Communications in Nonlinear Science and Numerical Simulation 104 (2022) 106047

Contents lists available at ScienceDirect

Communications in Nonlinear Science and Numerical Simulation

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Analysis and discretization of a variable-order fractional wave equation

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ARTICLE INFO

Article history:
Received 8 November 2020
Received in revised form 31 July 2021
Accepted 12 September 2021
Available online 14 September 2021

Keywords:
Fractional wave equation of variable order
Viscoelastic modeling
Well-posedness and regularity
Finite element method and error estimate

ABSTRACT

We analyze a variable-order time-fractional wave equation, which models, e.g., the vibration of a membrane in a viscoelastic environment. We prove that the solutions to the variable-order ordinary differential equations in the spectral decomposition of the solution to the fractional wave equation exhibit power-law decaying characteristics and overcome the difficulty that its solution operator does not have an exponential decay in contrast to its variable-order fractional diffusion analogue.

We prove an optimal-order error estimate of a numerical discretization of the variable-order fractional wave equation only under regularity assumptions of the data of the model but with no smoothness assumption of its solution. As the solution exhibits initial weak singularity, the local truncation error is suboptimal. A conventional analysis gives a suboptimal-order estimate. We develop a new technique to derive the desired optimal-order convergence rate. We also conduct numerical experiments to substantiate the mathematically proved findings.

1. Introduction

Fractional partial differential equations provide adequate descriptions for complex dynamics in many disciplines and applications that exhibit power-law decaying behavior [1–14]. For instance, fractional wave equations were used to model the viscoelastic phenomena [1,7,9,15–21]. The corresponding numerical and mathematical analysis relies heavily on special-function-based solution representations [3,5,7,9,22,23]. Similar to its diffusion analogue [24–26], the time-fractional wave equation (FWE) exhibits initial weak singularity that does not seem physically relevant. Consequently, many results that were proved assuming that the solutions are smooth are deemed inappropriate.

The inherent cause is these equations exhibit two time scales. In the former, a diffusive process in a heterogeneous material is Brownian initially and then becomes anomalous as time evolves. The field diffusion equation [27–29] has the attractive property that it has the long-term subdiffusion behavior typical of fractional diffusion models yet no nonphysical initial weak singularity. In the vibration of membrane in viscoelastic media, the vibration process exhibits instantaneous elastic behavior near the initial time and viscoelastic behavior as time evolves. The corresponding FWE contains both integer-order and fractional-order time derivatives that account for the inertial and viscoelastic behavior, respectively [30,31]. Further, as the body undergoes vibrations due to external forces, the frictions convert some of the kinetic energy into heat and damps the vibrations of the body. These processes change the structures and properties of the

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https://doi.org/10.1016/j.cnsns.2021.106047
1007-5704/

medium, which gives rises to the variation of the fractal dimension of the medium that has an impact on the fractional order of the field equation [32,33]. This leads to the variable-order FWE (cf. Section 2)

$$\big(\partial_t^2 + k(t)\partial_t^{\alpha(t)} - K\Delta\big)u(\boldsymbol{x},t) = f(\boldsymbol{x},t) \tag{1}$$

for $\boldsymbol{x} \in \Omega$ and $t \in (0,T]$, equipped with the zero boundary condition and the initial conditions $u(\boldsymbol{x},0) = u_0(\boldsymbol{x})$ and $\partial_t u(\boldsymbol{x},0) = \check{u}_0(\boldsymbol{x})$. Here Ω is a l-dimensional convex polytope for $1 \le l \le 3$ with the boundary $\partial\Omega$, Δ is the Laplacian, and the fractional deriative reads [34–39]

$$\partial_t^{\alpha(t)} g := I_t^{1-\alpha(t)}\partial_t g(t), \quad 0 < \alpha(t) < 1; \quad I_t^{\alpha(t)} g := \int_0^t \frac{(t-s)^{\alpha(t)-1}}{\Gamma(\alpha(t))} g(s)ds.$$

Mathematical analysis of well-posedness and regularity of variable-order FWEs is rare due to the following reasons: **(i)** They cannot be analyzed as their constant-order counterparts via the closed-form analytical expressions in terms of special functions [3,5,7,9,22,23]. **(ii)** These problems cannot be analyzed as their variable-order time-fractional diffusion analogue [38,40,41]. The reason is that the analysis in [38,40] relies heavily on the exponential decaying of the solution operator of the variable-order diffusion equation, which are used to balance the exponential growth of the solutions to the variable-order ordinary differential equations (ODEs) in the spectral decomposition with respect to the eigenvalues λ_i (cf. Section 4). However, the solution operator of problem (1) does not decay exponentially, and so cannot balance the exponential growth on λ_i in the estimates of solutions to variable-order wave differential equations [38]. **(iii)** Convergence estimates of numerical approximations to variable-order FWEs were proved under the false assumptions of sufficiently smoothness of the solutions. In fact, the existence of the solution to problem (1) was never proved.

In this paper we develop a new technique to prove a greatly improved polynomial growth of the solutions of the variable-order wave ordinary differential equations in the spectral decomposition of problem (1) with respect to the eigenvalues λ_i from the previously proved exponential growth. We then accordingly prove the well-posedness and regularity estimates of the variable-order FWE (1). We prove an optimal convergence rate of a numerical discretization to problem (1) with no regularity presumption on its solution. As the solution exhibits initial weak singularity, the local truncation error is suboptimal. A conventional analysis gives a suboptimal-order estimate. We develop a new technique to prove its optimal convergence rate. We numerically substantiate the theoretical findings.

The paper is organized as follows: In Section 2 we present a variable-order FWE to model the vibration of a membrane in the viscoelastic medium. In Section 3 we present preliminaries and introduce notations, spaces and assumptions. In Section 4 we mathematically analyze problem (1). In Section 5 we adopt the order reduction approach [42,43] and employ the L1 discretization [44,45] to derive a numerical scheme and prove its optimal convergence rate. In Section 6 we numerically investigate the discretization and substantiate the mathematically proved findings.

2. Modeling issues

To motivate the model problem (1) we consider the vibration of a perfectly elastic membrane tautly stretched along the boundary $\partial\Omega$ of the physical domain Ω in the xy plane as the equilibrium position of the membrane and construct a u axis perpendicular to the xy plane. We assume that the membrane is so flexible that it does not resist deformations and the tension in the membrane acts tangentially, and that the vibrations are so small that a membrane at a point (x,y) in the equilibrium position moves vertically.

Let $u(x,y,t)$ denote the position of the membrane of the initial position (x,y) at time t. Let $\hat{\rho}(x,y)$ denote the density per unit area of the membrane in the equilibrium position and $\rho(x,y,t)$ the density at time t. Denote by $S(A)$ the piece of the membrane with the initial position $A := [x_1,x_2]\times[y_1,y_2]$. Then mass conservation principle yields

$$\int_A \hat{\rho}(x,y)dA = \int_{S(A)} \rho(x,y,t)dS = \int_A \rho(x,y,t)\big[1 + (\partial_x u)^2 + (\partial_y u)^2\big]^{\frac{1}{2}} dA. \tag{2}$$

Since the subregion A is arbitrary, we conclude from (2) that

$$\hat{\rho}(x,y) = \rho(x,y,t)\big[1 + (\partial_x u)^2 + (\partial_y u)^2\big]^{\frac{1}{2}}. \tag{3}$$

Let $T(x,y,t)$ be the linear density of tension, and $\theta_x(x,y,t)$ and $\theta_y(x,y,t)$ be the angles between the x-axis and the tangent of the surface $S(A)$ in the x direction and between the y-axis and the tangent of the surface in the y-axis, respectively. Since only vertical motion takes place, the forces in the horizontal directions balance

$$\int_{y_1}^{y_2} T(x_2,y,t)\cos\theta_x(x_2,y,t)dy - \int_{y_1}^{y_2} T(x_1,y,t)\cos\theta_x(x_1,y,t)dy = 0,$$
$$\int_{x_1}^{x_2} T(x,y_2,t)\cos\theta_y(x,y_2,t)dx - \int_{x_1}^{x_2} T(x,y_1,t)\cos\theta_y(x,y_1,t)dx = 0.$$

