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Analysis and Numerical Approximation for a Nonlinear Hidden-Memory Variable-Order Fractional Stochastic Differential Equation

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Abstract. We prove the wellposedness of a nonlinear hidden-memory variable-order fractional stochastic differential equation driven by a multiplicative white noise, in which the hidden-memory type variable order describes the memory of a fractional order. We then present a Euler-Maruyama scheme for the proposed model and prove its strong convergence rate. Numerical experiments are performed to substantiate the theoretical results.

AMS subject classifications: 60H20, 65L20

Key words: Variable-order fractional stochastic differential equation, hidden memory, Euler-Maruyama method, strong convergence.

1. Introduction

Stochastic differential equations (SDEs) provide a prominent modeling tool for many stochastic phenomena in sciences and engineering like biology, physics, chemistry and finance [6–8, 11, 15, 16, 19, 20, 24, 32]. In the processes containing nonlocal or memory effects, fractional derivatives provide a better description than integer-order derivatives do, which leads to the fractional SDEs (fSDEs). However, there is a large class of physical, biological and physiological diffusion phenomena that relate processes exhibiting accelerating or decelerating diffusion behaviors that cannot be characterized by the constant-order fractional diffusion equations. Typical features of these phenomena are that they are complex to analysis and the diffusion behavior depends on the time evolution, space variation

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or even system parameters. Since the orders of fractional derivatives in fSDEs are closely related to the fractal dimension of the media determined via the Hurst index [22], the variable fractional order derivatives are introduced to accommodate the structure change of the surroundings, which in turn leads to the variable-order (VO) fSDEs [12–14, 31, 33, 36, 37]. Works [9, 17, 27] introduced the space dependent VO into differential equations under the assumption that the probability density function is space dependent in the continuous time rand walk, which indicates that the memory rate depends on the space location in the considered system. Papers [27–29] proved that the mean square displacement is $\langle x^2(t)\rangle \propto t^{\alpha(t)}$, where $\alpha(t)$ is the order of the fractional diffusion equation. Measurement data also show that the diffusion behavior changing with the time evolution can be modeled by a time dependent VO fractional model. Thereby, it is more reasonable to investigate the VO fractional equations, and so further theoretical and numerical investigations of variable-order fSDEs are required for describing more complicated stochastic diffusion process.

Motivated by the preceding discussions, we study the following nonlinear Caputo fractional SDE with a hidden-memory variable order:

$$du = \big(-\lambda\, {}^{C}_{0}D^{\alpha(t)}_{t} u + f(t,u)\big)dt + b(t,u)dW, \quad t \in (0,T], \quad u(0) = u_0. \tag{1.1}$$

Here $\lambda \geq 0$, $0 \leq \alpha(t) \leq \alpha^* \leq 1/2$, and the hidden-memory variable-order fractional differential operator ${}^{C}_{0}D^{\alpha(t)}_{t}$ is defined in terms of the corresponding fractional integral via the Gamma function Γ [21, 28, 34, 35]

$${}^{C}_{0}D^{\alpha(t)}_{t} g(t) := {}_{0}I^{1-\alpha(t)}_{t} g'(t), \quad {}_{0}I^{1-\alpha(t)}_{t} g(t) := \int_0^t \frac{(t-s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))} g(s)ds. \tag{1.2}$$

Note that in the fractional integral, the power α assumes its historical state at the historical time instant s, which represents the memory of the order history and is named as hidden memory in order to distinguish it from the fading memory property of the fractional operators [27, 28].

FSDEs have attracted extensive attentions mathematically and numerically [1, 3–5, 10, 23, 25, 26, 36], while the corresponding investigations for variable-order FSDEs are meager. In a very recent work the well-posedness of a variable-order FSDE was analyzed, in which the variable-order fractional derivative is defined by (1.2) with $\alpha(s)$ replaced by $\alpha(t)$. Note that in such definition, the kernel becomes $(t-s)^{-\alpha(t)}/\Gamma(1-\alpha(t))$, which can be integrated into a close-form expression that significantly facilitates the mathematical analysis. However, the definition (1.2) does not enjoy this benefit, which shows the salient feature of the hidden-memory variable-order fractional problems and complicates the corresponding mathematical and numerical analysis.

We aim to prove the existence and uniqueness of the strong solution for (1.1), based on which we propose a Euler-Maruyama approximation and prove its optimal error estimates. The rest of this paper is organized as follows. In Section 2 we present preliminaries and the reformulation of the problem be used subsequently. In Section 3 we prove the wellposedness and moment estimate of the governing equation (1.1). In Section 4 we establish the

Euler-Maruyama scheme and prove its strong convergence results. Numerical experiments are presented to substantiate the theoretical results in the last section.

2. Problem Formulation

2.1. Preliminaries

We first present assumptions on the data of the problem (1.1).

A1. $\mathbb{E}[u_0^2] < \infty$ and α belongs to the space $C^1[0,T]$ of continuously differentiable functions on $[0,T]$.

A2. There exist $L > 0$ and $\beta, \gamma \in [0,1]$ such that

$$\begin{aligned} &|f(t,v_1)-f(t,v_2)| \le L|v_1-v_2|, && |b(t,v_1)-b(t,v_2)| \le L|v_1-v_2|,\\ &|f(t_1,u)-f(t_2,u)| \le L|t_1-t_2|^{\beta}, && |b(t_1,u)-b(t_2,u)| \le L_2|t_1-t_2|^{\gamma},\\ &|f(t,v)| \le L(1+|v|), && |b(t,v)| \le L(1+|v|) \end{aligned}$$

for $v, v_1, v_2 \in \mathbb{R}$ and for $t, t_1, t_2 \in [0,T]$.

Let us recall useful auxiliary results.

Lemma 2.1 (The Burkhölder-Davis-Gundy Inequality, cf. Refs. [18,32]). *If Y is a continuous martingale on $[0,T]$, there exists a positive constant $\bar{Q}_1$ such that*

$$\mathbb{E}\Big[\Big(\sup_{t\in[0,T]}|Y(t)|\Big)^2\Big] \le \bar{Q}_1\mathbb{E}\big[|Y(T)|^2\big]. \tag{2.1}$$

Lemma 2.2 (Jensen Inequality, cf. Le Gall [18]). *If $a_i, p \in \mathbb{R}$ with $p \ge 1$ and $m \in \mathbb{N}^+$, then*

$$\Big|\sum_{i=1}^m a_i\Big|^p \le m^{p-1}\sum_{i=1}^m |a_i|^p. \tag{2.2}$$

Lemma 2.3 (Generalized Gronwall Inequality, cf. Ye *et al.* [30]). *Let $\bar{Q}_2(t)$ be a non-negative and non-decreasing locally integrable function on $(a,b]$ and $\bar{Q}_3$ be a non-negative constant. Suppose $g(t)$ is a non-negative locally integrable function on $(a,b]$ with*

$$g(t) \le \bar{Q}_2(t) + \bar{Q}_3\int_a^t \frac{g(s)}{(t-s)^{1-\beta}}ds, \quad \forall t \in (a,b], \quad 0<\beta<1,$$

then

$$g(t) \le \bar{Q}_2(t)E_{\beta,1}\big(\bar{Q}_3\Gamma(\beta)(t-a)^{\beta}\big), \quad \forall t \in (a,b]. \tag{2.3}$$

Here $E_{p,q}(z)$ is the Mittag-Leffler functions defined by [25]

$$E_{p,q}(z) := \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(pk+q)}, \quad z \in \mathbb{R}, \quad p \in \mathbb{R}^+, \quad q \in \mathbb{R}.$$

Lemma 2.4 (Generalized discrete Gronwall Inequality, cf. Brunner [2]). *Suppose that a non-negative sequence $\{z_n\}_{n=1}^N$ and a non-negative non-decreasing sequence $\{y_n\}_{n=1}^N$ satisfy the following relation:*

$$z_n \le y_n + \bar{Q}_4 \tau^{\nu} \sum_{i=1}^{n-1} \frac{z_i}{(n-i)^{1-\nu}}, \quad 1 \le n \le N, \quad 0 < \nu < 1, \tag{2.4}$$

where $\tau = T/N$ and $\bar{Q}_4 \ge 0$. Then the sequence $\{z_n\}_{n=1}^N$ can be bounded by

$$z_n \le y_n E_{\nu,1}\big(\bar{Q}_4 \Gamma(\nu)(n\tau)^{\nu}\big), \quad 1 \le n \le N.$$

