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TEMPORAL SECOND-ORDER FINITE DIFFERENCE SCHEMES FOR VARIABLE-ORDER TIME-FRACTIONAL WAVE EQUATIONS*

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Abstract. We develop a temporal second-order finite difference scheme for a variable-order time-fractional wave partial differential equation in multiple space dimensions via the order reduction. We base on this scheme to develop an alternating direction implicit (ADI) finite difference scheme and a compact ADI finite difference scheme. We prove that all the schemes are unconditionally stable, and that the finite difference scheme and the ADI scheme have second-order convergence rates in space and time while the compact ADI scheme, which has the same stencil as the other two schemes, has a fourth-order convergence rate in space and second-order convergence rate in time. Numerical experiments are presented to substantiate the theoretical analysis and to demonstrate the computational efficiency of the schemes.

Key words. variable-order time-fractional wave equation, stability, error estimate, finite difference, ADI scheme

AMS subject classifications. 65M06, 65M12, 65M15

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1. Introduction.

Fractional partial differential equations (FPDEs) have been shown to provide a competitive means to accurately describe phenomena with nonlocal hereditary and memory properties in the real world [8], and have been successfully applied to model a wide variety of applications, such as dispersive anomalous diffusion [42], Pipkin's viscoelasticity [22], biological systems [18], finance [19], and quantum mechanics [10]. Recent studies showed that in many dynamic processes the properties of the materials or systems are not only anomalous but may also evolve with time [17], which makes variable-order FPDEs a natural choice and a feasible model tool to describe complex dynamic phenomena in these applications [24, 25, 26]. Since then, variable-order FPDEs have attracted an increasing number of research activities, ranging from their mathematical analysis and numerical approximations to their applications to more and more disciplines [3, 28, 29, 32, 41].

The complexities of variable-order FPDEs make it virtually impossible to find their analytical solutions in a closed form [4]. Therefore, the development of accurate and efficient numerical methods for variable-order FPDEs gain increasing attention. Variable-order fractional differential operators are nonlocal and weakly singular as their constant-order analogues do, but lose the convolution structures of constant-order fractional differential operators that played a crucial rule in the mathematical and numerical analysis as well as the numerical approximations to constant-order

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FPDEs. Consequently, variable-order FPDEs impose significant challenges. Zhuang et al. [46] and Chen et al. [2] developed and analyzed a numerical approximation to variable-order fractional advection-diffusion equations via Fourier analysis and proved a first-order temporal convergence rate. Zeng, Mao, and Karniadakis [43] developed a spectral collocation method for variable-order fractional diffusion PDEs. Shekari, Tayebi, and Heydari [27] and Tayebi, Shekari, and Heydari [37] developed a meshless method based on the moving least squares approximation and the finite difference scheme to solve variable-order time-fractional advection-diffusion and diffusion-wave equations in two space dimensions. Haq, Ghafoor, and Hussain [9] proposed a numerical scheme based on Haar wavelets coupled with the finite difference method to investigate variable-order time FPDEs. They proved the convergence of the scheme via asymptotic expansion. Several numerical schemes were proposed for solving variable-order time-fractional diffusion-wave PDEs [6, 11, 45]. Nevertheless, most of the numerical methods for variable-order FPDEs are first-order accurate in time.

In this paper we develop and analyze temporal second-order finite difference schemes for variable-order time-fractional wave equations

$$u_{tt}(\mathbf{x},t) + {}_0^C D_t^{\beta(t)} u(\mathbf{x},t) = \sum_{k=1}^{d} \partial^2_{x^{(k)}} u(\mathbf{x},t) + f(\mathbf{x},t), \quad \mathbf{x} \in \Omega, \quad t \in (0,T], \tag{1.1}$$

which is subject to the initial boundary value conditions

$$u(\mathbf{x},0) = \phi(\mathbf{x}),\ u_t(\mathbf{x},0) = \psi(\mathbf{x}), \quad \mathbf{x} \in \Omega; \qquad u(\mathbf{x},t) = 0, \quad \mathbf{x} \in \partial\Omega,\ t \in [0,T]. \tag{1.2}$$

Here $\Omega \subset \mathbb{R}^d$ is a simply connected bounded, convex domain with the piecewise smooth boundary $\partial\Omega$, $\mathbf{x} = (x^{(1)}, x^{(2)}, \dots, x^{(d)})$, and $f(\mathbf{x},t)$, $\phi(\mathbf{x})$, and $\psi(\mathbf{x})$ represent the external loading, the initial displacement, and the initial velocity, respectively, which are assumed to be sufficiently smooth. The variable-order Caputo fractional differential operator ${}_0^C D_t^{\beta(t)}$ of order $1 \le \beta(t) \le \beta_* < 2$ is defined as [2, 32]

$$ {}_0^C D_t^{\beta(t)} \xi(t) := \frac{1}{\Gamma(2-\beta(t))} \int_0^t (t-s)^{1-\beta(t)} \xi''(s) ds.$$

Note that (1.1) without the fractional derivative term reduces to the standard second-order hyperbolic PDE [21] that models the undamped motion of the perfectly elastic material. In this case, the u_{tt} term presents the inertial force of elastic material of the unit density, the Laplacian term accounts for the impact of the internal force, while f represents the external loading. Furthermore, the medium where the vibrations are taking place (e.g., when the elastic material is immersed in water) may impede the motion. A law of friction must be provided (often determined empirically). A linear law of friction is often assumed for the damped vibration of the perfectly elastic material, introducing a cu_t term on the left-hand side of (1.1).

Many experiments reported in the literature showed viscoelastic behavior of materials. That is, the materials do not behave purely elastically but also demonstrate certain internal dissipation mechanisms, and so exhibit both stored and dissipative energy components with nonlocal memory effect. Hence, the conventional model given by the u_t term does not properly describe the damping effect in the current context. Instead, a fractional time derivative term was adopted to improve the mathematical model [30].

Furthermore, as they undergo vibrations due to cyclic stresses, the materials may experience structural change that in turn leads to the change of material properties. Consequently, the order of the fractional differential operator changes, leading to variable-order FPDEs (e.g., of the form (1.1) [30, 38]).

We now turn to the numerical discretizations of variable-order FPDEs. It is well known that finite difference methods are relatively simple to implement, apply to a wide range of problems, and enjoy good numerical and mathematical properties (e.g., the maximum principle), and have attracted active research activities in the FPDE community [1, 7, 12, 44]. In this paper we base our work on the high-order temporal discretizations such as the L1-2 formula [7] and L2-1_σ formula [1], which were developed recently for constant-order FPDEs, to develop temporal second-order finite difference schemes for the initial boundary value problem of the variable-order time-fractional wave equation (1.1)–(1.2) via the order reduction. We prove the unconditional stability and provide error estimates of the finite difference schemes via the discrete energy technique.

In the development of the finite difference schemes, we assume that the spatial domain Ω is a d-dimensional rectangular domain. Furthermore, for notational simplicity, we restrict our presentation to problem (1.1) in two space dimensions. We denote $\mathbf{x} = (x, y)$ and the spatial domain $\Omega = (l_1, r_1) \times (l_2, r_2)$. For problem (1.1)–(1.2) on a general spatial domain, in principle a finite difference discretization, e.g., with a fictitious domain technique [13], or a finite element discretization may be used to discretize the spatial Laplacian operator, which is omitted here. The rest of the paper is organized as follows. In section 2 we introduce notations and cite some lemmas in the literature to be used subsequently. In section 3 we first transform the problem (1.1)–(1.2) into an equivalent one by the method of order reduction, and then derive some temporal second-order finite difference schemes for the problem (1.1)–(1.2), in which an alternating direction implicit (ADI) scheme is proposed in order to reduce CPU time for solving the problem (1.1)–(1.2) efficiently. In addition, for the purpose of reducing the storage requirement and improving the accuracy of the scheme, a compact ADI scheme is also presented. In section 4 we utilize the discrete energy technique to prove the unconditional stability and convergence of the finite difference schemes. Some numerical experiments are carried out to investigate the numerical accuracy, reliability, and efficiency in section 5. The paper ends with some concluding remarks.

2. Preliminaries. We introduce some preliminary notions and notations to be used in the subsequent sections. For a positive integer N, define a uniform partition on the time interval $[0, T]$ by $t_n := n\tau$ for $n = 0, 1, \ldots, N$ with $\tau := T/N$. Assume that $\alpha \in C^1[0, T]$, the space of continuously differentiable functions defined on $[0, T]$, satisfies $0 \le \alpha \le \alpha_* < 1$. For a sufficiently small $\tau > 0$, the equation

$$F(\sigma) := \sigma - \Big[1 - \frac{1}{2}\alpha(t_n + \sigma\tau)\Big] = 0 \tag{2.1}$$

has a unique root $\sigma_n = \sigma_n(\tau) \in (\frac{1}{2}, 1)$, which can be conveniently calculated, e.g., by Newton's method [5]. We accordingly define a partial time step $t_{n+\sigma_n}$ by

$$t_{n+\sigma_n} := t_n + \sigma_n\tau, \quad n = 1, 2, \ldots, N-1. \tag{2.2}$$

Let $\mathfrak{T} := \{v^n\}_{n=0}^N$ be a temporal grid space; we define the temporal operators

(2.3)

$$v^{\frac{1}{2}} := \frac{v^1 + v^0}{2}, \qquad v^{n+\sigma_n} := \sigma_n v^{n+1} + (1 - \sigma_n)v^n, \quad 1 \le n \le N-1;$$

$$\delta_t v^{n+\frac{1}{2}} := \frac{v^{n+1} - v^n}{\tau}, \quad \delta_t v^n := \frac{(2\sigma_n + 1)v^{n+1} - 4\sigma_n v^n + (2\sigma_n - 1)v^{n-1}}{2\tau}, \quad 1 \le n \le N-1.$$

For $\alpha \in C[0,T]$ and $g \in C^1[0,T]$, the spaces of continuous functions and continuously differentiable functions defined on $[0,T]$, respectively. Let

$$\alpha_{n+\sigma_n} := \alpha(t_{n+\sigma_n}),\ \ g^n := g(t_n),\ \ s_n := \begin{cases} 2^{1-\alpha_{1/2}}\tau^{\alpha_{1/2}}\Gamma(2-\alpha_{1/2}), & n=0, \\ \tau^{\alpha_{n+\sigma_n}}\Gamma(2-\alpha_{n+\sigma_n}), & 1 \le n \le N-1. \end{cases} \tag{2.4}$$

We discretize ${}_0^C D_t^{\alpha(t)} g(t)|_{t=t_{n+\sigma_n}}$ $(1 \le n \le N-1)$ by approximating $g(s)$ on the interval $[t_n, t_{n+\sigma_n}]$ via the linear interpolation polynomial of function $g(t)$ using two points $(t_n, g(t_n))$ and $(t_{n+1}, g(t_{n+1}))$, and by approximating $g(s)$ on the interval $[t_{k-1}, t_k]$ for $k = 1, 2, \ldots, n$ via the quadratic interpolation polynomial of function $g(s)$ using three points $(t_{k-1}, g(t_{k-1}))$, $(t_k, g(t_k))$, and $(t_{k+1}, g(t_{k+1}))$ to obtain

$$\begin{aligned} {}_0^C D_t^{\alpha(t)} g(t)\Big|_{t=t_{n+\sigma_n}} &= \frac{1}{\Gamma(1-\alpha_{n+\sigma_n})}\left[\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} \frac{g'(s)ds}{(t_{n+\sigma_n}-s)^{\alpha_{n+\sigma_n}}} + \int_{t_n}^{t_{n+\sigma_n}} \frac{g'(s)ds}{(t_{n+\sigma_n}-s)^{\alpha_{n+\sigma_n}}}\right] \\ &\approx \frac{1}{\Gamma(1-\alpha_{n+\sigma_n})}\left(\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} \frac{\frac{t_{k+\frac12}-s}{\tau}\delta_t g^{k-\frac12} + \frac{s-t_{k-\frac12}}{\tau}\delta_t g^{k+\frac12}}{(t_{n+\sigma_n}-s)^{\alpha_{n+\sigma_n}}}\right. \\ &\qquad\qquad \left. + \int_{t_n}^{t_{n+\sigma_n}} \frac{\delta_t g^{n+\frac12}}{(t_{n+\sigma_n)}-s)^{\alpha_{n+\sigma_n}}} ds\right) \\ &= \frac{1}{s_n}\sum_{k=0}^{n} c_{n-k}^{(n)}\left(g^{k+1}-g^k\right) =: \mathcal{D}^{\alpha_{n+\sigma_n}} g^{n+\sigma_n}. \end{aligned} \tag{2.5}$$

Here, for clarity, we leave the derivation (2.5) and the explicit formula for $c_{n-k}^{(n)}$ to Appendix A.

Lemma 2.1 (see [5]). *If $0 < \alpha(t) < 1$, then $\{c_k^{(n)}\}_{k=0}^n$ satisfy the following relations:*

$$c_0^{(n)} > c_1^{(n)} > \cdots > c_{n-1}^{(n)} > c_n^{(n)} > \frac{1-\alpha_{n+\sigma_n}}{2(n+\sigma_n)^{\alpha_{n+\sigma_n}}} > 0, \quad (2\sigma_n - 1)c_0^{(n)} - \sigma_n c_1^{(n)} > 0.$$

Lemma 2.2 (see [5, 31, 34, 35]). *The following expansions hold for $v \in C^3[0,T]$ and $0 < \alpha(t) < 1$:*

$$\begin{aligned} &v(t_{1/2}) = v^{\frac12} + O(\tau^2), \quad v_t(t_{1/2}) = \delta_t v^{\frac12} + O(\tau^2), \\ &{}_0^C D_t^{\alpha(t)} v(t)\big|_{t=t_{\frac12}} = \frac{v^1 - v^0}{s_0} + O\big(\tau^{2-\alpha_{1/2}}\big), \\ &v(t_{n+\sigma_n}) = v^{n+\sigma_n} + O(\tau^2), \quad v_t(t_{n+\sigma_n}) = \delta_t v^n + O(\tau^2), \quad 1 \le n \le N-1, \\ &{}_0^C D_t^{\alpha(t)} v(t)|_{t=t_{n+\sigma_n}} = \mathcal{D}^{\alpha_{n+\sigma_n}} v^{n+\sigma_n} + O(\tau^{3-\alpha_{n+\sigma_n}}), \qquad 1 \le n \le N-1, \end{aligned}$$

where s_0 is defined in (2.4).