These lead to the force relations in the x and y directions

$$T(x,y,t)\cos\theta_x(x,y,t) = \tau_x, \qquad T(x,y,t)\cos\theta_y(x,y,t) = \tau_y. \tag{4}$$

The time rate of change of the momentum of the vertical motion of the surface $S(A)$ of the membrane is given by

$$\begin{aligned}\frac{d}{dt}\int_{S(A)}\rho(x,y,t)\partial_t u dS &= \frac{d}{dt}\int_A \rho(x,y,t)\big[1+(\partial_x u)^2+(\partial_y u)^2\big]^{\frac{1}{2}}\partial_t u\, dA\\ &= \frac{d}{dt}\int_A \hat{\rho}(x,y)\partial_t u\, dA = \int_A \hat{\rho}(x,y)\partial_t^2 u\, dA.\end{aligned} \tag{5}$$

We use relations (4) to express the vertical component of the tension acting on the boundary ∂A as follows

$$\begin{aligned}&\int_{y_1}^{y_2} T(x_2,y,t)\sin\theta_x(x_2,y,t)dy - \int_{y_1}^{y_2} T(x_1,y,t)\sin\theta_x(x_1,y,t)dy\\ &\quad + \int_{x_1}^{x_2} T(x,y_2,t)\sin\theta_y(x,y_2,t)dx - \int_{x_1}^{x_2} T(x,y_1,t)\sin\theta_y(x,y_1,t)dx\\ &= \int_{y_1}^{y_2}\tau_x\int_{x_1}^{x_2}\partial_z\tan\theta_x(z,y,t)dzdy + \int_{x_1}^{x_2}\tau_y\int_{y_1}^{y_2}\partial_z\tan\theta_y(x,z,t)dzdx\\ &= \int_{y_1}^{y_2}\tau_x\big[\partial_x u(x_2,y,t)-\partial_x u(x_1,y,t)\big]dy\\ &\quad + \int_{x_1}^{x_2}\tau_y\big[\partial_y u(x,y_2,t)-\partial_y u(x,y_1,t)\big]dx = \int_A\big(\tau_x\partial_x^2 u+\tau_y\partial_y^2 u\big)dA.\end{aligned} \tag{6}$$

Let $f(x,y,t)$ be an external force density acting on the surface per unit area. We model the resultant force by

$$\begin{aligned}\int_{S(A)}\rho(x,y,t)f dS &= \int_A \rho(x,y,t)\big[1+(\partial_x u)^2+(\partial_y u)^2\big]^{\frac{1}{2}} f dA\\ &= \int_A \hat{\rho}(x,y) f dA.\end{aligned} \tag{7}$$

We combine (5), (6), and (7) and apply Newton's second law to obtain

$$\int_A \hat{\rho}(x,y)\partial_t^2 u\, dA = \int_A\big(\tau_x\partial_x^2 u+\tau_y\partial_y^2 u\big)dA + \int_A \hat{\rho}(x,y) f dA. \tag{8}$$

Since A is arbitrary, we derive from (8) the following wave equation describing the vibration of membrane by assuming $\tau_x = \tau_y$ and so $K := \tau_x/\hat{\rho}$

$$\partial_t^2 u(x,y,t) = K\Delta u(x,y,t) + f(x,y,t) \tag{9}$$

In case that the external force $f = 0$, Eq. (9) reduces to the damping-free wave equation that conserves total mechanical energy and the magnitude of the vibrations does not decay in time, which contradicts to the physical reality.

One of the fundamental reasons is that the model did not take into account for the fact that the surrounding medium where the vibrations are taking place will impede the motion. A conventional form of the law of friction

$$\begin{aligned}-\int_{S(A)}\rho(x,y,t)k\partial_t u dS &= -\int_A \rho(x,y,t)\big[1+(\partial_x u)^2+(\partial_y u)^2\big]^{\frac{1}{2}} k\partial_t u dA\\ &= -\int_A \hat{\rho}(x,y)k\partial_t u\, dA.\end{aligned} \tag{10}$$

Putting (10) into the right side of (8) yields the damped wave equation

$$\partial_t^2 u(x,y,t) + k\partial_t u(x,y,t) = K\Delta u(x,y,t) + f(x,y,t). \tag{11}$$

Eq. (11) describes the damping effect of the vibrations of membrane in a surrounding viscous medium, e.g., in water. Many materials, such as biological tissues, polymers, and hydrocarbon, may simultaneously exhibit the properties of both elastic materials and viscous materials, and so are viscoelastic and the law of friction is modeled by time-fractional derivative [1,7,9]

$$\begin{aligned}-\int_{S(A)}\rho(x,y,t)k\partial_t^\alpha u dS &= -\int_A \rho(x,y,t)\big[1+(\partial_x u)^2+(\partial_y u)^2\big]^{\frac{1}{2}} k\partial_t^\alpha u dA\\ &= -\int_A \hat{\rho}(x,y)k\partial_t^\alpha u dA,\quad 0<\alpha<1.\end{aligned}$$

A similar derivation to (11) yields the FWE

$$\partial_t^2 u(x,y,t) + k\partial_t^\alpha u(x,y,t) = K\Delta u(x,y,t) + f(x,y,t). \tag{12}$$

As the membrane undergoes vibrations due to external forces, the frictions with the surrounding viscoelastic medium convert some of the kinetic energy into heat and damps the vibrations of the membrane. Then the fractional order in the constitutive relation of the friction (12) will change in response to the varying of the micro-structure of the materials. In other words, the mathematical model is better described by the variable-order FWE (1).

3. Preliminaries

For a non-negative integer m and a real number $p \geq 1$, define the Sobolev space $W^{m,p}(\Omega)$ in a standard manner [46] with $L^p(\Omega) := W^{0,p}(\Omega)$ and $H^m(\Omega) := W^{m,2}(\Omega)$. Let $C^m(\Omega)$ be the space of mth continuously differentiable functions. The definitions of their norms follows [46] and we could replace Ω by an interval $\mathcal{I}$ to define corresponding spaces in time. We also define the time-dependent space $C^m(\mathcal{I};\mathcal{Y})$ for some Banach space $\mathcal{Y}$ endowed with [46,47]

$$\|g\|_{C^m(\mathcal{I};\mathcal{Y})} := \max_{1\leq s\leq m} \sup_{t\in\mathcal{I}} \|\partial_t^s g(\cdot,t)\|_{\mathcal{Y}}.$$

Denote the eigen-pairs of $-\Delta$ with zero boundary conditions by $\{\lambda_i^2, \varphi_i\}_{i=1}^\infty$ where $\{\varphi_i\}_{i=1}^\infty$ serves as an orthonormal L^2 basis and $0 < \lambda_1 < \lambda_2 < \cdots$ [47]. Furthermore, for any $\gamma \geq 0$ we introduce the Sobolev space [25,48]

$$\check{H}^\gamma(\Omega) := \left\{ v \in L^2(\Omega) : \|v\|^2_{\check{H}^\gamma} := \sum_{i=1}^\infty \lambda_i^{2\gamma}(v,\varphi_i)^2 < \infty \right\}.$$

We impose the **Assumption A:** $\alpha \in C[0,T]$ and $\alpha \in (0,1)$, which implies that there exist constants $0 < \alpha_* < \alpha^* < 1$ such that $\alpha \in [\alpha_*, \alpha^*]$. In the rest of the paper $Q > 0$ refers to a generic constant and we simply set $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$ and omit the notation Ω in the aforementioned spaces and their norms.

