

Mean Convex Mean Curvature Flow with Free Boundary

NICK EDELEN

University of Notre Dame

ROBERT HASLHOFER

University of Toronto

MOHAMMAD N. IVAKI

Technische Universität Wien

AND

JONATHAN J. ZHU

Princeton University

Abstract

In this paper, we generalize White’s regularity and structure theory for mean-convex mean curvature flow [45, 46, 48] to the setting with free boundary. A major new challenge in the free boundary setting is to derive an a priori bound for the ratio between the norm of the second fundamental form and the mean curvature. We establish such a bound via the maximum principle for a triple-approximation scheme, which combines ideas from Edelen [9], Haslhofer-Hershkovits [16], and Volkmann [43]. Other important new ingredients are a Bernstein-type theorem and a sheeting theorem for low-entropy free boundary flows in a half-slab, which allow us to rule out multiplicity 2 (half-)planes as possible tangent flows and, for mean-convex domains, as possible limit flows. © 2021 Wiley Periodicals LLC.

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1 Introduction

A family of closed embedded hypersurfaces $\{M_t\}_{t \geq 0}$ in a Riemannian manifold (N^{n+1}, g) moves by mean curvature flow if the normal velocity at each point on the (hyper)surface is given by the mean curvature vector at that point. If the initial surface is mean-convex, i.e., if everywhere on the surface the mean curvature vector points inwards, then the flow can be continued uniquely through all singularities [11, 45]. This evolution can be described both in the language of partial differential equations as a level set flow [8, 11] and in the language of geometric measure theory as an integral Brakke flow [3].

White and Huisken-Sinestrari developed a far-reaching structure theory for the mean curvature flow of such mean-convex hypersurfaces. Their papers [21, 22, 45, 46, 48] provide a package of estimates and structural results that yield a precise description of singularities and of high curvature regions (more recently, an alternative treatment of this theory has been given by Haslhofer-Kleiner [18]). In particular, their work implies that the parabolic Hausdorff dimension of the singular set is at most $n - 1$, and all blowup limits are convex, noncollapsed, and smooth until they become extinct.¹

The goal of the present paper is to generalize this theory to hypersurface with free-boundary, i.e., hypersurfaces satisfying a Neumann-type boundary condition. In the smooth setting, mean curvature flow with free-boundary has been studied in important work by Stahl [41, 42] and Buckland [7]. In the singular setting, in analogy to the flow of closed hypersurfaces, the free-boundary flow can be described as a level-set flow [13] or as a Brakke-type flow [10]. We prove here the existence of a unique mean-convex free boundary flow for all time and establish sharp results on the size and structure of singularities for this flow. Recent work of White [49] has developed an existence and regularity theory for flows with prescribed (Dirichlet) data, and in particular showed that the resulting flow is always regular near its Dirichlet data. This is very different from the free-boundary setting, where one expects singularities on the boundary to be modelled on interior singularities.

1.1 Mean convex free boundary flow

To describe the setup, fix a smooth compact domain $D \subset N^{n+1}$. A smooth domain $K \subset D$ has *free boundary* if $\partial K := K \setminus \text{Int}_D(K)$ meets ∂D orthogonally;² in other words,

$$(1.1) \quad \langle N(x), v(x) \rangle = 0 \quad \text{for all } x \in \partial K \cap \partial D,$$

where v denotes the outward unit normal of ∂K and N denotes the outward unit normal to ∂D . These notions are illustrated in Figure 1.1. We assume that K

¹ By Brendle [5] and Haslhofer-Hershkovits [16] this applies without dimensional restrictions and in general ambient manifolds.

² It is more convenient to phrase everything in terms of compact domains K instead of hypersurfaces $M = \partial K$. These points of view are of course equivalent.

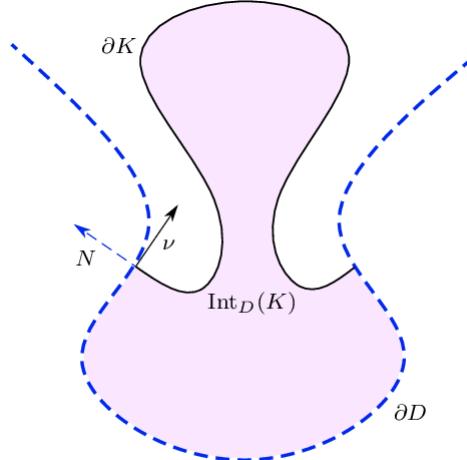


FIGURE 1.1. Mean curvature flow with free boundary

is *mean-convex*, i.e., that the mean curvature vector everywhere on ∂K points inside K .

Given K , there is a unique evolution by the *free boundary level set flow* $F_t(K)$ from Giga-Sato [13]. This has been described originally in terms of viscosity solutions of degenerate partial differential equations. Using the more geometric interpretation from Ilmanen [24], $K_t := F_t(K) \subset D$ simply is the maximal family of closed sets starting at K that does not bump into any smooth free boundary subsolution (see Section 3 for more detailed definitions). Mean convexity is preserved under the free boundary level set flow (see again Section 3), i.e.,

$$(1.2) \quad K_{t_2} \subseteq \text{Int}_D K_{t_1} \quad \text{for all } t_2 > t_1 \geq 0.$$

We denote by

$$(1.3) \quad \mathcal{K} := \bigcup_{t \geq 0} K_t \times \{t\} \subset D \times \mathbb{R}_+$$

the space-time track of the free boundary level set flow. Moreover, it can be checked (see Theorem 1.6 and Remark 1.7 below) that

$$(1.4) \quad \mathcal{M} = \{\mathcal{H}^n \lfloor \partial K_t\}_{t \geq 0}$$

is a *free boundary Brakke flow* as defined in [10] (see also Section 5). The pair $(\mathcal{M}, \mathcal{K})$ is called a *mean-convex free boundary flow*.

1.2 Size of the singular set

Given a mean-convex free boundary flow $(\mathcal{M}, \mathcal{K})$ in a domain $D \subset \mathbb{N}^{n+1}$, its singular set $\mathcal{S} \subset D \times \mathbb{R}_+$ consists by definition of all points $X = (x, t)$ in the support of the flow such that there is no two-sided space-time neighborhood of X in which evolution can be described as smooth free boundary flow. The parabolic

Hausdorff dimension of a subset of spacetime is the Hausdorff dimension with respect to the parabolic metric

$$(1.5) \quad d((x, t), (y, s)) = \max\{|x - y|, |t - s|^{1/2}\}.$$

For example, the parabolic Hausdorff dimension of $D \times \mathbb{R}_+$ is $n + 3$. Our first main theorem shows that the singular set in any mean-convex free boundary flow is quite small:

THEOREM 1.1 (Size of the singular set). *Let $(\mathcal{M}, \mathcal{K})$ be a mean-convex free boundary flow in $D \subset N^{n+1}$. Then the parabolic Hausdorff dimension of its singular set \mathcal{S} is*

- at most $n - 1$ if the domain D is mean-convex, and
- at most n if the domain D is arbitrary.

Simple examples show that the result is sharp. Indeed, rotationally symmetric thin shrinking tori or half-tori in a ball $D \subset \mathbb{R}^{n+1}$ show that an $(n-1)$ -dimensional singular set can occur both in the interior and at the boundary. Likewise, the number n is sharp for general barriers, since the surface can pop when it hits ∂D , cf. [10]. For example, if $D = \bar{B}_4 \setminus B_1 \subset \mathbb{R}^{n+1}$ is an annulus and $K = \bar{B}_2 \setminus B_1$, then the shrinking sphere pops when it hits the inner boundary of the annulus and produces an n -dimensional singular set.

1.3 Structure of singularities

Let $(\mathcal{M}, \mathcal{K})$ be a mean-convex free boundary flow in a mean-convex domain D .

Given $X_i = (x_i, t_i) \rightarrow X = (x, t)$ in the support of the flow and $\lambda_i \rightarrow \infty$, we consider the *blowup sequence* $(\mathcal{M}^i, \mathcal{K}^i)$ that is obtained from $(\mathcal{M}, \mathcal{K})$ by translating X_i to the origin and parabolically rescaling by λ_i . It is always possible to find a convergent subsequence where \mathcal{M}^i converges in the sense of (free boundary) Brakke flows [10, 25] and \mathcal{K}^i converges in the Hausdorff sense. Any subsequential limit $(\mathcal{M}', \mathcal{K}')$ is called a *limit flow at X* . Here, $(\mathcal{M}', \mathcal{K}')$ is either a flow without boundary in \mathbb{R}^{n+1} or a flow with free boundary defined in a half-space $\mathbb{H} \subset \mathbb{R}^{n+1}$. If $X_i = X$, we call it a *tangent flow at X* . Tangent flows are always backwardly self-similar [10, 20].

THEOREM 1.2 (Structure of singularities). *Let $(\mathcal{M}, \mathcal{K})$ be a mean-convex free boundary flow in a mean-convex domain $D \subset N^{n+1}$. Let $(\mathcal{M}', \mathcal{K}')$ be a limit flow, and denote by $T_{\text{ext}} = T_{\text{ext}}(\mathcal{M}', \mathcal{K}') \in (-\infty, +\infty]$ its extinction time, i.e., the supremum of all t with $\mathcal{K}'_t \neq \emptyset$. Then:*

- \mathcal{M}' has multiplicity 1 and its support equals $\partial \mathcal{K}'$.
- $(\mathcal{M}', \mathcal{K}') \cap \{t < T_{\text{ext}}\}$ is smooth.
- \mathcal{K}' has convex time slices, i.e., \mathcal{K}'_t is convex for every t .
- \mathcal{K}' is either a static half-space or quarter-space, or it has strictly positive mean curvature and sweeps out the whole space or half-space, i.e., $\bigcup_t \partial \mathcal{K}'_t = \mathbb{R}^{n+1}$ or $\bigcup_t \partial \mathcal{K}'_t = \mathbb{H}$, respectively.

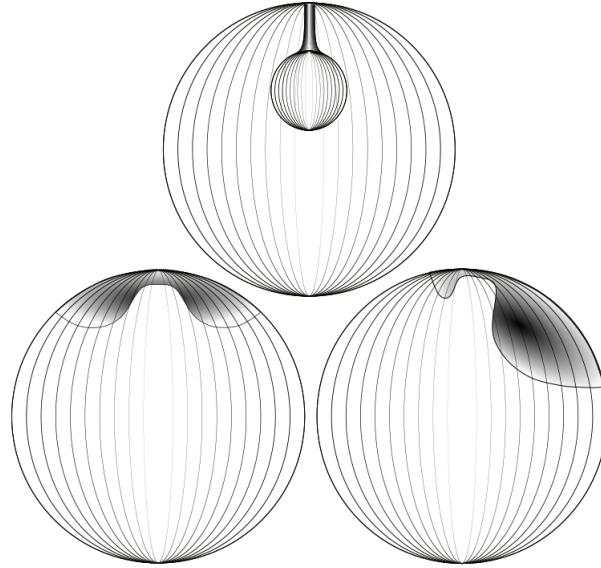


FIGURE 1.2. Neckpinches at the boundary

- \mathcal{K}' is 1-noncollapsed; i.e., every $p \in \partial K'_t$ admits interior and exterior balls of radius $1/H(p, t)$.

Furthermore, if $(\mathcal{M}', \mathcal{K}')$ is backwardly self-similar, then it is either (i) a static multiplicity 1 plane or half-plane, (ii) a round shrinking sphere or half-sphere, or (iii) a round shrinking cylinder or half-cylinder.

Examples where half-cylinders occur as tangent flows are neckpinches at the boundary. Note that there are two types of half-cylinders with free boundary, depending on whether the axis is contained in $\partial \mathbb{H}$ or perpendicular to it. An illustrative example where a limit flow is not backwardly self-similar is a degenerate neckpinch at the boundary, in which case some limit flow $(\mathcal{M}', \mathcal{K}')$ would be a half-bowl. These examples are illustrated in Figure 1.2.

Together with the recent breakthroughs by Brendle-Choi [6] and Angenent-Daskalopoulos-Sesum [2], we obtain for $n = 2$:

COROLLARY 1.3 (Classification of limit flows). *Let $(\mathcal{M}, \mathcal{K})$ be a two-dimensional mean-convex free boundary flow in a mean-convex domain $D \subset \mathbb{N}^3$. Then any limit flow $(\mathcal{M}', \mathcal{K}')$ is one of the following:*

- a static multiplicity 1 plane or half-plane,
- a round shrinking sphere or half-sphere,
- a round shrinking cylinder or half-cylinder,
- a translating bowl or half-bowl,
- an ancient oval or half-oval.

In any dimension, Theorem 1.2 (structure of singularities) in particular implies the following estimates for the flow $(\mathcal{M}, \mathcal{K})$ itself.

COROLLARY 1.4 (Estimates for mean-convex free boundary flow). *Let $(\mathcal{M}, \mathcal{K})$ be a mean-convex free boundary flow in a mean-convex domain $D \subset \mathbb{R}^{n+1}$.*

- Sharp noncollapsing. *For any $\alpha < 1$ and $T < \infty$, there exists an $H_0 < \infty$ with the following significance. For any $t \leq T$, at any point $p \in \partial K_t$ with $H(p, t) \geq H_0$ there exist interior and exterior balls of radius at least $\alpha/H(p, t)$.*
- Convexity estimate. *For all $\varepsilon > 0$ and $T < \infty$, there exists an $H_0 < \infty$ with the following significance. For any $t \leq T$, at any point $p \in \partial K_t$ with $H(p, t) \geq H_0$, we have $\lambda_1(p, t) \geq -\varepsilon H(p, t)$.*
- Gradient estimate. *For all $T < \infty$ there is a $C < \infty$ such that at all times $t \leq T$ and at all points we have $|\nabla H| \leq CH^2$.*

We mention that a related convexity estimate was proved in [9]. On the other hand, the gradient estimate for mean-convex free boundary flows only follows after establishing our main structure theorem (Theorem 1.2), and we are not aware of any shorter path towards establishing such a gradient estimate directly. This is similar in spirit to the fact that Perelman obtained the gradient estimate for 3-D Ricci flow as a corollary after establishing his canonical neighborhood theorem [36][37].

1.4 Long-time behavior

Mean curvature flow of closed surfaces in Euclidean space always becomes extinct in finite time, since there are no closed minimal surfaces it can converge to. The mean curvature flow with free boundary can have more interesting long-time behavior even for $D \subset \mathbb{R}^{n+1}$:

THEOREM 1.5 (Long-time behavior). *Let $(\mathcal{M}, \mathcal{K})$ be a mean-convex free boundary flow in a mean-convex domain $D \subset \mathbb{R}^{n+1}$, and set*

$$(1.6) \quad K_\infty := \bigcap_t K_t.$$

Then either $K_\infty = \emptyset$ or K_∞ has finitely many connected components. The boundary of each component is a stable free boundary minimal surface whose singular set has Hausdorff dimension at most $n - 7$. Furthermore, ∂K_t converges smoothly (either locally one-sheeted or two-sheeted depending on whether or not the component has interior points) to ∂K_∞ away from the singular set of ∂K_∞ .

For example, consider the case where D looks like a dumbbell, which contains a strictly stable free boundary minimal disc Σ . Fix $\varepsilon > 0$ small enough. If $K = \{x \in D : d(x, \Sigma) \leq \varepsilon\}$, then we get two-sheeted convergence to Σ . On the other hand, letting C be one of the components of $D \setminus \Sigma$ and setting $K = \{x \in D : d(x, C) \leq \varepsilon\}$, we get one-sheeted convergence.

1.5 Some remarks on the proofs

As the reader will expect, one of the key steps is of course to rule out blowup limits of higher multiplicity. In the case of mean-convex surfaces without boundary there is a short maximum principle argument by Andrews [1]. Unfortunately, we have not been able to generalize the argument by Andrews to the free boundary setting (related to this, Brendle's proof of the Lawson conjecture [4] does not seem to generalize in any obvious way to establish uniqueness of the critical catenoid). We thus follow White's original approach from [45]. To this end, we prove free boundary versions of the expanding hole lemma, the Bernstein-type theorem for low entropy flows in a slab, and the sheeting theorem, and use them to rule out static and quasi-static planes or free boundary half-planes with density 2 as potential tangent flows or limit flows (see Section 6).

Another major issue is to rule out nontrivial minimal cones, such as the Simons cone, as potential blowup limits. This requires an a priori bound for the ratio between the norm of the second fundamental form and the mean curvature, which turns out to be substantially more involved than in the case without boundary. The technical heart of the present paper is the following:

THEOREM 1.6 (Elliptic regularization and consequences). *Let $K_t := F_t(K)$ be a free boundary level set flow with smooth strictly mean-convex initial data $K \subset D$. Then:*

- $\{K_t \times \mathbb{R}\}_{t>0}$ arises as a limit of smooth free boundary flows.
- $t \mapsto \mathcal{H}^n \lfloor \partial^* K_t$ is a free boundary Brakke flow.
- K_t is one-sided minimizing, i.e., $\mathcal{H}^n(\partial^* K_t) \leq \mathcal{H}^n(\partial^* F)$ whenever $K_t \subseteq F \subseteq K$.
- There exist constants $c = c(K, D) > 0$ and $\rho = \rho(K, D) < \infty$ such that for all $t > 0$ we have

$$\inf_{\partial K_t^{\text{reg}}} H \geq ce^{-\rho t}.$$

- For every $\delta > 0$ there exist constants $C = C(K, D, \delta) < \infty$ and $\rho = \rho(K, D, \delta) < \infty$ such that for every $t \geq \delta$ we have

$$\sup_{\partial K_t^{\text{reg}}} \frac{|A|}{H} \leq Ce^{\rho t}.$$

Remark 1.7. A priori, one has to work with the reduced boundary $\partial^* K_t$. A posteriori, Theorem 1.1 will imply that $\mathcal{H}^n \lfloor \partial K_t = \mathcal{H}^n \lfloor \partial^* K_t$ for all $t \geq 0$. The distinction is immaterial for the argument itself.

To prove Theorem 1.6 (elliptic regularization and consequences) we use a triple-approximate elliptic regularization scheme. We use a capillary parameter to deal with the lack of an L^∞ -estimate in case the flow does not become extinct in finite time (cf. Haslhofer-Hershkovits [16]). Next, we bend the corners of K to deal with the mixed Dirichlet-Neumann problem (cf. Volkmann [43]). Moreover, we have to

perturb the second fundamental form to get a reasonable normal derivative at the barrier (cf. Edelen [9]). Finally, we need an additional weight function to force the extrema away from the barrier (cf. Edelen [9]). We carry this out in Section 4.

Once Theorem 1.6 (free boundary level set flow) and multiplicity 1 (see Section 6) are established, all the remaining steps proceed broadly following the proof strategy of [45, 46], the main point being to rule out density 2 (planar) limit flows. In addition to multiplicity 2 planes and free boundary half-planes, there is the new possibility of a (quasi)static limit flow defined in a half-space and coincident with the barrier plane, which is not directly analogous to the case without boundary. We rule out this potential scenario if the barrier is mean-convex.³

2 Notation and Conventions

Let us fix a Riemannian manifold (N^{n+1}, g) and a compact domain $D \subset (N^{n+1}, g)$ with smooth boundary ∂D . For any domain $K \subset D$ we write \mathring{K} for the interior of K viewed as a subset of N , and $\text{Int}_D K$ for the interior of K viewed as a subset of the topological space D . We call $\partial K := K \setminus \text{Int}_D K$ the Dirichlet boundary of K , and $\delta K := K \setminus (\mathring{K} \cup \partial K)$ the Neumann boundary of K . We let ν be the outward unit normal of ∂K and use the sign convention that $\mathbf{H} = -H\nu$. We write N for the outward unit normal to ∂D .

