),\cdots,f(x_{N+2})]^\top$. Scheme (4.1) can be formulated as a matrix form

$$\mathbf{A}\mathbf{v}=-\mathbf{f}.\tag{4.6}$$

THEOREM 4.1. *If Assumption (A) and condition* (3.4) *hold, then* $\mathbf{A}$ *is strictly diagonally dominant, so scheme* (4.1) *has a unique solution* $v_h\in S_h$.

P r o o f. By (4.3)-(4.5), we have

$$a_{1,1}-\sum_{i=2}^{N+2}|a_{1,i}|=1-(1-\gamma)d(x_1)\int_{\underline{\alpha}(x_1)}^{\overline{\alpha}(x_1)}\frac{\omega(\alpha,x_1)}{\Gamma(\alpha+1)}d\alpha\ge 1-\mu>0,$$

and for $2\le n\le N+1$,

$$\begin{aligned}a_{n,n}-\sum_{i\ne n}|a_{n,i}|&=1-d(x_n)\int_{\underline{\alpha}(x_n)}^{\overline{\alpha}(x_n)}\frac{\omega(\alpha,x_n)h^\alpha}{\Gamma(\alpha+2)}\Big[\gamma(n-1)^\alpha\\ &\quad+(1-\gamma)(N+2-n)^\alpha\Big]d\alpha+2d(x_n)\int_{\underline{\alpha}(x_n)}^{\overline{\alpha}(x_n)}\frac{\omega(\alpha,x_n)h^\alpha}{\Gamma(\alpha+2)}d\alpha\\ &>1-d(x_n)\int_{\underline{\alpha}(x_n)}^{\overline{\alpha}(x_n)}\frac{\omega(\alpha,x_n)}{\Gamma(\alpha+1)}d\alpha+2d(x_n)\int_{\underline{\alpha}(x_n)}^{\overline{\alpha}(x_n)}\frac{\omega(\alpha,x_n)h^\alpha}{\Gamma(\alpha+2)}d\alpha\\ &\ge 1-\mu>0.\end{aligned}$$

Similarly, we have $a_{N+2,N+2}-\sum_{i=1}^{N+1}|a_{N+2,i}|\ge 1-\mu$. □

THEOREM 4.2. *Suppose Assumption (A) and (3.4) hold. If $d, f \in C^{2,\lambda}[0,1]$ with λ given in Theorem 3.1, the error estimates hold for h sufficiently small*

$$\|v - v_h\|_{\hat{L}^\infty} \leq Q\|f\|_{C^{2,\lambda}[0,1]}h^{2\lambda}, \quad \|u - u_h\|_{\hat{L}^\infty} \leq Q\|f\|_{C^{2,\lambda}[0,1]}h^{2\lambda}.$$

Furthermore, if $v \in C^2[0,1]$, the optimal-order error estimates hold

$$\|v - v_h\|_{\hat{L}^\infty} \leq Q\|v\|_{C^2[0,1]}h^2, \ \|u - u_h\|_{\hat{L}^\infty} \leq Q\|v\|_{C^2[0,1]}h^2.$$

Here the discrete norm $\|\cdot\|_{\hat{L}^\infty}$ is defined by $\|g\|_{\hat{L}^\infty} := \max_{1\leq n\leq N+2}\big|g(x_n)\big|$.

P r o o f. By the proof of Theorem 3.1, $K \in \mathcal{L}(C[0,1], C[0,1])$ is compact. By setting $X = C[0,1]$ in Lemma 7.2, we conclude that $(I+K)^{-1} \in \mathcal{L}(C[0,1], C[0,1])$. Let $\Pi_h : C[0,1] \to S_h$ be the piecewise linear interpolation operator. We subtract the interpolation of (2.2) $\Pi_h v + \Pi_h(Kv) = -\Pi_h f$ from (4.1) to get $\Pi_h v - v_h = -\Pi_h K(v - v_h)$, which implies

$$(I + \Pi_h K)(\Pi_h v - v_h) = -\Pi_h K(v - \Pi_h v).$$

We use Lemma 7.3 to get

$$\|(I + \Pi_h K)^{-1}\|_{\mathcal{L}(C[0,1],C[0,1])} \leq Q\|(I + K)^{-1}\|_{\mathcal{L}(C[0,1],C[0,1])},$$

which implies that

$$\|\Pi_h v - v_h\|_{C[0,1]} \leq Q\|\Pi_h K(v - v_h)\|_{C[0,1]}.$$

As $\|\cdot\|_{C[0,1]}$ and $\|\cdot\|_{\hat{L}^\infty}$ are equivalent for functions in S_h, we obtain

$$\|\Pi_h v - v_h\|_{\hat{L}^\infty} \leq Q\|\Pi_h K(v - \Pi_h v)\|_{\hat{L}^\infty} := Q\|R\|_{\hat{L}^\infty},$$

Then we remain to estimate $\|R\|_{\hat{L}^\infty}$ according to Theorem 3.4. We begin with the case $d, f \in C^{2,\lambda}[0,1]$. Let $\eta := v - \Pi_h v$, we have

$$R(x_n) = -(K\eta)(x_n) = -\int_{\underline{\alpha}(x_n)}^{\overline{\alpha}(x_n)} \frac{\omega(\alpha, x_n)}{\Gamma(\alpha)}\big(\gamma I^\alpha_{x_n}\eta + (1-\gamma)\hat{I}^\alpha_{x_n}\eta\big)d\alpha, \quad (4.7)$$

it suffices to estimate the first right-hand side term of (4.7) by symmetry

$$R^L_n = \int_{\underline{\alpha}(x_n)}^{\overline{\alpha}(x_n)} \omega(\alpha, x_n)I^\alpha_{x_n}\eta d\alpha = \int_{\underline{\alpha}(x_n)}^{\overline{\alpha}(x_n)} \frac{\omega(\alpha, x_n)}{\Gamma(\alpha)}\sum_{i=2}^{n}\int_{x_{i-1}}^{x_i}\frac{\eta(s)ds}{(x_n - s)^{1-\alpha}}d\alpha \quad (4.8)$$

for $n \geq 2$. We start with $2 \leq n \leq \lfloor N/2\rfloor + 1$. For $i = 2$, we apply the integral type residue of the interpolation (cf. [42, Equation 5.12]) to bound the integral on $[0, x_2]$ in (4.8) by

$$Q\int_0^{x_2}\frac{\int_0^{x_2}|v'(y)|dy}{(x_n - s)^{1-\alpha}}ds \leq Q\|f\|_{C^{1,\lambda}}\int_0^{x_2}\frac{\int_0^{x_2}y^{\lambda-1}dy}{(x_n - s)^{1-\alpha}}ds \leq Q\|f\|_{C^{1,\lambda}[0,1]}h^{2\lambda},$$

here we use the fact that for $\alpha \in \big[\underline{\alpha}(x_n), \overline{\alpha}(x_n)\big]$,

$$\int_0^{x_2}(x_n-s)^{\alpha-1}ds \le Q\int_0^{x_2}(x_n-s)^{\underline{\alpha}(s)-1}ds \le Q\int_0^{x_2}(x_2-s)^{\underline{\alpha}(s)-1}ds$$
$$\le Q\int_0^{x_2}(x_2-s)^{\underline{\alpha}(x_2)-1}ds \le Qx_2^{\underline{\alpha}(x_2)} \le Qh^{\lambda}.$$

For $3 \le i \le n$, we have

$$\Big|\int_{x_{i-1}}^{x_i}\frac{\eta(s)}{(x_n-s)^{1-\alpha}}ds\Big| \le Qh\|f\|_{C^{2,\lambda}[0,1]}\int_{x_{i-1}}^{x_i}\frac{\int_{x_{i-1}}^{x_i}y^{\lambda-2}dy}{(x_n-s)^{1-\alpha}}ds$$
$$\le Q\|f\|_{C^{2,\lambda}[0,1]}h^2x_{i-1}^{\lambda-2}\big[(x_n-x_{i-1})^{\alpha}-(x_n-x_i)^{\alpha}\big].$$