2.2. Reformulation of (1.1)

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, W be a Brownian motion and u_0 be a second-order random variable that is independent of W. Let $\mathscr{F}(t) := \mathscr{U}(W(s)(0 \le s \le t), u_0)$ denote the σ-algebra generated by u_0 and the history of the Brownian motion up to time t. We integrate problem (1.1) from 0 to t to get

$$u(t) = u_0 + \int_0^t {}_0^C D^{\alpha(s)} u(s) ds + \int_0^t f\big(s, u(s)\big) ds + \int_0^t b\big(s, u(s)\big) dW(s). \tag{2.5}$$

The second term in the right-hand side of (2.5) can be rewritten by integration by parts such that

$$\begin{aligned}
&\int_0^t {}_0^C D_s^{\alpha(s)} u(s) ds = \int_0^t \int_0^s \frac{u'(x)dx}{\Gamma(1-\alpha(x))(s-x)^{\alpha(x)}} ds \\
&= \int_0^t \frac{u'(s)}{\Gamma(1-\alpha(s))} \int_s^t \frac{dx}{(x-s)^{\alpha(s)}} ds = \int_0^t \frac{(t-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} u'(s) ds \\
&= \frac{(t-s)^{1-\alpha(s)} u(s)}{\Gamma(2-\alpha(s))}\bigg|_{s=0}^t - \int_0^t u(s) \frac{\partial}{\partial s} \left(\frac{(t-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} \right) ds \\
&= -\frac{t^{1-\alpha(0)} u_0}{\Gamma(2-\alpha(0))} - \int_0^t k(t,s) u(s) ds,
\end{aligned} \tag{2.6}$$

where the kernel $k(t,s)$ is defined by

$$k(t,s) := \frac{\partial}{\partial s} \left(\frac{(t-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} \right). \tag{2.7}$$

We apply (2.6) to formally reformulate the proposed model as follows [8, 24]: Find a stochastic process u on $[0, T]$ that is progressively measurable with respect to $\mathscr{F}(\cdot)$ such that

$$\begin{aligned}
u(t) = &\left(1 + \frac{\lambda t^{1-\alpha(0)}}{\Gamma(2-\alpha(0))} \right) u_0 + \lambda \int_0^t k(t,s) u(s) ds \\
&+ \int_0^t f\big(s, u(s)\big) ds + \int_0^t b\big(s, u(s)\big) dW(s).
\end{aligned} \tag{2.8}$$

3. Wellposedness and Moment Estimates

In this section, we prove the wellposedness of problem (1.1) and establish moment estimates for its solutions.

3.1. Auxiliary lemmas

Lemma 3.1. *Let* $0 < \beta < 1$*, then the following estimate holds:*

$$\sup_{0\le x\le t}\int_0^x (x-s)^{-\beta}|\xi(s)|ds \le \int_0^t (t-s)^{-\beta}\sup_{0\le r\le s}|\xi(r)|ds.$$

Proof. By the substitution $s=\theta x$, direct calculations yield

$$\begin{aligned}&\sup_{0\le x\le t}\int_0^x (x-s)^{-\beta}|\xi(s)|ds = \sup_{0\le x\le t}\int_0^1 x^{-\beta}(1-\theta)^{-\beta}|\xi(\theta x)|x d\theta \\ &\le t^{1-\beta}\int_0^1 (1-\theta)^{-\beta}\sup_{0\le r\le \theta t}|\xi(r)|d\theta = \int_0^t (t-s)^{-\beta}\sup_{0\le r\le s}|\xi(r)|ds,\end{aligned}$$

which completes the proof. □

Lemma 3.2. *Suppose assumption* A1 *holds, then for any* $0\le s<t\le T$

$$|k(t,s)| \le Q_0(t-s)^{-\alpha^*} \tag{3.1}$$

for some constant $Q_0 \ge 0$ *depending on* $\|\alpha\|_{C^1[0,T]}$, α^* *and* T.

Proof. By the definition of $k(t,s)$ in (2.7), we have

$$k(t,s) = -\frac{(1-\alpha(s))+(t-s)\alpha'(s)\ln(t-s)}{\Gamma(2-\alpha(s))(t-s)^{\alpha(s)}} + \frac{\alpha'(s)\Gamma'(2-\alpha(s))}{\Gamma^2(2-\alpha(s))}(t-s)^{1-\alpha(s)}.$$

Since

$$|(t-s)\ln(t-s)| \le \max\left\{e^{-1}, T\ln T\right\},$$

we estimate $k(t,s)$ as

$$|k(t,s)| \le Q(t-s)^{-\alpha(s)} = Q(t-s)^{-\alpha^*}(t-s)^{\alpha^*-\alpha(s)} \le Q\max\{1,T\}(t-s)^{-\alpha^*},$$

which completes the proof. □

3.2. Analysis of variable-order fSDE

Theorem 3.1. *If the assumptions* A1 *and* A2 *hold, then the problem* (1.1) *has a unique solution u such that*

$$\mathbb{E}\Big[\sup_{0\le s\le t}|u(s)|^2\Big]\le Q_2E_{1-\alpha^*,1}\big(Q_2'\Gamma(1-\alpha^*)t^{1-\alpha^*}\big)<\infty,\quad t\in[0,T]. \tag{3.2}$$

Here, constants Q_2 *and* Q_2' *are given by*

$$Q_2=4C_0^2\mathbb{E}\big[u_0^2\big]+8\bar{Q}_1L^2T,\quad Q_2'=\frac{4\lambda^2Q_0^2T^{1-\alpha^*}}{1-\alpha^*}+8\bar{Q}_1L^2T^{\alpha^*}$$

with Q_0,C_0 *and* $\bar{Q}_1$ *are the constants defined in Lemma* 3.2, (3.8) *and Lemma* 2.1*, respectively.*

Proof. We define a functional sequence $\{v_n\}_{n=0}^{\infty}$ by

$$\begin{aligned}v_0(t)&=\left(1+\frac{\lambda t^{1-\alpha(0)}}{\Gamma\big(2-\alpha(0)\big)}\right)u_0,\\ v_n(t)&=v_0(t)+\lambda\int_0^t k(t,s)v_{n-1}(s)ds+\int_0^t f(s,v_{n-1}(s))ds\\ &\quad+\int_0^t b\big(s,v_{n-1}(s)\big)dW(s).\end{aligned} \tag{3.3}$$

Then for $x\in[0,T]$, we have

$$\begin{aligned}(v_{n+1}-v_n)(x)=\lambda\int_0^x k(x,s)(v_n-v_{n-1})(s)ds+\int_0^x\Big(f\big(s,v_n(s)\big)-f\big(s,v_{n-1}(s)\big)\Big)ds\\ +\int_0^x\Big(b\big(s,v_n(s)\big)-b\big(s,v_{n-1}(s)\big)\Big)dW(s).\end{aligned}$$

Applying the Jensen inequality (2.2) yields

$$\begin{aligned}&\mathbb{E}\Big[\sup_{0\le x\le t}|v_{n+1}(x)-v_n(x)|^2\Big]\\ \le\,&3\mathbb{E}\left[\lambda^2\sup_{0\le x\le t}\left|\int_0^x k(x,s)\big(v_{n+1}(x)-v_n(x)\big)ds\right|^2\right.\\ &\quad+\sup_{0\le x\le t}\left|\int_0^x\Big(f\big(s,v_n(s)\big)-f\big(s,v_{n-1}(s)\big)\Big)ds\right|^2\\ &\quad\left.+\sup_{0\le x\le t}\left|\int_0^x\Big(b\big(s,v_n(s)\big)-b\big(s,v_{n-1}(s)\big)\Big)dW(s)\right|^2\right]=I_1+I_2+I_3.\end{aligned}$$

By the Cauchy inequality (3.1) and Lemma 3.1, I_1 can be bounded by

$$\begin{aligned}
I_1 &\le 3\lambda^2\mathbb{E}\left[\sup_{0\le x\le t}\left(\int_0^x |k(x,s)|ds\int_0^x |k(x,s)||v_n(s)-v_{n-1}(s)|^2 ds\right)\right]\\
&\le \frac{3\lambda^2Q_0^2t^{1-\alpha^*}}{1-\alpha^*}\mathbb{E}\left[\sup_{0\le x\le t}\int_0^x (x-s)^{-\alpha^*}|v_n(s)-v_{n-1}(s)|^2ds\right]\\
&\le \frac{3\lambda^2Q_0^2t^{1-\alpha^*}}{1-\alpha^*}\int_0^t (t-s)^{-\alpha^*}\mathbb{E}\Big[\sup_{0\le r\le s}|v_n(r)-v_{n-1}(r)|^2\Big]ds.
\end{aligned}\tag{3.4}$$