4. Analysis of the variable-order FWE

Let $\{u_i(t), u_{0,i}, \check{u}_{0,i}, f_i(t)\}_{i=1}^\infty$ be the Fourier coefficients of u, u_0, $\check{u}_0$, and f, respectively, in terms of $\{\phi_i\}_{i=1}^\infty$. Then model (1) could be reduced to a system of ODEs [25,26,38,40]

$$\begin{aligned} &\left(u_i'' + k(t)\partial_t^{\alpha(t)} u_i + \lambda_i^2 u_i\right)(t) = f_i(t), \quad t \in (0,T], \\ &\quad u_i(0) = u_{0,i}, \quad u_i'(0) = \check{u}_{0,i}, \quad i = 1,2,\ldots. \end{aligned} \tag{13}$$

In principle, we could analyze (1) as those for the variable-order time-fractional diffusion PDEs [38,40]. However, the analysis in [38,40] relies heavily on the exponential decaying property of their solution operators, which are used to balance the exponential growth of the stability and regularity estimates of $u_i(t)$ on λ_i in the time-fractional diffusion ODE analogue of problem (13) in [49]. As the solution operator of (1) does not have the exponential decaying, and so cannot be used to balance the exponential growth of the stability and regularity estimates of u_i on λ_i in the estimates of solutions to variable-order time-fractional wODEs [38].

4.1. Analysis of the variable-order time-fractional wODE (13)

We analyze the variable-order time-fractional wODE

$$\left(v'' + k(t)\partial_t^{\alpha(t)} v + \lambda^2 v\right)(t) = g(t), \quad t \in (0,T]; \quad v(0) = v_0, \ v'(0) = \check{v}_0. \tag{14}$$

Here $\lambda > 0$ is a prescribed constant, v_0, $\check{v}_0$ and $g(t)$ are given data.

Theorem 4.1. *If k, $\alpha \in C^1[0,T]$, $g \in H^1(0,T)$, and Assumption A holds, then the fractional ODE (14) has a unique solution in $C^2[0,T]$ and*

$$\|v\|_{C^m[0,T]} \leq Q\left(\lambda^m|v_0| + \lambda^{-1+m}|\check{v}_0| + \lambda^{-2+m}\|g\|_{H^1(0,T)}\right), \quad m = 1,2. \tag{15}$$

If further $g \in C^1[0,T]$, then $u''' \in C(0,T]$ and for any $t \in (0,T]$

$$|v'''(t)| \leq Q\left(\lambda^3|v_0| + \lambda^2|\check{v}_0| + \lambda\|g\|_{H^1(0,T)} + |g'| + t^{-\alpha(0)}|\check{v}_0|\right). \tag{16}$$

Here the constant Q is independent of λ, v_0, $\check{v}_0$, or g.

Proof. We move $k(t)\partial_t^{\alpha(t)}v(t)$ in (14) to the right side and treat the equation formally as a second-order constant-coefficient inhomogeneous ODE of the form

$$v''(t) + \lambda^2 v(t) = G(t) := g(t) - k(t)\partial_t^{\alpha(t)} v(t); \quad v(0) = v_0, \ v'(0) = \check{v}_0, \tag{17}$$

which leads to

$$v(t) = v_0\cos(\lambda t) + \frac{\check{v}_0}{\lambda}\sin(\lambda t) + \frac{1}{\lambda}G(t) * \sin(\lambda t). \tag{18}$$

We combine (17)–(18) and integrate the convolution term in (18) by parts to obtain

$$\begin{aligned} v''(t) &= G(t) - v_0\lambda^2\cos(\lambda t) - \check{v}_0\lambda\sin(\lambda t) - \lambda G(t)*\sin(\lambda t)\\ &= -\lambda^2 v_0\cos(\lambda t) - \lambda\check{v}_0\sin(\lambda t) + G(0)\cos(\lambda t) + G'(t)*\cos(\lambda t). \end{aligned} \tag{19}$$

We evaluate $G'(s)$ in (19) as follows

$$\begin{aligned} G'(s) &= -\Big(k(s)\int_0^s \frac{v'(y)dy}{\Gamma(1-\alpha(s))(s-y)^{\alpha(s)}}\Big)' + g'(s)\\ &= -\Big(\mathcal{Q}(0,s)\check{v}_0 + \int_0^s \mathcal{Q}(y,s)v''(y)dy\Big)' + g'(s)\\ &= -\mathcal{Q}(0,s)'\check{v}_0 - \int_0^s \partial_s\mathcal{Q}(y,s)v''(y)dy + g'(s), \quad \mathcal{Q}(y,s) := k(s)\frac{(s-y)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))}. \end{aligned}$$

We evaluate the convolution of the second term in the above equation with $\cos(\lambda t)$ to get

$$\int_0^t \cos(\lambda(t-s))\int_0^s \partial_s\mathcal{Q}(y,s)v''(y)dyds = \int_0^t v''(y)\int_y^t \partial_s\mathcal{Q}(y,s)\cos(\lambda(t-s))dsdy.$$

We incorporate the preceding two equations into (19) and rewrite (19) in terms of $w = v''$

$$\begin{aligned} w(t) &= -\int_0^t\int_y^t \partial_s\mathcal{Q}(y,s)\cos(\lambda(t-s))ds\,w(y)dy - \lambda^2 v_0\cos(\lambda t)\\ &\quad -\lambda\check{v}_0\sin(\lambda t) + g(0)\cos(\lambda t) + \Big(-\mathcal{Q}(0,t)'\check{v}_0 + g'(t)\Big)*\cos(\lambda t). \end{aligned} \tag{20}$$

We apply the theory of integral equations ([50, Theorem 2.1.1]) and use $|g(0)| \le Q\|g\|_{H^1(0,T)}$ to conclude that (20) admits a unique solution $w \in C[0,T]$ and

$$\|w\|_{C[0,T]} \le Q\big(\lambda^2|v_0| + \lambda|\check{v}_0| + \|g\|_{H^1(0,T)}\big). \tag{21}$$

Consequently, $v(t) = w(t)*t + t\check{v}_0 + v_0 \in C^2[0,T]$ is the solution to (14). Hence, estimate (15) with $m = 2$ holds by (21). The uniqueness of the C^2 solution to (14) follows from the uniqueness of integral equation (18) in $C[0,T]$.

To estimate v', we differentiate (18) and use

$$\begin{aligned} &\big(-k(t)\partial_t^{\alpha(t)}v(t)\big)*\cos(\lambda t) = \int_0^t v'(y)\int_y^t \mathcal{H}(y,s)dsdy,\\ &\qquad \mathcal{H}(y,s) := \frac{-k(s)\cos(\lambda(t-s))}{\Gamma(1-\alpha(s))(s-y)^{\alpha(s)}},\\ &g(t)*\cos(\lambda t) = g(0)\lambda^{-1}\sin(\lambda t) + \lambda^{-1}g'(t)*\sin(\lambda t) \end{aligned}$$

to obtain

$$\begin{aligned} v'(t) &= -\lambda v_0\sin(\lambda t) + \check{v}_0\cos(\lambda t) + G(t)*\cos(\lambda t)\\ &= \int_0^t v'(y)\int_y^t \mathcal{H}(y,s)dsdy - \lambda v_0\sin(\lambda t)\\ &\quad + \check{v}_0\cos(\lambda t) + \frac{g(0)\sin(\lambda t)}{\lambda} + \frac{g'(t)*\sin(\lambda t)}{\lambda}. \end{aligned}$$

Applying Gronwall's inequality yields (15) with $m = 1$.