By the mean value theorem, we have

$$\Big|\int_{x_2}^{x_{\lfloor n/2\rfloor}}\frac{\eta(s)ds}{(x_n-s)^{1-\alpha}}\Big| \le Q\|f\|_{C^{2,\lambda}[0,1]}h^3\sum_{i=2}^{\lfloor n/2\rfloor}x_{i-1}^{\lambda-2}(x_n-x_i)^{\alpha-1}$$
$$\le Q\|f\|_{C^{2,\lambda}}h^3x_{\lfloor n/2\rfloor}^{\alpha-1}\sum_{i=2}^{\lfloor n/2\rfloor}x_{i-1}^{\lambda-2} \le Q\|f\|_{C^{2,\lambda}}h^{\lambda+1}x_n^{\underline{\alpha}(x_n)-1}\sum_{i=2}^{\lfloor n/2\rfloor}(i-1)^{\lambda-2}$$
$$\le Q\|f\|_{C^{2,\lambda}}h^{\lambda+1}x_n^{\underline{\alpha}(0)-1} \le Q\|f\|_{C^{2,\lambda}}h^{2\lambda},$$

and

$$\Big|\int_{x_{\lfloor n/2\rfloor+1}}^{x_n}\frac{\eta(s)ds}{(x_n-s)^{1-\alpha}}\Big|$$
$$\le Q\|f\|_{C^{2,\lambda}[0,1]}h^2x_{\lfloor n/2\rfloor}^{\lambda-2}\sum_{i=\lfloor n/2\rfloor+1}^{n}\big[(x_n-x_{i-1})^{\alpha}-(x_n-x_i)^{\alpha}\big]$$
$$\le Q\|f\|_{C^{2,\lambda}[0,1]}h^2x_n^{\lambda-2}(x_n-x_{\lfloor n/2\rfloor})^{\alpha} \le Q\|f\|_{C^{2,\lambda}[0,1]}h^2x_n^{\lambda+\underline{\alpha}(x_n)-2}$$
$$\le Q\|f\|_{C^{2,\lambda}[0,1]}h^2x_n^{\lambda+\underline{\alpha}(0)-2} \le Q\|f\|_{C^{2,\lambda}[0,1]}h^{2\lambda}.$$

Therefore, we have the following estimate for $2 \le n \le \lfloor N/2\rfloor+1$:

$$|R_n^L| \le Q\|f\|_{C^{2,\lambda}[0,1]}h^{2\lambda}. \tag{4.9}$$

Next, we prove the estimate of R_n^1 for $\lfloor N/2\rfloor+2 \le n \le N+1$. By the previous estimates, we immediately obtain

$$\Big|\sum_{i=2}^{\lfloor N/2\rfloor+1}\int_{x_{i-1}}^{x_i}\frac{\eta(s)ds}{(x_n-s)^{1-\alpha}}ds\Big| \le Q\|f\|_{C^{2,\lambda}[0,1]}h^{2\lambda}.$$

For $\lfloor N/2 \rfloor + 2 \le i \le n-1$, the integral on $[x_{i-1}, x_i]$ can be bounded by

$$\begin{aligned}
\Big| \int_{x_{i-1}}^{x_i} \frac{\eta(s)ds}{(x_n - s)^{1-\alpha}} \Big| &\le h \int_{x_{i-1}}^{x_i} \frac{\int_{x_{i-1}}^{x_i} |v''(y)|dy}{(x_n - s)^{1-\alpha}} ds \\
&\le Qh\|f\|_{C^{2,\lambda}[0,1]} \int_{x_{i-1}}^{x_i} \frac{\int_{x_{i-1}}^{x_i} (1-y)^{\lambda-2} dy}{(x_n - s)^{1-\alpha}} ds \\
&\le Qh^2 \|f\|_{C^{2,\lambda}[0,1]} (1-x_i)^{\lambda-2} \int_{x_{i-1}}^{x_i} (x_n - s)^{\underline{\alpha}(x_n)-1} ds \\
&\le Q\|f\|_{C^{2,\lambda}[0,1]} h^2 (1-x_i)^{\lambda-2} \big[(x_n - x_{i-1})^{\underline{\alpha}(x_n)} - (x_n - x_i)^{\underline{\alpha}(x_n)}\big].
\end{aligned}$$

If $\lfloor N/2 \rfloor + 2 \le i \le 2n - (N+2)$ (which implies $1 - x_n \le x_n - x_i$) we have

$$\begin{aligned}
&\Big| \sum_{i=\lfloor N/2 \rfloor+2}^{2n-(N+2)} \int_{x_{i-1}}^{x_i} \frac{\eta(s)ds}{(x_n - s)^{1-\alpha}} \Big| \\
&\le Q\|f\|_{C^{2,\lambda}[0,1]} h^2 \sum_{i=\lfloor N/2 \rfloor+2}^{2n-(N+2)} (1-x_i)^{\lambda-2} \big[(x_n - x_{i-1})^{\underline{\alpha}(x_n)} - (x_n - x_i)^{\underline{\alpha}(x_n)}\big] \\
&\le Q\|f\|_{C^{2,\lambda}[0,1]} h^3 \sum_{i=\lfloor N/2 \rfloor+2}^{2n-(N+2)} (1-x_i)^{\lambda-2} (x_n - x_i)^{\underline{\alpha}(x_n)-1} \\
&\le Q\|f\|_{C^{2,\lambda}[0,1]} h^3 (1-x_n)^{\underline{\alpha}(x_n)-1} \sum_{i=\lfloor N/2 \rfloor+2}^{2n-(N+2)} (1-x_i)^{\lambda-2} \\
&\qquad \le Q\|f\|_{C^{2,\lambda}[0,1]} h^{2\lambda} \sum_{i=\lfloor N/2 \rfloor+2}^{2n-(N+2)} (N-i+2)^{\lambda-2} \le Q\|f\|_{C^{2,\lambda}[0,1]} h^{2\lambda},
\end{aligned}$$

where we used the fact that $\underline{\alpha} \in C^1[0,1]$ and

$$\begin{aligned}
(1-x_n)^{\underline{\alpha}(x_n)-1} &= (x_{N+2} - x_n)^{\underline{\alpha}(x_{N+2})-1} (x_{N+2} - x_n)^{\underline{\alpha}(x_n) - \underline{\alpha}(x_{N+2})} \\
&\le Qh^{\lambda-1}.
\end{aligned}$$

Otherwise, for $2n - N - 1 \le i \le n$ we have

$$\begin{aligned}
&\Big| \sum_{i=2n-N-1}^{n} \int_{x_{i-1}}^{x_i} \frac{\eta(s)ds}{(x_n - s)^{1-\alpha}} \Big| \\
&\le Q\|f\|_{C^{2,\lambda}[0,1]} h^2 \sum_{i=2n-N-1}^{n-1} (1-x_i)^{\lambda-2} \big[(x_n - x_{i-1})^{\underline{\alpha}(x_n)} - (x_n - x_i)^{\underline{\alpha}(x_n)}\big] \\
&\le Q\|f\|_{C^{2,\lambda}[0,1]} h^2 (1-x_n)^{\lambda-2} (x_n - x_{2n-N-1})^{\underline{\alpha}(x_n)} \\
&\le Q\|f\|_{C^{2,\lambda}[0,1]} h^2 (1-x_n)^{\lambda+\underline{\alpha}(x_n)-2} \le Q\|f\|_{C^{2,\lambda}[0,1]} h^{2\lambda}.
\end{aligned}$$

Therefore, (4.9) holds for $\lfloor N/2 \rfloor + 2 \le n \le N+1$.