3 Free Boundary Level Set Flow

A smooth family of closed domains $\{L_t \subset D\}_{t \in [a, b]}$ is a *smooth free boundary subsolution* if ∂L_t moves inwards at least as fast as the mean curvature flow and hits ∂D in a convex fashion, i.e.,

$$(3.1) \quad \langle \mathbf{vel}, \nu \rangle \leq \langle \mathbf{H}, \nu \rangle, \quad N \cdot \nu \geq 0,$$

where \mathbf{vel} denotes the normal velocity of ∂L_t . A family of closed sets $\{K_t \subset D\}_{t \in [a, b]}$ is a *set theoretic subsolution of the free boundary mean curvature flow* if it avoids any smooth free boundary subsolution, i.e., whenever $\{L_t \subset D\}_{t \in [t_0, t_1]}$, where $a \leq t_0 \leq t_1 \leq b$ is a smooth free boundary subsolution with $K_{t_0} \cap L_{t_0} = \emptyset$, then $K_t \cap L_t = \emptyset$ for all $t \in [t_0, t_1]$. The *free boundary level set flow* $F_t(K)$ is the maximal set-theoretic subsolution with initial condition $F_0(K) = K$. By the work of Giga-Sato [13] the free boundary level set flow has the following basic properties:

- (1) Consistency with smooth flows: If $\{M_t\}_{t \in [0, T]}$ is a smooth free-boundary mean curvature flow, then $M_t = F_t(M_0)$.
- (2) Avoidance: If K and L are disjoint, then $F_t(K)$ and $F_t(L)$ are also disjoint and $F_t(K \cup L) = F_t(K) \cup F_t(L)$.
- (3) Inclusion: If $K \subseteq L$, then $F_t(K) \subseteq F_t(L)$.

³This assumption on the barrier is indeed necessary. For general barriers, the flow can “pop,” which is captured by quasi-static density 2 tangent flows.

- (4) Semigroup property: $F_{t_1+t_2}(K) = F_{t_2}(F_{t_1}(K))$.
- (5) Strict inclusion: If $K \subseteq \text{Int}_D L$, then $F_t(K) \subseteq \text{Int}_D F_t(L)$.

A closed subset $K \subset D$ is said to be *mean-convex* if for some $\delta > 0$, $F_t(K) \subseteq \text{Int}_D K$ for all $0 < t < \delta$. If K is smooth, this is equivalent to the condition that $H \geq 0$ everywhere and not identically zero.

PROPOSITION 3.1. *Let $K \subset D$ be mean-convex. Mean convexity is preserved under the free boundary level set flow, i.e., $F_{t_2}(K) \subseteq \text{Int}_D F_{t_1}(K)$ for all $t_2 > t_1 \geq 0$. Moreover, the boundaries $\partial F_t(K)$ form a possibly singular foliation of $K \setminus \cap_{t=0}^{\infty} F_t(K)$, and it holds that $F_t(\partial K) = \partial F_t(K)$. In particular, the flow $t \mapsto F_t(\partial K)$ is nonfattening.*

PROOF. This follows from the above basic properties, arguing as in [45, sec. 3]. Namely, using the definition of mean-convex and the basic properties (4) and (5) we infer that

$$(3.2) \quad F_{t+h}(K) \subseteq \text{Int}_D F_t(K)$$

for all $t \geq 0$ and all $0 < h < \delta$. By induction we conclude that (3.2) holds for all $t \geq 0$ and all $0 < h < j\delta$, where $j = 1, 2, \dots$; hence mean convexity is preserved. In particular, the sets $\{\partial F_t(K)\}_{t \geq 0}$ are disjoint. Next, given any $x \in K \setminus \cap_{t=0}^{\infty} F_t(K)$, denoting by $u(x)$ the last time such that $x \in F_{u(x)}(K)$, we have $x \in \partial F_{u(x)}(K)$. Hence

$$(3.3) \quad K \setminus \cap_{t=0}^{\infty} F_t(K) = \bigcup_{t \geq 0} \partial F_t(K).$$

Next, it follows from (2) and (5) that $t \mapsto \partial F_t(K)$ is a set-theoretic subsolution. Also, note that $F_t(\partial K) \subseteq F_t(K)$ is property (3). Together with the observation that $F_t(\partial K)$ is disjoint from

$$(3.4) \quad \bigcup_{h>0} F_{t+h}(K) = \text{Int}_D F_t(K)$$

by (5), we infer that $F_t(\partial K) \subseteq \partial F_t(K)$. Recalling that $t \mapsto \partial F_t(K)$ is a set-theoretic subsolution with initial condition ∂K , while $F_t(\partial K)$ is the maximal set-theoretic subsolution with initial condition ∂K , we conclude that $F_t(\partial K) = \partial F_t(K)$. Since $\text{Int}_D \partial F_t(K) = \emptyset$, this finishes the proof of the proposition. \square

4 Elliptic Regularization and Consequences

In this section, we prove Theorem 1.6 (elliptic regularization and consequences).

4.1 Triple approximation scheme

The aim now is to construct solutions of a mixed Dirichlet-Neumann problem:

THEOREM 4.1. *Given a strictly mean-convex domain $K \subset D$, and constants $\varepsilon, \sigma > 0$, there exists a unique solution $u_{\varepsilon, \sigma} \in C^\infty(K \setminus \partial K) \cap C^{0,1}(K)$ of the problem*

$$(4.1) \quad \begin{aligned} \operatorname{div} \left(\frac{Du_{\varepsilon, \sigma}}{\sqrt{\varepsilon^2 + |Du_{\varepsilon, \sigma}|^2}} \right) + \frac{1}{\sqrt{\varepsilon^2 + |Du_{\varepsilon, \sigma}|^2}} &= \sigma u_{\varepsilon, \sigma} \quad \text{in } \mathring{K}, \\ \langle N, Du_{\varepsilon, \sigma} \rangle &= 0 \quad \text{on } \delta K, \\ u_{\varepsilon, \sigma} &= 0 \quad \text{on } \partial K. \end{aligned}$$

To solve problem (4.1), we need some additional approximations. Namely, to deal with the mixed Dirichlet-Neumann boundary we approximate the initial domain K by domains K^τ ($\tau > 0$) as in Volkmann [43, p. 74], which in particular satisfy the strict angle condition

$$(4.2) \quad \langle v_{\partial K^\tau}, N \rangle \leq -\tau/2.$$

Given $\varepsilon, \sigma, \tau > 0$, we then consider the triple approximation problem

$$(4.3) \quad \begin{aligned} \operatorname{div} \left(\frac{Du_{\varepsilon, \sigma, \tau}}{\sqrt{\varepsilon^2 + |Du_{\varepsilon, \sigma, \tau}|^2}} \right) + \frac{1}{\sqrt{\varepsilon^2 + |Du_{\varepsilon, \sigma, \tau}|^2}} &= \sigma u_{\varepsilon, \sigma, \tau} \quad \text{in } \mathring{K}^\tau, \\ \langle N, Du_{\varepsilon, \sigma, \tau} \rangle &= 0 \quad \text{on } \delta K^\tau, \\ u_{\varepsilon, \sigma, \tau} &= 0 \quad \text{on } \partial K^\tau. \end{aligned}$$

To solve (4.3) we use the continuity method; i.e., we introduce yet another parameter $\kappa \in [0, 1]$ and consider the problem

$$(4.4) \quad \begin{aligned} \operatorname{div} \left(\frac{Du_{\varepsilon, \sigma, \tau, \kappa}}{\sqrt{\varepsilon^2 + |Du_{\varepsilon, \sigma, \tau, \kappa}|^2}} \right) + \frac{\kappa}{\sqrt{\varepsilon^2 + |Du_{\varepsilon, \sigma, \tau, \kappa}|^2}} &= \sigma u_{\varepsilon, \sigma, \tau, \kappa} \quad \text{in } \mathring{K}^\tau, \\ \langle N, Du_{\varepsilon, \sigma, \tau, \kappa} \rangle &= 0 \quad \text{on } \delta K^\tau, \\ u_{\varepsilon, \sigma, \tau, \kappa} &= 0 \quad \text{on } \partial K^\tau. \end{aligned}$$

For $\kappa = 0$ the problem (4.4) has the obvious solution $u_{\varepsilon, \sigma, \tau, 0} = 0$. We will now derive the needed a priori estimates for $\kappa \in (0, 1]$. Note first that we have the sup-bound

$$(4.5) \quad 0 \leq u_{\varepsilon, \sigma, \tau, \kappa} \leq \frac{\kappa}{\sigma \varepsilon},$$

which follows directly from the maximum principle. To proceed further, we consider the graph

$$(4.6) \quad M^{\varepsilon, \sigma, \tau, \kappa} = \operatorname{graph}(u_{\varepsilon, \sigma, \tau, \kappa} / \varepsilon) \subset N \times \mathbb{R}_+.$$

We write $\eta = \frac{\partial}{\partial z}$ for the unit vector in the \mathbb{R}_+ direction, and v for the upward pointing unit normal of M (here and in the following we drop the dependence

on $(\varepsilon, \sigma, \tau, \kappa)$ in the notation when there is no risk of confusion). Written more geometrically, problem (4.4) takes the form

$$(4.7) \quad M \text{ satisfies } H + \sigma u = \kappa V \text{ with free boundary on } \partial D \times \mathbb{R}_+ \text{ and Dirichlet boundary on } \partial K^\tau \times \{0\},$$

where H is the mean curvature of $M \subset N \times \mathbb{R}_+$, and

$$(4.8) \quad V = \frac{1}{\varepsilon} \langle \eta, v \rangle = \frac{1}{\sqrt{\varepsilon^2 + |Du|^2}}.$$

We recall from [16] lemma 2.5] that

$$(4.9) \quad \Delta V = \frac{1}{\varepsilon} \langle \eta, \nabla H \rangle - (|A|^2 + \text{Rc}(v, v))V.$$

Let $d : D \rightarrow \mathbb{R}_+$ be a C^2 -function satisfying

$$(4.10) \quad d|_{\partial D} \equiv 0, \quad N(d) = -1, \quad |Dd| \leq 1, \quad |D^2d| \leq 10C_{\partial D},$$

where $C_{\partial D}$ denotes the maximal curvature of the barrier. We will sometimes tacitly view d as a function on M that is independent of the z -coordinate.

LEMMA 4.2. *The weight function $w = e^{mz-bd}$, where m and b are constants, satisfies*

$$(4.11) \quad \nabla \log w = m\eta^\top - b\nabla d$$

and

$$(4.12) \quad \Delta w = (|m\eta^\top - b\nabla d|^2 - b \text{tr}_{TM} D^2d - (m\varepsilon V - b\nu(d))H)w.$$

PROOF. The first formula is immediate. To prove the second formula choosing an orthonormal frame $\{e_i\}$ with $\nabla_{e_i} e_j = 0$ at the point in consideration, we compute

$$\begin{aligned} \Delta w &= \nabla_{e_i} (\langle m\eta - bDd, e_i \rangle w) \\ &= \langle m\eta - bDd, e_i \rangle \langle m\eta - bDd, e_i \rangle w \\ &\quad - b \langle D^2d, e_i \otimes e_i \rangle w - H \langle m\eta - bDd, v \rangle w. \end{aligned}$$

This implies the assertion. \square

LEMMA 4.3. *The function $Vw : M^{\varepsilon, \sigma, \tau, \kappa} \rightarrow \mathbb{R}_+$ satisfies*

$$(4.13) \quad N(\log(Vw)) = b + A_{\partial D}(v, v)$$

and

$$\begin{aligned} (4.14) \quad \Delta(Vw) &= 2\langle \nabla \log w, \nabla(Vw) \rangle + \frac{1}{\varepsilon} \langle \eta, \kappa \nabla V - \sigma \nabla u \rangle w \\ &\quad - (|m\eta^\top - b\nabla d|^2 + |A|^2 + \text{Rc}(v, v) + b \text{tr}_{TM} D^2d) Vw \\ &\quad - (\kappa V - \sigma u)(m\varepsilon V - b\nu(d))Vw. \end{aligned}$$

PROOF. Differentiating $\langle v, N \rangle = 0$ and using (4.7) one obtains that

$$(4.15) \quad N(V) = A_{\partial D}(v, v)V.$$

Together with $N(\log w) = b$, this yields (4.13). Using that

$$\Delta(Vw) = w\Delta V + V\Delta w + 2\langle \nabla \log w, \nabla(Vw) \rangle - 2Vw|\nabla \log w|^2,$$

the second formula follows from (4.9) and (4.12). \square

PROPOSITION 4.4. *Let $b = 2C_{\partial D}$ and $m = \max(20C_{\partial D}, 2 \max_D |\text{Rc}|^{1/2})$. Then the function $V : M^{\varepsilon, \sigma, \tau, \kappa} \rightarrow \mathbb{R}$ satisfies*

$$(4.16) \quad V(x, z) \geq e^{-b \max d} \min\left(\frac{1}{2\varepsilon}, \min_{\partial K^\tau} V\right) e^{-mz}.$$

PROOF. By equation (4.13), the normal derivative of Vw is positive at the barrier for $b = 2C_{\partial D}$. Therefore, the minimum of Vw is attained in $K^\tau \setminus \delta K^\tau$. If the minimum is attained on ∂K^τ or if the minimum is at least $\frac{1}{2\varepsilon}e^{-b \max d}$, then we are done.

Suppose now towards a contradiction that the minimum of Vw is attained in \mathring{K}^τ and is less than $\frac{1}{2\varepsilon}e^{-b \max d}$. By Lemma 4.2 and Lemma 4.3 at such an interior point we get

$$(4.17) \quad \nabla \log V = b \nabla d - m \eta^\top,$$

and

$$(4.18) \quad \begin{aligned} 0 &\geq |m \eta^\top - b \nabla d|^2 + |A|^2 + \text{Rc}(v, v) + b \text{tr}_{TM} D^2 d \\ &\quad + (\kappa V - \sigma u)(m \varepsilon V - b v(d)) + \frac{\kappa}{\varepsilon} \langle m \eta^\top - b \nabla d, \eta \rangle, \end{aligned}$$

where we also used that $\langle \nabla u, \eta \rangle = \varepsilon |\eta^\top|^2 \geq 0$. Furthermore, taking also into account the graphical identity

$$(4.19) \quad \langle \nabla d, \eta \rangle = -\varepsilon V v(d)$$

implies

$$(4.20) \quad \begin{aligned} 0 &\geq m^2 |\eta^\top|^2 + m \left(\frac{\kappa}{\varepsilon} |\eta^\top|^2 - \sigma u \varepsilon V \right) \\ &\quad + \text{Rc}(v, v) - b(2m \varepsilon V + \sigma u + 10C_{\partial D}). \end{aligned}$$

Since $\varepsilon V < \frac{1}{2}$, we have $|\eta^\top|^2 \geq \frac{3}{4}$. Hence, $m \geq 2 \max_D |\text{Rc}|^{1/2}$ yields

$$(4.21) \quad 0 \geq \frac{m^2}{2} + \frac{m\kappa}{4\varepsilon} - 2C_{\partial D} \left(m + \frac{\kappa}{\varepsilon} + 10C_{\partial D} \right),$$

but this contradicts $m \geq 20C_{\partial D}$. \square

Note that a lower bound for V is equivalent to an upper bound for $|Du|$. The next lemma provides a uniform lower bound for $\min_{\partial K^\tau} V$.

LEMMA 4.5. *There exists a constant $C = C(\varepsilon, \sigma, K) < \infty$ such that*

$$(4.22) \quad \sup_{\partial K^\tau} |Du_{\varepsilon, \sigma, \tau, \kappa}| \leq C.$$

PROOF. We argue as in the proof of [43, lemma 3.9] and seek a supersolution of the form

$$(4.23) \quad v = \alpha r \cdot g(s),$$

where r is the distance from ∂K^τ and s is the distance from ∂D . Here, $g(s) = 1 + f(\beta s)$, with $f(t)$ denoting a smooth mollification of $\max(0, 1 - t)$. We work in the region $T_\rho := \{x \in K^\tau : r(x) < \rho\}$, where $\rho > 0$ is in particular small enough to ensure that r is smooth on T_ρ . Due to the sup-bound (4.5) we have $v \geq u_{\varepsilon, \sigma, \tau, \kappa}$ on ∂T_ρ provided

$$(4.24) \quad \alpha \geq \frac{1}{\sigma \varepsilon \rho}.$$

Let $\gamma = \gamma(D, K) > 0$ be a constant such that for any $\tau \in [0, 1]$ we have $N(\log \text{dist}(\cdot, \partial K^\tau)) \geq -\gamma$ on ∂K^τ . Then the normal derivative of v on the Neumann-boundary ∂T_ρ satisfies

$$(4.25) \quad N(v) = \alpha g(s)N(r) + \alpha r g'(s)N(s) > 0$$

provided

$$(4.26) \quad \beta > 4\gamma.$$

By the maximum principle, it is thus enough to show that

$$(4.27) \quad \mathcal{D}v := \text{div} \left(\frac{Dv}{\sqrt{\varepsilon^2 + |Dv|^2}} \right) + \frac{\kappa}{\sqrt{\varepsilon^2 + |Dv|^2}} - \sigma v \leq 0$$

in $\overset{\circ}{T}_\rho$ (once this is done, one concludes that $u_{\varepsilon, \sigma, \tau, \kappa} \leq v$ in T_ρ , which yields the assertion of the lemma).

Let us now show that (4.27) indeed holds for a suitable choice of constants α, β, ρ . To this end, we start by estimating

$$(4.28) \quad \mathcal{D}v \leq \frac{1}{\sqrt{\varepsilon^2 + |Dv|^2}} \left(\Delta v + 1 - \frac{1}{2(\varepsilon^2 + |Dv|^2)} \langle D|Dv|^2, Dv \rangle \right).$$

If ρ is small enough, then by the Riccati equation we have

$$(4.29) \quad \Delta r \leq -\frac{1}{2} \min_{\partial K} H.$$

Thus, we obtain

$$(4.30) \quad \begin{aligned} \Delta v &= \alpha g \Delta r + 2\alpha \langle Dr, Dg \rangle + \alpha r \Delta g \\ &\leq -\frac{\alpha}{2} \min_{\partial K} H + 2\alpha \beta f' \langle Dr, Ds \rangle + C(\beta) \alpha \rho. \end{aligned}$$

Next, we calculate

$$(4.31) \quad \frac{1}{\alpha} Dv = gDr + \beta r f' Ds$$

and

$$(4.32) \quad \frac{1}{\alpha^2} |Dv|^2 = g^2 + \beta^2 r^2 f'^2 + 2\beta r f' g \langle Dr, Ds \rangle.$$

In particular, we see that

$$(4.33) \quad |Dv|^2 \geq \frac{\alpha^2}{2}$$

for ρ small enough. Taking another derivative of (4.32), we get

$$(4.34) \quad \frac{1}{\alpha^2} D|Dv|^2 = 2\beta f' g Ds + 2\beta f' g \langle Dr, Ds \rangle Dr + \mathcal{R},$$

where the remainder satisfies $|\mathcal{R}| \leq C(\beta)\rho$. This yields

$$(4.35) \quad \left| \frac{1}{\alpha^3} \langle D|Dv|^2, Dv \rangle - 4\beta f' g^2 \langle Dr, Ds \rangle \right| \leq C(\beta)\rho.$$

Finally, using again (4.32) we see that

$$(4.36) \quad 2\alpha\beta f' \langle Dr, Ds \rangle - \frac{4\alpha^3\beta f' g^2 \langle Dr, Ds \rangle}{2(\varepsilon^2 + |Dv|^2)} \leq \frac{\beta}{\alpha} + C(\beta)\alpha\rho.$$

Putting everything together we conclude that

$$(4.37) \quad \sqrt{\varepsilon^2 + |Dv|^2} Dv \leq -\frac{\alpha}{2} \min_{\partial K} H + 1 + \frac{\beta}{\alpha} + C(\beta)\alpha\rho < 0,$$

provided we first fix β large enough, and then choose α very large and set $\rho = \frac{1}{\sigma\varepsilon\alpha}$. This proves the lemma. \square

We can now prove existence for our triple approximate problem.