Apply the Cauchy inequality and assumption A2 to bound I_2 with

$$\begin{aligned}
I_2 &\le 3L^2\mathbb{E}\left[\sup_{0\le x\le t}\int_0^x 1^2ds\int_0^x |v_n(s)-v_{n-1}(s)|^2ds\right]\\
&\le 3L^2T\int_0^t \mathbb{E}\Big[\sup_{0\le r\le s}|v_n(r)-v_{n-1}(r)|^2\Big]ds.
\end{aligned}$$

The term I_3 can be bounded by the Burkhölder-Davis-Gundy inequality (2.1), assumption A2 and Itô's isometry as

$$\begin{aligned}
I_3 &= 3\mathbb{E}\left[\sup_{0\le x\le t}\left|\int_0^x b\Big(s,v_n(s)-b\big(s,v_{n-1}(s)\big)\Big)dW(s)\right|^2\right]\\
&\le 3\bar{Q}_1\mathbb{E}\left[\left|\int_0^t \Big(b\big(s,v_n(s)\big)-b\big(s,v_{n-1}(s)\big)\Big)dW(s)\right|^2\right]\\
&\le 3\bar{Q}_1L^2\mathbb{E}\left[\left|\int_0^t |v_n(s)-v_{n-1}(s)|dW(s)\right|^2\right]\\
&\le 3\bar{Q}_1L^2\int_0^t \mathbb{E}\Big[\sup_{0\le r\le s}|v_n(r)-v_{n-1}(r)|^2\Big]ds.
\end{aligned}\tag{3.5}$$

Combining (3.4)-(3.5) gives

$$\mathbb{E}\Big[\sup_{0\le x\le t}|v_{n+1}(x)-v_n(x)|^2\Big]\le Q_1\int_0^t \frac{\mathbb{E}\Big[\sup_{0\le r\le s}|v_n(r)-v_{n-1}(r)|^2\Big]}{\Gamma(1-\alpha^*)(t-s)^{\alpha^*}}ds \tag{3.6}$$

with

$$Q_1 = 3\Gamma(1-\alpha^*)T^{\alpha^*}\left(\frac{\lambda^2Q_0^2T^{1-2\alpha^*}}{1-\alpha^*}+L^2T+\bar{Q}_1L^2\right).$$

For $n\ge 1$, (3.6) can be rewritten as

$$\mu_n(t):=\mathbb{E}\Big[\sup_{0\le x\le t}|v_{n+1}(x)-v_n(x)|^2\Big]\le Q_1\int_0^t \frac{\mu_{n-1}(s)}{\Gamma(1-\alpha^*)(t-s)^{\alpha^*}}ds. \tag{3.7}$$

By the definition of v_0 in (3.3), we have

$$|v_0(t)| \le C_0|u_0|, \quad C_0 = 1 + \frac{\lambda T^{1-\alpha(0)}}{\Gamma(2-\alpha(0))}, \tag{3.8}$$

which indicates that $\mathbb{E}[v_0^2] \le C_0^2\mathbb{E}[u_0^2]$. In a similar way, we can bound $\mu_0(t)$ by

$$\begin{aligned}
\mu_0(t) &= \mathbb{E}\Big[\sup_{0\le x\le t} |v_1(x) - v_0(x)|^2\Big] \\
&\le 3\mathbb{E}\Bigg[\lambda^2 \sup_{0\le x\le t}\left(\int_0^x k(x,s)v_0(s)ds\right)^2 + \sup_{0\le x\le t}\left(\int_0^x f\big(s, v_0(s)\big)ds\right)^2 \\
&\qquad + \sup_{0\le x\le t}\left(\int_0^x b\big(s, v_0(s)\big)dW(s)\right)^2\Bigg] \\
&\le \frac{3\lambda^2 Q_0^2 T^{2(1-\alpha^*)}C_0^2}{(1-\alpha^*)^2}\mathbb{E}\big[u_0^2\big] + 6L^2T\big(T + \bar{Q}_1\big)\big(1 + C_0^2\mathbb{E}\big[u_0^2\big]\big) =: Q_1'.
\end{aligned} \tag{3.9}$$

Combining (3.7)-(3.9), we obtain

$$\mu_1(t) \le Q_1\int_0^t \frac{(t-s)^{-\alpha^*}}{\Gamma(1-\alpha^*)}\mu_0(s)ds \le Q_1Q_1'\int_0^t \frac{(t-s)^{-\alpha^*}}{\Gamma(1-\alpha^*)}ds = \frac{Q_1Q_1't^{1-\alpha^*}}{\Gamma((1-\alpha^*)+1)}$$

for $t \in (0, T]$. Suppose for any $n \ge 1$, $\mu_n(t)$ can be bounded by

$$\mu_n(t) \le \frac{Q_1'Q_1^n t^{n(1-\alpha^*)}}{\Gamma(n(1-\alpha^*)+1)}, \quad t \in (0, T]. \tag{3.10}$$

Substituting (3.10) into (3.7) gives

$$\begin{aligned}
\mu_{n+1}(t) &\le Q_1\int_0^t \frac{\mu_n(s)}{\Gamma(1-\alpha^*)(t-s)^{\alpha^*}}ds \\
&\le \frac{Q_1'Q_1^{n+1}}{\Gamma(1-\alpha^*)\Gamma(n(1-\alpha^*)+1)}\int_0^t \frac{s^{n(1-\alpha^*)}}{(t-s)^{\alpha^*}}ds \\
&= \frac{Q_1'Q_1^{n+1}t^{(n+1)(1-\alpha^*)}B(n(1-\alpha^*)+1, 1-\alpha^*)}{\Gamma(n(1-\alpha^*)+1)\Gamma(1-\alpha^*)} \\
&= \frac{Q_1'Q_1^{n+1}t^{(n+1)(1-\alpha^*)}}{\Gamma((n+1)(1-\alpha^*)+1)}.
\end{aligned}$$

By mathematical induction, (3.10) holds for any $n \in \mathbb{N}$. The series defined in the right-hand side of (3.10) converges to the Mittag-Leffler function

$$\sum_{n=0}^{\infty} \frac{Q_1'Q_1^n t^{n(1-\alpha^*)}}{\Gamma(n(1-\alpha^*)+1)} = Q_1'E_{1-\alpha^*,1}\big(Q_1t^{1-\alpha^*}\big) < \infty, \quad t \in (0, T],$$

which implies that

$$\sum_{n=0}^{\infty}\mathbb{E}\Big[\sup_{0\le t\le T}|v_{n+1}(t)-v_n(t)|^2\Big]<\infty.$$

By Chebyshev's inequality, we conclude that

$$\begin{aligned}&P\bigg(\sup_{t\in[0,T]}|v_{n+1}(t)-v_n(t)|^2\ge 2^{-n}\bigg)\\&\le 2^{2n}\mathbb{E}\Big[\sup_{t\in[0,T]}|v_{n+1}(t)-v_n(t)|^2\Big]\\&\le\frac{Q_1'(4Q_1t^{1-\alpha^*})^n}{\Gamma(n(1-\alpha^*)+1)}\le\frac{Q_1'(4Q_1T^{1-\alpha^*})^n}{\Gamma(n(1-\alpha^*)+1)}\to 0,\quad n\to\infty.\end{aligned}$$

By the Borel-Cantelli lemma, the sequence

$$v_n(t)=\sum_{j=1}^{n}\big(v_j(t)-v_{j-1}(t)\big)+v_0(t)$$

converges uniformly to a continuous function $u(t)$ that solves (2.6).

Let $\tilde{u}$ be another solution to (2.6) and $e(t)=\tilde{u}-u$. Then a similar derivation to (3.6) yields for all $t\in[0,T]$

$$\mathbb{E}\Big[\sup_{0\le x\le t}|e(x)|^2\Big]\le Q_1\int_0^t\frac{\mathbb{E}\Big[\sup_{0\le r\le s}|e(r)|^2\Big]}{\Gamma(1-\alpha^*)(t-s)^{\alpha^*}}ds.$$

We apply Gronwall inequality to conclude that $e(t)=0$ a.s., which proves the uniqueness of the solution.