Finally, if $n = N+2$, it follows directly from previous estimates that

$$\Big| \sum_{i=2}^{N+1} \int_{x_{i-1}}^{x_i} \frac{\eta(s)ds}{(x_{N+2}-s)^{1-\alpha}} \Big| \le Q\|f\|_{C^{2,\lambda}[0,1]} h^{2\lambda}$$

and

$$\Big| \int_{x_{N+1}}^{x_{N+2}} \frac{\eta(s)ds}{(x_{N+2}-s)^{1-\alpha}} \Big| \le \int_{x_{N+1}}^{x_{N+2}} \frac{\int_{x_{N+1}}^{x_{N+2}} |v'(y)|dy}{(x_{N+2}-s)^{1-\alpha}} ds$$
$$\le Qh^\lambda \int_{x_{N+1}}^{x_{N+2}} (x_{N+2}-s)^{\alpha-1} ds \le Qh^\lambda (1-x_{N+1})^\alpha \le Q\|f\|_{C^{1,\lambda}[0,1]} h^{2\lambda},$$

where we used the fact that $\alpha \ge \underline{\alpha}(x_{N+2}) = \underline{\alpha}(1) \ge \lambda (:= \min\{\underline{\alpha}(0), \underline{\alpha}(1)\})$. That is, (4.9) holds for $n = N+2$ and thus we complete the proof of R_n^L for $n \ge 2$ and thus the estimate of $\|R\|_{\hat{L}^\infty}$.

In particular, if $v \in C^2[0,1]$, which implies $\|v - \Pi_h v\|_{\hat{L}^\infty} \le Q\|v\|_{C^2[0,1]} h^2$, then $R(x_n)$ can be bounded by

$$|R(x_n)| \le Q\|v\|_{C^2[0,1]} h^2 \int_{\underline{\alpha}(x_n)}^{\overline{\alpha}(x_n)} \frac{\omega(\alpha, x_n)}{\Gamma(\alpha+1)} d\alpha \le Q\|f\|_{C^2[0,1]} h^2,$$

which finishes the proof of the estimates of v. Finally we subtract the expression of u in (2.2) from (4.2) to obtain

$$u - u_h = \int_0^x (x-s)(v(s) - v_h(s))ds - x\int_0^1 (1-s)(v(s)-v_h(s))ds.$$

Plugging the estimates of v in the statement of this theorem into this equation leads to the estimate of $u - u_h$. □

5. A fast solution method and its error estimate

In the previous section, we prove error estimates of the proposed numerical scheme, e.g., the $O(h^{2\lambda})$ accuracy of the error under the $\hat{L}^\infty$ norm, based only on the regularity assumptions on the data. For implementation, a straightforward idea is to discretize the distributed-order integral by the composite trapezoidal quadrature. If we apply this quadrature on the fixed interval $[\check{\alpha}, \hat{\alpha}]$, which always contains the range of α for any $x \in [0,1]$, the integrand may be discontinuous if $\omega \ne 0$ on end points of the support of α. In this case, the accuracy of the quadrature may be reduced. If we alternatively discretize the distributed-order integral exactly on the support of α for each spatial node, the resulting stiffness matrix may lose its Toeplitz structure due to the varying of the range of the fractional order at different spatial nodes. In order to retain the $O(h^{2\lambda})$ accuracy of the numerical scheme, we set the mesh size of the distributed-order integral σ as $\sigma = O(h^\lambda)$, which leads to $O(N^\lambda)$ terms in the discretization and thus the

$O(N^{2+\lambda})$ operations to generate the stiffness matrix and $O(N^2)$ memory for storage. To solve the linear system, Gaussian elimination requires $O(N^3)$ operations while the Krylov subspace method requires $O(N^2)$ operations for each iteration, which is expensive for large N.

In this section we propose and analyze a fast method to reduce the storage and computations. The key is to approximately expand the stiffness matrix by a sum of $O(\log N)$ Toeplitz matrices multiplied by diagonal matrices, which can be employed to develop the fast solver for the approximated system without affecting the accuracy of the numerical discretization.

5.1. **An approximated scheme and error estimate.** Let $\tilde{\alpha} = (\check{\alpha} + \hat{\alpha})/2$, $3 \le \nu \in \mathbb{N}^+$ (the value of ν will be determined later), and $i \le n-\nu-1$. By the Taylor expansion we have

$$\begin{aligned}&(n-i+1)^{\alpha+1} - 2(n-i)^{\alpha+1} + (n-i-1)^{\alpha+1}\\ &= \sum_{k=0}^{s} \frac{(\alpha-\tilde{\alpha})^k}{k!}\Big((n-i+1)^{\tilde{\alpha}+1}\ln^k(n-i+1) - 2(n-i)^{\tilde{\alpha}+1}\ln^k(n-i)\\ &\quad +(n-i-1)^{\tilde{\alpha}+1}\ln^k(n-i-1)\Big) + T^{s,l}_{n,i},\end{aligned}$$

where $T^{s,l}_{n,i}$ is the local truncation error given by

$$\begin{aligned}T^{s,l}_{n,i} = &\frac{(\alpha-\tilde{\alpha})^{s+1}}{(s+1)!}\Big((n-i+1)^{\theta+1}\ln^{s+1}(n-i+1)\\ &-2(n-i)^{\theta+1}\ln^{s+1}(n-i) + (n-i-1)^{\theta+1}\ln^{s+1}(n-i-1)\Big)\end{aligned}$$

with θ lying between α and $\tilde{\alpha}$. We apply this expansion to approximate $a_{n,i}$ defined in (4.4) by $\bar{a}_{n,i}$

$$\begin{aligned}\bar{a}_{n,i} = \gamma d(x_n)\sum_{k=0}^{s} c_{n,k}&\Big((n-i+1)^{1+\tilde{\alpha}}\ln^k(n-i+1)\\ &-2(n-i)^{1+\tilde{\alpha}}\ln^k(n-i) + (n-i-1)^{1+\tilde{\alpha}}\ln^k(n-i-1)\Big),\end{aligned} \tag{5.1}$$

where

$$c_{n,k} = \int_{\underline{\alpha}(x_n)}^{\overline{\alpha}(x_n)} \frac{\omega(\alpha, x_n)(\alpha-\tilde{\alpha})^k}{k!\Gamma(\alpha+2)}\, d\alpha.$$

By symmetry, $a_{n,i}$ for $i \ge n+\nu+1$ can be approximated in a similar manner with corresponding local truncation error $T^{s,r}_{n,i}$ defined for $i \ge n+\nu+1$. Therefore, the matrix $\mathbf{A}$ can be approximated by $\bar{\mathbf{A}} = (\bar{a}_{n,i})_{n,i=1}^{N+2}$ with $\bar{a}_{n,i} = a_{n,i}$ for $|i-n| \le \nu$ or $n, i = 1, N+2$,

$$\bar{a}_{n,i} = \gamma \sum_{k=0}^{s} c_{n,k} \Big((n-i+1)^{\tilde{\alpha}+1} \ln^k (n-i+1) - 2(n-i)^{\tilde{\alpha}+1} \ln^k (n-i) + (n-i-1)^{\tilde{\alpha}+1} \ln^k (n-i-1) \Big), \quad 2 \le i \le n-\nu-1,$$

and for $n+\nu+1 \le i \le N+1$,

$$\bar{a}_{n,i} = (1-\gamma) \sum_{k=0}^{s} c_{n,k} \Big((i-n+1)^{\tilde{\alpha}+1} \ln^k (i-n+1) - 2(i-n)^{\tilde{\alpha}+1} \ln^k (i-n) + (i-n-1)^{\tilde{\alpha}+1} \ln^k (i-n-1) \Big).$$

The local truncation error is given by

$$T_n^s = d(x_n) \int_{\underline{\alpha}(x_n)}^{\overline{\alpha}(x_n)} \frac{\omega(\alpha, x_n) h^\alpha}{\Gamma(\alpha+2)} \Big(\gamma \sum_{i=2}^{n-\nu-1} T_{n,i}^{s,l} + (1-\gamma) \sum_{i=n+\nu+1}^{N+1} T_{n,i}^{s,r} \Big) d\alpha. \tag{5.2}$$

Then the approximated linear system can be written as

$$\bar{\mathbf{A}} \bar{\mathbf{v}} = -\mathbf{f}, \ \bar{\mathbf{v}} := [\bar{v}_h(x_1), \cdots, \bar{v}_h(x_{N+2})]^\top \tag{5.3}$$

for some $\bar{v}_h \in S_h$. The corresponding approximation $\bar{u}_h$ of u is defined by the second equation of (2.2) with v replaced by $\bar{v}_h$.