Lemma 2.3 (see [23]). *Let $v, w, g^n \in$ be $\in \mathfrak{T}$ be nonnegative temporal grid functions, g^n be nondecreasing, and A and B be nonnegative constants.* (I) *If $v^n \le$*

$(1+\tau B)v^{n-1}+\tau w^{n-1}$ *for* $1\le n\le N$, *then*

$$v^n \le \exp(Bn\tau)\Big[v^0+\tau\sum_{l=0}^{n-1} w^l\Big],\quad 0\le n\le N.$$

(II) *If* $v^n \le g^n + B\tau\sum\limits_{l=1}^{n-1} v^l$ *for* $0\le n\le N$, *then*

$$v^n \le g^n\exp(Bn\tau),\quad 0\le n\le N.$$

Let M_1 and M_2 be positive integers. Define a uniform partition of Ω by $x_i := l_1+ih_1$ for $0\le i\le M_1+1$ with $h_1:=(r_1-l_1)/(M_1+1)$ and $y_j:=l_2+jh_2$ for $0\le j\le M_2+1$ with $h_2:=(r_2-l_2)/(M_2+1)$. We assume that the partition is quasi-uniform, i.e., $0<Q_1h_1\le h_2\le Q_2h_1<\infty$ with Q_1 and Q_2 being positive constants independent of h_1, h_2, or τ. We denote $h:=\max\{h_1,h_2\}$. In addition, let $\omega:=\big\{(i,j)\mid 1\le i\le M, \quad 1,\ 1\le j\le M_2\big\}$, $\bar\omega:=\big\{(i,j)\mid 0\le i\le M_1+1,\ 0\le j\le M_2+1\big\}$, and $\partial\omega:=\bar\omega\backslash\omega$. Define spatial grid function spaces

$$\mathfrak{U}:=\{u=\{u_{i,j}\,|,(i,j)\}\in\},\quad \mathring{\mathfrak{U}}:=\{u\,|\,u\in\mathfrak{U},u_{i,j}=0,\,(i,j)\in\partial\omega\}.$$

We introduce the following discrete operators in the grid space $\mathfrak{U}$:

$$\delta_x u_{i+\frac12,j}:=\frac{u_{i+1,j}-u_{i,j}}{h_1},\quad \delta_x^2 u_{i,j}:=\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h_1^2},$$
$$\delta_y u_{i,j+\frac12}:=\frac{u_{i,j+1}-u_{i,j}}{h_2},\quad \delta_y^2 u_{i,j}:=\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{h_2^2},$$

$$\mathcal{A}_1u_{i,j}:=\begin{cases}\dfrac{u_{i-1,j}+10u_{i,j}+u_{i+1,j}}{12}, & 1\le i\le M_1,\qquad 0\le j\le M_2+1,\\ u_{i,j}, & i=0,M_1+1,\quad 0\le j\le M_2+1;\end{cases}$$

$$\mathcal{A}_2u_{i,j}:=\begin{cases}\dfrac{u_{i,j-1}+10u_{i,j}+u_{i,j+1}}{12}, & 1\le j\le M_2,\qquad 0\le i\le M_1+1,\\ u_{i,j}, & j=0,M_2+1,\quad 0\le i\le M_1+1;\end{cases}$$

$$\Delta_h u_{i,j}:=\delta_x^2u_{i,j}+\delta_y^2u_{i,j},\quad \mathcal{A}_hu_{i,j}:=\mathcal{A}_1\mathcal{A}_2u_{i,j},\quad \Lambda_hu_{i,j}:=\mathcal{A}_2\delta_x^2+\mathcal{A}_1\delta_y^2.$$

In the grid function space $\mathring{\mathfrak{U}}$, define the discrete inner products and norms

$$(u,w):=h_1h_2\sum_{i=1}^{M_1}\sum_{j=1}^{M_2}u_{i,j}w_{i,j},\qquad \|u\|:=\sqrt{(u,u)}\,,$$
$$(u,w)_{\mathcal{A}_h}:=(u,\mathcal{A}_hw),\quad (u,w)_{\mathcal{A}_1}:=(u,\mathcal{A}_1w),\qquad (u,w)_{\mathcal{A}_2}:=(u,\mathcal{A}_2w),$$
$$\|u\|_{\mathcal{A}_h}:=\sqrt{(u,\mathcal{A}_hu)}\,,\quad \|u\|_{\mathcal{A}_1}:=\sqrt{(u,\mathcal{A}_1u)}\,,\qquad \|u\|_{\mathcal{A}_2}:=\sqrt{(u,\mathcal{A}_2u)}\,,$$
$$(\delta_xu,\delta_xw):=h_1h_2\sum_{i=0}^{M_1}\sum_{j=1}^{M_2}\delta_xu_{i+\frac12,j}\,\delta_xw_{i+\frac12,j},\qquad \|\delta_xu\|:=\sqrt{(\delta_xu,\delta_xu)}\,,$$
$$(\delta_yu,\delta_yw):=h_1h_2\sum_{i=1}^{M_1}\sum_{j=0}^{M_2}\delta_yu_{i,j+\frac12}\,\delta_yw_{i,j+\frac12},\qquad \|\delta_yu\|:=\sqrt{(\delta_yu,\delta_yu)}\,,$$
$$|u|_1:=\sqrt{\|\delta_xu\|^2+\|\delta_yu\|^2}\,.$$

3. Derivation of finite difference schemes. Let $\alpha(t) := \beta(t)-1$ and $v(x,y,t) := u_t(x,y,t)$. It is straightforward to verify that

$$ {}_0^C D_t^{\beta(t)} u(x,y,t) = {}_0^C D_t^{\alpha(t)} v(x,y,t). $$

We use the method of order reduction to reformulate the initial boundary value problem of the variable-order time fractional wave PDEs (1.1)–(1.2) to the following initial boundary value problem of the variable-order FPDE system

$$ v_t(x,y,t) + {}_0^C D_t^{\alpha(t)} v(x,y,t) = u_{xx}(x,y,t) + u_{yy}(x,y,t) + f(x,y,t), \quad (x,y) \in \Omega,\ t \in (0,T], \tag{3.1} $$

$$ u_t(x,y,t) = v(x,y,t), \quad (x,y) \in \bar{\Omega},\ t \in (0,T], \tag{3.2} $$

$$ u(x,y,0) = \phi(x,y), \quad v(x,y,0) = \psi(x,y), \quad (x,y) \in \Omega, \tag{3.3} $$

$$ u(x,y,t) = 0, \quad (x,y) \in \partial\Omega,\ t \in [0,T]. \tag{3.4} $$

We turn to the derivation of temporal second-order finite difference schemes for problem (3.1)–(3.4).

3.1. A second-order finite difference scheme. Let $u_{i,j}^n := u(x_i, y_j, t_n)$, $v_{i,j}^n := v(x_i, y_j, t_n)$, and $f_{i,j}^n := f(x_i, y_j, t_n)$ for $(x_i, y_j) \in \bar{\Omega}$ and $0 \le n \le N$. We assume that the solution $u \in C^4(\bar{\Omega} \times [0,T])$. We apply the first three expansions in Lemma 2.2 to discretize the two terms on the left-hand side and the source term in (3.1) at the space-time location $(x_i, y_j, t_{1/2})$ and use the five-point finite difference approximation $\Delta_h u$ to $u_{xx} + u_{yy}$ at $(x_i, y_j, t_{1/2})$ to reformulate (3.1) in the following form:

$$ \delta_t v_{i,j}^{\frac{1}{2}} + \frac{v_{i,j}^1 - v_{i,j}^0}{s_0} = \Delta_h u_{i,j}^{\frac{1}{2}} + f_{i,j}^{\frac{1}{2}} + R_{i,j}^0, \quad (i,j) \in \omega. \tag{3.5} $$

Here s_0 is defined in (2.4) and $R_{i,j}^0 = O(\tau^{2-\alpha_{1/2}} + h_1^2 + h_2^2)$ represents the local truncation error resulting from these approximations.

At later time steps, we utilize the last three expansions in Lemma 2.2 to get the expansion

$$ \delta_t v_{i,j}^n + \frac{1}{s_n} \sum_{k=0}^{n} c_{n-k}^{(n)} \big(v_{i,j}^{k+1} - v_{i,j}^k\big) = \Delta_h u_{i,j}^{n+\sigma_n} + f_{i,j}^{n+\sigma_n} + R_{i,j}^n, \quad (i,j) \in \omega,\ 1 \le n \le N-1, \tag{3.6} $$

where to similarly discretize (3.1) at the space-time location $(x_i, y_j, t_{n+\sigma_n})$ for $n \ge 1$ is given by

$$ u_{xx}(x_i, y_j, t_{n+\sigma_n}) + u_{yy}(x_i, y_j, t_{n+\sigma_n}) = \Delta_h u_{i,j}^{n+\sigma_n} + O(\tau^2 + h_1^2 + h_2^2). $$

Here $R_{i,j}^n = O(\tau^2 + h_1^2 + h_2^2)$ is the local truncation error for $1 \le n \le N-1$.

We use Lemma 2.2 to discretize (3.2) at the points $(x_i, y_j, t_{\frac{1}{2}})$ and $(x_i, y_j, t_{n+\sigma_n})$ for $n \ge 1$:

$$ \delta_t u_{i,j}^{\frac{1}{2}} = v_{i,j}^{\frac{1}{2}} + r_{i,j}^0, \quad \delta_t u_{i,j}^n = v_{i,j}^{n+\sigma_n} + r_{i,j}^n, \quad (i,j) \in \bar{\omega}. \tag{3.7} $$

Here the local truncation error $r_{i,j}^n$ with $0 \le n \le N-1$ can be bounded by

$$r_{i,j}^n = O(\tau^2), \quad \Delta_h r_{i,j}^n = O(\tau^2), \quad \Lambda_h r_{i,j}^n = O(\tau^2), \quad (i,j) \in \omega; \quad r_{i,j}^n = 0, \quad (i,j) \in \partial\omega. \tag{3.8}$$

The initial value condition (3.3) and boundary value condition (3.4) can be formulated as

$$u_{i,j}^0 = \phi(x_i, y_j), \quad v_{i,j}^0 = \psi(x_i, y_j), \quad (i,j) \in \omega; \qquad u_{i,j}^n = 0, \quad (i,j) \in \partial\omega, \ 0 \le n \le N. \tag{3.9}$$

Let $U_{i,j}^n$ and $V_{i,j}^n$ be the finite difference approximations to $u_{i,j}^n$ and $v_{i,j}^n$, respectively. We drop the local truncation error terms in (3.5)–(3.7) to obtain the following second-order finite difference scheme:

$$\delta_t V_{i,j}^{\frac{1}{2}} + \frac{V_{i,j}^1 - V_{i,j}^0}{s_0} = \Delta_h U_{i,j}^{\frac{1}{2}} + f_{i,j}^{\frac{1}{2}}, \quad (i,j) \in \omega, \tag{3.10}$$

$$\delta_t V_{i,j}^n + \frac{1}{s_n} \sum_{k=0}^{n} c_{n-k}^{(n)} \big(V_{i,j}^{k+1} - V_{i,j}^k\big) = \Delta_h U_{i,j}^{n+\sigma_n} + f_{i,j}^{n+\sigma_n}, \quad (i,j) \in \omega, \ 1 \le n \le N-1, \tag{3.11}$$

$$\delta_t U_{i,j}^{\frac{1}{2}} = V_{i,j}^{\frac{1}{2}}, \quad \delta_t U_{i,j}^n = V_{i,j}^{n+\sigma_n}, \quad (i,j) \in \bar{\omega}, \quad 1 \le n \le N-1, \tag{3.12}$$

$$U_{i,j}^0 = \phi(x_i, y_j), \ V_{i,j}^0 = \psi(x_i, y_j), \ (i,j) \in \omega; \ U_{i,j}^n = V_{i,j}^n = 0, \ (i,j) \in \partial\omega, \ 0 \le n \le N. \tag{3.13}$$

Equations in (3.12) can be reformulated for $(i,j) \in \bar{\omega}$ and $1 \le n \le N-1$ as

$$U_{i,j}^1 = \tau V_{i,j}^{\frac{1}{2}} + U_{i,j}^0, \qquad U_{i,j}^{n+1} = \frac{2\tau V_{i,j}^{n+\sigma_n} - (2\sigma_n - 1) U_{i,j}^{n-1} + 4\sigma_n U_{i,j}^n}{2\sigma_n + 1}. \tag{3.14}$$

With the given initial value condition and boundary value condition in (3.13), (3.10) and the first family of equations in (3.14) are solved for $U_{i,j}^1$ and $V_{i,j}^1$ for $(i,j) \in \omega$. One way to achieve this goal is to use a block Gaussian elimination by substituting the first family of equations in (3.14) into (3.10) to yield a tridiagonal block algebraic system for $V_{i,j}^1$. Once $V_{i,j}^1$ is obtained, $U_{i,j}^1$ can be obtained easily from the first family of equations in (3.14). Once $U_{i,j}^1$ and $V_{i,j}^1$ are obtained, the finite difference scheme can be solved for $U_{i,j}^{n+1}$ and $V_{i,j}^{n+1}$ for $(i,j) \in \omega$ at time step t_n for $1 \le n \le N-1$ in a time-stepping procedure. This would eliminate $U_{i,j}^{n+1}$ in (3.11) by the second family of equations in (3.14) to arrive at a linear system for $V_{i,j}^{n+1}$ for $(i,j) \in \omega$. Once, $V_{i,j}^{n+1}$ for $(i,j) \in \omega$ are obtained, $U_{i,j}^{n+1}$ are obtained from the second family of equations in (3.14).

3.2. An ADI finite difference scheme. Due to the nonlocal nature of fractional differential operators, FPDEs generate numerical discretizations that have significantly increased computational complexity [39, 44]. Different acceleration techniques were developed for the efficient numerical solutions of FPDEs [14, 15, 20, 36].

In this paper we analyze an ADI acceleration for the finite difference scheme developed in subsection 3.1 for its great computational advantage, relative simplicity, and wide application. For simplicity of exposition, we outline the idea of the ADI scheme assuming that $M_1 = M_2$ so the total number of unknowns is $M = M_1^2$ at each time step.

For a one-dimensional analogue of the finite difference scheme, the resulting linear system can be inverted in $O(M_1)$ computations by the Thomas algorithm each time step, which is of optimal order computational complexity. However, for the two-dimensional problem, the resulting algebraic linear system is inverted in $O(M^2)$ computations at each time step due to the fact that the stencil of the finite difference discretization is widespread. The idea of ADI is to reformulate the two-dimensional finite difference scheme as two families of one-dimensional linear systems in the x and y directions, respectively [20, 36]. The computational cost of inverting the resulting linear systems is $M_2O(M_1) + M_1O(M_2) = O(M_1M_2) = O(M)$, i.e., in the optimal-order computational complexity at each time step. Although it has been successfully applied to the efficient solutions of integer-order PDEs, the ADI technique has never been developed and, in particular, analyzed for variable-order fractional wave PDEs. Motivated by all the considerations, we develop and analyze an ADI finite difference scheme in this paper.