THEOREM 4.6. *There exists $u_{\varepsilon, \sigma, \tau} \in C^\infty(K^\tau \setminus \partial K^\tau) \cap C^{1,\alpha}(K^\tau)$, where $\alpha = \alpha(\tau) > 0$, which solves problem (4.3).*

PROOF. As in [29, 33, 43] we work with the weighted Hölder space

$$(4.38) \quad H_{2,\alpha}^{(-1-\alpha)}(K^\tau) := \{u : \|u\|_{2,\alpha}^{(-1-\alpha)} < \infty\},$$

equipped with the norm

$$(4.39) \quad \|u\|_{2,\alpha}^{(-1-\alpha)} := \sup_{\delta>0} \delta \|u\|_{C^{2,\alpha}(\text{Int}_\delta K^\tau)},$$

where

$$(4.40) \quad \text{Int}_\delta K^\tau := \{x \in K^\tau : d(x, \partial K^\tau) \geq \delta\}.$$

It follows directly from the definitions that

$$(4.41) \quad H_{2,\alpha}^{(-1-\alpha)}(K^\tau) \subseteq C_{\text{loc}}^{2,\alpha}(\text{Int}_D K^\tau) \cap C^{1,\alpha}(K^\tau).$$

Fix $\varepsilon, \sigma, \tau > 0$, and consider

$$(4.42) \quad I := \{\kappa \in [0, 1] \mid (4.4) \text{ has a solution in } H_{2,\alpha}^{(-1-\alpha)}(K^\tau)\}.$$

We want to show that $1 \in I$, provided $\alpha = \alpha(\tau) > 0$ is sufficiently small. Since $0 \in I$, it suffices to show that $I \subseteq [0, 1]$ is open and closed.

Note that by equation (4.5), Proposition 4.4, and Lemma 4.5, we have the a priori estimate

$$(4.43) \quad \sup_{K^\tau} (u + |Du|) \leq C,$$

where $C < \infty$ is independent of κ . Since by (4.2) the corners of the domain K^τ have angles strictly less than $\pi/2$, for $\alpha = \alpha(\tau) > 0$ small enough, we can now apply Lieberman's estimates for mixed boundary value problems [29, 30], to get the a priori estimate

$$(4.44) \quad \|u\|_{2,\alpha}^{(-1-\alpha)} \leq C,$$

where $C < \infty$ is independent of κ . It follows that I is closed.

Next, observe that the linearization of (4.4) is given by

$$(4.45) \quad \begin{aligned} \mathcal{L}(v) = \operatorname{div} \left(\frac{Dv}{\sqrt{\varepsilon^2 + |Du_{\varepsilon,\sigma,\tau,\kappa}|^2}} - \frac{\langle Du_{\varepsilon,\sigma,\tau,\kappa}, Dv \rangle Du_{\varepsilon,\sigma,\tau,\kappa}}{(\varepsilon^2 + |Du_{\varepsilon,\sigma,\tau,\kappa}|^2)^{3/2}} \right) \\ - \frac{\kappa \langle Du_{\varepsilon,\sigma,\kappa}, Dv \rangle}{(\varepsilon^2 + |Du_{\varepsilon,\sigma,\tau,\kappa}|^2)^{3/2}} - \sigma v. \end{aligned}$$

By the maximum principle and the Hopf lemma the only solution of $\mathcal{L}(v) = 0$ with zero boundary conditions is $v = 0$. Together with the Fredholm alternative for mixed boundary value problems [29, 30] and the inverse function theorem, it follows that I is open.

Finally, by standard elliptic estimates, the solution is smooth away from the corners. \square

PROOF OF THEOREM 4.1 Existence of solutions $u_{\varepsilon,\sigma}$ for the problem (4.1) now follows by taking the solution $u_{\varepsilon,\sigma,\tau}$ of problem (4.3) from Theorem 4.6 and sending $\tau \rightarrow 0$. Uniqueness is a consequence of the maximum principle and the Hopf lemma. \square

4.2 Double approximate estimate for H

The goal of this subsection is to prove a lower bound for $V = H + \sigma u_{\varepsilon,\sigma}$. We start by giving a uniform sup-bound for $u_{\varepsilon,\sigma}$ in a neighborhood of ∂K .

LEMMA 4.7. *There exist constants $\delta_\star = \delta_\star(K, D) > 0$ and $C = C(K, D) < \infty$ such that $u_{\varepsilon,\sigma} \leq C$ in $K \setminus K_{\delta_\star}$.*

PROOF. We will construct a suitable supersolution. Let u be the arrival time function the free boundary mean curvature flow $\{\partial K_t\}$. Then $u : K \setminus K_{\delta_0} \rightarrow \mathbb{R}$ for δ_0 sufficiently small is smooth and satisfies

$$(4.46) \quad \operatorname{div} \left(\frac{Du}{|Du|} \right) + \frac{1}{|Du|} = 0, \quad \langle N, Du \rangle = 0,$$

and

$$(4.47) \quad C^{-1} \leq |Du| \leq C, \quad |D^2u| \leq C,$$

for some $C < \infty$. For $\delta \in (0, \delta_0)$ consider the function $\phi(t) = \frac{1}{\delta-t} - \frac{1}{\delta}$. A straightforward calculation as in [16, lemma 3.8] shows that $\phi(u)$ is a supersolution of (4.1) provided δ is small enough. Hence, by the maximum principle we conclude that

$$(4.48) \quad u_{\varepsilon, \sigma} \leq \phi(u) \leq \frac{1}{\delta} \quad \text{in } K_0 \setminus K_{\frac{\delta}{2}}.$$

This proves the lemma. \square

PROPOSITION 4.8. *Let $b = 2C_{\partial D}$. Then for $a = a(K, D) < \infty$ sufficiently large, the function $V : M^{\varepsilon, \sigma, \tau} \rightarrow \mathbb{R}$ satisfies*

$$(4.49) \quad V(x, z) \geq e^{-b \max d} \min\left(\frac{1}{2}, \min_{\partial K^\tau} V\right) e^{-\varepsilon a z}.$$

PROOF. Consider the function Vw where $w = \exp(\varepsilon a z - bd)$. Suppose towards a contradiction that the minimum of Vw is attained in $\overset{\circ}{K}{}^\tau$ and is less than $\frac{1}{2}e^{-b \max d}$. Setting $\kappa = 1$ and $m = \varepsilon a$, equations (4.18) and (4.19) imply

$$(4.50) \quad \begin{aligned} 0 &\geq (a + \varepsilon^2 a^2) |\eta^\top|^2 + |A|^2 + \text{Rc}(\nu, \nu) \\ &\quad - b(\sigma u + 2a\varepsilon^2 V - \text{tr}_{TM} D^2 d) - \varepsilon^2 a \sigma u V. \end{aligned}$$

Using $\min V < \frac{1}{2}$ we get $|\eta^\top|^2 \geq \frac{3}{4}$ and $|\sigma u| \leq \frac{1}{2} + \sqrt{n}|A|$. Hence, for a sufficiently large, the positive term $(a + \varepsilon^2 a^2) |\eta^\top|^2 + |A|^2$ dominates all other terms in (4.50), and we obtain a contradiction. \square

THEOREM 4.9. *There exist constants $a = a(K, D) < \infty$ and $c = c(K, D) > 0$ such that for all $\varepsilon, \sigma > 0$ we have the estimate*

$$(4.51) \quad H\left(x, \frac{1}{\varepsilon} u_{\varepsilon, \sigma}(x)\right) + \sigma u_{\varepsilon, \sigma}(x) \geq c e^{-a u_{\varepsilon, \sigma}(x)}$$

for all $x \in K \setminus \partial K$.

PROOF. Fix ε and σ . In Section 4.1, we have proved that $|Du_{\varepsilon, \sigma, \tau}| \leq C(\varepsilon, \sigma, K)$ in K^τ and that $u_{\varepsilon, \sigma, \tau}$ converges uniformly in K to the unique solution $u_{\varepsilon, \sigma}$ of (4.1) as $\tau \rightarrow 0$. Hence, by Lemma 4.7 we get $u_{\varepsilon, \sigma, \tau} \leq 2\Lambda$ in $K^\tau \setminus K_{\delta_*}$ for τ sufficiently small. Now due to this new $(\varepsilon, \sigma, \tau)$ -independent sup-bound, we can choose the constant α in the proof of Lemma 4.5 (for $\kappa = 1$) to be also independent of $\varepsilon, \sigma, \tau$. This in turn implies an $(\varepsilon, \sigma, \tau)$ -independent gradient bound for $u_{\varepsilon, \sigma, \tau}$ on ∂K^τ . Hence, by Proposition 4.8 we get

$$(4.52) \quad H\left(x, \frac{1}{\varepsilon} u_{\varepsilon, \sigma, \tau}(x)\right) + \sigma u_{\varepsilon, \sigma, \tau}(x) \geq c(K, D) e^{-a u_{\varepsilon, \sigma, \tau}(x)}$$

for all $x \in K^\tau$. Taking $\tau \rightarrow 0$, this estimate passes to the limit in $K \setminus \partial K$. \square

4.3 Double approximate estimate for $|A|/H$

The goal of this section is to prove the following estimate:

THEOREM 4.10. *There exist constants $a = a(K, D) < \infty$ and $C = C(K, D) < \infty$ such that for any $\delta \in (0, \delta_*)$ we have*

$$(4.53) \quad \begin{aligned} & \frac{|A|(x, \frac{1}{\varepsilon}u_{\varepsilon,\sigma}(x))}{H(x, \frac{1}{\varepsilon}u_{\varepsilon,\sigma}(x)) + \sigma u_{\varepsilon,\sigma}(x)} \\ & \leq C \left(1 + \max_{y \in \partial K_\delta} |A|(y, \frac{1}{\varepsilon}u_{\varepsilon,\sigma}(y)) \right) e^{au_{\varepsilon,\sigma}(x)} \end{aligned}$$

for all $x \in K_\delta$.

To prove Theorem 4.10 we will apply the maximum principle to the function

$$(4.54) \quad f := \frac{|B| + \Lambda \sigma u + \Theta}{Vw},$$

where $w = e^{\varepsilon az - bd}$ and $a, b, \Lambda, \Theta < \infty$ are constants to be chosen below. Here B denotes a certain perturbation of the second fundamental form; cf. Edelen [9]. To define B , fix some smooth extensions k and N of the second fundamental form and the unit normal vector of the barrier $\partial D \times \mathbb{R}_+$ to $D \times \mathbb{R}_+$. The perturbed second fundamental form is then defined by

$$(4.55) \quad B_{ij} = A_{ij} + T_{ij\nu},$$

where A is the second fundamental form of the graph of $u_{\varepsilon,\sigma}/\varepsilon$, and T is a 3-tensor on $D \times \mathbb{R}_+$ defined by

$$(4.56) \quad T(X, Y, Z) = k(X, Z)\langle Y, N \rangle + k(Y, Z)\langle X, N \rangle.$$

Since the graph of u is perpendicular to $\partial D \times \mathbb{R}_+$, we have $\langle N, \nu \rangle = 0$. Thus, we get $A(X, N) = -k(X, \nu)$ for any tangent vector X perpendicular to N , which in turn implies

$$(4.57) \quad B(X, N) = 0$$

whenever X is perpendicular to N .

LEMMA 4.11. *The perturbed second fundamental form satisfies*

$$N|B| \leq C(|B| + \sigma u + 1).$$

PROOF. Let $\{e_i\}$ be an orthonormal frame field for $T\partial M$ with $e_1 = N$. A calculation as in [9, lemma 6.1] shows that

$$(4.58) \quad \nabla_N h_{ij} = O(|A| + 1) \quad \forall i, j > 1,$$

$$(4.59) \quad \nabla_N h_{11} = k(\nu, \nu)V + O(|A| + 1).$$

Together with the fact that $b_{1j} = 0$ for $j > 1$ by (4.57), and the formula $\nabla T = DT + A * T$ (cf. [9, p. 13]), this implies the assertion. \square

Moreover, the perturbed second fundamental form controls the second fundamental form. Namely, at any point with $|B| \geq 1$ we have

$$(4.60) \quad |A| \leq C|B|$$

and

$$(4.61) \quad |\nabla A| \leq C(|B| + |\nabla B|).$$

Indeed, (4.60) follows from the fact that T is bounded, and (4.61) follows from the formula $\nabla T = DT + A * T$.

PROPOSITION 4.12. *Whenever $|B| \geq 1$, then*

$$(4.62) \quad \left(\frac{1}{\varepsilon}\nabla_{\eta^\top} - \Delta\right)|B| \leq -\frac{|\nabla B|^2 - |\nabla|B||^2}{|B|} + |B|^3 + C(1 + \sigma u)|B|^2.$$

PROOF. To begin with, by [16, eqs. (4.8), (4.9)] we have

$$(4.63) \quad \frac{1}{\varepsilon}\nabla_{\eta^\top} A = \nabla^2 H + (A^2 - \varepsilon^2 \sigma A + \text{Rm}(\nu, \cdot, \nu, \cdot))(H + \sigma u).$$

Together with Simon's identity

$$(4.64) \quad \Delta A = \nabla^2 H + A^2 H - |A|^2 A + O(1 + |A|),$$

this yields

$$(4.65) \quad \left(\frac{1}{\varepsilon}\nabla_{\eta^\top} - \Delta\right)A \leq |B|^2 B + C(1 + \sigma u)|B|^2.$$

Next, the tensor T satisfies the identities

$$(4.66) \quad \frac{1}{\varepsilon}\nabla_{\eta^\top} T_{ijv} = \partial_k V T_{ijk} - \partial_i V T_{vjk} - \partial_j V T_{ivv} + O(|B|^2),$$

$$(4.67) \quad \Delta T_{ijv} = \partial_k H T_{ijk} - \partial_i H T_{vjk} - \partial_j H T_{ivv} + O(|B|^2);$$

cf. [9, p. 14]. Taking the difference and using $\nabla u = \varepsilon\eta^\top = O(1)$, we infer that

$$(4.68) \quad \left(\frac{1}{\varepsilon}\nabla_{\eta^\top} - \Delta\right)T \leq C|B|^2.$$

Combining (4.65) and (4.68), we conclude that

$$(4.69) \quad \frac{1}{2}\left(\frac{1}{\varepsilon}\nabla_{\eta^\top} - \Delta\right)|B|^2 \leq -|\nabla B|^2 + |B|^4 + C(1 + \sigma u)|B|^3.$$

From this the claim follows. \square

LEMMA 4.13. *There exists a constant $\gamma = \gamma(n) < 1$ such that*

$$(4.70) \quad |\nabla|B|| \leq \gamma|\nabla B| + 2|\nabla H| + C(1 + |A|).$$

PROOF. The proof is essentially the same as the one for [16, prop. 4.12]. The only change is that due to $|\nabla T| = O(1 + |A|)$, we need to add $O(1 + |A|)$ to the right-hand sides of [16, eqs. (4.16), (4.18)]. The rest of the argument remains intact. \square

PROOF OF THEOREM 4.10. Consider the function f as defined in (4.54). The constants a, b, Λ, Θ will be chosen below. Throughout the proof, we tacitly assume that ε and σ are small enough such that

$$(4.71) \quad \varepsilon a < 1, \quad \sigma \Lambda < 1.$$

Using Lemma 4.11 we see that

$$(4.72) \quad N(\log f) \leq \frac{C(|B| + \sigma u + 1)}{|B| + \Lambda \sigma u + \Theta} - k(v, v) - b < 0$$

provided b is large enough. Hence the maximum of f over K_δ is either attained in the interior \mathring{K}_δ or on the Dirichlet boundary ∂K_δ . Suppose now the maximum is attained at $x_0 \in \mathring{K}_\delta$, and let $z_0 = \frac{1}{\varepsilon} u_{\varepsilon, \sigma}(x_0)$.

CLAIM 4.14. *There exists $\delta = \delta(n) > 0$ such that at (x_0, z_0) we have*

$$(4.73) \quad \left(\frac{1}{\varepsilon} \nabla_{\eta^\top} - \Delta\right) |B| \leq -\delta \frac{|\nabla|B||^2}{|B|} + |B|^3 + C(1 + \sigma u)|B|^2$$

provided that $|B| \geq \max\{1, \frac{4}{1-\gamma} V\}$ at (x_0, z_0) .

PROOF. At (x_0, z_0) we have $\nabla \log f = 0$, or equivalently

$$(4.74) \quad \nabla H = \left(\frac{\nabla|B| + \Lambda \sigma \varepsilon \eta^\top}{|B| + \Lambda \sigma u + \Theta} - \varepsilon a \eta^\top + b \nabla d \right) V - \varepsilon \sigma \eta^\top.$$

Using (4.71) and $|B| \geq \max\{1, \frac{4}{1-\gamma} V\}$ we deduce that

$$(4.75) \quad |\nabla H| \leq \frac{1-\gamma}{4} |\nabla B| + C|B|.$$

Therefore, in view of Lemma 4.13 we get the estimate

$$(4.76) \quad |\nabla B|^2 - |\nabla|B||^2 \geq \delta |\nabla|B||^2 - C|B|^2,$$

where $\delta = \delta(n) > 0$. Together with Proposition 4.12 this implies the claim. \square

Continuing the proof of the theorem, we compute

$$(4.77) \quad \left(\frac{1}{\varepsilon} \nabla_{\eta^\top} - \Delta\right) u \leq 1$$

and

$$(4.78) \quad \begin{aligned} \left(\frac{1}{\varepsilon} \nabla_{\eta^\top} - \Delta\right)(Vw) &\geq (|B|^2 + a - C(|B| + \sigma u))Vw \\ &\quad - 2\langle \nabla \log w, \nabla(Vw) \rangle; \end{aligned}$$

cf. Lemma 4.3.