Using (3.3), the second moment of $v_n(t)$ for $n\ge 1$ can be bounded by

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le x\le t}|v_n(x)|^2\Big]\le{}&4\mathbb{E}\big[v_0^2\big]+4\lambda^2\mathbb{E}\left[\sup_{0\le x\le t}\left(\int_0^x k(x,s)v_{n-1}(s)ds\right)^2\right]\\&+4\mathbb{E}\left[\sup_{0\le x\le t}\left(\int_0^x f\big(s,v_{n-1}(s)\big)ds\right)^2\right]\\&+4\mathbb{E}\left[\sup_{0\le x\le t}\left(\int_0^x b(s,v_{n-1}(s))dW(s)\right)^2\right]\\\le{}&Q_2+Q_2'\int_0^t\frac{\mathbb{E}\Big[\sup_{0\le r\le s}|v_{n-1}(r)|^2\Big]}{(t-s)^{\alpha^*}}ds,\end{aligned}$$

here

$$Q_2=4C_0^2\mathbb{E}\big[u_0^2\big]+8(1+\bar{Q}_1)L^2T,\quad Q_2'=\frac{4\lambda^2Q_0^2T^{1-\alpha^*}}{1-\alpha^*}+8(1+\bar{Q}_1)L^2T^{\alpha^*}.$$

Passing to the limit as $n\to\infty$ and applying Gronwall inequality (2.3) we obtain (3.2) and thus finish the proof. $\square$

4. Euler-Maruyama Approximation and Its Strong Convergence

4.1. Derivation and stability of scheme

Define the uniform partition of $[0,T]$ by $t_n := n\tau$ for $0 \le n \le N$ with $\tau := T/N$. At t_n we discretize the last three right-hand side terms in (2.8) by

$$
\begin{aligned}
&\int_0^{t_n} k(t_n,s)u(s)ds \approx \sum_{l=0}^{n-1}\left(\int_{t_l}^{t_{l+1}} k(t_n,s)ds\right)u(t_l)\\
&=\sum_{l=0}^{n-1}\left(\frac{(t_n-t_{l+1})^{1-\alpha(t_{l+1})}}{\Gamma(2-\alpha(t_{l+1}))}-\frac{(t_n-t_l)^{1-\alpha(t_l)}}{\Gamma(2-\alpha(t_l))}\right)u(t_l)=:\sum_{l=0}^{n-1}c_{n,l}u(t_l),\\
&\int_0^{t_n} f\big(s,u(s)\big)ds \approx \sum_{l=0}^{n-1}\int_{t_l}^{t_{l+1}} f\big(t_l,u(t_l)\big)ds=\tau\sum_{l=0}^{n-1}f\big(t_l,u(t_l)\big),\\
&\int_0^{t_n} b\big(s,u(s)\big)dW(s) \approx \sum_{l=0}^{n-1}\int_{t_l}^{t_{l+1}} b\big(t_l,u(t_l)\big)dW(s)=\sum_{l=0}^{n-1}b\big(t_l,u(t_l)\big)\Delta W_l,
\end{aligned}
$$

where $\Delta W_l := W(t_{l+1})-W(t_l) \sim N(0,\tau)$ is a Gaussian random variable.

We invoke these discretizations into Eq. (2.8) to obtain a Euler-Maruyama scheme to the variable-order fSDE (1.1): Given the initial data u_0 in problem (1.1), find y_n for $1 \le n \le N$ such that

$$
y_n=\left(1+\frac{\lambda t_n^{1-\alpha(0)}}{\Gamma(2-\alpha(0))}\right)u_0+\lambda\sum_{l=0}^{n-1}c_{n,l}y_l+\tau\sum_{l=0}^{n-1}f(t_l,y_l)+\sum_{l=0}^{n-1}b(t_l,y_l)\Delta W_l. \tag{4.1}
$$

Theorem 4.1. *Under the assumptions* A1-A2*, the solution y_n to the Euler-Maruyama scheme* (4.1) *satisfies the moment estimate for $1 \le n \le N$, i.e.*

$$
\begin{aligned}
\mathbb{E}\big[y_n^2\big] &\le Q_3 E_{1-\alpha^*,1}\left(Q_{4,\lambda}\Gamma(1-\alpha^*)t_n^{1-\alpha^*}\right)\\
&\le Q_3 E_{1-\alpha^*,1}\left(Q_{4,\lambda}\Gamma(1-\alpha^*)T^{1-\alpha^*}\right)=:M_{1,\lambda},
\end{aligned} \tag{4.2}
$$

where Q_3 and $Q_{4,\lambda}$ are the following constants:

$$
\begin{aligned}
&Q_3=4C_0\mathbb{E}\big[u_0^2\big]+8L^2T(T+1),\\
&Q_{4,\lambda}=8L^2(T+1)+\frac{4\lambda^2Q_0^2\big(2^{\alpha^*}+1/(1-\alpha^*)\big)T^{1-\alpha^*}}{1-\alpha^*}.
\end{aligned} \tag{4.3}
$$

Proof. We apply Jensen inequality (2.2) with $m = 4$ and Cauchy inequality to bound the moment of y_n in (4.1) by

$$
\begin{aligned}
\mathbb{E}\big[y_n^2\big] \le\; & 4C_0^2\mathbb{E}\big[u_0^2\big]+4\lambda^2\mathbb{E}\left[\left(\sum_{l=0}^{n-1}c_{n,l}y_l\right)^2\right]+4\tau^2\mathbb{E}\left[\left(\sum_{l=0}^{n-1}f(t_l,y_l)\right)^2\right]\\
&+4\mathbb{E}\left[\left(\sum_{l=0}^{n-1}b(t_l,y_l)\Delta W_l\right)^2\right].
\end{aligned} \tag{4.4}
$$

For $0 \le l \le n-2$, the coefficients $c_{n,l}$ can be bounded by

$$|c_{n,l}| \le \int_{t_l}^{t_{l+1}} |k(t_n,s)|ds \le \frac{Q_0 \tau}{(t_n - t_{l+1})^{\alpha^*}} \le \frac{2^{\alpha^*} Q_0}{(n-l)^{\alpha^*}} \tau^{1-\alpha^*},$$

where we used the fact $n-l \le 2(n-l-1)$ for $n-l \ge 2$. For $l = n-1$, we have

$$|c_{n,n-1}| = \int_{t_{n-1}}^{t_n} |k(t_n,s)|ds \le \frac{Q_0}{1-\alpha^*} \tau^{1-\alpha^*}.$$

Therefore, $c_{n,l}$ for $1 \le l \le n-1$ can be bounded by

$$|c_{n,l}| \le \frac{Q_0\big(2^{\alpha^*} + 1/(1-\alpha^*)\big)\tau^{1-\alpha^*}}{(n-l)^{\alpha^*}}, \quad 0 \le l \le n-1.$$

Lemma 3.2 yields

$$\sum_{l=0}^{n-1} |c_{n,l}| \le \int_0^{t_n} |k(t_n,s)|ds \le \frac{Q_0 t_n^{1-\alpha^*}}{1-\alpha^*}.$$

Combining these estimates with the Cauchy inequality, we estimate the second term in the right side of (4.4) as follows:

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{l=0}^{n-1} c_{n,l} y_l\right)^2\right] &\le \sum_{l=0}^{n-1} |c_{n,l}|\, \mathbb{E}\big[y_l^2\big] \sum_{l=0}^{n-1} |c_{n,l}| \\ &\le \frac{Q_0^2\big(2^{\alpha^*} + 1/(1-\alpha^*)\big)T^{1-\alpha^*}}{(1-\alpha^*)} \tau^{1-\alpha^*} \sum_{l=0}^{n-1} \frac{\mathbb{E}\big[y_l^2\big]}{(n-l)^{\alpha^*}}. \end{aligned} \tag{4.5}$$

By the Cauchy inequality and assumption A2, the third term in the right-hand side of (4.4) can be bounded by

$$\begin{aligned}\tau^2 \mathbb{E}\left[\left(\sum_{l=0}^{n-1} f(t_l,y_l)\right)^2\right] &\le T\tau \sum_{l=0}^{n-1} \mathbb{E}\big[f(t_l,y_l)^2\big] \le 2L^2 T\tau \sum_{l=0}^{n-1} \mathbb{E}\big[\big(1+y_l^2\big)\big] \\ &\le 2L^2T^2 + 2L^2 T\tau \sum_{l=0}^{n-1} \mathbb{E}\big[y_l^2\big]. \end{aligned}$$

We use Itô's isometry and assumption A2 to bound the last term on the right-hand side of (4.4) by