We substitute the first family of equations in (3.7) into (3.5) to eliminate $u_{i,j}^1$ in (3.5) to reformulate (3.5) as follows:

$$\left(\frac{1}{s_0} + \frac{1}{\tau}\right)(v_{i,j}^1 - v_{i,j}^0) = \frac{\tau}{2}\Delta_h v_{i,j}^{\frac{1}{2}} + \Delta_h u_{i,j}^0 + f_{i,j}^{\frac{1}{2}} + \hat{R}_{i,j}^0, \quad (i,j) \in \omega. \tag{3.15}$$

Here $\hat{R}_{i,j}^0 := \frac{\tau}{2}\Delta_h r_{i,j}^0 + R_{i,j}^0 = O(\tau^{2-\alpha_{1/2}} + h_1^2 + h_2^2)$ with $R_{i,j}^0$ and $r_{i,j}^0$ being introduced in (3.5) and (3.7), respectively. A direct calculation shows that (3.15) can be reformulated as follows:

$$\left(\frac{1}{s_0} + \frac{1}{\tau}\right)(v_{i,j}^1 - v_{i,j}^0) + \frac{s_0\tau^4\delta_x^2\delta_y^2\delta_t v_{i,j}^{\frac{1}{2}}}{16(s_0+\tau)} = \frac{\tau}{2}\Delta_h v_{i,j}^{\frac{1}{2}} + \Delta_h u_{i,j}^0 + f_{i,j}^{\frac{1}{2}} + \tilde{R}_{i,j}^0, \quad (i,j) \in \omega. \tag{3.16}$$

Here

$$\tilde{R}_{i,j}^0 := \hat{R}_{i,j}^0 + \frac{s_0\tau^4}{16(s_0+\tau)}\delta_x^2\delta_y^2\delta_t v_{i,j}^{\frac{1}{2}} = O(\tau^{2-\alpha_{1/2}} + h_1^2 + h_2^2). \tag{3.17}$$

We substitute the second family of equations in (3.7) into (3.6) to reformulate (3.6) as follows:

$$\begin{aligned}&\delta_t v_{i,j}^n + \frac{1}{s_n}\left[c_0^{(n)} v_{i,j}^{n+1} - \sum_{k=1}^{n}(c_{n-k}^{(n)} - c_{n-k+1}^{(n)})v_{i,j}^k - c_n^{(n)} v_{i,j}^0\right]\\ &= \frac{2\sigma_n\tau\Delta_h v_{i,j}^{n+\sigma_n}}{2\sigma_n+1} + \left(\frac{4\sigma_n^2}{2\sigma_n+1} + 1 - \sigma_n\right)\Delta_h u_{i,j}^n - \frac{(2\sigma_n-1)\sigma_n\Delta_h u_{i,j}^{n-1}}{2\sigma_n+1} + f_{i,j}^{n+\sigma_n} + \hat{R}_{i,j}^n\end{aligned} \tag{3.18}$$

for $(i,j) \in \omega$ and $1 \le n \le N-1$. Here

$$\hat{R}_{i,j}^n := \frac{2\tau\sigma_n}{2\sigma_n+1}\Delta_h r_{i,j}^n + R_{i,j}^n = O(\tau^2 + h_1^2 + h_2^2) \tag{3.19}$$

with $R_{i,j}^n$ and $r_{i,j}^n$ being introduced in (3.6) and (3.7), respectively.

Equation (3.18) can be reformulated as follows:

$$
\begin{aligned}
&\delta_t v_{i,j}^n + \frac{1}{s_n}\Big[c_0^{(n)} v_{i,j}^{n+1} - \sum_{k=1}^{n}(c_{n-k}^{(n)} - c_{n-k+1}^{(n)}) v_{i,j}^k - c_n^{(n)} v_{i,j}^0\Big] \\
&\quad + \frac{8\sigma_n^4 s_n \tau^3 \delta_x^2 \delta_y^2 v_{i,j}^{n+1}}{(2\sigma_n+1)^2\big[(2\sigma_n+1)s_n + 2\tau c_0^{(n)}\big]} \\
&= \frac{2\sigma_n\tau}{2\sigma_n+1}\Delta_h v_{i,j}^{n+\sigma_n} + \Big(\frac{4\sigma_n^2}{2\sigma_n+1} + 1 - \sigma_n\Big)\Delta_h u_{i,j}^n \\
&\quad - \frac{(2\sigma_n-1)\sigma_n}{2\sigma_n+1}\Delta_h u_{i,j}^{n-1} + f_{i,j}^{n+\sigma_n} + \tilde{R}_{i,j}^n
\end{aligned}
\tag{3.20}
$$

for $(i,j)\in\omega$ and $1\le n\le N-1$. Here

$$
\tilde{R}_{i,j}^n := \hat{R}_{i,j}^n + \frac{8\sigma_n^4 s_n\tau^3}{(2\sigma_n+1)^2\big[(2\sigma_n+1)s_n + 2\tau c_0^{(n)}\big]}\delta_x^2\delta_y^2 v_{i,j}^{n+1} = O(\tau^2 + h_1^2 + h_2^2).
\tag{3.21}
$$

Equations (3.16) and (3.20) can be enclosed by the following equations:

$$
u_{i,j}^1 = \frac{\tau}{2}(v_{i,j}^1 + v_{i,j}^0) + u_{i,j}^0, \quad u_{i,j}^{n+1} = \frac{2\tau v_{i,j}^{n+\sigma_n} + 4\sigma_n u_{i,j}^n - (2\sigma_n-1)u_{i,j}^{n-1}}{2\sigma_n+1}
\tag{3.22}
$$

for $(i,j)\in\omega$ and $1\le n\le N-1$ and the initial boundary value conditions

$$
u_{i,j}^0 = \phi(x_i,y_j),\ \ v_{i,j}^0 = \psi(x_i,y_j),\ \ (i,j)\in\omega,\ \ u_{i,j}^n = v_{i,j}^n = 0,\ \ (i,j)\in\partial\omega,\ 0\le n\le N.
\tag{3.23}
$$

Denote

$$
\eta_0 := \frac{s_0\tau^2}{4(s_0+\tau)}, \qquad \eta_n := \frac{4\sigma_n^2 s_n\tau^2}{(2\sigma_n+1)\big[(2\sigma_n+1)s_n + 2\tau c_0^{(n)}\big]}, \quad n\ge 1.
\tag{3.24}
$$

We multiply (3.16) by $s_0\tau/(s_0+\tau)$ to rewrite the equation as

$$
(v_{i,j}^1 - v_{i,j}^0) + \tau\eta_0^2\delta_x^2\delta_y^2\delta_t v_{i,j}^{\frac12} = 2\eta_0\Delta_h v_{i,j}^{\frac12} + \frac{s_0\tau}{s_0+\tau}\big(\Delta_h u_{i,j}^0 + f_{i,j}^{\frac12} + \tilde{R}_{i,j}^0\big).
\tag{3.25}
$$

Let I be the identity operator. We can split (3.25) for $(i,j)\in\omega$ as follows:

$$
(I-\eta_0\delta_x^2)(I-\eta_0\delta_y^2)v_{i,j}^1 = (I+\eta_0\delta_x^2)(I+\eta_0\delta_y^2)v_{i,j}^0 + \frac{s_0\tau}{s_0+\tau}\big(\Delta_h u_{i,j}^0 + f_{i,j}^{\frac12} + \tilde{R}_{i,j}^0\big).
\tag{3.26}
$$

Let $U_{i,j}^1$ and $V_{i,j}^1$ be the finite difference approximations to $u_{i,j}^1$ and $v_{i,j}^1$, respectively. An ADI finite difference scheme is formulated as follows: For any fixed $1\le j\le M_2$, solve the following one-dimensional linear system in the x-direction for $V_{i,j}^{1,*}$:

$$
\begin{gathered}
(I-\eta_0\delta_x^2)V_{i,j}^{1,*} = (I+\eta_0\delta_x^2)(I+\eta_0\delta_y^2)V_{i,j}^0 + \frac{s_0\tau}{s_0+\tau}\big(\Delta_h U_{i,j}^0 + f_{i,j}^{\frac12}\big), \quad 1\le i\le M_1, \\
V_{0,j}^{1,*} = V_{M_1+1,j}^{1,*} = 0.
\end{gathered}
\tag{3.27}
$$

For each fixed $1\le i\le M_1$, solve the following one-dimensional linear system in the y-direction for $V_{i,j}^1$ and then find all the $U_{i,j}^1$ as follows:

$$
\begin{aligned}
&(I-\eta_0\delta_y^2)V_{i,j}^1 = V_{i,j}^{1,*}, \quad 1\le j\le M_2, \qquad V_{i,0}^1 = V_{i,M_2+1}^1 = 0; \\
&U_{i,j}^1 = \frac{\tau}{2}(V_{i,j}^1 + V_{i,j}^0) + U_{i,j}^0, \quad (i,j)\in\bar{\omega}.
\end{aligned}
\tag{3.28}
$$

Similarly, at each subsequent time step t_n for $n = 1, 2, \cdots, N-1$, for any fixed $1 \le j \le M_2$, solve the following one-dimensional system in the x-direction for $V_{i,j}^{n,*}$:

$$
\begin{aligned}
&(I - \eta_n \delta_x^2) V_{i,j}^{n,*} = b_{i,j}^n, \quad 1 \le i \le M_1; \qquad V_{0,j}^{n,*} = V_{M_1+1,j}^{n,*} = 0;\\
&b_{i,j}^n := \frac{2\tau s_n}{(2\sigma_n+1)s_n + 2\tau c_0^{(n)}} \bigg\{ \frac{4\sigma_n V_{i,j}^n - (2\sigma_n - 1) V_{i,j}^{n-1}}{2\tau}\\
&\qquad + \frac{1}{s_n}\bigg[\sum_{k=1}^{n} \big(c_{n-k}^{(n)} - c_{n-k+1}^{(n)}\big) V_{i,j}^k + c_n^{(n)} V_{i,j}^0\bigg] + \frac{2\sigma_n(1-\sigma_n)\tau}{2\sigma_n+1} \Delta_h V_{i,j}^n\\
&\qquad + \Big(\frac{4\sigma_n^2}{2\sigma_n+1} + 1 - \sigma_n\Big) \Delta_h U_{i,j}^n - \frac{(2\sigma_n-1)\sigma_n}{2\sigma_n+1} \Delta_h U_{i,j}^{n-1} + f_{i,j}^{n+\sigma_n} \bigg\}.
\end{aligned} \tag{3.29}
$$

For each fixed $1 \le i \le M_1$, solve the following one-dimensional linear system in the y-direction for $V_{i,j}^{n+1}$ and then find all the $U_{i,j}^{n+1}$:

$$
\begin{aligned}
&(I - \eta_n \delta_y^2) V_{i,j}^{n+1} = V_{i,j}^{n,*}, \quad 1 \le j \le M_2; \qquad V_{i,0}^{n+1} = V_{i,M_2+1}^{n+1} = 0;\\
&U_{i,j}^{n+1} = \big(2\tau V_{i,j}^{n+\sigma_n} + 4\sigma_n U_{i,j}^n - (2\sigma_n - 1) U_{i,j}^{n-1}\big)/(2\sigma_n + 1), \quad (i,j) \in \bar{\omega}.
\end{aligned} \tag{3.30}
$$

3.3. A compact ADI finite difference scheme. Under the assumption that the solution u is six time continuously differentiable in space, we derive a fourth-order compact finite difference approximation in space. The key advantage of the compact ADI scheme is that it uses the same stencil as the ADI scheme just described earlier to achieve a fourth-order spatial accuracy, which greatly reduces computational cost and memory requirement. We apply the averaging operator $\mathcal{A}_h$ on both sides of (3.1) evaluated at $(x_i, y_j, t_{\frac{1}{2}})$ and $(x_i, y_j, t_{n+\sigma_n})$, respectively, to obtain the following equations:

$$
\begin{aligned}
&\mathcal{A}_h \delta_t v_{i,j}^{\frac{1}{2}} + \frac{1}{s_0} \mathcal{A}_h (v_{i,j}^1 - v_{i,j}^0) = \Lambda_h u_{i,j}^{\frac{1}{2}} + \mathcal{A}_h f_{i,j}^{\frac{1}{2}} + \check{R}_{i,j}^0,\\
&\mathcal{A}_h \delta_t v_{i,j}^n + \frac{1}{s_n} \sum_{k=0}^{n} c_{n-k}^{(n)} \big(\mathcal{A}_h v_{i,j}^{k+1} - \mathcal{A}_h v_{i,j}^k\big) = \Lambda_h u_{i,j}^{n+\sigma_n} + \mathcal{A}_h f_{i,j}^{n+\sigma_n} + \check{R}_{i,j}^n
\end{aligned} \tag{3.31}
$$

for $(i,j) \in \omega$ and $1 \le n \le N-1$. Here $\check{R}_{i,j}^0 = O(\tau^{2-\alpha_{1/2}} + h_1^4 + h_2^4)$, $\check{R}_{i,j}^n = O(\tau^2 + h_1^4 + h_2^4)$.

A similar derivation to subsection 3.2 yields the finite difference approximation

$$
\begin{aligned}
&(\mathcal{A}_1 - \eta_0 \delta_x^2)(\mathcal{A}_2 - \eta_0 \delta_y^2) V_{i,j}^1\\
&\quad = (\mathcal{A}_1 + \eta_0 \delta_x^2)(\mathcal{A}_2 + \eta_0 \delta_y^2) V_{i,j}^0 + \frac{s_0 \tau}{s_0 + \tau} \big(\Lambda_h U_{i,j}^0 + \mathcal{A}_h f_{i,j}^{\frac{1}{2}}\big),\\
&(\mathcal{A}_1 - \eta_n \delta_x^2)(\mathcal{A}_2 - \eta_n \delta_y^2) V_{i,j}^{n+1}\\
&\quad = \frac{2\tau s_n}{(2\sigma_n+1)s_n + 2\tau c_0^{(n)}} \bigg\{ \frac{1}{2\tau} \mathcal{A}_1 \mathcal{A}_2 (4\sigma_n V_{i,j}^n - (2\sigma_n - 1) V_{i,j}^{n-1})\\
&\qquad + \frac{1}{s_n} \mathcal{A}_1 \mathcal{A}_2 \bigg[\sum_{k=1}^{n} (c_{n-k}^{(n)} - c_{n-k+1}^{(n)}) V_{i,j}^k + c_n^{(n)} V_{i,j}^0\bigg] + \frac{2\sigma_n(1-\sigma_n)\tau}{2\sigma_n+1} \Lambda_h V_{i,j}^n\\
&\qquad + \Big(\frac{4\sigma_n^2}{2\sigma_n+1} + 1 - \sigma_n\Big) \Lambda_h U_{i,j}^n - \frac{(2\sigma_n-1)\sigma_n}{2\sigma_n+1} \Lambda_h U_{i,j}^{n-1} + \mathcal{A}_h f_{i,j}^{n+\sigma_n} \bigg\}
\end{aligned} \tag{3.32}
$$

for $(i,j) \in \omega$ and $1 \le n \le N-1$, which are closed by (3.13)–(3.14).

4. Stability analysis and error estimate. In this section we analyze the stability and convergence of the finite difference schemes developed in section 3.