If $k, \alpha \in C^1[0,T]$ and $g \in C^1[0,T]$, the kernel of the first term and the remaining two terms on the right side of (20) are continuously differentiable on $(0,T]$. This implies $w \in C^1(0,T]$. We differentiate (20) on $(0,T]$ to obtain

$$\begin{aligned} w'(t) &= -\int_0^t \partial_t\mathcal{Q}(y,t)w(y)dy + \lambda\int_0^t\int_y^t \partial_s\mathcal{Q}(y,s)\sin(\lambda(t-s))ds\,w(y)dy\\ &\quad +\lambda^3 v_0\sin(\lambda t) - \lambda^2\check{v}_0\cos(\lambda t) - \lambda g(0)\sin(\lambda t) - \mathcal{Q}'(0,t)\check{v}_0\\ &\quad +g'(t) - \lambda\big(-\mathcal{Q}'(0,t)\check{v}_0 + g'(t)\big)*\sin(\lambda t). \end{aligned}$$

We incorporate (21) into the preceding equation and use the fact

$$\begin{aligned} \big|\big(t^{1-\alpha(t)}\big)'\big| &= \big|t^{1-\alpha(t)}\big(-\alpha'(t)\ln t + (1-\alpha(t))/t\big)\big| \le Qt^{-\alpha(t)}\\ &= Qt^{-\alpha(0)}t^{\alpha(0)-\alpha(t)} = Qt^{-\alpha(0)}e^{(\alpha(0)-\alpha(t))\ln t} \le Qt^{-\alpha(0)} \end{aligned}$$

to get

$$|v'''(t)| = |w'(t)| \le Q\big(\lambda\|w\|_{C[0,T]} + \lambda^3|v_0| + \lambda^2|\check{v}_0| + \lambda\|g\|_{H^1(0,T)} + |g'| + t^{-\alpha(0)}|\check{v}_0|\big) \le Q\big(\lambda^3|v_0| + \lambda^2|\check{v}_0| + \lambda\|g\|_{H^1(0,T)} + |g'| + t^{-\alpha(0)}|\check{v}_0|\big).$$

We thus complete the proof. ■

4.2. Analysis of model (1)

Theorem 4.2. *Suppose that $\alpha, k \in C^1[0,T]$, $u_0 \in \check{H}^2$, $\check{u}_0 \in \check{H}^1$, $f \in H^1(0,T;L^2)$, and Assumption A holds. Then there exists a unique solution for model (1) in $C^2([0,T];L^2) \cap C([0,T];\check{H}^2)$ and*

$$\|u\|_{C^2([0,T];L^2)} + \|u\|_{C([0,T];\check{H}^2)} \le Q\big(\|u_0\|_{\check{H}^2} + \|\check{u}_0\|_{\check{H}^1} + \|f\|_{H^1(0,T;L^2)}\big).$$

If $f \in H^1(0,T;\check{H}^m)$ and u_0, $\check{u}_0$ belong to $\check{H}^{m+2}$, $\check{H}^{m+1}$, respectively, for $m = 1, 2$, then

$$\|u\|_{C^m([0,T];\check{H}^2)} \le Q\big(\|u_0\|_{\check{H}^{m+2}} + \|\check{u}_0\|_{\check{H}^{m+1}} + \|f\|_{H^1(0,T;\check{H}^m)}\big). \tag{22}$$

Suppose k, $\alpha \in C^2[0,T]$, $u_0 \in \check{H}^3$, $\check{u}_0 \in \check{H}^2$, and $f \in H^1(0,T;\check{H}^1) \cap C^1([0,T];L^2)$, then $u \in C^3((0,T];L^2)$. Moreover, for any $0 < \varepsilon \ll 1$

$$\|u\|_{C^3([\varepsilon,T];L^2)} \le Q\big(\varepsilon^{-\alpha(0)}\|\check{u}_0\|_{L^2} + \|u_0\|_{\check{H}^3} + \|\check{u}_0\|_{\check{H}^2} + \|f\|_{H^1(0,T;\check{H}^1)} + \|f\|_{C^1([0,T];L^2)}\big). \tag{23}$$

Here the constant Q does not depend on u_0, $\check{u}_0$, or f.

Proof. By $u = \sum_{i=1}^\infty u_i(t)\varphi_i(\boldsymbol{x})$ we have

$$u(\boldsymbol{x},t) - u_0(\boldsymbol{x})t - \check{u}_0(\boldsymbol{x}) = \int_0^t (t-s)\bigg(\sum_{i=1}^\infty u_i''(s)\varphi(\boldsymbol{x})\bigg)ds. \tag{24}$$

By Theorem 4.1, (13) admits a unique solution $u_i \in C^2[0,T]$ for $i = 1, 2, \ldots$, and (15) holds with $m = 2$ and λ, v, v_0, $\check{v}_0$ and g replaced by λ_i, u_i, $u_{0,i}$, $\check{u}_{0,i}$ and f_i, respectively, which lead to the following estimate

$$\begin{aligned}\|\partial_t^2 u\|^2_{C([0,T];L^2)} &= \max_{t\in[0,T]}\sum_{i=1}^\infty |u_i''(t)|^2 \le \sum_{i=1}^\infty \|u_i\|^2_{C^2[0,T]} \\ &\le Q\sum_{i=1}^\infty\big(\lambda_i^4|u_{0,i}|^2 + \lambda_i^2|\check{u}_{0,i}|^2 + \|f_i\|^2_{H^1(0,T)}\big) \\ &\le Q\big(\|u_0\|^2_{\check{H}^2} + \|\check{u}_0\|^2_{\check{H}^1} + \|f\|^2_{H^1(0,T;L^2)}\big).\end{aligned} \tag{25}$$

Furthermore, we use (1), (25) and

$$\|u\|_{C([0,T];\check{H}^2)} = \|\Delta u\|_{C([0,T];L^2)} = \frac{1}{|K|}\big\|\partial_t^2 u + k(t)\partial_t^{\alpha(t)}u - f\big\|_{C([0,T];L^2)}$$

to get th estimate of $\|u\|_{C([0,T];\check{H}^2)}$ in this theorem. The uniqueness of (1) in $C^2([0,T];L^2)$ follows from that of (13).

If $u_0 \in \check{H}^3$, $\check{u}_0 \in \check{H}^2$ and $f \in H^1(0,T;\check{H}^1)$, we use (15) with $m = 1$ to obtain

$$\begin{aligned}\|u\|^2_{C^1([0,T];\check{H}^2)} &\le Q\sum_{i=1}^\infty \lambda_i^4\|u_i\|^2_{C^1[0,T]} \\ &\le Q\sum_{i=1}^\infty \lambda_i^4\Big(\lambda_i^2|u_{0,i}|^2 + |\check{u}_{0,i}|^2 + \lambda_i^{-2}\|f_i\|^2_{H^1(0,T)}\Big) \\ &\le Q\big(\|u_0\|^2_{\check{H}^3} + \|\check{u}_0\|^2_{\check{H}^2} + \|f\|^2_{H^1(0,T;\check{H}^1)}\big).\end{aligned}$$

We thus prove (22) with $m = 1$. We similarly prove (22) with $m = 2$ by (15) with $m = 2$ using the corresponding assumptions on the data.