Since $0 \leq \left(\frac{1}{\varepsilon} \nabla_{\eta^\top} - \Delta\right)f - 2\langle \nabla \log(Vw), \nabla f \rangle$ at (x_0, z_0) , combining the inequalities (4.73), (4.77), and (4.78) we obtain

$$(4.79) \quad \begin{aligned} 0 &\leq \frac{1}{Vw} \left(-\delta \frac{|\nabla|B||^2}{|B|} + |B|^3 + C\sigma u|B|^2 \right) + Cf(|B| + \sigma u) \\ &\quad - f(|B|^2 + a - 2\langle \nabla \log w, \nabla \log(Vw) \rangle) \end{aligned}$$

provided that $|B| \geq \left\{1, \frac{4}{1-\gamma}V\right\}$. Due to $|\nabla u| \leq 1$, the gradient term can be estimated by

$$\begin{aligned} 2f|\langle \nabla \log w, \nabla \log(Vw) \rangle| &= \frac{2}{Vw}|\langle \nabla \log w, \nabla(|B| + \Lambda\sigma u) \rangle| \\ (4.80) \quad &\leq \frac{1}{Vw}\left(\delta \frac{|\nabla|B||^2}{|B|} + C|B|\right). \end{aligned}$$

Using this, and assuming $|B| \geq \max\{1, \frac{4}{1-\gamma}V, \frac{\sigma}{2n}u\}$, we infer that

$$(4.81) \quad 0 \leq (C - \Lambda)\sigma u|B|^2 + (C\Lambda - \Theta)|B|^2 + (C\Theta - a)|B|,$$

which yields a contradiction, provided $\Lambda > C$, $\Theta > C\Lambda$ and $a > C\Theta$.

So far we have shown that there exist constants a, b, Λ, Θ , depending only on D and K , such that at any interior maximum of f we have

$$(4.82) \quad |B| \leq \max\left\{1, \frac{4}{1-\gamma}V, \frac{\sigma}{2n}u\right\}.$$

The last part of the proof is to show that $f(x_0, z_0)$ is uniformly bounded. To this end, let us point out first that (4.82) obviously implies

$$(4.83) \quad |B| \leq \max\left\{1 + 2|\text{tr } T|, \frac{4}{1-\gamma}V, \frac{\sigma}{2n}u\right\}.$$

If the maximum on the right-hand side of (4.83) equals $1 + 2|\text{tr } T|$, then we get

$$(4.84) \quad f(x_0, z_0) \leq \frac{(1 + 2n\Lambda)(1 + 2|\text{tr } T|) + \Theta}{Vw} \leq C,$$

since Vw is bounded below by Theorem 4.9. If the maximum on the right-hand side of (4.83) equals $\frac{4}{1-\gamma}V$, then

$$(4.85) \quad f(x_0, z_0) \leq \frac{4 + 8n}{(1-\gamma)w} + \frac{\Theta}{Vw} \leq C,$$

since w and Vw are bounded below. Finally, if the maximum on the right-hand side of (4.83) equals $\frac{\sigma}{2n}u$, then $|\text{tr } T| \leq \frac{1}{4}\sigma u$ and $|H + \text{tr } T| \leq \frac{\sigma}{2}u$. This implies $V \geq \frac{\sigma}{4}u$; hence again

$$(4.86) \quad f(x_0, z_0) \leq \frac{4(\Lambda + 1)}{w} + \frac{\Theta}{Vw} \leq C.$$

Since $\sigma u \leq C$ on $K \setminus K_{\delta_*}$ by Lemma 4.7 this finishes the proof of the theorem. \square

4.4 Passing to limits and one-sided minimization

We will first send $\sigma \rightarrow 0$, then prove one-sided minimization, and then send $\varepsilon \rightarrow 0$.

LEMMA 4.15. *There exists a relatively open set $\Omega^\varepsilon \subseteq K$ containing ∂K such that the following holds. The (improper) limit⁴ $u_{\varepsilon,\sigma} \rightarrow u_\varepsilon$ as $\sigma \rightarrow 0$ is in $C_{\text{loc}}^{0,1}(\Omega^\varepsilon) \cap C_{\text{loc}}^\infty(\Omega^\varepsilon \setminus \partial K)$ and solves*

$$\begin{aligned}
 \text{div} \left(\frac{Du_\varepsilon}{\sqrt{\varepsilon^2 + |Du_\varepsilon|^2}} \right) + \frac{1}{\sqrt{\varepsilon^2 + |Du_\varepsilon|^2}} &= 0 && \text{in } \overset{\circ}{\Omega}{}^\varepsilon, \\
 \langle N, Du_\varepsilon \rangle &= 0 && \text{on } \partial \Omega^\varepsilon, \\
 u_\varepsilon &= 0 && \text{on } \partial K, \\
 u_\varepsilon(x) &\rightarrow \infty && x \rightarrow \partial \Omega^\varepsilon \setminus \partial K.
 \end{aligned} \tag{4.87}$$

Moreover, for $x \in K \setminus \partial K$ we have estimates

$$H(x, \frac{1}{\varepsilon} u_\varepsilon(x)) \geq c e^{-a u_\varepsilon(x)}, \tag{4.88}$$

where $c = c(K, D) > 0$ and $a = a(K, D) < \infty$, and for any $\delta \in (0, \delta_*)$ for $x \in K_\delta$ we have the estimate

$$\frac{|A|}{H}(x, \frac{1}{\varepsilon} u_\varepsilon(x)) \leq C \left(1 + \sup_{\partial K_\delta} \frac{|A|}{H}(y, \frac{1}{\varepsilon} u_\varepsilon(y)) \right) e^{\rho u_\varepsilon(x)}, \tag{4.89}$$

where $C = C(K, D) < \infty$ and $\rho = \rho(K, D) < \infty$.

PROOF. Suppose $\sigma_1 \leq \sigma_2$. Since u_{ε,σ_1} is a supersolution of (4.1) for $\sigma = \sigma_2$, by the maximum principle we have $u_{\varepsilon,\sigma_1} \geq u_{\varepsilon,\sigma_2}$. Therefore we can take a pointwise (improper) limit $u_\varepsilon(x) = \lim_{\sigma \rightarrow 0} u_{\varepsilon,\sigma}(x) \in [0, \infty]$ for each $x \in K$. Obviously $u_\varepsilon = 0$ on ∂K . Define

$$\Omega^\varepsilon = \{x \in K : u_\varepsilon(x) < \infty\}. \tag{4.90}$$

By Theorem 4.9, we have the uniform Lipschitz estimate

$$|De^{-a u_{\varepsilon,\sigma}}| \leq C; \tag{4.91}$$

hence $\Omega^\varepsilon \subseteq K$ is open, $u_\varepsilon(x) \rightarrow \infty$ as $x \rightarrow \partial \Omega^\varepsilon \setminus \partial K$, and (4.88) holds. Finally, by standard elliptic estimates (cf. the proof of Theorem 4.6), the convergence $u_{\varepsilon,\sigma} \rightarrow u_\varepsilon$ is in $C_{\text{loc}}^\infty(\Omega^\varepsilon \setminus \partial K)$, and passing the estimates from Theorem 4.9 and Theorem 4.10 to the limit yields (4.88) and (4.89). \square

Consider

$$L_t^\varepsilon := \{(x, z) \in \Omega^\varepsilon \times \mathbb{R} : \varepsilon z \leq u_\varepsilon(x) - t\}. \tag{4.92}$$

The geometric meaning of (4.87) is that $\{L_t^\varepsilon\}$ is a smooth self-similar solution of the free boundary mean curvature flow, translating downwards with speed $1/\varepsilon$.

⁴Here, the word *improper* just indicates that $u_\varepsilon(x) = \infty$ for $x \in K \setminus \Omega^\varepsilon$.

PROPOSITION 4.16. *The sets L_t^ε are one-sided minimizing. Namely, given any compact set $E \subset \text{Int}_{D \times \mathbb{R}} L_0^\varepsilon$ we have the estimate*

$$(4.93) \quad \mathcal{H}^{n+1}(\partial L_t^\varepsilon \cap E) \leq \mathcal{H}^{n+1}(\partial F \cap E)$$

whenever $L_t^\varepsilon \subseteq F$ and $F \setminus L_t^\varepsilon \subseteq E$.

PROOF. Since $\{L_t^\varepsilon\}_{t>0}$ is mean-convex, we can use the unit normal ν as calibration and evaluate

$$(4.94) \quad 0 \leq \int_{F \setminus L_t^\varepsilon} \text{div } \nu \, d\mathcal{H}^{n+2}$$

using the divergence theorem. In general, the boundary of $F \setminus L_t^\varepsilon$ consists of three parts. By the free boundary condition, the part contained in $\partial D \times \mathbb{R}$ does not contribute to the flux integral. From this, the assertion follows. \square

PROOF OF THEOREM 1.6. Let $u_\varepsilon : \Omega^\varepsilon \rightarrow \mathbb{R}$ be as in Lemma 4.15. By (4.88), we have

$$(4.95) \quad |De^{-au_\varepsilon}| \leq C.$$

Thus, for any sequence $\varepsilon_k \rightarrow 0$ there is a subsequence $\varepsilon_{k'} \rightarrow 0$ and a relatively open set $\Omega \subseteq K$ containing a neighborhood of ∂K such that $u_{\varepsilon_{k'}} \rightarrow \hat{u}$ locally uniformly in Ω and $u_{\varepsilon_{k'}}(x) \rightarrow \infty$ as $x \rightarrow \partial\Omega \setminus \partial K$. Since \hat{u} arises as a limit of locally uniform Lipschitz functions, it solves

$$\begin{aligned} \text{div} \left(\frac{D\hat{u}}{|D\hat{u}|} \right) + \frac{1}{|D\hat{u}|} &= 0 \quad \text{in } \overset{\circ}{\Omega}, \\ \langle N, D\hat{u} \rangle &= 0 \quad \text{on } \partial\Omega, \\ \hat{u} &= 0 \quad \text{on } \partial K, \end{aligned}$$

in the viscosity sense. By the definition of viscosity solutions, the family of closed sets $\hat{M}_t := \{x \in K : \hat{u}(x) = t\}$ satisfies the avoidance principle, and thus is a set-theoretic subsolution of the mean curvature flow with free boundary. Hence, by the same argument as in [16, p. 1154] the limit \hat{u} agrees with the arrival time function u of the free boundary level set flow. In particular, $\Omega = K \setminus \bigcap_{t \geq 0} K_t$, and the subsequential convergence entails a full limit.

Recall that $\{L_t^\varepsilon\}$ as defined in (4.92) is a smooth self-similar solution of the free boundary mean curvature flow in $D \times \mathbb{R}$, translating downwards with speed $1/\varepsilon$. The arrival time function of $\{L_t^\varepsilon\}$ is given by $U_\varepsilon(x, z) = u_\varepsilon(x) - \varepsilon z$ and converges locally uniformly to $U(x, z) = u(x)$, which is the arrival time function of $\{K_t \times \mathbb{R}\}$. Hence, for any $\delta > 0$ the flow $\{L_t^\varepsilon\}_{t \geq \delta}$ converges to $\{K_t \times \mathbb{R}\}_{t \geq \delta}$ in the strong Hausdorff sense as defined in [18, def. 4.10], i.e., the space-time track

$$(4.96) \quad \mathcal{L}_\delta^\varepsilon = \bigcup_{t \geq \delta} (\{(x, z) \in D \times \mathbb{R} : U_\varepsilon(x, z) \geq t\} \times \{t\})$$

Hausdorff converges to

$$(4.97) \quad \mathcal{K}_\delta = \overline{\bigcup_{t \geq \delta} (\{(x, z) \in D \times \mathbb{R} : u(x) \geq t\} \times \{t\})},$$

and the closure of the complement of the space-time track, namely

$$(4.98) \quad \overline{(\mathcal{L}_\delta^\varepsilon)^c} = \overline{\bigcup_{t \geq \delta} (\{(x, z) \in D \times \mathbb{R} : U_\varepsilon(x, z) < t\} \times \{t\})}.$$

Hausdorff converges to

$$(4.99) \quad \overline{(\mathcal{K}_\delta)^c} = \overline{\bigcup_{t \geq \delta} (\{(x, z) \in D \times \mathbb{R} : u(x) < t\} \times \{t\})}.$$

Next, we will pass to a limit in the sense of free-boundary Brakke flows and relate this to the strong Hausdorff limit from above. The following argument uses some basic definitions and facts about free-boundary Brakke flows that will be reviewed in Section 5 below.

By the compactness theorem for Brakke flow with free boundary [10], which can be applied thanks to the uniform area-ratio bounds implied by Proposition 4.16, for any sequence $\varepsilon_k \rightarrow 0$ we can pass to a subsequence so that $\mathcal{M}^k = \{\tilde{\mu}_t^k := \mathcal{H}^{n+1} \llcorner \partial L_t^{\varepsilon_k}\}_{t \geq 0}$ converges in $\{t > 0\}$ in the sense of free-boundary Brakke flows to an $(n+1)$ -dimensional unit-regular free-boundary Brakke flow $\tilde{\mathcal{M}} = \{\tilde{\mu}_t\}_{t > 0}$ in $D \times \mathbb{R}$.

We now want to relate the Dirichlet boundary of

$$(4.100) \quad \tilde{\mathcal{K}} := \bigcup_{t \geq 0} (K_t \times \{t\})$$

with the support of $\tilde{\mathcal{M}}$. Convergence as free-boundary Brakke flows implies that $\text{spt}(\mathcal{M}^k)$ Hausdorff converges to $\text{spt}(\tilde{\mathcal{M}})$. On the other hand, $\partial \mathcal{L}^{\varepsilon_k}$ Hausdorff converges to $\partial \mathcal{K}$ thanks to the strong Hausdorff convergence from above. Since $\text{spt}(\mathcal{M}^k) = \partial \mathcal{L}^{\varepsilon_k}$, we infer that

$$(4.101) \quad \text{spt}(\tilde{\mathcal{M}}) = \partial \tilde{\mathcal{K}}.$$

Next, we can pass the one-sided minimization property from Proposition 4.16 to the limit (cf. [38] sec. 5), which yields

$$(4.102) \quad \partial^* K_t \text{ is one-sided minimizing for every } t > 0.$$

Together with (4.101), by the same argument as in [38, sec. 5] this implies

$$(4.103) \quad \tilde{\mu}_t = \mathcal{H}^{n+1} \llcorner (\partial^* K_t \times \mathbb{R}) \quad \text{for a.e. } t.$$

Furthermore, thanks to our Lipschitz estimate for the arrival time, comparing the co-area formula for the reduced boundary and the topological boundary (cf. again [38, sec. 5]) gives

$$(4.104) \quad \mathcal{H}^{n+1} \llcorner (\partial^* K_t \times \mathbb{R}) = \mathcal{H}^{n+1} \llcorner (\partial K_t \times \mathbb{R}) \quad \text{for a.e. } t.$$

In fact, we can improve (4.103) as follows. Given any ball $B_r(x) \subset \text{Int}_D K_0$, for every $s > t > 0$ we get by one-sided minimization

$$\mathcal{H}^n(\partial^* K_s \cap B_r(x)) \leq \mathcal{H}^n(\partial^* K_t \cap B_r(x)) + \mathcal{H}^n(\partial B_r(x) \cap (K_t \setminus K_s)).$$

Since $K_{t+h} \rightarrow K_t$ in L^1 as $h \rightarrow 0$ (a consequence of nonfattening), we deduce that for every $t > 0$ we have the inequalities

$$(4.105) \quad \lim_{i \rightarrow \infty} \mathcal{H}^n \llcorner \partial^* K_{t+h_i} \leq \mathcal{H}^n \llcorner \partial^* K_t \leq \lim_{i \rightarrow \infty} \mathcal{H}^n \llcorner \partial^* K_{t-h_i},$$

whenever the limits on the left/right-hand side exist (here $h_i > 0$). On the other hand, by standard semidecreasing properties for free-boundary Brakke flows ([10, prop. 4.13]), the limits $\lim_{s \rightarrow t^\pm} \tilde{\mu}_s$ exist for every $t > 0$, and agree on a co-countable set of times. Hence,

$$\lim_{s \rightarrow t^+} \tilde{\mu}_s \leq \mathcal{H}^{n+1} \llcorner (\partial^* K_t \times \mathbb{R}) \leq \lim_{s \rightarrow t^-} \tilde{\mu}_s.$$

Therefore, we can replace $\tilde{\mu}_t$ with $\mathcal{H}^{n+1} \llcorner (\partial^* K_t \times \mathbb{R})$ whenever they don't agree (which is only a countable collection of times), and still have a unit-regular free-boundary Brakke flow. Moreover, the support $\text{spt } \tilde{\mathcal{M}}$ does not change.

Splitting off a line, the above shows that $\mathcal{M} = \{\mu_t := \mathcal{H}^n \llcorner \partial^* K_t\}_{t>0}$ is a unit-regular free-boundary Brakke flow with $\text{spt } \mathcal{M} = \partial \mathcal{K}$ on $\{t > 0\}$. Since $\{\partial K_t\}_t$ is a smooth free-boundary flow for short time $t \in [0, \delta^*)$, we have $\lim_{t \rightarrow 0} \mu_t = \mathcal{H}^n \llcorner \partial^* K_0 = \mathcal{H}^n \llcorner \partial K_0$ and $\text{spt } \mathcal{M} = \partial \mathcal{K}$.

Finally, passing the estimates (4.88) and (4.89) to the limit at smooth points via the local regularity theorem [10, 47], we infer that

$$(4.106) \quad \inf_{\partial K_t^{\text{reg}}} H \geq c e^{-at} \quad (t > 0),$$

and, taking also into account that ∂K_δ is smooth with bounded curvature for $\delta > 0$ small enough, that

$$(4.107) \quad \sup_{\partial K_t^{\text{reg}}} \frac{|A|}{H} \leq C e^{\rho t} \quad (t \geq \delta).$$

This finishes the proof of Theorem 1.6. □

5 Free Boundary Brakke Flow and Limit Flows

For ease of notation let us pretend that $(N^{n+1}, g) = (\mathbb{R}^{n+1}, \delta)$ (everything generalizes in a straightforward way to other ambient manifolds).

Let $K_t = F_t(K)$ be a free boundary level set flow with smooth, strictly mean-convex initial data $K \subset D$. As before, we write

$$(5.1) \quad \mathcal{K} = \bigcup_{t \geq 0} K_t \times \{t\} \subset D \times \mathbb{R}_+$$

for its space-time track. By Theorem 1.6 (elliptic regularization and consequences), we can also consider the associated free boundary Brakke flow

$$(5.2) \quad \mathcal{M} = \{\mathcal{H}^n \llcorner \partial^* K_t\}_{t \geq 0}.$$

We recall that the pair $(\mathcal{M}, \mathcal{K})$ is called a *mean-convex free boundary flow*.

5.1 Free boundary Brakke flow

As in [10, sec. 3], for any Borel-measurable vector field X , define the vector field

$$(5.3) \quad X_* = X - 1_{\partial D} \langle X, N \rangle N.$$

We recall from [10, sec. 4] that in general a free boundary Brakke flow $\mathcal{M} = \{\mu_t\}_{t \geq 0}$ is given by a family of Radon measures in D that are integer n -rectifiable for almost all times such that

$$(5.4) \quad \frac{d}{dt} \int \phi \, d\mu_t \leq \int (-\phi H_*^2 + \langle \nabla \phi, H_* \rangle + \partial_t \phi) \, d\mu_t$$

for all nonnegative C^1 -functions ϕ . Here $\frac{d}{dt}$ denotes the limsup of difference quotients, and it is assumed that for almost every time the first variation of the associated varifold V_{μ_t} is represented by a function $H \in L^2((D, \mu_t); \mathbb{R}^{n+1})$, namely,

$$(5.5) \quad \delta V_{\mu_t}(X) = - \int H_* \cdot X \, d\mu_t$$

for all C^1 -vector fields X that are tangential along ∂D ⁵

Remark 5.1. Thanks to Theorem 1.6 (elliptic regularization and consequences) all free boundary Brakke flows that we encounter in the present paper, or more precisely their stabilized version obtained by crossing with a line, are limits of smooth free boundary flows.