4.1. Auxiliary lemmas. We cite some auxiliary lemmas to be used in the subsequent proofs.

LEMMA 4.1 ([33]). *Let $w, z \in \mathring{\mathfrak{U}}$ be spatial grid functions satisfying the homogeneous boundary condition. The following identities and inequalities hold:*

$$-(w, \Delta_h z) = (\delta_x w, \delta_x z) + (\delta_y w, \delta_y z), \qquad -(w, \Delta_h w) = |w|_1^2, \quad -(w, \Lambda_h w) \geq \frac{2}{3}|w|_1^2;$$
$$\|\mathcal{A}_h w\| \leq \|w\|, \quad \|w\| \leq Q_0 |w|_1, \quad Q_0 := \left(\frac{6}{(r_1 - l_1)^2} + \frac{6}{(r_2 - l_2)^2}\right)^{-1/2};$$
$$\frac{2}{3}\|w\|^2 \leq \|w\|_{\mathcal{A}_1}^2 \leq \|w\|^2, \quad \frac{2}{3}\|w\|^2 \leq \|w\|_{\mathcal{A}_2}^2 \leq \|w\|^2, \quad \frac{1}{3}\|w\|^2 \leq \|w\|_{\mathcal{A}_h}^2 \leq \|w\|^2. \tag{4.1}$$

LEMMA 4.2. *For any space-time grid function $v \in \mathring{\mathfrak{U}} \times \mathfrak{T}$, the following estimates hold:*

$$\sum_{k=0}^{n} c_{n-k}^{(n)} \big(v^{k+1} - v^k, v^{n+\sigma_n}\big) \geq \frac{1}{2}\left[c_0^{(n)}\|v^{n+1}\|^2 - \sum_{k=1}^{n}\big(c_{n-k}^{(n)} - c_{n-k+1}^{(n)}\big)\|v^k\|^2 - c_n^{(n)}\|v^0\|^2\right], \tag{4.2}$$

$$\big\|v^{n+\sigma_n}\big\|^2 \leq \big\|v^{n+1}\big\|^2 + \big\|v^n\big\|^2. \tag{4.3}$$

Proof. Inequality (4.2) can be deduced by

$$\begin{aligned}\sum_{k=0}^{n} c_{n-k}^{(n)}\big(v^{k+1} - v^k, v^{n+\sigma_n}\big) &\geq \frac{1}{2}\sum_{k=0}^{n} c_{n-k}^{(n)}\big(\|v^{k+1}\|^2 - \|v^k\|^2\big)\\ &= \frac{1}{2}\left[c_0^{(n)}\|v^{n+1}\|^2 - \sum_{k=1}^{n}\big(c_{n-k}^{(n)} - c_{n-k+1}^{(n)}\big)\|v^k\|^2 - c_n^{(n)}\|v^0\|^2\right];\end{aligned}$$

see more details in [1, Lemma 1]. As seen, the proof mainly relies on Lemma 2.1.

A direct evaluation shows

$$\begin{aligned}\big\|v^{n+\sigma_n}\big\|^2 &= \sigma_n^2\big\|v^{n+1}\big\|^2 + (1-\sigma_n)^2\big\|v^n\big\|^2 + 2\sigma_n(1-\sigma_n)\big(v^n, v^{n+1}\big)\\ &\leq \big[\sigma_n^2 + (1-\sigma_n)^2\big]\big(\big\|v^{n+1}\big\|^2 + \big\|v^n\big\|^2\big) \leq \big\|v^{n+1}\big\|^2 + \big\|v^n\big\|^2.\end{aligned}$$

Thus we complete the proof. ∎

LEMMA 4.3. *Suppose that $\alpha'(t) \leq 0$ when $0 \leq t \leq T$. For any fixed $n (2 \leq n \leq N)$, we have*

$$\frac{c_k^{(n)}}{s_n} \leq (1 + Q_3\tau)\frac{c_k^{(n-1)}}{s_{n-1}}, \quad 0 \leq k \leq n-2.$$

In order not to break the flow of the paper, we move its proof to Appendix B.

LEMMA 4.4. *For fixed n, we have*

$$\tau \sum_{k=1}^{n} \frac{c_{k-1}^{(k)} - c_k^{(k)}}{s_k} \le Q_4 < \infty, \tag{4.4}$$

$$\tau \sum_{k=1}^{n} \frac{c_k^{(k)}}{s_k} \le Q_5 < \infty. \tag{4.5}$$

Proof. To estimate the above inequalities, we employ the integral definition of the coefficients $c_k^{(n)}$ defined in (A.1).

For the inequality (4.4), it can be verified that for $2 \le k \le n$

$$\begin{aligned}
&\frac{c_{k-1}^{(k)} - c_k^{(k)}}{1-\alpha_{k+\sigma_k}} \\
&= \int_0^1 \left(\frac{3}{2} - \theta\right)\left[(k+\sigma_k-1-\theta)^{-\alpha_{k+\sigma_k}} - (k+\sigma_k-\theta)^{-\alpha_{k+\sigma_k}}\right]\mathrm{d}\theta \\
&\quad + \int_0^{\frac{1}{2}} \theta\left(\alpha_{k+\sigma_k}\int_{-\theta}^{\theta}\left(k-\frac{1}{2}+\sigma_k+\xi\right)^{-\alpha_{k+\sigma_k}-1} d\xi\right)\mathrm{d}\theta \\
&\le \frac{3}{2}\int_0^1 (k+\sigma_k-1-\theta)^{-\alpha_{k+\sigma_k}}\mathrm{d}\theta + 2\alpha_{k+\sigma_k}\int_0^{\frac{1}{2}}\left(\frac{1}{8}-\frac{1}{2}\xi^2\right)\left(k-\frac{1}{2}+\sigma_k+\xi\right)^{-\alpha_{k+\sigma_k}-1}\mathrm{d}\xi \\
&\le \frac{3}{2}\int_0^1 (k+\sigma_k-1-\theta)^{-\alpha_{k+\sigma_k}}\mathrm{d}\theta + \frac{1}{4}\left(k+\sigma_k-\frac{1}{2}\right)^{-\alpha_{k+\sigma_k}} \le \frac{7}{4}(k+\sigma_k-2)^{-\alpha_{k+\sigma_k}}.
\end{aligned}$$

Thus we have

$$\tau \sum_{k=2}^{n} \frac{c_{k-1}^{(k)} - c_k^{(k)}}{s_k} \le \frac{7}{4}\tau \sum_{k=2}^{n} t_{k+\sigma_k-2}^{-\alpha_{k+\sigma_k}}.$$

Here a few cases are considered separately.

(1) If $t_{n+\sigma_n-2} \le 1$, then $t_{k+\sigma_k-2}^{-\alpha_{k+\sigma_k}} \le t_{k+\sigma_k-2}^{-\alpha_*}$. It follows that

$$\begin{aligned}
\tau \sum_{k=2}^{n} t_{k+\sigma_k-2}^{-\alpha_{k+\sigma_k}} &\le \tau^{1-\alpha_*}\sum_{k=2}^{n}(k+\sigma_k-2)^{-\alpha_*} = \tau^{1-\alpha_*}\left[\sigma_2^{-\alpha_*} + \sum_{k=3}^{n}(k+\sigma_k-2)^{-\alpha_*}\right] \\
&\le \tau^{1-\alpha_*}\left[2^{\alpha_*} + \sum_{k=3}^{n}\left(k-\frac{3}{2}\right)^{-\alpha_*}\right] \le \tau^{1-\alpha_*}\left[2^{\alpha_*} + \sum_{k=3}^{n}\int_{k-1}^{k}\left(s-\frac{3}{2}\right)^{-\alpha_*} ds\right] \\
&\le 2^{\alpha_*}\tau^{1-\alpha_*} + \frac{t_{n-\frac{3}{2}}^{1-\alpha_*}}{1-\alpha_*}.
\end{aligned}$$

(2) If $t_{n+\sigma_n-2} > 1$, then there exists an integer k_* such that $t_{k+\sigma_k-2} < 1$ for $1 \le k \le k_*$ and $t_{k+\sigma_k-2} > 1$ for $k_*+1 \le k \le n$. Then we have

$$\tau \sum_{k=2}^{n} t_{k+\sigma_k-2}^{-\alpha_{k+\sigma_k}} = \tau \sum_{k=2}^{k_*} t_{k+\sigma_k-2}^{-\alpha_{k+\sigma_k}} + \tau \sum_{k=k_*+1}^{n} t_{k+\sigma_k-2}^{-\alpha_{k+\sigma_k}} \le 2^{\alpha_*}\tau^{1-\alpha_*} + \frac{t_{k_*-\frac{3}{2}}^{1-\alpha_*}}{1-\alpha_*} + (n-k_*)\tau.$$

In addition, we have

$$\tau \frac{c_0^{(1)} - c_1^{(1)}}{s_1} \le \frac{1}{4}\tau^{1-\alpha_{1+\sigma_1}}\left(\frac{1}{2}+\sigma_1\right)^{-\alpha_{1+\sigma_1}} + \sigma_1^{1-\alpha_{1+\sigma_1}} \le \frac{5}{4}\tau^{1-\alpha_*}.$$

From the above analysis, the inequality (4.4) is proved.

For the second inequality (4.5), we have

$$\frac{c_k^{(k)}}{s_k} = \frac{\int_0^1 (\frac{3}{2}-\theta)(k+\sigma_k-\theta)^{-\alpha_{k+\sigma_k}}\,\mathrm{d}\theta}{\tau^{\alpha_{k+\sigma_k}}\Gamma(1-\alpha_{k+\sigma_k})} \le \frac{1}{\tau^{\alpha_{k+\sigma_k}}}(k+\sigma_k-1)^{-\alpha_{k+\sigma_k}} \int_0^1 (\frac{3}{2}-\theta)\mathrm{d}\theta$$
$$= t_{k+\sigma_k-1}^{-\alpha_{k+\sigma_k}}.$$

Through a similar treatment to (4.4), we can easily get that $\tau\sum_{k=1}^n \frac{c_k^{(k)}}{s_k}$ is also bounded. This completes the proof. ◻

LEMMA 4.5. *Let $u, v \in \mathring{\mathfrak{U}} \times \mathfrak{T}$ be space-time grid functions and E^n, F^n, and G^n be defined by*

(4.6)
$$E^n := (2\sigma_{n-1}+1)\|v^n\|^2 - (2\sigma_{n-1}-1)\|v^{n-1}\|^2 + (2\sigma_{n-1}^2+\sigma_{n-1}-1)\|v^n - v^{n-1}\|^2,$$
$$F^n := (2\sigma_{n-1}+1)|u^n|_1^2 - (2\sigma_{n-1}-1)|u^{n-1}|_1^2 + (2\sigma_{n-1}^2+\sigma_{n-1}-1)|u^n - u^{n-1}|_1^2,$$
$$G^n := E^n + F^n + \frac{2\tau}{s_{n-1}}\sum_{k=2}^n c_{n-k}^{(n-1)}\|v^k\|^2, \qquad 1 \le n \le N.$$

The following estimates hold for $1 \le n \le N-1$:

$$E^{n+1} \ge \frac{1}{\sigma_n}\|v^{n+1}\|^2, \quad F^{n+1} \ge \frac{1}{\sigma_n}|u^{n+1}|_1^2, \tag{4.7}$$
$$\left(\delta_t v^n, v^{n+\sigma_n}\right) \ge \frac{E^{n+1}-E^n}{4\tau}, \quad -\left(\delta_t\Delta_h u^n, u^{n+\sigma_n}\right) \ge \frac{F^{n+1}-F^n}{4\tau},$$

$$G^1 \le (4\sigma_0^2+4\sigma_0-1)\left(\|v^1\|^2 + |u^1|_1^2\right) + (4\sigma_0^2-1)\left(\|v^0\|^2 + |u^0|_1^2\right),$$
$$G^{n+1} \ge \frac{1}{\sigma_n}\left(\|v^{n+1}\|^2 + |u^{n+1}|_1^2\right) + \frac{\tau}{\Gamma(1-\alpha_{n+\sigma_n})T^{\alpha_{n+\sigma_n}}}\sum_{k=2}^{n+1}\|v^k\|^2. \tag{4.8}$$

Proof. As pointed out early in [31, Lemma 3.5], it is easy to know that the first three inequalities in (4.7) hold. With the help of (4.1), we further have

$$-\left(\delta_t\Delta_h u^n, u^{n+\sigma_n}\right) = \left(\delta_t\delta_x u^n, \delta_x u^{n+\sigma_n}\right) + \left(\delta_t\delta_y u^n, \delta_y u^{n+\sigma_n}\right) \ge \frac{F^{n+1}-F^n}{4\tau}.$$

From the definition of E^n and F^n in (4.6), we get

$$E^1 \le (4\sigma_0^2+4\sigma_0-1)\|v^1\|^2 + (4\sigma_0^2-1)\|v^0\|^2,$$
$$F^1 \le (4\sigma_0^2+4\sigma_0-1)|u^1|_1^2 + (4\sigma_0^2-1)|u^0|_1^2.$$

Recalling Lemma 2.1 and (4.7), (4.8) is verified. This completes the proof. ◻

4.2. Stability and convergence of the finite difference scheme (3.10)–(3.13). The goal of this subsection is to prove the following theorems.

THEOREM 4.6. *Assume that $\alpha'(t) \le 0$ when $0 \le t \le T$. Then the finite difference scheme* (3.10)–(3.13) *is unconditionally stable. Namely, suppose that $\{U_{i,j}^n, V_{i,j}^n \,|\, (i,j) \in \bar{\omega},\, 0 \le n \le N\}$ satisfies*

(4.9)
$$\delta_t V_{i,j}^{\frac{1}{2}} + \frac{V_{i,j}^1 - V_{i,j}^0}{s_0} = \Delta_h U_{i,j}^{\frac{1}{2}} + f_{i,j}^{\frac{1}{2}}, \quad (i,j) \in \omega,$$

(4.10)
$$\delta_t V_{i,j}^n + \frac{1}{s_n}\sum_{k=0}^{n} c_{n-k}^{(n)}\big(V_{i,j}^{k+1} - V_{i,j}^k\big) = \Delta_h U_{i,j}^{n+\sigma_n} + f_{i,j}^{n+\sigma_n}, \quad (i,j) \in \omega,\ 1 \le n \le N-1,$$

(4.11)
$$\delta_t U_{i,j}^{\frac{1}{2}} = V_{i,j}^{\frac{1}{2}} + g_{i,j}^{\frac{1}{2}}, \quad \delta_t U_{i,j}^n = V_{i,j}^{n+\sigma_n} + g_{i,j}^{n+\sigma_n}, \quad (i,j) \in \bar{\omega},\ 1 \le n \le N-1,$$

(4.12)
$$U_{i,j}^0 = \phi(x_i, y_j), \quad V_{i,j}^0 = \psi(x_i, y_j), \quad (i,j) \in \omega;\ U_{i,j}^n = V_{i,j}^n = 0,\ (i,j) \in \partial\omega,\ 0 \le n \le N.$$

Then it holds that

(4.13)
$$\begin{aligned} \|V^1\| + |U^1|_1 &\le Q\big(\|V^0\| + |U^0|_1 + \tau^{(1+\alpha_{1/2})/2}\big\|f^{\frac{1}{2}}\big\| + \tau\|\Delta_h g^{\frac{1}{2}}\|\big), \\ \|V^{n+1}\| + |U^{n+1}|_1 &\le Q\Bigg[\|V^0\| + |U^0|_1 + \tau^{(1+\alpha_{1/2})/2}\big\|f^{\frac{1}{2}}\big\| + \tau\|\Delta_h g^{\frac{1}{2}}\| \\ &\quad + \Big(\tau\sum_{k=1}^{n}\|f^{k+\sigma_k}\|^2\Big)^{1/2} + \Big(\tau\sum_{k=1}^{n}\|\Delta_h g^{k+\sigma_k}\|^2\Big)^{1/2}\Bigg],\ 1 \le n \le N-1. \end{aligned}$$

Here Q is a positive constant that is independent of τ and h.

Proof. We carry out the proof in two steps.