If k, $\alpha \in C^2[0,T]$, $u_0 \in \check{H}^3$, $\check{u}_0 \in \check{H}^2$ and $f \in H^1(0,T;\check{H}^1) \cap C^1([0,T];L^2)$, we apply the estimate (16) in Theorem 4.1 to conclude that $u \in C^3((0,T];L^2)$. Finally, we arrive at the following estimate

$$\begin{aligned}\|\partial_t^3 u(\cdot,t)\|^2 &\le Q\sum_{i=1}^\infty\big(\lambda_i^3|u_{0,i}| + \lambda_i^2|\check{u}_{0,i}| + \lambda_i\|f_i\|_{H^1(0,T)} + |f_i'| + t^{-\alpha(0)}|\check{u}_{0,i}|\big)^2 \\ &\le Q\big(\|u_0\|^2_{\check{H}^3} + \|\check{u}_0\|^2_{\check{H}^2} + \|f\|^2_{H^1(0,T;\check{H}^1)} + \|f\|^2_{C^1([0,T];L^2)} + t^{-2\alpha(0)}\|\check{u}_0\|^2_{L^2}\big).\end{aligned}$$

We thus prove the estimate (23) and completes the proof of the theorem. ■

5. A finite element approximation and its error estimates

To develop and analyze a finite element approximation to problem (1), we adopt the order reduction approach [42,43] to reformulate problem (1) as the following first-order system

$$\partial_t z + k(t) I_t^{1-\alpha(t)} z + \mathcal{L} u = f, \qquad \partial_t u = z. \tag{26}$$

5.1. Discretization

Define a uniform partition $0 = t_0 < t_1 < \cdots < t_N = T$ of $[0, T]$ for some $N > 0$ with the mesh size τ, and let $u_n := u(\boldsymbol{x}, t_n)$, $z_n := v(\boldsymbol{x}, t_n)$, $k_n := k(t_n)$ and $f_n := f(\boldsymbol{x}, t_n)$. Then for $1 \le n \le N$

$$\begin{aligned}
\partial_t u(\boldsymbol{x}, t_n) &= \delta_\tau u_n + \hat{E}_n := \frac{u_n - u_{n-1}}{\tau} + \frac{1}{\tau}\int_{t_{n-1}}^{t_n} \partial_t^2 u(\boldsymbol{x}, t)(t - t_{n-1}) dt, \\
\partial_t z(\boldsymbol{x}, t_n) &= \delta_\tau z_n + E_n := \frac{z_n - z_{n-1}}{\tau} + \frac{1}{\tau}\int_{t_{n-1}}^{t_n} \partial_t^2 z(\boldsymbol{x}, t)(t - t_{n-1}) dt, \\
I_t^{1-\alpha(t_n)} z(\boldsymbol{x}, t_n) &= \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \frac{z(\boldsymbol{x}, s) ds}{\Gamma(1-\alpha(t_n))(t_n - s)^{\alpha(t_n)}} = I_\tau^{1-\alpha(t_n)} z_n + R_n.
\end{aligned} \tag{27}$$

Here $I_\tau^{1-\alpha(t_n)} z_n$ and R_n are defined by

$$\begin{aligned}
I_\tau^{1-\alpha(t_n)} z_n &:= \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \frac{z_{k-1}\, ds}{\Gamma(1-\alpha(t_n))(t_n - s)^{\alpha(t_n)}} = \sum_{k=1}^n b_{n,k} z_{k-1}, \\
b_{n,k} &:= \frac{1}{\Gamma(1-\alpha(t_n))} \int_{t_{k-1}}^{t_k} \frac{1}{(t_n - s)^{\alpha(t_n)}} ds, \\
R_n &= \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \frac{z(\boldsymbol{x}, s) - z_{k-1}}{\Gamma(1-\alpha(t_n))(t_n - s)^{\alpha(t_n)}} ds.
\end{aligned} \tag{28}$$

Then we perform the inner product of the first equation in (26) and $\chi \in H_0^1(\Omega)$ on Ω to get for $n = 1, 2, \ldots, N$

$$\big(\delta_\tau z_n + k_n I_\tau^{1-\alpha(t_n)} z_n, \chi\big) + \big(K \nabla u_n, \nabla \chi\big) = \big(f_n, \chi\big) - \big(k_n R_n + E_n, \chi\big), \tag{29}$$

$$\delta_\tau u_n = z_n - \hat{E}_n. \tag{30}$$

Let S_h be the space of continuous and piecewise linear functions on Ω with respect to its quasi-uniform partition with the mesh diameter h. Define the Ritz projection operator $\Pi_h : H_0^1(\Omega) \to S_h$ by

$$\big(K \nabla \Pi_h g, \nabla \chi_h\big) = \big(K \nabla g, \nabla \chi_h\big), \quad \forall \chi_h \in S_h. \tag{31}$$

The Ritz projection Π_h has the approximation property [48,51]

$$\|g - \Pi_h g\| \le Q h^2 \|g\|_{H^2}, \quad \forall g \in H^2(\Omega) \cap H_0^1(\Omega). \tag{32}$$

If Ω and its partition are rectangular, the following superconvergence holds [52]

$$\|\nabla(\Pi_h g - g)\|_{\hat{L}^2} := \bigg(\sum_{\boldsymbol{i}} \big|\nabla(\Pi_h g - g)(\boldsymbol{x}_{\boldsymbol{i}-\boldsymbol{e}/2})\big|^2 h^d\bigg)^{\frac{1}{2}} \le Q h^2 \|g\|_{H^3}. \tag{33}$$

Here $\boldsymbol{e} := (1, 1, \ldots, 1)$ and $\boldsymbol{i} - \boldsymbol{e}/2$ refers to the cell center for every cell in Ω.

We also define a discrete operator $\mathcal{L}_h : S_h \to S_h$ such that for any $\zeta \in S_h$

$$(\mathcal{L}_h \zeta, \chi_h) = (K \nabla \zeta, \nabla \chi_h), \quad \forall \chi_h \in S_h. \tag{34}$$

We drop the local truncation errors in (29)–(30) to obtain the finite element discretization of problem (26): find U_n, $Z_n \in S_h$ such that for $1 \le n \le N$

$$\big(\delta_\tau Z_n + k_n I_\tau^{1-\alpha(t_n)} Z_n, \chi_h\big) + \big(K \nabla U_n, \nabla \chi_h\big) = \big(f_n, \chi_h\big), \quad \forall \chi_h \in S_h; \tag{35}$$

$$\delta_\tau U_n = Z_n, \qquad U_0 := \Pi_h u_0, \qquad Z_0 := \check{u}_0. \tag{36}$$

5.2. Auxiliary estimates

We bound temporal truncation errors E_n, $\hat{E}_n$, R_n and spatial truncation errors in terms of $\eta(\boldsymbol{x},t) := z(\boldsymbol{x},t) - \Pi_h z(\boldsymbol{x},t)$ and $\hat{\eta}(\boldsymbol{x},t) := u(\boldsymbol{x},t) - \Pi_h u(\boldsymbol{x},t)$.

Theorem 5.1. *Suppose $k, \alpha \in C^2[0,T]$, $u_0 \in \check{H}^4$, $\check{u}_0 \in \check{H}^3$ and $f \in H^1(0,T;\check{H}^2) \cap C^1([0,T];L^2)$. Then the following estimates hold*

$$\|\hat{E}\|_{\hat{L}^\infty(L^2)} + \|\nabla \hat{E}\|_{\hat{L}^\infty(L^2)} + \|R\|_{\hat{L}^\infty(L^2)} \le QM(\tau + h^2),$$
$$\|\eta\|_{\hat{L}^\infty(L^2)} + \|\delta_\tau \eta\|_{\hat{L}^\infty(L^2)} + \|\delta_\tau \hat{\eta}\|_{\hat{L}^\infty(L^2)} + \|I_\tau^{1-\alpha}\eta\|_{\hat{L}^\infty(L^2)} \le QM(\tau + h^2),$$
$$\|E\|_{\hat{L}^\infty(L^2)} \le QM\tau^{1-\alpha(0)}, \qquad \|E\|_{\hat{L}^1(L^2)} := \tau \sum_{k=1}^N \|E_k\| \le QM\tau.$$

Here $M := \|u_0\|_{\check{H}^4} + \|\check{u}_0\|_{\check{H}^3} + \|f\|_{H^1(0,T;\check{H}^2)} + \|f\|_{C^1([0,T];L^2)}$, $\|R\|_{\hat{L}^\infty(L^2)} := \max_{1\le n\le N} \|R_n\|$, and the constant Q is independent of n, u_0, $\check{u}_0$, or f.