Let $(\mathcal{M}, \mathcal{K})$ be a mean-convex free boundary flow. The *support* of \mathcal{M} consists by definition of all points $X = (x, t)$ that have Gaussian density (see Section 5.3 below) at least 1. We write $\partial \mathcal{K} := \mathcal{K} \setminus \text{Int}_{D \times \mathbb{R}} \mathcal{K}$ for the Dirichlet boundary of \mathcal{K} as a subset of space-time.

PROPOSITION 5.2 (support). *If $(\mathcal{M}, \mathcal{K})$ is a free boundary Brakke flow, then the support of \mathcal{M} is equal to $\partial \mathcal{K}$ for $t > 0$.*

PROOF. This already has been shown in the proof of Theorem 1.6. □

⁵In particular, by (5.5), V_{μ_t} has free boundary in the sense of integral varifolds.

5.2 Blowup sequences and limit flows

Let $(\mathcal{M}, \mathcal{K})$ be a mean-convex free boundary flow in D . Given $X_i = (x_i, t_i) \in \partial\mathcal{K}$ (with $\liminf_{i \rightarrow \infty} t_i > 0$ and $\limsup_{i \rightarrow \infty} t_i < \infty$) and $\lambda_i \rightarrow \infty$, we consider the *blowup sequence* $(\mathcal{M}^i, \mathcal{K}^i)$, which is obtained from $(\mathcal{M}, \mathcal{K})$ by translating X_i to the origin and parabolically rescaling by λ_i . After passing to a subsequence, we can assume that either

$$(5.6) \quad \lim_{i \rightarrow \infty} \lambda_i d(x_i, \partial D) = \infty \quad (\text{interior case}),$$

or

$$(5.7) \quad \lim_{i \rightarrow \infty} \lambda_i d(x_i, \partial D) < \infty \quad (\text{boundary case}).$$

Note that $(\mathcal{M}^i, \mathcal{K}^i)$ is defined in the domain D^i , which is obtained from D by shifting x_i to the origin and rescaling by λ_i . For $i \rightarrow \infty$ the domains D^i converge locally smoothly to \mathbb{R}^{n+1} in the interior case, and to a half-space, which we denote by \mathbb{H} , in the boundary case. By the area bounds from one-sided minimization (Theorem 1.6) and the compactness theorem for free boundary Brakke flows [10, theorem 4.10] after passing to a subsequence, we can assume that \mathcal{M}^i converges to a limit \mathcal{M}' , which is a Brakke flow in the interior case and a free boundary Brakke flow in the boundary case. After passing to a further subsequence we can also assume that \mathcal{K}^i converges in the Hausdorff sense to a limit \mathcal{K}' . Any such pair $(\mathcal{M}', \mathcal{K}')$ is called a *limit flow*.

In the boundary case, when $(\mathcal{M}', \mathcal{K}')$ is a free boundary flow in \mathbb{H} , then one can reflect $(\mathcal{M}', \mathcal{K}')$ around the planar barrier to obtain a boundaryless flow $(\widetilde{\mathcal{M}}', \widetilde{\mathcal{K}}')$ (cf. [10, prop. 4.4]), called the *reflected limit flow*.

PROPOSITION 5.3 (Characterization of planar limit flows). *Let $(\mathcal{M}, \mathcal{K})$ be a mean-convex free boundary flow in D , and let $(\mathcal{M}^i, \mathcal{K}^i)$ be a blowup sequence converging to a limit flow $(\mathcal{M}', \mathcal{K}')$. Suppose $\text{spt } \mathcal{M}'$ is contained in some static plane. Then one of the following six cases occurs:*

- (1) \mathcal{K}' is a static half-space in \mathbb{R}^{n+1} , and \mathcal{M}' is the static plane $\partial\mathcal{K}'$.
- (2) \mathcal{K}' is a (quasi)static plane in \mathbb{R}^{n+1} , and \mathcal{M}' is the (quasi)static plane $\partial\mathcal{K}'$ with multiplicity 2.
- (3) \mathcal{K}' is a static quarter-space in \mathbb{H} , and \mathcal{M}' is the static half-plane $\partial\mathcal{K}'$ with multiplicity 1.
- (4) \mathcal{K}' is a (quasi)static half-plane in \mathbb{H} , and \mathcal{M}' is a (quasi)static half-plane $\partial\mathcal{K}'$ with multiplicity 2.
- (5) \mathcal{K}' is a static slab in \mathbb{H} containing $\partial\mathbb{H}$, and \mathcal{M}' is the static plane $\partial\mathcal{K}'$ with multiplicity 1.
- (6) \mathcal{K}' is the (quasi)static plane coincident with $\partial\mathbb{H}$, and \mathcal{M}' is the (quasi)static plane coincident with $\partial\mathbb{H}$ having multiplicity 1.

PROOF. We adapt the proof from [45, theorem 5.4] to our setting. The hypothesis implies that

$$(5.8) \quad \text{spt } \mathcal{M}' = P \times (-\infty, T^*]$$

where P is either a plane or a free boundary half-plane, and $T^* \leq \infty$. Since $\text{spt } \mathcal{M}^i = \partial \mathcal{K}^i$ it follows that $\mathcal{K}' = \lim \mathcal{K}^i$ must be either

- (a) $Q \times (-\infty, T^*]$, where Q is either a half-space bounded by P , or a quarter-space or slab bounded by P and $\partial \mathbb{H}$, or
- (b) $P \times (-\infty, T^*]$.

Let us first suppose that our limit flow is defined in entire \mathbb{R}^{n+1} . Then, in scenario (a) the only possibility is that Q is a half-space and $T^* = \infty$, and by one-sided minimization the multiplicity of $P = \partial Q$ must be 1; i.e., we are in case (1). In scenario (b) by one-sided minimization the multiplicity of P is at most 2. If the multiplicity was equal to 1, then by the local regularity theorem [47] the convergence would be smooth and \mathcal{K}' would have nonempty interior, contradicting (b). Hence, the multiplicity must be 2 and we are in case (2).

Suppose now our limit flow is defined in a half-space \mathbb{H} and P is a free boundary half-plane. Then, arguing as above, with the only difference that we now also use the free boundary version of the local regularity theorem [10], we see that we are in case (3) or (4).

Suppose now our limit flow is defined in a half-space \mathbb{H} and P is a plane, which must be necessarily parallel to $\partial \mathbb{H}$.

If $P \neq \partial \mathbb{H}$, then a priori there are the following three possibilities:

- (i) $\mathcal{K}' = Q \times (-\infty, T^*]$, where Q is the slab between P and $\partial \mathbb{H}$.
- (ii) $\mathcal{K}' = Q \times (-\infty, T^*]$, where Q is half-space bounded by P .
- (iii) $\mathcal{K}' = P \times (-\infty, T^*]$.

In scenario (i) by one-sided minimization the multiplicity of P must be 1, and we are in case (5). In scenario (ii) by one-sided minimization the multiplicity of P would be 0⁶ contradicting the fact that the multiplicity is at least 1. Hence, scenario (ii) cannot occur. In scenario (iii) by one-sided minimization the multiplicity of P would be 1. But then by the local regularity theorem [10, 47] the convergence would be smooth and \mathcal{K}' would have nonempty interior; a contradiction. Hence, scenario (iii) cannot occur either.

If $P = \partial \mathbb{H}$, then the above possibilities (i) and (iii) degenerate into the same possibility; i.e., a priori we have the two following possibilities:

- (i') $\mathcal{K}' = P \times (-\infty, T^*]$.
- (ii') $\mathcal{K}' = Q \times (-\infty, T^*]$, where Q is a half-space bounded by P .

In scenario (i') by one-sided minimization the multiplicity must be 1, and we are in case (6). Finally, scenario (ii') can be ruled out similarly as scenario (ii). This concludes the proof of the proposition. \square

⁶ After doubling, this corresponds to the case $k = 2$ in [45, theorem 3.9].

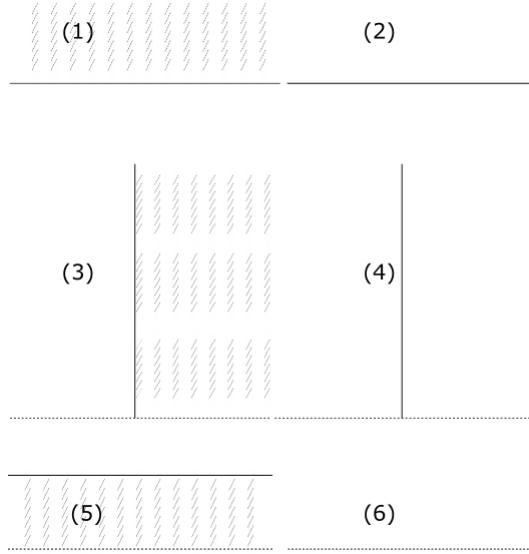


FIGURE 5.1. Planar limit flows of Proposition 5.3

THEOREM 5.4 (Properties of limit flows). *Let $(\mathcal{M}, \mathcal{K})$ be a mean-convex free boundary flow in D , and let $(\mathcal{M}^i, \mathcal{K}^i)$ be a blowup sequence converging to a limit flow $(\mathcal{M}', \mathcal{K}')$. Then:*

- (1) \mathcal{K}' is weakly mean-convex; i.e., $K'_{t_2} \subseteq K'_{t_1}$ whenever $t_2 \geq t_1$.
- (2) The support of \mathcal{M}' equals $\partial\mathcal{K}'$.
- (3) The sets $\partial\mathcal{K}^i$ Hausdorff converge to $\partial\mathcal{K}'$.
- (4) $(\mathcal{M}', \mathcal{K}')$ is one-sided minimizing.

PROOF. The first assertion is trivial. Assertions (2) and (3) follow as in [45, proof of theorem 5.5], using Proposition 5.3 in place of [45, theorem 5.4]. Assertion (4) follows from the one-sided-minimization of \mathcal{K}^i as in [45, theorem 6.1]. \square

5.3 Tangent flows and Gaussian density

For $x \in D$ close enough to ∂D there is a well-defined projection $\xi(x)$ to the nearest point in ∂D . Denote by

$$(5.9) \quad \tilde{x} = 2\xi(x) - x$$

the point that is obtained by reflection across ∂D . Using this, one can define the almost monotone quantity

$$(5.10) \quad \Theta_{\text{refl}(\partial D)}(\mathcal{M}, X, r)$$

as in [10, def. 5.1.1], which interpolates between Huisken's monotone quantity in the interior [20] and an almost monotone reflected quantity close to the boundary.

Let X in the support of $(\mathcal{M}, \mathcal{K})$ be fixed, and $\lambda_i \rightarrow \infty$. Let $(\mathcal{M}^i, \mathcal{K}^i)$ be the sequence of flows that is obtained from $(\mathcal{M}, \mathcal{K})$ by translating X to the origin and parabolically rescaling by λ_i . Any subsequential limit $(\mathcal{M}', \mathcal{K}')$ is called a *tangent flow at X* . By the almost monotonicity formula from [10, theorem 5.1], tangent flows are always backwardly self-similar, i.e., $(\mathcal{M}', \mathcal{K}') \cap \{t \leq 0\}$ is invariant under parabolic dilation. In particular, they have a well-defined *reflected Gaussian density*

$$(5.11) \quad \Theta_{\text{refl}(\partial D)}(\mathcal{M}, X) := \lim_{r \rightarrow 0} \Theta_{\text{refl}(\partial D)}(\mathcal{M}, X, r).$$

Tangent flows are either *shrinking*, *static*, or *quasi-static*; see [44].

If $(\mathcal{M}', \mathcal{K}')$ is a limit flow at $X \in \partial D$, which is defined in a half-space, and $(\tilde{\mathcal{M}}', \tilde{\mathcal{K}}')$ is the doubled flow, then by [10, theorem 6.4 and lemma 7.1] we have

$$(5.12) \quad \text{Ent}[\tilde{\mathcal{M}}'] \leq \Theta_{\text{refl}(\partial D)}(\mathcal{M}, X),$$

with equality in the case of tangent flows. Here,

$$(5.13) \quad \text{Ent}[\tilde{\mathcal{M}}'] = \lim_{t \rightarrow -\infty} \int \frac{1}{(4\pi|t|)^{n/2}} e^{-\frac{|x|^2}{4|t|}} d\mu'_t(x)$$

denotes the *entropy* (aka density at ∞) of $\tilde{\mathcal{M}}'$.

6 Multiplicity 1

The goal of this section is to prove that static and quasi-static density-2 planes and half-planes cannot occur as tangent flows or limit flows (note also that planes and half-planes of density ≥ 3 are immediately ruled out by one-sided minimization).

6.1 Limit flows with entropy at most 2

In this section, we consider the following class of limit flows.

DEFINITION 6.1 (Class of limit flows). Let $(\mathcal{M}, \mathcal{K})$ be a mean-convex flow in D . Let \mathcal{C} be the class of all limit flows $(\mathcal{M}', \mathcal{K}')$ such that

- (1) $\text{Ent}[\mathcal{M}'] \leq 2$ in case $(\mathcal{M}', \mathcal{K}')$ is defined in entire space, and $\text{Ent}[\tilde{\mathcal{M}}'] \leq 2$ in case $(\mathcal{M}', \mathcal{K}')$ is defined in a half-space.
- (2) $(\mathcal{M}', \mathcal{K}')$ is not a static or quasi-static multiplicity-2 plane or half-plane.

We recall that all limits flows are either free boundary flows in a half-space or flows without boundary in entire space. In the above definition, in case $(\mathcal{M}', \mathcal{K}')$ is defined in a half-space, $(\tilde{\mathcal{M}}', \tilde{\mathcal{K}}')$ is reflected flow \mathbb{R}^{n+1} .

PROPOSITION 6.2 (Partial regularity). *If $(\mathcal{M}', \mathcal{K}') \in \mathcal{C}$, then no tangent flow at a singular point can be static or quasi-static. In particular, the parabolic Hausdorff dimension of the singular set is at most $n - 1$.*

PROOF. In case $(\mathcal{M}', \mathcal{K}')$ is defined in a half-space, we consider its double $(\tilde{\mathcal{M}}', \tilde{\mathcal{K}}')$. By the equality case of the monotonicity formula and the definition of the class \mathcal{C} , no tangent flow can be a static or quasi-static plane of multiplicity 2. Hence, by the stratification of the singular set from [44, sec. 9], each time slice of the singular set has Hausdorff dimension at most $n - 1$.

Now, if a tangent flow is a stationary cone, then arguing as above we see that its singular set has dimension at most $n - 1$. Since our flow arises as the limit of smooth flows with $|A| \leq CH$ by Theorem 1.6 (elliptic regularization) using the local regularity theorem, we infer that A vanishes identically on the regular part; i.e., the cone is flat. Furthermore, the cone cannot be a union of three or four half-planes by one-sided minimization (Theorem 5.4). Hence the cone is a static multiplicity 1 plane, and by the local regularity theorem the point is regular.

Summing up, all tangent flows at singular points are shrinking. Hence, again by [44, sec. 9], the parabolic Hausdorff dimension of the singular set is at most $n - 1$. \square

COROLLARY 6.3 (Static limit flows). *If $(\mathcal{M}', \mathcal{K}') \in \mathcal{C}$ is static (or quasi-static), then one of the following five cases occurs:*

- (1) \mathcal{K}' is a static half-space in \mathbb{R}^{n+1} , and \mathcal{M}' is the static plane $\partial\mathcal{K}'$.
- (2) \mathcal{M}' is a pair of two static, parallel, multiplicity 1 planes in \mathbb{R}^{n+1} , and \mathcal{K}' is the region in between.
- (3) \mathcal{K}' is a static quarter-space in \mathbb{H} , and \mathcal{M}' is the static half-plane $\partial\mathcal{K}'$ with multiplicity 1.
- (4) \mathcal{M}' is a pair of static multiplicity 1 half-planes in \mathbb{H} with free boundary, and \mathcal{K}' is the region in between.
- (5) \mathcal{M}' is a static multiplicity 1 plane in \mathbb{H} parallel to the barrier plane $\partial\mathbb{H}$, and \mathcal{K}' is the region in between.

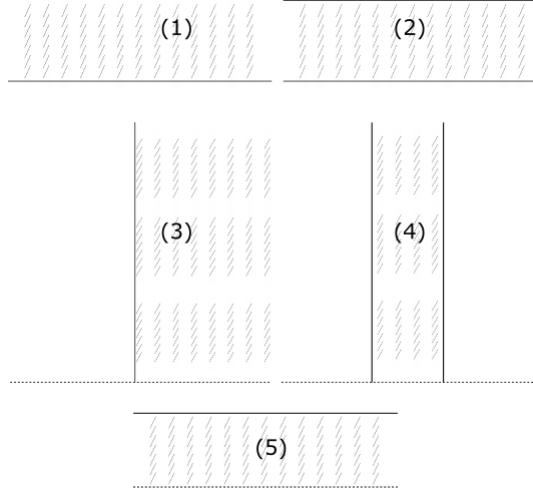
PROOF. The argument above shows that $(\mathcal{M}', \mathcal{K}')$ must be smooth and flat. Hence, it is the union of one or two planes or half-planes. Together with the one-sided minimization (Theorem 5.4) and unit regularity, the assertion follows. \square

THEOREM 6.4 (Separation theorem). *Let $(\mathcal{M}', \mathcal{K}') \in \mathcal{C}$. In case the flow is defined in a half-space, suppose there is a half-plane H perpendicular to the barrier plane such that*

$$(6.1) \quad H \subseteq \bigcap_t K'_t,$$

and suppose the complement of $\bigcap_t K'_t$ contains points on each side of H . Then $(\mathcal{M}', \mathcal{K}')$ is static, and \mathcal{K}' is the region between two parallel half-planes perpendicular to the barrier (a similar statement holds in case the limit flow is defined in entire space and contains a plane).

PROOF. Consider the doubled flow $(\tilde{\mathcal{M}}', \tilde{\mathcal{K}}')$. Using the partial regularity result (Proposition 6.2) and one-sided minimization (Theorem 5.4), the same argument

FIGURE 6.1. (Quasi-)static limit flows in \mathcal{C} .

as in [45, proof of theorem 7.4] shows that $\tilde{\mathcal{M}}'$ splits into two components. Since the entropy is at most 2, and each nonplanar component contributes strictly more than 1, this implies the assertion. \square

6.2 Bernstein-type theorem

For a set $S \subseteq D$, a point $x \in D$, and a radius $r > 0$, the *relative thickness* of S in $B(x, r)$ is

$$(6.2) \quad \text{Th}(S, x, r) = \inf_{|v|=1} \text{Th}_v(S, x, r),$$

where

$$(6.3) \quad \text{Th}_v(S, x, r) = \frac{1}{r} \sup_{y \in S \cap B(x, r)} |\langle v, y - x \rangle|.$$

LEMMA 6.5 (Expanding hole lemma). *For every $A < \infty$ there exists $\delta = \delta_A > 0$ with the following significance. Suppose \mathcal{K} is a set-theoretic subsolution of the free boundary mean curvature flow in our compact domain D or in a half-space \mathbb{H} , or a set-theoretic subsolution of the mean curvature flow in \mathbb{R}^{n+1} ; suppose that $R > 0$ is less than $\text{diam}(D)$ in the former case and arbitrary in the other cases. If there exists (x, t) such that*

$$(6.4) \quad \text{Th}(K_t, x, r) < \delta \quad \text{for } r \leq R$$

and

$$(6.5) \quad x \notin K_t,$$

then

$$(6.6) \quad \text{dist}(K_{t+r^2}, x) \geq Ar \quad \text{for } 0 \leq r \leq \delta R.$$

PROOF. We follow the strategy of the proof of [45, theorem 4.1]. By translation we may assume $(x, t) = (0, 0)$. Suppose that the result is false for some A . Then for every $\delta_j = 1/j$ there is a set-theoretic subsolution \mathcal{K}^j satisfying the first two conditions but not the last. Let

$$(6.7) \quad \rho_j = \inf \{r \mid \text{dist}(K_{r^2}^j, 0) \leq Ar\}.$$

Since $0 \notin K_0^j$, we certainly have $\rho_j > 0$, and the failure of the last condition implies that $\rho_j \leq \delta_j R_j$. In particular, in the case where \mathcal{K}^j is defined in D , we get $\rho_j \rightarrow 0$.