Step 1. We prove the first estimate in (4.13). Take the inner product of (4.9) with $V^{\frac{1}{2}}$ to get

$$\big(\delta_t V^{\frac{1}{2}}, V^{\frac{1}{2}}\big) + \frac{1}{s_0}\big(V^1 - V^0, V^{\frac{1}{2}}\big) = \big(\Delta_h U^{\frac{1}{2}}, V^{\frac{1}{2}}\big) + \big(f^{\frac{1}{2}}, V^{\frac{1}{2}}\big).$$

Use (2.3) and Lemma 4.1 to rewrite the equation as follows:

$$\Big(\frac{1}{2\tau} + \frac{1}{2s_0}\Big)\big(\|V^1\|^2 - \|V^0\|^2\big) = -\big(\delta_x U^{\frac{1}{2}}, \delta_x V^{\frac{1}{2}}\big) - \big(\delta_y U^{\frac{1}{2}}, \delta_y V^{\frac{1}{2}}\big) + \big(f^{\frac{1}{2}}, V^{\frac{1}{2}}\big). \tag{4.14}$$

Apply Δ_h to the first equation in (4.11) to get

$$\delta_t \Delta_h U_{i,j}^{\frac{1}{2}} = \Delta_h V_{i,j}^{\frac{1}{2}} + \Delta_h g_{i,j}^{\frac{1}{2}}, \quad (i,j) \in \omega. \tag{4.15}$$

Take the inner product of (4.15) with $-U^{\frac{1}{2}}$ and use Lemma 4.1 to obtain

$$\frac{1}{2\tau}\big(|U^1|_1^2 - |U^0|_1^2\big) = \big(\delta_x V^{\frac{1}{2}}, \delta_x U^{\frac{1}{2}}\big) + \big(\delta_y V^{\frac{1}{2}}, \delta_y U^{\frac{1}{2}}\big) - \big(\Delta_h g^{\frac{1}{2}}, U^{\frac{1}{2}}\big). \tag{4.16}$$

Combine (4.14) and (4.16), and use Lemma 4.1 to obtain

$$\begin{aligned}
&\Big(\frac{1}{2\tau}+\frac{1}{2s_0}\Big)\big(\|V^1\|^2-\|V^0\|^2\big)+\frac{1}{2\tau}\big(|U^1|_1^2-|U^0|_1^2\big)=\big(f^{\frac12},V^{\frac12}\big)-\big(\Delta_h g^{\frac12},U^{\frac12}\big)\\
&\le \frac{1}{s_0}\big\|V^{\frac12}\big\|^2+\frac{s_0}{4}\big\|f^{\frac12}\big\|^2+\frac{1}{2Q_0^2\tau}\|U^{\frac12}\|^2+\frac{Q_0^2\tau}{2}\|\Delta_h g^{\frac12}\|^2\\
&\le \frac{1}{2s_0}\big(\|V^1\|^2+\|V^0\|^2\big)+\frac{s_0}{4}\big\|f^{\frac12}\big\|^2+\frac{1}{4\tau}(|U^1|_1^2+|U^0|_1^2)+\frac{Q_0^2\tau}{2}\|\Delta_h g^{\frac12}\|^2.
\end{aligned}$$

We recollect terms to obtain the following estimate:

$$\begin{aligned}
\|V^1\|^2+\tfrac12|U^1|_1^2 &\le \Big(\frac{2^{\alpha_{1/2}}\tau^{1-\alpha_{1/2}}}{\Gamma(2-\alpha_{1/2})}+1\Big)\|V^0\|^2+\frac32|U^0|_1^2\\
&\quad+2^{-\alpha_{1/2}}\Gamma(2-\alpha_{1/2})\tau^{1+\alpha_{1/2}}\|f^{\frac12}\|^2+Q_0^2\tau^2\|\Delta_h g^{\frac12}\|^2.
\end{aligned}\tag{4.17}$$

We thus prove the first estimate in (4.13).

Step 2. We next turn to the second estimate in (4.13). For $1\le n\le N-1$, take the inner product of (4.10) with $V^{n+\sigma_n}$ and use (4.2) to get

$$\begin{aligned}
&\big(\delta_t V^n,V^{n+\sigma_n}\big)+\frac{1}{2s_n}\Big[c_0^{(n)}\|V^{n+1}\|^2-\sum_{k=1}^{n}(c_{n-k}^{(n)}-c_{n-k+1}^{(n)})\|V^k\|^2-c_n^{(n)}\|V^0\|^2\Big]\\
&\quad\le \big(\delta_t V^n,V^{n+\sigma_n}\big)+\frac{1}{2s_n}\sum_{k=0}^{n}c_{n-k}^{(n)}\big(V^{k+1}-V^k,V^{n+\sigma_n}\big)\\
&\quad=\big(\Delta_h U^{n+\sigma_n},V^{n+\sigma_n}\big)+\big(f^{n+\sigma_n},V^{n+\sigma_n}\big)\\
&\quad=-\big(\delta_x U^{n+\sigma_n},\delta_x V^{n+\sigma_n}\big)-\big(\delta_y U^{n+\sigma_n},\delta_y V^{n+\sigma_n}\big)+\big(f^{n+\sigma_n},V^{n+\sigma_n}\big).
\end{aligned}\tag{4.18}$$

Apply Δ_h to the second equation in (4.11), take the inner product of the resulting equation with $-U^{n+\sigma_n}$, and use Lemma 4.1 to obtain the following equation for $1\le n\le N-1$:

$$-\big(\delta_t\Delta_h U^n,U^{n+\sigma_n}\big)=\big(\delta_x V^{n+\sigma_n},\delta_x U^{n+\sigma_n}\big)+\big(\delta_y V^{n+\sigma_n},\delta_y U^{n+\sigma_n}\big)-\big(\Delta_h g^{n+\sigma_n},U^{n+\sigma_n}\big).\tag{4.19}$$

Add (4.18) and (4.19), and use the estimate (4.7) to conclude

$$\begin{aligned}
&\frac{1}{4\tau}\big[(E^{n+1}+F^{n+1})-(E^n+F^n)\big]\\
&\qquad+\frac{1}{2s_n}\Big[c_0^{(n)}\|V^{n+1}\|^2-\sum_{k=1}^{n}(c_{n-k}^{(n)}-c_{n-k+1}^{(n)})\|V^k\|^2-c_n^{(n)}\|V^0\|^2\Big]\\
&\quad\le\big(\delta_t V^n,V^{n+\sigma_n}\big)-\big(\delta_t\Delta_h U^n,U^{n+\sigma_n}\big)\\
&\qquad+\frac{1}{2s_n}\Big[c_0^{(n)}\|V^{n+1}\|^2-\sum_{k=1}^{n}(c_{n-k}^{(n)}-c_{n-k+1}^{(n)})\|V^k\|^2-c_n^{(n)}\|V^0\|^2\Big]\\
&\quad\le\big(f^{n+\sigma_n},V^{n+\sigma_n}\big)-\big(\Delta_h g^{n+\sigma_n},U^{n+\sigma_n}\big),\quad 1\le n\le N-1.
\end{aligned}$$

We multiply the inequality by 4τ and rearrange the terms to get, for $1 \leq n \leq N-1$,

$$
\begin{aligned}
G^{n+1} &= E^{n+1} + F^{n+1} + \frac{2\tau}{s_n}\sum_{k=2}^{n+1} c_{n-k+1}^{(n)}\|V^k\|^2 \\
&\leq E^n + F^n + \frac{2\tau}{s_n}\Big[\sum_{k=2}^{n} c_{n-k}^{(n)}\|V^k\|^2 + c_{n-1}^{(n)}\|V^1\|^2 + c_n^{(n)}\|V^0\|^2\Big] \\
&\quad - \frac{2\tau}{s_n}c_n^{(n)}\|V^1\|^2 + 4\tau\big(f^{n+\sigma_n}, V^{n+\sigma_n}\big) - 4\tau\big(\Delta_h g^{n+\sigma_n}, U^{n+\sigma_n}\big).
\end{aligned} \tag{4.20}
$$

Lemma 4.3 implies

$$
\frac{2\tau}{s_n}\sum_{k=2}^{n} c_{n-k}^{(n)}\|V^k\|^2 \leq (1+Q_3\tau)\frac{2\tau}{s_{n-1}}\sum_{k=2}^{n} c_{n-k}^{(n-1)}\|V^k\|^2. \tag{4.21}
$$

We incorporate (4.21) into (4.20) so that for $1 \leq n \leq N-1$

$$
\begin{aligned}
G^{n+1} &\leq (1+Q_3\tau)G^n + \frac{2\tau}{s_n}\big[c_{n-1}^{(n)} - c_n^{(n)}\big]\|V^1\|^2 \\
&\quad + \frac{2\tau}{s_n}c_n^{(n)}\|V^0\|^2 + 4\tau\big(f^{n+\sigma_n}, V^{n+\sigma_n}\big) - 4\tau\big(\Delta_h g^{n+\sigma_n}, U^{n+\sigma_n}\big).
\end{aligned}
$$

Apply the first Gronwall inequality in Lemma 2.3 to deduce that for $1 \leq n \leq N-1$

$$
\begin{aligned}
G^{n+1} \leq e^{Q_3 n\tau}\Big[&G^1 + 2\tau\sum_{k=1}^{n}\frac{c_{k-1}^{(k)} - c_k^{(k)}}{s_k}\|V^1\|^2 + 2\tau\sum_{k=1}^{n}\frac{c_k^{(k)}}{s_k}\|V^0\|^2 \\
&+ 4\tau\sum_{k=1}^{n}\big(f^{k+\sigma_k}, V^{k+\sigma_k}\big) - 4\tau\sum_{k=1}^{n}\big(\Delta_h g^{k+\sigma_k}, U^{k+\sigma_k}\big)\Big].
\end{aligned} \tag{4.22}
$$

Note that Γ is decreasing on the interval $(0,1]$. Since $0 < 1-\alpha_* \leq 1-\alpha(t) \leq 1$, $\Gamma(1-\alpha(t))^{-1} \leq 1$. As $T^{-\alpha(t)} \leq \max\{1, T^{-1}\}$ for $t \in [0,T]$, we conclude that $Q_6 := \min_{0\leq t\leq T}\big[\Gamma(1-\alpha(t))T^{\alpha(t)}\big]^{-1} \leq \max\{1, T^{-1}\} < \infty$. We use the second estimate in (4.8) to bound G^{n+1} from below by

$$
G^{n+1} \geq \frac{1}{\sigma_n}(\|V^{n+1}\|^2 + |U^{n+1}|_1^2) + Q_6\tau\sum_{k=2}^{n+1}\|V^k\|^2. \tag{4.23}
$$

We combine estimates (4.22) and (4.23) to conclude that for $1 \leq n \leq N-1$

$$
\begin{aligned}
&\|V^{n+1}\|^2 + |U^{n+1}|_1^2 + \frac{Q_6\tau}{2}\sum_{k=2}^{n+1}\|V^k\|^2 \leq \|V^{n+1}\|^2 + |U^{n+1}|_1^2 + \sigma_n Q_6\tau\sum_{k=2}^{n+1}\|V^k\|^2 \\
&\leq e^{Q_3 T}\Big[G^1 + 2\tau\|V^1\|^2\sum_{k=1}^{n}\frac{c_{k-1}^{(k)} - c_k^{(k)}}{s_k} + 2\tau\|V^0\|^2\sum_{k=1}^{n}\frac{c_k^{(k)}}{s_k} \\
&\quad + \frac{\varepsilon\tau}{2Q_0^2}\sum_{k=1}^{n}\|U^{k+\sigma_k}\|^2 + \frac{8Q_0^2\tau}{\varepsilon}\sum_{k=1}^{n}\|\Delta_h g^{k+\sigma_k}\|^2\Big] \\
&\quad + \frac{\varepsilon Q_6\tau}{2}\sum_{k=1}^{n}\|V^{k+\sigma_k}\|^2 + \frac{8e^{2Q_3T}\tau}{\varepsilon Q_6}\sum_{k=1}^{n}\|f^{k+\sigma_k}\|^2, \quad 0 < \varepsilon < 1.
\end{aligned} \tag{4.24}
$$

We use (4.1), (4.3), and the first inequality in (4.8) and fix an ε such that $\varepsilon \le \frac{1}{2}$ to obtain

(4.25)
$$\begin{aligned}
&\|V^{n+1}\|^2 + |U^{n+1}|_1^2 \\
&\le Q_7 e^{Q_3 T}\Bigg[(4\sigma_0^2 + 4\sigma_0 - 1)(\|V^1\|^2 + |U^1|_1^2) + (4\sigma_0^2 - 1)(\|V^0\|^2 + |U^0|_1^2) \\
&+ 2\tau\|V^0\|^2 \sum_{k=1}^{n} \frac{c_k^{(k)}}{s_k} + 2\tau\|V^1\|^2 \sum_{k=1}^{n} \frac{c_{k-1}^{(k)} - c_k^{(k)}}{s_k} + \tau\sum_{k=1}^{n} |U^k|_1^2 + 8Q_0^2\tau \sum_{k=1}^{n} \|\Delta_h g^{k+\sigma_k}\|^2\Bigg] \\
&+ Q_6\tau\|V^1\|^2 + Q_7\tau\sum_{k=1}^{n} \|f^{k+\sigma_k}\|^2.
\end{aligned}$$

With the help of (4.4) and (4.5) in Lemma 4.4, we know that $\tau\sum_{k=1}^n \frac{c_{k-1}^{(k)} - c_k^{(k)}}{s_k}$ and $\tau\sum_{k=1}^n \frac{c_k^{(k)}}{s_k}$ are bounded. Further since $\|V^1\|$ and $|U^1|_1$ were already bounded in (4.17), we deduce from (4.25) that

(4.26)
$$\begin{aligned}
\|V^{n+1}\|^2 + |U^{n+1}|_1^2 \le Q_8\tau\sum_{k=1}^{n}(\|V^k\|^2 + |U^k|_1^2) + Q_8\Bigg[&\|V^0\|^2 + |U^0|_1^2 + \tau^{1+\alpha_{1/2}}\big\|f^{\frac{1}{2}}\big\|^2 \\
&+ \tau^2\|\Delta_h g^{\frac{1}{2}}\|^2 + \tau\sum_{k=0}^{n}\|f^{k+\sigma_k}\|^2 + \tau\sum_{k=1}^{n}\|\Delta_h g^{k+\sigma_k}\|^2\Bigg].
\end{aligned}$$

We apply the second Gronwall inequality in Lemma 2.3 to complete the proof of the second estimate in (4.13). ∎

THEOREM 4.7. *Suppose the problem* (3.1)–(3.4) *has a unique smooth solution* $\{u(\mathbf{x}, t), v(\mathbf{x}, t)\}$. *Let* $\{U_{i,j}^n, V_{i,j}^n\}$ *be the solution of the difference scheme* (3.10)–(3.13). *Denote*

$$e_{i,j}^n := u_{i,j}^n - U_{i,j}^n, \quad \tilde{e}_{i,j}^n := v_{i,j}^n - V_{i,j}^n, \quad (i,j) \in \bar{\omega},\ 0 \le n \le N.$$

Then, the optimal-order error estimate holds:

$$|e^n|_1 + \|e^n\| \le Q(\tau^2 + h^2), \quad 1 \le n \le N. \tag{4.27}$$

Proof. Subtracting (3.10)–(3.13) from (3.5)–(3.7) and (3.9), respectively, we get the system of error equations as follows:

(4.28)
$$\delta_t \tilde{e}_{i,j}^{\frac{1}{2}} + \frac{\tilde{e}_{i,j}^1 - \tilde{e}_{i,j}^0}{s_0} = \Delta_h e_{i,j}^{\frac{1}{2}} + R_{i,j}^0, \quad (i,j) \in \omega,$$

(4.29)
$$\delta_t \tilde{e}_{i,j}^n + \frac{1}{s_n}\sum_{k=0}^{n} c_{n-k}^{(n)}(\tilde{e}_{i,j}^{k+1} - \tilde{e}_{i,j}^k) = \Delta_h e_{i,j}^{n+\sigma_n} + R_{i,j}^n, \quad (i,j) \in \omega,\ 1 \le n \le N-1,$$

(4.30)
$$\delta_t e_{i,j}^{\frac{1}{2}} = \tilde{e}_{i,j}^{\frac{1}{2}} + r_{i,j}^0, \quad \delta_t e_{i,j}^n = \tilde{e}_{i,j}^{n+\sigma_n} + r_{i,j}^n, \quad (i,j) \in \bar{\omega},\ 1 \le n \le N-1,$$

(4.31)
$$e_{i,j}^0 = 0, \quad \tilde{e}_{i,j}^0 = 0, \quad (i,j) \in \omega; \qquad e_{i,j}^n = 0, \quad (i,j) \in \partial\omega,\ 0 \le n \le N.$$

Apply the stability estimate (4.13) to the system of error equations to deduce the error estimate (4.27). ∎

Remark 4.8. We have proved the stability and convergence of the difference scheme with the condition that $\alpha'(t) \le 0$ when $0 \le t \le T$. The numerical example demonstrated that this condition is unnecessary, which needs further consideration.