THEOREM 5.1. *By setting* $s \ge \Big\lfloor \dfrac{(1+\hat{\alpha}+2\lambda)\log(N+1)}{\log 2 - \log(\hat{\alpha}-\check{\alpha})} \Big\rfloor + 1,$ *the local truncation error* T_n^s *can be bounded by*

$$|T_n^s| \le Q h^{2\lambda}, \ 1 \le n \le N+2.$$

P r o o f. By Theorem 5 in [17, 18], $T_{n,i}^{s,l}$ and $T_{n,i}^{s,r}$ with suitable indexes n and i can be bounded by

$$|T_{n,i}^{s,l}| \le Q \frac{(\hat{\alpha}-\check{\alpha})^s (n-i+1)^{\hat{\alpha}}}{2^s \sqrt{s}}, \quad |T_{n,i}^{s,r}| \le Q \frac{(\hat{\alpha}-\check{\alpha})^s (i-n+1)^{\hat{\alpha}}}{2^s \sqrt{s}}.$$

Therefore, T_n^s defined in (5.2) can be bounded by

$$|T_n^s| \le Q \frac{(\hat{\alpha}-\check{\alpha})^s}{2^s \sqrt{s} h^{\hat{\alpha}+1}},$$

By setting $s = \Big\lfloor \dfrac{(1+\hat{\alpha}+2\lambda)\log(N+1)}{\log 2 - \log(\hat{\alpha}-\check{\alpha})} \Big\rfloor + 1$, we get $\big((\hat{\alpha}-\check{\alpha})/2\big)^s \le h^{1+\hat{\alpha}+2\lambda}$, which completes the proof. □

THEOREM 5.2. *Suppose the Assumption (A) and (3.4) hold, $d, f \in C^{2,\lambda}[0,1]$ and $s \geq \left\lfloor \frac{(1+\hat{\alpha}+2\lambda)\log(N+1)}{\log 2 - \log(\hat{\alpha}-\check{\alpha})} \right\rfloor + 1$, then the following estimates hold*

$$\|v - \bar{v}_h\|_{\hat{L}^\infty} \leq Q\|f\|_{C^{2,\lambda}[0,1]} h^{2\lambda}, \quad \|u - \bar{u}_h\|_{\hat{L}^\infty} \leq Q\|f\|_{C^{2,\lambda}[0,1]} h^{2\lambda}.$$

Here $\bar{u}_h$ is defined by the second equation of (2.2) with v replaced by $\bar{v}_h$.

P r o o f. By Theorem 5.1 we get $\|\mathbf{A} - \bar{\mathbf{A}}\|_\infty \leq Qh^{2\lambda}$, where $\|\cdot\|_\infty$ denote the l^∞ norm of a vector or a matrix. Combining this with Theorem 4.1, we obtain $\|\mathbf{A}^{-1}\|_\infty \leq 1/(1-\mu)$ [34] and thus

$$\|\mathbf{A}^{-1}\|_\infty \|\mathbf{A} - \bar{\mathbf{A}}\|_\infty \leq Qh^{2\lambda}/(1-\mu).$$

We subtract (5.3) from (4.6) to get

$$\mathbf{v} - \bar{\mathbf{v}} = \mathbf{A}^{-1}(\mathbf{A} - \bar{\mathbf{A}})(\mathbf{v} - \bar{\mathbf{v}}) - \mathbf{A}^{-1}(\mathbf{A} - \bar{\mathbf{A}})\mathbf{v},$$

which yields

$$\|\mathbf{v} - \bar{\mathbf{v}}\|_\infty \leq \|\mathbf{A}^{-1}\|_\infty \|\mathbf{A} - \bar{\mathbf{A}}\|_\infty (\|\mathbf{v} - \bar{\mathbf{v}}\|_\infty + \|\mathbf{v}\|_\infty).$$

For h sufficiently small we have $\|\mathbf{A}^{-1}\|_\infty \|\mathbf{A} - \bar{\mathbf{A}}\|_\infty \leq 1/2$, which leads to

$$\frac{\|\mathbf{v} - \bar{\mathbf{v}}\|_\infty}{\|\mathbf{v}\|_\infty} \leq 2\|\mathbf{A}^{-1}\|_\infty \|\mathbf{A} - \bar{\mathbf{A}}\|_\infty \leq Qh^{2\lambda}. \tag{5.4}$$

We combine Theorem 3.1 and Theorem 4.2 to bound $\mathbf{v}$ by

$$\|\mathbf{v}\|_{\hat{L}^\infty} \leq \|\mathbf{v}\|_{\hat{L}^\infty} + \|\mathbf{v} - \bar{\mathbf{v}}\|_{\hat{L}^\infty} \leq Q\|f\|_{C[0,1]} + Q\|f\|_{C^{2,\lambda}} h^{2\lambda} \leq Q\|f\|_{C^{2,\lambda}},$$

which, together with (5.4), yields $\|\mathbf{v} - \bar{\mathbf{v}}\|_\infty \leq Q\|f\|_{C^{2,\lambda}} h^{2\lambda}$. Combining this with the estimate of $e = v - v_h$ proved in Theorem 4.2 we obtain the estimate of $v - \bar{v}_h$ and apply this to find that of $u - \bar{u}_h$. □

5.2. **Matrix structure and fast method.** We observe from Theorem 5.2 that we may solve the approximated linear system (5.3) instead of (4.1) without loss of accuracy. Then we remain to reduce the memory requirement and computational cost of solving (5.3), or more specifically, performing the matrix-vector multiplication $\bar{\mathbf{A}}\mathbf{w}$ for $\mathbf{w} \in \mathbb{R}^{N+2}$. We divide the approximated matrix $\bar{\mathbf{A}}$ into the following block form

$$\bar{\mathbf{A}} = \begin{bmatrix} a_{1,1} & \mathbf{A}_{1,c} & a_{1,N+2} \\ \mathbf{A}_{c,1} & \bar{\mathbf{A}}_c & \mathbf{A}_{c,N+2} \\ a_{N+2,1} & \mathbf{A}_{N+2,c} & a_{N+2,N+2} \end{bmatrix}. \tag{5.5}$$

For any vector $\mathbf{w} = [w_1, w_2, \cdots, w_{N+2}]^T = [w_1, \mathbf{w}_c, w_{N+2}]^T \in \mathbb{R}^{N+2}$, direct calculations yield

$$\bar{\mathbf{A}}\mathbf{w} = \begin{bmatrix} a_{11}w_1 + \mathbf{A}_{1,c}\mathbf{w}_c + a_{1,N+2}w_{N+2} \\ w_1\mathbf{A}_{c,1} + \bar{\mathbf{A}}_c\mathbf{w}_c + w_{N+2}\mathbf{A}_{c,N+2} \\ a_{N+2,1}w_1 + \mathbf{A}_{N+2,c}\mathbf{w}_c + a_{N+2,N+2}w_{N+2} \end{bmatrix}. \tag{5.6}$$