Parabolically rescale by ρ_j^{-1} and pass to a subsequential limit. The limiting domain is either \mathbb{R}^{n+1} or a half-space \mathbb{H} , and the limiting set-theoretic subsolution \mathcal{K}' satisfies $\text{Th}(K'_0, 0, r) = 0$ for all r , as well as $\text{dist}(K'_{r^2}, 0) \geq Ar$ for $r \leq 1$, and $\text{dist}(K'_1, 0) = A$.

Since its thickness is 0, K'_t must be contained in a plane P , and in the case the domain is \mathbb{H} the half-plane $H = P \cap \mathbb{H}$ must meet $\partial\mathbb{H}$ orthogonally. In either case, the distance condition implies that $K'_{1/2}$ is a proper subset of the static plane or half-plane solution, and must therefore vanish instantly. In particular, $K'_1 = \emptyset$, which is a contradiction. \square

Similarly as in [45, sec. 4] the expanding hole lemma (Lemma 6.5) implies the following two corollaries.

COROLLARY 6.6. *Let \mathcal{K} be as above and assume in addition that it is weakly mean-convex. If (x, t) is a point such that*

$$(6.8) \quad \limsup_{r \rightarrow 0} \text{Th}(K_t, x, r) < \delta_A$$

and such that $x \notin K_{t+h}$ for all $h > 0$, then

$$(6.9) \quad \liminf_{r \rightarrow 0} \frac{\text{dist}(K_{t+r^2}, x)}{r} \geq A.$$

COROLLARY 6.7. *If \mathcal{K} is a weakly mean-convex set-theoretic subsolution of the free boundary mean curvature flow in a half-space \mathbb{H} or of the mean curvature flow in \mathbb{R}^{n+1} , and there is a point x such that*

$$(6.10) \quad \limsup_{r \rightarrow \infty} \text{Th}(K_0, x, r) < \delta_A,$$

then either

$$(6.11) \quad \liminf_{r \rightarrow \infty} \frac{\text{dist}(K_{r^2}, x)}{r} \geq A$$

or

$$(6.12) \quad \bigcap_{t \geq 0} K_t \neq \emptyset.$$

The following generalizes White's Bernstein-type theorem [45, theorem 7.5] to limit flows defined in a half-space. Importantly, we do not assume a priori at which angle the thin slab meets the barrier.

THEOREM 6.8 (Bernstein-type theorem). *There exists an $\varepsilon > 0$ with the following significance. If $(\mathcal{M}', \mathcal{K}') \in \mathcal{C}$ is defined in a half-space \mathbb{H} and there is a point x such that*

$$(6.13) \quad \liminf_{r \rightarrow \infty} \frac{\text{dist}(K'_{r^2}, x)}{r} < 1$$

and

$$(6.14) \quad \limsup_{r \rightarrow \infty} \text{Th}(K'_{-r^2}, x, r) < \varepsilon,$$

then \mathcal{M}' is either a pair of static, parallel, multiplicity 1 half-planes with free boundary in \mathbb{H} , or a static multiplicity 1 plane parallel to the barrier in \mathbb{H} . In either case \mathcal{K}' is the region in between the planes of \mathcal{M}' and the barrier.

Similarly, if $(\mathcal{M}', \mathcal{K}')$ is defined in \mathbb{R}^{n+1} , then under the same assumptions \mathcal{M}' is a pair of static, parallel, multiplicity 1 planes, with \mathcal{K}' the region in between.

PROOF. The statement for flows in \mathbb{R}^{n+1} follows from the proof of [45, theorem 7.5], so we focus on the case of free boundary flows in \mathbb{H} .

Take $\Sigma = \bigcap_t K'_t$, which for small enough ε must be nonempty by Corollary 6.7. Consider the flows obtained by translating $(\mathcal{M}', \mathcal{K}')$ by $(0, -T)$ and let $(\mathcal{M}'', \mathcal{K}'')$ be a limit as $T \rightarrow \infty$. Then $(\mathcal{M}'', \mathcal{K}'')$ is a static flow, and $K'' = \Sigma$ at any time t .

By the classification of static limit flows from Corollary 6.3, we see that K'' is either a static multiplicity 2 half-plane; the region in between a pair of multiplicity 1 free boundary half-planes and the barrier; or finally the region bounded by the barrier plane and a parallel multiplicity 1 plane. In the first two cases, Theorem 6.4 (separation theorem) immediately implies the result. For the final case, the reflected flow $\tilde{\mathcal{K}}''$ will be the static flow in \mathbb{R}^{n+1} between two parallel multiplicity 1 planes, so applying the entire case of Theorem 6.4 (separation theorem) shows that the reflected flow $(\tilde{\mathcal{K}}', \tilde{\mathcal{M}}')$ consists of the region between two multiplicity 1 planes parallel to the barrier, which implies the result. \square

6.3 Sheeting theorem

The goal of this section is to prove a sheeting theorem (Theorem 6.10, Corollary 6.12, and Corollary 6.13).

Let $(\mathcal{M}, \mathcal{K})$ be a mean-convex free boundary flow in a compact domain D . Recall that for any $X = (x, t) \in \mathbb{R}^{n+1} \times \mathbb{R}$ and any $r > 0$ we denote by

$$(6.15) \quad B(X, r) = B(x, r) \times (t - r^2, t + r^2)$$

the two-sided parabolic ball with center X and radius r .

The following lemma shows that if a blowup sequence Hausdorff converges to a multiplicity 2 plane or half-plane, then we can find some sequence of rescaling

factors for which one has smooth convergence to a pair of parallel planes or half-planes.

LEMMA 6.9. *Let $(\mathcal{M}^i, \mathcal{K}^i)$ be a blowup sequence (see Section 5.2), and suppose that*

$$(6.16) \quad d(\mathcal{K}^i \cap B(0, 2), (P \times \mathbb{R}) \cap B(0, 2)) \rightarrow 0,$$

where P is either a plane or a half-plane. Then, there exists some sequence $\rho_i > 0$ converging to 0 such that the parabolic dilates $\mathcal{D}_{\rho_i^{-1}} \mathcal{M}^i$ converge smoothly to either

- (a) a pair of parallel planes in \mathbb{R}^{n+1} , or
- (b) a pair of parallel half-planes with free boundary in a half-space \mathbb{H} , or
- (c) a multiplicity 1 plane parallel to the boundary of \mathbb{H} .

Furthermore, in all cases $\mathcal{D}_{\rho_i^{-1}} \mathcal{K}^i$ converges to the enclosed region.

PROOF. Fix $\varepsilon > 0$ small enough. Take ρ_i to be the least number such that for all $r \in [\rho_i, 1]$ we have

$$(6.17) \quad \text{Th}(K_{-r^2}^i, 0, r) \leq \varepsilon \quad \text{and} \quad \text{dist}(K_{r^2}^i, 0) \leq r.$$

By Corollary 6.6 (expanding holes), we have $\rho_i > 0$ for each i . Moreover, assumption (6.16) easily implies that $\rho_i \rightarrow 0$.

Consider the dilates $\mathcal{D}_{\rho_i^{-1}}(\mathcal{M}^i, \mathcal{K}^i)$ and take a subsequential limit. Any such limit $(\mathcal{M}', \mathcal{K}')$ satisfies

$$(6.18) \quad \text{Th}(K'_{-r^2}, r) \leq \varepsilon \quad \text{and} \quad \text{dist}(K'_{r^2}, 0) \leq r \quad \text{for all } r \geq 1,$$

with at least one inequality being nonstrict for $r = 1$, namely,

$$(6.19) \quad \text{Th}(K'_{-1}, 1) = \varepsilon \quad \text{or} \quad \text{dist}(K'_1, 0) = 1.$$

Moreover, by one-sided minimization (Theorem 1.6) the (reflected) density at infinity of \mathcal{M}' is at most 2. Now due to (6.19) we can apply Theorem 6.8 (Bernstein-type theorem) to deduce that \mathcal{M}' consists of separate multiplicity 1 (half-)planes, and then the local regularity theorem [10, 47] gives smooth convergence of $\mathcal{D}_{\rho_i^{-1}} \mathcal{M}^i$ to \mathcal{M}' . This proves the lemma. \square

The above lemma essentially gives that the \mathcal{M}^i are eventually smooth, but to get smooth convergence at the original scale the strategy is to find a separating surface as follows. Recall that D^i denotes the domain of the rescaled flow $(\mathcal{K}^i, \mathcal{M}^i)$. If we are in case (a) or (b) of the above lemma, then we let S_t^i be the set of centers of open balls B such that $B \cap D^i \subseteq K_t^i$ and $\bar{B} \cap D^i$ touches ∂K_t^i at two or more points. Set

$$(6.20) \quad \mathcal{S}^i = \bigcup_t S_t^i \cap D^i.$$

THEOREM 6.10 (Sheeting theorem). *Let $(\mathcal{M}^i, \mathcal{K}^i)$ be a blowup sequence (see Section 5.2) and suppose that*

$$(6.21) \quad d(\mathcal{K}^i \cap B(0, 4), (P \times \mathbb{R}) \cap B(0, 4)) \rightarrow 0,$$

where either $\lim D^i = \mathbb{R}^{n+1}$ and P is a plane, or $\lim D^i = \mathbb{H}$ and P is a free boundary half-plane. Then, for all i large $\mathcal{S}^i \cap B(0, 1)$ is a C^1 -hypersurface that divides $\partial \mathcal{K}^i$ into two nonempty components $\partial \mathcal{K}_1^i$ and $\partial \mathcal{K}_2^i$. In particular, any convergent subsequence of \mathcal{M}^i in $B(0, 1)$ converges smoothly to a plane or half-plane with multiplicity 2.

PROOF. We claim that $\mathcal{S}^i \cap B(0, 1)$ is a C^1 properly embedded hypersurface of $B(0, 1)$, which therefore divides $B(0, 1)$ into two disjoint open subsets each bounded by \mathcal{S}^i (once this is shown, $\partial \mathcal{K}_1^i$ and $\partial \mathcal{K}_2^i$ can be defined as the respective portions of $\partial \mathcal{K}^i$ in each open subset).

Suppose the claim is false, so that there are points $X_i \in \mathcal{S}^i \cap B(0, 1)$ about which \mathcal{S}^i fails to be a C^1 embedded hypersurface. Then we may consider the translates $(\tilde{\mathcal{M}}^i, \tilde{\mathcal{K}}^i) := (\mathcal{M}^i - X_i, \mathcal{K}^i - X_i)$. Up to taking a subsequence, the $\tilde{\mathcal{K}}^i \cap B(0, 2)$ will Hausdorff-converge locally to a translate of $P \times \mathbb{R}$.

Now, Lemma 6.9 (note that outcome (c) is excluded by the hypotheses on P) implies that for large i there are radii $r_i > 0$ for which $\mathcal{K}^i \cap B(X_i, r_i)$ splits as the region in $B(X_i, r_i)$ bounded either by two smooth, disjoint hypersurfaces $\mathcal{S}_1^i, \mathcal{S}_2^i$ (without boundary), or by smooth disjoint hypersurfaces $\mathcal{S}_1^i, \mathcal{S}_2^i$ (with boundary on ∂D^i) together with the barrier ∂D^i . In either case, each \mathcal{S}_α^i is graphical over (the plane containing) P , with uniformly small $C^{1,1}$ norm, and with the C^0 norm tending to 0 as $i \rightarrow \infty$.

But the distance function from a smooth submanifold \mathcal{S} with boundary is $C^{1,1}$ in a small neighborhood $\mathcal{U} \setminus \mathcal{S}$, indeed with nonzero gradient $\nabla d_{\mathcal{S}}(x) = \frac{x - \pi_{\mathcal{S}}(x)}{d_{\mathcal{S}}}$; see, e.g., [35]. Since $\mathcal{S}^i \cap B(X_i, r_i)$ is clearly given by the locus $d(\cdot, \mathcal{S}_1^i) = d(\cdot, \mathcal{S}_2^i)$ and both \mathcal{S}_α^i Hausdorff-converge locally to $P \times \mathbb{R}$, the implicit function theorem then implies that $\mathcal{S}^i \cap B(X_i, r_i)$ is in fact a C^1 embedded hypersurface. This provides the desired contradiction, and thus proves the claim.

It remains to show that the $\partial \mathcal{K}_\alpha^i$ converge locally smoothly and separately to $P \times \mathbb{R}$. For each $\alpha = 1, 2$, by the assumption we have that $\partial \mathcal{K}_\alpha^i \cap B(0, 1)$ Hausdorff converges to $(P \times \mathbb{R}) \cap B(0, 1)$ with some multiplicity. But one-sided minimization (Theorem 1.6) implies that the sum of (reflected) densities is at most 2. Therefore each must converge with multiplicity 1, and the partial regularity theorem [10, 47] then implies the smooth convergence. \square

To reformulate Theorem 6.10 (sheeting theorem) in a more geometric way, it will be convenient to fix certain boundary-straightening maps Φ_y centered about a point $y \in \partial D$, as in [10]. Locally these maps may be described as follows: Up to a translation we may take $y = 0$, and up to a rotation and reflection we may take $T_y \partial D$ to be $\{0\} \times \mathbb{R}^n$, with the inward normal of ∂D at 0 pointing in the positive

x_1 -direction. Near y the barrier ∂D is then locally given by a graph u over $T_y \partial D$, and we define a map $\Phi_y : [0, \rho) \times B^n(\rho) \rightarrow D$ by

$$(6.22) \quad \Phi_y(s, x') = (u(x'), x') + sv_{\partial D}(u(x'), x').$$

The key properties are that for every small enough ρ , we have uniformly

$$(6.23) \quad |\Phi - \text{id}| \leq C\rho^2, \quad |D\Phi - \text{id}| \leq C\rho, \quad |D^k\Phi| \leq C_k\rho^{2-k}.$$

In particular, define $D^i = \lambda_i(D - x_i)$ to be a sequence of rescalings of D , with $\lambda_i d(x_i, \partial D) < \infty$ so that D^i converges to a half-space \mathbb{H} . Let y_i be the nearest-point projection of x_i to ∂D^i ; up to translation we may assume $y = \lim y_i = 0$. Then the boundary-straightening maps for D^i are given by $\Phi_i : [0, \lambda_i\rho) \times B^n(\lambda_i\rho) \rightarrow D^i \hookrightarrow \mathbb{R}^{n+1}$ by

$$(6.24) \quad \Phi_i(p) = \lambda_i(\Phi_{y_i}(p/\lambda_i) - x_i).$$

Hence,

$$(6.25) \quad \Phi_i \rightarrow \text{id}_{\mathbb{H}} \quad \text{locally smoothly.}$$

Finally, for a function f defined in $B(0, r)$, define the scale-invariant $C^{2,1}$ -norm of f to be the usual $C^{2,1}$ -norm of the function

$$(6.26) \quad (x, t) \in B^{n,1}(0, 1) \mapsto rf(x/r, t/r^2).$$

Remark 6.11. If for some r , $M_t \cap B^{n,1}(y, r)$ is parametrized by

$$(6.27) \quad B^{n,1}(y, r) \ni (x, t) \mapsto \Phi(f(x, t), x),$$

then $f(x, t)$ satisfies the graphical mean curvature flow equation (with respect to $\Phi^*\delta$), given by

$$(6.28) \quad \partial_t f = \gamma^{ij}(Df, f, x)D_{ij}^2 f + E,$$

where γ is the pullback of δ under the map $x \mapsto \Phi(f(x, t), x)$, and E is an analytic function of f , Df , $D\Phi$, $D^2\Phi$, and $D^3\Phi$ such that $E = 0$ when $\Phi = \text{id}$.

Using the above notions, the sheeting theorem (Theorem 6.10) implies the following two corollaries.

COROLLARY 6.12 (Sheeting at the boundary). *Let D be a compact domain or the half-space $\mathbb{H} = \{x_1 \geq 0\}$. Assume that $0 \in \partial D$, and that the inward unit normal at 0 points in the positive x_1 -direction. Let $\Phi = \Phi_0$ be the boundary-straightening map of D centered at 0 . Then, for any $\eta > 0$ there exists an $\varepsilon > 0$ with the following significance.*

Let $(\mathcal{M}', \mathcal{K}')$ be a free boundary flow in D , which is either a mean-convex flow in a compact domain D , or a limit flow in $D = \mathbb{H}$. Set

$$(6.29) \quad \mathcal{H} := \{(x, t) \in \mathbb{R}^{n+1,1} \mid x_{n+1} = 0, x_1 \geq 0\},$$

and suppose that

$$(6.30) \quad d(\mathcal{K}' \cap B(0, 4r), \mathcal{H} \cap B(0, 4r)) < \varepsilon r$$

for some $r > 0$. Then, there exist functions

$$(6.31) \quad f, g : B^{n,1}(0, 3r) \cap \{x_1 \geq 0\} \rightarrow \mathbb{R},$$

such that:

- (1) $f \leq g$.
- (2) f and g have scale-invariant $C^{2,1}$ norms $\leq \eta$.
- (3) Inside $B(0, 2r) \cap D$, the set \mathcal{K}' coincides with the region that is 0 between $\Phi(\text{graph}(f))$ and $\Phi(\text{graph}(g))$.
- (4) f, g satisfy the graphical mean curvature flow equation (6.28).
- (5) f, g satisfy the Neumann boundary condition $\partial_{x_1} f = \partial_{x_1} g = 0$ on

$$B^{n,1}(0, r) \cap \{x_1 = 0\}.$$

- (6) For any fixed x , the functions $t \mapsto f(x, t)$ and $t \mapsto g(x, t)$ are increasing and decreasing, respectively.

Note that for a mean-convex flow, in order for (6.30) to hold for some scale r , it must be the case that $r \leq r_0(\varepsilon)$, where $\lim_{\varepsilon \rightarrow 0} r_0(\varepsilon) = 0$. With this observation the above corollary follows after scaling from the second case of Theorem 6.10. The first case immediately yields the following:

COROLLARY 6.13 (Sheeting in the interior). *Let D be a compact domain, or a half-space \mathbb{H} , or entire space \mathbb{R}^{n+1} . Assume that 0 is an interior point of D and let*

$$(6.32) \quad \mathcal{V} := \{(x, t) \in \mathbb{R}^{n+1,1} \mid x_{n+1} = 0\}.$$

Then, for any $\eta > 0$ there exists an $\varepsilon > 0$ with the following significance: Let $(\mathcal{M}', \mathcal{K}')$ be either a mean-convex free boundary flow in a compact domain D or a limit flow in \mathbb{H} or \mathbb{R}^{n+1} . Suppose that

$$(6.33) \quad d(\mathcal{K}' \cap B(0, 4r), \mathcal{V} \cap B(0, 4r)) < \varepsilon r$$

for some $r < \frac{1}{4}d(0, \partial D)$. Then, there exist functions

$$(6.34) \quad f, g : B^{n,1}(0, 3r) \rightarrow \mathbb{R}$$

such that:

- (1) $f \leq g$.
- (2) f, g have scale-invariant $C^{2,1}$ norms $\leq \eta$.
- (3) Inside $B(0, 2r)$, the set \mathcal{K}' coincides with the region between $\text{graph}(f)$ and $\text{graph}(g)$.
- (4) f, g satisfy the graphical mean curvature flow equation.
- (5) For any fixed x , the functions $t \mapsto f(x, t)$ and $t \mapsto g(x, t)$ are increasing and decreasing, respectively.