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{l=0}^{n-1} b(t_l,y_l)\Delta W_l\right)^2\right] &= \tau \sum_{l=0}^{n-1} \mathbb{E}\big[b(t_l,y_l)^2\big] \le 2L^2 \tau \sum_{l=0}^{n-1} \mathbb{E}\big[1 + y_l^2\big] \\ &\le 2L^2 T + 2L^2 \tau \sum_{l=0}^{n-1} \mathbb{E}\big[y_l^2\big]. \end{aligned} \tag{4.6}$$

We invoke the estimates (4.5)-(4.6) into (4.4) to obtain

$$\begin{aligned}\mathbb{E}\big[y_n^2\big] &\le 4C_0^2\mathbb{E}\big[u_0^2\big]+8L^2T(T+1)+8L^2(T+1)\tau\sum_{l=0}^{n-1}\mathbb{E}\big[y_l^2\big]\\&\quad+\frac{4\lambda^2Q_0^2\big(2^{\alpha^*}+\frac{1}{1-\alpha^*}\big)T^{1-\alpha^*}}{(1-\alpha^*)}\tau^{1-\alpha^*}\sum_{l=0}^{n-1}\frac{\mathbb{E}[y_l]^2}{(n-l)^{\alpha^*}}\\&\le Q_3+Q_{4,\lambda}\tau^{1-\alpha^*}\sum_{l=0}^{n-1}\frac{\mathbb{E}[y_l]^2}{(n-l)^{\alpha^*}}\end{aligned}$$

with Q_3 and $Q_{4,\lambda}$ being given in (4.3). Applying the generalized discrete Gronwall's inequality (2.4) with $\nu=1-\alpha^*$ leads to the estimate (4.2). □

4.2. Auxiliary equation estimates

To analyze the strong convergence of the Euler-Maruyama scheme, we define an auxiliary continuous time stochastic process $y(t)$ on $[0,T]$ using the step function $\hat{s}=\hat{s}(s)$ such that $\hat{s}:=t_n$ for $s\in[t_n,t_{n+1})$ and $0\le n\le N-1$

$$\begin{aligned}y(t)&=\left(1+\frac{\lambda t^{1-\alpha(0)}}{\Gamma(2-\alpha(0))}\right)u_0+\lambda\int_0^t k(t,s)y(\hat{s})ds\\&\quad+\int_0^t f\big(s,y(\hat{s})\big)ds+\int_0^t b\big(s,y(\hat{s})\big)dW.\end{aligned}\tag{4.7}$$

Lemma 4.1. *Under the assumptions* A1-A2*, the error between* $\{y_n\}_{n=0}^N$ *and* $y(t)$ *can be bounded by*

$$\mathbb{E}\Big[\big(y(t_n)-y_n\big)^2\Big]\le 4L^2T^2\big(\tau^{2\beta}+\tau^{2\gamma}\big),\quad 0\le n\le N.\tag{4.8}$$

Proof. It is clear that $y(0)=y_0=u_0$. For $0<n\le N$ we decompose the integrals on the right-hand side of (4.7) by

$$\begin{aligned}y(t_n)&=\left(1+\frac{\lambda t_n^{1-\alpha(0)}}{\Gamma(2-\alpha(0))}\right)u_0+\lambda\sum_{m=0}^{n-1}c_{n,m}y(t_m)+\tau\sum_{m=0}^{n-1}f\big(t_m,y(t_m)\big)\\&\quad+\sum_{m=0}^{n-1}\int_{t_m}^{t_{m+1}}\Big(f\big(s,y(t_m)\big)-f\big(t_m,y(t_m)\big)\Big)ds+\sum_{m=0}^{n-1}b\big(t_m,y(t_m)\big)\Delta W_m\\&\quad+\sum_{m=0}^{n-1}\int_{t_m}^{t_{m+1}}\Big(b\big(s,y(t_m)\big)-b\big(t_m,y(t_m)\big)\Big)dW(s)\\&=y_n+\sum_{m=0}^{n-1}\int_{t_m}^{t_{m+1}}\Big(f\big(s,y(t_m)\big)-f\big(t_m,y(t_m)\big)\Big)ds\\&\quad+\sum_{m=0}^{n-1}\int_{t_m}^{t_{m+1}}\Big(b\big(s,y(t_m)\big)-b\big(t_m,y(t_m)\big)\Big)dW(s).\end{aligned}$$

By the assumption A2 and the Itô isometry, we get

$$
\begin{aligned}
\mathbb{E}\Big[\big(y(t_n)-y_n\big)^2\Big] &\le 2\mathbb{E}\left[\left(\sum_{m=0}^{n-1}\int_{t_m}^{t_{m+1}}\Big(f\big(s,y(t_m)\big)-f\big(t_m,y(t_m)\big)\Big)ds\right)^2\right]\\
&\quad+2\mathbb{E}\left[\left(\sum_{m=0}^{n-1}\int_{t_m}^{t_{m+1}}\Big(b\big(s,y(t_m)\big)-b\big(t_m,y(t_m)\big)\Big)dW(s)\right)^2\right]\\
&\le 2TL^2\sum_{m=0}^{n-1}\left(\int_{t_m}^{t_{m+1}}|s-t_m|^{2\beta}ds+\int_{t_m}^{t_{m+1}}|s-t_m|^{2\gamma}ds\right)\\
&\le 4L^2T^2\big(\tau^{2\beta}+\tau^{2\gamma}\big).
\end{aligned}
$$

Thus we finish the proof. □

Lemma 4.2. *Under the assumption* A1*, it holds for* $t\in[t_n,t_{n+1})$

$$
\left(\int_0^{t_n}|k(t,s)-k(t_n,s)|ds\right)^2\le Q_5\tau^{2(1-\alpha^*)},
$$

where

$$
\begin{aligned}
Q_5=3\Bigg\{&\max\big\{1,T^2\big\}\left(\frac{1}{1-\alpha^*}+\frac{\|\alpha\|_{C[0,1]}}{\alpha^*}\right)^2+\frac{T^2\|\alpha\|^2_{C^1[0,T]}}{(1-\alpha^*)^2}\\
&+4\|\alpha\|^2_{C^1[0,T]}\max\big\{1,T^2\big\}\left(\frac{4T^{\frac{1-\alpha^*}{2}}}{(1-\alpha^*)^2e}+\frac{T^{1-\alpha^*}}{1-\alpha^*}\right)^2\Bigg\}.
\end{aligned}
$$

Proof. Using the mean value theorem and the inequalities $1/\Gamma(\beta)<1$, $|\Gamma'(1+\beta)|/\Gamma^2(1+\beta)\le 1$ for $1/2\le\beta\le 1$, we estimate $|k(t,s)-k(t_n,s)|$ as follows:

$$
\begin{aligned}
|k(t,s)-k(t_n,s)| &= \left|\frac{\partial}{\partial s}\left(\frac{(t-s)^{1-\alpha(s)}-(t_n-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))}\right)\right|\\
&\le\left|\frac{(1-\alpha(s))((t-s)^{-\alpha(s)}-(t_n-s)^{-\alpha(s)})}{\Gamma(2-\alpha(s))}\right|\\
&\quad+\left|\frac{\alpha'(s)((t-s)^{1-\alpha(s)}\ln(t-s)-(t_n-s)^{1-\alpha(s)}\ln(t_n-s))}{\Gamma(2-\alpha(s))}\right|\\
&\quad+\left|\frac{\Gamma'(2-\alpha(s))\alpha'(s)((t-s)^{1-\alpha(s)}-(t_n-s)^{1-\alpha(s)})}{\Gamma^2(2-\alpha(s))}\right|\\
&\le\left|(t_n-s)^{-\alpha(s)}-(t-s)^{-\alpha(s)}\right|+\|\alpha\|_{C^1[0,T]}(t_n-s)^{-\alpha(s)}\tau\\
&\quad+2\|\alpha\|_{C^1[0,T]}\left|(t-s)^{1-\alpha(s)}\ln(t-s)-(t_n-s)^{1-\alpha(s)}\ln(t_n-s)\right|.
\end{aligned}
$$