4.3. Stability and convergence of the ADI scheme and the compact ADI scheme. We similarly prove the following theorem on the stability and convergence of the ADI scheme and the compact ADI scheme.

THEOREM 4.9. *The ADI finite difference scheme and the compact ADI finite difference scheme are unconditionally stable. Moreover, the ADI finite difference scheme has a second-order convergence rate in both space and time and the compact ADI finite difference scheme has a fourth-order convergence rate in space and second-order convergence rate in time.*

Proof. The analysis of the ADI scheme and compact ADI scheme can be carried out using a similar technique to the proof of Theorem 4.6, which we outline here. As a matter of fact, the ADI scheme (3.27)–(3.30) can be expressed in the following equivalent form:

(4.32)
$$\begin{aligned}
&\delta_t V_{i,j}^{\frac{1}{2}} + \frac{V_{i,j}^1 - V_{i,j}^0}{s_0} + \frac{s_0\tau^4\delta_x^2\delta_y^2\delta_t V_{i,j}^{\frac{1}{2}}}{16(s_0+\tau)} = \Delta_h U_{i,j}^{\frac{1}{2}} + f_{i,j}^{\frac{1}{2}}, \quad (i,j)\in\omega,\\
&\delta_t V_{i,j}^n + \frac{1}{s_n}\sum_{k=0}^{n} c_{n-k}^{(n)}(V_{i,j}^{k+1} - V_{i,j}^k) + \frac{8\sigma_n^4 s_n\tau^3\delta_x^2\delta_y^2 V_{i,j}^{n+1}}{(2\sigma_n+1)^2[(2\sigma_n+1)s_n + 2\tau c_0^{(n)}]}\\
&\qquad = \Delta_h U_{i,j}^{n+\sigma_n} + f_{i,j}^{n+\sigma_n}, \quad (i,j)\in\omega,\ 1\le n\le N-1,\\
&\delta_t U_{i,j}^{\frac{1}{2}} = V_{i,j}^{\frac{1}{2}}, \quad \delta_t U_{i,j}^n = V_{i,j}^{n+\sigma_n}, \quad (i,j)\in\bar\omega,\ 1\le n\le N-1,\\
&U_{i,j}^0 = \phi(x_i,y_j), \quad V_{i,j}^0 = \psi(x_i,y_j), \quad (i,j)\in\omega;\ U_{i,j}^n = 0, \quad (i,j)\in\partial\omega,\ 0\le n\le N.
\end{aligned}$$

We compare (4.32) with the corresponding equations for the finite difference scheme (3.10)–(3.13) to find out that the last term on the left-hand side of the first two equations in (4.32) are the only extra terms to the corresponding equations in the finite difference scheme (3.10)–(3.13). We use the corresponding procedures leading to (4.14) and (4.18) to rewrite these two terms as follows:

$$(4.33)\qquad (\delta_x^2\delta_y^2\delta_t V^{\frac{1}{2}}, V^{\frac{1}{2}}) = (\delta_t\delta_x\delta_y V^{\frac{1}{2}}, \delta_x\delta_y V^{\frac{1}{2}}) = \frac{1}{2\tau}(\|\delta_x\delta_y V^1\|^2 - \|\delta_x\delta_y V^0\|^2),$$

$$\begin{aligned}
(\delta_x^2\delta_y^2 V^{n+1}, V^{n+\sigma_n}) &= (\delta_x\delta_y V^{n+1}, \delta_x\delta_y V^{n+\sigma_n})\\
&\ge \frac{3\sigma_n - 1}{2}\|\delta_x\delta_y V^{n+1}\|^2 - \frac{1-\sigma_n}{2}\|\delta_x\delta_y V^n\|^2\\
&\ge \frac{3\sigma_n - 1}{2}\big(\|\delta_x\delta_y V^{n+1}\|^2 - \|\delta_x\delta_y V^n\|^2\big).
\end{aligned}$$

We use a similar procedure to the proof of the stability of the finite difference scheme (3.10)–(3.13) in Theorem 4.6 to prove the stability and second-order spatial and temporal convergence rates of the ADI scheme and omit the details.

We finally turn to the analysis of the compact ADI scheme. Similarly to the ADI scheme (4.32), we rewrite the compact ADI scheme (3.32) as the following equivalent

form:
(4.34)

$$
\begin{aligned}
&\mathcal{A}_h\delta_t V_{i,j}^{\frac{1}{2}} + \frac{\mathcal{A}_h V_{i,j}^1 - \mathcal{A}_h V_{i,j}^0}{s_0} + \frac{s_0\tau^4\delta_x^2\delta_y^2\delta_t V_{i,j}^{\frac{1}{2}}}{16(s_0+\tau)} = \Lambda_h U_{i,j}^{\frac{1}{2}} + \mathcal{A}_h f_{i,j}^{\frac{1}{2}}, \quad (i,j)\in\omega,\\
&\mathcal{A}_h\delta_t V_{i,j}^n + \frac{1}{s_n}\sum_{k=0}^{n} c_{n-k}^{(n)}(\mathcal{A}_h V_{i,j}^{k+1} - \mathcal{A}_h V_{i,j}^k) + \frac{8\sigma_n^4 s_n\tau^3\delta_x^2\delta_y^2 V_{i,j}^{n+1}}{(2\sigma_n+1)^2[(2\sigma_n+1)s_n + 2\tau c_0^{(n)}]}\\
&\quad = \Lambda_h U_{i,j}^{n+\sigma_n} + \mathcal{A}_h f_{i,j}^{n+\sigma_n}, \quad (i,j)\in\omega,\ 1\le n\le N-1,\\
&\delta_t U_{i,j}^{\frac{1}{2}} = V_{i,j}^{\frac{1}{2}}, \quad \delta_t U_{i,j}^n = V_{i,j}^{n+\sigma_n}, \quad (i,j)\in\bar{\omega},\ 1\le n\le N-1,\\
&U_{i,j}^0 = \phi(x_i,y_j), \quad V_{i,j}^0 = \psi(x_i,y_j), \quad (i,j)\in\omega;\ U_{i,j}^n = 0, \quad (i,j)\in\partial\omega,\ 0\le n\le N.
\end{aligned}
$$

We compare (4.34) with the corresponding equations for the ADI scheme (4.32) to find out that the first two equations in (4.34) are compact operator notations to the corresponding equations in the ADI scheme (4.32). Fortunately, with the help of the good properties of the compact operator given in Lemma 4.1, by a similar technique to the proof of Theorem 4.6, we can easily prove the stability and fourth-order spatial convergence rate and second-order temporal convergence rate of the compact ADI scheme. ∎

5. Numerical experiments.

We carry out numerical experiments to investigate the accuracy and efficiency of the finite difference scheme (3.10)–(3.13), and ADI finite difference scheme (3.27)–(3.30), and the compact ADI finite difference scheme (3.32). All the schemes are implemented in MATLAB (R2019b). The data are chosen as follows: $\Omega = (0,\pi)\times(0,\pi)$ and $[0,T]=[0,1]$, the initial data are $\phi(x,y)=\sin x\sin y$ and $\psi(x,y)=0$, three different types of variable order $\beta(t)=(6+\sin t)/4$, $2-t^2$ and $1+\exp(-t)$ are chosen, and the forcing term is

$$
\begin{aligned}
&f(x,y,t)\\
&= \Big(6(t+1) + \frac{6}{\Gamma(4-\beta(t))}t^{3-\beta(t)} + \frac{6}{\Gamma(3-\beta(t))}t^{2-\beta(t)} + 2(t^3+3t^2+1)\Big)\sin x\sin y.
\end{aligned}
$$

It can be verified directly that the true solution to problem (1.1)–(1.2) with the data given above is

$$u(x,y,t) = (t^3+3t^2+1)\sin x\sin y.$$

In the numerical example runs, we choose $h := h_1 = h_2$ and measure the error of the numerical solution in the H_1-norm

$$F(h,\tau) := \max_{0\le n\le N} |U^n - u^n|_1,$$

and fit the convergence rate

$$
\begin{aligned}
&Order_\tau := \log_2\Big(\frac{F(h,2\tau)}{F(h,\tau)}\Big), \quad Order_h := \log_2\Big(\frac{F(2h,\tau)}{F(h,\tau)}\Big),\\
&Order := \log_2\Big(\frac{F(2h,2\tau)}{F(h,\tau)}\Big).
\end{aligned}
$$

Since both the finite difference scheme (3.10)–(3.13) and the ADI scheme (3.27)–(3.30) have second-order convergence rates in space and time, we choose $\tau \approx h$ in the numerical experiments and present the results in Table 1. We observe that both schemes generate numerical solutions with the same accuracy and that the ADI scheme

TABLE 1
Convergence rates and the CPU time of the finite difference scheme and the ADI scheme.

$\beta(t)$	h	τ	ADI scheme			The finite difference scheme		
			$F(h,\tau)$	Order	CPU time (s)	$F(h,\tau)$	Order	CPU time (s)
$(6+\sin t)/4$	$\pi/20$	1/20	1.9530e-3		0.13	1.9616e-3		0.48
	$\pi/40$	1/40	5.4903e-4	1.83	0.41	5.5014e-4	1.83	2.16
	$\pi/80$	1/80	1.4679e-4	1.90	2.82	1.4693e-4	1.90	86.16
	$\pi/160$	1/160	3.8217e-5	1.94	23.17	3.8236e-5	1.94	4579
$2-t^2$	$\pi/20$	1/20	2.5310e-3		0.06	2.5542e-3		0.12
	$\pi/40$	1/40	6.3357e-4	2.00	0.38	6.3651e-4	2.00	2.08
	$\pi/80$	1/80	1.5858e-4	2.00	2.81	1.5895e-4	2.00	83.58
	$\pi/160$	1/160	3.9674e-5	2.00	23.28	3.9722e-5	2.00	4606
$1+\exp(-t)$	$\pi/20$	1/20	2.4211e-3		0.07	2.4459e-3		0.10
	$\pi/40$	1/40	6.2297e-4	1.96	0.39	6.2615e-4	1.97	2.03
	$\pi/80$	1/80	1.5799e-4	1.98	2.84	1.5840e-4	1.98	85.40
	$\pi/160$	1/160	3.9784e-5	1.99	23.13	3.9836e-5	1.99	4614

TABLE 2
The temporal convergence rates of the ADI scheme and the compact ADI scheme with $h = \pi/2000$.

$\beta(t)$	τ	The ADI scheme		The compact ADI scheme	
		$F(h,\tau)$	Order_τ	$F(h,\tau)$	Order_τ
$(6+\sin t)/4$	1/20	1.9530e-3		1.9528e-3	
	1/40	5.4922e-4	1.83	5.4904e-4	1.83
	1/80	1.4696e-4	1.90	1.4679e-4	1.90
	1/160	3.8399e-5	1.94	3.8217e-5	1.94
$2-t^2$	1/20	2.5292e-3		2.5288e-3	
	1/40	6.3382e-4	2.00	6.3344e-4	2.00
	1/80	1.5895e-4	2.00	1.5857e-4	2.00
	1/160	4.0054e-5	1.99	3.9674e-5	2.00
$1+\exp(-t)$	1/20	2.4191e-3		2.4187e-3	
	1/40	6.2322e-4	1.96	6.2283e-4	1.96
	1/80	1.5838e-4	1.98	1.5798e-4	1.98
	1/160	4.0176e-5	1.98	3.9784e-5	1.99

uses much less CPU time than the finite difference scheme does as the spatial mesh sizes and time step size gets finer and finer. This clearly demonstrates the numerical and computational advantage of the ADI scheme and coincides with the analysis.

Next we investigate the performance of the ADI scheme and the compact ADI scheme. In Table 2, we compute the temporal convergence rate of both schemes with a very fine spatial grid size so that the spatial error can be neglected. We observe from the numerical results that both schemes have second-order temporal convergence rates, which is consistent with the theoretical results.

In Table 3 we investigate the spatial convergence rates for both the ADI scheme and the compact ADI scheme, where we use a very fine temporal step size and test the spatial convergence rate. Since the ADI scheme has the second-order spatial convergence rate while the compact ADI scheme has the fourth-order spatial convergence rate, we fix the temporal step size $\tau = 1/2000$ for the ADI scheme and $\tau = 1/5000$ for the compact ADI scheme, but use the same spatial step size refined from $h = \pi/5$ to $\pi/40$. We observe from these results that the ADI scheme has the second-order spatial accuracy and the compact ADI scheme has the fourth-order spatial accuracy, which substantiates the theoretical analysis. We note that the spatial convergence rate of the compact ADI scheme deteriorates as the mesh size h is refined, which

TABLE 3
The spatial convergence rates of the ADI scheme and the compact ADI scheme.

$\beta(t)$	h	The ADI scheme			The compact ADI scheme		
		τ	$F(h,\tau)$	Order$_h$	τ	$F(h,\tau)$	Order$_h$
$(6+\sin t)/4$	$\pi/5$	1/2000	6.4232e-2		1/5000	1.3006e-3	
	$\pi/10$		1.6384e-2	1.97		8.1410e-5	4.00
	$\pi/20$		4.1168e-3	1.99		5.1216e-6	3.99
	$\pi/40$		1.0307e-3	2.00		3.5271e-7	3.86
$2-t^2$	$\pi/5$		5.9244e-2			1.2000e-3	
	$\pi/10$		1.5115e-2	1.97		7.5120e-5	4.00
	$\pi/20$		3.7983e-3	1.99		4.7339e-6	3.99
	$\pi/40$		9.5096e-4	2.00		3.3399e-7	3.83
$1+\exp(-t)$	$\pi/5$		6.1150e-2			1.2384e-3	
	$\pi/10$		1.5600e-2	1.97		7.7525e-5	4.00
	$\pi/20$		3.9200e-3	1.99		4.8846e-6	3.99
	$\pi/40$		9.8142e-4	2.00		3.4376e-7	3.83

TABLE 4
The CPU time of the ADI scheme and the compact ADI scheme.