In both cases the strong maximum principle implies that in fact either f is strictly dominated by g or f is identically equal to g (the latter can only happen for limit flows).

Finally, we also have a version parallel to the barrier.

COROLLARY 6.14 (Graphs above the barrier). *Let $(\mathcal{M}, \mathcal{K})$ be a mean-convex free boundary flow in a compact domain D . Assume without loss of generality that $0 \in \partial D$ and that the inward unit normal at 0 points in the positive x_1 -direction. Denote $\mathbb{H} := \mathbb{R}^{n+1} \cap \{x_1 \geq 0\}$ and*

$$(6.35) \quad \mathcal{V} := \{(x, t) \in \mathbb{R}^{n+1, 1} \mid x_1 = 0\}.$$

Then, for any $\eta > 0$ there exists an $\varepsilon > 0$ with the following significance. Suppose that

$$(6.36) \quad d(\mathcal{K} \cap B(0, 4r), \mathcal{V} \cap B(0, 4r)) < \varepsilon r$$

for some $r > 0$. Then there exists a function

$$(6.37) \quad g : B^{n, 1}(0, 3r) \rightarrow \mathbb{R}$$

such that:

- (1) $g \geq 0$.
- (2) g has scale-invariant $C^{2,1}$ norm $\leq \eta$.
- (3) Inside $B(0, 2r)$, the set \mathcal{K} coincides with the region between $\Phi(\text{graph}(0))$ and $\Phi(\text{graph}(g))$, where the graphs are over $\partial\mathbb{H}$.
- (4) g satisfies the graphical mean curvature flow equation (6.28).
- (5) For any fixed x , the function $t \mapsto g(x, t)$ is decreasing.

PROOF. By one-sided minimization (Theorem 1.6), locally \mathcal{K} must be between ∂D and $\partial\mathcal{K}$. From this, the assertion follows easily. \square

In particular, we can rule out static density two planes or half-planes as potential tangent flows.

COROLLARY 6.15. *Static density 2 planes and half-planes cannot occur as tangent flows.*

PROOF. Suppose that $X = (x, t)$ is a point of density 2 with a static (half)plane as tangent flow. Applying Corollary 6.12, Corollary 6.13 or Corollary 6.14, respectively, and using the strong maximum principle (in the first two cases) or the fact that $H > 0$ and $\text{spt } \mathcal{M} \subset D \times \mathbb{R}$ (in the last case), we see that all points in a neighborhood of X are regular points of multiplicity 1; a contradiction. \square

6.4 Ruling out density two tangent flows

In this section, assuming D is mean-convex, we rule out quasi-static density 2 planes and half-planes as potential tangent flows (recall that the static case has already been ruled out in Corollary 6.15).

For any $X = (x, t) \in \mathbb{R}^{n+1} \times \mathbb{R}$ and any $r > 0$, we denote by

$$(6.38) \quad P(X, r) = B(x, r) \times (t - r^2, t]$$

, the backwards parabolic ball of radius r with center X .

Given any closed space-time subset \mathcal{K}' of $\mathbb{R}^{n+1} \times \mathbb{R}$, we define two quantities to measure Hausdorff-closeness to a quasi-static plane or half-plane, respectively. Recall that in [45], sec. 9, White states

$$(6.39) \quad \begin{aligned} \phi(\mathcal{K}') & \text{ is the infimum of } s > 0 \text{ such that} \\ & \inf_{\mathcal{V}} d(\mathcal{K}' \cap P(0, s^{-1}), \mathcal{V} \cap P(0, s^{-1})) < s, \end{aligned}$$

where $\inf_{\mathcal{V}}$ is taken over static planes \mathcal{V} through the origin. Similarly, we define

$$(6.40) \quad \begin{aligned} \phi_+(\mathcal{K}') & \text{ is the infimum of } s > 0 \text{ such that} \\ & \inf_{\mathcal{H}} d(\mathcal{K}' \cap P(0, s^{-1}), \mathcal{H} \cap P(0, s^{-1})) < s, \end{aligned}$$

where $\inf_{\mathcal{H}}$ is taken over static half-planes $\mathcal{H} = \mathcal{V} \cap \{x_1 \geq 0\}$ that intersect $\{x_1 = 0\}$ orthogonally at 0 (that is, $(0, 0) \in \partial\mathcal{H}$ and the inner conormal is e_1).

The following lemma gives sheeting sequences and their limiting behaviour for blowups at the boundary:

LEMMA 6.16 (Sheeting sequence at the boundary). *Let D be a compact domain. Assume without loss of generality that $0 \in \partial D$, and that the inward unit normal at 0 is given by e_1 . Let $(\mathcal{M}^i, \mathcal{K}^i)$ be either a blowup sequence at $0 \in \partial D$ or a sequence of tangent flows at 0, and suppose that $\phi_+(\mathcal{K}^i) \rightarrow 0$.*

Then, for large enough i there are functions f_i and g_i , defined on an exhaustion of $\mathbb{R}_+^n \times (-\infty, 0)$ such that:

- (1) *Either $f_i < g_i$ everywhere, or $f_i \equiv g_i$.*
- (2) *For any $U \Subset \mathbb{H}$ and $[a, b] \subset (-\infty, 0)$, for i large enough the region K_t^i coincides in U with the region between $\Phi_i(\text{graph}(f_i))$ and $\Phi_i(\text{graph}(g_i))$ for all $t \in [a, b]$, where Φ_i denotes the boundary-straightening map for $D^i = \lambda_i D$ as in (6.24).*
- (3) *f_i and g_i converge smoothly on compact subsets to 0.*
- (4) *f_i and g_i solve the graphical mean curvature flow equation in the pullback metric $\Phi_i^* \delta$.*
- (5) *f_i and g_i satisfy the zero Neumann boundary condition.*
- (6) *f_i and g_i are increasing and decreasing in time, respectively.*

Furthermore, if $f_i < g_i$ for infinitely many i , then there exist constants $c_i > 0$ and a subsequence $c_i \cdot (g_i - f_i)$ that converges smoothly on compact subsets to the constant function $u(x, t) \equiv 1$ on $\mathbb{R}_+^n \times (-\infty, 0)$.

A similar statement holds for blowups in the interior:

LEMMA 6.17 (Sheeting sequence in the interior). *Let D be a compact domain. Let $(\mathcal{M}^i, \mathcal{K}^i)$ be either a blowup sequence about a fixed interior point or a sequence of tangent flows at a fixed interior point, and suppose that $\phi(\mathcal{K}^i) \rightarrow 0$.*

Then, for large enough i there are functions f_i and g_i , defined on an exhaustion of $\mathbb{R}^n \times (-\infty, 0)$ such that:

- (1) *Either $f_i < g_i$ everywhere, or $f_i \equiv g_i$.*

⁷ Here we denote $\mathbb{R}_+^n = \{x \in \mathbb{R}^n \mid x_1 \geq 0\}$ and $\mathbb{H} = \{x \in \mathbb{R}^{n+1} \mid x_1 \geq 0\}$.

- (2) For any $U \Subset \mathbb{R}^{n+1}$ and $[a, b] \subset (-\infty, 0)$, for i large enough the region K_t^i coincides in U , after a suitable rotation, with the region between $\text{graph}(f_i)$ and $\text{graph}(g_i)$ for all $t \in [a, b]$.
- (3) Both sequences f_i and g_i converge smoothly on compact subsets to 0.
- (4) f_i and g_i are solutions to the graphical mean curvature flow equation.
- (5) f_i and g_i are increasing and decreasing in time, respectively.

Furthermore, if $f_i < g_i$ for infinitely many i , then there exist constants $c_i > 0$ and a subsequence $c_i(g_i - f_i)$ that converges smoothly on compact subsets to the constant function $u(x, t) \equiv 1$ on $\mathbb{R}^n \times (-\infty, 0)$.

PROOF OF LEMMAS 6.16 AND 6.17 Taking $i \rightarrow \infty$, we get convergence to some tangent flow $(\mathcal{M}', \mathcal{K}')$ that is either defined in \mathbb{H} and supported in a free boundary half-plane, or defined in \mathbb{R}^{n+1} and supported in a plane. By Proposition 5.3 these must be the static or quasi-static flows with multiplicity 2. We can therefore apply Corollary 6.13 and Corollary 6.12 to obtain the required f_i and g_i , respectively.

Suppose now there is a subsequence with $f_i < g_i$. Since $\Phi_i \rightarrow \text{id}$ locally smoothly, the difference

$$(6.41) \quad u_i := g_i - f_i > 0$$

satisfies a linear parabolic equation with coefficients converging locally smoothly as $i \rightarrow \infty$ to those of the ordinary heat equation.

Let us first analyze the interior case. Since the functions u_i on the one hand are decreasing by mean-convexity, but on the other hand want to become increasing driven by the Harnack inequality, the argument can be concluded as in the proof of [45, theorem 9.1].

In the boundary case, we consider the sequence of functions \tilde{u}_i that is obtained from u_i via doubling at the boundary of \mathbb{R}_+^n . Applying the same argument to \tilde{u}_i , the proof can be concluded in this case also. \square

Finally, we also have a version for blowups parallel to the barrier. Before stating it, we need an auxiliary result concerning mean-convex domains.

LEMMA 6.18. *Assume D is mean-convex. Then for any point $x \in \partial D$, there exists $r > 0$ and a smooth minimal surface in $B_r(x) \setminus \text{int } D$ that passes through x .*

PROOF. Assume without loss of generality that $x = 0$, $\mathbb{R}^n = T_0 \partial D$. The result follows from the implicit function theorem: Consider for instance the map from $C^{2,\alpha}(B_1) \times S^2_+(B_1)$ that maps

$$(u, g) \rightarrow (u(0), Du(0), D^2u(0), H_g u),$$

where $H_g u \in C^\alpha(B_1)$ is the mean curvature of $\text{graph}(u)$ in the metric g . The linearization in u at the Euclidean disk $(0, \delta)$ is clearly surjective, so in particular for any diagonal, trace-free matrix A with $|A| < \varepsilon$ (and any metric $|g - \delta| < \varepsilon$) we are able to find a smooth (g -)minimal surface Σ tangent to D with second fundamental form equal to A at x .

To complete the proof, let B be the second fundamental form of ∂D at x ; then B is diagonal in some orthonormal basis and has $\text{tr}(B) > 0$, so we may fix a diagonal, trace-free matrix $A < B$ and apply the above after appropriate scaling. Note that $A < B$ ensures that Σ is disjoint from $\text{int } D$ in a small enough ball. \square

LEMMA 6.19 (Graphical sequence above the barrier). *Let D be a compact domain. Assume without loss of generality that $0 \in \partial D$. Let $(\mathcal{M}^i, \mathcal{K}^i)$ be a blowup sequence at 0 or a sequence of tangent flows at 0, and suppose that $\phi(\mathcal{K}^i) \rightarrow 0$. Let Σ_i be the corresponding dilates of the minimal surface passing through 0 as constructed in Lemma 6.18, or if the $(\mathcal{M}^i, \mathcal{K}^i)$ are tangent flows, take $\Sigma_i \equiv \partial \mathbb{H}$.*

Then, for large enough i there are functions g_i , defined on an exhaustion of $\partial \mathbb{H} \times (-\infty, 0)$, and f_i defined on an exhaustion of $\partial \mathbb{H}$, where $\mathbb{H} = \{x \in \mathbb{R}^{n+1} \mid x_1 \geq 0\}$ such that setting $f_i(\cdot, t) \equiv f_i(\cdot)$, we have:

- (1) *Either $g_i > 0 \geq f_i$ or $g_i \equiv 0 \equiv f_i$.*
- (2) *For any $U \Subset \mathbb{H}$ and $[a, b] \subset (-\infty, 0)$ and for i large enough, the region K_t^i coincides in U with the region between $\Phi_i(\text{graph}(0))$ and $\Phi_i(\text{graph}(g_i))$, where the graphs here are over $\partial \mathbb{H}$.*
- (3) *For any $U \Subset \mathbb{R}^{n+1}$ and for any i large enough, Σ_i coincides in U with $\Phi_i(\text{graph}(f_i))$.*
- (4) *g_i and f_i converge smoothly on compact subsets to 0.*
- (5) *g_i and f_i are solutions to the graphical mean curvature flow equation in the pullback metric $\Phi_i^* \delta$.*
- (6) *g_i is decreasing in time.*

Furthermore, if $g_i > 0$ for infinitely many i , then there exist constants $c_i > 0$ and a subsequence $c_i(g_i - f_i)$ that converges smoothly on compact subsets to the constant function $u(x, t) \equiv 1$ on $\partial \mathbb{H} \times (-\infty, 0)$.

PROOF. Any subsequential limit of the \mathcal{K}^i must be a plane, and hence must be the barrier plane $\partial \mathbb{H}$ with multiplicity 1 (reflected density 2). The existence of the required g_i follows from Corollary 6.14, and the f_i exist since Σ_i is smooth and converging smoothly to the barrier plane at 0. The convergence of $c_i(g_i - f_i)$ follows as in the interior and free boundary cases. \square

The following theorem shows that density 2 planes and half-planes are isolated:

THEOREM 6.20 (Isolation). *Let $(\mathcal{M}, \mathcal{K})$ be a mean-convex flow in a compact domain D or a limit flow of such a flow. In case $0 \in \partial D$, assume without loss of generality that the inward unit normal at 0 points in positive x_1 -direction. Then there exists $\delta > 0$ such that for any tangent flow $(\mathcal{M}', \mathcal{K}')$ to $(\mathcal{M}, \mathcal{K})$ at $X = (0, t)$, we have*

- (1) *If 0 is a boundary point and $\phi_+(\mathcal{K}') < \delta$, then $\phi_+(\mathcal{K}') = 0$.*
- (2) *If 0 is an interior point and $\phi(\mathcal{K}') < \delta$, then $\phi(\mathcal{K}') = 0$.*
- (3) *If 0 is a boundary point and $\phi(\mathcal{K}') < \delta$, then $\phi(\mathcal{K}') = 0$.*

PROOF. Suppose towards a contradiction that there is a sequence of tangent flows $(\mathcal{M}^i, \mathcal{K}^i)$ at $X = (0, t)$ with $\phi(\mathcal{K}^i) \rightarrow 0$ or $\phi_+(\mathcal{K}^i) \rightarrow 0$. We must show that for large enough i , we have $\phi(\mathcal{K}^i) = 0$, respectively $\phi_+(\mathcal{K}^i) = 0$.

In each case (1)–(3) consider the functions f_i and g_i given by Lemma 6.16, Lemma 6.17, and Lemma 6.19, respectively, taking $f_i = 0$ for case (3). Since we are dealing with sequences of tangent flows, all boundary-straightening maps are trivial, so \mathcal{K}^i corresponds to the region between $\text{graph}(f_i)$ and $\text{graph}(g_i)$. Moreover, since the $(\mathcal{K}^i, \mathcal{M}^i)$ are tangent flows, they are backwardly self-similar; i.e., it holds that

$$(6.42) \quad f_i(rx, r^2t) = rf_i(x, t) \quad \text{and} \quad g_i(rx, r^2t) = rg_i(x, t)$$

for all $r > 0$ and all (x, t) with $t < 0$.

If $f_i < g_i$ for infinitely many i , then by the conclusions of the lemmata above, there exist $c_i > 0$ so that $c_i(g_i - f_i)$ converges smoothly to the constant function $u(x, t) \equiv 1$. However, equation (6.42) implies that $u(rx, r^2t) = ru(x, t)$, which is absurd.

So $f_i \equiv g_i$ for all sufficiently large i . But then $f_i = g_i$ are both increasing and decreasing, and hence constant in t . The self-similarity above then states that

$$(6.43) \quad g_i(rx) = rg_i(x).$$

Since g_i is smooth and 1-homogeneous, it must be linear. In cases (2) and (3), we conclude that \mathcal{K}^i is a plane for i large enough; hence $\phi(\mathcal{K}^i) = 0$. In case (1), taking also into account the vanishing Neumann boundary data, we conclude that \mathcal{K}^i is a half-plane orthogonal to $\partial\mathbb{H}$ for i large enough; hence $\phi_+(\mathcal{K}^i) = 0$. This finishes the proof of the theorem. \square

We can now rule out quasi-static density-2 tangent flows:

THEOREM 6.21 (Tangent flows). *Quasi-static multiplicity 2 planes, respectively half-planes, cannot occur as tangent flows to a mean-convex free boundary flow $(\mathcal{M}, \mathcal{K})$. If additionally D is mean-convex, then quasi-static density 2 planes also cannot occur as tangent flows.*

Note the last statement excludes case (6) of Proposition 5.3.

PROOF. We follow the proof strategy in [45, theorem 9.2], with some adjustments.

First, suppose, for the sake of contradiction, that one tangent flow at a point $X = 0$ is a quasi-static multiplicity 2 plane or half-plane. In particular, we are in case (4) or (2), respectively, of Proposition 5.3. Recall that by one-sided minimization, any planar limit flow must have (reflected) density either 1 or 2. In particular, if a limit flow is a multiplicity 2 plane, then it must be defined in \mathbb{R}^{n+1} (that is, it cannot be a plane parallel to the barrier in \mathbb{H}).

We may assume without loss of generality that $X = 0$, and in case 0 is a boundary point that the inward normal of ∂D points in the positive x_1 -direction. Consider

the parabolic dilations $\mathcal{D}_\lambda \mathcal{K}$. By the discussion above, we are in case (1) or (2) of Theorem 6.20, so we must have

$$(6.44) \quad \lim_{\lambda \rightarrow \infty} \phi(\mathcal{D}_\lambda \mathcal{K}) = 0 \quad \text{or} \quad \lim_{\lambda \rightarrow \infty} \phi_+(\mathcal{D}_\lambda \mathcal{K}) = 0,$$

respectively, at $X = 0$. In particular, by the definitions of ϕ and ϕ_+ , every tangent flow at $X = 0$ must be of the same type; that is, a quasi-static multiplicity 2 plane and half-plane, respectively.