By the Jensen inequality we get

$$
\begin{aligned}
&\left(\int_0^{t_n}|k(t,s)-k(t_n,s)|ds\right)^2\\
&\leq 3\left(\int_0^{t_n}\left|(t_n-s)^{-\alpha(s)}-(t-s)^{-\alpha(s)}\right|ds\right)^2\\
&\quad+12\|\alpha\|^2_{C^1[0,T]}\left(\int_0^{t_n}\left|(t-s)^{1-\alpha(s)}\ln(t-s)-(t_n-s)^{1-\alpha(s)}\ln(t_n-s)\right|ds\right)^2\\
&\quad+3\|\alpha\|^2_{C^1[0,T]}\left(\int_0^{t_n}(t_n-s)^{-\alpha(s)}ds\right)^2\tau^2=:3J_1^2+12\|\alpha\|^2_{C^1}J_2^2+3\|\alpha\|^2_{C^1}J_3^2. \qquad (4.9)
\end{aligned}
$$

We are now in the position to estimate J_1-J_3. If $s\in[0,t_{n-1}]$, the mean value theorem yields

$$
\begin{aligned}
&\left|(t_n-s)^{-\alpha(s)}-(t-s)^{-\alpha(s)}\right|\\
&\leq\alpha(s)(t_n-s)^{-\alpha(s)-1}\tau\leq\max\{1,T\}\|\alpha\|_{C[0,T]}(t_n-s)^{-\alpha^*-1}\tau,
\end{aligned}
$$

so that

$$
\begin{aligned}
&\int_0^{t_{n-1}}\left|(t_n-s)^{-\alpha(s)}-(t-s)^{-\alpha(s)}\right|ds\\
&\leq\max\{1,T\}\frac{\|\alpha\|_{C^1[0,T]}}{\alpha^*}\left(\tau^{-\alpha^*}-t_n^{-\alpha^*}\right)\tau\leq\max\{1,T\}\frac{\|\alpha\|_{C[0,T]}}{\alpha^*}\tau^{1-\alpha^*}.
\end{aligned}
$$

If $s\in(t_{n-1},t_n)$, then

$$
\begin{aligned}
&\int_{t_{n-1}}^{t_n}\left|(t_n-s)^{-\alpha(s)}-(t-s)^{-\alpha(s)}\right|ds\\
&\leq\int_{t_{n-1}}^{t_n}(t_n-s)^{\alpha^*-\alpha(s)}(t_n-s)^{-\alpha^*}ds\leq\max\{1,T\}\frac{\tau^{1-\alpha^*}}{1-\alpha^*}.
\end{aligned}
$$

We incorporate these estimates to bound J_1 by

$$
|J_1|\leq\max\{1,T\}\left(\frac{1}{1-\alpha^*}+\frac{\|\alpha\|_{C[0,T]}}{\alpha^*}\right)\tau^{1-\alpha^*}.
$$

We apply the mean value theorem to bound J_2 by

$$
\begin{aligned}
|J_2|&=\int_0^{t_n}\left|\partial_z(z-s)^{1-\alpha(s)}\ln(z-s)\right|\Big|_{z=\xi}(t-t_n)ds\\
&\leq\int_0^{t_n}\left|(1-\alpha(s))(\xi-s)^{-\alpha(s)}\ln(\xi-s)+(\xi-s)^{-\alpha(s)}\right|ds\tau
\end{aligned}
$$

$$
\begin{aligned}
&\le \max\{1,T\}\left(\frac{2}{(1-\alpha^*)e}\int_0^{t_n}(\xi-s)^{-\frac{1+\alpha^*}{2}}ds+\int_0^{t_n}(\xi-s)^{-\alpha^*}ds\right)\\
&\le \max\{1,T\}\left(\frac{4T^{\frac{1-\alpha^*}{2}}}{(1-\alpha^*)^2e}+\frac{T^{1-\alpha^*}}{1-\alpha^*}\right)\tau.
\end{aligned}
$$

Similarly, J_3 can be bounded by

$$
|J_3|\le T^{\alpha^*}\left(\int_0^{t_n}(t_n-s)^{-\alpha^*}ds\right)\tau\le\frac{T}{1-\alpha^*}\tau.
$$

Applying the preceding estimates in (4.9) finishes the proof. □

Theorem 4.2. *Under the assumptions* A1-A2*, the following estimate holds for* $t\in[t_n,t_{n+1})$*:*

$$
\mathbb{E}\Big[\big(y(t)-y(t_n)\big)^2\Big]\le M_{2,\lambda}\tau,\quad 0\le n\le N-1,
$$

where $M_{2,\lambda}$ *is given by*

$$
M_{2,\lambda}=\frac{4\lambda^2\mathbb{E}\big[u_0^2\big]}{\Gamma^2(2-\alpha(0))}+8\lambda^2M_{1,\lambda}\left(\frac{Q_0^2}{(1-\alpha^*)^2}+Q_5\right)+16L^2(1+M_{1,\lambda}).
$$

Proof. For $t\in[t_n,t_{n+1})$ with $0\le n\le N-1$, we subtract Eq. (4.7) at time t_n from (4.7) and apply the Jensen inequality to obtain

$$
\begin{aligned}
\mathbb{E}\Big[\big(y(t)-y(t_n)\big)^2\Big]\le{}&\frac{4\lambda^2\big(t^{1-\alpha(0)}-t_n^{1-\alpha(0)}\big)^2}{\Gamma^2(2-\alpha(0))}\mathbb{E}\big[u_0^2\big]+4\mathbb{E}\left[\left(\int_{t_n}^t f\big(s,y(\hat{s})\big)ds\right)^2\right]\\
&+4\mathbb{E}\left[\left(\int_{t_n}^t b\big(s,y(\hat{s})\big)dW(s)\right)^2\right]+8\lambda^2\mathbb{E}\left[\left(\int_{t_n}^t k(t,s)y(\hat{s})ds\right)^2\right]\\
&+8\lambda^2\mathbb{E}\left[\left(\int_0^{t_n}\big(k(t,s)-k(t_n,s)\big)y(\hat{s})ds\right)^2\right].
\end{aligned}\tag{4.10}
$$

The first term in the right-hand side of (4.10) can be bounded by

$$
\left|\frac{4\lambda^2\big(t^{1-\alpha(0)}-t_n^{1-\alpha(0)}\big)^2}{\Gamma^2(2-\alpha(0))}\right|\mathbb{E}\big[u_0^2\big]\le\frac{4\lambda^2\mathbb{E}[u_0^2]}{\Gamma^2(2-\alpha(0))}\tau^{2(1-\alpha(0))}.
$$

The assumption A2 and Theorem 4.1 allows to estimate the second term as follows:

$$
\mathbb{E}\left[\left(\int_{t_n}^t f\big(s,y(\hat{s})\big)ds\right)^2\right]\le 2L^2\tau\int_{t_n}^t\big(1+\mathbb{E}\big[y_n^2\big]\big)ds\le 2L^2(1+M_{1,\lambda})\tau^2.
$$

We use Itô isometry and assumption A2 to bound the third right-hand side term by

$$\mathbb{E}\left[\left(\int_{t_n}^{t} b\big(s,y(\hat{s})\big)dW(s)\right)^2\right]$$
$$=\int_{t_n}^{t}\mathbb{E}\Big[b\big(s,y(\hat{s})\big)^2\Big]dt \le 2L^2\big(1+\mathbb{E}\big[y_n^2\big]\big)\tau \le 2L^2(1+M_{1,\lambda})\tau.$$

We employ the Cauchy inequality to bound the fourth term as

$$\mathbb{E}\left[\left(\int_{t_n}^{t} k(t,s)y(\hat{s})ds\right)^2\right] \le \int_{t_n}^{t}|k(t,s)|\,\mathbb{E}\big[y(\hat{s})^2\big]ds\int_{t_n}^{t}|k(t,s)|ds$$
$$\le M_{1,\lambda}\left(\int_{t_n}^{t}|k(t,s)|ds\right)^2 \le \frac{M_{1,\lambda}Q_0^2\tau^{2(1-\alpha^*)}}{(1-\alpha^*)^2}.$$

Finally we apply Lemma 4.2 to estimate the last term in the right-hand side of (4.10)

$$\mathbb{E}\left[\left(\int_0^{t_n}\big(k(t,s)-k(t_n,s)\big)y(\hat{s})^2ds\right)\right]$$
$$\le \int_0^{t_n}|k(t,s)-k(t_n,s)|\,\mathbb{E}\big[y(\hat{s})\big]^2ds\int_0^{t_n}|k(t,s)-k(t_n,s)|ds$$
$$\le M_{1,\lambda}\left(\int_0^{t_n}|k(t,s)-k(t_n,s)|ds\right)^2 \le Q_5M_{1,\lambda}\tau^{2(1-\alpha^*)}.$$

We incorporate the preceding estimates to complete the proof. □

Remark 4.1. We remark that the strong convergence order is $\mathscr{O}(\tau^{1-\alpha^*}+\tau^{0.5})$, accurately, where the error $\mathscr{O}(\tau^{1-\alpha^*})$ comes from the fractional term and $\mathscr{O}(\tau^{0.5})$ comes from the Brownian motion term. Since it is assumed that $0\le\alpha(t)\le 0.5$, the error is dominated by $O(\tau^{0.5})$ as expressed in Theorem 4.2.