Proof. We use Theorem 4.2 to bound E and $\hat{E}$ in (27) by

$$\begin{aligned}
\|\hat{E}\|_{\hat{L}^\infty(L^2)} &\le \max_{1\le n\le N} \int_{t_{n-1}}^{t_n} \|\partial_t^2 u(\cdot,t)\| dt \le Q\|u\|_{C^2([0,T];L^2)}\tau \le QM\tau,\\
\|E\|_{\hat{L}^\infty(L^2)} &\le \max_{1\le n\le N} \int_{t_{n-1}}^{t_n} \|\partial_t^2 z(\cdot,t)\| dt \le QM \int_{t_{n-1}}^{t_n} t^{-\alpha(0)} dt\\
&\le QM \max_{1\le n\le N} \big(t_n^{1-\alpha(0)} - t_{n-1}^{1-\alpha(0)}\big) \le QM\tau^{1-\alpha(0)},\\
\|E\|_{\hat{L}^1(L^2)} &\le \tau \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \|\partial_t^2 z(\cdot,t)\| dt \le \tau \|\partial_t^2 z(\cdot,t)\|_{L^1(0,T;L^2)} \le QM\tau.
\end{aligned} \tag{37}$$

We may bound $\nabla \hat{E}$ similarly. We apply Theorem 4.2 to bound R by

$$\begin{aligned}
\|R\|_{\hat{L}^\infty(L^2)} &\le Q \max_{1\le n\le N} \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \frac{\int_{t_{k-1}}^s \|\partial_s z(\cdot,s)\| ds}{(t_n - s)^{\alpha(t_n)}} ds\\
&\le Q\tau \|z\|_{C^1([0,T];L^2)} \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \frac{1}{(t_n - s)^{\alpha(t_n)}} ds \le QM\tau.
\end{aligned}$$

We finally use Theorem 4.2 and (32) to obtain

$$\begin{aligned}
\|\delta_\tau \eta\|_{\hat{L}^\infty(L^2)} &= \max_{1\le n\le N} \frac{1}{\tau} \left\| \int_{t_{n-1}}^{t_n} \partial_t \eta dt \right\| \le \frac{1}{\tau} \max_{1\le n\le N} \int_{t_{n-1}}^{t_n} \|\Pi_h \partial_t z - \partial_t z\| dt\\
&\le Qh^2 \|z\|_{C^1([0,T];H^2)} \le QMh^2,\\
\left\|I_\tau^{1-\alpha}\eta\right\|_{\hat{L}^\infty(L^2)} &\le Q \max_{1\le n\le N} \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \frac{\|\eta_k\| ds}{(t_n - s)^{\alpha(t_n)}}\\
&\le Qh^2 \|z\|_{C([0,T];H^2)} \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \frac{ds}{(t_n - s)^{\alpha(t_n)}} \le QMh^2.
\end{aligned}$$

We bound η and $\delta_\tau \hat{\eta}$ similarly and thus complete the proof. ■

We prove the following lemma to be used in the subsequent estimates.

Lemma 5.2. *Suppose the non-negative sequences $\{\rho_n\}_{n=1}^N$ and $\{\sigma_n\}_{n=1}^N$ satisfy*

$$\sigma_n^2 \le M_0 + \tau \sum_{k=1}^{n-1} \rho_k \sigma_k, \quad 1 \le n \le N; \qquad \sum_{k=1}^N \rho_k \le M_1 \tag{38}$$

for some non-negative constants M_0 and M_1. Then $\{\sigma_n\}_{n=1}^N$ satisfy the estimate

$$\sigma_n \le \sqrt{M_0} + M_1\tau, \quad 1 \le n \le N.$$

Proof. Setting $n = 1$ in (38) yields $\sigma_n \le \sqrt{M_0} \le \sqrt{M_0} + M_1\tau$. Suppose the following estimates hold

$$\sigma_k \le \sqrt{M_0} + M_1\tau, \quad 1 \le k \le n-1. \tag{39}$$

We plug (39) into the first equation in (38) and invoke the second estimate in (38) to obtain

$$\begin{aligned}\sigma_n^2 &\le M_0 + (\sqrt{M_0} + M_1\tau)\tau\sum_{k=1}^{n-1}\rho_k \le M_0 + (\sqrt{M_0} + M_1\tau)M_1\tau \\ &= M_0 + \sqrt{M_0}M_1\tau + M_1^2\tau^2 \le (\sqrt{M_0} + M_1\tau)^2.\end{aligned}$$

Thus, (39) holds for $k = n$ and so for any $1 \le k \le N$ by induction. ■

5.3. Analysis of the scheme

Theorem 5.3. *Suppose $k, \alpha \in C^2[0,T]$, $u_0 \in \check{H}^4$, $\check{u}_0 \in \check{H}^3$, and $f \in H^1(0,T;\check{H}^2) \cap C^1([0,T];L^2)$. The following error estimate holds for $0 < \tau \le \tau_0$ and for some $\tau_0 > 0$*

$$\|u - U\|_{\hat{L}^\infty(L^2)} + \|\partial_t u - Z\|_{\hat{L}^\infty(L^2)} \le QM(\tau + h^2). \tag{40}$$

Here M is given in Theorem 5.1 and Q is independent of u_0, $\check{u}_0$, f, τ, or h.

Further, if $u \in L^\infty(0,T;H^3)$ and Ω and its partition are rectangular, the following superconvergence holds

$$\|\nabla(u - U)\|_{\hat{L}^\infty(\hat{L}^2)} \le QM(\tau + h^2). \tag{41}$$

Remark 5.4. By similar techniques in Theorem 4.1, we can prove from (18) that $|v|$ can be bounded by the data $|v_0| + \lambda^{-1}|\check{v}_0| + \lambda^{-1}\|g\|_{L^1(0,T)}$. Consequently, the assumption $u \in L^\infty(0,T;H^3)$ in this theorem can be fulfilled by imposing $u_0 \in \check{H}^3$, $\check{u}_0 \in \check{H}^2$ and $f \in L^2(0,T;\check{H}^2)$, and the corresponding norms are included in the constant M.

Proof. We decompose $u_n - U_n = \hat{\xi}_n + \hat{\eta}_n$ with $\hat{\xi}_n := \Pi_h u_n - U_n \in S_h$ and $z_n - Z_n = \xi_n + \eta_n$ with $\xi_n := \Pi_h z_n - Z_n \in S_h$. We take $\chi = \chi_h = \xi_n$ in (29) and (35), subtract (35) from (29), and use (31) to write the resulting error equation in terms of $\hat{\xi}$, $\hat{\eta}$, ξ and η as follows

$$\begin{aligned}&\big(\delta_\tau\xi_n, \xi_n\big) + \big(K\nabla\hat{\xi}_n, \nabla\xi_n\big) \\ &\quad = -\big(k_n I_\tau^{1-\alpha(t_n)}(\xi_n + \eta_n), \xi_n\big) - \big(\delta_\tau\eta_n, \xi_n\big) - \big(k_n R_n + E_n, \xi_n\big).\end{aligned} \tag{42}$$

We subtract scheme (36) from reference equation (30) to obtain

$$\hat{\xi}_n = \hat{\xi}_{n-1} + \tau\big(-\delta_\tau\hat{\eta}_n + \xi_n + \eta_n - \hat{E}_n\big). \tag{43}$$

We multiply (43) by $\mathcal{L}_h\hat{\xi}_n$, integrate the resulting equation on Ω, and use Ritz projection (31) and (34) to arrive at the following equation

$$(K\nabla\hat{\xi}_n, \nabla\hat{\xi}_n) - \tau(K\nabla\xi_n, \nabla\hat{\xi}_n) = (K\nabla\hat{\xi}_{n-1}, \nabla\hat{\xi}_n) - \tau(K\nabla\hat{E}_n, \nabla\hat{\xi}_n). \tag{44}$$