It suffices to study the fast implementation of $\bar{\mathbf{A}}_c\mathbf{w}_c$ as the rest of the entries in (5.6) can be evaluated in $O(N)$ operations. By (5.1), matrix $\bar{\mathbf{A}}_c$ can be decomposed by

$$\bar{\mathbf{A}}_c = \mathbf{B}^\nu + \mathrm{diag}(\mathbf{D})\sum_{k=0}^{s}\mathrm{diag}(\mathbf{C}^k)\big(\gamma\mathbf{T}^{(k)} + (1-\gamma)(\mathbf{T}^{(k)})^\top\big). \tag{5.7}$$

Here $\mathbf{B}^\nu = \big(b_{n,i}\big)_{n,i=1}^N$ is a band matrix defined by $b_{n,i} = a_{n,i}$ for $|n-i| \le \nu$ and zeros otherwise. The vectors $\mathbf{C}^k := (c_{n+1,k})_{n=1}^N$ for $0 \le k \le s$ and $\mathbf{D} := \mathrm{diag}\big(\{d(x_n)\}_{i=1}^{N+2}\big)$. The Toeplitz matrices $\mathbf{T}^{(k)}$ for $0 \le k \le s$ are generated with $\mathbf{t}^k$ and $\mathbf{0}$ being the first columns and rows where $\mathbf{t}^k = (t_i^k)_{i=1}^N$ are given by $t_i^k = 0$ for $1 \le i \le \nu$ and

$$t_i^k = (i-2)^{1+\tilde{\alpha}}\ln^k(i-2) - 2(i-1)^{1+\tilde{\alpha}}\ln^k(i-1) + i^{1+\tilde{\alpha}}\ln^k i$$

for $\nu+1 \le i \le N$.

THEOREM 5.3. *Let $\nu = s = O(\log N)$. Then $\bar{\mathbf{A}}_c$ can be stored in $O(N\log N)$ memory and the matrix-vector multiplication of $\bar{\mathbf{A}}_c\mathbf{w}_c$ for $\mathbf{w}_c \in \mathbb{R}^N$ requires $O(N\log^2 N)$ operations. Furthermore, $O(N^{1+\lambda}\log N)$ operations are needed to generate components $\mathbf{D}, \mathbf{C}^k$ and $\mathbf{T}^k$ for $0 \le k \le s$ in $\bar{\mathbf{A}}$ in order to keep the $O(h^{2\lambda})$ accuracy.*

P r o o f. We observe from (5.7) that we require $O(\nu N)$ storage for $\mathbf{B}^\nu$ and $O(\nu N)$ computations for evaluating $\mathbf{B}^\nu\mathbf{w}_c$. To keep the order of magnitude of computations, we set $\nu = s$ in this paper. Composite trapezoidal formula with N^λ points can be applied to evaluate the integrals in $\mathbf{C}^k$ for $0 \le k \le s$ with the local truncation error $O(h^{2\lambda})$, and the total computation is $O(N^{1+\lambda}\log N)$. The matrix $\bar{\mathbf{A}}_c^{(k)} = \gamma\mathbf{T}^{(k)} + (1-\gamma)(\mathbf{T}^{(k)})^\top$ is a Toeplitz matrix, which requires $O(N)$ storage and $O(N\log N)$ operations for evaluating $\bar{\mathbf{A}}_c^{(k)}\mathbf{w}$ via the fast Fourier transform (FFT). These observations lead to the conclusions of this theorem. □

6. Numerical experiments

We investigate the performances of the approximated fast conjugate gradient squared (FCGS) method for solving model (1.2), which employs the standard CGS incorporated with the FFT to solve the approximated linear system (5.3), by comparing it with the traditional Gaussian elimination (Gauss) method and the CGS. All these methods are implemented on Matlab 2016b on a computer with Intel(R) Corel i7-9700 and Ram 16GB.

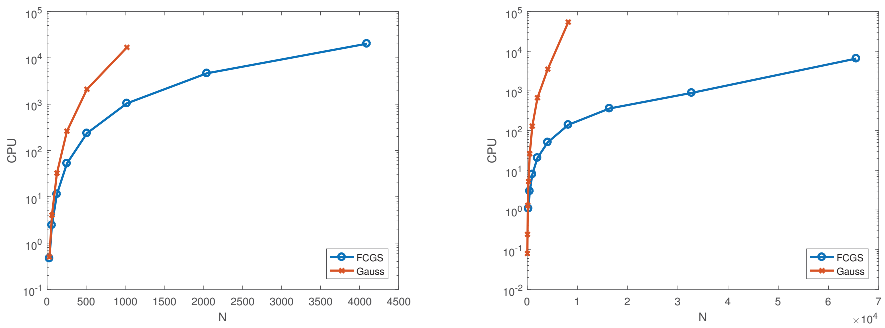

FIGURE 1. CPU of Gauss and FCGS for Experiment 1 (left) and Experiment 2 (right)

The symbol '-' in tables implies that the running time of the program exceeds one day. We measure the CPU times (CPU) of generating coefficients and solving the linear systems and the relative errors in the norm $\|\cdot\|_{\tilde{L}^\infty}$ defined by, e.g., $\|v - v_h\|_{\tilde{L}^\infty} := \|v - v_h\|_{\hat{L}^\infty}/\|v\|_{\hat{L}^\infty}$, as well as the corresponding convergence rates (CR).

6.1. **Experiment 1: Smooth solutions.** We test the aforementioned methods for (1.2) with $d(x) = 1$, $\omega = 2$, $\underline{\alpha}(x) = 1/8+x/4$, $\overline{\alpha}(x) = 5/8+x/4$, $u(x) = 16x^2(1-x)^2$ and $f(x)$ is evaluated accordingly. As u is smooth, we expect $O(h^2)$ accuracy of the numerical scheme, which requires the N-points composite trapezoidal formula to distcretize the distributed-order integral in order to keep the accuracy. Thus it takes $O(N^3)$ operations to compute the matrix entries and to solve the linear system for in traditional methods, while in the FCGS, only $O(sN^2) = O(N^2 \log N)$ operations are needed to evaluate the integrals in $\mathbf{C}^k$ and $O(N \log^2 N)$ operations are required in each iteration to solve the approximated linear system. Numerical results are presented in the left figure of Figure 1 and Table 1, which coincide with the theoretical analysis.

6.2. **Experiment 2: Non-smooth solutions.** Let $\gamma = 0.5$, $f(x) = 1$, $d(x) = (1 + x)$, $\underline{\alpha}(x) = 1/4 + x/10$, $\overline{\alpha}(x) = 3/4 + x/10$ and $\omega(\alpha, x) = 20\alpha/(5 + x)$. As the exact solution is not available, we use the numerical solution of FCGS with $N = 2^{16}$ as the reference solution. In this case $\lambda = 0.25$ and we thus use the composite trapezoidal formula of $N^{\frac{1}{4}}$ points, which leads to the $O(N^{1/4+2})$ computational cost of generating the entries of $\mathbf{A}$ and $O(N^3)$ operations of solving the linear system in the traditional method. Instead, the FCGS only takes $O(N^{5/4} \log N + N \log^2 N)$ for computing the components of $\bar{\mathbf{A}}$ and $O(N \log^2 N)$ for solving the approximated linear system in each iteration, respectively. Numerical result are presented