$\beta(t)$	τ	The ADI scheme			The compact ADI scheme		
		h	$F(h,\tau)$	CPU time (s)	h	$F(h,\tau)$	CPU time (s)
$(6+\sin t)/4$	1/16	$\pi/16$	8.5657e-3	0.02	$\pi/4$	5.2455e-3	0.13
	1/64	$\pi/64$	5.8217e-4	0.30	$\pi/8$	3.7699e-4	0.24
	1/256	$\pi/256$	3.7609e-5	16.96	$\pi/16$	2.4840e-5	5.52
	1/1024	$\pi/1024$	2.3846e-6	1831	$\pi/32$	1.5875e-6	172
$2-t^2$	1/16	$\pi/16$	9.9110e-3	0.02	$\pi/4$	6.7933e-3	0.02
	1/64	$\pi/64$	6.1928e-4	0.29	$\pi/8$	4.2944e-4	0.20
	1/256	$\pi/256$	3.8723e-5	16.58	$\pi/16$	2.6936e-5	5.56
	1/1024	$\pi/1024$	2.4206e-6	1817	$\pi/32$	1.6851e-6	170
$1+\exp(-t)$	1/16	$\pi/16$	9.8694e-3	0.02	$\pi/4$	6.6637e-3	0.02
	1/64	$\pi/64$	6.2944e-4	0.30	$\pi/8$	4.3359e-4	0.24
	1/256	$\pi/256$	3.9548e-5	16.74	$\pi/16$	2.7383e-5	5.47
	1/1024	$\pi/1024$	2.4751e-6	1825	$\pi/32$	1.7160e-6	172

implies that with the spatial mesh size $h = \pi/40$ the impact of the temporal error of the compact ADI scheme cannot be safely neglected as h^2 is already comparable to $\tau = 1/5000$. This indicates the fourth-order convergence rate of the compact ADI scheme from another point of view.

Finally, in Table 4 we investigate the CPU time of the ADI scheme and the compact ADI scheme. We let $\tau \approx h$ for the ADI scheme and $\tau \approx h^2$ for the compact ADI scheme to take full advantage of the convergence rates of the schemes. We observe that the compact ADI scheme can use a much coarser spatial mesh size and the same temporal step size as those used by the ADI scheme to achieve the same accuracy. Consequently, the compact ADI scheme has an improved computational efficiency.

6. Concluding remarks. We developed a temporal second-order finite difference scheme for the variable-order time-fractional wave PDE (1.1) by generalizing Alikhanov's $L2$-1_σ formula and applying the method of order reduction to the variable-order fractional differential operator. We further developed an ADI finite difference scheme and a compact ADI finite difference scheme to the variable-order time-fractional wave PDE (1.1). We then used the discrete energy technique to prove the unconditional stability of these schemes. We proved that the finite difference scheme and the ADI finite difference scheme have the second-order spatial and temporal convergence rates, and that the compact ADI finite difference scheme has the fourth-order spatial convergence rate and the second-order temporal convergence rate. Numerical experiments were presented to substantiate the theoretical analysis, and to demonstrate the computational efficiency of the ADI scheme and the compact ADI scheme.

Because of the complexities of variable-order time-FPDEs, their mathematical analysis has been very meager in the literature. The well-posedness and regularity of the initial-value problem of a nonlinear variable-order fractional ordinary differential equation were analyzed in [40] along with the error estimates of its numerical approximation. These results were then combined with a spectral analysis technique to prove the well-posedness and regularity estimate of the initial boundary value problem of a variable-order time-fractional diffusion PDEs in [38]. The well-posedness and regularity estimate of the initial boundary value problem of a variable-order time-fractional diffusion PDE with a space-dependent variable order were analyzed in [16].

The well-posedness and regularity estimate were proved in [45] for the initial-value problem of a nonlinear variable-order fractional wave ordinary differential equation, which is probably the most closely related well-posedness and regularity estimate results to problem (1.1)-(1.2). To the best knowledge of the authors, the well-posedness and related regularity estimates of the initial boundary value problem of the variable-order time-fractional wave PDE (1.1)-(1.2) are yet to be proved, and are currently under investigation.

Appendix A. The derivation of (2.5). We reformulate (2.5) to obtain

$$
\begin{aligned}
&{}_0^C D_t^{\alpha(t)} g(t)\Big|_{t=t_{n+\sigma_n}} = \frac{1}{\Gamma(1-\alpha_{n+\sigma_n})}\Bigg\{\bigg[\int_{t_0}^{t_1}\frac{t_{\frac{3}{2}}-s}{\tau}\,(t_{n+\sigma_n}-s)^{-\alpha_{n+\sigma_n}}\,\mathrm{d}s\bigg]\delta_t g^{\frac{1}{2}} \\
&+\sum_{k=1}^{n-1}\bigg[\int_{t_k}^{t_{k+1}}\frac{t_{k+\frac{3}{2}}-s}{\tau}\,(t_{n+\sigma_n}-s)^{-\alpha_{n+\sigma_n}}\,\mathrm{d}s \\
&+\int_{t_{k-1}}^{t_k}\frac{s-t_{k-\frac{1}{2}}}{\tau}\,(t_{n+\sigma_n}-s)^{-\alpha_{n+\sigma_n}}\,\mathrm{d}s\bigg]\delta_t g^{k+\frac{1}{2}} \\
&+\bigg[\int_{t_{n-1}}^{t_n}\frac{s-t_{n-\frac{1}{2}}}{\tau}\,(t_{n+\sigma_n}-s)^{-\alpha_{n+\sigma_n}}\,\mathrm{d}s+\int_{t_n}^{t_{n+\sigma_n}}(t_{n+\sigma_n}-s)^{-\alpha_{n+\sigma_n}}\,\mathrm{d}s\bigg]\delta_t g^{n+\frac{1}{2}}\Bigg\} \\
&=\frac{\tau^{1-\alpha_{n+\sigma_n}}}{\Gamma(1-\alpha_{n+\sigma_n})}\Bigg\{\bigg[\int_0^1\Big(\frac{3}{2}-\theta\Big)(n+\sigma_n-\theta)^{-\alpha_{n+\sigma_n}}\,\mathrm{d}\theta\bigg]\delta_t g^{\frac{1}{2}} \\
&+\sum_{k=1}^{n-1}\bigg[\int_0^1\Big(\frac{3}{2}-\theta\Big)(n-k+\sigma_n-\theta)^{-\alpha_{n+\sigma_n}}\,\mathrm{d}\theta \\
&+\int_0^{\frac{1}{2}}\theta\Big(\alpha_{n+\sigma_n}\int_{-\theta}^{\theta}(n-k+\frac{1}{2}+\sigma_n+\xi)^{-\alpha_{n+\sigma_n}-1}d\xi\Big)d\theta\bigg]\delta_t g^{k+\frac{1}{2}} \\
&+\bigg[\int_0^{\frac{1}{2}}\theta\Big(\alpha_{n+\sigma_n}\int_{-\theta}^{\theta}(\frac{1}{2}+\sigma_n+\xi)^{-\alpha_{n+\sigma_n}-1}d\xi\Big)d\theta+\frac{\sigma_n^{1-\alpha_{n+\sigma_n}}}{1-\alpha_{n+\sigma_n}}\bigg]\delta_t g^{n+\frac{1}{2}}\Bigg\} \\
&=\frac{1}{s_n}\sum_{k=0}^{n} c_{n-k}^{(n)}\big(g^{k+1}-g^k\big).
\end{aligned}
$$

Here the coefficients $\{c_k^{(n)}\}_{k=0}^n$ for $n \ge 1$ are given by

(A.1)
$$c_k^{(n)} = (1-\alpha_{n+\sigma_n})\begin{cases} \displaystyle\int_0^{\frac{1}{2}} \theta\Big(\alpha_{n+\sigma_n}\int_{-\theta}^{\theta}\Big(\frac{1}{2}+\sigma_n+\xi\Big)^{-\alpha_{n+\sigma_n}-1}d\xi\Big)d\theta + \frac{\sigma_n^{1-\alpha_{n+\sigma_n}}}{1-\alpha_{n+\sigma_n}}, & k=0, \\ \displaystyle\int_0^1 \Big(\frac{3}{2}-\theta\Big)\big(k+\sigma_n-\theta\big)^{-\alpha_{n+\sigma_n}}d\theta & \\ \displaystyle\quad + \int_0^{\frac{1}{2}} \theta\Big(\alpha_{n+\sigma_n}\int_{-\theta}^{\theta}\Big(k+\frac{1}{2}+\sigma_n+\xi\Big)^{-\alpha_{n+\sigma_n}-1}d\xi\Big)d\theta, & 1\le k\le n-1, \\ \displaystyle\int_0^1 \Big(\frac{3}{2}-\theta\Big)\big(n+\sigma_n-\theta\big)^{-\alpha_{n+\sigma_n}}d\theta, & k=n. \end{cases}$$

Further computing yields for $n \ge 1$

(A.2)
$$c_k^{(n)} = \begin{cases} \dfrac{(1+\sigma_n)^{2-\alpha_{n+\sigma_n}} - \sigma_n^{2-\alpha_{n+\sigma_n}}}{2-\alpha_{n+\sigma_n}} - \dfrac{1}{2}\big[(1+\sigma_n)^{1-\alpha_{n+\sigma_n}} - \sigma_n^{1-\alpha_{n+\sigma_n}}\big], & k=0, \\ \dfrac{(k+\sigma_n+1)^{2-\alpha_{n+\sigma_n}} - 2(k+\sigma_n)^{2-\alpha_{n+\sigma_n}} + (k+\sigma_n-1)^{2-\alpha_{n+\sigma_n}}}{2-\alpha_{n+\sigma_n}} & \\ \quad - \dfrac{(k+\sigma_n+1)^{1-\alpha_{n+\sigma_n}} - 2(k+\sigma_n)^{1-\alpha_{n+\sigma_n}} + (k+\sigma_n-1)^{1-\alpha_{n+\sigma_n}}}{2}, & 1\le k\le n-1, \\ \dfrac{3(k+\sigma_n)^{1-\alpha_{n+\sigma_n}} - (k+\sigma_n-1)^{1-\alpha_{n+\sigma_n}}}{2} & \\ \quad - \dfrac{(k+\sigma_n)^{2-\alpha_{n+\sigma_n}} - (k+\sigma_n-1)^{2-\alpha_{n+\sigma_n}}}{2-\alpha_{n+\sigma_n}} & k=n. \end{cases}$$

Appendix B. Proof of Lemma 4.3. Before giving the proof, we first present two lemmas that need be used.

LEMMA B.1. *Denote* $\|\alpha'\|_\infty = \max\limits_{0\le t\le T}|\alpha'(t)|$. *Then we have*

$$\big|\alpha_{n+\sigma_n} - \alpha_{n-1+\sigma_{n-1}}\big| \le \frac{3\tau}{2}\|\alpha'\|_\infty, \qquad \big|\sigma_n - \sigma_{n-1}\big| \le \frac{3}{4}\tau\|\alpha'\|_\infty.$$

In addition, if $\alpha'(t) \le 0$ *when* $t \in (t_{n-\frac{1}{2}}, t_{n+1})$, *then* $\sigma_n \ge \sigma_{n-1}$ *and* $\alpha_{n+\sigma_n} \le \alpha_{n-1+\sigma_{n-1}}$.

Proof. (a) Using the Taylor expansion, we have

$$\begin{aligned} \alpha_{n+\sigma_n} - \alpha_{n-1+\sigma_{n-1}} &= \alpha'(\xi_n)\big[t_n - t_{n-1} + (\sigma_n - \sigma_{n-1})\tau\big] \\ &= \alpha'(\xi_n)\big(1+\sigma_n-\sigma_{n-1}\big)\tau, \quad \xi_n \in \big(t_{n-1+\sigma_{n-1}}, t_{n+\sigma_n}\big). \end{aligned} \tag{B.1}$$

Noting the fact that $\sigma_{n-1}, \sigma_n \in (\frac{1}{2}, 1)$, we have

$$\left|\alpha_{n+\sigma_n} - \alpha_{n-1+\sigma_{n-1}}\right| \le \frac{3\tau}{2}\|\alpha'\|_\infty.$$

(b) By the definition of (2.1), we have $\sigma_n = 1 - \frac{1}{2}\alpha(t_n + \sigma_n\tau)$ and then

$$\sigma_n - \sigma_{n-1} = -\frac{1}{2}\Big[\alpha(t_n+\sigma_n\tau) - \alpha(t_{n-1}+\sigma_{n-1}\tau)\Big] = -\frac{1}{2}\alpha'(\xi_n)\big(1+\sigma_n-\sigma_{n-1}\big)\tau. \tag{B.2}$$

Further we have $|\sigma_n - \sigma_{n-1}| \le \frac{3}{4}\tau\|\alpha'\|_\infty$.

(c) From (B.1) and (B.2), we see that $\sigma_n \ge \sigma_{n-1}$ and $\alpha_{n+\sigma_n} \le \alpha_{n-1+\sigma_{n-1}}$ if $\alpha'(t) \le 0$ when $t \in (t_{n-\frac{1}{2}}, t_{n+1})$. ∎

Lemma B.2. *Suppose that $\alpha'(t) \le 0$, $t \in [0, T]$. For fixed n and $k \le n-2$, denote*

$$P_n = \frac{\Gamma(1-\alpha_{n-1+\sigma_{n-1}})}{\Gamma(1-\alpha_{n+\sigma_n})}, \quad S_n = \frac{\sigma_n}{\sigma_{n-1}} \cdot \frac{1-\alpha_{n-1+\sigma_{n-1}}}{1-\alpha_{n+\sigma_n}} \cdot \frac{t_{n-1}^{\alpha_{n-1+\sigma_{n-1}}}}{t_n^{\alpha_{n+\sigma_n}}},$$

$$G^{(n)}(k,\theta) = \frac{\big[t_k + (\sigma_{n-1}-\theta)\tau\big]^{\alpha_{n-1+\sigma_{n-1}}}}{\big[t_k + (\sigma_n-\theta)\tau\big]^{\alpha_{n+\sigma_n}}}, \quad \frac{1}{2} \le k \le n - \frac{3}{2}, \ -\frac{1}{2} \le \theta \le 1.$$

Then we have

(I)

$$P_n \le 1 + Q_9\tau. \tag{B.3}$$

(II) *If $\alpha'(t) \le 0$ when $0 \le t \le T$, we have*

$$G^{(n)}\left(k+\frac{1}{2}, \theta\right) \le G^{(n)}\left(k+\frac{1}{2}, -\theta\right), \quad 0 \le k \le n-2, \ 0 \le \theta \le \frac{1}{2}, \tag{B.4}$$

$$S_n \le 1 + Q_{10}\tau, \tag{B.5}$$

$$\max_{-\frac{1}{2}\le\theta\le 1} G^{(n)}(k,\theta) \le 1 + Q_{11}\tau, \quad \frac{1}{2} \le k \le n-2, \tag{B.6}$$

where Q_i $(i = 9, 10, 11)$ are positive constants independent of τ and h.