We sum (42) multiplied by τ and (44) to get the following equation

$$\begin{aligned}&\|\xi_n\|^2 + (K\nabla\hat{\xi}_n, \nabla\hat{\xi}_n) \\ &\quad = (\xi_{n-1}, \xi_n) + (K\nabla\hat{\xi}_{n-1}, \nabla\hat{\xi}_n) - \tau(K\nabla\hat{E}_n, \nabla\hat{\xi}_n) \\ &\qquad -\tau\big(k_n I_\tau^{1-\alpha(t_n)}(\xi_n + \eta_n), \xi_n\big) - \tau\big(\delta_\tau\eta_n, \xi_n\big) - \tau\big(k_n R_n + E_n, \xi_n\big).\end{aligned}$$

We use Cauchy inequality and the geometric–arithmetic mean inequality to cancel $\|\xi_n\|^2/2$ and $\|\sqrt{K}\nabla\hat{\xi}_n\|^2/2$ on both sides and multiply the resulting inequality by 2 to obtain

$$\begin{aligned}\|\xi_n\|^2 + \|\sqrt{K}\nabla\hat{\xi}_n\|^2 \le \|\xi_{n-1}\|^2 + \|\sqrt{K}\nabla\hat{\xi}_{n-1}\|^2 + \tau\Big(&\|k_n I_\tau^{1-\alpha(t_n)}\xi_n\|^2 \\ &+4\|\xi_n\|^2 + \|\sqrt{K}\nabla\hat{\xi}_n\|^2 + \|\delta_\tau\eta_n + k_n I_\tau^{1-\alpha(t_n)}\eta_n\|^2 \\ &+\|\sqrt{K}\nabla\hat{E}_n\|^2 + \|k_n R_n\|^2 + \|E_n\|\|\xi_n\|\Big).\end{aligned} \tag{45}$$

Note that by the second estimate in (37), $\|E_n\|$ only has a suboptimal-order error estimate. Thus, if we treat the last term on the right side as other terms, we will end up with a suboptimal-order error estimate of the scheme, which does not fully reflect the optimal-order convergence rate observed numerically.

Therefore, we have to carry out the error estimate in an alternative manner. We sum the inequality (45) from $n=1$ to n_* for $1 \le n_* \le N$, cancel the like terms, and use the fact that $\xi_0 = \hat{\xi}_0 \equiv 0$ to get

$$\begin{aligned} \|\xi_{n_*}\|^2 + \|\sqrt{K}\nabla\hat{\xi}_{n_*}\|^2 \le \tau \sum_{n=1}^{n_*} \big(& \|k_n I_\tau^{1-\alpha(t_n)}\xi_n\|^2 + 4\|\xi_n\|^2 + \|\sqrt{K}\nabla\hat{\xi}_n\|^2 \\ & + \|\delta_\tau \eta_n + k_n I_\tau^{1-\alpha(t_n)}\eta_n\|^2 + \|\sqrt{K}\nabla\hat{E}_n\|^2 \\ & + \|k_n R_n\|^2 + \|E_n\|\|\xi_n\|\big). \end{aligned} \tag{46}$$

We estimate the right side of (46) term by term. We use (27)–(28) to bound

$$\begin{aligned} &\sum_{k=1}^{n} b_{n,k} = \int_0^{t_n} \frac{ds}{\Gamma(1-\alpha(t_n))(t_n-s)^{\alpha(t_n)}} = \frac{t_n^{1-\alpha(t_n)}}{\Gamma(2-\alpha(t_n))} \le Q, \\ &\sum_{n=k}^{n_*} b_{n,k} \le Q \sum_{n=k}^{n_*} \int_{t_{k-1}}^{t_k} \frac{ds}{(t_n-s)^{\alpha^*}} \\ &\quad = \frac{Q}{1-\alpha^*} \sum_{n=k}^{n_*} \big[(t_n-t_{k-1})^{1-\alpha^*} - (t_n-t_k)^{1-\alpha^*}\big] \le Q. \end{aligned}$$

We use the preceding estimates, (28), Cauchy inequality and interchange the order of summation to bound the first term on the right side of (46) by

$$\begin{aligned} &\tau \sum_{n=1}^{n_*} \|k_n I_\tau^{1-\alpha(t_n)}\xi_n\|^2 \le \tau \|k\|_{C[0,T]}^2 \sum_{n=1}^{n_*} \Big(\sum_{k=1}^{n} b_{n,k}\|\xi_{k-1}\| \Big)^2 \\ &\quad \le Q\tau \sum_{n=1}^{n_*} \sum_{k=1}^{n} b_{n,k}\|\xi_{k-1}\|^2 \sum_{k=1}^{n} b_{n,k} \le Q\tau \sum_{k=1}^{n_*} \|\xi_{k-1}\|^2 \sum_{n=k}^{n_*} b_{n,k} \le Q\tau \sum_{k=1}^{n_*} \|\xi_{k-1}\|^2. \end{aligned}$$

We use Theorem 5.1 to bound all the remaining terms except for the last on the right side of (46) by $QM^2(\tau^2+h^4)$. We incorporate the proceeding two estimates to rewrite (46) for $1 \le n_* \le N$ as

$$\begin{aligned} \|\xi_{n_*}\|^2 + \|\sqrt{K}\nabla\hat{\xi}_{n_*}\|^2 \le Q_1 \tau & \sum_{n=1}^{n_*} \big(\|\xi_n\|^2 + \|\sqrt{K}\nabla\hat{\xi}_n\|^2\big) \\ & + Q_1 M^2(\tau^2+h^4) + \tau \sum_{n=1}^{n_*} \|E_n\|\|\xi_n\|. \end{aligned} \tag{47}$$

We choose $\tau > 0$ sufficiently small such that $Q_1\tau \le 1/2$ and apply Gronwall's inequality to (47) to obtain that for $1 \le n_* \le N$

$$\|\xi_{n_*}\|^2 + \|\sqrt{K}\nabla\hat{\xi}_{n_*}\|^2 \le Q_2 M^2(\tau^2+h^4) + Q_2\tau \sum_{n=1}^{n_*} \|E_n\|\|\xi_n\|. \tag{48}$$

We use Theorem 5.1 and the geometric–arithmetic mean inequality to get

$$\tau\|E_{n_*}\|\|\xi_{n_*}\| \le \frac{\tau^{2\alpha(0)}\|E_{n_*}\|^2 + \tau^{2(1-\alpha(0))}\|\xi_{n_*}\|^2}{2} \le QM^2\tau^2 + \frac{\tau^{2(1-\alpha(0))}\|\xi_{n_*}\|^2}{2}.$$

We choose τ sufficiently small such that $Q_2\tau^{2(1-\alpha(0))} \le 1$ to rewrite (48)

$$\|\xi_{n_*}\|^2 \le Q_3 M^2(\tau^2+h^4) + Q_3\tau \sum_{n=1}^{n_*-1} \|E_n\|\|\xi_n\|, \quad 1 \le n_* \le N. \tag{49}$$

By Theorem 5.1, we have

$$\sum_{k=1}^{N} \|E_k\| \le Q_4 M.$$

We apply Lemma 5.2 to (49) to conclude that

$$\|\xi_n\| \le \sqrt{Q_3 M^2(\tau^2+h^4)} + Q_3 Q_4 M\tau \le QM(\tau+h^2), \quad 1 \le n \le N. \tag{50}$$

To bound $\hat{\xi}_n$, we use (43) to find

$$\|\hat{\xi}_n\| \le \|\hat{\xi}_{n-1}\| + \tau\big(\|\delta_\tau\hat{\eta}_n\| + \|\xi_n\| + \|\eta_n\| + \|\hat{E}_n\|\big), \quad 1 \le n \le N.$$

Table 1
Accuracy of the discretization for one-dimensional problem of cases (i)–(iii).