Table 1. Errors and CPU times (seconds) of Experiment 1

	N	$\|v-v_h\|_{\tilde{L}^\infty}$	CR_v	$\|u-u_h\|_{\tilde{L}^\infty}$	CR_u	CPU
	2^5	4.1818e-4	–	2.5084e-3	–	0.49
	2^6	1.0310e-4	2.02	5.9940e-4	2.07	4.01
Gauss	2^7	2.5741e-5	2.00	1.4599e-4	2.04	32
	2^8	6.4604e-6	1.99	3.5923e-5	2.02	217
	2^9	1.6245e-6	1.99	8.8902e-6	2.01	2096
	2^{10}	4.0862e-7	1.99	2.2075e-6	2.01	16704
	2^{11}	–	–	–	–	> 1 *day*
	2^5	4.1818e-4	–	2.5084e-3	–	0.50
	2^6	1.0310e-4	2.02	5.9940e-4	2.07	4.00
	2^7	2.5741e-5	2.00	1.4599e-4	2.04	32
CGS	2^8	6.4604e-6	1.99	3.5923e-5	2.02	260
	2^9	1.6245e-6	1.99	8.8902e-6	2.01	2092
	2^{10}	4.0862e-7	1.99	2.2075e-6	2.01	16813
	2^{11}	–	–	–	–	> 1 *day*
	2^5	4.1818e-4	–	2.5084e-3	–	0.47
	2^6	1.0310e-4	2.02	5.9940e-4	2.07	2.44
	2^7	2.5741e-5	2.00	1.4599e-4	2.04	11.4
FCGS	2^8	6.4604e-6	1.99	3.5923e-5	2.02	52
	2^9	1.6245e-6	1.99	8.8902e-6	2.01	235
	2^{10}	4.0862e-7	1.99	2.2075e-6	2.01	1042
	2^{11}	1.0275e-7	1.99	5.4924e-7	2.00	4642
	2^{12}	2.5826e-8	1.99	1.3683e-7	2.00	20210

in the right figure of Figure 1 and Tables 2-3, which show that the convergence rates of v are consistent with the theoretical results while that of u are higher than the expectations that needs further investigation.

7. Appendix

We refer several lemmas from [14] to support the preceding estimates.

Lemma 7.1. *If k in (2.2) satisfies $|l(x,s)| \leq Q$ and $|l'_x(x,s)| \leq Q/|x-s|$ for $s \neq x$, then $K \in \mathcal{L}(C[0,1], C^{\breve{\alpha}}[0,1])$. $K \in \mathcal{L}(C[0,1], C^{\beta}[0,1])$ for $0 < \beta < 1$ belongs to $\mathcal{K}(C^{\mu}[0,1], C^{\mu}[0,1])$ for $0 \leq \mu \leq \beta$.*

Lemma 7.2. *If the homogeneous integral equation $(I+K)v = 0$ has only the trivial solution for $K \in \mathcal{K}(X,X)$, then $(I+K)^{-1} \in \mathcal{L}(X,X)$ and equation (2.2) has a unique solution for each $-f \in X$ given by $v = -(I+K)^{-1} f \in X$.*

Table 2. Errors and convergence rates of Experiment 2

	N	$\|v-v_h\|_{\tilde{L}^\infty}$	CR_v	$\|u-u_h\|_{\tilde{L}^\infty}$	CR_u
	2^5	5.7179e-2	–	1.0398e-1	–
	2^6	4.4607e-2	0.36	5.5503e-2	0.91
	2^7	3.3856e-2	0.40	2.8673e-2	0.95
Gauss	2^8	2.5095e-2	0.42	1.4539e-2	0.98
	2^9	1.8442e-2	0.44	7.3034e-3	0.99
	2^{10}	1.3434e-2	0.46	3.6387e-3	1.01
	2^{11}	9.7176e-3	0.47	1.7944e-3	1.02
	2^{12}	6.9687e-2	0.48	8.6924e-4	1.05
	2^5	5.7179e-2	–	1.0398e-1	–
	2^6	4.4607e-2	0.36	5.5503e-2	0.91
	2^7	3.3856e-2	0.40	2.8973e-2	0.95
FCGS	2^8	2.5095e-2	0.43	1.4539e-2	0.98
	2^9	1.8442e-2	0.44	7.3034e-3	0.99
	2^{10}	1.3434e-2	0.46	3.6387e-3	1.01
	2^{11}	9.7176e-3	0.47	1.7944e-3	1.02
	2^{12}	6.9687e-3	0.48	8.6924e-4	1.05

Table 3. CPU times (seconds) for Experiment 2

N	2^8	2^9	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}	2^{15}	2^{16}
Gauss	5.2	26	130	670	3533	54392	–	–	–
CGS	4.1	20	98	455	2083	10388	–	–	–
FCGS	1.1	3.0	7.9	21	51	140	361	894	6546

Lemma 7.3. *If $K \in \mathcal{K}(X,X)$, then $\Pi_h K$ converges to K in $\mathcal{L}(X,X)$ as $h \to 0$. In particular, if $(I+K)^{-1} \in \mathcal{L}(X,X)$, then for h sufficiently small*

$$\|(I+\Pi_h K)^{-1}\|_{\mathcal{L}(X,X)} \le Q\|(I+K)^{-1}\|_{\mathcal{L}(X,X)}.$$

Lemma 7.4. *Suppose Assumption (A) holds, then $p(x) \in C^{\underline{\alpha}(0)}[0,1]$, $q(x) \in C^{\underline{\alpha}(1)}[0,1]$, here*

$$p(x) = \int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \omega(\alpha,x)x^\alpha d\alpha, \quad q(x) = \int_{\underline{\alpha}(x)}^{\overline{\alpha}(x)} \omega(\alpha,x)(1-x)^\alpha d\alpha.$$

P r o o f. By symmetry, it suffices to analyze p. For $0 \le x_1 < x_2 \le 1$ we decompose $p(x_2)-p(x_1)$ by

$$p(x_2)-p(x_1)=\Big(\int_{\underline{\alpha}(x_2)}^{\overline{\alpha}(x_2)}\omega(\alpha,x_2)x_2^\alpha d\alpha-\int_{\underline{\alpha}(x_1)}^{\overline{\alpha}(x_1)}\omega(\alpha,x_2)x_2^\alpha d\alpha\Big)$$
$$+\Big(\int_{\underline{\alpha}(x_1)}^{\overline{\alpha}(x_1)}\omega(\alpha,x_2)x_2^\alpha d\alpha-\int_{\underline{\alpha}(x_1)}^{\overline{\alpha}(x_1)}\omega(\alpha,x_1)x_2^\alpha d\alpha\Big)$$
$$+\Big(\int_{\underline{\alpha}(x_1)}^{\overline{\alpha}(x_1)}\omega(\alpha,x_1)x_2^\alpha d\alpha-\int_{\underline{\alpha}(x_1)}^{\overline{\alpha}(x_1)}\omega(\alpha,x_1)x_1^\alpha d\alpha\Big)=J_1+J_2+J_3.$$

By Assumption (A), J_1 and J_2 can be simply bounded by $Q(x_2-x_1)$ and we remain to bound J_3. If $x_2-x_1\geq x_1$, we apply $x_2^\alpha-x_1^\alpha\leq(x_2-x_1)^\alpha$ and $\alpha\in[\underline{\alpha}(x_1),\overline{\alpha}(x_1)]$ to obtain

$$(x_1-x_1)^\alpha\leq(x_2-x_1)^{\underline{\alpha}(x_1)}\leq(x_2-x_1)^{\underline{\alpha}(0)}(x_2-x_1)^{\alpha(x_1)-\underline{\alpha}(0)}$$
$$\leq(x_2-x_1)^{\underline{\alpha}(x_0)}x_1^{-\|\underline{\alpha}\|_{C^1[0,1]}(x_1-0)}\leq Q(x_2-x_1)^{\underline{\alpha}(0)}.$$

Otherwise we have

$$x_2^\alpha-x_1^\alpha=\alpha\xi^{\alpha-1}(x_2-x_1)\leq\alpha x_1^{\alpha-1}(x_2-x_1)$$
$$=\alpha x_1^{\underline{\alpha}(0)-1}x_1^{\underline{\alpha}(x_1)-\underline{\alpha}(0)}(x_2-x_1)\leq Q(x_2-x_1)^{\underline{\alpha}(0)}.$$

Thus we bound $|J_3|$ by $Q(x_2-x_1)^{\underline{\alpha}(0)}$ and thus complete the proof. □