Proof. (a) Notice the fact that $0 \le \alpha(t) \le \alpha_* < 1$, and $\Gamma(x)$ is a decreasing and differentiated function on the interval $[1-\alpha_*, 1]$. Denote $\Gamma_* = \max_{1-\alpha_*\le t\le 1}|\Gamma'(t)|$. Then we obtain

$$\begin{aligned} P_n &= \frac{\Gamma(1-\alpha_{n-1+\sigma_{n-1}})}{\Gamma(1-\alpha_{n+\sigma_n})} = \frac{\Gamma(1-\alpha_{n+\sigma_n}) + \Gamma'(\eta_n)(\alpha_{n+\sigma_n} - \alpha_{n-1+\sigma_{n-1}})}{\Gamma(1-\alpha_{n+\sigma_n})} \\ &\le 1 + \frac{\max\limits_{1-\alpha_*\le s\le 1}|\Gamma'(s)|}{\min\limits_{1-\alpha^*\le s\le 1}\Gamma(s)} \cdot |\alpha_{n+\sigma_n} - \alpha_{n-1+\sigma_{n-1}}| \le 1 + \frac{3}{2}\tau\Gamma_*\|\alpha'\|_\infty. \end{aligned}$$

(b) Since $\sigma_n \geq \sigma_{n-1}$ and $\alpha_{n-1+\sigma_{n-1}} \geq \alpha_{n+\sigma_n}$ for $0 \leq k \leq n-2,\ 0 \leq \theta \leq \frac{1}{2}$, we have

$$\frac{\partial G^{(n)}(k+\frac{1}{2},\theta)}{\partial\theta} = \frac{\tau\big[t_{k+\frac{1}{2}}+(\sigma_{n-1}-\theta)\tau\big]^{\alpha_{n-1+\sigma_{n-1}}-1}\big[t_{k+\frac{1}{2}}+(\sigma_n-\theta)\tau\big]^{\alpha_{n+\sigma_n}-1}}{\big[t_{k+\frac{1}{2}}+(\sigma_n-\theta)\tau\big]^{2\alpha_{n+\sigma_n}}}A_k^{(n)}(\theta)$$

and where

$$A_k^{(n)}(\theta) = \big(\alpha_{n+\sigma_n}-\alpha_{n-1+\sigma_{n-1}}\big)\big(t_{k+\frac{1}{2}}-\theta\tau\big)+\big(\alpha_{n+\sigma_n}\sigma_{n-1}-\alpha_{n-1+\sigma_{n-1}}\sigma_n\big)\tau \leq 0.$$

Thus, $G^{(n)}(k+\frac{1}{2},\theta)$ is a monotonically decreasing sequence with respect to θ. Hence, (B.4) holds.

(c) Denote $\tilde{\alpha}_n = \big(\alpha_{n-1+\sigma_{n-1}}+\alpha_{n+\sigma_n}\big)/2$. Let's analyze each term in S_n.

It is easy to obtain

$$\frac{\sigma_n}{\sigma_{n-1}} = 1+\frac{\sigma_n-\sigma_{n-1}}{\sigma_{n-1}} \leq 1+\frac{|\sigma_n-\sigma_{n-1}|}{\sigma_{n-1}} \leq 1+\frac{3}{2}\|\alpha'\|_\infty\tau$$

and

$$\frac{1-\alpha_{n-1+\sigma_{n-1}}}{1-\alpha_{n+\sigma_n}} = 1+\frac{\alpha_{n+\sigma_n}-\alpha_{n-1+\sigma_{n-1}}}{1-\alpha_{n+\sigma_n}} \leq 1+\frac{|\alpha_{n+\sigma_n}-\alpha_{n-1+\sigma_{n-1}}|}{1-\alpha_{n+\sigma_n}} \leq 1+\frac{\frac{3}{2}\tau\|\alpha'\|_\infty}{1-\alpha_*}.$$

On the other hand,

$$\begin{aligned}\frac{t_{n-1}^{\alpha_{n-1+\sigma_{n-1}}}}{t_n^{\alpha_{n+\sigma_n}}} &= \Big(\frac{t_{n-1}}{t_n}\Big)^{\tilde{\alpha}_n}\cdot\frac{t_{n-1}^{\alpha_{n-1+\sigma_{n-1}}-\tilde{\alpha}_n}}{t_n^{\alpha_{n+\sigma_n}-\tilde{\alpha}_n}}\\ &= \Big(\frac{t_{n-1}}{t_n}\Big)^{\tilde{\alpha}_n}\cdot\Big(t_{n-1}t_n\Big)^{\frac{\alpha_{n-1+\sigma_{n-1}}-\alpha_{n+\sigma_n}}{2}}\\ &\leq \Big(t_{n-1}t_n\Big)^{\frac{\alpha_{n-1+\sigma_{n-1}}-\alpha_{n+\sigma_n}}{2}}.\end{aligned}$$

If $t_n \geq \frac{1}{2}$, then

$$\Big(t_{n-1}t_n\Big)^{\frac{\alpha_{n-1+\sigma_{n-1}}-\alpha_{n+\sigma_n}}{2}} = 1+O(\tau).$$

If $t_n < \frac{1}{2}$, noting $\alpha_{n-1+\sigma_{n-1}} \geq \alpha_{n+\sigma_n}$, we have

$$\Big(t_{n-1}t_n\Big)^{\frac{\alpha_{n-1+\sigma_{n-1}}-\alpha_{n+\sigma_n}}{2}} \leq 1.$$

Summarizing the above results, we can obtain

$$S_n \leq 1+Q_{10}\tau.$$

(d) For $\frac{1}{2} \le k \le n-2, \ -\frac{1}{2} \le \theta \le 1$, we have

$$
\begin{aligned}
&G^{(n)}(k,\theta)\\
&= \frac{\big[t_k + (\sigma_{n-1}-\theta)\tau\big]^{\alpha_{n-1+\sigma_{n-1}}}}{\big[t_k + (\sigma_n-\theta)\tau\big]^{\alpha_{n+\sigma_n}}}\\
&= \Big[\frac{t_k + (\sigma_{n-1}-\theta)\tau}{t_k + (\sigma_n-\theta)\tau}\Big]^{\tilde{\alpha}_n} \cdot \frac{\big[t_k + (\sigma_{n-1}-\theta)\tau\big]^{\alpha_{n-1+\sigma_{n-1}}-\tilde{\alpha}_n}}{\big[t_k + (\sigma_n-\theta)\tau\big]^{\alpha_{n+\sigma_n}-\tilde{\alpha}_n}}\\
&= \Big[1 + \frac{(\sigma_{n-1}-\sigma_n)\tau}{t_k + (\sigma_n-\theta)\tau}\Big]^{\tilde{\alpha}_n} \cdot \Big\{\big[t_k + (\sigma_{n-1}-\theta)\tau\big]\cdot\big[t_k + (\sigma_n-\theta)\tau\big]\Big\}^{\frac{\alpha_{n-1+\sigma_{n-1}}-\alpha_{n+\sigma_n}}{2}}\\
&= \Big[1 + \frac{\sigma_{n-1}-\sigma_n}{k+\sigma_n-\theta}\Big]^{\tilde{\alpha}_n} \cdot \Big\{\big[t_k + (\sigma_{n-1}-\theta)\tau\big]\cdot\big[t_k + (\sigma_n-\theta)\tau\big]\Big\}^{\frac{\alpha_{n-1+\sigma_{n-1}}-\alpha_{n+\sigma_n}}{2}}\\
&\le \big(1 + 2\|\alpha'\|_\infty\tau\big)^{\tilde{\alpha}_n} \cdot \Big\{\big[t_k + (\sigma_{n-1}-\theta)\tau\big]\cdot\big[t_k + (\sigma_n-\theta)\tau\big]\Big\}^{\frac{\alpha_{n-1+\sigma_{n-1}}-\alpha_{n+\sigma_n}}{2}}.
\end{aligned}
$$

If $t_k \ge \frac{1}{2}$, noting $|\alpha_{n-1+\sigma_{n-1}} - \alpha_{n+\sigma_n}| \le \frac{3}{2}\tau\|\alpha'\|_\infty$, we have

$$
\big[t_k + (\sigma_{n-1}-\theta)\tau\big]\cdot\big[t_k + (\sigma_n-\theta)\tau\big]^{\frac{\alpha_{n-1+\sigma_{n-1}}-\alpha_{n+\sigma_n}}{2}} = 1 + O(\tau).
$$

If $t_k < \frac{1}{2}$, then $\big[t_k+(\sigma_{n-1}-\theta)\tau\big]\cdot\big[t_k+(\sigma_n-\theta)\tau\big] < 1$. Noting $\alpha_{n-1+\sigma_{n-1}} - \alpha_{n+\sigma_n} \ge 0$, we have

$$
\big[t_k + (\sigma_{n-1}-\theta)\tau\big]\cdot\big[t_k + (\sigma_n-\theta)\tau\big]^{\frac{\alpha_{n-1+\sigma_{n-1}}-\alpha_{n+\sigma_n}}{2}} \le 1.
$$

Thus

$$
G^{(n)}(k,\theta) \le 1 + Q_{11}\tau.
$$

This completes the proof. ∎

Now we give the proof of Lemma 4.3. A careful calculation with an application of (B.4) shows that for $k=0$

$$
\begin{aligned}
\frac{c_k^{(n)}}{s_n} &= \frac{1}{\Gamma(1-\alpha_{n+\sigma_n})}\Big\{\int_0^{\frac{1}{2}} \theta\Big[\big(t_{\frac{1}{2}} + (\sigma_n-\theta)\tau\big)^{-\alpha_{n+\sigma_n}} - \big(t_{\frac{1}{2}} + (\sigma_n+\theta)\tau\big)^{-\alpha_{n+\sigma_n}}\Big]\mathrm{d}\theta\\
&\quad + \frac{\sigma_n}{1-\alpha_{n+\sigma_n}} t_n^{-\alpha_{n+\sigma_n}}\Big\}\\
&= \frac{P_n}{\Gamma(1-\alpha_{n-1+\sigma_{n-1}})}\Big\{\int_0^{\frac{1}{2}} \theta\Big[\frac{G^{(n)}(\frac{1}{2},\theta)}{\big(t_{\frac{1}{2}} + (\sigma_{n-1}-\theta)\tau\big)^{\alpha_{n-1+\sigma_{n-1}}}}\\
&\quad - \frac{G^{(n)}(\frac{1}{2},-\theta)}{\Big(t_{\frac{1}{2}} + (\sigma_{n-1}+\theta)\tau\big)^{\alpha_{n-1+\sigma_{n-1}}}}\Big]\mathrm{d}\theta + S_n\frac{\sigma_{n-1}}{1-\alpha_{n-1+\sigma_{n-1}}}\, t_{n-1}^{-\alpha_{n-1+\sigma_{n-1}}}\Big\}\\
&\le \frac{P_n}{\Gamma(1-\alpha_{n-1+\sigma_{n-1}})}\Big\{\int_0^{\frac{1}{2}} G^{(n)}\Big(\frac{1}{2},-\theta\Big)\cdot\theta\Big[\big(t_{\frac{1}{2}} + (\sigma_{n-1}-\theta)\tau\big)^{-\alpha_{n-1+\sigma_{n-1}}}\\
&\quad - \big(t_{\frac{1}{2}} + (\sigma_{n-1}+\theta)\tau\big)^{-\alpha_{n-1+\sigma_{n-1}}}\Big]\mathrm{d}\theta + S_n\frac{\sigma_{n-1}}{1-\alpha_{n-1+\sigma_{n-1}}}\, t_{n-1}^{-\alpha_{n-1+\sigma_{n-1}}}\Big\}\\
&\le P_n \max\Big\{\max_{0\le\theta\le\frac{1}{2}} G^{(n)}\Big(\frac{1}{2},-\theta\Big), S_n\Big\}\cdot\frac{c_0^{(n-1)}}{s_{n-1}}
\end{aligned}
$$

and, for $1 \le k \le n-2$, we have

$$\begin{aligned}
\frac{c_k^{(n)}}{s_n} &= \frac{1}{\Gamma(1-\alpha_{n+\sigma_n})}\Big\{ \int_0^1 \Big(\frac{3}{2}-\theta\Big)\big(t_k+(\sigma_n-\theta)\tau\big)^{-\alpha_{n+\sigma_n}} \mathrm{d}\theta \\
&\quad + \int_0^{\frac{1}{2}} \theta\Big[\big(t_{k+\frac{1}{2}}+(\sigma_n-\theta)\tau\big)^{-\alpha_{n+\sigma_n}} - \big(t_{k+\frac{1}{2}}+(\sigma_n+\theta)\tau\big)^{-\alpha_{n+\sigma_n}}\Big]\mathrm{d}\theta\Big\} \\
&= \frac{P_n}{\Gamma(1-\alpha_{n-1+\sigma_{n-1}})}\Big\{ \int_0^1 \Big(\frac{3}{2}-\theta\Big)\frac{G^{(n)}(k,\theta)}{\big(t_k+(\sigma_{n-1}-\theta)\tau\big)^{\alpha_{n-1+\sigma_{n-1}}}}\mathrm{d}\theta \\
&\quad + \int_0^{\frac{1}{2}}\theta\Big[\frac{G^{(n)}(k+\frac{1}{2},\theta)}{\big(t_{k+\frac{1}{2}}+(\sigma_{n-1}-\theta)\tau\big)^{\alpha_{n-1+\sigma_{n-1}}}} - \frac{G^{(n)}(k+\frac{1}{2},-\theta)}{\big(t_{k+\frac{1}{2}}+(\sigma_{n-1}+\theta)\tau\big)^{\alpha_{n-1+\sigma_{n-1}}}}\Big]\mathrm{d}\theta\Big\} \\
&\le \frac{P_n}{\Gamma(1-\alpha_{n-1+\sigma_{n-1}})}\Big\{ \int_0^1 \Big(\frac{3}{2}-\theta\Big)\frac{G^{(n)}(k,\theta)}{\big(t_k+(\sigma_{n-1}-\theta)\tau\big)^{\alpha_{n-1+\sigma_{n-1}}}}\mathrm{d}\theta \\
&\quad + \int_0^{\frac{1}{2}} G^{(n)}\Big(k+\frac{1}{2},-\theta\Big) \\
&\quad \cdot \theta\Big[\big(t_{k+\frac{1}{2}}+(\sigma_{n-1}-\theta)\tau\big)^{-\alpha_{n-1+\sigma_{n-1}}} - \big(t_{k+\frac{1}{2}}+(\sigma_{n-1}+\theta)\tau\big)^{-\alpha_{n-1+\sigma_{n-1}}}\Big]\mathrm{d}\theta\Big\} \\
&\le P_n \max\Big\{\max_{0\le\theta\le 1} G^{(n)}(k,\theta), \max_{0\le\theta\le\frac{1}{2}} G^{(n)}(k+\frac{1}{2},-\theta)\Big\}\cdot\frac{c_k^{(n-1)}}{s_{n-1}}.
\end{aligned}$$

Thanks to the inequalities (B.3) and (B.5)–(B.6), there exists a constant Q_3 independent of k and n such that

$$\frac{c_k^{(n)}}{s_n} \le (1+Q_3\tau)\frac{c_k^{(n-1)}}{s_{n-1}}, \quad 0 \le k \le n-2.$$

This completes the proof. □

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