τ	1/16	1/24	1/36	1/48	υ
(i)	1.02E−02	6.71E−03	4.40E−03	3.25E−03	1.04
(ii)	1.09E−02	7.34E−03	4.90E−03	3.65E−03	0.99
(iii)	6.18E−03	4.23E−03	2.88E−03	2.17E−03	0.95
h	1/120	1/144	1/180	1/240	ι
(i)	4.35E−04	3.05E−04	1.95E−04	1.06E−04	2.03
(ii)	4.12E−04	2.88E−04	1.84E−04	1.00E−04	2.04
(iii)	4.25E−04	2.98E−04	1.90E−04	1.04E−04	2.04

Table 2
Numerical results of $\|u-U\|_{\hat{L}^\infty(L^2)}$ for the discretization for two-dimensional problem of cases (i)–(iii).

τ	1/32	1/48	1/64	1/80	υ
(i)	4.39E−02	2.91E−02	2.17E−02	1.73E−02	1.01
(ii)	3.87E−02	2.57E−02	1.93E−02	1.54E−02	1.01
(iii)	2.99E−02	2.00E−02	1.51E−02	1.21E−02	0.99
h	1/16	1/24	1/32	1/40	ι
(i)	1.08E−02	4.79E−03	2.69E−03	1.72E−03	2.00
(ii)	9.60E−03	4.26E−03	2.40E−03	1.53E−03	2.00
(iii)	7.59E−03	3.41E−03	1.94E−03	1.25E−03	1.97

We add this equation from $n=1$ to $n_*\le N$ to obtain

$$\|\hat{\hat{\xi}}_{n_*}\| \le \tau \sum_{n=1}^{n_*}\big(\|\delta_\tau \hat{\eta}_n\| + \|\xi_n\| + \|\eta_n\| + \|\hat{E}_n\|\big), \quad 1\le n_*\le N.$$

We combine Theorem 5.1 with (50) to get

$$\|\hat{\hat{\xi}}_{n_*}\| \le QM(\tau + h^2), \quad 1\le n_*\le N. \tag{51}$$

We Combine the estimates (50) and (51) with Theorem 5.1 to prove (40).

We deduce from (48), (50) and Theorem 5.1 that

$$\begin{aligned}\big(\|\xi_{n_*}\| + \|\sqrt{K}\nabla\hat{\xi}_{n_*}\|\big)^2 &\le 2\big(\|\xi_{n_*}\|^2 + \|\sqrt{K}\nabla\hat{\xi}_{n_*}\|^2\big)\\ &\le 2Q_2M^2(\tau^2+h^4) + 2Q_2\tau\sum_{n=1}^{n_*}\|E_n\|\|\xi_n\|\\ &\le 2Q_2M^2(\tau^2+h^4) + 2QM^2\tau(\tau+h^2) \le QM^2(\tau^2+h^4),\end{aligned}$$

from which we conclude that

$$\sqrt{K_*}\|\nabla\hat{\xi}_{n_*}\| \le \|\sqrt{K}\nabla\hat{\xi}_{n_*}\| \le QM(\tau+h^2), \quad 1\le n\le N. \tag{52}$$

We combine (52) with (33) to complete the proof of (41). ∎

6. Numerical experiments

We carry numerical experiments to substantiate the theoretical results.

6.1. The approximation to model (1) in one space dimension

In the numerical example runs the data are as follows: $\Omega=(0,1)$, $T=k(t)=f(\boldsymbol{x},t)\equiv 1$, $K=0.01$, $u_0(x)=\sin(\pi x)$, $\check{u}_0=\sin(\pi x)$, and

$$\text{(i) } \alpha(t)=0.4t+0.2, \quad \text{(ii) } \alpha(t)=0.5+0.2\sin(\pi t), \quad \text{(iii) } \alpha(t)=0.8\exp(-t). \tag{53}$$

As model (1) with $f\equiv 1$ could not be solved analytically, we use the numerical solution U_{ref} under $(\tau_f,h_f)=(1/720,1/360)$ as the reference solution to test the temporal convergence rate υ and $(\tau_f,h_f)=(1/720,1/720)$ for the spatial convergence rate ι. When measuring υ, we adopt the same mesh size for h as used for the reference solution. We similarly measure ι. We present the numerical results of $\|U_{ref}-U\|_{\hat{L}^\infty(L^2)}$ in Table 1 and observe second-order accuracy on h and first-order convergence on τ as proved in Theorem 5.3.

Table 3
Numerical results of $\|\partial_t u - Z\|_{\hat{L}^\infty(L^2)}$ for the discretization for two-dimensional problem of cases (i)–(iii).

τ	1/32	1/48	1/64	1/80	υ
(i)	3.79E−02	2.51E−02	1.88E−02	1.50E−02	1.01
(ii)	2.84E−02	1.88E−02	1.40E−02	1.12E−02	1.02
(iii)	1.67E−02	1.11E−02	8.36E−03	6.69E−03	1.00
h	1/16	1/24	1/32	1/40	ι
(i)	9.26E−03	4.09E−03	2.30E−03	1.47E−03	2.01
(ii)	6.88E−03	3.03E−03	1.70E−03	1.08E−03	2.02
(iii)	4.12E−03	1.84E−03	1.04E−03	6.68E−04	1.98

Table 4
Numerical results of $\|\nabla(u - U)\|_{\hat{L}^\infty(\hat{L}^2)}$ for the discretization for two-dimensional problem of cases (i)–(iii).

τ	1/32	1/48	1/64	1/80	υ
(i)	1.91E−01	1.25E−01	9.28E−02	7.34E−02	1.04
(ii)	1.68E−01	1.11E−01	8.19E−02	6.47E−02	1.04
(iii)	1.29E−01	8.52E−02	6.32E−02	5.00E−02	1.03
h	1/16	1/24	1/32	1/40	ι
(i)	3.34E−02	1.49E−02	8.36E−03	5.35E−03	2.00
(ii)	2.81E−02	1.25E−02	7.07E−03	4.53E−03	2.00
(iii)	1.92E−02	8.78E−03	5.02E−03	3.25E−03	1.94

6.2. The approximation to model (1) in two space dimensions

We let $\Omega = (0, 1)^2$, $T = k(t) \equiv 1$, $\boldsymbol{K} = \text{diag}(0.01, 0.01)$ and let $\alpha(t)$ be given by (53). According to Theorem 4.2, the solution is chosen to be $u(\boldsymbol{x}, t) = t^{3-\alpha(0)} \sin(\pi x_1) \sin(\pi x_2)$, and f is calculated accordingly. Mesh size of $h = 1/32$ is used to measure the temporal convergence rate υ, while $\tau = 2h^2$ is used to measure the spatial convergence rate ι. We present the numerical results of $\|u - U\|_{\hat{L}^\infty(L^2)}$, $\|\partial_t u - Z\|_{\hat{L}^\infty(L^2)}$ and $\|\nabla(u - U)\|_{\hat{L}^\infty(\hat{L}^2)}$ in Tables 2–4, which again substantiate the second-order accuracy of ι and first-order convergence of υ as proved in Theorem 5.3.

CRediT authorship contribution statement

Xiangcheng Zheng: Methodology, Formal analysis, Writing - original draft, Funding acquisition. **Hong Wang:** Conceptualization, Methodology, Writing - review & editing, Supervision, Funding acquisition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The authors would like to express their most sincere thanks to the referees for their very helpful comments and suggestions, which greatly improved the quality of this paper. This work was partly funded by ARO MURI, USA Grant W911NF-15-1-0562, by the National Science Foundation, USA under Grant DMS-2012291, by the International Postdoctoral Exchange Fellowship Program (Talent-Introduction Program), USA YJ20210019, and by the China Postdoctoral Science Foundation 2021TQ0017.

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