LEMMA 7.5. *If Assumption (A) holds, $g\in C^\beta[0,1]$ for $\beta\geq0$ and $0<\check{\alpha}+\beta<1$, then $\phi_g^{l_2}\in C^{\underline{\alpha}(0)}[0,1],\phi_g^{r_2}\in C^{\underline{\alpha}(1)}[0,1]$, and $\phi_g^{l_1},\phi_g^{r_1}\in C^{\check{\alpha}+\beta}[0,1]$ with*

$$\|\phi_g^{l_1}\|_{C^{\check{\alpha}+\beta}[0,1]}\leq Q\|g\|_{C^\beta[0,1]},\quad \|\phi_g^{r_1}\|_{C^{\check{\alpha}+\beta}[0,1]}\leq Q\|g\|_{C^\beta[0,1]}. \tag{7.1}$$

P r o o f. By Lemma 7.4 we have $\phi_g^{r_2}\in C^{\underline{\alpha}(0)}[0,1]$ and $\phi_g^{r_2}\in C^{\underline{\alpha}(1)}[0,1]$ and we remain to estimate $\phi_g^{l_1}$ by symmetry. For $0\leq x_1<x_2\leq1$ we decompose

$$\phi_g^{l_1}(x_2)-\phi_g^{l_1}(x_1)=J_4+J_5+J_6$$
$$:=\Big(\int_{\underline{\alpha}(x_2)}^{\overline{\alpha}(x_2)}\omega(\alpha,x_2)\psi(\alpha,x_2)d\alpha-\int_{\underline{\alpha}(x_1)}^{\overline{\alpha}(x_1)}\omega(\alpha,x_2)\psi(\alpha,x_2)d\alpha\Big)$$
$$+\Big(\int_{\underline{\alpha}(x_1)}^{\overline{\alpha}(x_1)}\omega(\alpha,x_2)\psi(\alpha,x_2)d\alpha-\int_{\underline{\alpha}(x_1)}^{\overline{\alpha}(x_1)}\omega(\alpha,x_1)\psi(\alpha,x_2)d\alpha\Big)$$
$$+\Big(\int_{\underline{\alpha}(x_1)}^{\overline{\alpha}(x_1)}\omega(\alpha,x_1)\psi(\alpha,x_2)d\alpha-\int_{\underline{\alpha}(x_1)}^{\overline{\alpha}(x_1)}\omega(\alpha,x_1)\psi(\alpha,x_1)d\alpha\Big).$$

J_4 and J_5 can be simply bounded by $Q\|\psi\|_{C[0,1]}|x_2-x_1|$. To estimate J_6, we define a function $\psi(x)=\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^x\frac{g(s)-g(0)}{(x-s)^{1-\alpha}}ds$. As is proved in [40], if $\alpha + \beta < 1$ and $g \in C^{\beta}[0,1]$, then $\psi(x) \in C^{\alpha+\beta}[0,1] \subset C^{\check{\alpha}+\beta}[0,1]$ with the estimate $\|\psi\|_{C^{\check{\alpha}+\beta}} \leq Q\|g\|_{C^{\beta}[0,1]}$. If $\alpha + \beta = 1$, we may slightly reduce β to ensure the above conclusion. If $\alpha + \beta > 1$, by Theorem 3.3 in [40], we have $\|\psi\|_{C^{\check{\alpha}+\beta}} \leq Q\|\psi\|_{C^1} \leq Q\|\psi\|_{C^{\beta}}$. We conclude from these estimates that $|J_6|$ can be bounded by $Q|x_2 - x_1|^{\check{\alpha}+\beta}$, which completes the proof. □

LEMMA 7.6. *Suppose that* $g \in C^{\beta}[0,1]$ *for* $\beta > 0$ *and* $\check{\alpha} + \beta > 1$ *and Assumption (A) holds, then* $\phi_g^{l_1}, \phi_g^{r_1} \in C^1[0,1]$ *and*

$$\|\phi_g^{l_1}\|_{C^1[0,1]} \leq Q\|g\|_{C^{\beta}[0,1]}, \quad \|\phi_g^{r_1}\|_{C^1[0,1]} \leq Q\|g\|_{C^{\beta}[0,1]}. \tag{7.2}$$

P r o o f. The proof can be performed in parallel with that of Theorem 3.3 in [40] and thus be omitted. □

Acknowledgements

The authors would like to express their most sincere thanks to the referees for their very helpful comments and suggestions, which greatly improved the quality of this paper. This work was partially funded by the National Natural Science Foundation of China under Grants 11971272 and 12001337, by the ARO MURI Grant W911NF-15-1-0562, by the National Science Foundation under Grant DMS-2012291, by the China Postdoctoral Science Foundation 2021TQ0017, by the International Postdoctoral Exchange Fellowship Program (Talent-Introduction Program) YJ20210019, and by the Natural Science Foundation of Shandong Province under Grant ZR2019BA026.

References

[1] R. Adams and J. Fournier, *Sobolev Spaces*. Elsevier (2003).

[2] R. Bagley and P. Torvik, On the existence of the order domain and the solution of distributed order equations–Part I. *Int. J. Appl. Math.* **2**, No 7 (2000), 865–882.

[3] D.A. Benson, R.A. Schumer, M.M. Meerschaert, and S.M. Wheetcraft, Fractional dispersion, Lévy motion, and the MADE tracer tests. *Transport Porous Med.* **42**, No 1 (2001), 211–240; DOI: 10.1023/A:1006733002131.

[4] M. Caputo, Diffusion with space memory modelled with distributed order space fractional differential equations. *Ann. Geophys.* **46**, No 2 (2009), 223–234; DOI: 10.4401/ag-3395.

[5] A.V. Chechkin, R. Gorenflo, I.M. Sokolov and V.Y. Gonchar, Distributed order time fractional diffusion equation. *Fract. Calc. Appl. Anal.* **6**, No 3 (2003), 259–279.

[6] D. del-Castillo-Negrete, Front propagation in reaction-diffusion systems with anomalous diffusion. *Bol. Soc. Mat. Mex.* **20**, No 1 (2014), 87–105; DOI: 10.1007/s40590-014-0008-8.

[7] K. Diethelm and N.J. Ford, Numerical solution methods for distributed order differential equations. *Fract. Calc. Appl. Anal.* **4**, No 4 (2001), 531–542.

[8] K. Diethelm and N.J. Ford, Numerical analysis for distributed-order differential equations. *J. Comput. Appl. Math.* **225**, No 1 (2009), 96–104; DOI: 10.1016/j.cam.2008.07.018.

[9] V.J. Ervin, N. Heuer and J.P. Roop, Regularity of the solution to $1-D$ fractional order diffusion equations. *Math. Comput.* **87**, No 313 (2018), 2273–2294; DOI: 10.1090/mcom/3295.

[10] V.J. Ervin, Regularity of the solution to fractional diffusion, advection, reaction equations in weighted Sobolev spaces. *J. Differ. Equ.* **278** (2021), 294–325; DOI: 10.1016/j.jde.2020.12.034.

[11] G. Gao, H. Sun, and Z. Sun, Some high-order difference schemes for the distributed-order differential equations. *J. Comput. Phys.* **298** (2015), 337–359; DOI: 10.1016/j.jcp.2015.05.047.

[12] R. Garrappa and E. Kaslik, Stability of fractional-order systems with Prabhakar derivatives. *Nonlinear Dyn.* **102** (2020), 567–578; DOI: 10.1007/s11071-020-05897-9.

[13] R. Gorenflo, Y. Luchko and M. Stojanović. Fundamental solution of a distributed order time-fractional diffusion-wave equation as probability density. *Fract. Calc. Appl. Anal.* **16**, No 2 (2013), 297–316;
DOI: 10.2478/s13540-013-0019-6;
https://www.degruyter.com/journal/key/fca/16/2/html.

[14] W. Hackbusch, *Integral Equations: Theory and Numerical Treatment.* Birkhauser Verlag (1995).