In either case we may consider the quantity

$$(6.45) \quad V(r) = \sup\{\rho \mid 0 \in (\overline{B^{n+1}(y, \rho)} \cap D') \subset K_{-r^2} \text{ for some } y\},$$

where D' is \mathbb{R}^{n+1} in the planar case and $\mathbb{R}^{n+1} \cap \{x_1 \geq 0\}$ in the half-planar case. The observation above implies that

$$(6.46) \quad \lim_{r \rightarrow 0} \frac{V(r)}{r} = 0.$$

In particular, we may take a sequence $r_i \rightarrow 0$ such that

$$(6.47) \quad \frac{V(r_i)}{r_i} < 2 \frac{V(3r_i)}{3r_i}.$$

Consider the parabolic dilates $\mathcal{K}^i = \mathcal{D}_{r_i^{-1}} \mathcal{K}$. Applying Lemma 6.16 and Lemma 6.17 gives functions $f_i \leq g_i$ such that \mathcal{K}^i corresponds to the region between $\Phi_i(\text{graph}(f_i))$ and $\Phi_i(\text{graph}(g_i))$ for some diffeomorphisms Φ_i that converge smoothly to id . In particular, for large i we have $V(\lambda r_i) \simeq u_i(0, -\lambda^2)$, where $u_i := g_i - f_i$, and hence

$$(6.48) \quad u_i(0, -1) \leq \frac{3}{4} u_i(0, -9).$$

Since $(\mathcal{M}, \mathcal{K})$ is a mean-convex flow, we must have $u_i > 0$. Therefore, by the above lemmata there exist $c_i > 0$ so that $c_i u_i$ converges uniformly to $u(x, t) \equiv 1$. This is incompatible with the inequality (6.48).

For the remaining density 2 case, suppose a tangent flow at $X = 0$ is a quasi-static multiplicity 1 plane coincident with the barrier tangent plane $\partial\mathbb{H}$. Let Σ be the minimal surface in $B_r(0) \setminus \text{int } D$ passing through 0 as in Lemma 6.18 for some $r > 0$. Set \mathcal{M}_Σ to be the mean curvature flow in $B_r(0)$ obtained by keeping Σ static, and let \mathcal{K}_Σ be the region in $B_r(0)$ between Σ and D .

By Theorem 6.20(isolation), every tangent flow of \mathcal{M} at $X = 0$ coincides with $\partial\mathbb{H}$, with multiplicity 1. Therefore by Corollary 6.14, there is a parabolic region

$$\mathcal{U} = \{\varepsilon(t) |x|^2 \leq |t|, t_0 < t < 0\} \quad \text{for some } t_0 < 0,$$

where $\varepsilon(t) \rightarrow 0$ as $t \rightarrow 0$, so that in \mathcal{U} the flow $(\mathcal{M} + \mathcal{M}_\Sigma, \mathcal{K} \cup \mathcal{K}_\Sigma)$ is a graphical, weakly mean-convex mean curvature flow without boundary, with the property that any blowup sequence centered at $(0, 0)$ converges smoothly to a multiplicity 2 plane. We can then use the same argument as in the interior, using Lemma 6.19 in place of Lemma 6.17 to deduce a contradiction. \square

6.5 Ruling out density 2 limit flows

In this section, for mean-convex free boundary flow $(\mathcal{M}, \mathcal{K})$ in a mean-convex domain D , we rule density 2 (quasi)static planes or half-planes as potential limit flows (in the case of tangent flows this has already been done in the previous section). Adapting [45, sec. 12], we start with the following lemma:

LEMMA 6.22. *Let $(\mathcal{M}', \mathcal{K}')$ be a limit flow of $(\mathcal{M}, \mathcal{K})$, defined in the limiting domain $D' = \mathbb{R}^{n+1}$ or $D' = \mathbb{H}$. Let $(\mathcal{M}'', \mathcal{K}'')$ be a tangent flow of $(\mathcal{M}', \mathcal{K}')$ taken at a density 2 point $X = (x, t)$. Then:*

- (1) *If x is an interior point of D' , and \mathcal{K}'' is a static or quasi-static plane, then there is an open neighborhood U of x in \mathbb{R}^{n+1} , an open interval (a, b) , and a properly embedded smooth minimal hypersurface Σ in U such that $K'_\tau \cap U = \Sigma$ for all $\tau \in (a, b)$.*
- (2) *If x is a boundary point of D' , and \mathcal{K}'' is a static or quasi-static half-plane, then there is an open neighborhood U of x in \mathbb{H} , an open interval (a, b) , and a properly embedded smooth free boundary minimal hypersurface Σ in U such that $K'_\tau \cap U = \Sigma$ for all $\tau \in (a, b)$.*
- (3) *If x is a boundary point of D' , and \mathcal{K}'' is a static or quasi-static plane, then there exists a neighborhood U of 0 in \mathbb{H} and an open interval (a, b) such that $K'_\tau \cap U = \partial\mathbb{H} \cap U$ for all $\tau \in (a, b)$.*

Furthermore, in case (1) we have $\Theta(\mathcal{M}', \cdot) \geq 2$ on all of $\Sigma \times (-\infty, b]$, and in cases (2) and (3) we have $\Theta(\tilde{\mathcal{M}}', \cdot) \geq 2$ on all of $\tilde{\Sigma} \times (-\infty, b]$ or $\partial\mathbb{H} \times (-\infty, b]$, respectively.

PROOF. We proceed as in the proof of Theorem 6.21 (tangent flows). In the limit flow setting, we have $D' = \mathbb{R}^{n+1}$ or $D' = \mathbb{H}$, so the dilates $\mathcal{D}_{r_i^{-1}}\mathcal{K}'$ correspond in case (1) and (2) to the region between the graphs of $f_i \leq g_i$. However, it is possible to have $f_i \equiv g_i$ for limit flows, and in fact the proof of Theorem 6.21 shows that this must be the case for sufficiently large i . This immediately implies that, in some backwards parabolic neighborhood of X , the flow \mathcal{K}' is a smooth static (free boundary) mean curvature flow, which yields (1) and (2).

Similarly, in case (3) we see that $g_i \equiv 0$ for large enough i , which yields that \mathcal{K}' equals $\partial\mathbb{H}$ in a backwards parabolic neighborhood of X .

The final assertion follows from arguing similarly as in [45, proof of theorem 12.2]. \square

THEOREM 6.23 (Limit flows). *Let $(\mathcal{M}, \mathcal{K})$ be a mean-convex free boundary flow in a mean-convex domain D . Then (quasi)static density-2 planes or half-planes do not occur as limit flows.*

PROOF. We follow the proof strategy of [45, theorem 12.3], with some adjustments. Suppose towards a contradiction that there is a blowup sequence $(\mathcal{M}^i, \mathcal{K}^i)$ that converges to a (quasi)static density-2 plane or half-plane $(\mathcal{M}^\infty, \mathcal{K}^\infty)$. Fix $\delta > 0$ as in Theorem 6.20 (isolation).

Adjusting the sequence a bit we can assume that $0 \in \partial D^i \cap \partial K_0^i$, and that the inward unit normal of ∂D^i at 0 is given by e_1 .

Case 1. $(\mathcal{M}^\infty, \mathcal{K}^\infty)$ is a density-2 half-plane.

Let $\mu_i > 0$ be the smallest number for which

$$(6.49) \quad \phi_+(\mathcal{D}_{\mu_i} \mathcal{K}^i) \geq \delta/2.$$

Note that $\mu_i \rightarrow 0$ by hypothesis. Let $(\mathcal{K}', \mathcal{M}')$ be a subsequential limit of $(\mathcal{D}_{\mu_i} \mathcal{K}^i, \mathcal{D}_{\mu_i} \mathcal{M}^i)$. The limit satisfies

$$\phi_+(\mathcal{K}') \geq \delta/2,$$

but

$$(6.50) \quad \phi_+(\mathcal{D}_\mu \mathcal{K}') \leq \delta/2 \quad \text{for } \mu > 1.$$

By equation (6.50) and Theorem 6.20 (isolation) any tangent flow $(\mathcal{M}'', \mathcal{K}'')$ to $(\mathcal{M}', \mathcal{K}')$ at the origin must be a multiplicity 2 half-plane. In particular, by Theorem 6.21, $(\mathcal{M}', \mathcal{K}')$ must be a limit flow, not a dilate of $(\mathcal{M}, \mathcal{K})$.

Now, by Lemma 6.22 there exists a $b \in \mathbb{R}$ and a free boundary minimal hypersurface Σ in an open neighborhood $U \subset \mathbb{H}$ of the origin, so that

$$(6.51) \quad \Theta(\tilde{\mathcal{M}}', \cdot) \geq 2 \quad \text{on } \tilde{\Sigma} \times (-\infty, b].$$

Consider the time translates of $(\tilde{\mathcal{M}}', \tilde{\mathcal{K}}')$ by $(x, t) \rightarrow (x, t + j)$; sending $j \rightarrow \infty$ we get a static limit flow $(\hat{\mathcal{M}}, \hat{\mathcal{K}})$, with

$$(6.52) \quad \hat{K} = \hat{K}_t \equiv \bigcup_{\tau} K'_\tau$$

and

$$(6.53) \quad \Theta(\hat{\mathcal{M}}, \cdot) \geq 2 \quad \text{on } \Sigma \times \mathbb{R}.$$

Together with (6.49) and monotonicity this implies

$$(6.54) \quad \text{Ent}[\hat{\mathcal{M}}] > 2.$$

Case 2. $(\mathcal{M}^i, \mathcal{K}^i)$ converges to a (quasi)static density 2 plane.

When scaling down along the sequence there is the potential scenario that the barrier comes back in from infinity. To deal with this, instead of ϕ_+ , we consider the more general quantity

$$(6.55) \quad \begin{aligned} \phi_h(\mathcal{K}') &\text{ is the infimum of } s > 0 \text{ such that} \\ &\inf_{\mathcal{H}} d(\mathcal{K}' \cap P(0, s^{-1}), \mathcal{H} \cap P(0, s^{-1})) < s, \end{aligned}$$

where $\inf_{\mathcal{H}}$ is now taken over all half-planes $\mathcal{H} = \mathcal{V} \cap \{x_1 \geq a\}$, $a \leq 0$, and \mathcal{V} is a static plane intersecting $\{x_1 = 0\}$ orthogonally at 0 (in particular, $\mathcal{H} \ni (0, 0)$ and the inner conormal on $\partial \mathcal{H}$ is given by e_1).

Consider the quantity

$$(6.56) \quad \psi = \min(\phi, \phi_h),$$

where ϕ is defined in (6.39) and ϕ_h is defined in (6.55). Let $\mu_i > 0$ be the smallest number for which

$$(6.57) \quad \psi(\mathcal{D}_{\mu_i} \mathcal{K}^i) \geq \delta/2.$$

Note that $\mu_i \rightarrow 0$. Let $(\mathcal{K}', \mathcal{M}')$ be a subsequential limit, defined in D' , of $\mathcal{D}_{\mu_i}(\mathcal{K}^i, \mathcal{M}^i)$. The limit satisfies

$$(6.58) \quad \psi(\mathcal{K}') \geq \delta/2,$$

but

$$(6.59) \quad \psi(\mathcal{D}_{\mu} \mathcal{K}') \leq \delta/2 \quad \text{for } \mu > 1.$$

In particular, if $(\mathcal{K}'', \mathcal{M}'')$ is a tangent flow to $(\mathcal{K}', \mathcal{M}')$ at 0, then

$$(6.60) \quad \psi(\mathcal{K}'') \leq \delta/2.$$

We consider the following subcases:

Case 2a. 0 is an interior point of D' .

In this case, it follows that

$$(6.61) \quad \phi(\mathcal{K}'') \leq \delta/2,$$

and thus \mathcal{K}'' is a multiplicity 2 plane by Theorem 6.20 (isolation).

Case 2b. 0 is a boundary point of $D' = \mathbb{H}$, and $\phi_h(\mathcal{K}'') \leq \delta/2$.

In this case, it follows that $\phi_+(\mathcal{K}'') \leq \delta/2$, so by Theorem 6.20 (isolation) \mathcal{K}'' must be a multiplicity 2 half-plane.

Case 2c. 0 is a boundary point of $D' = \mathbb{H}$, and $\phi(\mathcal{K}'') \leq \delta/2$.

Then, by Theorem 6.20 (isolation), we have $\phi(\mathcal{K}'') = 0$ and \mathcal{K}'' is the barrier plane.

In all cases 2a, 2b, and 2c, Theorem 6.21 (tangent flows) implies that $(\mathcal{M}', \mathcal{K}')$ must be a limit flow, not a dilate of $(\mathcal{M}, \mathcal{K})$. By applying Lemma 6.22 (at different centers), it follows that there exists $b \in \mathbb{R}$ and Σ containing the origin such that \mathcal{M}' has (reflected) density at least 2 on $\Sigma \times (-\infty, b]$, where Σ is given by either:

- a minimal hypersurface in \mathbb{R}^{n+1} (Case 2a),
- a free boundary minimal hypersurface in \mathbb{H} (Case 2a or 2b), or
- the barrier plane $\partial\mathbb{H}$ (Case 2c).

In particular, the (possibly reflected) flow satisfies

$$(6.62) \quad \Theta(\tilde{\mathcal{M}}', \cdot) \geq 2 \quad \text{on } \tilde{\Sigma} \times (-\infty, b],$$

where $\tilde{\Sigma}$ is a minimal hypersurface in \mathbb{R}^{n+1} .

Again taking translates of $(\mathcal{M}', \mathcal{K}')$ and $(\tilde{\mathcal{M}}', \tilde{\mathcal{K}}')$ by $(x, t) \mapsto (x, t + j)$ and sending $j \rightarrow \infty$, we get a static limit flow $(\hat{\mathcal{M}}, \hat{\mathcal{K}})$ defined in \mathbb{R}^{n+1} . In each Case 2a, 2b, and 2c, (6.58) prevents this flow from being planar, so by again monotonicity it must satisfy

$$(6.63) \quad \text{Ent}[\hat{\mathcal{M}}] > 2.$$

In all cases 1 and 2, we have thus constructed a static (possibly reflected) limit flow $(\widehat{\mathcal{M}}, \widehat{\mathcal{K}})$ defined in \mathbb{R}^{n+1} that has entropy strictly larger than 2. Let $(\mathcal{M}^*, \mathcal{K}^*)$ be a blowdown limit (i.e., a tangent flow at infinity) of $(\widehat{\mathcal{M}}, \widehat{\mathcal{K}})$. Then $(\mathcal{M}^*, \mathcal{K}^*)$ is a static cone of multiplicity strictly larger than 2. Hence, by one-sided minimization it must be nonflat, contradicting the bound $|A| \leq CH$. This finishes the proof of the theorem. \square

7 Conclusion

In this final section, we explain how to use Theorem 1.6 (elliptic regularization and consequences) and multiplicity 1 (see Section 6) to conclude the proofs of Theorem 1.1 (size of the singular set), Theorem 1.2 (structure of singularities) and Theorem 1.5 (long-time behavior).

7.1 Size of the singular set

PROOF OF THEOREM 1.1. By the local regularity theorem [10, 47] a point X is regular if and only if one sees a density 1 plane or half-plane as tangent flow at X . By the results from Section 6 (multiplicity 1) we have several restrictions on the possible tangent flows. Namely, in case the barrier is mean-convex the tangent flows at singular points cannot be static or quasi-static, but must be self-similar shrinkers or self-similar shrinkers with free boundary, respectively. For a general barrier, the only additional case that can occur is quasi-static density-2 planes. Hence, the assertion follows from dimension reduction; cf. [44, sec. 9]. \square

7.2 Structure of singularities

PROOF OF THEOREM 1.2. Let $(\mathcal{M}, \mathcal{K})$ be a mean-convex free boundary flow in (D^{n+1}, g) , and assume that D is mean-convex.

Given a point X_0 in the support of the flow we consider

$$(7.1) \quad \begin{aligned} \mathcal{F}_1 := \{(\mathcal{M}', \mathcal{K}') \mid & (\mathcal{M}', \mathcal{K}') \text{ is a limit flow at } X_0, \\ & \text{which is defined in entire space}\} \end{aligned}$$

and

$$(7.2) \quad \begin{aligned} \mathcal{F}_2 := \{(\widetilde{\mathcal{M}}', \widetilde{\mathcal{K}}') \mid & (\mathcal{M}', \mathcal{K}') \text{ is a limit flow at } X_0, \\ & \text{which is defined in a half-space}\}. \end{aligned}$$

Let

$$(7.3) \quad \mathcal{F} := \mathcal{F}_1 \cup \mathcal{F}_2.$$

As explained in the proof of Theorem 1.1 (size of the singular set), the class \mathcal{F} does not contain any singular stationary cones. Moreover, whenever a tangent flow of some flow in the class \mathcal{F} at some point is a static or quasi-static plane, then it is in fact a static multiplicity 1 plane. Hence the arguments from White's second paper [46] apply to the class \mathcal{F} (actually with some simplifications thanks to the a priori bound $|A| \leq CH$ from Theorem 1.6), yielding all assertions of Theorem

[1.2] except for the last bullet point. In particular, note that the class \mathcal{F} cannot contain any doubling of a (quasi)-static plane in a half-space parallel to $\partial\mathbb{H}$, since otherwise it would also contain $\partial\mathbb{H}$. Finally, the assertion in the last bullet point follows from [17, cor. 1.5]. \square

7.3 Long-time behavior

PROOF OF THEOREM [1.5]. We first observe that the proof of the Bernstein-type theorem (Theorem 6.8) goes through if instead of the $|A| \leq CH$ bound (which unfortunately degenerates as $t \rightarrow \infty$) one uses the Schoen-Simon result about (mildly singular) minimal hypersurfaces in a slab [39] similarly as in [45, cor. 7.3]. Consequently, the sheeting theorem (Theorem 6.10) can also be applied as $t \rightarrow \infty$.

Now, suppose $(\mathcal{M}, \mathcal{K})$ is a mean-convex free boundary flow in a mean-convex domain D such that

$$(7.4) \quad K_\infty := \bigcap_t K_t \neq \emptyset.$$

We argue as in [45, sec. 11]. Let $(\mathcal{M}^T, \mathcal{K}^T)$ be the result of translating $(\mathcal{M}, \mathcal{K})$ by $(x, t) \mapsto (x, t - T)$. Then a subsequence will converge to a limit $(\mathcal{M}', \mathcal{K}')$. Note that

$$(7.5) \quad \mathcal{K}' = \bigcap_t K_t \times \mathbb{R}.$$

This is independent of the sequence of times going to infinity. Together with the fact that \mathcal{M}' is determined by \mathcal{K}' (the support of \mathcal{M}' equals $\partial\mathcal{K}'$, and multiplicity 1 versus-2 is determined by whether the component of \mathcal{K}' under consideration has interior points), we infer that

$$(7.6) \quad (\mathcal{M}^T, \mathcal{K}^T) \rightarrow (\mathcal{M}', \mathcal{K}')$$

as $T \rightarrow \infty$ (i.e., it is not necessary to pass to a subsequence). By (7.5) the limit $(\mathcal{M}', \mathcal{K}')$ is static, i.e.,

$$(7.7) \quad \text{spt}(\mathcal{M}') = \partial K_\infty \times \mathbb{R}, \quad K'_t = K_\infty.$$

By one-sided minimization (Theorem 5.4) and the sheeting theorem (Theorem 6.10), ∂K_∞ is a union of finitely many stable free boundary minimal surfaces. Hence, by Simons [40] and Grüter [15] the dimension of the singular set is at most $n - 7$. This finishes the proof of Theorem 1.5. \square

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NICK EDELEN
University of Notre Dame
255 Hurley Bldg
Notre Dame, IN 46556
USA
E-mail: nedelen@nd.edu

ROBERT HASLHOFER
University of Toronto
40 St George St
Toronto, ON M5S 2E4
CANADA
E-mail: roberth@math.toronto.edu

MOHAMMAD N. IVAKI
Technische Universität Wien
Wiedner Hauptstr 8-10
1040 Wien
AUSTRIA
E-mail: mohammad.ivaki@tuwien.ac.at

JONATHAN J. ZHU
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544
USA
E-mail: jjzhu@math.princeton.edu

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