4.3. Errors of Euler-Maruyama scheme (4.1)

We now prove the strong convergence of Euler-Maruyama scheme (4.1).

Theorem 4.3. *Suppose the assumptions* A1-A2 *hold. Then the following error estimate holds*

$$\max_{0\le n\le N}\mathbb{E}\big[|u(t_n)-y_n|^2\big]\le M_{3,\lambda}\tau+4L^2T\left(\tau^{2\beta}+\tau^{2\gamma}\right),$$

where

$$M_{3,\lambda}=Q_6E_{1-\alpha^*,1}\left(Q_7\Gamma(1-\alpha^*)T^{1-\alpha^*}\right)M_{2,\lambda},$$

$$Q_6 = \frac{6\lambda^2 Q_0^2 T^{2(1-\alpha^*)}}{(1-\alpha^*)^2} + 6L^2T(T+1),$$
$$Q_7 = \frac{6\lambda^2 Q_0^2 T^{1-\alpha^*}}{1-\alpha^*} + 6L^2T^{\alpha^*}(T+1). \tag{4.11}$$

Proof. Let $t \in [t_n, t_{n+1})$ for some $0 \le n \le N-1$. We subtract Eq. (4.7) from Eq. (2.8) to obtain

$$\begin{aligned}\mathbb{E}\Big[\big(u(t)-y(t)\big)^2\Big] \le{}& 3\lambda^2\mathbb{E}\left[\left(\int_0^t k(t,s)\big(u(s)-y(\hat{s})\big)ds\right)^2\right]\\ &+3\mathbb{E}\left[\left(\int_0^t f\big(s,u(s)\big)-f\big(s,y(\hat{s})\big)ds\right)^2\right]\\ &+3\mathbb{E}\left[\left(\int_0^t b\big(s,u(s)\big)-b\big(s,y(\hat{s})\big)dW\right)^2\right] =: \sum_{j=1}^3 I_j. \end{aligned}\tag{4.12}$$

Using the assumption A2, Cauchy inequality, Itô isometry, Theorem 4.2 and $u(s)-y(\hat{s}) = (u(s)-y(s)) + (y(s)-y(\hat{s}))$, we estimate I_2 and I_3 as follows:

$$\begin{aligned} I_2 &\le 6TL^2\int_0^t \Big(\mathbb{E}\big[|u(s)-y(s)|^2\big]+\mathbb{E}\big[|y(s)-y(\hat{s})|^2\big]\Big)ds\\ &\le 6TL^2\int_0^t \mathbb{E}\big[|u(s)-y(s)|^2\big]ds + 6T^2L^2M_{2,\lambda}\tau, \end{aligned}\tag{4.13}$$

and

$$\begin{aligned} I_3 &\le 6L^2\int_0^t \Big(\mathbb{E}\big[|u(s)-y(s)|^2\big]+\mathbb{E}\big[|y(s)-y(\hat{s})|^2\big]\Big)ds\\ &\le 6L^2\int_0^t \mathbb{E}\big[|u(s)-y(s)|^2\big]ds + 6TL^2M_{2,\lambda}\tau. \end{aligned}$$

Considering I_1, we represent it in the form

$$\begin{aligned} I_1 \le{}& 6\lambda^2\mathbb{E}\left[\left(\int_0^t k(t,s)\big(u(s)-y(s)\big)ds\right)^2\right]\\ &+6\lambda^2\mathbb{E}\left[\left(\int_0^t k(t,s)\big(y(s)-y(\hat{s})\big)ds\right)^2\right]\\ =:{}& I_{1,1}+I_{1,2} \end{aligned}$$

and use the Cauchy inequality and Theorem 4.2, thus obtaining

$$I_{1,1} = 6\lambda^2\int_0^t \mathbb{E}\big[|u(s)-y(s)|^2\big]|k(t,s)|ds\int_0^t |k(t,s)|ds$$

$$\leq \frac{6\lambda^2 Q_0^2 T^{1-\alpha^*}}{1-\alpha^*}\int_0^t \frac{\mathbb{E}\big[|u(s)-y(s)|^2\big]}{(t-s)^{\alpha^*}}ds,$$

$$I_{1,2} \leq 6\lambda^2 \int_0^t \mathbb{E}\big[|y(s)-y(\hat{s})|^2\big]|k(t,s)|ds \int_0^t |k(t,s)|ds$$

$$\leq \frac{6\lambda^2 Q_0^2 T^{2(1-\alpha^*)} M_{2,\lambda}}{(1-\alpha^*)^2}\tau. \tag{4.14}$$

Combining (4.13)-(4.14) and (4.12) yields

$$\mathbb{E}\Big[\big(u(t)-y(t)\big)^2\Big] \leq Q_6 M_{2,\lambda}\tau + Q_7\int_0^t \frac{\mathbb{E}\big[(u(s)-y(s))^2\big]}{(t-s)^{\alpha^*}}ds,$$

where Q_6 and Q_7 are given by (4.11). We apply the generalized Gronwall's inequality (2.3) and set $t = t_n$ to obtain

$$\max_{t\in[0,T]} \mathbb{E}\big[|u(t)-y(t)|^2\big] \leq M_{3,\lambda}\tau.$$

This and the estimate (4.8) complete the proof. □

5. Numerical Experiments

We carry out numerical experiments to investigate the performance of the Euler-Maruyama scheme (4.1).

5.1. Strong convergence of the Euler-Maruyama scheme

Let $u(t_n,\omega_j)$ be the j-th independent sample path of the fSDE (1.1) evaluated at t_n with the numerical approximation $y_n(\omega_j)$ by the Euler-Maruyama scheme (4.1) for $n = 0,1,2,\ldots,N$ and $j = 1,2,\ldots,M$. The sample mean of the error and the convergence rate κ are defined by

$$e_\tau := \max_{0\leq n\leq N}\left(\frac{1}{2^{10}}\sum_{j=1}^{2^{10}}\Big|u(t_n,\omega_j)-y_n(\omega_j)\Big|^2\right)^{\frac{1}{2}} = QN^{-\kappa}$$

with $\kappa = \log_2(e_\tau/e_{\tau/2})$. In numerical experiments we let $[0,T] = [0,1]$, $\lambda = 1$, and the variable order α is chosen as

$$\alpha(t) = \alpha(1) + \big(\alpha(0)-\alpha(1)\big)\left((1-t) - \frac{\sin(2\pi(1-t))}{2\pi}\right). \tag{5.1}$$

Since the true solution is not available, we use the fine mesh size of $N_{ref} = 2^{10}$ to compute reference solutions.

Example 5.1 (Linear VO fSDE). We choose $f(t,u) = b(t,u) = -tu$ and $u_0 = 0.1$ in the problem (1.1). The errors e_τ for different mesh sizes N and difference choices of $\alpha(0)$ and $\alpha(1)$ are shown in Fig. 1. Note that the numerical scheme (4.1) converges strongly with the order $\mathscr{O}(\tau^{0.5})$, consistent with Theorem 4.3.

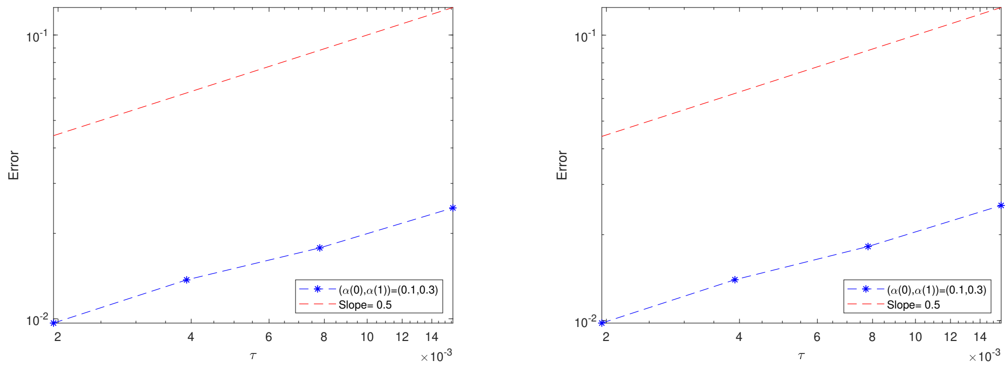

Figure 1: Example 5.1. Errors. Left: $(\alpha(0),\alpha(1)) = (0.1, 0.3)$. Right: $(\alpha(0),\alpha(1)) = (0.5, 0.2)$.