[15] J. Jia and H. Wang, A fast finite difference method for distributed-order space-fractional partial differential equations on convex domains. *Comput. Math. Appl.* **75**, No 6 (2018), 2031–2043;
DOI: 10.1016/j.camwa.2017.09.003.

[16] J. Jia and H. Wang, A fast finite volume method for conservative space-time fractional diffusion equations discretized on space-time locally refined meshes. *Comput. Math. Appl.* **78**, No 1 (2019), 1345–1356;
DOI: 10.1016/j.camwa.2019.04.003.

[17] J. Jia, H. Wang and X. Zheng, A fast collocation approximation to a two-sided variable-order space-fractional diffusion equation and its analysis. *J. Comput. Appl. Math.* **388** (2021), Art. 113234;
DOI: 10.1016/j.cam.2020.113234.

[18] J. Jia, X. Zheng, H. Fu, P. Dai and H. Wang, A fast method for variable-order space-fractional diffusion equations. *Numer. Algor.* **85** (2020), 1519–1540; DOI: 10.1007/s11075-020-00875-z.

[19] B. Jin, R. Lazarov, J. Pasciak and W. Rundell, Variational formulation of problems involving fractional order differential operators. *Math. Comp.* **84**, No 296 (2015), 2665–2700; DOI: 10.1090/mcom/2960.

[20] B. Jin, R. Lazarov, D. Sheen and Z. Zhou, Error estimates for approximations of distributed-order time fractional diffusion with nonsmooth data. *Fract. Calc. Appl. Anal.* **19**, No 1 (2016), 69–93; DOI: 10.1515/fca-2016-0005; https://www.degruyter.com/journal/key/fca/19/1/html.

[21] J. Li, F. Liu, L. Feng, I. Turner, A novel finite volume method for the Riesz space distributed-order diffusion equation. *Comput. Math. Appl.* **74**, No 4 (2017), 772–783; DOI: 10.1016/j.camwa.2017.05.017.

[22] X. Li, Z. Mao, N. Wang, F. Song, H. Wang, and G.E. Karniadakis, A fast solver for spectral elements applied to fractional differential equations using hierarchical matrix approximation. *Comput. Methods Appl. Mech. Engrg.* **366** (2020), Art. 113053; DOI: 10.1016/j.cma.2020.113053.

[23] Z. Li, Y. Luchko, M. Yamamoto, Asymptotic estimates of solutions to initial-boundary-value problems for distributed order time-fractional diffusion equations. *Fract. Calc. Appl. Anal.* **17**, No 4 (2014), 1114–1136; DOI: 10.2478/s13540-014-0217-x;
https://www.degruyter.com/journal/key/fca/17/4/html.

[24] Z. Li, Y. Luchko, M. Yamamoto, Analyticity of solutions to a distributed order time-fractional diffusion equation and its application to an inverse problem. *Comput. & Math. Appl.* **73**, No 6 (2017), 1041–1052; DOI: 10.1016/j.camwa.2016.06.030.

[25] X. Lin, M. Ng, and H. Sun, A splitting preconditioner for Toeplitz-like linear systems arising from fractional diffusion equations. *SIAM J. Matrix Anal. Appl.* **38**, No 4 (2017), 1580–1614; DOI: 10.1137/17M1115447.

[26] C. Lorenzo and T. Hartley, Variable order and distributed order fractional operators. *Nonlinear Dyn.* **29**, No 1 (2002), 57–98; DOI: 10.1023/A:1016586905654.

[27] Y. Luchko, Boundary value problems for the generalized time-fractional diffusion equation of distributed order. *Fract. Calc. Appl. Anal.* **12**, No 4 (2009), 409–422.

[28] F. Mainardi, G. Pagnini, and R. Gorenflo, Some aspects of fractional diffusion equations of single and distributed order. *Appl. Math. Comput.* **187**, No 1 (2007), 295–305; DOI: 10.1016/j.amc.2006.08.126.

[29] M. Meerschaert and A. Sikorskii, *Stochastic Models for Fractional Calculus.* De Gruyter (2012).

[30] R. Metzler and J. Klafter, The random walk's guide to anomalous diffusion: a frational dynamics approach. *Phys. Rep.* **339**, No 1 (2000), 1–77; DOI: 10.1016/S0370-1573(00)00070-3.

[31] M. Morgado and M. Rebelo, Numerical approximation of distributed order reaction-diffusion equations. *J. Comput. Appl. Math.* **275** (2015), 216–227; DOI: 10.1016/j.cam.2014.07.029.

[32] I. Podlubny, *Fractional Differential Equations.* Academic Press (1999).

[33] H. Sun, W. Chen, H. Wei and Y. Chen, A comparative study of constant-order and variable-order fractional models in characterizing memory property of systems. *Eur. Phys. J. Spec. Top.* **193**, No 1 (2011), 185–192; DOI: 10.1140/epjst/e2011-01390-6.

[34] J. Varah, A lower bound for the smallest singular value of a matrix. *Linear Algebra Appl.* **11**, No 1 (1975), 3–5; DOI: 10.1016/0024-3795(75)90112-3.

[35] H. Wang and T.S. Basu, A fast finite difference method for two-dimensional space-fractional diffusion equations. *SIAM J. Sci. Comput.* **34** (2012), A2444–A2458; DOI: 10.1137/12086491X.

[36] H. Wang, K. Wang, and T. Sircar, A direct $O(N \log^2 N)$ finite difference method for fractional diffusion equations. J. Comput. Phys. **229**, No 21 (2010), 8095–8104; DOI: 10.1016/j.jcp.2010.07.011.

[37] H. Wang, D. Yang, S. Zhu, Inhomogeneous Dirichlet boundary-value problems of space-fractional diffusion equations and their finite element approximations. *SIAM J. Numer. Anal.* **52** (2014), 1292-1310; DOI: 10.1137/130932776.

[38] X. Zhao, X. Hu, W. Cai, and G.E. Karniadakis, Adaptive finite element method for fractional differential equations using hierarchical matrices. *Comput. Methods Appl. Mech. Engrg.* **325** (2017), 56–76; DOI: 10.1016/j.cma.2017.06.017.

[39] X. Zheng, V.J. Ervin, H. Wang, Optimal Petrov-Galerkin spectral approximation method for the fractional diffusion, advection, reaction equation on a bounded interval. *J. Sci. Comput.* **86** (2021), Art. 29; DOI: 10.1007/s10915-020-01366-y.

[40] X. Zheng and H. Wang, Variable-order space-fractional diffusion equations and a variable-order modification of constant-order fractional problems. *Appl. Anal.* (2020); DOI: 10.1080/00036811.2020.1789596.

[41] X. Zheng, H. Liu, H. Wang, H. Fu, Optimal-order finite element approximations to variable-coefficient two-sided space-fractional advection-reaction-diffusion equation in three space dimensions. *Appl. Numer. Math.* **161**, No 2 (2021), 1–12; DOI: 10.1016/j.apnum.2020.10.022.

[42] X. Zheng and H. Wang, An optimal-order numerical approximation to variable-order space-fractional diffusion equations on uniform or graded meshes. *SIAM J. Numer. Anal.* **58**, No 1 (2020), 330–352; DOI: 10.1137/19M1245621.

[1] *School of Mathematics and Statistics*
Shandong Normal University
Jinan, Shandong Province 250358, CHINA
e-mail: jhjia@sdnu.edu.cn

[2] *School of Mathematical Sciences*
Peking University
Beijing 100871, CHINA
e-mail: zhengxch@math.pku.edu.cn

[3] *Department of Mathematics*
University of South Carolina
Columbia, South Carolina 29208, USA
e-mail: hwang@math.sc.edu (Corresponding author)

Received: December 20, 2020 , Revised: August 29, 2021

Please cite to this paper as published in:
Fract. Calc. Appl. Anal., Vol. **24**, No ?? (2021), pp. xxxx–xxxx,
DOI: 10.1515/fca-2021-yyyy