Example 5.2 (Nonlinear VO fSDE). Let $f(t,u) = b(t,u) = -t\sin(u)$ and $u_0 = 0.1$ in problem (1.1). The numerical results presented in Fig. 2 lead to the same conclusion as in Example 5.1.

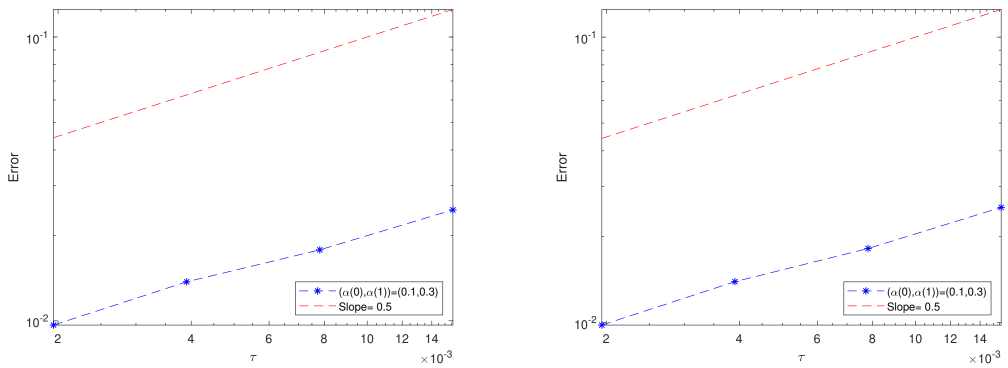

Figure 2: Example 5.2. Errors. Left: $(\alpha(0),\alpha(1)) = (0.1, 0.3)$. Right: $(\alpha(0),\alpha(1)) = (0.5, 0.2)$

5.2. Performance of the variable-order fSDE

We compare the performance of the variable-order fSDE (1.1) and the conventional SDE — i.e. the model (1.1) with $\lambda = 0$. Let $[0,T] = [0,1]$, $f(t,u) = b(t,u) = -tu$, $u_0 = 1$, and variable α is given by (5.1). We use the Euler-Maruyama scheme (4.1) with $N = 2^{10}$ to compute the solutions for different $(\alpha(0),\alpha(1))$ and present the results in Figs. 3 and 4. Note that:

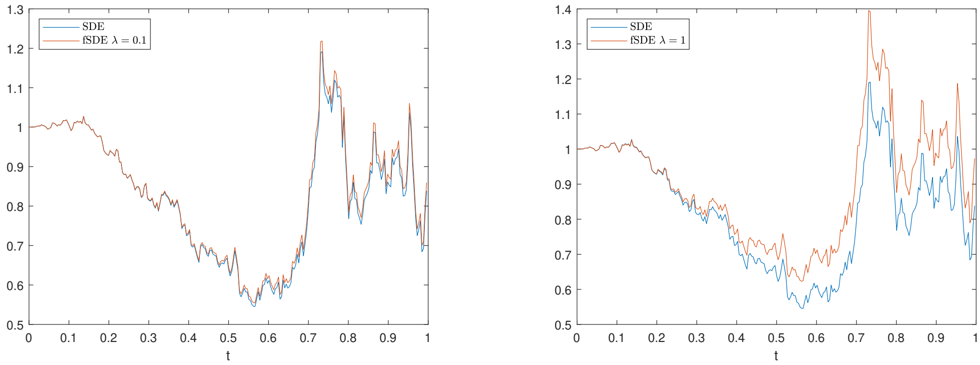

Figure 3: Solution curves for the same sample. Blue: Integer-order SDE. Red: Variable-order fSDE, $(\alpha(0),\alpha(1)) = (0.1, 0.3)$. Left: $\lambda = 0.1$. Right: $\lambda = 1$.

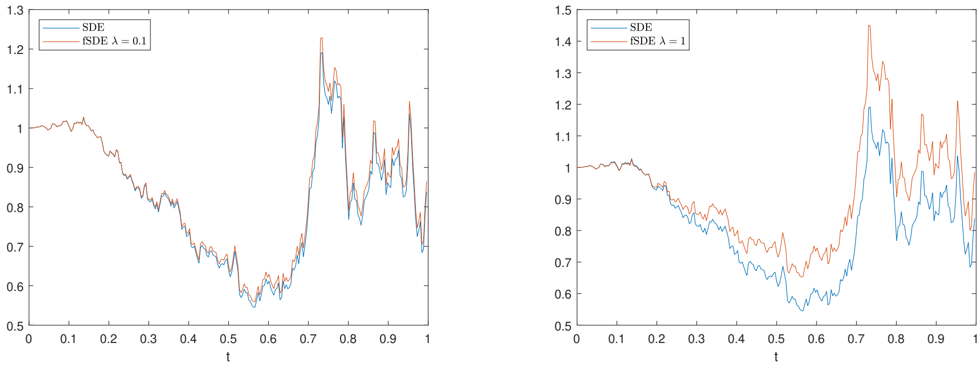

Figure 4: Solution curves for the same sample. Blue: Integer-order SDE. Red: Variable-order fSDE, $(\alpha(0),\alpha(1)) = (0.5, 0.2)$. Left: $\lambda = 0.1$. Right: $\lambda = 1$.

(i) The solutions to model (1.1) with small λ are close to the solutions to integer-order SDE, while there exhibits considerable differences between them if λ is not small enough.

(ii) Due to the history memory property of the fractional derivative, the discrepancy of solutions of VO fSDEs to SDEs are accumulated.

In order to show the effect of $\alpha(t)$ on the solutions, we choose two more functions $\alpha(t) = 0.2 + 0.2e^{-t}$ and $\alpha(t) = 0.5t^2$ as our variable-order functions in Fig. 5. Note that different variable-order functions will lead to drastic change in the image of the solutions.

6. Conclusion

In this paper, we focus on time-dependent VO fSDEs. We prove the wellposedness of a hidden-memory VO fSDE and establishe the following error estimate $\mathscr{O}(\tau^{1-\alpha^*} + \tau^{0.5}) =$

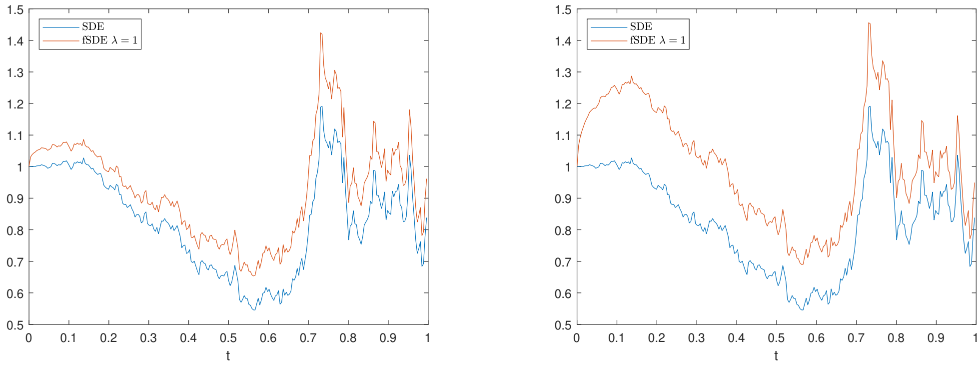

Figure 5: Solution curves for the same sample. Blue: Integer-order SDE. Red: Variable-order fSDE. Left: $\alpha(t) = 0.2 + 0.2e^{-t}$. Right: $\alpha(t) = 0.5t^2$.

$\mathscr{O}(\tau^{0.5})$ for $\alpha^* \leq 0.5$ for the Euler-Maruyama scheme, with $\tau^{1-\alpha^*}$ determined by the approximation of the fractional term and with dominated term $\mathscr{O}(\tau^{0.5})$ is defined by the discretization of Browian motion. In the future work, we will investigate the stochastic partial differential equation with space dependent variable order $\alpha(x)$ or space-time dependent variable order $\alpha(x, t)$.

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